

On an Optimal Extraction Problem with Regime Switching*

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Abstract. This paper studies a finite-fuel two-dimensional degenerate singular stochastic control problem under regime switching that is motivated by the optimal irreversible extraction problem of an exhaustible commodity. A company extracts a natural resource from a reserve with finite capacity, and sells it in the market at a spot price that evolves according to a Brownian motion with volatility modulated by a two-state Markov chain. In this setting, the company aims at finding the extraction rule that maximizes its expected discounted cash flow, net of the costs of extraction and maintenance of the reserve. We provide expressions both for the value function and for the optimal control. On the one hand, if the running cost for the maintenance of the reserve is a convex function of the reserve level, the optimal extraction rule prescribes a Skorokhod reflection of the (optimally) controlled state process at a certain state and price dependent threshold. On the other hand, in presence of a concave running cost function it is optimal to instantaneously deplete the reserve at the time at which the commodity's price exceeds an endogenously determined critical level. In both cases, the threshold triggering the optimal control is given in terms of the optimal stopping boundary of an auxiliary family of perpetual optimal selling problems with regime switching.

Key words: singular stochastic control, optimal stopping, regime switching, Hamilton-Jacobi-Bellman equation, free boundary, commodity extraction, optimal selling.

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1 Introduction

Since the seminal work [8], both the literature in Applied Mathematics and that in Economics have seen numerous papers on optimal extraction problems of non-renewable resources under uncertainty. Some of these works formulate the extraction problem as an optimal timing problem (see, e.g., [13], [33] and references therein); some as a combined absolutely continuous/impulse stochastic control problem (e.g., [7] and [25]); and some others as a stochastic optimal control problem only with classical absolutely continuous controls (cf. [1] and [15], among many others),

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but with commodity price dynamics possibly described by a Markov regime switching model (cf., e.g., [23]). The latter kind of dynamics, firstly introduced in [22], may indeed help to explain boom and bust periods of commodity prices in terms of different regimes in a unique stochastic process.

In this paper we provide the solution to a stochastic irreversible extraction problem in presence of regime shifts in the underlying commodity spot price process. The problem we have in mind is that of a company extracting continuously in time a commodity from a reserve with finite capacity, and selling the natural resource in the spot market. The reserve level can be decreased at any time at a given proportional cost, following extraction policies which do not need to be rates. Moreover, the company faces a running cost for the extraction procedures (e.g. a cost for the reserve's maintenance) that is dependent on the reserve level. The company aims at finding the extraction rule that maximizes the expected discounted net cash flow in presence of market uncertainty and macroeconomic cycles. The latter are described through regime shifts in the volatility of the commodity spot price dynamics.

We set up the optimal extraction problem as a finite-fuel two-dimensional degenerate singular stochastic control problem under Markov regime switching. It is two-dimensional because for any regime i the state variable consists of the value of the spot price, x , and the level of the reserve, y . It is a problem of singular stochastic control with finite-fuel since extraction does not need to be performed at rates, and the commodity reserve has a finite capacity. Finally, it is degenerate since the state variable describing the level of the reserve is purely controlled, and does not have any diffusive component.

While the literature on optimal stopping problems under regime switching is relatively rich (see, e.g., [5], [9], [18], [19], [35], among others), that on singular stochastic control problems with regime switching is still quite limited. We refer, e.g., to [26], [27], [32] and [37] where the optimal dividend problem of actuarial science is formulated as a one-dimensional problem under Markov regime switching. If we then further restrict our attention to singular stochastic control problems with a two-dimensional state space and regime shifts, to the best of our knowledge [20] is the only other paper available in the literature. That work addresses an optimal irreversible investment problem in which the growth and the volatility of the decision variable jump between two states at independent exponentially distributed random times. However, although in [20] the authors provide a detailed discussion on the structure of the candidate solution and on the economic implications of regime switching for capital accumulation and growth, they do not confirm their guess by a verification theorem.

In this paper, with the aim of a complete analytical study, we assume that the commodity spot price X evolves according to a Bachelier model¹ with regime switching between two states. We show that the optimal extraction rule is of threshold type, and we provide the expression of the value function.

The Hamilton-Jacobi-Bellman (HJB) equation for our optimal extraction problem takes the form of a system of two coupled variational inequalities with state dependent gradient constraints. The coupling is through the transition rates of the underlying continuous-time Markov chain ε , and it makes the problem of finding an explicit solution much harder than in the standard case without regime switching. We associate to the singular control problem a family of auxiliary optimal stopping problems for the Markov process (X, ε) . Such family is parametrized through the initial reserve level y . We solve the related free-boundary problem, and we char-

¹The choice of an arithmetic dynamics can be justified also the modeling stage. Indeed, it has been shown in [17] that for certain commodities an arithmetic dynamics fits historical time series better than a mean-reverting one. Moreover, it has been recently observed that some commodities can be traded at negative prices (see [14]). This happened, e.g., to propane prices in Edmonton (Canada) in June 2015.

acterize the geometry of stopping and continuation regions. As it is usual in optimal stopping theory, we show that the first time at which the underlying process leaves the continuation region is an optimal stopping rule. For any given and fixed y , such time takes the form of the first hitting time of X to a regime dependent boundary $x_i^*(y)$, $i = 1, 2$. These boundaries are the unique solutions to a system of nonlinear algebraic equations derived by imposing the smooth-fit principle.

Under the assumption that the running cost function is either strictly convex or strictly concave in the reserve level, we show that the value function of the optimal extraction problem is given in terms of that of the auxiliary family of optimal stopping problems. Moreover, we prove that the optimal extraction policy is triggered by the optimal stopping boundaries $x_i^*(y)$, $i = 1, 2$. However, the behavior of the optimal control, and the regularity of the value function, significantly change when passing from a strictly convex running cost to a strictly concave one.

On the one hand, if the running cost is a strictly convex function of the reserve level, we show that the optimal extraction policy keeps at any time the optimally controlled reserve level below a certain critical value b^* with minimal effort, i.e. according to a *Skorokhod reflection*. Such threshold depends on the spot price and on the market regime, and it is the inverse of the optimal stopping boundary $x_i^*(\cdot)$ previously determined. Also, we prove that, for any regime $i = 1, 2$, the value function of the optimal extraction problem is a $C^{2,1}$ -solution of the associated HJB equation, and it is given as the integral, with respect to the controlled state variable, of the value function of the auxiliary optimal stopping problem.

On the other hand, if the running cost is a strictly concave function of the reserve, the optimal extraction rule prescribes the instantaneous depletion of the reserve at the time at which the commodity's price in regime $i = 1, 2$ exceeds the critical level $x_i^*(y)$. As a consequence of such *bang-bang* nature of the optimal policy - not extract or extract all - for any regime $i = 1, 2$ the value function only belongs to the class $C^0(\mathbb{R} \times [0, 1]) \cap C^{1,1}(\mathbb{R} \times (0, 1])$, with second order derivatives with respect to x that are bounded on any compact subset of $\mathbb{R} \times (0, 1]$.

Although optimal controls of reflecting and bang-bang type already appeared in the literature on two-dimensional degenerate singular stochastic control problems (see, e.g., the recent [10], [11] and references therein), to the best of our knowledge, this is the first paper in which these two different behaviors of the optimal control arise in a model with Markov regime switching.

The study of the auxiliary family of optimal stopping problems performed in this paper is of interest on its own as well. Each stopping problem takes indeed the form of a perpetual optimal selling problem under regime switching that we completely solve. It is worth noticing that most of the papers dealing with optimal stopping problems with regime switching, and following a guess and verify approach, assume existence of a solution of the smooth-fit equations and additional properties of the candidate value function in order to perform a verification theorem (see, e.g., Theorem 3.1 in [19], and Theorems 3 and 5 in [35]). An abstract and nonconstructive approach, based on a thorough analysis of the involved variational inequality, is adopted in [5]. Here, instead, we construct a solution to the free-boundary problem, and we then prove all the properties needed to verify that such solution is actually the value function of our optimal stopping problem with regime switching (see our Theorems 3.4 and 3.5 below). We believe that also such result represents an interesting contribution to the literature.

Although not solvable in closed form, the system of nonlinear algebraic equations characterizing the optimal stopping boundaries - hence the optimal extraction policy - can be easily solved numerically. This fact allows us, e.g., to compare the optimal extraction boundaries in the case with and without regime switching, and thus to draw interesting economic conclusions (see Section 5). In particular, we show that in presence of macroeconomic cycles, the company is more reluctant (resp. favourable) to extract and then sell the commodity, relative to the case

in which the market were always in the good (resp. bad) regime with the lowest (resp. highest) volatility.

The rest of the paper is organized as follows. In Section 2 we formulate the optimal extraction problem, we introduce the associated HJB equation and we discuss the solution approach. The family of optimal stopping problems is then solved in Section 3, whereas the optimal control is provided in Section 4. A comparison with the optimal extraction rule that one would find in the no-regime-switching case, and some economic conclusions, are contained in Section 5. Appendix A collects the proofs of some results of Section 3, whereas in Appendix B one can find auxiliary results needed in the paper.

2 Problem Formulation and Solution Approach

2.1 The Optimal Extraction Problem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, rich enough to accommodate a one-dimensional Brownian motion $\{W_t, t \geq 0\}$ and a continuous-time Markov chain $\{\varepsilon_t, t \geq 0\}$ with state space $E := \{1, 2\}$, and with irreducible generator matrix

$$Q := \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix}, \quad (2.1)$$

for some $\lambda_1, \lambda_2 > 0$. The Markov chain ε jumps between the two states at exponentially distributed random times, and the constant λ_i gives the rate of leaving state $i = 1, 2$. We take ε independent of W and denote by $\mathbb{F} := \{\mathcal{F}_t, t \geq 0\}$ the filtration jointly generated by W and ε , as usual augmented by \mathbb{P} -null sets.

We assume that the spot price of the commodity evolves according to a Bachelier model [3] with regime switching; i.e.

$$dX_t = \sigma_{\varepsilon_t} dW_t, \quad t > 0, \quad X_0 = x \in \mathbb{R}, \quad (2.2)$$

where for every state $i = 1, 2$ $\sigma_i > 0$ is a known finite constant. From the modeling point of view, the choice of an arithmetic dynamics might be justified by noticing that certain commodities can be treated at negative spot prices (see, e.g., [14]), and do not show a mean-reverting behavior (cf. [17], among others).

(X, ε) is a strong Markov process (see [36], Remark 3.11) and we set $\mathbb{P}_{(x,i)}(\cdot) := \mathbb{P}(\cdot | X_0 = x, \varepsilon_0 = i)$, and we denote by $\mathbb{E}_{(x,i)}$ the corresponding expectation operator. From Section 3.1 in [36] we also know that (X, ε) is regular, in the sense that the sequence of stopping times $\{\beta_n, n \in \mathbb{N}\}$, with $\beta_n := \inf\{t \geq 0 : |X_t| = n\}$, is such that $\lim_{n \uparrow \infty} \beta_n = +\infty$, $\mathbb{P}_{(x,i)}$ -a.s.

The level of the commodity reserve is such that

$$dY_t^\nu = -d\nu_t, \quad t > 0, \quad Y_0^\nu = y \in [0, 1]. \quad (2.3)$$

Taking $y \leq 1$ we model the fact that the reserve has a finite capacity, normalized to 1 without loss of generality. Here ν_t represents the cumulative amount of commodity extracted up to time $t \geq 0$. We say that an extraction policy is admissible if, given $y \in [0, 1]$, it belongs to the nonempty convex set

$$\mathcal{A}_y := \{\nu : \Omega \times \mathbb{R}_+ \mapsto \mathbb{R}_+, (\nu_t(\omega) := \nu(\omega, t))_{t \geq 0} \text{ is nondecreasing, left-continuous, } \mathbb{F}\text{-adapted with } y - \nu_t \geq 0 \ \forall t \geq 0, \ \nu_0 = 0 \ \mathbb{P}\text{-a.s.}\}. \quad (2.4)$$

Moreover, we let $\mathbb{P}_{(x,y,i)}(\cdot) := \mathbb{P}(\cdot | X_0 = x, Y_0 = y, \varepsilon_0 = i)$ and $\mathbb{E}_{(x,y,i)}$ the corresponding expectation operator.

While extracting, the company faces two types of costs: the first one is an extraction cost that we take proportional through a constant $c > 0$ to the amount of commodity extracted; the second one is a running cost for the extraction procedures (e.g. a cost for the maintenance of the reserve). The latter is measured by a function f of the reserve level satisfying the following assumption.

Assumption 2.1. $f : \mathbb{R} \rightarrow \mathbb{R}_+$ is strictly increasing, continuous on $[0, 1]$ and such that $f(0) = 0$. Moreover, either of the following two conditions are satisfied:

- (I) $y \mapsto f(y)$ is strictly convex and continuously differentiable on $[0, 1]$;
- (II) $y \mapsto f(y)$ is strictly concave on $[0, 1]$, continuously differentiable on $(0, 1]$, and such that $\lim_{y \downarrow 0} f'(y) = +\infty$ (Inada condition).

Assumption 2.1 will be standing throughout this paper.

Remark 2.2.

1. From an economic point of view, a running cost function that is concave on $[0, 1]$ reflects economies of scale in the size of the operation. On the other hand, a running cost function convex on $[0, 1]$ seems to be more appropriate for a company facing diseconomies of scale.
2. The requirement $f(0) = 0$ is without loss of generality, since if $f(0) = f_o > 0$ then one can always set $\hat{f}(y) := f(y) - f(0)$ and write $f(y) = \hat{f}(y) + f_o$, so that the firms's optimization problem (cf. (2.6) below) remains unchanged up to an additive constant.
3. Cost functions of the form $f(y) = \alpha_o y^2 + \beta_o y$ for some $\alpha_o, \beta_o > 0$, or $f(y) = y^{\gamma_o}$, for some $\gamma_o \in (0, 1)$, clearly meet Assumption 2.1.

Following an extraction policy $\nu \in \mathcal{A}_y$ and selling the extracted amount in the spot market at price X , the expected discounted cash flow of the company, net of extraction and maintenance costs, is

$$\mathcal{J}_{x,y,i}(\nu) := \mathbb{E}_{(x,y,i)} \left[\int_0^\infty e^{-\rho t} (X_t - c) d\nu_t - \int_0^\infty e^{-\rho t} f(Y_t^\nu) dt \right], \quad (x, y, i) \in \mathcal{O}, \quad (2.5)$$

where $\rho > 0$ is a given discount factor and $\mathcal{O} := \mathbb{R} \times [0, 1] \times \{1, 2\}$. Throughout this paper, for $t > 0$ and $\nu \in \mathcal{A}_y$ we will make use of the notation $\int_0^t e^{-\rho s} (X_s^x - c) d\nu_s$ to indicate the Stieltjes integral $\int_{[0,t)} e^{-\rho s} (X_s^x - c) d\nu_s$ with respect to ν . As a byproduct of Lemma B.4 in Appendix B, the functional (2.5) is well-defined and finite for any $\nu \in \mathcal{A}_y$.

The company aims at choosing an admissible extraction rule that maximizes (2.5); that is, it faces the optimization problem

$$V(x, y, i) := \sup_{\nu \in \mathcal{A}_y} \mathcal{J}_{x,y,i}(\nu), \quad (x, y, i) \in \mathcal{O}. \quad (2.6)$$

Notice that if $y = 0$ then no control can be exerted, i.e. $\mathcal{A}_0 = \{\nu \equiv 0\}$, and therefore $V(x, 0, i) = \mathcal{J}_{x,0,i}(0) = 0$, for any $(x, i) \in \mathbb{R} \times \{1, 2\}$.

Remark 2.3. *Due to the convexity of \mathcal{A}_y , and the linearity of $\nu \mapsto Y^\nu$, if $y \mapsto f(y)$ is strictly convex on $[0, 1]$, then the functional $\mathcal{J}_{x,y,i}(\cdot)$ is strictly concave on \mathcal{A}_y , and (2.6) is a well-posed maximization problem of a concave functional. On the other hand, if $y \mapsto f(y)$ is strictly concave on $[0, 1]$, then $\mathcal{J}_{x,y,i}(\cdot)$ is strictly convex on \mathcal{A}_y . We will see in Section 4 how the convexity/concavity of f will impact on the behavior of the optimal control, and on the regularity of the value function.*

Problem (2.6) falls into the class of singular stochastic control problems, i.e. problems in which admissible controls do not need to be absolutely continuous with respect to the Lebesgue measure, as functions of time (see [30] and Chapter VIII in [16] for an introduction). In particular, it is a finite-fuel two-dimensional degenerate singular stochastic control problem under Markov regime switching. It is degenerate because the state process Y is purely controlled, and does not have a diffusive component. Moreover, it is of finite-fuel since the controls stay bounded.

Remark 2.4. *Since the extraction rule adopted by the company does not affect the price of the commodity, our model takes into consideration a price-taker company. Allowing for a direct instantaneous effect of the extraction policy on the price dynamics, our problem would share a similar mathematical structure with the problem of optimal execution in algorithm trading, where an investor sells a large number of stock shares over a given time horizon and her actions have impact on the stock price (see, e.g., [21] for a recent formulation of the optimal execution problem involving singular controls). We leave the analysis of the optimal extraction problem with price impact as an interesting future research topic.*

2.2 The Hamilton-Jacobi-Bellman Equation, a First Verification Theorem, and the Solution Approach

In light of classical results in stochastic control (see, e.g., Chapter VIII in [16]), we expect that for any $i = 1, 2$ the value function $V(\cdot, \cdot, i)$ suitably satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$\max \left\{ (\mathcal{G} - \rho)U(x, y, i) - f(y), (x - c) - U_y(x, y, i) \right\} = 0, \quad (2.7)$$

for $(x, y) \in \mathbb{R} \times (0, 1]$ and with boundary condition $U(x, 0, i) = 0$. Here \mathcal{G} is the infinitesimal generator of (X, ε) . It acts on functions $h : \mathbb{R} \times \{1, 2\} \rightarrow \mathbb{R}$ with $h(\cdot, i) \in C^2(\mathbb{R})$ for any given and fixed $i = 1, 2$ as

$$\mathcal{G}h(x, i) := \frac{1}{2}\sigma_i^2 h_{xx}(x, i) + \lambda_i(h(x, 3-i) - h(x, i)). \quad (2.8)$$

It is worth noting that, due to (2.8), equation (2.7) is actually a system of two variational inequalities with state-dependent gradient constraints, coupled through the transition rates λ_1, λ_2 . The next preliminary verification result shows that any suitable solution to (2.7) provides an upper bound for the value function V .

Theorem 2.5. *For $i = 1, 2$, let $U(\cdot, \cdot, i) \in W_{loc}^{2,1,\infty}(\mathbb{R} \times (0, 1))$ be a solution in the a.e. sense of (2.7) such that $U(x, 0, i) = 0$, $x \in \mathbb{R}$, and $|U(x, y, i)| \leq K(1 + |x|)$, $(x, y, i) \in \mathcal{O}$, for some $K > 0$. Then one has $U \geq V$ on \mathcal{O} .*

Proof. Fix $(x, y, i) \in \mathcal{O}$, and take arbitrary $R > 0$ and $T > 0$. Set $\tau_R := \inf \{t \geq 0 : X_t \notin (-R, R)\}$, and let $0 \leq \eta_1 < \eta_2 < \dots < \eta_N \leq \tau_R \wedge T$ be the random times of jumps of ε in the interval $[0, \tau_R \wedge T)$ (clearly, the number N of those jumps is random as well). Then pick an

admissible control ν , and notice that the regularity of U allows us to apply a generalized version of Itô's formula (see, e.g., [16], Ch. VIII, Sec. VIII.4, Th. 4.1) on each of the intervals $[0, \eta_1)$, $(\eta_1, \eta_2), \dots, (\eta_N, \tau_R \wedge T)$. Piecing together all the terms as in the proof of Lemma 3 at p. 104 of [31] (see also Lemma 2.4 in [34] for a similar idea of proof) one finds

$$\begin{aligned} U(x, y, i) = & \mathbb{E}_{(x, y, i)} \left[e^{-\rho(\tau_R \wedge T)} U(X_{\tau_R \wedge T}^x, Y_{\tau_R \wedge T}^\nu, \varepsilon_{\tau_R \wedge T}) - \int_0^{\tau_R \wedge T} e^{-\rho s} (\mathcal{G} - \rho) U(X_s, Y_s^\nu, \varepsilon_s) ds \right] \\ & + \mathbb{E}_{(x, y, i)} \left[\int_0^{\tau_R \wedge T} e^{-\rho s} U_y(X_s, Y_s^\nu, \varepsilon_s) d\nu_s \right] \\ & - \mathbb{E}_{(x, y, i)} \left[\sum_{0 \leq s < \tau_R \wedge T} e^{-\rho s} (U(X_s, Y_{s+}^\nu, \varepsilon_s) - U(X_s, Y_s^\nu, \varepsilon_s) - U_y(X_s, Y_s^\nu, \varepsilon_s) \Delta Y_s) \right], \end{aligned}$$

where $\Delta Y_s := Y_{s+} - Y_s = -\Delta \nu_s := -(\nu_{s+} - \nu_s)$, and the expectation of the stochastic integral vanishes since U_x is bounded on $(x, y, i) \in [-R, R] \times [0, 1] \times \{1, 2\}$.

Now, noticing that any admissible control ν can be written as the sum of its continuous part and of its pure jump part, i.e. $d\nu = d\nu^{cont} + \Delta \nu$ (see again [16], Chapter 8, Section VIII.4, Theorem 4.1 at pp. 301-302), one has

$$\begin{aligned} U(x, y, i) = & \mathbb{E}_{(x, y, i)} \left[e^{-\rho(\tau_R \wedge T)} U(X_{\tau_R \wedge T}^x, Y_{\tau_R \wedge T}^\nu, \varepsilon_{\tau_R \wedge T}) - \int_0^{\tau_R \wedge T} e^{-\rho s} (\mathcal{G} - \rho) U(X_s, Y_s^\nu, \varepsilon_s) ds \right] \\ & + \mathbb{E}_{(x, y, i)} \left[\int_0^{\tau_R \wedge T} e^{-\rho s} U_y(X_s, Y_s^\nu, \varepsilon_s) d\nu_s^{cont} \right] \\ & - \mathbb{E}_{(x, y, i)} \left[\sum_{0 \leq s < \tau_R \wedge T} e^{-\rho s} (U(X_s, Y_{s+}^\nu, \varepsilon_s) - U(X_s, Y_s^\nu, \varepsilon_s)) \right]. \end{aligned}$$

Since U satisfies the HJB equation (2.7), and because

$$U(X_s, Y_{s+}^\nu, \varepsilon_s) - U(X_s, Y_s^\nu, \varepsilon_s) = - \int_0^{\Delta \nu_s} U_y(X_s, Y_s^\nu - z, \varepsilon_s) dz, \quad (2.9)$$

one obtains

$$\begin{aligned} U(x, y, i) \geq & \mathbb{E}_{(x, y, i)} \left[e^{-\rho(\tau_R \wedge T)} U(X_{\tau_R \wedge T}^x, Y_{\tau_R \wedge T}^\nu, \varepsilon_{\tau_R \wedge T}) \right] - \mathbb{E}_{(x, y, i)} \left[\int_0^{\tau_R \wedge T} e^{-\rho s} f(Y_s^\nu) ds \right] \\ & + \mathbb{E}_{(x, y, i)} \left[\int_0^{\tau_R \wedge T} e^{-\rho s} (X_s - c) d\nu_s^{cont} \right] + \mathbb{E}_{(x, y, i)} \left[\sum_{0 \leq s < \tau_R \wedge T} e^{-\rho s} (X_s - c) \Delta \nu_s \right] \\ = & \mathbb{E}_{(x, y, i)} \left[e^{-\rho(\tau_R \wedge T)} U(X_{\tau_R \wedge T}^x, Y_{\tau_R \wedge T}^\nu, \varepsilon_{\tau_R \wedge T}) + \int_0^{\tau_R \wedge T} e^{-\rho s} (X_s^x - c) d\nu_s \right] \\ & - \mathbb{E}_{(x, y, i)} \left[\int_0^{\tau_R \wedge T} e^{-\rho s} f(Y_s^\nu) ds \right]. \end{aligned} \quad (2.10)$$

By Hölder's inequality, (2.2), and Itô's isometry we have

$$\begin{aligned} \mathbb{E}_{(x, y, i)} \left[e^{-\rho(\tau_R \wedge T)} |X_{\tau_R \wedge T}| \right] & \leq \mathbb{E}_{(x, y, i)} \left[e^{-2\rho(\tau_R \wedge T)} \right]^{\frac{1}{2}} \mathbb{E}_{(x, y, i)} \left[|X_{\tau_R \wedge T}|^2 \right]^{\frac{1}{2}} \\ & \leq \sqrt{2} \mathbb{E}_{(x, y, i)} \left[e^{-2\rho(\tau_R \wedge T)} \right]^{\frac{1}{2}} \left(|x|^2 + \mathbb{E}_{(x, y, i)} \left[\left| \int_0^{\tau_R \wedge T} \sigma_{\varepsilon_u} dW_u \right|^2 \right] \right)^{\frac{1}{2}} \\ & \leq \sqrt{2} \mathbb{E}_{(x, y, i)} \left[e^{-2\rho(\tau_R \wedge T)} \right]^{\frac{1}{2}} \left(|x|^2 + (\sigma_1^2 \vee \sigma_2^2) T \right)^{\frac{1}{2}}. \end{aligned}$$

The previous estimate, together with the sublinear growth property of U , then imply

$$\begin{aligned} \mathbb{E}_{(x,y,i)} \left[e^{-\rho(\tau_R \wedge T)} U(X_{\tau_R \wedge T}, Y_{\tau_R \wedge T}^\nu, \varepsilon_{\tau_R \wedge T}) \right] &\geq -C \mathbb{E}_{(x,y,i)} \left[e^{-\rho(\tau_R \wedge T)} \right] \\ &\quad - \sqrt{2} C \mathbb{E}_{(x,y,i)} \left[e^{-2\rho(\tau_R \wedge T)} \right]^{\frac{1}{2}} \left(|x|^2 + (\sigma_1^2 \vee \sigma_2^2) T \right)^{\frac{1}{2}}, \end{aligned}$$

for some constant $C > 0$. Hence

$$\begin{aligned} U(x, y, i) &\geq -C \mathbb{E}_{(x,y,i)} \left[e^{-\rho(\tau_R \wedge T)} \right] - \sqrt{2} C \mathbb{E}_{(x,y,i)} \left[e^{-2\rho(\tau_R \wedge T)} \right]^{\frac{1}{2}} \left(|x|^2 + (\sigma_1^2 \vee \sigma_2^2) T \right)^{\frac{1}{2}} \\ &\quad + \mathbb{E}_{(x,y,i)} \left[\int_0^{\tau_R \wedge T} e^{-\rho s} (X_s^x - c) d\nu_s \right] - \mathbb{E}_{(x,y,i)} \left[\int_0^{\tau_R \wedge T} e^{-\rho s} f(Y_s^\nu) ds \right]. \end{aligned} \quad (2.11)$$

When taking limits as $R \rightarrow \infty$ we have $\tau_R \wedge T \rightarrow T$, $\mathbb{P}_{(x,y,i)}$ -a.s. by regularity of (X, ε) . By Lemma B.4 in Appendix B, the integrals on the right-hand side of (2.11) are uniformly integrable. We can thus invoke Vitali's convergence theorem to take limits as $R \uparrow \infty$ in (2.11), and then as $T \uparrow \infty$, and obtain

$$U(x, y, i) \geq \mathbb{E}_{(x,y,i)} \left[\int_0^\infty e^{-\rho s} (X_s - c) d\nu_s - \int_0^\infty e^{-\rho s} f(Y_s^\nu) ds \right]. \quad (2.12)$$

Since (2.12) holds for any $\nu \in \mathcal{A}_y$, we have $U(x, y, i) \geq V(x, y, i)$. \square

In this paper we solve problem (2.6) in the two following cases (cf. Assumption 2.1 and Remark 2.3):

- (I) $y \mapsto f(y)$ is strictly convex on $[0, 1]$ (cf. Section 4.1);
- (II) $y \mapsto f(y)$ is strictly concave on $[0, 1]$ (cf. Section 4.2).

The case of a running cost that is neither convex nor concave on $[0, 1]$ clearly needs a separate analysis, and it is left as an interesting open problem.

In cases (I) and (II) we construct a solution of the HJB equation (2.7), and we provide a candidate optimal extraction policy. The optimality of the latter is verified through a verification argument. We find that even if the optimal policy qualitatively differs among the two cases, in both cases (I) and (II) the value function is given in terms of that of the parameter-dependent (as $y \in (0, 1]$ enters only as a parameter) optimal stopping problem with regime switching

$$u(x, i; y) := \sup_{\tau \geq 0} \mathbb{E}_{(x,i)} \left[e^{-\rho\tau} (X_\tau - \theta(y)) \right]. \quad (2.13)$$

In (2.13) the optimization is taken over all $\mathbb{P}_{(x,i)}$ -a.s. finite \mathbb{F} -stopping times; moreover, $\theta(y)$ is a given suitable real number whose specific dependence on y will be different among the two cases (I) and (II).

In the next section we solve (2.13) and we show that its solution is triggered by suitable regime-dependent stopping boundaries $x_i^*(y)$, $y \in (0, 1]$. These boundaries are completely characterized as unique solutions to a system of nonlinear algebraic equations.

3 The Associated Family of Optimal Selling Problems

In this section we solve the parameter-dependent optimal stopping problem with regime switching (2.13). This result is of interest on its own since problem (2.13) takes the form of an optimal selling problem in a Bachelier model with regime switching, and with a transaction cost $\theta(y)$ that parametrically depends on $y \in (0, 1]$. In the rest of this section $y \in (0, 1]$ is given and fixed.

Some preliminary properties of u are stated in the next proposition. These will be important in the following when constructing the solution to (2.13).

Proposition 3.1. *Recall (2.13). There exists a constant $K(y) > 0$ such that for any $(x, i) \in \mathbb{R} \times \{1, 2\}$*

1. $u(x, i; y) \geq x - \theta(y)$;
2. $|u(x, i; y)| \leq K(y)(1 + |x|)$.

Proof. The first claim immediately follows by taking the admissible $\tau = 0$. As for the second property, let τ be an \mathbb{F} -stopping time and notice that by an integration by parts we can write

$$e^{-\rho\tau}(X_\tau - \theta(y)) = (x - \theta(y)) - \int_0^\tau \rho e^{-\rho s} (X_s - \theta(y)) ds + \int_0^\tau e^{-\rho s} \sigma_{\varepsilon_s} dW_s. \quad (3.1)$$

Denoting $M_t := \int_0^t e^{-\rho s} \sigma_{\varepsilon_s} dW_s$, $t \geq 0$, and recalling the boundedness of σ_{ε_s} , M is uniformly bounded in $L^2(\Omega, \mathbb{P}_{(x,i)})$, and therefore $\mathbb{P}_{(x,i)}$ -uniformly integrable. Hence, taking expectations in (3.1), applying the optional stopping theorem (see Theorem 3.2 in [29]), and then taking absolute values we obtain

$$\begin{aligned} \left| \mathbb{E}_{(x,i)} \left[e^{-\rho\tau} (X_\tau - \theta(y)) \right] \right| &\leq |x| + |\theta(y)| + \mathbb{E}_{(x,i)} \left[\int_0^\infty \rho e^{-\rho s} |X_s - \theta(y)| ds \right] \\ &\leq 2(|x| + |\theta(y)|) + \int_0^\infty \rho e^{-\rho s} \mathbb{E}_{(x,i)} \left[\left| \int_0^s \sigma_{\varepsilon_u} dW_u \right|^2 \right]^{\frac{1}{2}} ds \\ &\leq 2(|x| + |\theta(y)|) + (\sigma_1^2 \vee \sigma_2^2)^{\frac{1}{2}} \int_0^\infty \rho \sqrt{s} e^{-\rho s} ds \leq K(y)(1 + |x|), \end{aligned} \quad (3.2)$$

for some $K(y) > 0$. Equation (2.2), Tonelli's Theorem and Hölder's inequality imply the second step above, whereas the third step is guaranteed by Itô's isometry. The second claim of the proposition then easily follows from (3.2). \square

In line with the standard theory of optimal stopping (see, e.g., [28]) we expect u of (2.13) to suitably satisfy the variational inequality

$$\max \left\{ (\mathcal{G} - \rho)w(x, i; y), x - \theta(y) - w(x, i; y) \right\} = 0, \quad (3.3)$$

for $(x, i) \in \mathbb{R} \times \{1, 2\}$ and for any given $y \in (0, 1]$. Also, we define the continuation and stopping regions of (2.13) as

$$\mathcal{C} := \left\{ (x, i) \in \mathbb{R} \times \{1, 2\} : u(x, i; y) > x - \theta(y) \right\}, \quad \mathcal{S} := \left\{ (x, i) \in \mathbb{R} \times \{1, 2\} : u(x, i; y) = x - \theta(y) \right\},$$

respectively. Given the structure of optimal stopping problem (2.13) we expect that

$$\mathcal{C} := \left\{ (x, 1) : x < x_1^*(y) \right\} \cup \left\{ (x, 2) : x < x_2^*(y) \right\}, \quad (3.4)$$

for some thresholds, $x_i^*(y)$, $i = 1, 2$, such that $x_i^*(y) \geq \theta(y)$, $i = 1, 2$, and depending parametrically on $y \in (0, 1]$.

According to this conjecture three configurations are possible: (A) $x_1^*(y) < x_2^*(y)$, (B) $x_1^*(y) = x_2^*(y)$, and (C) $x_1^*(y) > x_2^*(y)$. We now solve (3.3) in cases (A) and (B). Case (C) is completely symmetric to case (A), and it can be treated with similar arguments. We therefore omit its discussion in this paper in the interest of length. In a second step, by a verification argument, we will show that the solution w to (3.3) satisfies $w \equiv u$. As a byproduct we will also provide the optimal stopping rule τ^* .

3.1 Case (A): $x_1^*(y) < x_2^*(y)$

We rewrite (3.3) in the form of a free-boundary problem to find $(w(x, 1; y), w(x, 2; y), x_1^*(y), x_2^*(y))$, with $w(\cdot, i; y) \in C^1(\mathbb{R})$ and $w_{xx}(\cdot, i; y) \in L_{loc}^\infty(\mathbb{R})$ for any $i = 1, 2$, solving

$$\left\{ \begin{array}{ll} \frac{1}{2}\sigma_i^2 w_{xx}(x, i; y) - \rho w(x, i; y) + \lambda_i(w(x, 3-i; y) - w(x, i; y)) = 0 & \text{for } x < x_1^*(y) \text{ and } i = 1, 2 \\ \frac{1}{2}\sigma_2^2 w_{xx}(x, 2; y) - \rho w(x, 2; y) + \lambda_2(w(x, 1; y) - w(x, 2; y)) = 0 & \text{for } x_1^*(y) < x < x_2^*(y) \\ w(x, 1; y) = x - \theta(y) & \text{for } x_1^*(y) \leq x \leq x_2^*(y) \\ w(x, 1; y) = x - \theta(y) = w(x, 2; y) & \text{for } x \geq x_2^*(y) \\ w_x(x, 1; y) = 1 & \text{for } x = x_1^*(y) \\ w(x-, 2; y) = w(x+, 2; y) & \text{for } x = x_1^*(y) \\ w_x(x-, 2; y) = w_x(x+, 2; y) & \text{for } x = x_1^*(y) \\ w_x(x, 2; y) = 1 & \text{for } x = x_2^*(y) \\ \frac{1}{2}\sigma_i^2 w_{xx}(x, i; y) - \rho w(x, i; y) + \lambda_i(w(x, 3-i; y) - w(x, i; y)) \leq 0 & \text{for a.e. } x \in \mathbb{R} \text{ and } i = 1, 2 \\ w(x, i; y) \geq x - \theta(y), & \text{for } x \in \mathbb{R} \text{ and } i = 1, 2 \end{array} \right. \quad (3.5)$$

where $w(\cdot \pm, i; y) := \lim_{h \downarrow 0} w(\cdot \pm h, i; y)$ and $w_x(\cdot \pm, i; y) := \lim_{h \downarrow 0} w_x(\cdot \pm h, i; y)$.

Recalling that $\sigma_i > 0$ and $\lambda_i > 0$, $i = 1, 2$, let $\alpha_1 < \alpha_2 < 0 < \alpha_3 < \alpha_4$ be the roots of the fourth-order equation $\Phi_1(\alpha)\Phi_2(\alpha) - \lambda_1\lambda_2 = 0$, with $\Phi_i(\alpha) := -\frac{1}{2}\sigma_i^2\alpha^2 + \rho + \lambda_i$, $i = 1, 2$ (see Lemma B.1 in Appendix B). Then notice that the first equation of (3.5) is actually a system of two second-order ordinary differential equations (ODEs). Hence, transforming such a system into a system of four first-order ODEs, one finds that its general solution is given by

$$\left\{ \begin{array}{l} w(x, 1; y) = A_1(y)e^{\alpha_1 x} + A_2(y)e^{\alpha_2 x} + A_3(y)e^{\alpha_3 x} + A_4(y)e^{\alpha_4 x} \\ w(x, 2; y) = B_1(y)e^{\alpha_1 x} + B_2(y)e^{\alpha_2 x} + B_3(y)e^{\alpha_3 x} + B_4(y)e^{\alpha_4 x}, \end{array} \right. \quad (3.6)$$

for any $x < x_1^*(y)$, $x_1^*(y)$ to be found, and where $B_j(y) := \frac{\Phi_1(\alpha_j)}{\lambda_1} A_j(y) = \frac{\lambda_2}{\Phi_2(\alpha_j)} A_j(y)$, $j = 1, 2, 3, 4$, with $A_j(y)$ to be determined. Since the value function (2.13) diverges at most linearly (cf. Proposition 3.1) we set $A_1(y) = 0 = A_2(y)$, so that also $B_1(y) = 0 = B_2(y)$.

On the other hand, the general solution to the second and the third equations of (3.5) is given on $(x_1^*(y), x_2^*(y))$ by

$$\left\{ \begin{array}{l} w(x, 1; y) = x - \theta(y) \\ w(x, 2; y) = B_5(y)e^{\alpha_5 x} + B_6(y)e^{-\alpha_5 x} + \lambda_2 \left(\frac{x - \theta(y)}{\rho + \lambda_2} \right), \end{array} \right. \quad (3.7)$$

with $\alpha_5 = \sqrt{\frac{2(\rho+\lambda_2)}{\sigma_2^2}}$, and for some $B_5(y)$ and $B_6(y)$ to be found.

Finally, for any $x \geq x_2^*(y)$ we have (cf. the fourth equation of (3.5))

$$w(x, 1; y) = x - \theta(y) = w(x, 2; y). \quad (3.8)$$

It now remains to find the constants $A_3(y), A_4(y), B_5(y), B_6(y)$ and the two threshold values $x_1^*(y), x_2^*(y)$. To accomplish that we impose that $w(\cdot, 1; y)$ is continuous with continuous first-order derivative at $x = x_1^*(y)$, and that $w(\cdot, 2; y)$ is continuous with continuous first-order derivative at $x = x_1^*(y)$ and $x = x_2^*(y)$. Then we find from (3.6)–(3.8) the nonlinear system

$$\begin{cases} A_3(y)e^{\alpha_3 x_1^*(y)} + A_4(y)e^{\alpha_4 x_1^*(y)} = x_1^*(y) - \theta(y) \\ \alpha_3 A_3(y)e^{\alpha_3 x_1^*(y)} + \alpha_4 A_4(y)e^{\alpha_4 x_1^*(y)} = 1 \\ B_3(y)e^{\alpha_3 x_1^*(y)} + B_4(y)e^{\alpha_4 x_1^*(y)} = B_5(y)e^{\alpha_5 x_1^*(y)} + B_6(y)e^{-\alpha_5 x_1^*(y)} + \lambda_2 \left(\frac{x_1^*(y) - \theta(y)}{\rho + \lambda_2} \right) \\ \alpha_3 B_3(y)e^{\alpha_3 x_1^*(y)} + \alpha_4 B_4(y)e^{\alpha_4 x_1^*(y)} = \alpha_5 B_5(y)e^{\alpha_5 x_1^*(y)} - \alpha_5 B_6(y)e^{-\alpha_5 x_1^*(y)} + \frac{\lambda_2}{\rho + \lambda_2} \\ B_5(y)e^{\alpha_5 x_2^*(y)} + B_6(y)e^{-\alpha_5 x_2^*(y)} + \lambda_2 \left(\frac{x_2^*(y) - \theta(y)}{\rho + \lambda_2} \right) = x_2^*(y) - \theta(y) \\ \alpha_5 B_5(y)e^{\alpha_5 x_2^*(y)} - \alpha_5 B_6(y)e^{-\alpha_5 x_2^*(y)} + \frac{\lambda_2}{\rho + \lambda_2} = 1. \end{cases} \quad (3.9)$$

Solving the first two equations of (3.9) with respect to $A_3(y)$ and $A_4(y)$ we obtain after some simple algebra

$$A_3(y) = \left[\frac{\alpha_4(x_1^*(y) - \theta(y)) - 1}{\alpha_4 - \alpha_3} \right] e^{-\alpha_3 x_1^*(y)}, \quad A_4(y) = \left[\frac{1 - \alpha_3(x_1^*(y) - \theta(y))}{\alpha_4 - \alpha_3} \right] e^{-\alpha_4 x_1^*(y)}. \quad (3.10)$$

Analogously, the solution to the fifth and the sixth equations of (3.9) is given in terms of the unknown $x_2^*(y)$ as

$$\begin{cases} B_5(y) = \frac{\rho}{\rho + \lambda_2} \left[\frac{e^{-\alpha_5 x_2^*(y)}(1 + \alpha_5(x_2^*(y) - \theta(y)))}{2\alpha_5} \right] \\ B_6(y) = \frac{\rho}{\rho + \lambda_2} \left[\frac{e^{\alpha_5 x_2^*(y)}(\alpha_5(x_2^*(y) - \theta(y)) - 1)}{2\alpha_5} \right]. \end{cases} \quad (3.11)$$

Finally, plugging (3.10) and (3.11) into the third and the fourth equations of (3.9), recalling that $B_3(y) = \frac{\Phi_1(\alpha_3)}{\lambda_1} A_3(y)$ and $B_4(y) = \frac{\Phi_1(\alpha_4)}{\lambda_1} A_4(y)$, we find after some algebra that $(x_1^*(y), x_2^*(y))$ should satisfy

$$F_1(x_1^*(y), x_2^*(y); y) = 0 \quad \text{and} \quad F_2(x_1^*(y), x_2^*(y); y) = 0, \quad (3.12)$$

where we have set

$$\begin{cases} F_1(u, v; y) := \frac{\rho}{\rho + \lambda_2} \left[(v - \theta(y)) \cosh(\alpha_5(v - u)) - \frac{1}{\alpha_5} \sinh(\alpha_5(v - u)) \right] + a_1(u - \theta(y)) + a_2 \\ F_2(u, v; y) := \frac{\rho}{\rho + \lambda_2} \left[\cosh(\alpha_5(v - u)) - \alpha_5(v - \theta(y)) \sinh(\alpha_5(v - u)) \right] + a_3(u - \theta(y)) + a_4, \end{cases} \quad (3.13)$$

with $a_i := a_i(\rho, \lambda_1, \lambda_2, \sigma_1, \sigma_2)$, $i = 1, 2, 3, 4$, given by

$$\begin{cases} a_1 := -\frac{\alpha_4 \Phi_1(\alpha_3) - \alpha_3 \Phi_1(\alpha_4)}{\lambda_1(\alpha_4 - \alpha_3)} + \frac{\lambda_2}{\rho + \lambda_2}, & a_2 := \frac{\Phi_1(\alpha_3) - \Phi_1(\alpha_4)}{\lambda_1(\alpha_4 - \alpha_3)}, \\ a_3 := \frac{\alpha_3 \alpha_4}{\lambda_1(\alpha_4 - \alpha_3)} [\Phi_1(\alpha_4) - \Phi_1(\alpha_3)], & a_4 := \frac{\alpha_3 \Phi_1(\alpha_3) - \alpha_4 \Phi_1(\alpha_4)}{\lambda_1(\alpha_4 - \alpha_3)} + \frac{\lambda_2}{\rho + \lambda_2}. \end{cases} \quad (3.14)$$

Notice that $a_1 < 0$, $a_2 > 0$, $a_3 < 0$ and $a_4 > 0$ by Lemma B.2 in Appendix B.

Since we expect from (2.13) that $x_i^*(y)$, $i = 1, 2$, are such that $x_2^*(y) > x_1^*(y) \geq \theta(y)$, it is natural to check if (3.12) admits a solution in $(\theta(y), \infty) \times (\theta(y), \infty)$. So far we do not know about existence, and in case uniqueness, of such a solution. To investigate this fact we define $z_1^*(y) := x_1^*(y) - \theta(y)$ and $z_2^*(y) := x_2^*(y) - x_1^*(y)$, so that $x_2^*(y) - \theta(y) = z_1^*(y) + z_2^*(y)$, and we notice that with such definition the explicit dependence with respect to y disappears in (3.12). We can thus drop the y -dependence in $z_i^*(y)$, $i = 1, 2$, and set (z_1^*, z_2^*) as the solution, if it does exist, of the equivalent system

$$G_1(u, v) = 0 \quad \text{and} \quad G_2(u, v) = 0, \quad (3.15)$$

with

$$\begin{cases} G_1(u, v) := (a_1 + \frac{\rho}{\rho+\lambda_2} \cosh(\alpha_5 v))u - \frac{\rho}{\rho+\lambda_2} [\frac{1}{\alpha_5} \sinh(\alpha_5 v) - v \cosh(\alpha_5 v)] + a_2 \\ G_2(u, v) := (a_3 - \frac{\rho\alpha_5}{\rho+\lambda_2} \sinh(\alpha_5 v))u - \frac{\rho}{\rho+\lambda_2} [v\alpha_5 \sinh(\alpha_5 v) - \cosh(\alpha_5 v)] + a_4, \end{cases} \quad (3.16)$$

for $u, v \geq 0$.

Proposition 3.2. *Let \hat{z}_2 be the unique positive solution to the equation*

$$a_1 + \frac{\rho}{\rho+\lambda_2} \cosh(\alpha_5 v) = 0, \quad v \geq 0,$$

with a_1 as in (3.14) and $\alpha_5 = \sqrt{\frac{2(\rho+\lambda_2)}{\sigma_2^2}}$. Then there exists a unique couple (z_1^*, z_2^*) solving (3.15) in $(0, \infty) \times (0, \hat{z}_2)$ if and only if $\sigma_1^2 < \sigma_2^2$. Moreover z_1^* is such that

$$-\frac{a_2}{a_1 + \frac{\rho}{\rho+\lambda_2}} < z_1^* < -\frac{\frac{\rho}{\rho+\lambda_2} + a_4}{a_3}.$$

Proof. Step 1. We preliminary notice that the function $r(v) := \frac{\rho}{\rho+\lambda_2} [\frac{1}{\alpha_5} \sinh(\alpha_5 v) - v \cosh(\alpha_5 v)] - a_2$, $v \geq 0$, is strictly decreasing, and therefore strictly negative for any $v \geq 0$ since $r(0) = -a_2 < 0$ (cf. Lemma B.2 in Appendix B).

Step 2. We here prove that the equation $h(v) = 0$ with $h(v) := a_1 + \frac{\rho}{\rho+\lambda_2} \cosh(\alpha_5 v)$, $v \geq 0$, admits a unique solution $\hat{z}_2 > 0$. For this it suffices to notice that $v \mapsto h(v)$ is strictly increasing with $\lim_{v \rightarrow \infty} h(v) = +\infty$, and that $h(0) = a_1 + \frac{\rho}{\rho+\lambda_2} = -\frac{\rho + \frac{1}{2}\sigma_1^2\alpha_3\alpha_4}{\lambda_1} < 0$. The last equality in the previous formula follows by using (B-4) in Appendix B.

For any $v \in [0, \hat{z}_2)$ we can thus rewrite (3.15) in the equivalent form

$$u = M_1(v), \quad M_1(v) - M_2(v) = 0,$$

with

$$\begin{cases} M_1(v) := \frac{\frac{\rho}{\rho+\lambda_2} [\frac{1}{\alpha_5} \sinh(\alpha_5 v) - v \cosh(\alpha_5 v)] - a_2}{a_1 + \frac{\rho}{\rho+\lambda_2} \cosh(\alpha_5 v)} \\ M_2(v) := \frac{\frac{\rho}{\rho+\lambda_2} [v\alpha_5 \sinh(\alpha_5 v) - \cosh(\alpha_5 v)] - a_4}{a_3 - \frac{\rho\alpha_5}{\rho+\lambda_2} \sinh(\alpha_5 v)}, \end{cases} \quad (3.17)$$

where we have also used the fact that $a_3 - \frac{\rho\alpha_5}{\rho+\lambda_2} \sinh(\alpha_5 v) \neq 0$ on $[0, \infty)$ being $a_3 < 0$ (see again Lemma B.2 in Appendix B).

The numerator of M_1 in (3.17) is strictly negative on $v \geq 0$ by *Step 1*. Using this fact, and noticing that $a_1 + \frac{\rho}{\rho+\lambda_2} \cosh(\alpha_5 v) < 0$ on $[0, \hat{z}_2)$, by direct calculations one can observe that $v \mapsto M_1(v)$ strictly increases on $[0, \hat{z}_2)$, and it is such that $\lim_{z \uparrow \hat{z}_2} M_1(v) = +\infty$.

Also, one can check by employing (B-3) and (B-5) of Lemma B.2, and the definitions of α_3 and α_4 , that $M_1(0) - M_2(0) = \frac{1}{a_3} \left(\frac{\rho}{\rho+\lambda_2} + a_4 \right) - \frac{a_2}{a_1 + \frac{\rho}{\rho+\lambda_2}} < 0$ if and only if $\sigma_1^2 < \sigma_2^2$. We now claim (and prove later) that $v \mapsto M_2(v)$ strictly decreases in $[0, \hat{z}_2]$, so that $v \mapsto M_1(v) - M_2(v)$ strictly increases on $[0, \hat{z}_2)$ and diverges to $+\infty$ as z approaches \hat{z}_2 . Combining all these facts we conclude that there exists a unique $z_2^* \in (0, \hat{z}_2)$ solving $M_1(v) - M_2(v) = 0$. Hence, $z_1^* = M_1(z_2^*)$ (or, equivalently, $z_1^* = M_2(z_2^*)$), and $z_1^* > 0$ because $M_1(v) \geq M_1(0) > 0$ on $[0, \hat{z}_2)$.

Moreover, since $M_1(\cdot)$ is strictly increasing, $M_2(\cdot)$ is strictly decreasing on $[0, \hat{z}_2)$, and $z_2^* < \hat{z}_2$, one has $M_1(0) < z_1^* < M_2(0)$, i.e.

$$0 < -\frac{a_2}{a_1 + \frac{\rho}{\rho+\lambda_2}} < z_1^* < -\frac{\frac{\rho}{\rho+\lambda_2} + a_4}{a_3}. \quad (3.18)$$

Step 3. To complete the proof we need to show that $v \mapsto M_2(v)$ is strictly decreasing in $[0, \hat{z}_2]$. By direct calculations one can see that the latter monotonicity property holds if

$$-\frac{\rho}{\rho + \lambda_2} \cosh(\alpha_5 v) + a_3 v < 0$$

on $[0, \hat{z}_2]$. But this is true since $a_3 < 0$. \square

Because by Proposition 3.2 there exists a unique couple (z_1^*, z_2^*) solving (3.15) in $(0, \infty) \times (0, \hat{z}_2)$ if and only if $\sigma_1^2 < \sigma_2^2$, the latter condition is taken as a standing assumption throughout the rest of this section.

Corollary 3.3. *There exists a unique couple $(x_1^*(y), x_2^*(y)) \in (\theta(y), +\infty) \times (\theta(y), +\infty)$ solving (3.12). Moreover, it is such that $x_2^*(y) > x_1^*(y)$.*

Proof. By Proposition 3.2 there exists a unique couple (z_1^*, z_2^*) solving (3.15) in $(0, \infty) \times (0, \hat{z}_2)$. Since $z_1^* = x_1^*(y) - \theta(y)$ and $z_2^* = x_2^*(y) - x_1^*(y)$, one has $x_1^*(y) = z_1^* + \theta(y) > \theta(y)$ and $x_2^*(y) = z_2^* + x_1^*(y) > x_1^*(y) > \theta(y)$. \square

Theorem 3.4 below proves that $(w(x, 1; y), w(x, 2; y), x_1^*(y), x_2^*(y))$ solves free-boundary problem (3.5). Its proof is quite long and technical, and for this reason it is postponed to Appendix A.

Theorem 3.4. [The Candidate Value Function] *Let $(x_1^*(y), x_2^*(y))$ with $x_2^*(y) > x_1^*(y)$ be the unique solution to (3.12) in $(\theta(y), +\infty) \times (\theta(y), +\infty)$. Define $A_3(y)$ and $A_4(y)$ as in (3.10), $B_3(y) := \frac{\Phi_1(\alpha_3)}{\lambda_1} A_3(y)$ and $B_4(y) := \frac{\Phi_1(\alpha_4)}{\lambda_1} A_4(y)$, and $B_5(y)$ and $B_6(y)$ as in (3.11). Then the functions*

$$w(x, 1; y) := \begin{cases} A_3(y)e^{\alpha_3 x} + A_4(y)e^{\alpha_4 x}, & x \leq x_1^*(y) \\ x - \theta(y), & x \geq x_1^*(y), \end{cases} \quad (3.19)$$

and

$$w(x, 2; y) := \begin{cases} B_3(y)e^{\alpha_3 x} + B_4(y)e^{\alpha_4 x}, & x \leq x_1^*(y) \\ B_5(y)e^{\alpha_5 x} + B_6(y)e^{-\alpha_5 x} + \lambda_2 \left(\frac{x - \theta(y)}{\rho + \lambda_2} \right), & x_1^*(y) \leq x \leq x_2^*(y) \\ x - \theta(y), & x \geq x_2^*(y), \end{cases} \quad (3.20)$$

are such that $w(\cdot, i; y) \in C^1(\mathbb{R})$ with $w_{xx}(\cdot, i; y) \in L_{loc}^\infty(\mathbb{R})$ for any $i = 1, 2$, and $|w(x, i; y)| \leq \kappa_i(y)(1 + |x|)$ for some $\kappa_i(y) > 0$. Moreover, $(w(x, 1; y), w(x, 2; y), x_1^*(y), x_2^*(y))$ solves free-boundary problem (3.5).

We now verify the actual optimality of the candidate value function of Theorem 3.4. The proof of this result is contained in Appendix A.

Theorem 3.5. [The Verification Theorem] *Let $\mathcal{C} = \{(x, 1) : x < x_1^*(y)\} \cup \{(x, 2) : x < x_2^*(y)\}$. Then, for w as in Theorem 3.4 and for u as in (2.13), one has that $w = u$ on $\mathbb{R} \times \{1, 2\}$ and*

$$\tau^* := \inf\{t \geq 0 : (X_t, \varepsilon_t) \notin \mathcal{C}\}, \quad \mathbb{P}_{(x, i)} - a.s., \quad (3.21)$$

is an optimal stopping time.

3.2 Case (B): $x_1^*(y) = x_2^*(y)$

In this section we study the case in which the two boundaries $x_1^*(y)$ and $x_2^*(y)$ coincide and are equal to some $x^*(y)$ to be found.

We rewrite (3.3) in the form of a free-boundary problem to find $(w(x, 1; y), w(x, 2; y), x^*(y))$, with $w(\cdot, i; y) \in C^1(\mathbb{R})$ and $w_{xx}(\cdot, i; y) \in L_{loc}^\infty(\mathbb{R})$ for any $i = 1, 2$, solving

$$\begin{cases} \frac{1}{2}\sigma_i^2 w_{xx}(x, i; y) - \rho w(x, i; y) + \lambda_i(w(x, 3-i; y) - w(x, i; y)) = 0 & \text{for } x < x^*(y) \text{ and } i = 1, 2 \\ w(x, i; y) = x - \theta(y) & \text{for } x \geq x^*(y) \\ w_x(x, i; y) = 1 & \text{for } x = x^*(y) \\ \frac{1}{2}\sigma_i^2 w_{xx}(x, i; y) - \rho w(x, i; y) + \lambda_i(w(x, 3-i; y) - w(x, i; y)) \leq 0 & \text{for a.e. } x \in \mathbb{R} \text{ and } i = 1, 2 \\ w(x, i; y) \geq x - \theta(y), & \text{for } x \in \mathbb{R} \text{ and } i = 1, 2. \end{cases} \quad (3.22)$$

Recall

$$\Phi_i(\alpha) = -\frac{1}{2}\sigma_i^2\alpha^2 + \rho + \lambda_i, \quad i = 1, 2, \quad (3.23)$$

and that $\alpha_1 < \alpha_2 < 0 < \alpha_3 < \alpha_4$ denote the solutions to the fourth-order equation $\Phi_1(\alpha)\Phi_2(\alpha) - \lambda_1\lambda_2 = 0$ (cf. Lemma B.1 in Appendix B). Then the general solution to the system of two second-order ODEs appearing in the first equation of (3.22) is given for any $x < x^*(y)$ by

$$\begin{cases} w(x, 1; y) = \tilde{A}_1(y)e^{\alpha_1 x} + \tilde{A}_2(y)e^{\alpha_2 x} + \tilde{A}_3(y)e^{\alpha_3 x} + \tilde{A}_4(y)e^{\alpha_4 x} \\ w(x, 2; y) = \tilde{B}_1(y)e^{\alpha_1 x} + \tilde{B}_2(y)e^{\alpha_2 x} + \tilde{B}_3(y)e^{\alpha_3 x} + \tilde{B}_4(y)e^{\alpha_4 x}, \end{cases} \quad (3.24)$$

with

$$\tilde{B}_j(y) = \frac{\Phi_1(\alpha_j)}{\lambda_1} \tilde{A}_j(y) = \frac{\lambda_2}{\Phi_2(\alpha_j)} \tilde{A}_j(y), \quad j = 1, 2, 3, 4. \quad (3.25)$$

Notice that from the expressions of α_3 and α_4 (see the proof of Lemma B.1 in Appendix B) one has $\Phi_1(\alpha_3) > 0$ and $\Phi_1(\alpha_4) < 0$. Since for $x \rightarrow -\infty$ the value function diverges at most with linear growth (cf. Proposition 3.1) we set $\tilde{A}_1(y) = \tilde{A}_2(y) = 0 = \tilde{B}_1(y) = \tilde{B}_2(y)$.

For $x \in [x^*(y), +\infty)$ we have from (3.22)

$$w(x, 1; y) = x - \theta(y) = w(x, 2; y). \quad (3.26)$$

It now only remains to find $\tilde{A}_3(y)$, $\tilde{A}_4(y)$ and $x^*(y)$, since $\tilde{B}_3(y)$ and $\tilde{B}_4(y)$ are given in terms of $\tilde{A}_3(y)$ and $\tilde{A}_4(y)$ through (3.25). To do so, we impose that $w(\cdot, i; y)$, $i = 1, 2$, is continuous across $x^*(y)$ together with its first derivative, and we obtain

$$\begin{cases} \tilde{A}_3(y)e^{\alpha_3 x^*(y)} + \tilde{A}_4(y)e^{\alpha_4 x^*(y)} = x^*(y) - \theta(y) \\ \alpha_3 \tilde{A}_3(y)e^{\alpha_3 x^*(y)} + \alpha_4 \tilde{A}_4(y)e^{\alpha_4 x^*(y)} = 1 \\ \tilde{B}_3(y)e^{\alpha_3 x^*(y)} + \tilde{B}_4(y)e^{\alpha_4 x^*(y)} = x^*(y) - \theta(y) \\ \alpha_3 \tilde{B}_3(y)e^{\alpha_3 x^*(y)} + \alpha_4 \tilde{B}_4(y)e^{\alpha_4 x^*(y)} = 1. \end{cases} \quad (3.27)$$

Solving the first two equations of (3.27) for $\tilde{A}_3(y)$ and $\tilde{A}_4(y)$, one has

$$\tilde{A}_3(y) = \left[\frac{\alpha_4(x^*(y) - \theta(y)) - 1}{(\alpha_4 - \alpha_3)} \right] e^{-\alpha_3 x^*(y)} \quad \tilde{A}_4(y) = \left[\frac{1 - \alpha_3(x^*(y) - \theta(y))}{(\alpha_4 - \alpha_3)} \right] e^{-\alpha_4 x^*(y)}. \quad (3.28)$$

On the other hand, recalling (3.25) and plugging $\tilde{A}_3(y)$ and $\tilde{A}_4(y)$ from (3.28) into the third equation of (3.27), some simple algebra leads to

$$x^*(y) = \frac{\frac{1}{2}\sigma_1^2(\alpha_3 + \alpha_4)}{\rho + \frac{1}{2}\sigma_1^2\alpha_3\alpha_4} + \theta(y), \quad (3.29)$$

where (3.23) has also been used. Similarly, inserting $\tilde{A}_3(y)$ and $\tilde{A}_4(y)$ of (3.28) into the fourth equation of (3.27) and using (3.23) one obtains

$$x^*(y) = \frac{\frac{1}{2}\sigma_1^2(\alpha_3^2 + \alpha_4^2 + \alpha_3\alpha_4) - \rho}{\frac{1}{2}\sigma_1^2\alpha_3\alpha_4(\alpha_3 + \alpha_4)} + \theta(y). \quad (3.30)$$

Equations (3.29) and (3.30) then imply that system (3.27) admits a solution (which is then unique) if and only if

$$\frac{\frac{1}{2}\sigma_1^2(\alpha_3 + \alpha_4)}{\rho + \frac{1}{2}\sigma_1^2\alpha_3\alpha_4} = \frac{\frac{1}{2}\sigma_1^2(\alpha_3^2 + \alpha_4^2 + \alpha_3\alpha_4) - \rho}{\frac{1}{2}\sigma_1^2\alpha_3\alpha_4(\alpha_3 + \alpha_4)}. \quad (3.31)$$

Using that $(\alpha_3\alpha_4)^2 = 4[(\rho + \lambda_1)(\rho + \lambda_2) - \lambda_1\lambda_2]/\sigma_1^2\sigma_2^2$ and $\alpha_3^2 + \alpha_4^2 = 2\sigma_1^2(\rho + \lambda_2) + 2\sigma_2^2(\rho + \lambda_1)/\sigma_1^2\sigma_2^2$ by Vieta's formulas, one can show that (3.31) is equivalent to $\sigma_1^2 = \sigma_2^2 =: \sigma^2$. In such case, it is not hard to check by direct calculations that $\alpha_3^2 = 2\rho/\sigma^2$ and $\alpha_4^2 = 2(\rho + \lambda_1 + \lambda_2)/\sigma^2$. With regard to (3.25) this in turn gives

$$\tilde{B}_3(y) = \tilde{A}_3(y) \quad \text{and} \quad \tilde{B}_4(y) = -\frac{\lambda_2}{\lambda_1}\tilde{A}_4(y). \quad (3.32)$$

The proof of the next result follows by using the identities $\alpha_3^2 = 2\rho/\sigma^2$ and $\alpha_4^2 = 2(\rho + \lambda_1 + \lambda_2)/\sigma^2$ in (3.29).

Corollary 3.6. *One has*

$$x^*(y) = \frac{\sigma}{\sqrt{2\rho}} + \theta(y) > \theta(y) \quad (3.33)$$

Although the boundary x^* of (3.33) coincides with that one would obtain in a model without regime switching (see Section 5 for details), the next result - whose proof is contained in Appendix A - shows that the (candidate) optimal value function is regime-dependent, and differs from that one finds in a no-regime-switching case.

Theorem 3.7. [The Candidate Value Function] Assume $\sigma_1 = \sigma_2$. Let $x^*(y) \in (\theta(y), +\infty)$ be given by (3.33), define $\tilde{A}_3(y)$ and $\tilde{A}_4(y)$ as in (3.28) and recall (3.32). Then the functions

$$w(x, 1, y) := \begin{cases} \tilde{A}_3(y)e^{\alpha_3 x} + \tilde{A}_4(y)e^{\alpha_4 x}, & x \leq x^*(y) \\ x - \theta(y), & x \geq x^*(y), \end{cases} \quad (3.34)$$

and

$$w(x, 2, y) := \begin{cases} \tilde{A}_3(y)e^{\alpha_3 x} - \frac{\lambda_2}{\lambda_1} \tilde{A}_4(y)e^{\alpha_4 x}, & x \leq x^*(y) \\ x - \theta(y), & x \geq x^*(y), \end{cases} \quad (3.35)$$

are such that $w(\cdot, i, y) \in C^1(\mathbb{R})$ with $w_{xx}(\cdot, i, y) \in L_{loc}^\infty(\mathbb{R})$ for any $i = 1, 2$, and $|w(x, i, y)| \leq \kappa_i(y)(1 + |x|)$ for some $\kappa_i(y) > 0$. Moreover, $(w(x, 1, y), w(x, 2, y), x^*(y))$ solves free-boundary problem (3.22).

By arguing as in the proof of Theorem 3.5, one can also prove the following theorem.

Theorem 3.8. [The Verification Theorem] Let $\mathcal{C} = \{(x, i) \in \mathbb{R} \times \{1, 2\} : x < x^*(y)\}$. Under the same conditions of Theorem 3.7 and with u as in (2.13) one has that $w = u$ on $\mathbb{R} \times \{1, 2\}$ and

$$\tau^* := \inf\{t \geq 0 : (X_t, \varepsilon_t) \notin \mathcal{C}\}, \quad \mathbb{P}_{(x, i)} - a.s., \quad (3.36)$$

is an optimal stopping time.

4 The Optimal Extraction Policy

In this section we provide the solution to the finite-fuel singular stochastic control problem (2.6) in terms of the solution to the optimal stopping problem with regime switching (2.13). In particular, we consider separately the two cases (I) $y \mapsto f(y)$ strictly convex on $[0, 1]$, and (II) $y \mapsto f(y)$ strictly concave on $[0, 1]$ (cf. Assumption 2.1). It turns out that the optimal extraction rule is qualitatively different among these two cases.

4.1 Case (I): $y \mapsto f(y)$ strictly convex on $[0, 1]$

Assume that $y \mapsto f(y)$ fulfills condition (I) of Assumption 2.1. For any $y \in [0, 1]$, let $\theta(y)$ in (2.13) be such that

$$\theta(y) := c - \frac{f'(y)}{\rho},$$

and notice that with such a choice of θ all the results of Section 3 remains valid for $y \in [0, 1]$.

By Corollary 3.3 we know that $x_1^*(y) = z_1^* + \theta(y)$ and $x_2^*(y) = z_2^* + x_1^*(y)$ (see also Corollary 3.6 in the case $x_1^*(y) = x_2^*(y) = x^*(y)$). Because $y \mapsto f(y)$ is continuously differentiable and strictly convex on $[0, 1]$, it follows that for any $i = 1, 2$, $y \mapsto x_i^*(y)$ is continuous and strictly decreasing on $[0, 1]$, and it has an inverse with respect to y . For $i = 1, 2$, we then define

$$b_i^*(x) := \begin{cases} 1, & x \leq x_i^*(1) \\ (x_i^*)^{-1}(x), & x \in (x_i^*(1), x_i^*(0)) \\ 0, & x \geq x_i^*(0), \end{cases} \quad (4.1)$$

and we observe that $b_i^* : \mathbb{R} \rightarrow [0, 1]$ is continuous and decreasing. Notice that also the case in which $x_1^*(y) = x_2^*(y)$ (i.e. case (B) of Section 3) can be accommodated into (4.1). Indeed, in such case we simply have $b_1^* = b_2^*$.

We now aim at providing a candidate value function for problem (2.6). To this end, for u as in Theorems 3.5 or 3.8, we introduce the function

$$F(x, y, i) := \int_0^y u(x, i; z) dz - \frac{f(y)}{\rho}. \quad (4.2)$$

Proposition 4.1. *The function F of (4.2) is such that $F(\cdot, \cdot, i) \in C^{2,1}(\mathbb{R} \times [0, 1])$ for any $i = 1, 2$ and the following bounds hold*

$$|F(x, y, i)| + |F_y(x, y, i)| \leq C(1 + |x|), \quad |F_x(x, y, i)| + |F_{xx}(x, y, i)| \leq \kappa, \quad (4.3)$$

for $(x, y, i) \in \mathcal{O}$ and some positive constants C and κ (possibly depending on i).

Proof. It is easy to verify from (3.19) and (3.20), and from (3.34) and (3.35) (upon recalling also Theorems 3.5 and 3.8) that u is of the form $u(x, i; y) = \zeta_i(y)G_i(x) + \eta_i(y)H_i(x)$ for some continuous functions ζ_i , η_i , G_i and H_i . It thus follows that $(x, y) \mapsto F(x, y, i)$ and $(x, y) \mapsto F_y(x, y, i)$ are continuous on $\mathbb{R} \times [0, 1]$. Also, from (3.19) and (3.20), and from (3.34) and (3.35), one can see that for any x in a bounded set $\mathcal{K} \subset \mathbb{R}$ and for any $i = 1, 2$ the derivatives $|u_x|$ and $|u_{xx}|$ are at least bounded by a function $F_{\mathcal{K}}(y) \in L^1(0, 1)$. It follows that to determine F_x and F_{xx} one can invoke the dominate convergence theorem and evaluate derivatives inside the integral in (4.2) so to obtain

$$F_x(x, y, i) = \int_0^{b_1^*(x) \wedge y} u_x(x, i; z) dz + \int_{b_1^*(x) \wedge y}^{b_2^*(x) \wedge y} u_x(x, i; z) dz + \int_{b_2^*(x) \wedge y}^y u_x(x, i; z) dz \quad (4.4)$$

and

$$F_{xx}(x, y, i) = \int_0^{b_1^*(x) \wedge y} u_{xx}(x, i; z) dz + \int_{b_1^*(x) \wedge y}^{b_2^*(x) \wedge y} u_{xx}(x, i; z) dz, \quad (4.5)$$

where the second integrals on the right hand side of (4.4) and (4.5) equal zero in case $b_1^* = b_2^*$. Therefore $F(\cdot, \cdot, i) \in C^{2,1}(\mathbb{R} \times [0, 1])$ for $i = 1, 2$ by (3.19) and (3.20), (3.34) and (3.35), Theorems 3.5 and 3.8, and continuity of $b_i^*(\cdot)$ (cf. (4.1)). Finally, bounds (4.3) follow from (3.19) and (3.20), (3.34) and (3.35), (4.2), (4.4) and (4.5). \square

The next result shows that F solves the HJB equation (2.7).

Proposition 4.2. *For all $(x, y, i) \in \mathbb{R} \times (0, 1] \times \{1, 2\}$, F is a classical solution of (2.7). Moreover, it satisfies the boundary condition $F(x, 0, i) = 0$ for $(x, i) \in \mathbb{R} \times \{1, 2\}$.*

Proof. First of all we observe that for any $(x, y, i) \in \mathcal{O}$ one has by (4.2) that

$$F_y(x, y, i) = u(x, i; y) - \frac{f'(y)}{\rho} \geq x - c, \quad (4.6)$$

where the last inequality follows from the fact that $u(x, i; y) \geq x - \theta(y) = x - c + \frac{f'(y)}{\rho}$. In particular, for any $i = 1, 2$ one has equality in (4.6) on $\{(x, y) \in \mathbb{R} \times [0, 1] : x \geq x_i^*(y)\}$.

For any fixed $i = 1, 2$, take $y \in [0, 1]$ and $x \in \mathbb{R}$ such that $F_y(x, y, i) > x - c$, i.e. $y < b_i^*(x)$, and notice that, thanks to Proposition 4.1, one can write

$$(\mathcal{G} - \rho)F(x, y, 1) = \int_0^y (\mathcal{G} - \rho)u(x, 1; z) dz + f(y) = f(y),$$

and

$$(\mathcal{G} - \rho)F(x, y, 2) = \int_0^{y \wedge b_1^*(x)} (\mathcal{G} - \rho)u(x, 2; z)dz + \int_{y \wedge b_1^*(x)}^{y \wedge b_2^*(x)} (\mathcal{G} - \rho)u(x, 2; z)dz + f(y) = f(y).$$

The last equalities in the two equations above follow from the fact that u solves free-boundary problem (3.5) (cf. Theorems 3.4 and 3.5; see also Theorems 3.7 and 3.8 in the case $x_1^*(y) = x_2^*(y) = x^*(y)$).

On the other hand, for arbitrary $(x, y, i) \in \mathcal{O}$ we notice that (cf. (4.1))

$$\begin{aligned} (\mathcal{G} - \rho)F(x, y, i) &= \int_0^{b_1^*(x) \wedge y} (\mathcal{G} - \rho)u(x, i; z)dz + \int_{b_1^*(x) \wedge y}^{b_2^*(x) \wedge y} (\mathcal{G} - \rho)u(x, i; z)dz \\ &\quad + \int_{b_2^*(x) \wedge y}^y (\mathcal{G} - \rho)u(x, i; z)dz + f(y) \leq f(y), \end{aligned}$$

since, again, u solves free-boundary problem (3.5). Hence F solves (2.7). Moreover, recalling that $f(0) = 0$, it is straightforward to see from (4.2) that $F(x, 0, i) = 0$ for any $(x, i) \in \mathbb{R} \times \{1, 2\}$. \square

Satisfying (2.7) and the boundary condition $F(x, 0, i) = 0$ for $(x, i) \in \mathbb{R} \times \{1, 2\}$, F is clearly a candidate value function for problem (2.6). We now introduce a candidate optimal control process. Let $(x, y, i) \in \mathcal{O}$, recall b_i^* of (4.1) and consider the process

$$\nu_t^* = \left[y - \inf_{0 \leq s < t} b_{\varepsilon_s}^*(X_s) \right]^+, \quad t > 0, \quad \nu_0^* = 0, \quad (4.7)$$

where $[\cdot]^+$ denotes the positive part.

Proposition 4.3. *The process ν^* of (4.7) is an admissible control.*

Proof. Recall (2.4). For any given and fixed $\omega \in \Omega$, $t \mapsto \nu_t^*(\omega)$ is clearly nondecreasing and such that $Y_t^{\nu^*}(\omega) \geq 0$, for any $t \geq 0$, since $b_i^*(x) \in [0, 1]$ for any $x \in \mathbb{R}$. Moreover, since (X, ε) is right-continuous with left-limits (cf. Lemma 3.6 in [36]) and $(x, i) \mapsto b_i^*(x)$ is continuous, $t \mapsto \nu_t^*(\omega)$ is left-continuous. Finally, \mathbb{F} -progressive measurability of (X, ε) and measurability of b^* imply that ν^* is \mathbb{F} -progressively measurable by [12], Theorem IV.33, whence \mathbb{F} -adapted. \square

Process ν^* is the minimal effort needed to have $Y_t^{\nu^*} \leq b_{\varepsilon_t}^*(X_t)$ at any time t . In particular it is a standard result (see, e.g., Proposition 2.7 in [10] and references therein for a proof in a similar setting) that ν^* of (4.7) solves the Skorokhod reflection problem

1. $Y_t^* \leq b_{\varepsilon_t}^*(X_t)$, $\mathbb{P}_{(x, y, i)}$ -almost surely, for each $t > 0$;
2. $\int_0^T \mathbf{1}_{\{Y_t^* < b_{\varepsilon_t}^*(X_t)\}} d\nu_t^* = 0$ $\mathbb{P}_{(x, y, i)}$ -almost surely, for all $T \geq 0$,

where $Y^* := Y^{\nu^*}$. An illustration of the (candidate) optimal policy ν^* is provided in Figure 1.

Theorem 4.4. [The Verification Theorem] *The control ν^* of (4.7) is optimal for problem (2.6), and F of (4.2) is such that $F \equiv V$.*

Proof. Since F is a classical solution to the HJB equation due to Proposition 4.2, one has $F \geq V$ on \mathcal{O} by Theorem 2.5. We now show that one actually has $F = V$ on \mathcal{O} , and that ν^* of (4.7) is optimal for problem (2.6).

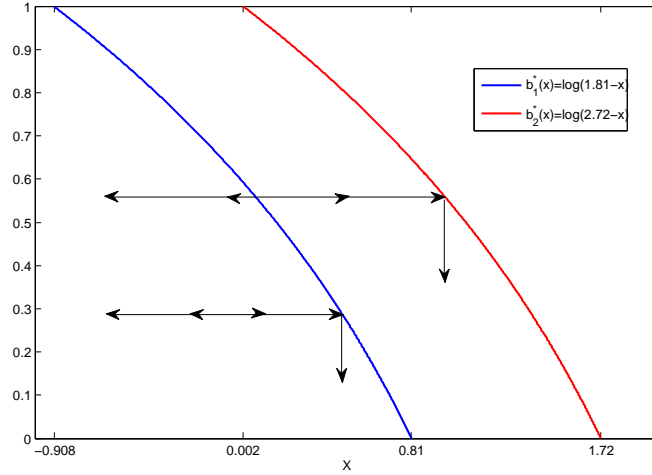


Figure 1: Adopting the terminology of [20], the boundaries b_i^* , $i = 1, 2$, split the state space into the *inaction region* ($y < b_1^*(x)$), *transient region* ($b_1^*(x) < y < b_2^*(x)$) and *action region* ($y > b_2^*(x)$). When the initial state is $(x, y, i) \in \mathcal{O}$ with $y < b_i^*(x)$ one observes a Skorokhod reflection of (X, Y^*, ε) at b_i^* in the vertical direction up to when all the fuel is spent. If the system is reflected at the upper boundary, at a time of regime switch ν^* prescribes an immediate jump of Y^* from the upper to the lower boundary (whenever they are different). This plot was obtained solving with Matlab the nonlinear system (3.15) when $f(y) = \frac{1}{3}(e^y - 1)$ and with $\sigma_1 = 0.38$, $\sigma_2 = 1.9$, $\lambda_1 = 1.7$, $\lambda_2 = 0.44$, $\rho = 1/3$ and $c = 0.5$.

If $y = 0$ then $F(x, 0, i) = 0 = V(x, 0, i)$. Take then $y \in (0, 1]$, set $Y^* := Y^{\nu^*}$ with ν^* as in (4.3) and define $\vartheta := \inf\{t \geq 0 : \nu_t^* = y\}$ and $\tau_R := \inf\{t \geq 0 : X_t^x \notin (-R, R)\}$, for some $R > 0$. Given the regularity of F , we can repeat the arguments at the beginning of the proof of Theorem 2.5, and apply Itô's formula so to obtain

$$\begin{aligned}
F(x, y, i) = & \mathbb{E}_{(x, y, i)} \left[e^{-\rho(\tau_R \wedge \vartheta)} F(X_{\tau_R \wedge \vartheta}, Y_{\tau_R \wedge \vartheta}^*, \varepsilon_{\tau_R \wedge \vartheta}) - \int_0^{\tau_R \wedge \vartheta} e^{-\rho s} (\mathcal{G} - \rho) F(X_s, Y_s^*, \varepsilon_s) ds \right] \\
& + \mathbb{E}_{(x, y, i)} \left[\int_0^{\tau_R \wedge \vartheta} e^{-\rho s} F_y(X_s, Y_s^*, \varepsilon_s) d\nu_s^{*, cont} \right] \\
& - \mathbb{E}_{(x, y, i)} \left[\sum_{0 \leq s < \tau_R \wedge \vartheta} e^{-\rho s} (F(X_s, Y_{s+}^*, \varepsilon_s) - F(X_s, Y_s^*, \varepsilon_s)) \right],
\end{aligned} \tag{4.8}$$

where $\nu^{*, cont}$ denotes the continuous part of ν^* .

Recall now (2.9), that $(\mathcal{G} - \rho)F(x, y, i) = -f(y)$ for $y < b_i^*(x)$ and $F_y(x, y, i) = x - c$ for $y \geq b_i^*(x)$, and the fact that ν^* solves the Skorokhod reflection problem, and therefore $\{t : d\nu_t^*(\omega) > 0\} \subseteq \{t : Y_t^*(\omega) \geq b_{\varepsilon_t(\omega)}^*(X_t(\omega))\}$ for any $\omega \in \Omega$. Then by using all these facts we obtain from (4.8)

$$\begin{aligned}
F(x, y, i) = & \mathbb{E}_{(x, y, i)} \left[e^{-\rho(\tau_R \wedge \vartheta)} F(X_{\tau_R \wedge \vartheta}, Y_{\tau_R \wedge \vartheta}^*, \varepsilon_{\tau_R \wedge \vartheta}) - \int_0^{\tau_R \wedge \vartheta} e^{-\rho s} f(Y_s^*) ds \right] \\
& + \int_0^{\tau_R \wedge \vartheta} e^{-\rho s} (X_s - c) d\nu_s^*.
\end{aligned} \tag{4.9}$$

As $R \rightarrow \infty$, $\tau_R \rightarrow \infty$, and clearly $\tau_R \wedge \vartheta \rightarrow \vartheta$, $\mathbb{P}_{(x, y, i)}$ -a.s. Moreover, we can use the sublinear growth property of F (cf. (4.3)) and Lemma B.3 in Appendix B to apply the dominated

convergence theorem and have

$$\lim_{R \uparrow \infty} \mathbb{E}_{(x,y,i)} \left[e^{-\rho(\tau_R \wedge \vartheta)} F(X_{\tau_R \wedge \vartheta}, Y_{\tau_R \wedge \vartheta}^*, \varepsilon_{\tau_R \wedge \vartheta}) \right] = \mathbb{E}_{(x,y,i)} \left[e^{-\rho\vartheta} F(X_\vartheta, Y_\vartheta^*, \varepsilon_\vartheta) \right] = 0.$$

Finally, we also notice that since $d\nu_s^* \equiv 0$ and $f(Y_s^*) \equiv 0$ for $s > \vartheta$ the integrals in (4.9) may be extended beyond ϑ up to $+\infty$ to get

$$F(x, y, i) = \mathbb{E}_{(x,y,i)} \left[\int_0^\infty e^{-\rho s} (X_s - c) d\nu_s^* - \int_0^\infty e^{-\rho s} f(Y_s^*) ds \right] = \mathcal{J}_{x,y,i}(\nu^*). \quad (4.10)$$

Then $F \equiv V$ and ν^* is optimal. \square

4.2 Case (II): $y \mapsto f(y)$ strictly concave on $[0, 1]$

Assume now that $y \mapsto f(y)$ satisfies condition (II) in Assumption 2.1, and for $y \in (0, 1]$ take $\theta(y)$ in (2.13) such that

$$\theta(y) := c - \frac{1}{\rho} \frac{f(y)}{y}.$$

By Corollary 3.3 we know that $x_1^*(y) = z_1^* + \theta(y)$ and $x_2^*(y) = z_2^* + x_1^*(y)$ (see also Corollary 3.6 in the case $x_1^*(y) = x_2^*(y) = x^*(y) = \frac{\sigma}{\sqrt{2\rho}} + \theta(y)$). Because $y \mapsto f(y)$ is continuously differentiable on $(0, 1]$, and strictly concave on $[0, 1]$, it follows that for any $i = 1, 2$ $y \mapsto x_i^*(y)$ is strictly increasing and continuously differentiable on $(0, 1]$. Moreover, by de l'Hôpital rule, $\lim_{y \downarrow 0} x_i^*(y) = -\infty$, where we have used the fact that $\lim_{y \downarrow 0} f'(y) = +\infty$ (cf. Assumption 2.1).

Recall now u of (2.13), and for any $(x, y, i) \in \mathcal{O}$ define the function

$$W(x, y, i) := yu(x, i; y) - \frac{1}{\rho} f(y). \quad (4.11)$$

The next result shows that W identifies with a suitable solution to the HJB equation (2.7).

Proposition 4.5. *One has that $W(x, 0, i) = 0$ for all $(x, i) \in \mathbb{R} \times \{1, 2\}$, and that there exists $K > 0$ such that $|W(x, y, i)| \leq K(1 + |x|)$ on \mathcal{O} . Moreover, $W(\cdot, \cdot, i) \in C^0(\mathbb{R} \times [0, 1]) \cap C^{1,1}(\mathbb{R} \times (0, 1])$ with $W_{xx}(\cdot, \cdot, i) \in L_{loc}^\infty(\mathbb{R} \times (0, 1])$, and it satisfies the HJB equation (2.7) in the a.e. sense.*

Proof. We provide a proof only for $W(x, y, 1)$ in the case $x_1^*(y) < x_2^*(y)$, since similar arguments can be employed to deal with all the other cases.

Step 1. By Proposition 3.1 (see in particular the last line in (3.2)) we can write

$$|W(x, y, 1)| \leq y|u(x, 1; y)| + \frac{1}{\rho} f(y) \leq y[2|\theta(y)| + \kappa(1 + |x|)] \leq y[2c + \kappa(1 + |x|)] + \frac{3}{\rho} f(y), \quad (4.12)$$

for some $\kappa > 0$. Taking limit as $y \downarrow 0$, and recalling that $f(0) = 0$, we obtain $W(x, 0, i) = 0$ for all $(x, i) \in \mathbb{R} \times \{1, 2\}$. Also, from (4.12) we see that the monotonicity of $f(\cdot)$ and the fact that $y \leq 1$ imply that there exists $K > 0$ such that $|W(x, y, i)| \leq K(1 + |x|)$ on \mathcal{O} .

Step 2. As for the claimed regularity of $W(\cdot, \cdot, 1)$, from (3.34) and Theorem 3.5 it follows that $W_x(\cdot, \cdot, 1)$ is uniformly continuous on open sets of the form $(-R, R) \times (\delta, 1)$ for $\delta > 0$ and arbitrary $R > 0$. Hence $W_x(\cdot, \cdot, 1)$ has a continuous extension to $\mathbb{R} \times (0, 1]$ that we denote again by $W_x(\cdot, \cdot, 1)$. Moreover, $W_{xx}(\cdot, \cdot, i) \in L_{loc}^\infty(\mathbb{R} \times (0, 1])$.

We now prove that $y \mapsto W(x, y, 1)$ belongs to $C^1(\mathbb{R} \times (0, 1])$. A direct differentiation of (4.11), and the use of (3.34) yield for any $y \in [\delta, 1]$, $\delta > 0$ arbitrary,

$$\begin{aligned} W_y(x, y, 1) &= u(x, 1; y) + y u_y(x, 1; y) - \frac{1}{\rho} f'(y) \\ &= \begin{cases} A_3(y) e^{\alpha_3 x} [1 - \alpha_3 y \theta'(y)] + A_4(y) e^{\alpha_4 x} [1 - \alpha_4 y \theta'(y)] - \frac{1}{\rho} f'(y) & \text{for } x < x_1^*(y) \\ x - c & \text{for } x > x_1^*(y). \end{cases} \end{aligned} \quad (4.13)$$

By using (3.10) and exploiting the continuity of $x_1^*(\cdot)$, it can be checked that $y \mapsto W_y(x, y, 1)$ is continuous on $[\delta, 1]$ for any $x \in \mathbb{R}$. Also, one has that $x \mapsto W_y(x, y, 1)$ is continuous on \mathbb{R} uniformly with respect to $y \in [\delta, 1]$. In particular, by using once more the expressions for $A_3(y)$ and $A_4(y)$ (cf. (3.10)), one has $\lim_{\zeta \downarrow 0} W_y(x_1^*(y) - \zeta, y, 1) = x_1^*(y) + c$, uniformly with respect to $y \in [\delta, 1]$. Hence $W_y(\cdot, \cdot, 1)$ is continuous on $\mathbb{R} \times (0, 1]$.

Step 3. We here show that $W_y(x, y, 1) \geq x - c$ for any $(x, y) \in \mathbb{R} \times (0, 1]$. Since this is clearly true on $x > x_1^*(y)$ (cf. (4.13)), we consider only $x < x_1^*(y)$. We show that $W_{yx}(x, y, 1) \leq 1$ on $\{(x, y) \in \mathbb{R} \times (0, 1] : x < x_1^*(y)\}$, as this fact together with $W_y(x_1^*(y) -, y, 1) = x_1^*(y) - c$ implies that $W_y(x, y, 1) \geq x - c$ on that set. By differentiating $W_y(x, y, 1)$ with respect to x on $\{(x, y) \in \mathbb{R} \times (0, 1] : x < x_1^*(y)\}$ one finds that

$$W_{yx}(x, y, 1) - 1 = u_x(x, 1; y) - 1 + y u_{yx}(x, 1; y).$$

Theorem 3.5 together with *Step 2* of the proof of Theorem 3.4 imply that $u_x(x, 1; y) - 1 \leq 0$ for any $x < x_1^*(y)$, $y \in (0, 1]$. Moreover, recalling that $x_1^*(y) = z_1^* + \theta(y)$ (cf. Corollary 3.3), from (3.34) it follows that $y u_{yx}(x, 1; y) = -y \theta'(y) u_{xx}(x, 1; y)$ for any $x < x_1^*(y)$ and $y \in (0, 1]$. However, by Theorem 3.5 and *Step 2* of the proof of Theorem 3.4 we have $u_{xx}(x, 1; y) \geq 0$ for $x < x_1^*(y)$, whereas

$$-y \theta'(y) = \frac{1}{\rho} \left[\frac{f'(y)y - f(y)}{y} \right] < 0,$$

by the assumed strict concavity of f . Hence $W_{yx}(x, y, 1) - 1 \leq 0$ on $\{(x, y) \in \mathbb{R} \times (0, 1] : x < x_1^*(y)\}$, and therefore $W_y(x, y, 1) \geq x - c$ on that set.

Step 4. By Theorems 3.4 and 3.5 one has that $(u(x, 1; y), u(x, 2; y), x_1^*(y), x_2^*(y))$ solves free-boundary problem (3.5), and in particular $(\mathcal{G} - \rho)u(x, 1; y) \leq 0$ for a.e. $x \in \mathbb{R}$ and all $y \in (0, 1]$, and with equality for $x < x_1^*(y)$. It thus follows from (4.11) that $(\mathcal{G} - \rho)W(x, 1; y) \leq f(y)$ for a.e. $x \in \mathbb{R}$ and for any $y \in (0, 1]$, with equality for $x < x_1^*(y)$.

Combining the results of the previous steps, the proof is completed. \square

Recall that the stopping time

$$\tau^* = \inf \{t \geq 0 : X_t^x \geq x_{\varepsilon_t}^*(y)\}, \quad (4.14)$$

is optimal for (2.13), and for any $y \in (0, 1]$ define the admissible extraction rule

$$\nu_t^* := \begin{cases} 0, & t \leq \tau^*, \\ y, & t > \tau^*. \end{cases} \quad (4.15)$$

Theorem 4.6. *The admissible control ν^* of (4.15) is optimal for problem (2.6) and $W \equiv V$.*

Proof. Since W solves the HJB equation in the a.e. sense due to Proposition 4.2, one has $W \geq V$ on \mathcal{O} by Theorem 2.5. We now show that one actually has $W = V$ on \mathcal{O} , and that ν^* of (4.15) is optimal for problem (2.6).

Taking $Y_t^* := Y_t^{y, \nu^*} = y - \nu_t^*$, $y \in (0, 1]$, with ν^* as in (4.15), and applying Itô's formula in the weak version of [16], Chapter 8, Section VIII.4, Theorem 4.1 (by employing the rationale at the beginning of the proof of Theorem 2.5, and possibly using localization arguments) we find

$$\begin{aligned} W(x, y, i) = & \mathbb{E}_{(x, y, i)} \left[e^{-\rho \tau^*} W(X_{\tau^*}, Y_{\tau^*}^*, \varepsilon_{\tau^*}) - \int_0^{\tau^*} e^{-\rho s} f(Y_s^*) ds \right] \\ & + \mathbb{E}_{(x, y, i)} \left[\int_0^{\tau^*} e^{-\rho s} W_y(X_s, Y_s^*, \varepsilon_s) d\nu_s^{*, \text{cont}} \right] \\ & - \mathbb{E}_{(x, y, i)} \left[\sum_{0 \leq s < \tau^*} e^{-\rho s} \left(W(X_s, Y_{s+}^*, \varepsilon_s) - W(X_s, Y_s^*, \varepsilon_s) \right) \right], \end{aligned} \quad (4.16)$$

where $\nu^{*, \text{cont}}$ denotes the continuous part of ν^* . The fact that $(\mathcal{G} - \rho)W(X_s, Y_s^*, \varepsilon_s) = f(Y_s^*)$ for any $s \leq \tau^*$ has been used in (4.16) above.

Since $(X_s^x, Y_s^*, \varepsilon_s) = (X_s^x, y, \varepsilon_s)$ for $s \leq \tau^*$, then the third and fourth term on the right-hand side of (4.16) equal zero. On the other hand, for the first term we have from (4.15) and the optimality of τ^* for problem (2.13) that

$$\begin{aligned} \mathbb{E}_{(x, y, i)} \left[e^{-\rho \tau^*} W(X_{\tau^*}, Y_{\tau^*}^*, \varepsilon_{\tau^*}) \right] &= \mathbb{E}_{(x, y, i)} \left[e^{-\rho \tau^*} W(X_{\tau^*}, y, \varepsilon_{\tau^*}) \right] \\ &= \mathbb{E}_{(x, y, i)} \left[e^{-\rho \tau^*} \left(y u(X_{\tau^*}, y, \varepsilon_{\tau^*}) - \frac{1}{\rho} f(y) \right) \right] = \mathbb{E}_{(x, y, i)} \left[e^{-\rho \tau^*} \left(y X_{\tau^*} - y \theta(y) - \frac{1}{\rho} f(y) \right) \right] \\ &= \mathbb{E}_{(x, y, i)} \left[e^{-\rho \tau^*} (X_{\tau^*} - c) y \right] = \mathbb{E}_{(x, y, i)} \left[\int_0^{\infty} e^{-\rho s} (X_s - c) d\nu_s^* \right]. \end{aligned} \quad (4.17)$$

Also, for the second term on the right-hand side of (4.16) we have

$$\mathbb{E}_{(x, y, i)} \left[\int_0^{\tau^*} e^{-\rho s} f(Y_s^*) ds \right] = \mathbb{E}_{(x, y, i)} \left[\int_0^{\infty} e^{-\rho s} f(Y_s^*) ds \right], \quad (4.18)$$

since $f(Y_s^*) = f(0)$ for any $s > \tau^*$, and $f(0) = 0$ by assumption. Now, (4.16), (4.17) and (4.18) give $W(x, y, i) = \mathcal{J}_{x, y, i}(\nu^*) \leq V(x, y, i)$. Hence, $W = V$ and ν^* is optimal. \square

Remark 4.7. Notice that because $\lim_{y \downarrow 0} x_i^*(y) = -\infty$, $i = 1, 2$ (see the discussion at the beginning of this section), the instantaneous short-term payoff $x_i^*(y) - c$ is strictly negative for y sufficiently small and for $i = 1, 2$. This in particular implies that for y small enough such that $x_i^*(y) - c < 0$, one has $V(x, y, i) = y \mathbb{E}_{(x, y, i)} [x_{\varepsilon_{\tau^*}}^*(y) - c] < 0$ for all $x \geq x^*(y)$ and $i = 1, 2$. However, $V(x, 0, i) = 0$.

5 A Comparison to the No-Regime-Switching Case

It is quite immediate to solve our optimal extraction problem when there is no regime switching. In particular, in this case it can be checked that for any $(0, 1]$ the optimal extraction boundary is

$$x^\#(y) := \frac{\sigma}{\sqrt{2\rho}} + c + \theta(y) = \begin{cases} \frac{\sigma}{\sqrt{2\rho}} + c - \frac{1}{\rho} f'(y) & \text{if } f \text{ satisfies (I) of Assumption 2.1,} \\ \frac{\sigma}{\sqrt{2\rho}} + c - \frac{1}{\rho} \frac{f(y)}{y} & \text{if } f \text{ satisfies (II) of Assumption 2.1,} \end{cases} \quad (5.1)$$

with $\lim_{y \downarrow 0} x^\#(y) = \frac{\sigma}{\sqrt{2\rho}} + c - \frac{1}{\rho}f'(0)$ and $\lim_{y \downarrow 0} x^\#(y) = -\infty$ in cases (I) and (II), respectively. Consequently, if f satisfies (I) of Assumption 2.1, and in particular it is strictly convex on $[0, 1]$, the optimal extraction rule reads as

$$\nu_t^\# := \left[y - \inf_{0 \leq s < t} b^\#(X_s) \right]^+, \quad t > 0, \quad \nu_0^\# = 0, \quad (5.2)$$

where $b^\#(\cdot)$ denotes the inverse of $x^\#(\cdot)$. On the other hand, if f satisfies (II) of Assumption 2.1, and therefore it is strictly concave on $[0, 1]$, it is optimal to extract according to the following policy

$$\nu_t^\# := \begin{cases} 0, & t \leq \tau^\#, \\ y, & t > \tau^\#, \end{cases} \quad (5.3)$$

with $\tau^\# := \inf \{t \geq 0 : X_t^x \geq x^\#(y)\}$.

A first observation that is worth making is that $x^\# = x^*$, with x^* as in (3.33). To understand this, recall that in Section 3.2 we have obtained that the two regime-dependent boundaries x_i^* , $i = 1, 2$, coincide and are given by (3.33) if and only if $\sigma_1 = \sigma_2$. In such case the price process does not jump and it therefore behaves as if we had not regime switching. It is then reasonable to obtain for such setting the same optimal selling price that we would obtain in absence of regime shifts, even if the associated optimal value is still regime-dependent.

Although qualitatively similar to (5.2), the optimal extraction rule (4.7) shows an important feature which is not present in the single regime case. Indeed, ν^* of (4.7) jumps at the moments of regime shifts from state 2 to state 1, thus implying a lump-sum extraction at those instants. This fact is not observed in (5.2) where a jump can happen only at initial time. We also refer to the detailed discussion in [20].

It is also interesting to see how the presence of regime shifts is reflected into the optimal extraction boundaries. We study this in case (I) (i.e. for a strictly convex running cost function), and our findings are illustrated in Figure 2. There we take the strictly convex running cost

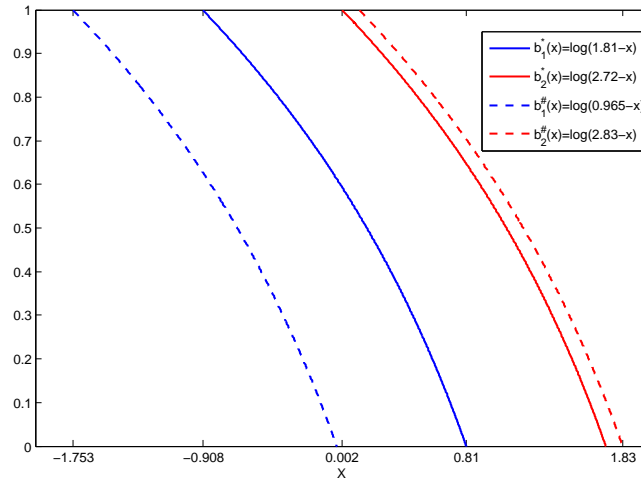


Figure 2: The dashed curve $b_i^\#(x)$, $i = 1, 2$, is the optimal extraction boundary (5.1) of the single regime case when the volatility is σ_i . The solid curves are the optimal extraction boundaries (b_1^* , b_2^*) when there is regime switching in the spot price process. To generate this plot with Matlab we have taken $f(y) = \frac{1}{3}(e^y - 1)$ and with $\sigma_1 = 0.38$, $\sigma_2 = 1.9$, $\lambda_1 = 1.7$, $\lambda_2 = 0.44$, $\rho = 1/3$ and $c = 0.5$.

$f(y) = \frac{1}{3}(e^y - 1)$, and we plot the optimal boundaries in the case of regime switching, b_i^* , $i = 1, 2$

(solid curves), and in the case of a single regime, $b_i^\#$ with volatility σ_i (dashed curves), $i = 1, 2$. Taking $\sigma_1 < \sigma_2$ we observe, that under macroeconomic cycles, the value at which the reserve level should be kept is higher than the one at which it would be kept if the volatility were always σ_1 . On the other hand, the value at which the reserve level should be maintained when business cycles are present, is lower than the one at which it would be kept if the volatility were always σ_2 . To some extent, this fact can be thought of as an *average effect* of the regime switching. For example, if the market volatility assumes at any time the highest value possible (i.e. it is always equal to σ_2), then the company would be more reluctant to extract and sell the commodity in the spot market relative to the case in which the volatility could jump to the lower value σ_1 . A symmetric argument applies to explain $b_1^\# < b_i^*$, $i = 1, 2$.

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A Some Proofs from Section 3

Proof of Theorem 3.4

Step 1. The fact that $w(\cdot, i; y) \in C^1(\mathbb{R})$ for $i = 1, 2$ follows by construction. It is also easy to verify from (3.19) and (3.20) that $w(\cdot, i; y)$, $i = 1, 2$, grows at most linearly and that $w_{xx}(\cdot, i; y)$ are bounded on any compact subset of \mathbb{R} .

We now show that $(w(x, 1; y), w(x, 2; y), x_1^*(y), x_2^*(y))$ solves free-boundary problem (3.5). Since by construction $(w(x, 1; y), w(x, 2; y), x_1^*(y), x_2^*(y))$ solves the first eight conditions of (3.5), then it suffices to prove that also the last two of (3.5) are fulfilled. This part of the proof is organized in the next steps.

Step 2. Here we show that $w(x, 1; y) \geq x - \theta(y)$ for any $x \in \mathbb{R}$. This is clearly true with equality by (3.19) for any $x \geq x_1^*(y)$. To prove the claim when $x < x_1^*(y)$ we show that $w(\cdot, 1; y)$ is convex therein. Indeed such property, together with the fact that $w_x(x_1^*(y), 1; y) - 1 = 0$, implies that $w_x(x, 1; y) - 1 \leq 0$ for any $x < x_1^*(y)$ and therefore that $w(x, 1; y) \geq x - \theta(y)$ for $x < x_1^*(y)$ since also $w(x_1^*(y), 1; y) - (x_1^*(y) - \theta(y)) = 0$.

To complete, we need to show that $w(\cdot, 1; y)$ is convex on $x < x_1^*(y)$. To this end note that for any $x < x_1^*(y)$ we have from (3.19)

$$w_{xx}(x, 1; y)(\alpha_4 - \alpha_3) = \alpha_3^2(\alpha_4(x_1^*(y) - \theta(y)) - 1)e^{\alpha_3(x - x_1^*(y))} + \alpha_4^2(1 - \alpha_3(x_1^*(y) - \theta(y)))e^{\alpha_4(x - x_1^*(y))}. \quad (\text{A-1})$$

Moreover, some algebra gives

$$\alpha_3^2(\alpha_4(x_1^*(y) - \theta(y)) - 1) + \alpha_4^2(1 - \alpha_3(x_1^*(y) - \theta(y))) = (\alpha_4 - \alpha_3) \left[\alpha_4 + \alpha_3 - \alpha_3\alpha_4(x_1^*(y) - \theta(y)) \right]. \quad (\text{A-2})$$

Also (by simple but tedious algebra) one has that

$$-\frac{1}{a_3} \left(\frac{\rho}{\rho + \lambda_2} + a_4 \right) - \frac{1}{\alpha_3} \leq \frac{1}{\alpha_4}. \quad (\text{A-3})$$

Then recalling that $x_1^*(y) - \theta(y) = z_1^*$, using the upper bound for z_1^* in (3.18) (cf. proof of Proposition 3.2) and (A-3) into (A-2) we get $(\alpha_4 - \alpha_3)[\alpha_4 + \alpha_3 - \alpha_3\alpha_4(x_1^*(y) - \theta(y))] \geq 0$, whence

$$\alpha_3^2(\alpha_4(x_1^*(y) - \theta(y)) - 1) + \alpha_4^2(1 - \alpha_3(x_1^*(y) - \theta(y))) = (\alpha_4 - \alpha_3) \left[\alpha_4 + \alpha_3 - \alpha_3\alpha_4(x_1^*(y) - \theta(y)) \right] \geq 0.$$

Therefore

$$\alpha_4^2(1 - \alpha_3(x_1^*(y) - \theta(y))) \geq -\alpha_3^2(\alpha_4(x_1^*(y) - \theta(y)) - 1),$$

which substituted into (A-1) yields

$$w_{xx}(x, 1; y)(\alpha_4 - \alpha_3) \geq \alpha_3^2(\alpha_4(x_1^*(y) - \theta(y)) - 1) \left[e^{\alpha_3(x - x_1^*(y))} - e^{\alpha_4(x - x_1^*(y))} \right] \geq 0, \quad (\text{A-4})$$

where (3.18), (A-3), and the fact that $\alpha_3 < \alpha_4$ but $x < x_1^*(y)$ have been employed for the last estimate.

Step 3. In this step we prove that $w(x_1^*(y), 2; y) \geq x_1^*(y) - \theta(y)$ and $w_x(x_1^*(y), 2; y) \leq 1$. These estimates will be needed to show that $w(x, 2; y) \geq x - \theta(y)$ for any $x \in \mathbb{R}$. From (3.20) and using that $B_3(y) = \frac{\Phi_1(\alpha_3)}{\lambda_1} A_3(y)$, $B_4(y) = \frac{\Phi_1(\alpha_4)}{\lambda_1} A_4(y)$, with $A_3(y)$ and $A_4(y)$ as in (3.10), one easily finds

$$w(x_1^*(y), 2; y) = \frac{\Phi_1(\alpha_3)[\alpha_4(x_1^*(y) - \theta(y)) - 1]}{\lambda_1(\alpha_4 - \alpha_3)} + \frac{\Phi_1(\alpha_4)[1 - \alpha_3(x_1^*(y) - \theta(y))]}{\lambda_1(\alpha_4 - \alpha_3)}$$

and

$$w_x(x_1^*(y), 2; y) = \frac{\alpha_3 \Phi_1(\alpha_3)[\alpha_4(x_1^*(y) - \theta(y)) - 1]}{\lambda_1(\alpha_4 - \alpha_3)} + \frac{\alpha_4 \Phi_1(\alpha_4)[1 - \alpha_3(x_1^*(y) - \theta(y))]}{\lambda_1(\alpha_4 - \alpha_3)}.$$

Recalling that $\Phi_i(z) = -\frac{1}{2}\sigma_i^2 z^2 + \rho + \lambda_i$, $i = 1, 2$, a simple calculation yields

$$w(x_1^*(y), 2; y) = \frac{-\frac{1}{2}\sigma_1^2(\alpha_3 + \alpha_4) + (x_1^*(y) - \theta(y))(\frac{1}{2}\sigma_1^2\alpha_3\alpha_4 + \rho + \lambda_1)}{\lambda_1} \quad (\text{A-5})$$

and

$$w_x(x_1^*(y), 2; y) = \frac{\alpha_4 \Phi(\alpha_4) - \alpha_3 \Phi(\alpha_3)}{\lambda_1(\alpha_4 - \alpha_3)} + \frac{\alpha_3 \alpha_4 \sigma_1^2 (x_1^*(y) - \theta(y))(\alpha_4 + \alpha_3)}{2\lambda_1}. \quad (\text{A-6})$$

It is now matter of algebraic manipulations to show that

$$\frac{\sigma_1^2(\alpha_3 + \alpha_4)}{\sigma_1^2\alpha_3\alpha_4 + 2\rho} = -\frac{a_2}{a_1 + \frac{\rho}{\rho + \lambda_2}}, \quad (\text{A-7})$$

and

$$-\frac{\frac{\rho}{\rho + \lambda_2} + a_4}{a_3} = \frac{2\lambda_1}{\alpha_3\alpha_4\sigma_1^2(\alpha_4 + \alpha_3)} \left[1 + \frac{\alpha_3\Phi_1(\alpha_3) - \alpha_4\Phi_1(\alpha_4)}{\lambda_1(\alpha_4 - \alpha_3)} \right]. \quad (\text{A-8})$$

Then by (3.18), (A-7) and (A-8) we obtain

$$\frac{\sigma_1^2(\alpha_3 + \alpha_4)}{\sigma_1^2\alpha_3\alpha_4 + 2\rho} \leq x_1^*(y) - \theta(y) \leq \frac{2\lambda_1}{\alpha_3\alpha_4\sigma_1^2(\alpha_4 + \alpha_3)} \left[1 + \frac{\alpha_3\Phi(\alpha_3) - \alpha_4\Phi(\alpha_4)}{\lambda_1(\alpha_4 - \alpha_3)} \right],$$

which, used into (A-5) and (A-6) yield $w(x_1^*(y), 2; y) \geq x_1^*(y) - \theta(y)$ and $w_x(x_1^*(y), 2; y) \leq 1$, respectively.

Step 4. We now show that $w(x, 2; y) \geq x - \theta(y)$ for $x < x_2^*(y)$ and therefore for any $x \in \mathbb{R}$ due to the fourth of (3.5). On $x \in (-\infty, x_1^*(y)) \cup (x_1^*(y), x_2^*(y))$ one has from (3.5)

$$\frac{1}{2}\sigma_2^2 w_{xx}(x, 2; y) + \lambda_2(w(x, 1; y) - w(x, 2; y)) - \rho w(x, 2; y) = 0. \quad (\text{A-9})$$

Setting $\widehat{w}(x, i; y) = w(x, i; y) - (x - \theta(y))$, $i = 1, 2$, it follows that on $(-\infty, x_1^*(y)) \cup (x_1^*(y), x_2^*(y))$

$$\frac{1}{2}\sigma_2^2\widehat{w}_{xx}(x, 2; y) + \lambda_2(\widehat{w}(x, 1; y) - \widehat{w}(x, 2; y)) - \rho\widehat{w}(x, 2; y) = \rho(x - \theta(y)). \quad (\text{A-10})$$

We now consider separately the two cases $x \in (-\infty, x_1^*(y))$ and $x \in (x_1^*(y), x_2^*(y))$. For $x \in (-\infty, x_1^*(y))$ we can differentiate (A-10) once more with respect to x so to obtain

$$\frac{1}{2}\sigma_2^2\widehat{w}_{xxx}(x, 2; y) + \lambda_2(\widehat{w}_x(x, 1; y) - \widehat{w}_x(x, 2; y)) - \rho\widehat{w}_x(x, 2; y) = \rho.$$

Setting $\tau_1 := \inf\{t \geq 0 : (X, \varepsilon) \notin \mathcal{D}_1\}$ $\mathbb{P}_{(x,i)}$ -a.s., where $\mathcal{D}_1 := \{(x, i) \in \mathbb{R} \times \{1, 2\} : x < x_1^*(y)\}$, an application of Itô's formula (possibly with a standard localization argument) leads to

$$\begin{aligned} \widehat{w}_x(x, 2; y) &= \mathbb{E}_{(x,i)} \left[e^{-\rho\tau_1} \widehat{w}_x(X_{\tau_1}, \varepsilon_{\tau_1}; y) - \int_0^{\tau_1} e^{-\rho s} \rho ds \right] \leq \mathbb{E}_{(x,i)} \left[e^{-\rho\tau_1} \widehat{w}_x(X_{\tau_1}, \varepsilon_{\tau_1}; y) \right] \\ &= \mathbb{E}_{(x,i)} \left[e^{-\rho\tau_1} \widehat{w}_x(X_{\tau_1}, \varepsilon_{\tau_1}; y) \mathbf{1}_{\{\varepsilon_{\tau_1}=1\}} \right] + \mathbb{E}_{(x,i)} \left[e^{-\rho\tau_1} \widehat{w}_x(X_{\tau_1}, \varepsilon_{\tau_1}; y) \mathbf{1}_{\{\varepsilon_{\tau_1}=2\}} \right] \end{aligned} \quad (\text{A-11})$$

for any $x < x_1^*(y)$. Since $\widehat{w}_x(x_1^*(y), i; y) = w_x(x_1^*(y), i; y) - 1 \leq 0$, $i = 1, 2$, by the fifth of (3.5) and by Step 3., and because $\tau_1 < +\infty$ $\mathbb{P}_{(x,i)}$ -a.s. due to recurrence property of (X, ε) (see (i) of Theorem 4.4 of [36] with $k > 0$, $\alpha \in (0, 1)$, $c_1 = c_2$ therein), we conclude from (A-11) that $\widehat{w}_x(x, 2; y) \leq 0$ for any $x < x_1^*(y)$. In turn this implies $w(x, 2; y) \geq x - \theta(y)$ for any $x < x_1^*(y)$ since $w(x_1^*(y), 2; y) \geq x_1^*(y) - \theta(y)$ again by the results of Step 3.

Take now $x \in (x_1^*(y), x_2^*(y))$ and define $\tau_{1,2} := \inf\{t \geq 0 : (X, \varepsilon) \notin \mathcal{D}_{1,2}\}$ $\mathbb{P}_{(x,i)}$ -a.s., where $\mathcal{D}_{1,2} := \{(x, i) \in \mathbb{R} \times \{1, 2\} : x_1^*(y) < x < x_2^*(y)\}$. Employing the same rationale of the arguments above, using that $\widehat{w}_x(x_2^*(y), 2; y) = 0$ and $\widehat{w}_x(x_1^*(y), 2; y) \leq 0$ (cf. end of Step 3), and that $\widehat{w}_x(x_2^*(y), 1; y) = 0 = \widehat{w}_x(x_1^*(y), 1; y)$, we obtain $\widehat{w}_x(x, 2; y) \leq 0$ for any $x \in (x_1^*(y), x_2^*(y))$. Hence $\widehat{w}(x, 2; y) \geq 0$ for any $x \in (x_1^*(y), x_2^*(y))$ since $\widehat{w}(x_2^*(y), 2; y) = 0$. Therefore $w(x, 2; y) \geq x - \theta(y)$ for any $x < x_2^*(y)$, by combining the previous results and the fact that $w(x_1^*(y), 2; y) \geq x_1^*(y) - \theta(y)$ proved in Step 3.

Steps 2, 3 and 4 above show that $w(x, i; y) \geq x - \theta(y)$ for $x \in \mathbb{R}$ and $i = 1, 2$. We now turn to prove that one also has $\frac{1}{2}\sigma_i^2 w_{xx}(x, i; y) - \rho w(x, i; y) + \lambda_i(w(x, 3-i; y) - w(x, i; y)) \leq 0$ for a.e. $x \in \mathbb{R}$ and $i = 1, 2$.

Step 5. We start showing that

$$\frac{1}{2}\sigma_2^2 w_{xx}(x, 2; y) - \rho w(x, 2; y) + \lambda_2(w(x, 1; y) - w(x, 2; y)) \leq 0 \quad (\text{A-12})$$

for a.e. $x \in \mathbb{R}$. This is true with equality for any $x < x_2^*(y)$ by construction. For $x > x_2^*(y)$ we have $w(x, 1; y) = x - \theta(y) = w(x, 2; y)$. On that interval (A-12) thus reads $-\rho(x - \theta(y)) \leq 0$, which is fulfilled since $\rho > 0$ and $x_2^*(y) > \theta(y)$ by Corollary 3.3.

We now check that one also has

$$\frac{1}{2}\sigma_1^2 w_{xx}(x, 1; y) - \rho w(x, 1; y) + \lambda_1(w(x, 2; y) - w(x, 1; y)) \leq 0 \quad (\text{A-13})$$

for a.e. $x \in \mathbb{R}$. Again, it suffices to show that the previous is true for $x > x_1^*(y)$, as it is verified with equality by construction on $(-\infty, x_1^*(y))$.

If $x > x_2^*(y)$ then $w(x, 2; y) = x - \theta(y) = w(x, 1; y)$ and (A-13) holds since $\rho > 0$ and $x_2^*(y) > \theta(y)$ by Corollary 3.3. To complete the proof we consider the case $x \in (x_1^*(y), x_2^*(y))$. On such interval we have again $w(x, 1; y) = x - \theta(y)$, and therefore (A-13) is verified on $(x_1^*(y), x_2^*(y))$ if

$$w(x, 2; y) \leq \frac{\rho + \lambda_1}{\lambda_1} w(x, 1; y), \quad (\text{A-14})$$

where we have used that $w_{xx}(x, 1; y) = 0$ on $(x_1^*(y), x_2^*(y))$. In Step 4. we have shown that $w_x(x, 2; y) - 1 \leq 0$ for any $x \in (x_1^*(y), x_2^*(y))$, from which one has $w(x, 2; y) - w(x, 1; y) = w(x, 2; y) - (x - \theta(y)) \leq w(x_1^*(y), 2; y) - (x_1^*(y) - \theta(y)) = w(x_1^*(y), 2; y) - w(x_1^*(y), 1; y)$, where the fact that $w(x, 1; y) = x - \theta(y)$ for any $x \geq x_1^*(y)$ has been used. Therefore on $(x_1^*(y), x_2^*(y))$

$$w(x, 2; y) \leq w(x_1^*(y), 2; y) - w(x_1^*(y), 1; y) + w(x, 1; y), \quad (\text{A-15})$$

Also $\frac{1}{2}\sigma_1^2 w_{xx}(x, 1; y) - \rho w(x, 1; y) + \lambda_1(w(x, 2; y) - w(x, 1; y)) = 0$, for any $x < x_1^*(y)$, and therefore

$$w(x, 2; y) \leq \frac{\rho + \lambda_1}{\lambda_1} w(x, 1; y), \quad x < x_1^*(y), \quad (\text{A-16})$$

by convexity of $w(x, 1; y)$ proved in Step 2. Then, taking limits as $x \uparrow x_1^*(y)$ we get from (A-16) and continuity of $w(\cdot, i; y)$

$$w(x_1^*(y), 2; y) \leq \frac{\rho + \lambda_1}{\lambda_1} w(x_1^*(y), 1; y), \quad (\text{A-17})$$

and we conclude from (A-15) and (A-17) that for any $x \in (x_1^*(y), x_2^*(y))$

$$w(x, 2; y) \leq \frac{\rho + \lambda_1}{\lambda_1} w(x_1^*(y), 1; y) - w(x_1^*(y), 1; y) + w(x, 1; y) \leq \frac{\rho + \lambda_1}{\lambda_1} w(x, 1; y),$$

where the fact that $w(x_1^*(y), 1; y) = x_1^*(y) - \theta(y) \leq (x - \theta(y)) = w(x, 1; y)$ for any $x > x_1^*(y)$ implies the last step. Hence (A-14) holds on $(x_1^*(y), x_2^*(y))$ and therefore also (A-13) is satisfied on that interval. \square

Proof of Theorem 3.5

Step 1. Fix $(x, i) \in \mathbb{R} \times \{1, 2\}$, let τ be an arbitrary $\mathbb{P}_{(x,i)}$ -a.s. finite stopping time, and set $\tau_R := \inf\{t \geq 0 : X_t \notin (-R, R)\}$ $\mathbb{P}_{(x,i)}$ -a.s. for $R > 0$. Then, let $0 \leq \eta_1 < \eta_2 < \dots < \eta_N \leq \tau \wedge \tau_R$ be the random times of jumps of ε in the interval $[0, \tau \wedge \tau_R)$ (clearly, the number N of these jumps is random as well) and, given the regularity of $w(\cdot, i; y)$ for any $i = 1, 2$ (cf. Theorem 3.4), apply Itô-Tanaka's formula (see, e.g., [29], Chapter VI, Proposition 1.5, Corollary 1.6 and following Remarks) between consecutive jumps of ε from time 0 up to time $\tau \wedge \tau_R$. Piecing together all the terms as in the proof of Lemma 3 at p. 104 of [31] (see also Lemma 2.4 and its idea of proof in [34]) we find

$$\begin{aligned} w(x, i; y) &= \mathbb{E}_{(x,i)} \left[e^{-\rho(\tau \wedge \tau_R)} w(X_{\tau \wedge \tau_R}, \varepsilon_{\tau \wedge \tau_R}; y) - \int_0^{\tau \wedge \tau_R} e^{-\rho s} (\mathcal{G} - \rho) w(X_s, \varepsilon_s; y) ds \right] \\ &\geq \mathbb{E}_{(x,i)} \left[e^{-\rho(\tau \wedge \tau_R)} w(X_{\tau \wedge \tau_R}, \varepsilon_{\tau \wedge \tau_R}; y) \right] \geq \mathbb{E}_{(x,i)} \left[e^{-\rho(\tau \wedge \tau_R)} (X_{\tau \wedge \tau_R} - \theta(y)) \right]. \end{aligned} \quad (\text{A-18})$$

In (A-18) we have used that w solves free-boundary problem (3.5) (cf. Theorem 3.4), and the fact that the stochastic integral over the interval $[0, \tau \wedge \tau_R)$ vanishes under expectation since w_x is bounded for $(x, i, y) \in [-R, R] \times \{1, 2\} \times [0, 1]$.

But now $\{e^{-\rho(\tau \wedge \tau_R)} X_{\tau \wedge \tau_R}, R > 0\}$ is a $\mathbb{P}_{(x,i)}$ -uniformly integrable family by Lemma B.3 in Appendix B, hence observing that if $R \uparrow \infty$ one has $\tau \wedge \tau_R \uparrow \tau$ a.s. by regularity of (X, ε) (cf. [36], Section 3.1), we can take limits as $R \uparrow \infty$ in (A-18), invoke Vitali's convergence theorem and obtain

$$w(x, i; y) \geq \mathbb{E}_{(x,i)} \left[e^{-\rho\tau} (X_\tau - \theta(y)) \right].$$

Since τ was arbitrary, $w(x, i; y) \geq \sup_{\tau \geq 0} \mathbb{E}_{(x,i)} [e^{-\rho\tau} (X_\tau - \theta(y))] = u(x, i; y)$.

Step 2. To prove the reverse inequality, i.e. $w(x, i; y) \leq u(x, i; y)$, take $\tau = \tau^*$, in the previous arguments and notice that one has $(\mathcal{G} - \rho)w(x, i; y) = 0$ on \mathcal{C} . Then taking limits as $R \uparrow \infty$ one finds

$$w(x, i; y) = \mathbb{E}_{(x, i)} \left[e^{-\rho\tau^*} w(X_{\tau^*}, \varepsilon_{\tau^*}; y) \right] = \mathbb{E}_{(x, i)} \left[e^{-\rho\tau^*} (X_{\tau^*} - \theta(y)) \right], \quad (\text{A-19})$$

where the last equality follows from the fact that $\tau^* < +\infty$ $\mathbb{P}_{(x, i)}$ -a.s. by recurrence of (X, ε) (cf. Theorem 4.4 in [36]). Therefore $w(x, i; y) \leq u(x, i; y)$, whence $w(x, i; y) = u(x, i; y)$ and optimality of τ^* . \square

Proof of Theorem 3.7

The claimed regularity follows by construction, whereas the growth property can be easily checked from (3.34) and (3.35). Moreover, by construction we also have that for any $i = 1, 2$ $\frac{1}{2}\sigma_i^2 w_{xx}(x, i; y) - \rho w(x, i; y) + \lambda_i(w(x, 3-i; y) - w(x, i; y)) = 0$ on $x < x^*(y)$ and $w(x, i; y) = x - \theta(y)$ on $x \geq x^*(y)$. It thus remains only to check that for any $i = 1, 2$ a) $\frac{1}{2}\sigma_i^2 w_{xx}(x, i; y) - \rho w(x, i; y) + \lambda_i(w(x, 3-i; y) - w(x, i; y)) \leq 0$ on $x > x^*(y)$ and b) $w(x, i; y) \geq x - \theta(y)$ on $x < x^*(y)$.

Since on $x > x^*(y)$ one has $w(x, 1; y) = x - \theta(y) = w(x, 2; y)$, property a) above follows by noticing that $x^*(y) \geq \theta(y)$ (cf. (3.29) or equivalently (3.30)) and recalling that $\rho > 0$.

As for b), it suffices to show that $w_x(x, i; y) - 1 \leq 0$ for any $x < x^*(y)$ since $w(x^*(y), i; y) - (x^*(y) - \theta(y)) = 0$. To this end define $\hat{w}(x, i; y) := w(x, i; y) - (x - \theta(y))$, $i = 1, 2$, and notice that we can differentiate once more the first of (3.22) with respect to x inside $(-\infty, x^*(y))$ to obtain

$$\frac{1}{2}\sigma_i^2 \hat{w}_{xxx}(x, i; y) + \lambda_i(\hat{w}_x(x, 3-i; y) - \hat{w}_x(x, i; y)) - \rho \hat{w}_x(x, i; y) = \rho.$$

Then an argument based on Dynkin's formula as that employed in Step 4. of the proof of Theorem 3.4 yields $\hat{w}_x(x, i; y) \leq 0$ for any $x < x^*(y)$, which in turn gives $w(x, i; y) \geq x - \theta(y)$ for any $x < x^*(y)$, because $w(x^*(y), i; y) = x^*(y) - \theta(y)$. \square

B Some Auxiliary Results

Lemma B.1. *For $i = 1, 2$ and $\alpha \in \mathbb{R}$, let $\Phi_i(\alpha) := -\frac{1}{2}\sigma_i^2\alpha^2 + \rho + \lambda_i$. Then there exist unique $\alpha_1 < \alpha_2 < 0 < \alpha_3 < \alpha_4$ satisfying the fourth-order equation*

$$\Phi_1(\alpha)\Phi_2(\alpha) - \lambda_1\lambda_2 = 0. \quad (\text{B-1})$$

Proof. We provide here a proof of this claim in our setting for the sake of completeness (see also [18], Remark 2.1, and [32], Lemma 3.1 for related results). Using the definition of Φ_i , $i = 1, 2$, equation (B-1) reads

$$\frac{1}{4}\sigma_1^2\sigma_2^2\alpha^4 - \left[\frac{1}{2}\sigma_1^2(\rho + \lambda_2) + \frac{1}{2}\sigma_2^2(\rho + \lambda_1) \right] \alpha^2 + (\rho + \lambda_1)(\rho + \lambda_2) - \lambda_1\lambda_2 = 0,$$

and letting

$$a_o := \frac{1}{4}\sigma_1^2\sigma_2^2, \quad b_o := \frac{1}{2}\sigma_1^2(\rho + \lambda_2) + \frac{1}{2}\sigma_2^2(\rho + \lambda_1), \quad c_o := (\rho + \lambda_1)(\rho + \lambda_2) - \lambda_1\lambda_2,$$

one can check that

$$b_o^2 - 4a_o c_o = \left[\frac{\sigma_1^2(\rho + \lambda_2) - \sigma_2^2(\rho + \lambda_1)}{2} \right]^2 + \lambda_1\lambda_2\sigma_1^2\sigma_2^2 \geq 0.$$

Hence there exists two solutions β_1 and β_2 to the second-order equation $a_o\beta^2 - b_o\beta + c_o = 0$, and they are such that $0 < \beta_2 < \beta_1$ since $a_o c_o > 0$. It thus follows that

$$-\alpha_1 := \sqrt{\beta_1} =: \alpha_4, \quad -\alpha_2 := \sqrt{\beta_2} =: \alpha_3$$

solve (B-1) and satisfy $\alpha_1 < \alpha_2 < 0 < \alpha_3 < \alpha_4$.

□

Lemma B.2. *Let a_i , $i = 1, 2, 3, 4$, be defined as in (3.14). Then one has $a_1 < 0$, $a_2 > 0$, $a_3 < 0$ and $a_4 > 0$.*

Proof. Noticing that $\Phi_i(\alpha) = -\frac{1}{2}\sigma_i^2\alpha^2 + \rho + \lambda_i$, $i = 1, 2$, is a strictly decreasing function of α , the fact that $\alpha_3 < \alpha_4$ imply $a_2 > 0$ and $a_3 < 0$.

As for a_1 , recall that from (3.14) one has

$$a_1 = -\frac{\alpha_4\Phi_1(\alpha_3) - \alpha_3\Phi_1(\alpha_4)}{\lambda_1(\alpha_4 - \alpha_3)} + \frac{\lambda_2}{\rho + \lambda_2}. \quad (\text{B-2})$$

By using the explicit expression of $\Phi_i(\alpha)$, $i = 1, 2$, direct calculations lead to

$$\alpha_4\Phi_1(\alpha_3) - \alpha_3\Phi_1(\alpha_4) = \left(\frac{1}{2}\sigma_1^2\alpha_3\alpha_4 + \rho + \lambda_1\right)(\alpha_4 - \alpha_3), \quad (\text{B-3})$$

which substituted into (B-2) yields

$$a_1 = -\frac{\frac{1}{2}\sigma_1^2\alpha_3\alpha_4 + \rho + \lambda_1}{\lambda_1} + \frac{\lambda_2}{\rho + \lambda_2} < -\frac{\frac{1}{2}\sigma_1^2\alpha_3\alpha_4 + \rho}{\lambda_1} < 0. \quad (\text{B-4})$$

We conclude showing that $a_4 > 0$. It is matter of simple algebra to show that

$$\alpha_3\Phi_1(\alpha_3) - \alpha_4\Phi_1(\alpha_4) = (\alpha_4 - \alpha_3)\left[\frac{1}{2}\sigma_1^2(\alpha_3\alpha_4 + \alpha_3^2 + \alpha_4^2) - (\rho + \lambda_1)\right], \quad (\text{B-5})$$

which used in the expression for a_4 of (3.14) allows to write

$$a_4 = \frac{\frac{1}{2}\sigma_1^2(\alpha_3\alpha_4 + \alpha_3^2 + \alpha_4^2) - (\rho + \lambda_1)}{\lambda_1} + \frac{\lambda_2}{\rho + \lambda_2}. \quad (\text{B-6})$$

Since α_3 and α_4 solve $\Phi_1(\alpha)\Phi_2(\alpha) = \lambda_1\lambda_2$, by Vieta's formulas we deduce that

$$\alpha_3^2 + \alpha_4^2 = \frac{2\sigma_1^2(\rho + \lambda_2) + 2\sigma_2^2(\rho + \lambda_1)}{\sigma_1^2\sigma_2^2}. \quad (\text{B-7})$$

Noticing that $\alpha_3\alpha_4 > 0$, and using (B-7) in (B-6) we obtain

$$a_4 > \frac{\frac{1}{2}\sigma_1^2(\alpha_3^2 + \alpha_4^2) - (\rho + \lambda_1)}{\lambda_1} > \frac{1}{\lambda_1}\left[\frac{\sigma_1^2\sigma_2^2(\rho + \lambda_1)}{\sigma_1^2\sigma_2^2} - (\rho + \lambda_1)\right] = 0,$$

thus completing the proof. □

Lemma B.3. *Fix $(x, i) \in \mathbb{R} \times \{1, 2\}$, let τ be an arbitrary $\mathbb{P}_{(x,i)}$ -a.s. finite stopping time, and for $R > 0$ set $\tau_R := \inf\{t \geq 0 : X_t \notin (-R, R)\}$ $\mathbb{P}_{(x,i)}$ -a.s. Then the family of random variables $\{e^{-\rho(\tau \wedge \tau_R)} X_{\tau \wedge \tau_R}, R > 0\}$ is $\mathbb{P}_{(x,i)}$ -uniformly integrable.*

Proof. By an integration by parts we have due to (2.2)

$$e^{-\rho(\tau \wedge \tau_R)} X_{\tau \wedge \tau_R} = x - \int_0^{\tau \wedge \tau_R} \rho e^{-\rho s} X_s ds + \int_0^{\tau \wedge \tau_R} e^{-\rho s} \sigma_{\varepsilon_s} dW_s.$$

On the one hand, by Hölder's inequality and Itô's isometry one has

$$\begin{aligned} \mathbb{E}_{(x,i)} \left[\int_0^\infty \rho e^{-\rho s} |X_s| ds \right] &\leq |x| + \int_0^\infty \rho e^{-\rho s} \mathbb{E}_{(x,i)} \left[\left| \int_0^s \sigma_{\varepsilon_u} dW_u \right|^2 \right]^{\frac{1}{2}} \\ &\leq |x| + (\sigma_1^2 \vee \sigma_2^2)^{\frac{1}{2}} \int_0^\infty \rho \sqrt{s} e^{-\rho s} ds < \infty, \end{aligned} \quad (\text{B-8})$$

for some $K > 0$. Hence $\int_0^\infty \rho e^{-\rho s} |X_s| ds \in L^1(\Omega, \mathbb{P}_{(x,i)})$. On the other hand, the continuous martingale $\{\int_0^t e^{-\rho s} \sigma_{\varepsilon_s} dW_s, t \geq 0\}$ is bounded in $L^2(\Omega, \mathbb{P}_{(x,i)})$ since $\mathbb{E}_{(x,i)}[\int_0^t e^{-\rho s} \sigma_{\varepsilon_s} dW_s]^2 \leq (\sigma_1^2 \vee \sigma_2^2) \int_0^\infty e^{-2\rho s} ds$, and therefore (cf. [29], Chapter IV, Proposition 1.23)

$$\mathbb{E}_{(x,i)} \left[\left| \int_0^{\tau \wedge \tau_R} e^{-\rho s} \sigma_{\varepsilon_s} dW_s \right|^2 \right] = \mathbb{E}_{(x,i)} \left[\int_0^{\tau \wedge \tau_R} e^{-2\rho s} \sigma_{\varepsilon_s}^2 ds \right] \leq (\sigma_1^2 \vee \sigma_2^2) \int_0^\infty e^{-2\rho s} ds, \quad R > 0.$$

Hence, the family $\{|\int_0^{\tau \wedge \tau_R} e^{-\rho s} \sigma_{\varepsilon_s} dW_s|, R > 0\}$ is bounded in $L^2(\Omega, \mathbb{P}_{(x,i)})$ as well, thus uniformly integrable. This fact, together with (B-8), in turn imply uniform integrability of the family $\{e^{-\rho(\tau \wedge \tau_R)} X_{\tau \wedge \tau_R}, R > 0\}$. \square

Lemma B.4. *Let \mathcal{T} denote the set of \mathbb{F} -stopping times. Then for any $\nu \in \mathcal{A}_y$, the families of random variables*

$$\left\{ \int_0^\tau e^{-\rho u} (X_u - c) d\nu_u, \tau \in \mathcal{T} \right\} \quad \text{and} \quad \left\{ \int_0^\tau e^{-\rho u} f(Y_u^\nu) du, \tau \in \mathcal{T} \right\}$$

are $\mathbb{P}_{(x,y,i)}$ -uniformly integrable.

Proof. We prove the uniform integrability of the first family of random variables by showing that it is uniformly bounded in $L^2(\Omega, \mathbb{P}_{(x,y,i)})$. Let τ be any given and fixed stopping time of \mathbb{F} , take any $\nu \in \mathcal{A}_y$, and notice that an integration by parts leads to

$$\int_0^\tau e^{-\rho u} (X_u - c) d\nu_u = e^{-\rho\tau} (X_\tau - c) \nu_\tau + \int_0^\tau \rho e^{-\rho u} (X_u - c) \nu_u du - \int_0^\tau e^{-\rho u} \nu_u \sigma_{\varepsilon_u} dW_u, \quad (\text{B-9})$$

where (2.2) has been employed. However we also have

$$e^{-\rho\tau} (X_\tau - c) \nu_\tau = \nu_\tau \left[x - ce^{-\rho\tau} - \int_0^\tau \rho e^{-\rho u} X_u du + \int_0^\tau e^{-\rho u} \sigma_{\varepsilon_u} dW_u \right]. \quad (\text{B-10})$$

Denoting by $K > 0$ a suitable constant possibly depending on x and y , but not on τ , and that may change from line to line, we obtain from (B-9) and (B-10)

$$\begin{aligned} \left| \int_0^\tau e^{-\rho u} (X_u - c) d\nu_u \right|^2 &\leq K \left[1 + \int_0^\infty \rho e^{-\rho u} |X_u|^2 du + \left| \int_0^\tau e^{-\rho u} \sigma_{\varepsilon_u} dW_u \right|^2 \right. \\ &\quad \left. + \left| \int_0^\tau e^{-\rho u} \nu_u \sigma_{\varepsilon_u} dW_u \right|^2 \right] \leq K \left[1 + \int_0^\infty \rho e^{-\rho u} \left| \int_0^u e^{-\rho s} \sigma_{\varepsilon_s} dW_s \right|^2 du \right. \\ &\quad \left. + \left| \int_0^\tau e^{-\rho u} \sigma_{\varepsilon_u} dW_u \right|^2 + \left| \int_0^\tau e^{-\rho u} \nu_u \sigma_{\varepsilon_u} dW_u \right|^2 \right], \end{aligned} \quad (\text{B-11})$$

where the boundedness of $\nu \in \mathcal{A}_y$ has been exploited. In (B-11) Jensen's inequality has been used in the first step for the integrals with respect to $\rho e^{-\rho u} du$, whereas the last step employs (2.2). Taking expectations in (B-11), using Itô's isometry, and noticing that $\sigma_{\varepsilon_t}^2 \leq \sigma_1^2 \vee \sigma_2^2$ a.s. and that any admissible control is bounded by one, we obtain

$$\mathbb{E}_{(x,y,i)} \left[\left| \int_0^\tau e^{-\rho u} (X_u - c) d\nu_u \right|^2 \right] \leq K \left[1 + (\sigma_1^2 \vee \sigma_2^2) \int_0^\infty \rho e^{-\rho u} (1 + u) du \right], \quad (\text{B-12})$$

which in turn proves the first claim.

Uniform integrability of the second family follows by noticing that for any \mathbb{F} -stopping time τ and any $\nu \in \mathcal{A}_y$ we have

$$0 \leq \int_0^\tau e^{-\rho u} f(Y_u^\nu) du \leq \int_0^\infty e^{-\rho u} f(1) du \leq \frac{f(1)}{\rho},$$

where we have used the fact that $f(\cdot)$ is nonnegative and increasing, and that $Y_t^\nu \leq 1$ a.s. \square

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