Universität Bielefeld

SFB 1283

Analysis of nonlocal and nonsmooth models

25 – 29 March 2019

Bielefeld University

Lecture Room: V2-210/216

This workshop is part of the DFG-funded CRC 1283

Taming uncertainty and profiting from randomness and low regularity in analysis, stochastics and their applications

at Bielefeld University

Organisers: Michael Hinz and Moritz Kassmann

`https://www.sfb1283.uni-bielefeld.de/2019_NNSSA/`

Schedule

	Monday	Tuesday	Wednesday	Thursday	Friday
08:45–09:30	Registration				
09:30–10:10	K. Bogdan	A. Björn	J. Wang	M. Bonforte	E. Jakobsen
10:15–10:45			Tee and Coffee		
10:45–10:25	V. Wagner	J. Lehrbäck	M. Cozzi	T. Ghosh	B. Dyda
10:30–12:10	F. Cipriani	K. Pietruska-Pałuba	G. Palatucci	T. Mengesha	G. Karch
12:15–14:00			Lunch		
14:00–14:40	T. Grzywny	L. Lafleche	A. Rozanova-Pierrat	J. Scott	
14:45–15:25	A. Kulik	POSTER SESSION	M.R. Lancia	J. Chaker	
15:30–16:00			Tee and Coffee		

Schedule: Monday March 25

Lecture Room: **V2-210/216**

08:45 – 09:30 **Registration**

09:30 – 10:10 **Krzysztof Bogdan**
Semilinear Dirichlet problem for the fractional Laplacian

10:15 – 10:45 **Tee and Coffee**

10:45 – 11:25 **Vanja Wagner**
Nonlocal quadratic forms with visibility constraint

11:30 – 12:10 **Fabio Cipriani**
How to hear the shape of a drum

12:15 – 14:00 **Lunch**

14:00 – 14:40 **Tomasz Grzywny**
Dirichlet problem for nonlocal operators with kernels of variable orders

14:45 – 15:25 **Alexei Kulik**
Regression approximation for locally α-stable Lévy-type processes, with applications to statistics and simulation

15:30 – 16:00 **Tee and Coffee**

Schedule: Tuesday March 26

Lecture Room: **V2-210/216**

09:30 – 10:10 **Anders Björn**
Sobolev spaces and p-harmonic functions on metric spaces

10:15 – 10:45 **Tee and Coffee**

10:45 – 11:25 **Juha Lehrbäck**
Geometric conditions for fractional Hardy and Hardy-Sobolev inequalities

11:30 – 12:10 **Katarzyna Pietruska-Pałuba**
Integrated density of states for nonlocal Schrödinger operators on the Sierpiński gasket

12:15 – 14:00 **Lunch**

14:00 – 14:40 **Laurent Lafleche**
Convergence to Equilibrium for the Fractional Fokker-Planck Equation

14:45 – 15:25 **POSTER SESSION**

15:30 – 16:00 **Tee and Coffee**

Schedule: Wednesday March 27

Lecture Room: **V2-210/216**

09:30 – 10:10 **Jian Wang**
Periodic homogenization of non-symmetric Lévy-type processes

10:15 – 10:45 **Tee and Coffee**

10:45 – 11:25 **Matteo Cozzi**
Rigidity results for nonlocal minimal graphs

11:30 – 12:10 **Giampiero Palatucci**
Analysis of nonlocal and nonsmooth models

12:15 – 14:00 **Lunch**

14:00 – 14:40 **Anna Rozanova-Pierrat**
Dirichlet-to-Neuman operator on d-sets

14:45 – 15:25 **Maria Rosaria Lancia**
Nonlocal heat transfer across irregular interfaces

15:30 – 16:00 **Tee and Coffee**

Schedule: Thursday March 28

Lecture Room: **V2-210/216**

09:30 – 10:10 **Matteo Bonforte**
Nonlinear and Nonlocal Degenerate Diffusions on Bounded Domains

10:15 – 10:45 **Tee and Coffee**

10:45 – 11:25 **Tuhin Ghosh**
The Fractional Calderon problem

11:30 – 12:10 **Tadele Mengesha**
Solvability of a system of nonlocal equations related to peridynamics

12:15 – 14:00 **Lunch**

14:00 – 14:40 **James Scott**
New Characterizations of Sobolev and Potential Spaces

14:45 – 15:25 **Jamil Chaker**
Nonlocal Operators with singular anisotropic kernels

15:30 – 16:00 **Tee and Coffee**

Schedule: Friday March 29

Lecture Room: **V2-210/216**

09:30 – 10:10 **Espen Jakobsen**
The Liouville theorem and linear operators satisfying the max principle: classification in the constant coefficient case

10:15 – 10:45 **Tee and Coffee**

10:45 – 11:25 **Bartłomiej Dyda**
Muckenhoupt A_p-properties of distance functions and applications to Hardy-Sobolev -type inequalities

11:30 – 12:10 **Grzegorz Karch**
Stable discontinuous patterns in reaction-"diffusion" models

12:15 – 14:00 **Lunch**

Abstracts

Anders Björn (Linköping University)

Sobolev spaces and p-harmonic functions on metric spaces

A Borel function $g: X \to [0,\infty]$ is an *upper gradient* of u if

$$|u(\gamma(0)) - u(\gamma(l_\gamma))| \le \int_\gamma g ds$$

for every rectifiable curve γ in the metric space X. The first-order (so-called Newtonian) Sobolev space $N^{1,p}(X)$ can then be defined using the norm

$$\|u\|_{N^{1,p}} = \inf_g \left(\int_X (|u|^p + g^p) d\mu \right)^{1/p},$$

where the infimum is over all upper gradients g of u. I will discuss these Sobolev spaces and how they are used to define p-harmonic functions on metric spaces.

The p-harmonic functions are a nonlinear generalization of the classical harmonic functions. As the upper gradient is only scalar-valued we cannot define p-harmonic functions using an equation in this generality. Instead they are defined as minimizers of the p-energy integral

$$\int g^p d\mu.$$

This definition has some nonlocal aspects that I will try to explain. However, on $\mathbf{R}^n$ it coincides with the usual definition of p-harmonic functions, and is local.

In the last 20 years quite an extensive potential theory has been built for p-harmonic functions on metric spaces, and I will try to convey some of its features. Many of the recent results are new also on unweighted $\mathbf{R}^n$.

I may also discuss quasiminimizers which is a nonlocal notion even on $\mathbf{R}$. A p-harmonic function is a quasiminimizer, but quasiminimizers is a more general notion.

Krzysztof Bogdan (Wrocław University of Science and Technology)

Semilinear Dirichlet problem for the fractional Laplacian

We give a general framework and results on existence, representation, and uniqueness of solutions to the semilinear problem for the fractional Laplacian with Dirichlet conditions on the complement and at the boundary of general open sets.

Matteo Bonforte (Universidad Autónoma de Madrid)

Nonlinear and Nonlocal Degenerate Diffusions on Bounded Domains

We study quantitative properties of nonnegative solutions to a nonlinear and nonlocal diffusion equation posed in a bounded domain, with appropriate homogeneous Dirichlet boundary conditions. The diffusion is driven by a linear operator in a quite general class, that includes the three most common versions of the fractional Laplacians on a bounded domain with zero Dirichlet boundary conditions, as well as many other examples. The nonlinearity is allowed to be degenerate, the prototype being $|u|^{m-1}u$, with $m>1$.

We will shortly present some recent results about existence, uniqueness and a priori estimates for a quite large class of very weak solutions, that we call weak dual solutions. Then we will concentrate on the regularity theory: decay and positivity, boundary behavior, Harnack inequalities, interior and boundary regularity, and asymptotic behavior. All this is done in a quantitative way, based on sharp a priori estimates. Although our focus is on the fractional models, our techniques cover

also the local case $s=1$ and provide new results even in this setting. A surprising instance of this problem is the possible presence of nonmatching powers for the boundary behavior: this unexpected phenomenon is a completely new feature of the nonlocal nonlinear structure of this model, and it is not present in the semilinear elliptic case, for which we will shortly present the most recent results. The above results are contained on a series of recent papers in collaboration with A. Figalli, Y. Sire, X. Ros-Oton and J. L. Vazquez.

Jamil Chaker (University of Chicago)

Nonlocal Operators with singular anisotropic kernels

We study nonlocal operators that generate anisotropic jump processes, such as a jump process that behaves like a stable process in each direction but with a different index of stability. Its generator is the sum of one-dimensional fractional Laplace operators with different orders of differentiability. We study such operators in the general framework of bounded measurable coefficients. The objective of this talk is to provide regularity results for weak solutions to corresponding integro-differential equations.
Joint work with Moritz Kassmann.

Fabio Cipriani (Politecnico di Milano)

How to hear the shape of a drum

In a iconic 1912 paper Hermann Weyl, motivated by problems posed by the physicist H.A. Lorentz about J.H. Jeans's radiation theory, showed that the dimension and the volume of a Euclidean domain may be traced from the asymptotic distribution of the eigenvalues of its Laplace operator. In a as much famous 1966 paper titled "Can one hear the shape of a drum" MarcKac popularized this and related problems connecting geometry and spectrum. Henoticed that the hope to characterize *isometrically*, Euclidean domains or compact Riemannian manifolds by the spectrum of the Laplace operator, is vain: John Milnor had showed in 1964 non isometric 16 dimensional tori sharing a common (discrete) spectrum.
The aim of the talk is to show how to detect *conformal maps* $\gamma\in\hom(U,V)$ between Euclidean domains $U,V\subseteq\mathbb{R}^n$ as those homeomorphisms which transform multipliers $a\in\mathcal{M}(H^{1,2}(V))$ of the Sobolev-Dirichlet spaces $H^{1,2}(V)$ into multipliers $a\circ\gamma\in\mathcal{M}(H^{1,2}(U))$ of $H^{1,2}(U)$ and leave invariant *the fundamental tone*

$$\mu_1(a)=\mu_1(a\circ\gamma)$$

of the Dirichlet integral $\mathcal{D}[b]:=\int_V|\nabla b|^2\cdot dx$ with respect to the energy measures $\Gamma[a]:=|\nabla a|^2\cdot dx$ on V of the multiplier. Related results hold true for *quasiconformal and bounded distortion maps*. The methods involve potential theory of Dirichlet forms (changing of speed measure, multipliers) and the Li-Yau conformal volume of Riemannian manifolds.

Matteo Cozzi (University of Bath)

Rigidity results for nonlocal minimal graphs

Nonlocal minimal surfaces are hypersurfaces of Euclidean space that minimize the fractional perimeter, a geometric functional introduced in 2010 by Caffarelli, Roquejoffre & Savin in connection with phase transition and diffusion problems involving very long-range interactions. In this talk, I will focus on the class of nonlocal minimal surfaces that can be written as graphs over the whole space $\mathbb{R}^n$. I will present some recent advancements on the classification of these minimizers. Such rigidity results will be a consequence of a weak Harnack-type inequality for

non-negative supersolutions of integral equations posed on nonlocal minimal surfaces and on more general hypersurfaces of $\mathbb{R}^{n+1}$ having Euclidean volume growth.
The talk will be based on works done in collaboration with X. Cabré (ICREA & UPC Barcelona), A. Farina (Université de Picardie), and L. Lombardini (Università di Milano).

Bartłomiej Dyda (Wrocław University of Science and Technology)

Muckenhoupt A_p-properties of distance functions and applications to Hardy-Sobolev-type inequalities

Let X be a metric space equipped with a doubling measure. We consider weights $w(x) = \operatorname{dist}(x,E)^{-\alpha}$, where E is a closed set in X and $\alpha \in \mathbb{R}$. We present sharp conditions, based on the Assouad (co)dimension of E, for the inclusion of w in Muckenhoupt's A_p classes of weights, $1 \le p < \infty$. We will also present an application of these results: (global) fractional Hardy-Sobolev inequalities in the setting of metric spaces.
Joint work with Lizaveta Ihnatsyeva, Juha Lehrbäck, Heli Tuominen and Antti V. Vähäkangas.

Tuhin Ghosh (The Hong Kong University of Science and Technology)

The Fractional Calderon problem

We will be discussing the fractional Calderon problem, where one tries to determine an unknown potential in a fractional Schrödinger equation from the exterior measurements of solutions.

Tomasz Grzywny (Wrocław University of Science and Technology)

Dirichlet problem for nonlocal operators with kernels of variable orders

We consider the nonlocal Dirichlet problem

$$\begin{aligned} \mathcal{L}u &= f \quad \text{in } D, \\ u &= g \quad \text{in } D^c, \end{aligned}$$

where $\mathcal{L}$ is an integrodifferential operator of the form

$$\mathcal{L}u(x) = \lim_{\epsilon \to 0} \int_{|y|>\epsilon} (u(x+y) - u(x)) \nu(|y|) \mathrm{d}y,$$

where $\nu : (0,\infty) \mapsto [0;\infty)$ is a non-increasing function satisfying

$$\int_0^\infty \left(1 \wedge r^2\right) \nu(r) r^{d-1} \mathrm{d}r < \infty.$$

We present a classical representation formula for distributional solutions to the above problem and study the question under which assumptions distributional solutions are twice differentiable in the classical sense. Sufficient conditions and counterexamples are provided.

References

[1] K. Bogdan, T. Grzywny, K. Pietruska-Pałuba and A. Rutkowski, Extension theorem for nonlocal operators, arxiv: https://arxiv.org/abs/1710.05880

[2] T. Grzywny, M. Kassmann and Ł. Leżaj, Remarks on the nonlocal Dirichlet problem, arxiv: https://arxiv.org/abs/1807.03676

Espen Jakobsen (Norwegian University of Science and Technology)

The Liouville theorem and linear operators satisfying the max principle: classification in the constant coefficient case

I will present a classification of linear operators $\mathscr{L}$ satisfying the Liouville theorem: Bounded solutions of $\mathscr{L}u=0$ are constant. Our results give necessary and sufficient conditions for all the generators of Levy processes, or in other words, the constant coefficient linear operators satisfying the max principle. Some examples of such generators are the Laplace, fractional Laplace, and discrete finite differences operators. The main novelty is the inclusion of the nonlocal part of such operators. Our proofs are short and natural and differs from most proofs e.g. for the fractional Laplacian. This is joint work with Nathael Alibaud (Besancon, France), Felix del Teso, and Jørgen Endal (both NTNU, Norway).

Grzegorz Karch (University od Wrocław)

Stable discontinuous patterns in reaction-"diffusion" models

I will explain that certain large class of systems of reaction-diffusion equations, either with a local diffusion modeled by Laplacian or with a non-local "diffusion" given by an integral operator, may have stationary solutions with discontinuities of a jump type.
These are results obtained jointly with Anna Marciniak-Czochra from Heidelberg University, Kanako Suzuki from Ibaraki University, and Szymon Cygan from University of Wrocław.

Alexei Kulik (Wrocław University of Science and Technology)

Regression approximation for locally α-stable Lévy-type processes, with applications to statistics and simulation

We study a class of *locally α-stable* Lévy-type processes, which contain both a 'principal' α-stable part (with state dependent jump kernel and the velocity field) and a 'residual' lower order jump part. We construct an approximation in law for such processes by non-linear regressions of the form $\widetilde{X}^x_t = \mathfrak{f}_t(x)+t^{1/\alpha}U^x_t$ with a deterministic *regressor term* $\mathfrak{f}_t(x)$ and α-stable *innovation term* U^x_t. A case study is performed, revealing different types of assumptions which lead to various choices of regressor/innovation terms and various types of the estimates. The assumptions are quite general, cover the *super-critical case* $\alpha<1$, and allow non-symmetry of the Lévy kernel and unboundedness of the velocity field. Several applications to statistics and Monte-Carlo simulation methods are discussed.

Laurent Lafleche (Université Paris Dauphine)

Convergence to Equilibrium for the Fractional Fokker-Planck Equation

In the case of a space homogeneous distribution of particles, the Fokker-Planck equation is nothing but a diffusion equation with a conservative drift. When the drift is created by a confining potential, the rate of convergence is known and depends on the growth of the potential at infinity. In this talk, I will explore the case of a Fokker-Planck equation with fractional diffusion, i.e. $\partial_t f=\Delta^s f+\mathrm{div}(Ef)$ where Δ^s denotes the fractional Laplacian and E is a confining force field with polynomial growth at infinity. This result is motivated by the strong analogy between the fractional Laplacian and the linearized Boltzmann operator, and it can also be proved that such an equation appears as a diffusion limit of kinetic equations with fat tails equilibria. The difficulty, in comparison with the classical Laplacian, comes from the fact that when $s<1/2$, the effect of the drift becomes stronger at small scales, resulting in the loss of the strong regularization properties of the diffusion. Therefore, when the regularization is not strong enough, we use

instead the nonlocal behavior of the fractional Laplacian which implies a gain of positiveness of the solutions to get the convergence to the stationary state.

Maria Rosaria Lancia (Università di Roma Sapienza)

Nonlocal heat transfer across irregular interfaces

We consider a nonlocal heat flow across a highly conductive layer of fractal type. Mathematically it is described by a parabolic quasilinear non local Venttsel problem which is also known as boundary value problem with dynamical boundary conditions.
Fractal layers and fractal boundaries may be of great interest to describe those physical settings in which diffusion phenomena take place in small volumes with large surfaces; for instance, with the increasing miniaturization of electronic chips and increasingly larger heat dissipation rates, better design of cooling systems are necessary. Having in mind the application point of view, it is also important to consider a non local term. Indeed, in the framework of heat transfer problems the nonlocal term accounts for a non constant conductivity which scales in a suitable way. Existence and uniqueness results of the weak solution will be discussed as well the issues concerning the numerical approximation. As we will see, in order to have an optimal rate of convergence it is crucial to prove regularity results for the weak solution in weighted Sobolev spaces. The structure of the non local term, which behaves like a regional fractional Laplacian, will play a key role. Open problems will be discussed.
These results are obtained in collaboration with S.Creo, A.Nazarov, A.Velez-Santiago and P.Vernole.

Juha Lehrbäck (University of Jyväskylä)

Geometric conditions for fractional Hardy and Hardy-Sobolev inequalities

The purpose of this talk is to review various conditions which are sufficient and/or necessary for the validity of fractional Hardy and Hardy-Sobolev type inequalities in Euclidean spaces and in more general metric spaces. These inequalities resemble the usual Sobolev inequalities with respect to an open set Ω, but they also take into account the distance $\operatorname{dist}(x,\partial\Omega)$ to the boundary of Ω. The conditions for their validity can be formulated in several different (but equivalent) ways. Rather simple geometric versions, which also reveal an interesting duality between different cases of the inequalities, are obtained by using the so-called Assouad (co)dimensions.
This talk is based on joint works with Bartłomiej Dyda, Lizaveta Ihnazyeva, Heli Tuominen and Antti Vähäkangas.

Tadele Mengesha (The University of Tennessee, Knoxville)

Solvability of a system of nonlocal equations related to peridynamics

I will present recent results on solvability of a system of equations that is used in a nonlocal model in mechanics. The nonlocal model is made up of a strongly coupled system of integral equations and is a nonlocal analogue of the Navier-Lame' system of classical elasticity.
We will discuss the well-posedness of the system as well as demonstrate optimal local Sobolev regularity of solutions. Connections between the associated nonlocal energy spaces and corresponding classical function spaces will be established. Conditions that imply compact embedding of these spaces in classical spaces will be given.

Giampiero Palatucci (Universita di Parma)

Analysis of nonlocal and nonsmooth models

We present some regularity estimates for viscosity solutions to a class of possible degenerate and singular integro-differential equations whose leading operator switches between two different types of fractional elliptic phases, according to the zero set of a modulating coefficient $a=a(\cdot,\cdot)$. The model case is driven by the following nonlocal double phase operator,

$$\int \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}\mathrm{d}y+\int a(x,y)\frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))}{|x-y|^{n+tq}}\mathrm{d}y,$$

where $q\geq p$ and $a(\cdot,\cdot)\geqq 0$. Our results do also apply for inhomogeneous equations, for very general classes of measurable kernels. By simply assuming the boundedness of the modulating coefficient, we are able to prove that the solutions are Hölder continuous, whereas similar sharp results for the classical local case do require a to be Hölder continuous. To our knowledge, this is the first (regularity) result for nonlocal double phase problems.

References

[1] M. Colombo, G. Mingione: Regularity for Double Phase Variational Problems. *Arch. Rational Mech. Anal.* **215** (2015), 443–496.

[2] M. Colombo, G. Mingione: Bounded Minimisers of Double Phase Variational Integrals. *Arch. Rational Mech. Anal.* **218** (2015), 219–273.

[3] C. De Filippis, G. Palatucci: Hölder regularity for nonlocal double phase equations. *J. Differential Equations.* `https://doi.org/10.1016/j.jde.2019.01.017`

[4] G. Palatucci: The Dirichlet problem for the p-fractional Laplace equation. *Nonlinear Anal.* **177** (2018), 699–732.

Katarzyna Pietruska-Pałuba (University of Warsaw)

Integrated density of states for nonlocal Schrödinger operators on the Sierpiński gasket

For the Schrödinger operator based on the generator of a subordinate Brownian motion on the Sierpiński gasket influenced by an attractive random Poisson potential

$$H^\omega\psi=L\psi+V^\omega\psi,$$

we construct the integrated density of states (IDS). We then examine its asymptotical behaviour near zero (the bottom of the spectrum of the operator considered) and we prove the Lifschitz singularity of the IDS.
The results were obtained jointly with Kamil Kaleta (Wrocław University of Technology).

References

[1] K. Kaleta, K. Pietruska-Pałuba, *Integrated density of states for Poisson-Schrödinger processes on the Sierpiński gasket*, Stochastic Process. Appl. 125 (2015), 1244-1281.

[2] K. Kaleta, K. Pietruska-Pałuba, *Lifschitz singularity for subordinate Brownian motions in presence of the Poissonian potential on the Sierpiński gasket*, Stochastic Process. Appl. 128 (2018) 3897–3939.

Anna Rozanova-Pierrat (École Centrale Paris)

Dirichlet-to-Neuman operator on d-sets

In the framework of the Laplacian transport, described by a Robin boundary value problem in an exterior domain in $\mathbf{R}^n$, we generalize the definition of the Poincaré-Steklov operator to d-set boundaries, $n-2<d<n$, and give its spectral properties to compare to the spectra of the interior domain and also of a truncated domain, considered as an approximation of the exterior case. The well-posedness of the Robin boundary value problems for the truncated and exterior domains is given in the general framework of n-sets. The results are obtained thanks to a generalization of the continuity and compactness properties of the trace and extension operators in Sobolev, Lebesgue and Besov spaces, in particular, by a generalization of the classical Rellich-Kondrachov Theorem of compact embeddings for n and d-sets.

James Scott (The University of Tennessee, Knoxville)

New Characterizations of Sobolev and Potential Spaces

We show that a class of spaces of vector fields whose semi-norms involve the magnitude of "directional" difference quotients is in fact equivalent to the class of fractional Sobolev spaces. The equivalence can be considered a Korn-type characterization of fractional Sobolev spaces. We additionally show that the class of vector-valued Bessel potential spaces can be characterized by a Marcinkiewicz-type integral that that is – pointwise – smaller than the classical Marcinkiewicz integral, and does not resemble other classes of potential-type integrals found in the literature. In applications, these results are used to better understand spaces of vector fields associated to a strongly coupled system of nonlocal equations related to a continuum model of peridynamics. This talk is comprised of joint work with Tadele Mengesha.

Vanja Wagner (University of Zagreb)

Nonlocal quadratic forms with visibility constraint

Given a subset D of the Euclidean space, we study nonlocal quadratic forms that take into account tuples $(x,y)\in D\times D$ if and only if the line segment between x and y is contained in D. In this talk, we discuss regularity of the corresponding Dirichlet form for special classes of jumping kernels and domains D, leading to the existence of a pure jump process on D with, so called, visibility constraint. Furthermore, we study the corresponding Poincaré inequalities in dumbbell shaped domains and show that the forms satisfy a Poincaré inequality with diffusive scaling. Joint work with Moritz Kassmann.

Jian Wang (Fujian Normal University)

Periodic homogenization of non-symmetric Lévy-type processes

We consider the homogenization problem for a Lévy-type operator in a medium with a periodic structure. Under a proper scaling of this operator, we establish the limit behavior of the rescaled operator as the scaling parameter tends to zero. In particular, we can completely characterize the homogenized process associated with the following typical operator

$$Lf(x)=\text{p.v.}\int_{R^d}(f(x+z)-f(x))k(x,z)\left(\frac{1}{|z|^{d+\alpha}}1_{\{|z|\le 1\}}+\frac{1}{|z|^{d+\beta}}1_{\{|z|>1\}}\right)dz,$$

where $k(x,z)$ is uniformly elliptic such that the function $x\mapsto k(x,z)$ is periodic for each $z\in R^d$, and the indexes $\alpha\in(0,2)$ and $\beta\in(0,\infty)$.

Some recommendations for restaurants

(1) **Argentina-Steakhouse** (`https://argentina-steakhouse.de/`)
Argentinian beef at its best

(2) **Brauhaus** (`https://bielefeld.brauhaus-joh-albrecht.de/`)
Home made beer plus German style food

(3) **Kometsu** (`http://www.kometsu.de/index.html`)
Authentic Japanese place for sushi

(4) **KDW** (`http://www.kdw-restaurant.de/index.html`)
Fine Greek cuisine

(5) **Numa** (`http://www.numa.de/`)
Asia meets East-Westphalia

(6) **Wernings Weinstube** (`https://www.wernings-weinstube.de/`)
Some regional dishes plus a good selection of wines

(7) **Sparrenburg** (`https://www.restaurant-sparrenburg.de/`)
German style food at the castle above Bielefeld

(8) **Wilde Kuh/ Wilde Kuh 2** (`https://www.facebook.com/WildeKuhBurger/`)
Excellent “build your own burger" place

(9) **Three sixty** (`http://bielefeld.three-sixty.de/`)
Sports bar with burgers and other snacks

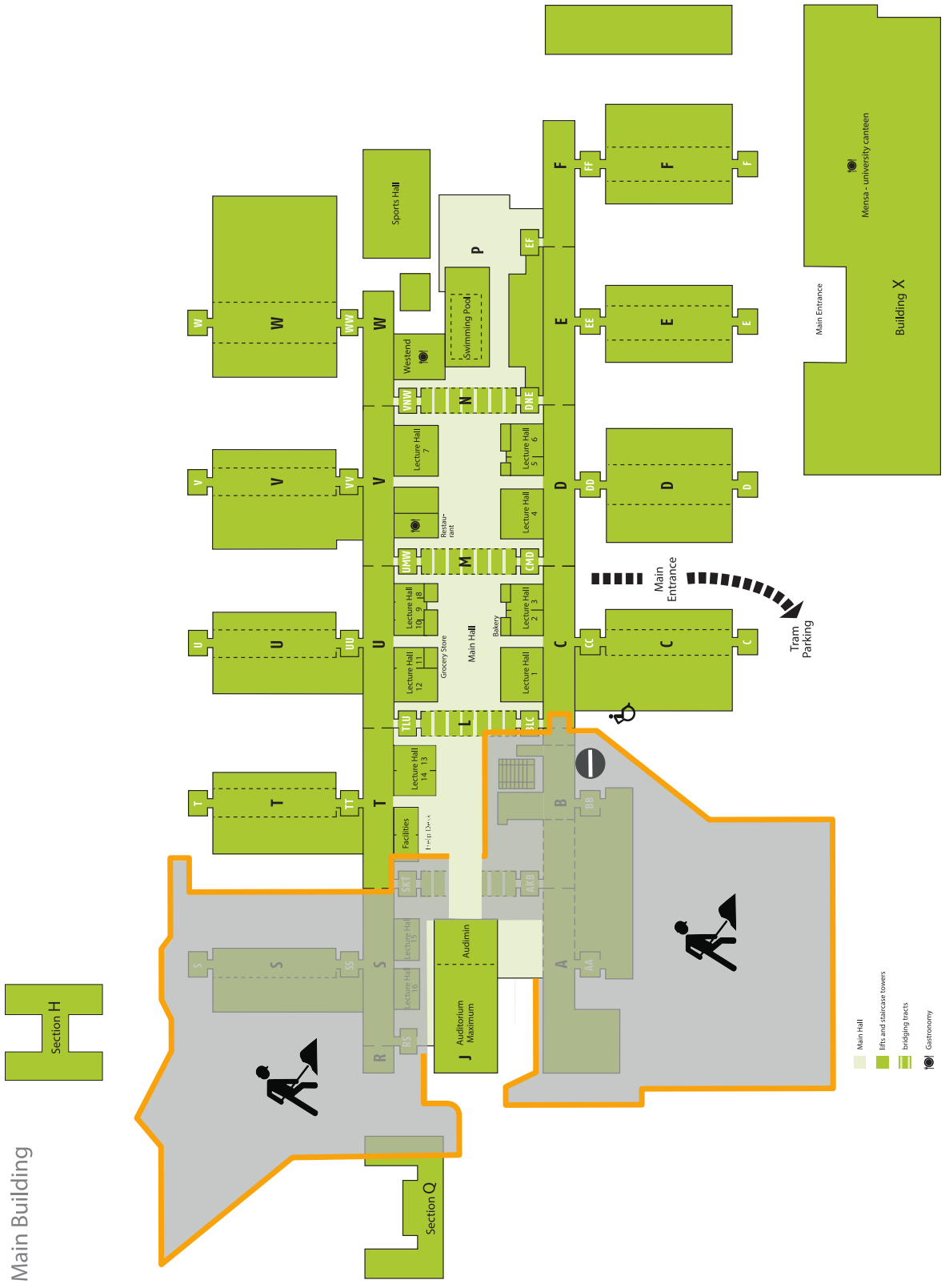
(10) **Jivino** (`http://www.jivino-enoteca.de/`)
Spanish tapas

(11) **Bernstein** (`https://www.the-bernstein.com/`)
Dinner plus cocktails in a fancy rooftop restaurant

Tram map


3
Babenhausen Süd
Voltmannstraße
Koblenzer Straße
Lange Straße
Auf der Hufe
Nordpark
Witte-kind-straße
1
Schildesche
Heidegärten
Kattenkamp
Deciusstraße
Johannesstift
Sudbrackstraße
2
Milse
Schelpmilser Weg
Baumheide
Seidenstickerstraße
Schüco
Ziegelstraße
Finkenstraße
Schillerstraße
Stadtheider Straße
Beckhausstraße
4
Lohmannshof
Wellensiek
Universität
Bültmannshof
Graf-von-Stauffenberg-Straße
Rudolf-Oetker-Halle
Siegfriedplatz
Hauptbahnhof
Jahnplatz
4
Rathaus
August-Schroeder-Straße
Ravensberger Straße
Krankenhaus Mitte
Oststraße
Hartlager Weg
Sieker Mitte
Luther-Kirche
Roggenkamp
Elpke
Gesamtschule Stieghorst
Stieghorst
3
Landgericht
Adenauerplatz
Bethel
Friedrich-List-Straße
Eggeweg
Brackwede Bahnhof
Gaswerkstraße
Normannenstraße
Brackwede Kirche
Windelsbleicher Straße
Rosenhöhe
Sennefriedhof
Senne
1
August-Bebel-Straße
Teutoburger Straße
Mozartstraße
Prießallee
Sieker
2
University
Line 4 - direction
'Lohmannshof'
1 Senne – Jahnplatz – Schildesche
2 Sieker – Jahnplatz – Milse
3 Stieghorst – Jahnplatz – Babenhausen Süd
4 Rathaus – Jahnplatz – Universität – Lohmannshof

Campus map


Main Building
Section H
Section Q
Sports Hall
Swimming Pool
Westend
Mensa - university canteen
Building X
Main Entrance
Tram Parking
Main Hall
Auditorium Maximum
Audimin
Lecture Hall 1
Lecture Hall 2
Lecture Hall 3
Lecture Hall 4
Lecture Hall 5
Lecture Hall 6
Lecture Hall 7
Lecture Hall 8
Lecture Hall 9
Lecture Hall 10
Lecture Hall 11
Lecture Hall 12
Lecture Hall 13
Lecture Hall 14
Lecture Hall 15
Lecture Hall 16
Facilities
Restaurant
Bakery
Grocery Store
Main Hall
lifts and staircase towers
bridging tracts
Gastronomy

Line 4

Rathaus

moBiel

Linie 4 Jahnplatz - Hauptbahnhof - Rudolf-Oetker-Halle - Universität - Lohmannshof

Linie 10 Jahnplatz - Hauptbahnhof - Rudolf-Oetker-Halle - Universität - Lohmannshof

	Montag - Freitag							Samstag						Sonn- u. Feiertag			
Std.	**Minuten**							**Minuten**						**Minuten**			
4	41	56															
5	11	26	41	56													
6	07	17	27	37	47	57		11	26	41	56						
7	07	17	27	37	47	57		11	26	41	56						
8	07	17	27	37	47	57		11	26	41	56			21	41	56	
9	07	17	27	37	47	57		11	26	41	56			11	26	41	56
10	07	17	27	37	47	57		11	26	41	50	56		11	26	41	56
11	07	17	27	37	47	51Ⓢ	57	07	17	27	37	47	57	11	26	41	56
12	07	17	27	37	47	57		07	17	27	37	47	57	11	26	41	56
13	07	17	27	37	47	57		07	17	27	37	47	57	11	26	41	56
14	07	17	27	37	47	57		07	17	27	37	47	57	11	26	41	56
15	07	17	27	37	47	57		07	17	27	37	47	57	11	26	41	56
16	07	17	27	37	47	57		07	17	27	37	47	57	11	26	41	56
17	07	17	27	37	47	57		07	17	27	37	47	57	11	26	41	56
18	07	17	27	37	47	57		07	17	27	37	47	57	11	26	41	56
19	07	17	27	42	56			07	17	27	37	47	57	11	26	41	56
20	11	26	41	56				11	26	41	56			11	26	41	56
21	11	26	41	56				11	26	41	56			11	26	41	56
22	11	26	41	56				11	26	41	56			11	26	41	56
23	11	25	41	55				11	26	41	56			11	25	41	55
0	11	25	41					11	26	41				11	25	41	

Ⓢ : fährt nur an Schultagen, Änderungen möglich

Diesen Fahrplan erhalten Sie in unseren Kundenzentren und unter www.moBiel.de.

Bitte beachten Sie die Sonderfahrpläne am 24.12. und 31.12.

Das Angebot ab 5.00 Uhr samstags, sonntags und an Feiertagen finden Sie in den NachtBus-Fahrplänen.

Nächste Verkaufsagentur:
ServiceCenter moBiel und Kundenzentrum Jahnplatz Nr. 5, Haltestelle Jahnplatz
Infotelefon moBiel, rund um die Uhr (05 21) 51-45 45

gültig ab: 17.06.2018

www.moBiel.de

Line 4

Universität

moBiel

Linie 4 Rudolf-Oetker-Halle - Hauptbahnhof - Jahnplatz - Rathaus

Linie 10 Rudolf-Oetker-Halle - Hauptbahnhof - Jahnplatz - Rathaus - Krankenhaus Mitte - Sieker Mitte - Stieghorst

	Montag - Freitag							Samstag						Sonn- u. Feiertag			
Std.	**Minuten**							**Minuten**						**Minuten**			
5	09	24	39	54													
6	09	19	29	39	49	59		39	54								
7	09	19	29	39	49	59		09	24	39	54						
8	09	19	29	39	49	59		09	24	39	54			39			
9	09	19	29	39	49	59		09	24	39	54			09	24	39	54
10	09	19	29	39	49	59		09	24	39	49	59		09	24	39	54
11	09	19	29	39	49	59		09	19	29	39	49	59	09	24	39	54
12	09	19	29	39	49	59		09	19	29	39	49	59	09	24	39	54
13	05A Ⓢ	09	19	29	39	49	59	09	19	29	39	49	59	09	24	39	54
14	09	19	29	39	49	59		09	19	29	39	49	59	09	24	39	54
15	09	19	29	39	49	59		09	19	29	39	49	59	09	24	39	54
16	09	19	29	39	49	59		09	19	29	39	49	59	09	24	39	54
17	09	19	29	39	49	59		09	19	29	39	49	59	09	24	39	54
18	09	19	29	39	49	59		09	19	29	39	49	59	09	24	39	54
19	09	19	29A	39	54			09	19	29	39	49	59A	09	24	39	54
20	09	24A	39A	54				09	24	39	54			09	24	39	54
21	09	24	39	54				09	24	39	54			09	24	39	54
22	09	24	39	54				09	24	39	54			09	24	39	54
23	09B	24	39C	54				09	09	24	39	54		09A	24	39E	54
0	09C	24	39D	54A				09	24	39A	54A			09E	24	39FE	54A
1	09A							09A						09A			

A : fährt als StadtBahn-Linie 2 nach Sieker

B : fährt freitags und vor Feiertagen als StadtBahn-Linie 4, montags bis donnerstags und an Sonn- und Feiertagen als StadtBahn-Linie 2 nach Sieker

C : fährt freitags und vor Feiertagen als StadtBahn-Linie 4, montags bis donnerstags und an Sonn- und Feiertagen als StadtBahn-Linie 10

D : fährt freitags und vor Feiertagen als StadtBahn-Linie 2 nach Sieker, montags bis donnerstags und sonn- und feiertags als StadtBahn-Linie 10 bis Sieker Mitte

E : StadtBahn-Linie 10

F : fährt nur bis Sieker Mitte

Ⓢ : fährt nur an Schultagen, Änderungen möglich

Diesen Fahrplan erhalten Sie in unseren Kundenzentren und unter www.moBiel.de.

Bitte beachten Sie die Sonderfahrpläne am 24.12. und 31.12.

Das Angebot ab 5.00 Uhr samstags, sonntags und an Feiertagen finden Sie in den NachtBus-Fahrplänen.

gültig ab: 17.06.2018

Infotelefon moBiel, rund um die Uhr (05 21) 51-45 45 **www.moBiel.de**