Nonlocal Equations

Analysis and Numerics

March 17–21, 2025

Bielefeld University

V2-205 (talks)

V2-210/216 (coffee breaks)

UNIVERSITÄT BIELEFELD
Faculty of Mathematics

SFB 1283

Linus Behn (Bielefeld)

De Giorgi's method for systems

We investigate regularity properties of solutions to nonlinear systems. We show that they satisfy a convex hull property. This is the generalization of the maximum principle to the case of systems. Additionally, we show local boundedness of solutions by using a vectorial version of the De Giorgi iteration. These results are obtained for both for local and nonlocal systems.

Joint work with Lars Diening, Simon Nowak, and Toni Scharle.

Verena Bögelein (Salzburg)

Regularity for (s,p)-harmonic functions

We report on higher Sobolev and Hölder regularity results for local weak solutions of the fractional p-Laplace equation of order $s \in (0,1)$ with $1 < p < \infty$. The relevant estimates are stable when the fractional order s reaches 1, and the known Sobolev regularity estimates for weak solutions of the local p-Laplace equation are recovered. As an application we establish Calderòn-Zygmund type estimates at the gradient level for the associated fractional p-Poisson equation.
The talk is based on joint work with Frank Duzaar (Salzburg), Kristian Moring (Salzburg), Naian Liao (Salzburg), Giovanni Molica Bisci (Urbino), and Raffaella Servadei (Urbino).

Matteo Bonforte (Madrid)

Sharp regularity estimates for nonlocal 0-order p-Laplacian evolution problems

We study regularity properties of solutions to nonlinear and nonlocal evolution problems driven by the so-called 0-*order fractional* $p-$*Laplacian* type operators:

$$\partial_t u(x,t) = J_p u(x,t) := \int_{\mathbb{R}^n} J(x-y)|u(y,t)-u(x,t)|^{p-2}(u(y,t)-u(x,t))\,dy\,,$$

where $n \geq 1$, $p > 1$, $J\colon \mathbb{R}^n \to \mathbb{R}$ is a bounded nonnegative function with compact support, $J(0) > 0$ and normalized such that $\|J\|_{L^1(\mathbb{R}^n)} = 1$, but not necessarily smooth. We deal with Cauchy problems on the whole space, and with Dirichlet and Neumann problems on bounded domains. Beside complementing the existing results about existence and uniqueness theory, we focus on sharp regularity results in the whole range $p \in (1,\infty)$. When $p > 2$, we find an unexpected $L^q - L^\infty$ regularization: the surprise comes from the fact that this result is false in the linear case $p = 2$.
We show next that bounded solutions automatically gain higher time regularity, more precisely that $u(x,\cdot) \in C^p_t$. We finally show that solutions preserve the regularity of the initial datum up to certain order, that we conjecture to be optimal (p-derivatives in space). When $p > 1$ is integer we can reach C^∞ regularity (gained in time, preserved in space) and even analyticity in time. The regularity estimates that we obtain are quantitative and constructive (all computable constants), and have a local character, allowing us to show further properties of the solutions: for instance, initial singularities do not move with time. We also study the asymptotic behavior for large times of solutions to Dirichlet and Neumann problems. Our results are new also in the linear case and are sharp when p is integer. We expect them to be optimal for all $p > 1$, supporting this claim with some numerical simulations.

Sun-Sig Byun (Seoul)

Calderon-Zygmund type estimates for local and nonlocal problems

This talk is concerned with an optimal regularity theory with Calderon-Zygmund type estimates of local and nonlocal problems with discontinuous coefficients in bounded domains

Andrea Cianchi (Florence)

Constrained differential operators, fractional integrals, and Sobolev inequalities

Inequalities for Riesz potentials are well-known to be equivalent to Sobolev inequalities of the same order for domain norms "far" from L^1, but to be weaker otherwise. Recent contributions by Van Schaftingen, by Hernandez, Raita and Spector, and by Stolyarov proved that this gap can be filled in Riesz potential inequalities for vector-valued functions in L^1 fulfilling a co-canceling differential condition. This work demonstrates that such a property is not just peculiar to the space L^1. Indeed, under the same differential constraint, a Riesz potential inequality is shown to hold for any domain and target rearrangement-invariant norms that render a Sobolev inequality of the same order true. This is based on a new interpolation inequality, which, via a kind of duality argument, yields a parallel property of Sobolev inequalities for any linear homogeneous elliptic canceling differential operator. Specifically, Sobolev inequalities involving the full gradient of a certain order share the same rearrangement-invariant domain and target spaces as their analogs for any other homogeneous elliptic canceling differential operator of equal order. As a consequence, Riesz potential inequalities under the co-canceling constraint and Sobolev inequalities for homogeneous elliptic canceling differential operators are offered for general families of rearrangement-invariant spaces, such as the Orlicz spaces and the Lorentz-Zygmund spaces. Especially relevant instances of inequalities for domain spaces neighboring L^1 are singled out. This is joint work with D.Breit and D.Spector.

Amiran Gogatishvili (Prague)

Nonlinear interpolation of α-Hölderian mappings with applications to quasilinear PDEs

We present some well-known and new results for identifying some interpolation spaces on nonlinear interpolation of α- Hölderian mappings between normed spaces. We apply these results to obtain some regularity results on the gradient of the weak or entropic-renormalized solution u to the homogeneous Dirichlet problem for the quasilinear equations of the form

$$-\operatorname{div}(\breve{a}(\nabla u)) + V(u) = f,$$

where, Ω is a bounded smooth domain of R^n, V is a nonlinear potential and f belongs to non-standard spaces like Lorentz-Zygmund spaces.

The presentation is based on the following papers:

1. I. Ahmed, A. Fiorenza, M.R. Formica, A. Gogatishvili, A. El Hamidi, J. M. Rakotoson, Quasilinear PDEs, interpolation spaces and Hölderian mappings, Anal. Math. 49 (2023), no. 4, 895–950.

2. I. Ahmed, A. Fiorenza, M.R. Formica, A. Gogatishvili, A. El Hamidi, J. M. Rakotoson,
Applications of Interpolation theory to the regularity of some quasilinear PDEs.. Proceedings
of the International Scientific Online Conference "Algebraic and geometric methods of analysis, AGMA 2024 May 27-30, 2024, Ukraine, Proceedings of the International Geometry Center, 2024, 35 pages.

Florian Grube (Bielefeld)

Exterior value problems for integro-differential operators

In this talk, we discuss two results on exterior value problems for integro-differential operators. In the first part, we study a variational setup to Dirichlet problems involving nonlocal operators like the fractional p-Laplacian in bounded domains. In particular, we answer the question for which exterior data weak solutions in an appropriate Sobolev-like function space exist. This entails appropriate trace and extension theorems. The fractional p-Laplacian converges to a classical differential operator as the order of differentiation increases to two. This phenomenon will be of particular interest to us. In the second part of the talk, we discuss existence, uniqueness, and regularity of distributional solutions to linear exterior-value problems involving $2s$-stable operators with square integrable exterior data. As we will have seen in the first part of the talk, these data are too irregular for the corresponding Dirichlet problem to be solved with variational methods.

Rubing Han (Beijing)

Local discontinuous Galerkin methods for the integral fractional Laplacian

We propose local discontinuous Galerkin (LDG) and minimal dissipation LDG (md-LDG) methods for solving the integral fractional Laplacian problem. Using the Riesz potential, we reformulate the problem in a 3-field mixed form. By the error equation, we establish a priori error estimates for the LDG scheme on polygonal meshes. For triangular meshes, we prove better a priori error estimates for both LDG and md-LDG schemes by utilizing continuous interpolation and special projection. Combining with the regularity theories, we demonstrate the convergence rates of these schemes on both quasi-uniform and graded meshes. Numerical experiments are provided to validate the effectiveness of our theoretical results.

Kyeongbae Kim (Seoul)

Pointwise gradient estimates for nonlocal equations

We discuss potential estimates of non-homogeneous nonlinear nonlocal equations.
We first recall previously known results about pointwise gradient estimates for solutions to the nonlinear generalization of Poisson's equation.
Then, we present that the gradient bounds of solutions to nonlinear nonlocal equations can be pointwise estimated by the Riesz potential of the right-hand side.
The talk is based on a joint work with Lars Diening, Ho-Sik Lee, and Simon Nowak.

Minhyun Kim (Seoul)

Removable singularities for nonlocal equations and nonlocal minimal graphs

In this talk, we study the local behavior of solutions, with possible singularities, of nonlocal nonlinear equations modeled on the fractional p-Laplace equation ($1 < p < \infty$) and nonlocal minimal surface equation (which corresponds to the case $p = 1$).
For the case $1 < p < \infty$, we first prove that sets of fractional capacity zero are removable for solutions under certain integrability conditions. We then characterize the asymptotic behavior of singular solutions near an isolated singularity in terms of the fundamental solution. This result is based on joint work with Se-Chan Lee.
For the case $p = 1$, we show that any nonlocal minimal graph in $\Omega \setminus K$, where $\Omega \subset \mathbb{R}^n$ is an open set and $K \subset \Omega$ is a compact set of $(s, 1)$-capacity zero, is indeed a nonlocal minimal graph in all of Ω.

Tomasz Klimsiak (Toruń)

Isolated singularities: Bôcher type theorem for elliptic equations with drift perturbed Lévy operator

A classical Bôcher's theorem asserts that any positive harmonic function (with respect to the Laplacian) in the unit punctured ball can be expressed, up to a multiplication constant, as the sum of the Newtonian kernel and a positive function that is harmonic in the whole unit ball. This theorem expresses one of the fundamental results in the theory of isolated singularities and it can be viewed as a statement on the asymptotic behavior of positive harmonic functions near their isolated singularities. We generalize this results to drift perturbed Lévy operators. We propose a new approach based on the probabilistic potential theory. It applies to Lévy operators for which the resolvent of its perturbation is strongly Feller. In particular our result encompasses drift perturbed fractional Laplacians with any stability index bounded between zero and two - the method therefore applies to subcritical and supercritical cases.

Naian Liao (Salzburg)

Harnack estimates for nonlocal diffusions

I will introduce new Harnack estimates for nonlocal diffusion equations that defy the waiting-time phenomenon – a joint work with Marvin Weidner. I will also report on Harnack estimates for nonlocal diffusion equations with a "drift".

Markus Melenk (Wien)

hp-FEM for the integral fractional Laplacian: quadrature

For the Dirichlet problem of the integral fractional Laplacian on intervals Ω and on polygons Ω, it has recently been shown that exponential convergence of the hp-FEM based on suitably designed meshes can be achieved, [Faustmann, Marcati, Melenk, Schwab, 2023]. These meshes are geometrically refined towards the edges and corners of Ω. The geometric refinement towards the edges results in anisotropic meshes away from corners. The use of such anisotropic elements is crucial for the exponential convergence result.

In this talk, we address the issue of setting the stiffness matrix. We show that a judicious combination of Duffy-like transformations and hp-quadrature techniques allow one to set up the matrix with work growing algebraically in the problem size while retaining the
exponential convergence of hp-FEM. The emphasis will be placed on the 1D fractional Laplacian, [Bahr, Faustmann, Melenk, 2024].

Tadele Mengesha (Knoxville)

Variational Analysis of a Parametrized Family of Transmission Problems Coupling Nonlocal and Fractional Models

I will present a work that examines the coupling of a model based on the regional fractional Laplacian and a nonlocal model employing a position-dependent interaction kernel. Both operators are inherently nonlocal and act on functions defined within their respective domains. The coupling occurs via a transmission condition across a hypersurface interface. The heterogeneous interaction kernel of the nonlocal operator leads to an energy space endowed with a well-defined trace operator. This, combined with well-established trace results of fractional Sobolev spaces, facilitates the imposition of a transmission condition across an interface. The family of problems will be parametrized by two key parameters that measure nonlocality and differentiability. For each pair of parameters, we demonstrate existence of a solution to the resulting variational problems. Furthermore, we investigate the limiting behavior of these solutions as the parameters approach their extreme.

Giuseppe Mingione (Parma)

Partial regularity in nonlocal problems

The theory of partial regular regularity for elliptic systems replaces the classical De Giorgi-Nash-Moser one for scalar equations asserting that solutions are regular outside a negligible closed subset called the singular set. Eventually, Hausdorff dimension estimates on such a set can be given. The singular set is in general non-empty. The theory is classical, started by Giusti & Miranda and Morrey, in turn relying on De Giorgi's seminal ideas for minimal surfaces. I shall present a few results aimed at extending the classical, local partial regularity theory to nonlinear integrodifferential systems and to provide a few basic, general tools in order to prove so called epsilon-regularity theorems in general non-local settings. From recent, joint work with Cristiana De Filippis (Parma) and Simon Nowak (Bielefeld).

Jihoon Ok (Seoul)

Nonlocal equations with various kernels

Nonlocal equations with p-growth is modeled by

$$P.V. \int_{\mathbb{R}^n} |u(x) - u(y)|^{p-2}(u(x) - u(y))k(x, y)\, dy = 0, \quad x \in \Omega,$$

where $1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ with $n \geq 2$. If the kernel $K : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ satisfies the following s-order uniform ellipticity condition for some $s \in (0, 1)$:

$$\frac{1}{\Lambda} \frac{1}{|x - y|^{n+sp}} \leq K(x, y) \leq \Lambda \frac{1}{|x - y|^{n+sp}},$$

where $\Lambda \geq 1$ is a constant, then weak solutions of these nonlocal equations are Hölder continuous and satisfy the Harnack inequality. Our main interest is to investigate conditions on kernels that do not satisfy the above uniform ellipticity but still lead to these regularity results. In this talk, I will introduce two classes of such kernels. The first class consists of degenerate or singular kernels associated with the Muckenhoupt class. The second class covers general kernels that include both the above uniformly elliptic cases and borderline cases as s approaches 1.

This is based on joint works with Linus Behn, Lars Diening, and Julian Rolfes (Bielefeld University), and Kyeong Song (KIAS, Seoul).

Mirco Piccinini (Pisa)

Nonlocal theory for fractional kinetic equations

We extend the De Giorgi-Nash-Moser theory to a class of nonlocal hypoelliptic equations naturally arising in kinetic theory, which combine a first-order transport term with an elliptic operator involving fractional derivatives along only part of the coordinates. Under sufficient integrability along the transport variables on the nonlocal tail, we prove the first local supremum estimate for this class of equations. Then, we establish the first full Harnack inequality for solutions to kinetic integral equations under the aforementioned tail summability assumption, which appears in clear accordance with the very recent counterexample by Kassmann and Weidner (Adv. in Math. 2024). This is based on series of papaers by F. Anceschi, M. Kassmann, A. Loher, G. Palatucci, M. Weidner and myself.

Luboš Pick (Prague)

Continuous and compact fractional Orlicz-Sobolev embeddings on domains

Necessary and sufficient conditions will be offered for compact embeddings of fractional Orlicz-Sobolev spaces into Orlicz spaces, or into rearrangement-invariant spaces, on bounded Lipschitz domains. The optimal Orlicz target space and the optimal rearrangement-invariant target space for merely continuous embeddings, also on domains, will be exhibited. Sharp embedding and compact embeddings of fractional Orlicz-Sobolev spaces into spaces of uniformly continuous functions governed by moduli of continuity will be characterized. This is a joint work with Angela Alberico, Andrea Cianchi and Lenka Slavíková.

Xavier Ros-Oton (Barcelona)

Regularity for the Boltzmann equation via nonlocal operators

The Boltzmann equation is the oldest nonlocal/fractional equation, dating back to 1872. It is a fundamental equation in statistical mechanics, modelling the evolution of a gas. In this talk we will present some recent results about the regularity of solutions to such an equation. Namely, we extend previous results of Imbert and Silvestre, showing that if some observables (mass and pressure) remain bounded, then solutions are smooth.

Abner J. Salgado (Knoxville)

Asymptotic compatibility of parametrized optimal design problems

We study optimal design problems where the design corresponds to a coefficient in
the principal part of the state equation. The state equation, in addition, is parameter dependent, and we allow it to change type in the limit of this (modeling) parameter. We develop a framework that guarantees asymptotic compatibility, that is unconditional convergence with respect to modeling and discretization parameters to the solution of the corresponding limiting problems. This framework is then applied to two distinct classes of problems where the modeling parameter represents the degree of nonlocality. Specifically, we show unconditional convergence of optimal design problems when the state equation is either a scalar-valued fractional equation, or a strongly coupled system of nonlocal equations derived from the bond-based model of peridynamics.
This is joint work with Tadele Mengesha (UTK) and Joshua Siktar (TAMU)

James Scott (New York)

Nonlocal Boundary Value Problems with Local Boundary Conditions

We state and analyze nonlocal problems with classically-defined, local boundary conditions. The model takes its horizon parameter to be spatially dependent, vanishing near the boundary of the domain. We establish a Green's identity for the nonlocal operator that recovers the classical boundary integral, which permits the use of variational techniques. Using this, we show the existence of weak solutions, as well as their variational convergence to classical counterparts as the bulk horizon parameter uniformly converges to zero. In certain circumstances, global regularity of solutions can be established, resulting in improved modes and rates of variational convergence. Generalizations of these results pertaining to models in continuum mechanics and Laplacian learning will also be presented.

Lenka Slavíková (Prague)

Classical multiplier theorems and their sharp variants

The question of finding good sufficient conditions on a bounded function m guaranteeing the L^p-boundedness of the associated Fourier multiplier operator is a long-standing open problem. In this talk, I will recall the classical multiplier theorems of Hörmander and Marcinkiewicz and present their sharp variants in which the multiplier belongs to a certain fractional Lorentz-Sobolev space. The talk is based on a joint work with L. Grafakos and M. Mastyło.

Xiaochuan Tian (San Diego)

Variational analysis of a Dirichlet energy of half-space nonlocal gradient

Inspired by recent studies in peridynamics for nonlocal mechanics, we study variational problems associated with the half-space nonlocal gradient operator. Specifically, a Dirichlet energy involving the nonlocal gradient operator is considered. The corresponding nonlocal function space is a Hilbert space with dense smooth functions and a nonlocal Poincaré inequality.

A key result is the convergence of variational solutions as the underlying kernel functions approach a limiting form. This relies on establishing compactness results in the spirit of Bourgain, Brezis, and Mironescu. These results lead to uniform Poincaré inequalities and the convergence of parameterized nonlocal problems, which further support the study of asymptotically compatible Galerkin approximations.

Marvin Weidner (Barcelona)

Regularity for the nonlocal Neumann problem

The boundary behavior for solutions to nonlocal equations with exterior Dirichlet boundary conditions has been extensively studied in recent years and it is well known that, in general, solutions are not better than C^s. In contrast, the Neumann problem for nonlocal equations has received much less attention, and the optimal boundary regularity of solutions remains unknown. In this talk, I will present recent progress on this question, based on a new classification result for solutions to general nonlocal equations in 1D. This is joint work with Serena Dipierro, Xavier Ros-Oton, and Enrico Valdinoci.

Tobias Weth (Frankfurt am Main)

Second radial eigenfunctions to a fractional Dirichlet problem and uniqueness for a semilinear equation

In this talk, I will report on joint work with Moustapha Fall on the shape of radial second Dirichlet eigenfunctions of fractional Schrödinger type operators in the unit ball B with a nondecreasing radial potential. Specifically, we show that the eigenspace corresponding to the second radial eigenvalue is simple and spanned by an eigenfunction which changes sign precisely once in the radial variable and has a nonvanishing fractional boundary derivative. We apply this result to prove uniqueness and nondegeneracy of positive ground state solutions to a semilinear fractional Dirichlet problem in B. If time permits, I will also address the uniqueness within the class of arbitrary positive solutions and the corresponding problem in the entire space.

Shuonan Wu (Beijing)

A monotone discretization for integral fractional Laplacian on bounded Lipschitz domains

We propose a monotone discretization method for the integral fractional Laplacian on bounded Lipschitz domains with homogeneous Dirichlet boundary conditions, specifically designed for solving fractional obstacle problems. Operating on unstructured grids in arbitrary dimensions, the method offers flexibility in approximating singular integrals over a domain that depends not only on the local grid size but also on the distance to the boundary, where the H"older regularity of the solution deteriorates. Using a discrete barrier function reflecting this distance, we establish optimal pointwise convergence rates in terms of the H"older regularity of the data on both quasi-uniform and graded grids. The method can be directly applied to fractional obstacle problems, and an improved policy iteration is proposed to achieve better numerical performance.

Rico Zacher (Ulm)

Differential Harnack inequalities for nonlocal diffusion problems

I will present recent results on differential Harnack inequalities of Li-Yau type for certain classes of nonlocal diffusion equations. This includes problems on infinite discrete structures (graphs) on which arbitrarily long jumps are possible and problems in Euclidean space with a fractional Laplace operator. One of the main difficulties is that the classical chain rule is not valid for the nonlocal operators under consideration. Additionally, if one wants to adopt Li and Yau's approach from their famous 1986 paper (Acta. Math.), new curvature-dimension (CD) inequalities are required, since the classical Bakry-Emery condition based on the Gamma calculus is no longer suitable. This also touches on the fundamental question of how to define lower curvature bounds on discrete structures in a meaningful way. In addition to the approach using CD inequalities, I will present another method which is based on heat kernel representations of the solutions and consists in reducing the problem to the heat kernel. This is partly joint work with S. Kräss, A. Spener and F. Weber.

Organizing committee

Anna Kh. Balci

Linus Behn

Lars Diening

Florian Grube

Moritz Kaßmann

Ho-Sik Lee

Simon Nowak

Harsh Prasad

Website

https://www.bi-nonlocal.de