SOBOLEV REGULARITY FOR THE POROUS MEDIUM EQUATION WITH A FORCE

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Abstract. We establish improved velocity averaging Lemmata with applications to non-isotropic parabolic-hyperbolic PDE. In particular, this leads to improved spatial regularity estimates for solutions to porous media equations with a force in fractional Sobolev spaces. Scaling arguments indicate that the obtained regularity is optimal and it is consistent with the optimal regularity in the linear limit. In particular, regularity estimates of an order of differentiability larger than one are obtained for forced porous media equations here for the first time. In addition, optimal regularity estimates for a degenerate parabolic Anderson model are shown in one spatial dimension.

1. Introduction

We consider the spatial regularity of solutions to porous media equations with a force, that is, to

$$\begin{aligned} \partial_t u - \Delta u^{[m]} &= S(t,x) \text{ on } (0,T)\times\mathbb{R}^d_x, \\ u(0) &= u_0 \text{ on } \mathbb{R}^d_x, \end{aligned} \tag{1.1}$$

with $u_0 \in L^1(\mathbb{R}^d_x)$, $S \in L^1([0,T]\times\mathbb{R}^d_x)$, $T \geq 0$, $m > 1$ and $u^{[m]} := |u|^{m-1}u$. We are particularly interested in the case $m \in (1,2)$ and in the limit case $m \downarrow 1$. To this end, we note that all available regularity results for (1.1) concerning spatial regularity in terms of Hölder or Sobolev spaces are restricted to a degree of differentiability of an order less than one. Clearly, this is in contrast to the limit case $m = 1$ where u is known to be twice (spatially) weakly differentiable. In contrast, in this paper we provide regularity estimates for (1.1) in (fractional) Sobolev spaces of order of differentiability $\frac{2}{m}-$. Scaling arguments suggest that this is the optimal regularity and, in particular, this is consistent with the optimal regularity in the linear limit $m \downarrow 1$.

The regularity of solutions to porous media equations in fractional Sobolev spaces has been previously analyzed by Ebmeyer in [16, Theorem 2.3], where it was shown that, for $S = 0$, $u_0 \in L^\infty$ and on bounded domains with zero Dirichlet boundary conditions,

$$u \in L^{m+1}([0,T]; W^{s,m+1}), \quad \forall s < \frac{2}{m+1}. \tag{1.2}$$

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By entirely different methods, Tadmor-Tao showed in [31] that, for all $t>0$ and $u_0 \in (L^1\cap L^\infty)(\mathbb{R}^d)$,

$$u(t)\in W^{s,1}_{loc}(\mathbb{R}^d),\quad \forall s<\frac{2}{m+1}.$$

A comparison with the Barenblatt solution

$$u_{BB}(t,x;a,\gamma)=(t+\gamma)^{-k}\left(a^2-\frac{k(m-1)}{2dm}\frac{|x|^2}{(t+\gamma)^{\frac{2k}{d}}}\right)_+^{\frac{1}{m-1}},$$

where γ,a,k are appropriately chosen constants (cf. [32]), which satisfies

$$u_{BB}(t)\in W^{\frac{2}{m+1}\gamma,m+1}_{loc}\quad \text{iff } \gamma<\frac{m}{m-1}$$

shows that (1.2) can be close to optimal only in the limit $m\uparrow\infty$. In particular, since $s<\frac{2}{m+1}<1$, no weak differentiability of an order higher than one can be deduced by the results of [16, Theorem 2.3]. This restriction is inherent to the methods employed in [16], for a more detailed discussion we refer to Remark 3.6 below. As a consequence, the regularity estimates obtained in [16,31] are not sharp for m close to one. In fact, in the linear case ($m=1$) we have $u\in L^1([0,T];W^{2,1}(\mathbb{R}^d_x))$ which may suggest that for (1.1) one has $u(t)\in W^{s_m,p_m}$ for some sequences $s_m\uparrow 2$, $p_m\downarrow 1$ for $m\downarrow 1$. Indeed, a simple scaling argument (cf. Section C) suggests an optimal regularity of order $\frac{2}{m}$. The proof of this fact is the main result of this paper: We prove that solutions of (1.1), with $u_0\in L^p(\mathbb{R}^d_x)$, $S\in L^p([0,T]\times\mathbb{R}^d_x)$ satisfy

$$u\in L^p([0,T];W^{s,p}_{loc}(\mathbb{R}^d_x)),\quad \forall s<\frac{2}{m},\ p\in[1,m), \tag{1.3}$$

which provides an optimal, consistent regularity estimate in the "linear limit" $m\downarrow 1$. We also provide quantitative estimates of (1.3) of the form, for every $\delta>0$ small enough,

$$\|u\|_{L^p_tW^{s,p}_{x,loc}}\le C\left(\|u_0\|^2_{L^1_x\cap L^{1+\delta}_x\cap L^p_x}+\|S\|^2_{L^1_{t,x}\cap L^{1+\delta}_{t,x}\cap L^p_{t,x}}+1\right).$$

In addition, we treat more general classes of equations, in particular including anisotropic porous media equations of the form

$$\partial_t u-\sum_{j=1}^d\partial_{x_jx_j}u^{[m_j]}=S(t,x)\text{ on }(0,T)\times\mathbb{R}^d_x, \tag{1.4}$$

with $1<\underline{m}:=\min\{m_j\}$ and let $\overline{m}:=\max\{m_j\}$. In this case we obtain that, for all $s<\frac{2}{\overline{m}}\left(\frac{\underline{m}-1}{\overline{m}-1}\right)$, $p<\frac{2\overline{m}}{\overline{m}+1}$,

$$\int_v f(t,x,v)\phi(v)\,dv\in L^p([0,T];W^{s,p}_{loc}(\mathbb{R}^d_x))$$

where $f(t,x,v):=1_{v<u(t,x)}-1_{v<0}$ and ϕ is an arbitrary cut-off function.

In a third main result, we consider the degenerate parabolic Anderson model

$$\begin{aligned}\partial_t u &= \partial_{xx} u^{[m]} + u\, S \text{ on } (0,T)\times I \\ u &= 0 \text{ on } (0,T)\times \partial I \\ u(0) &= u_0 \in L^{m+1}(I),\end{aligned} \tag{1.5}$$

on an open, bounded interval $I \subseteq \mathbb{R}$, with $m \in (1,2)$ and S being spatial white noise. The additional difficulty in this case is the irregularity of the source S, since spatial white noise is a distribution only. We prove the existence of a weak solution u to (1.5) satisfying

$$u \in L^p([0,T]; W^{s,p}_{loc}(I)), \quad \forall s < \frac{1}{m}\frac{3}{2},\, p \in [1,m). \tag{1.6}$$

The proof presented in this paper is based on Fourier analytic techniques and averaging Lemmata. The key first step is to pass to a kinetic formulation of (1.1). Introducing the kinetic function $f(t,x,v) := 1_{v<u(t,x)} - 1_{v<0}$ leads to the kinetic form of (1.1)

$$\partial_t f - m|v|^{m-1}\Delta f = \partial_v q + S(t,x)\delta_{u(t,x)}(v), \tag{1.7}$$

for some non-negative measure q. Since this constitutes a linear equation in f, the regularity of velocity averages $\int f\phi(v)\,dv$ for smooth cut-off functions ϕ can be analyzed by means of suitable micro-local decompositions in Fourier space. Up to this point our setup is in line with [31]. However, in the available literature, one of the drawbacks of analyzing regularity by means of averaging techniques is that it was unknown how to make use of the sign of the measure q. In fact, these estimates only made use of the fact that the total variation norm of q is finite (cf. e.g. [12, 13]). In this work, we make use of the additional fact that the entropy dissipation measure q has finite singular moments, meaning that $|v|^{-\gamma}q$ has finite mass for all $\gamma \in [0,1)$. In this way we are able to exploit, at least to some extent, the sign property of q. In addition, we introduce a new concept of isotropic truncation properties for Fourier multipliers, which allows us to obtain improved integrability exponents in the estimate (1.3). A further obstacle arising in classical averaging arguments is that they rely on a bootstrap technique. However, even if u is smooth, the kinetic function f will only have up to one spatial derivative. Therefore, the standard bootstrap argument is not well-suited in order to derive higher order differentiability estimates. In the anisotropic case, this difficulty is avoided in the current paper by directly exploiting the v-regularity of f. In the isotropic case these issues are overcome by introducing the isotropic truncation property mentioned above. In both cases this allows to fully avoid bootstrapping arguments. In order to underline the differences and improvements with regard to [31] we employ the notation and structure of [31] as far as possible. While, as usual in the theory of averaging techniques, our proof also relies on a micro-local decomposition in Fourier space, the order of decomposition and real-interpolation, the key Lemma A.3, the bootstrapping argument and the estimation of the entropy dissipation measure proceed differently, as outlined above.

1.1. **Short overview of the literature.** The study of regularity of solutions to porous media equations has a long history and we make no attempt to reproduce a complete account here. In the absence of external forces, the continuity of weak solutions to the porous medium equation has been first shown in general dimension by Caffarelli-Friedman in [8].

This result has been subsequently generalized to the case of forced porous media equations by Sacks in [29,30], based on arguments developed by Cafarelli-Evans in [7]. Further generalizations to more general classes of equations have been shown by DiBenedetto [14] and Ziemer [34]. A detailed account of these developments may be found in Vazquez [32]. Hölder continuity of solutions to the porous medium equation without force was first obtained by Caffarelli-Friedman [9], see also [32,33], where it is shown that bounded solutions to the porous medium equations are spatially α-Hölder continuous with $\alpha = \frac{1}{m} \in (0,1)$. We note that in the linear limit $m \downarrow 1$ this does not recover the optimal Hölder regularity of the linear case. A generalization to a more general class of degenerate PDE has been obtained by DiBenedetto-Friedman in [15]. In the recent work [25], the assumptions on the forcing have recently been relaxed and quantitative estimates are obtained. In particular, it is shown that the Hölder exponent α is bounded away uniformly from 0 for $m \downarrow 1$. In the nice recent works [5, 6] continuity estimates for the porous medium equation and inhomogeneous generalizations thereof with measure valued forcing have been derived.

A particular feature of the porous medium equation ($m > 1$) is the effect of finite speed of propagation and thus the occurrence of open interfaces. The regularity of the open interfaces has attracted a lot of attention in the literature, cf. e.g. Caffarelli-Friedman [9], Caffarelli-Vazquez-Wolansky [10], Koch [24] and the references therein.

In non-forced porous media equations also higher order regularity estimates have been obtained. In one spatial dimension Aronson-Vázquez [2] proved eventual C^∞ regularity of solutions. For recent progress in the general dimension case see Kienzler-Koch-Vazquez [23].

In terms of fractional Sobolev regularity of solutions to the porous medium equation less is known. As mentioned above, Ebmeyer [16] and Tadmor-Tao [31] proved for non forced porous media equations that

$$u \in L^{m+1}([0,T]; W^{s,m+1}_{loc}), \quad \forall s < \frac{2}{m+1}. \tag{1.8}$$

See also Appendix C for a slight improvement of these results. In the recent work [20], Gianazza-Schwarzacher proved higher integrability for nonnegative, local weak solutions to forced porous media equations in terms of a bound on

$$\|u^{\frac{m+1}{2}}\|_{L^{2+\varepsilon}_{loc}((0,T);W^{1,2+\varepsilon}_{loc})}$$

for all $\varepsilon > 0$ small enough. In the case of non-forced porous medium equations, Aronson-Benilan type estimates can be used to derive further regularity properties. For example, in [32, Theorem 8.7] it has been shown that $\Delta u^m \in L^1_{loc}((0,\infty); L^1)$.

1.2. **Structure of the paper.** In Section 2 we will consider the case of anisotropic, parabolic-hyperbolic second order PDE. The proof of certain multiplier estimates will be postponed to the Appendix A. In Section 3 we then treat the isotropic case in more detail, in particular introducing the concept of the isotropic truncation property for Fourier multipliers. We will then deduce our main regularity estimates for forced porous media equations. In Section 4 we treat the case of the one-dimensional degenerate parabolic Anderson model. A slight improvements of the results obtained by Ebmeyer [16] will be presented in Appendix C.

1.3. **Notation.** For $p \in [1,\infty)$ we let L^p be the usual Lebesgue spaces. The space of all locally finite Radon measures is $\mathcal{M}$, the subspace of all measures with finite total variation $\mathcal{M}_{TV}$. We let $\mathcal{M}^+ \subseteq \mathcal{M}$ be the set of all non-negative, locally finite Radon measures and $\mathcal{M}^+_{TV} = \mathcal{M}_{TV} \cap \mathcal{M}^+$. When convenient we will use the shorthand notation $L^1_x = L^1(\mathbb{R}^d)$, $L^1_{t,x} = L^1([0,T]\times\mathbb{R}^d)$. For $p \geq 1$ let p' be its conjugate, that is, $\frac{1}{p}+\frac{1}{p'} = 1$. We further let H^s_p be the fractional Sobolev spaces defined via their Fourier transform, that is, as in [22, Definition 6.2.2] and $W^{s,p}$ be the fractional Sobolev-Slobodeckij spaces (cf. [1, Section 7.35]). We follow the notation of [21, 22] and [3]. Let $\mathcal{N}^{s,p}(\mathbb{R}^d)$ be the Nikolskii spaces (cf. [27]) and $B^s_{p,q}$ Besov spaces (cf. [21]). We further let $\tilde{L}^p_t B^s_{p,q} = \tilde{L}^p([0,T]; B^s_{p,q}(\mathbb{R}^d))$ denote time-space nonhomogeneous Besov spaces as in [3, Definition 2.67]. We define the discrete increment operator by $\Delta^h_e u := u(x+he) - u(x)$. For results and standard notations in interpolation theory we refer to [4]. We let $\mathcal{S}^{d\times d}_+$ denote the space of symmetric, non-negative definite matrices. For $b = (b)_{i,j=1\dots d} \in \mathcal{S}^{d\times d}_+$ we set $\sigma = b^{\frac{1}{2}}$, that is, $b_{i,j} = \sum_{k=1}^d \sigma_{i,k}\sigma_{k,j}$. For a locally bounded function $b : \mathbb{R} \to \mathcal{S}^{d\times d}_+$ we let $\beta_{i,k}$ be such that $\beta'_{i,k}(v) = \sigma_{i,k}(v)$. Similarly, for $\psi \in C^\infty_c(\mathbb{R}_v)$ we let $\beta^\psi_{i,j}$ be such that $(\beta^\psi_{i,k})'(v) = \psi(v)\sigma_{i,k}(v)$. We further introduce the kinetic function

$$\chi(u,v) := 1_{v<u} - 1_{v<0}.$$

Analogously, for a function $u : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ we set $f(t,x,v) := \chi(u(t,x),v) := 1_{v<u(t,x)} - 1_{v<0}$. We use the short-hand notation $|\xi| \sim 2^j$ for the set $\{\xi \in \mathbb{R} : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$. For $u \in \mathbb{R}$ we set $u^{[m]} := |u|^{m-1}u$.

2. Anisotropic case

We consider equations of the form

$$\begin{aligned}\partial_t f(t,x,v) + a(v)\cdot\nabla_x f(t,x,v) - \operatorname{div}(b(v)\nabla_x f(t,x,v)) &=: \mathcal{L}(\partial_t,\nabla_x,v)f(t,x,v) \\ &= g_0(t,x,v) + \partial_v g_1(t,x,v),\end{aligned} \tag{2.1}$$

where $a : \mathbb{R} \to \mathbb{R}^d$, $b : \mathbb{R} \to \mathcal{S}^{d\times d}_+$ are C^1. The operator $\mathcal{L}$ is given by its symbol

$$\mathcal{L}(i\tau, i\xi, v) = i\tau + ia(v)\cdot\xi + (\xi, b(v)\xi). \tag{2.2}$$

In this section we will derive regularity estimates for the velocity average, for $\phi \in C^\infty_b(\mathbb{R}_v)$,

$$\bar{f}(t,x) := \int f(t,x,v)\phi(v)\,dv.$$

These regularity properties are obtained by using a suitable micro-local decomposition of f in Fourier space, which in turn relies on the so-called truncation property of the multiplier $\mathcal{L}$ (cf. Definition A.1 below). In contrast to previous results, we will make use of singular moments of g_1, that is, for $\gamma \in (0,1)$,

$$g_1(t,x,v)|v|^{-\gamma} \in \begin{cases} L^q(\mathbb{R}_t\times\mathbb{R}^d_x\times\mathbb{R}_v), & 1<q\leq 2 \\ \mathcal{M}_{TV}(\mathbb{R}_t\times\mathbb{R}^d_x\times\mathbb{R}_v), & q=1.\end{cases}$$

An additional difficulty arises in the use of bootstrapping arguments. In the theory of averaging Lemmata, optimal regularity estimates are typically obtained by bootstrapping a first non-optimal regularity estimate. This argument, however, can only be applied if the aspired final order of regularity is less than one. Therefore, we have to devise a proof which avoids the use of a bootstrapping argument, which is achieved in Section A by improving a fundamental L^p estimate on a class of Fourier multipliers by directly exploiting regularity of f in the velocity direction.

2.1. **Anisotropic averaging lemma.**

Lemma 2.1. *Let $f \in L^p_{t,x}(H^{\sigma,p}_v)$ for $1 < p \le 2$ solve, in the sense of distributions,*

$$\mathcal{L}(\partial_t, \nabla_x, v) f(t,x,v) = \Delta_x^{\frac{\eta}{2}} g_0(t,x,v) + \partial_v \Delta_x^{\frac{\eta}{2}} g_1(t,x,v) \text{ on } \mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}_v \tag{2.3}$$

with g_i being locally bounded measures satisfying

$$|g_0|(t,x,v) + |g_1|(t,x,v)|v|^{-\gamma} \in \begin{cases} L^q(\mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}_v), & 1 < q \le 2 \\ \mathcal{M}_{TV}(\mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}_v), & q = 1, \end{cases} \tag{2.4}$$

for some $\gamma \ge 0$, $\eta \ge 0$, $1 \le q \le p$ and $\mathcal{L}(\partial_t, \nabla_x, v)$ as in (2.1) with corresponding symbol $\mathcal{L}(i\tau, i\xi, v)$ as in (2.2). Assume that $\mathcal{L}(i\tau, i\xi, v)$ satisfies the truncation property uniformly in $v \in \mathbb{R}$ (cf. Appendix A). Let $I \subseteq \mathbb{R}$ be a not necessarily finite interval and set

$$\omega_{\mathcal{L}}(J;\delta) := \sup_{\tau\in\mathbb{R},\, \xi\in\mathbb{R}^d, |\xi|\sim J} |\Omega_{\mathcal{L}}(\tau,\xi;\delta)|, \quad \Omega_{\mathcal{L}}(\tau,\xi;\delta) = \{v \in I : |\mathcal{L}(i\tau, i\xi, v)| \le \delta\},$$

and suppose that the following non-degeneracy condition holds: There exist $\alpha, \beta > 0$ such that

$$\omega_{\mathcal{L}}(J;\delta) \lesssim \left(\frac{\delta}{J^\beta}\right)^\alpha \quad \forall \delta \ge 1,\ J \ge 1. \tag{2.5}$$

Moreover, assume that there exist $\lambda \ge 0$ and $\mu \in [0,1]$ such that, $\forall \delta \ge 1,\ J \ge 1$,

$$\sup_{\tau, |\xi|\sim J} \sup_{v \in \Omega_{\mathcal{L}}(\tau,\xi;\delta)} |\partial_v \mathcal{L}(i\tau, i\xi, v)||v|^\gamma \lesssim J^\lambda \delta^\mu \tag{2.6}$$

and $\frac{\alpha\beta}{q'} \le \lambda + \eta$. Then, for all $s \in [0, s^*)$, $\tilde p \in [1, p^*)$, $\phi \in C^\infty_b(I)$, $T \ge 0$ and $\mathcal{O} \subset\subset \mathbb{R}^d$, there is a $C \ge 0$ such that

$$\begin{aligned}\Big\| \int f(t,x,v)\phi(v)\,dv \Big\|_{L^{\tilde p}([0,T];W^{s,\tilde p}(\mathcal{O}))} \le C\big(&\|g_0\phi\|_{L^q_{t,x,v}} + \||v|^{-\gamma} g_1 \phi\|_{L^q_{t,x,v}} + \|g_1\phi'\|_{L^q_{t,x,v}} \\ &+ \|f\phi\|_{L^p_{t,x}(H^{\sigma,p}_v)} + \|f\phi\|_{L^q_{t,x}L^1_v} + \|f\phi\|_{L^{\tilde p}_{t,x}L^1_v}\big)\end{aligned}$$

with $s^* := (1-\theta)\frac{\alpha\beta}{r} + \theta(\frac{\alpha\beta}{q'} - \lambda - \eta)$, where $\theta = \theta_\alpha$ and p^* are given by

$$\theta := \frac{\frac{\alpha}{r}}{\alpha(\frac{1}{r} - \frac{1}{q'}) + 1} \in (0,1),\ \frac{1}{p^*} := \frac{1-\theta}{p} + \frac{\theta}{q},\ r \in \left(\frac{p'}{1+\sigma p'}, p'\right] \cap (1,\infty).$$

Proof. Let φ_0, φ_1 be smooth functions with φ_0 supported in $B_1(0)$ and φ_1 supported in the annulus $\{\xi \in \mathbb{R}^d : \frac{1}{2} \le |\xi| \le 2\}$ and

$$\varphi_0(\xi) + \sum_{j\in\mathbb{N}} \varphi_1(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^d.$$

By considering the decomposition $f = f_0 + f_1$ with

$$f_0 := \mathcal{F}_x^{-1}[\varphi_0(\xi)\mathcal{F}_x f],\ f_1 := \sum_{j\in\mathbb{N}} \mathcal{F}_x^{-1}[\varphi_1(\frac{\xi}{2^j})\mathcal{F}_x f],$$

we may assume without loss of generality that f has Fourier transform supported on $B_1(0)^c$, since for all $\eta \in [1,\infty)$

$$\|\int f_0\phi\, dv\|_{L_t^\eta W_x^{s,\eta}} \le \|f\phi\|_{L^\eta_{t,x}L^1_v}. \tag{2.7}$$

Partially inspired by [31, Averaging Lemma 2.3] we consider a micro-local decomposition of f with regard to the degeneracy of the operator $\mathcal{L}(\partial_t,\nabla_x,v)$. Let ψ_0, ψ_1 be smooth functions with ψ_0 supported in $B_1(0)$ and ψ_1 supported in the annulus $\{\xi\in\mathbb{C} : \frac{1}{2}\le|\xi|\le 2\}$ and

$$\psi_0(\xi) + \sum_{k\in\mathbb{N}}\psi_1(2^{-k}\xi) = 1, \quad \forall \xi\in\mathbb{C}.$$

For $\delta > 0$ to be specified later we write

$$\begin{aligned} f &= \psi_0\left(\frac{\mathcal{L}(\partial_t,\nabla_x,v)}{\delta}\right) f + \sum_{k\in\mathbb{N}}\psi_1\left(\frac{\mathcal{L}(\partial_t,\nabla_x,v)}{\delta 2^k}\right) f \\ &=: f^0 + f^1, \end{aligned}$$

where, for $k\in\mathbb{N}\cup\{0\}$,

$$\psi_i\left(\frac{\mathcal{L}(\partial_t,\nabla_x,v)}{\delta 2^k}\right) := \mathcal{F}^{-1}_{t,x}\psi_i\left(\frac{\mathcal{L}(i\tau,i\xi,v)}{\delta 2^k}\right)\mathcal{F}_{t,x}.$$

Since f solves (2.3) we have

$$\mathcal{L}(\partial_t,\nabla_x,v)f^1(t,x,v) = \sum_{k\in\mathbb{N}}\psi_1\left(\frac{\mathcal{L}(\partial_t,\nabla_x,v)}{\delta 2^k}\right)\left(\Delta_x^{\frac{\eta}{2}} g_0(t,x,v) + \Delta_x^{\frac{\eta}{2}}\partial_v g_1(t,x,v)\right) \tag{2.8}$$

and thus

$$\begin{aligned} f^1(t,x,v) =& \sum_{k\in\mathbb{N}}\frac{1}{\delta 2^k}\tilde{\psi}_1\left(\frac{\mathcal{L}(\partial_t,\nabla_x,v)}{\delta 2^k}\right)\Delta_x^{\frac{\eta}{2}} g_0(t,x,v) \\ &+ \sum_{k\in\mathbb{N}}\frac{1}{\delta 2^k}\tilde{\psi}_1\left(\frac{\mathcal{L}(\partial_t,\nabla_x,v)}{\delta 2^k}\right)\Delta_x^{\frac{\eta}{2}}\partial_v g_1(t,x,v) \\ =:& f^2(t,x,v) + f^3(t,x,v), \end{aligned} \tag{2.9}$$

where

$$\tilde{\psi}(z) = \frac{\psi(z)}{z}.$$

In conclusion, we have arrived at the decomposition

$$\bar{f} := \int f\phi\, dv = \int f^0\phi\, dv + \int f^2\phi\, dv + \int f^3\phi\, dv =: \bar{f}^0 + \bar{f}^2 + \bar{f}^3.$$

We aim to estimate the regularity of $\bar{f}^0$, $\bar{f}^2$, $\bar{f}^3$ in Besov spaces. Hence, we decompose each f^i into Littlewood-Paley pieces with respect to the x-variable. Let φ_0, φ_1 be as above. We set, for $i = 0, 2, 3$,

$$f_j^i := \mathcal{F}_x^{-1}[\varphi_1(\frac{\xi}{2^j})\mathcal{F}_x f^i], \quad \text{for } j \in \mathbb{N}.$$

Then, since f^i has Fourier transform supported on $B_1(0)^c$,

$$f^i = \sum_{j\geq 1} f_j^i,$$

where $\hat{f}_j^i(\tau, \xi, v)$ is supported on frequencies $|\xi| \sim 2^j$.

Step 1: f^0

Let $j \in \mathbb{N}$ arbitrary, fixed. Then, by Lemma A.3 for every $r \in (\frac{p'}{1+\sigma p'}, p'] \cap (1, \infty)$,

$$\begin{aligned}
\| \int f_j^0 \phi \, dv\|_{L^p_{t,x}} &\lesssim \|f_j \phi\|_{L^p_{t,x}(H_v^{\sigma,p})} \sup_{\tau, |\xi|\sim 2^j} |\Omega_{\mathcal{L}}(\tau, \xi, \delta)|^{\frac{1}{r}} \\
&\lesssim \|f\phi\|_{L^p_{t,x}(H_v^{\sigma,p})} \left(\frac{\delta}{(2^j)^\beta}\right)^{\frac{\alpha}{r}}.
\end{aligned}$$

Hence, $\bar{f}^0 = \int f^0\phi \, dv \in \tilde{L}^p_t B^{\frac{\alpha\beta}{r}}_{p,\infty}$ (cf. [3, Definition 2.67]) with

$$\| \int f^0 \phi \, dv\|_{\tilde{L}^p_t B^{\frac{\alpha\beta}{r}}_{p,\infty}} \lesssim \delta^{\frac{\alpha}{r}} \|f\phi\|_{L^p_x(H_v^{\sigma,p})}.$$

Step 2: f^2

Let $j \in \mathbb{N}$ arbitrary, fixed. We set

$$\begin{aligned}
f_j^{2,k} &= \frac{1}{\delta 2^k} \mathcal{F}_{t,x}^{-1} \varphi_1(\frac{\xi}{2^j}) \tilde{\psi}_1 \left(\frac{\mathcal{L}(i\tau, i\xi, v)}{\delta 2^k}\right) |\xi|^\eta \mathcal{F}_{t,x} g_0(x, v) \\
&= \frac{1}{\delta 2^k} \mathcal{F}_{t,x}^{-1} \tilde{\psi}_1 \left(\frac{\mathcal{L}(i\tau, i\xi, v)}{\delta 2^k}\right) |\xi|^\eta \mathcal{F}_{t,x} g_{0,j}(x, v)
\end{aligned}$$

Hence,

$$\int f_j^{2,k} \phi \, dv = \frac{1}{\delta 2^k} \int \tilde{\psi}_1 \left(\frac{\mathcal{L}(\partial_t, \nabla_x, v)}{\delta 2^k}\right) \Delta_x^{\frac{\eta}{2}} g_{0,j} \phi \, dv$$

and, by Lemma A.3 and since $|\xi|^\eta$ acts as a constant multiplier of order $(2^j)^\eta$ on $g_{0,j}$,

$$\begin{aligned}
\| \int f_j^{2,k} \phi \, dv\|_{L^q_{t,x}} &\lesssim \frac{1}{\delta 2^k} \Big(\| \int \mathcal{F}_{t,x}^{-1} \tilde{\psi}_1 \left(\frac{\mathcal{L}(i\tau, i\xi, v)}{\delta 2^k}\right) |\xi|^\eta \mathcal{F}_{t,x} g_{0,j} \phi \, dv\|_{L^q_{t,x}} \\
&\lesssim \frac{\sup_{\tau, |\xi|\sim 2^j} |\Omega(\tau, \xi, \delta 2^k)|^{\frac{1}{q'}}}{\delta 2^k} (2^j)^\eta \|g_{0,j}\phi\|_{L^q_{t,x,v}} \\
&\lesssim \frac{1}{\delta 2^k} \left(\frac{\delta 2^k}{(2^j)^\beta}\right)^{\frac{\alpha}{q'}} (2^j)^\eta \|g_{0,j}\phi\|_{L^q_{t,x,v}}.
\end{aligned}$$

Hence,

$$\|\int f_j^2\phi\,dv\|_{L^q_{t,x}} \lesssim \sum_{k\in\mathbb{N}} \frac{1}{\delta 2^k}\left(\frac{\delta 2^k}{(2^j)^\beta}\right)^{\frac{\alpha}{q'}}(2^j)^\eta\|g_{0,j}\phi\|_{L^q_{t,x,v}}$$
$$\lesssim \delta^{\frac{\alpha}{q'}-1}(2^j)^{\eta-\frac{\alpha\beta}{q'}}\|g_{0,j}\phi\|_{L^q_{t,x,v}}.$$

In conclusion, $\int f_j^2\phi\,dv \in \tilde{L}^q_t B^{\frac{\alpha\beta}{q'}-\eta}_{q,\infty}$ with

$$\|\int f^2\phi\,dv\|_{\tilde{L}^q_t B^{\frac{\alpha\beta}{q'}-\eta}_{q,\infty}} \lesssim \delta^{\frac{\alpha}{q'}-1}\|g_0\phi\|_{L^q_{t,x,v}}.$$

Step 3: f^3

Let $j\in\mathbb{N}$ arbitrary, fixed. We set

$$\begin{aligned} f_j^{3,k} &= \frac{1}{\delta 2^k}\mathcal{F}^{-1}_{t,x}\varphi_1(\frac{\xi}{2^j})\tilde{\psi}_1\left(\frac{\mathcal{L}(i\tau,i\xi,v)}{\delta 2^k}\right)|\xi|^\eta\mathcal{F}_{t,x}\partial_v g_1(t,x,v) \\ &= \frac{1}{\delta 2^k}\mathcal{F}^{-1}_{t,x}\tilde{\psi}_1\left(\frac{\mathcal{L}(i\tau,i\xi,v)}{\delta 2^k}\right)|\xi|^\eta\mathcal{F}_{t,x}\partial_v g_{1,j}(t,x,v)\end{aligned}$$

Hence,

$$\int f_j^{3,k}\phi\,dv = \frac{1}{\delta 2^k}\int\tilde{\psi}_1\left(\frac{\mathcal{L}(\partial_t,\nabla_x,v)}{\delta 2^k}\right)\Delta_x^{\frac{\eta}{2}}\phi\partial_v g_{1,j}dv$$

We observe

$$\begin{aligned}\int f_j^{3,k}\phi\,dv &= -\frac{1}{\delta 2^k}\int\partial_v\tilde{\psi}_1\left(\frac{\mathcal{L}(\partial_t,\nabla_x,v)}{\delta 2^k}\right)\Delta_x^{\frac{\eta}{2}}g_{1,j}\phi\,dv \\ &\quad-\frac{1}{\delta 2^k}\int\tilde{\psi}_1\left(\frac{\mathcal{L}(\partial_t,\nabla_x,v)}{\delta 2^k}\right)\Delta_x^{\frac{\eta}{2}}g_{1,j}\phi'\,dv \\ &= -\frac{1}{\delta 2^k}\int\mathcal{F}^{-1}_{t,x}\Big(\tilde{\psi}_1'\left(\frac{\mathcal{L}(i\tau,i\xi,v)}{\delta 2^k}\right)\frac{\partial_v\mathcal{L}(i\tau,i\xi,v)}{\delta 2^k}|\xi|^\eta\mathcal{F}_{t,x}g_{1,j}\phi\Big)dv \\ &\quad-\frac{1}{\delta 2^k}\int\tilde{\psi}_1\left(\frac{\mathcal{L}(\partial_t,\nabla_x,v)}{\delta 2^k}\right)\Delta_x^{\frac{\eta}{2}}g_{1,j}\phi'\,dv \\ &= \frac{1}{(\delta 2^k)^2}\int\mathcal{F}^{-1}_{t,x}\Big(\tilde{\psi}_1'\left(\frac{\mathcal{L}(i\tau,i\xi,v)}{\delta 2^k}\right)\partial_v\mathcal{L}(i\tau,i\xi,v)|v|^\gamma|\xi|^\eta\mathcal{F}_{t,x}|v|^{-\gamma}g_{1,j}\phi\Big)dv \\ &\quad-\frac{1}{\delta 2^k}\int\tilde{\psi}_1\left(\frac{\mathcal{L}(\partial_t,\nabla_x,v)}{\delta 2^k}\right)\Delta_x^{\frac{\eta}{2}}g_{1,j}\phi'\,dv.\end{aligned}$$

By the Marcinkiewicz Multiplier Theorem (cf. [21, Theorem 5.2.4]) and (2.6) we have that $\partial_v\mathcal{L}(i\tau,i\xi,v)|v|^\gamma$ acts as a constant multiplier on L^q of order $O((2^j)^\lambda\delta^\mu)$ on $g_{1,j}$. Hence, using Lemma A.3 yields

$$
\begin{aligned}
&\| \int f_j^{3,k} \phi \, dv \|_{L^q_{t,x}} \\
&\le \frac{1}{(\delta 2^k)^2} \| \int \mathcal{F}^{-1}_{t,x} \tilde{\psi}_1' \left(\frac{\mathcal{L}(i\tau, i\xi, v)}{\delta 2^k} \right) (\partial_v \mathcal{L})(i\tau, i\xi, v) |v|^\gamma |\xi|^\eta \mathcal{F}_{t,x} |v|^{-\gamma} g_{1,j} \phi \, dv \|_{L^q_{t,x}} \\
&+ \frac{1}{\delta 2^k} \| \int \tilde{\psi}_1 \left(\frac{\mathcal{L}(\partial_t, \nabla_x, v)}{\delta 2^k} \right) \Delta_x^{\frac{\eta}{2}} g_{1,j} \phi' \, dv \|_{L^q_{t,x}} \\
&\lesssim \frac{\sup_{\tau, |\xi| \sim J} |\Omega(\tau, \xi, \delta 2^k)|^{\frac{1}{q'}}}{(\delta 2^k)^2} (2^j)^{\lambda + \eta} \delta^\mu \| |v|^{-\gamma} g_{1,j} \phi \, dv \|_{L^q_{t,x,v}} \\
&+ \frac{\sup_{\tau, |\xi| \sim J} |\Omega(\tau, \xi, \delta 2^k)|^{\frac{1}{q'}}}{\delta 2^k} (2^j)^\eta \| g_{1,j} \phi' \|_{L^q_{t,x,v}} \\
&\lesssim \frac{1}{(\delta 2^k)^2} \left(\frac{\delta 2^k}{(2^j)^\beta} \right)^{\frac{\alpha}{q'}} (2^j)^\lambda (\delta 2^k)^\mu (2^j)^\eta \| |v|^{-\gamma} g_{1,j} \phi \|_{L^q_{t,x,v}} \\
&+ \frac{1}{\delta 2^k} \left(\frac{\delta 2^k}{(2^j)^\beta} \right)^{\frac{\alpha}{q'}} (2^j)^\eta \| g_{1,j} \phi' \|_{L^q_{t,x,v}} \\
&= (\delta 2^k)^{-2 + \frac{\alpha}{q'} + \mu} (2^j)^{\eta + \lambda - \frac{\alpha\beta}{q'}} \| |v|^{-\gamma} g_{1,j} \phi \|_{L^q_{t,x,v}} + (\delta 2^k)^{-1 + \frac{\alpha}{q'}} (2^j)^{\frac{\alpha\beta}{q'} + \eta} \| g_{1,j} \phi' \|_{L^q_{t,x,v}}
\end{aligned}
$$

Hence, for $\delta \ge 1$ and using $\mu \in [0,1]$,

$$
\begin{aligned}
\| \int f_j^3 \phi \, dv \|_{L^q_{t,x}} \lesssim & \sum_{k \in \mathbb{N}} (\delta 2^k)^{-2 + \frac{\alpha}{q'} + \mu} (2^j)^{\eta + \lambda - \frac{\alpha\beta}{q'}} \| |v|^{-\gamma} g_{1,j} \phi \|_{L^q_{t,x,v}} \\
& + (\delta 2^k)^{-1 + \frac{\alpha}{q'}} (2^j)^{\frac{\alpha\beta}{q'} + \eta} \| g_{1,j} \phi' \|_{L^q_{t,x,v}} \\
\lesssim & \delta^{-2 + \frac{\alpha}{q'} + \mu} (2^j)^{\eta + \lambda - \frac{\alpha\beta}{q'}} \| |v|^{-\gamma} g_{1,j} \phi \|_{L^q_{t,x,v}} + \delta^{-1 + \frac{\alpha}{q'}} (2^j)^{\frac{\alpha\beta}{q'} + \eta} \| g_{1,j} \phi' \|_{L^q_{t,x,v}} \\
\lesssim & \delta^{-1 + \frac{\alpha}{q'}} (2^j)^{\eta + \lambda - \frac{\alpha\beta}{q'}} (\| |v|^{-\gamma} g_{1,j} \phi \|_{L^q_{t,x,v}} + \| g_{1,j} \phi' \|_{L^q_{t,x,v}}).
\end{aligned}
$$

In conclusion, $\int f_j^3 \phi \, dv \in \tilde{L}^q_t B^{\frac{\alpha\beta}{q'} - \lambda - \eta}_{q,\infty}$ with

$$
\| \int f^3 \phi \, dv \|_{\tilde{L}^q_t B^{\frac{\alpha\beta}{q'} - \lambda - \eta}_{q,\infty}} \lesssim \delta^{-1 + \frac{\alpha}{q'}} (\| |v|^{-\gamma} g_{1,j} \phi \|_{L^q_{t,x,v}} + \| g_{1,j} \phi' \|_{L^q_{t,x,v}}).
$$

Step 4: Conclusion

Since $B^{\frac{\alpha\beta}{q'} - \eta}_{q,\infty} \hookrightarrow B^{\frac{\alpha\beta}{q'} - \lambda - \eta}_{q,\infty}$ we have

$$
\bar{f} = \bar{f}^0 + \bar{f}^1
$$

with $\bar f^0 \in \tilde L^p_t B^{\frac{\alpha\beta}{r}}_{p,\infty}$, $\bar f^1 = \bar f^2 + \bar f^3 \in \tilde L^q_t B^{\frac{\alpha\beta}{q'}-\lambda-\eta}_{q,\infty}$ and, for $\delta \geq 1$,

$$\|\bar f^0\|_{\tilde L^p_t B^{\frac{\alpha\beta}{r}}_{p,\infty}} \lesssim \delta^{\frac{\alpha}{r}} \|f\phi\|_{L^p_{t,x}(H^{\sigma,p}_v)},$$

$$\|\bar f^1\|_{\tilde L^q_t B^{\frac{\alpha\beta}{q'}-\lambda-\eta}_{q,\infty}} \lesssim \delta^{\frac{\alpha}{q'}-1}(\|g_0\phi\|_{L^q_{t,x,v}} + \||v|^{-\gamma} g_{1,j}\phi\|_{L^q_{t,x,v}} + \|g_{1,j}\phi'\|_{L^q_{t,x,v}}).$$

We aim to conclude by real interpolation. We set, for $z > 0$,

$$K(z,\overline{f}) := \inf\{\|\overline{f}^1\|_{\tilde L^q_t B^{\frac{\alpha\beta}{q'}-\lambda-\eta}_{q,\infty}} + z\|\overline{f}^0\|_{\tilde L^p_t B^{\frac{\alpha\beta}{r}}_{p,\infty}} : \overline{f}^0 \in \tilde L^p_t B^{\frac{\alpha\beta}{r}}_{p,\infty},$$
$$\overline{f}^1 \in \tilde L^q_t B^{\frac{\alpha\beta}{q'}-\lambda-\eta}_{q,\infty},\ \overline{f} = \overline{f}^0 + \overline{f}^1\}.$$

We first note the trivial estimate, since $\frac{\alpha\beta}{q'} - \lambda - \eta \leq 0$,

$$K(z,\overline{f}) \leq \|\overline{f}\|_{\tilde L^q_t B^{\frac{\alpha\beta}{q'}-\lambda-\eta}_{q,\infty}} \leq \|\overline{f}\|_{\tilde L^q_t L^q_x} \leq \|f\phi\|_{L^q_{t,x}L^1_v} \quad \forall z > 0.$$

Hence, it is enough to consider $z \leq 1$ in the estimates below. By the above estimates we obtain that, for $\delta \geq 1$,

$$K(z,\overline{f}) \leq \delta^{\frac{\alpha}{q'}-1}(\|g_0\phi\|_{L^q_{t,x,v}} + \||v|^{-\gamma} g_{1,j}\phi\|_{L^q_{t,x,v}} + \|g_{1,j}\phi'\|_{L^q_{t,x,v}}) + z\delta^{\frac{\alpha}{r}}\|f\phi\|_{L^p_{t,x}(H^{\sigma,p}_v)}.$$

We now equilibrate the first and the second term on the right hand side, that is, we set

$$\delta^{\frac{\alpha}{q'}-1} = z\delta^{\frac{\alpha}{r}},$$

which yields

$$\delta = z^{-\frac{1}{\alpha(\frac{1}{r}-\frac{1}{q'})+1}} \geq 1.$$

Hence, with

$$\theta = \frac{1-\frac{\alpha}{q'}}{\alpha(\frac{1}{r}-\frac{1}{q'})+1} = 1 - \frac{\frac{\alpha}{r}}{\alpha(\frac{1}{r}-\frac{1}{q'})+1}$$

we obtain, for $|z| \leq 1$,

$$K(z,\overline{f}) \leq z^{\theta}(\|g_0\phi\|_{L^q_{t,x,v}} + \||v|^{-\gamma} g_{1,j}\phi\|_{L^q_{t,x,v}} + \|g_{1,j}\phi'\|_{L^q_{t,x,v}} + \|f\phi\|_{L^p_{t,x}(H^{\sigma,p}_v)}).$$

Consequently, for $\tau \in (0,\theta)$ and $\frac{1}{p_\tau} = \frac{1-\tau}{q} + \frac{\tau}{p}$,

$$\begin{aligned}
\|\overline{f}\|^{p_\tau}_{(\tilde L^q_t B^{\frac{\alpha\beta}{q'}-\lambda-\eta}_{q,\infty}, \tilde L^p_t B^{\frac{\alpha\beta}{r}}_{p,\infty})_{\tau,p_\tau}} &= \|z^{-\tau}K(z,\overline{f})\|^{p_\tau}_{L^{p_\tau}_*(0,\infty)} \\
&= \|z^{-\tau}K(z,\overline{f})\|^{p_\tau}_{L^{p_\tau}_*(0,1)} + \|z^{-\tau}K(z,\overline{f})\|^{p_\tau}_{L^{p_\tau}_*(1,\infty)} \\
&\leq \|z^{\theta-\tau}\|^{p_\tau}_{L^{p_\tau}_*(0,1)}\big(\|g_0\phi\|_{L^q_{t,x,v}} + \||v|^{-\gamma} g_{1,j}\phi\|_{L^q_{t,x,v}} \\
&+ \|g_{1,j}\phi'\|_{L^q_{t,x,v}} + \|f\phi\|_{L^p_{t,x}(H^{\sigma,p}_v)}\big)^{p_\tau} + \|z^{-\tau}\|^{p_\tau}_{L^{p_\tau}_*(1,\infty)}\|f\phi\|^{p_\tau}_{L^q_{t,x}L^1_v} \\
&\lesssim \|g_0\phi\|^{p_\tau}_{L^q_{t,x,v}} + \||v|^{-\gamma} g_{1,j}\phi\|^{p_\tau}_{L^q_{t,x,v}} + \|g_{1,j}\phi'\|^{p_\tau}_{L^q_{t,x,v}} \\
&+ \|f\phi\|^{p_\tau}_{L^p_{t,x}(H^{\sigma,p}_v)} + \|f\phi\|^{p_\tau}_{L^q_{t,x}L^1_v}.
\end{aligned}$$

Let

$$s < s^* := (1-\theta)(\frac{\alpha\beta}{q'} - \lambda - \eta) + \theta\frac{\alpha\beta}{r}$$

From [3, p. 98] we recall, for $\varepsilon > 0$,

$$\tilde{L}_t^q B_{q,\infty}^{\frac{\alpha\beta}{q'}-\lambda-\eta} \hookrightarrow \tilde{L}_t^q B_{q,1}^{\frac{\alpha\beta}{q'}-\lambda-\eta-\varepsilon} \hookrightarrow L_t^q B_{q,1}^{\frac{\alpha\beta}{q'}-\lambda-\eta-\varepsilon}$$

and analogously for $\tilde{L}_t^p B_{p,\infty}^{\frac{\alpha\beta}{r}}$. Thus, using [4, Section 5.6 and Theorem 6.4.5] and choosing $\varepsilon > 0$ small enough yields

$$\begin{aligned}(\tilde{L}_t^q B_{q,\infty}^{\frac{\alpha\beta}{q'}-\lambda-\eta}, \tilde{L}_t^p B_{p,\infty}^{\frac{\alpha\beta}{r}})_{\tau,p_\tau} &\hookrightarrow L^{p_\tau}(B_{q,1}^{\frac{\alpha\beta}{q'}-\lambda-\eta-\varepsilon}, B_{p,1}^{\frac{\alpha\beta}{r}-\varepsilon})_{\tau,p_\tau} \\ &\hookrightarrow L_t^{p_\tau} B_{p_\tau,p_\tau}^s \hookrightarrow L_t^{p_\tau} W_x^{s,p_\tau}.\end{aligned}$$

Hence, choosing $\tau \in (0,\theta)$ large enough and recalling (2.7), for all $p < p^*$ with $\frac{1}{p^*} = \frac{1-\theta}{q} + \frac{\theta}{p}$ and all $\mathcal{O} \subset \mathbb{R}^d$ compact, we have

$$\begin{aligned}\|\bar{f}\|_{L^p([0,T];W^{s,p}(\mathcal{O}))} \lesssim\ & \|g_0\phi\|_{L^q_{t,x,v}} + \||v|^{-\gamma} g_{1,j}\phi\|_{L^q_{t,x,v}} + \|g_{1,j}\phi'\|_{L^q_{t,x,v}} \\ &+ \|f\phi\|_{L^p_{t,x}(H_v^{\sigma,p})} + \|f\phi\|_{L^q_{t,x}L^1_v} + \|f\phi\|_{L^p_{t,x}L^1_v}.\end{aligned}$$

□

Remark 2.2. In the above averaging Lemma we do not require ϕ to have compact support, nor I to be a bounded interval. We note that if I and supp ϕ are unbounded, then the non-degeneracy condition (2.5) entails a growth condition on $\mathcal{L}(i\tau, i\xi, v)$.

This becomes clear when looking at specific examples, such as porous media equations with nonlinearity $B(u)$, which in kinetic form corresponds to (2.1) with $a \equiv 0$, $b(v) = B'(v)Id$. In this case, $|\mathcal{L}(i\tau, i\xi, v)| \geq |\xi|^2 b(v)$ and thus

$$\begin{aligned}\omega_{\mathcal{L}}(J;\delta) &= \sup_{\tau,|\xi|\sim J} |\{v \in \operatorname{supp}\phi : |\mathcal{L}(i\tau,i\xi,v)| \leq \delta\}| \\ &\leq \sup_{|\xi|\sim J} |\{v \in \operatorname{supp}\phi : |b(v)| \leq \delta|\xi|^{-2}\}| \leq |b^{-1}(B_{\delta|J|^{-2}}(0)) \cap \operatorname{supp}\phi|.\end{aligned}$$

Hence, in the case supp $\phi = \mathbb{R}$ condition (2.5) becomes, roughly speaking, $|b^{-1}(B_r(0))| \lesssim r^\alpha$ for all $r > 0$.

2.2. **Anisotropic parabolic-hyperbolic equations.** In this section we consider parabolic-hyperbolic equations of the type

$$\begin{aligned}\partial_t u + \operatorname{div}A(u) - \operatorname{div}(b(u)\nabla u) &= S(t,x) \quad \text{on } (0,T)\times\mathbb{R}^d_x \\ u(0) &= u_0 \quad \text{on } \mathbb{R}^d_x,\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}u_0 \in L^1(\mathbb{R}^d_x),\ &S \in L^1([0,T]\times\mathbb{R}^d_x),\ T \geq 0, \\ a := A' \in\ & C(\mathbb{R};\mathbb{R}^d) \cap C^1(\mathbb{R}\setminus\{0\};\mathbb{R}^d), \\ b = (b_{jk})_{j,k=1\ldots d} \in\ & C(\mathbb{R};S_+^{d\times d}) \cap C^1(\mathbb{R}\setminus\{0\};S_+^{d\times d}).\end{aligned} \tag{2.11}$$

The corresponding kinetic form for

$$f(t,x,v) = \chi(u(t,x),v) \tag{2.12}$$

reads (cf. [11])

$$\begin{aligned}\mathcal{L}(\partial_t,\nabla_x,v)f(t,x,v) &= \partial_t f + a(v)\cdot\nabla_x f - \operatorname{div}(b(v)\nabla_x f)\\ &= \partial_v q + S(t,x)\delta_{u(t,x)=v}(v),\end{aligned} \tag{2.13}$$

where $q \in \mathcal{M}^+$ and $\mathcal{L}$ is identified with the symbol

$$\mathcal{L}(i\tau,i\xi,v) := i\tau + a(v)\cdot i\xi + (b(v)\xi,\xi). \tag{2.14}$$

Kinetic/entropy solutions to (2.10) are then defined analogously to [11, Definition 2.2] cf. Appendix B below. The existence and uniqueness of entropy solutions to (2.10) follows along the lines of the respective arguments of [11] with additional arguments concerning the forcing term to be found in [19]. We first establish the following a-priori bound

Lemma 2.3. *Let u be the unique entropy solution to* (2.10)*. Then, for all $\gamma \in (-\infty,1)$ there is a constant $C = C(T,\gamma) \geq 0$ such that*

$$\begin{aligned}&\sup_{t\in[0,T]}\|u(t)\|^{2-\gamma}_{L^{2-\gamma}_x} + (1-\gamma)\int_0^T\int_{\mathbb{R}^{d+1}}|v|^{-\gamma}q\,dvdxdr\\ &\leq C\big(\|u_0\|^{2-\gamma}_{L^{2-\gamma}_x} + \|S\|^{2-\gamma}_{L^{2-\gamma}_{t,x}}\big).\end{aligned} \tag{2.15}$$

Moreover, for $\eta \in C^\infty_c(\mathbb{R})$ convex we have

$$\begin{aligned}&\sup_{t\in[0,T]}\int_{\mathbb{R}^d_x}\eta(u(t))dx + \int_0^T\int_{\mathbb{R}^{d+1}}\eta''(v)q\,dvdxdr\\ &\leq \int_{\mathbb{R}^d_x}\eta(u_0)dx + \|\eta'\|_\infty\|S\|_{L^1_{t,x}}.\end{aligned} \tag{2.16}$$

Proof. Let $u_0^\varepsilon \in C_c^\infty(\mathbb{R}^d)$ with $u_0^\varepsilon \to u_0$ in $L^1(\mathbb{R}^d)$, $\|u_0^\varepsilon\|_{L^{2-\gamma}_x} \leq \|u_0\|_{L^{2-\gamma}_x}$ and $S^\varepsilon \in C_c^\infty([0,T]\times\mathbb{R}^d)$ with $\|S^\varepsilon\|_{L^{2-\gamma}_{t,x}} \leq \|S\|_{L^{2-\gamma}_{t,x}}$ and

$$S^\varepsilon \to S \quad \text{in } L^1([0,T]\times\mathbb{R}^d).$$

The unique entropy solution to (2.10) is obtained as a vanishing viscosity limit (see [11]), that is, we consider

$$\partial_t u^{\varepsilon,\eta} + \operatorname{div}A(u^{\varepsilon,\eta}) - \operatorname{div}(b(u^{\varepsilon,\eta})\nabla u^{\varepsilon,\eta}) - \eta\Delta u^{\varepsilon,\eta} = S^\varepsilon \text{ on } (0,T)\times\mathbb{R}^d_x, \tag{2.17}$$

with $u^\varepsilon(0) = u_0^\varepsilon$ and kinetic form

$$\mathcal{L}(\partial_t,\nabla_x,v)f^{\varepsilon,\eta} = \eta\Delta_x f^{\varepsilon,\eta} + \partial_v q^{\varepsilon,\eta} + S^{\varepsilon,\eta}(t,x)\delta_{u^{\varepsilon,\eta}(t,x)=v}(v), \tag{2.18}$$

where $q^{\varepsilon,\eta} = m^{\varepsilon,\eta} + n^{\varepsilon,\eta}$ is given by

$$m^{\varepsilon,\eta} = \delta_{v=u^{\varepsilon,\eta}}\eta|\nabla u^{\varepsilon,\eta}|^2, \quad n^{\varepsilon,\eta} = \delta_{v=u^{\varepsilon,\eta}}\sum_{k=1}^d\left(\sum_{i=1}^d\partial_{x_i}\beta_{ik}(u^{\varepsilon,\eta})\right)^2,$$

and β_{ik} was introduced in Section 1.3. Then, following [11] (cf. also Appendix B), one has $\lim_{\varepsilon\to0}\lim_{\eta\to0}u^{\varepsilon,\eta} \to u$ in $C([0,T];L^1_x)$, $f^{\varepsilon,\eta} \to f$ in $L^1_{t,x,v}$ and $q^{\varepsilon,\eta} \rightharpoonup^* q$ in $\mathcal{M}$. Due to weak lower semicontinuity of the left hand side of (2.15) it is sufficient to prove (2.15) for $u^{\varepsilon,\eta}$, $m^{\varepsilon,\eta}$, $n^{\varepsilon,\eta}$. Let $\eta\in C_c^\infty$ be convex. We multiply (2.18) by $\eta'(v)$ and integrate in v,x to get

$$\begin{aligned}\partial_t\int\eta(u^{\varepsilon,\eta})dx&=\int\eta'(u^{\varepsilon,\eta})S^\varepsilon(t,x)dx-\int\eta''(v)q^{\varepsilon,\eta}\,dvdx\\&\lesssim\int|\eta'(u^{\varepsilon,\eta})|^{\frac{2-\gamma}{1-\gamma}}dx+\int|S^\varepsilon(t,x)|^{2-\gamma}dx-\int\eta''(v)q^{\varepsilon,\eta}\,dvdx.\end{aligned}$$

Then (2.16) follows easily from the first line of the inequality above. Using a standard cut-off argument we may choose $\eta=\eta^\delta\in C^\infty$ with

$$(\eta^\delta)''(v):=(|v|^2+\delta)^{-\frac{\gamma}{2}}.$$

Then η^δ is convex and $(\eta^\delta)'(v)\le|v|^{1-\gamma}$. Hence,

$$\partial_t\int\eta^\delta(u^{\varepsilon,\eta})dx+\int(\eta^\delta)''(v)q^{\varepsilon,\eta}\,dvdx\lesssim\int|u^{\varepsilon,\eta}|^{2-\gamma}dx+\int|S^\varepsilon(t,x)|^{2-\gamma}dx.$$

Letting $\delta\to0$ yields, by Fatou's Lemma,

$$\partial_t\int|u^{\varepsilon,\eta}|^{2-\gamma}dx+\int\int|v|^{-\gamma}q^{\varepsilon,\eta}\,dvdx\lesssim\int|u^{\varepsilon,\eta}|^{2-\gamma}dx+\int|S^\varepsilon(t,x)|^{2-\gamma}dx.$$

Gronwall's inequality concludes the proof. □

Lemma 2.4. *Let u be the unique entropy solution to* (2.10) *and $\psi\in C^2(\mathbb{R})\cap Lip(\mathbb{R})$ be a convex function with $|\psi(r)|\le c|r|$, for some $c>0$. Then*

$$\int q(t,x,v)\psi''(v)dvdxdt\le C(\|u_0\|_{L^1_x}+\|S\|_{L^1_{t,x}}),$$

for some constant C depending only on c and $\sup_v|\psi'|(v)$.

Proof. We first note that multiplying (2.13) with a smooth approximation of $\mathrm{sgn}(v)$, integrating and taking the limit yields, for all $t\ge0$,

$$\int|u(t,x)|dx\le\int|u(0,x)|+\|S\|_{L^1([0,T]\times\mathbb{R}^d)}.$$

From (2.13) and a standard cut-off argument we further obtain

$$\begin{aligned}\partial_t\int\psi(u(t,x))dx&=\partial_t\int f(t,x,v)\psi'(v)dvdx\\&\le-\int\psi''(v)q(t,x,v)dvdx+\int S(t,x)\psi'(u(t,x))dx.\end{aligned}$$

Hence,

$$\begin{aligned}\int_0^T\int\psi''(v)q(t,x,v)dvdxdt&\le-\int\psi(u(\cdot,x))dx\Big|_0^T+\int_0^T\int S(r,x)\psi'(u(r,x))dxdr\\&\le c\int|u(0,x)|dx+c\int|u(T,x)|dx+C\|S\|_{L^1_{t,x}}\\&\le C(\|u_0\|_{L^1_x}+\|S\|_{L^1_{t,x}}).\end{aligned}$$

□

We may now apply Lemma 2.1 to obtain

Corollary 2.5. *Let $u_0 \in L^1(\mathbb{R}^d_x)$, $S \in L^1([0,T]\times\mathbb{R}^d_x)$, a, b satisfy (2.11) and let u be the entropy solution to (2.10). Further assume that the symbol $\mathcal{L}$ defined in (2.14) satisfies (2.5), (2.6) for all $\gamma \in [0,1)$ large enough. Then, for all*

$$s \in \big[0, \frac{\alpha}{\alpha+1}(\beta-\lambda)\big), \quad p \in \Big[1, \frac{2\alpha+2}{2\alpha+1}\Big),$$

all $\phi \in C^\infty_c(\mathbb{R}_v)$, $\gamma \in [0,1)$ large enough and $\mathcal{O} \subset\subset \mathbb{R}^d$, there is a constant $C \geq 0$ such that

$$\|\int f\phi\, dv\|_{L^p([0,T];W^{s,p}(\mathcal{O}))} \leq C(\|u_0\|_{L^1_x} + \|u_0\|^{2-\gamma}_{L^{2-\gamma}_x} + \|S\|_{L^1_{t,x}} + \|S\|^{2-\gamma}_{L^{2-\gamma}_{t,x}} + 1). \tag{2.19}$$

Proof. We will derive (2.19) on the level of the approximating equation (2.17). By convergence of the approximating solutions $u^{\varepsilon,\eta}$ and lower-semicontinuity of the norm this is sufficient. For notational simplicity we suppress the ε,η-dependency in the following, but note that all estimates are uniform with respect to these parameters. As in [11, Section 7] we observe the bound (uniformly in ε,η), for each $\psi \in C^\infty_c(\mathbb{R}_v)$, $k=1,\dots,d$,

$$\|\sum_{i=1}^d \partial_{x_i}\beta^\psi_{ik}(u)\|_{L^2_{t,x}} \lesssim \|u_0\|_{L^1_x} + \|S\|_{L^1_{t,x}} + 1.$$

We hence estimate, for any $\varphi \in C^\infty_c(\mathbb{R}_t\times\mathbb{R}^d_x\times\mathbb{R}_v)$ and $\psi \in C^\infty_c(\mathbb{R}_v)$ such that $\varphi\psi=\varphi$,

$$\begin{aligned}
\int_{t,x,v}|\nabla f\cdot b(v)\nabla\varphi| &\leq \sum_{k=1}^d\int_{t,x,v}\left|\left(\sum_{i=1}^d\partial_{x_i}f\sigma_{ik}(v)\right)\left(\sum_{j=1}^d\sigma_{kj}(v)\partial_{x_j}\varphi\right)\right| \\
&= \sum_{k=1}^d\int_{t,x,v}\left|\left(\sum_{i=1}^d\delta_{u(t,x)=v}\partial_{x_i}u\sigma_{ik}(v)\psi(v)\right)\left(\sum_{j=1}^d\sigma_{kj}(v)\partial_{x_j}\varphi\right)\right| \\
&= \sum_{k=1}^d\int_{t,x}\left|\left(\sum_{i=1}^d\partial_{x_i}\beta^\psi_{ik}(u)\right)\left(\sum_{j=1}^d\sigma_{kj}(u)(\partial_{x_j}\varphi)(t,x,u(t,x))\right)\right| \qquad (2.20)\\
&\leq \sum_{k=1}^d\|\sum_{i=1}^d\partial_{x_i}\beta^\psi_{ik}(u)\|_{L^2_{t,x}}\|\sum_{j=1}^d\sigma_{kj}(u)(\partial_{x_j}\varphi)(t,x,u(t,x))\|_{L^2_{t,x}} \\
&\lesssim \|u_0\|_{L^1_x} + \|S\|_{L^1_{t,x}} + 1.
\end{aligned}$$

We next note that due to (2.12) we have $\partial_v f(t,x,v) = -\delta_{u(t,x)=v}$ and thus $f \in L^\infty_{t,x}(\dot{BV}_v) \subseteq L^1_{t,x;loc}(\dot{BV}_v)$. Moreover,

$$\|f\|_{L^1_{t,x,v}} \leq \|u_0\|_{L^1_x} + \|S\|_{L^1_{t,x}} \tag{2.21}$$

and $|f| \leq 1$. Hence, $f \in L^1_{t,x,v}\cap L^\infty_{t,x,v}$ and by interpolation we obtain that $f \in L^2_{t,x;loc}(H^{\sigma,2}_v)$ for all $\sigma \in [0,\frac{1}{2})$ with

$$\|f\|_{L^2_{t,x;loc}(H^{\sigma,2}_v)} \lesssim 1 + \|u_0\|_{L^1_x} + \|S\|_{L^1_{t,x}}. \tag{2.22}$$

In order to apply Lemma 2.1 we hence have to localize f. Let $\varphi \in C_c^\infty((0,T)\times\mathbb{R}_x^d\times\mathbb{R}_v)$, $\eta^\delta \in C^\infty(\mathbb{R})$ satisfy $\eta^\delta(v)\in[0,1]$ for all $v\in\mathbb{R}$, $|(\eta^\delta)'|\lesssim \frac{1}{\delta}$,

$$\eta^\delta(v) = \begin{cases} 1 & \text{for } |v|\ge\delta \\ 0 & \text{for } |v|\le\frac{\delta}{2} \end{cases} \tag{2.23}$$

and set $\varphi^\delta = \varphi\eta^\delta$. For simplicity we suppress the δ-index in the following. Set $\tilde f := \varphi f \in L^2_{t,x}(W_v^{\sigma,2})$, $\tilde q := \varphi q$. Then

$$\begin{aligned}\partial_t \tilde f &= \varphi\big(-a(v)\cdot\nabla f + \operatorname{div}(b(v)\nabla f) + \partial_v q + S\delta_{u(t,x)=v}(v)\big) + f\partial_t\varphi \\ &= (-a(v)\cdot\nabla\tilde f + \operatorname{div}(b(v)\nabla\tilde f) + \partial_v\tilde q + \varphi S\delta_{u(t,x)=v}(v) \\ &\quad + a(v)\cdot f\nabla\varphi - 2\nabla f\cdot b(v)\nabla\varphi - f\operatorname{div}(b(v)\nabla\varphi) - (\partial_v\varphi)q + f\partial_t\varphi.\end{aligned} \tag{2.24}$$

Since φ is compactly supported and $q\in\mathcal{M}$, we have $\tilde q\in\mathcal{M}_{TV}$. Moreover, due to (2.20) and $S\in L^1_{t,x}$ we have

$$\begin{aligned} g_0 :=&\varphi S\delta_{u(t,x)=v}(v) + a(v)\cdot f\nabla\varphi - 2\nabla f\cdot b(v)\nabla\varphi - f\operatorname{div}(b(v)\nabla\varphi) \\ &- (\partial_v\varphi)q + f\partial_t\varphi \in \mathcal{M}_{TV}\end{aligned}$$

with

$$\|g_0\|_{\mathcal{M}_{TV}} \le \|u_0\|_{L^1_x} + \|S\|_{L^1_{t,x}} + \|\partial_v\varphi q\|_{\mathcal{M}_{TV}} + \|f\phi\partial_t\varphi\|_{L^1_{t,x,v}}. \tag{2.25}$$

Let $s\in[0,\frac{\alpha}{\alpha+1}(\beta-\lambda))$ and $p\in[1,\frac{2\alpha+2}{2\alpha+1})$. Choose $\gamma\in[0,1)$ large enough and $r>1$ small enough, such that $s<(1-\theta)\frac{\alpha\beta}{r}-\lambda\theta$ and $p<\frac{2}{1+\theta}$ where $\theta=\frac{\frac{\alpha}{r}}{\frac{\alpha}{r}+1}$. We may assume $u_0\in L^1_x\cap L^{2-\gamma}_x$, $S\in L^1_{t,x}\cap L^{2-\gamma}_{t,x}$, otherwise there is nothing to be shown. By Lemma 2.3 we have

$$\|u(t)\|^{2-\gamma}_{L^{2-\gamma}_x} + (1-\gamma)\int_0^t\int_{\mathbb{R}^{d+1}}|v|^{-\gamma}q\,dvdxdr \lesssim \|u_0\|^{2-\gamma}_{L^{2-\gamma}_x} + \|S\|^{2-\gamma}_{L^{2-\gamma}_{t,x}}.$$

We note that, due to (2.23) and (2.11) we may assume $a,b\in C^1$ without changing (2.24). We now apply Lemma 2.1 with $\eta=0$, $g_1=\tilde q$, $f=\tilde f$, $q=1$, $p=2$, $\sigma\in[0,\frac12)$ large enough, $T\ge0$, $\mathcal{O}\subseteq\mathbb{R}^d$ compact to obtain that there is a constant $C\ge0$ such that

$$\begin{aligned}\|\int f\varphi^\delta\phi\,dv\|_{L^p([0,T];W^{s,p}(\mathcal{O}))} &\lesssim \|g_0^\delta\phi\|_{\mathcal{M}_{TV}} + \||v|^{-\gamma}g_1^\delta\phi\|_{\mathcal{M}_{TV}} + \|g_1^\delta\phi'\|_{\mathcal{M}_{TV}} \\ &\quad + \|f\phi\|_{L^p_{t,x}(H_v^{\sigma,p})} + \|f\phi\|_{L^1_{t,x,v}} + \|f\phi\|_{L^p_{t,x}L^1_v}.\end{aligned}$$

By Lemma 2.3, (2.22), (2.21) and (2.25) we obtain that

$$\begin{aligned}\|\int f\varphi^\delta\phi\,dv\|_{L^p([0,T];W^{s,p}(\mathcal{O}))} &\lesssim \|u_0\|_{L^1_x} + \|S\|_{L^1_{t,x}} + \|\partial_v\varphi q\|_{\mathcal{M}_{TV}} + \|f\phi\partial_t\varphi\|_{L^1_{t,x,v}} \\ &\quad + \|u_0\|^{2-\gamma}_{L^{2-\gamma}_x} + \|S\|^{2-\gamma}_{L^{2-\gamma}_{t,x}} + 1.\end{aligned}$$

We next consider the limit $\delta\to0$. Since $|\eta^\delta|\le1$, the only nontrivial term appearing on the right hand side is $\|(\partial_v\eta^\delta)\varphi q\|_{\mathcal{M}_{TV}}$. Let ψ^δ be such that $(\psi^\delta)''=|\partial_v\eta^\delta|$ and $|\psi^\delta(r)|\le c|r|$.

Then ψ^δ satisfies the assumptions of Lemma 2.4 uniformly in δ which yields the required bound. Since φ is arbitrary, we conclude

$$\|\int f\phi\,dv\|_{L^p([0,T];W^{s,p}(\mathcal{O}))} \lesssim \|u_0\|_{L^1_x} + \|u_0\|_{L^{2-\gamma}_x}^{2-\gamma} + \|S\|_{L^1_{t,x}} + \|S\|_{L^{2-\gamma}_{t,x}}^{2-\gamma} + 1.$$

□

Example 2.6. Let $u_0 \in L^1(\mathbb{R}^d_x)$, $S \in L^1([0,T]\times\mathbb{R}^d_x)$, $m_j, n_j \geq 1$, $j = 1,\dots,d$ and let u be the entropy solution to

$$\partial_t u + \sum_{j=1}^d \partial_{x_j} u^{n_j} - \sum_{j=1}^d \partial^2_{x_j x_j} u^{[m_j]} = S(t,x) \quad \text{on } (0,T)\times\mathbb{R}^d \tag{2.26}$$
$$u(0) = u_0 \quad \text{on } \mathbb{R}^d.$$

We set $\underline{m} = \min(\{m_j : j = 1,\dots,d\})$, $\overline{m} = \max(\{m_j : j = 1,\dots,d\})$ and analogously $\underline{n}$, $\overline{n}$. Then, for all

$$s \in \big[1, \frac{2}{\overline{m}}\left(\frac{\underline{m}\wedge\underline{n} - 1}{\overline{m}-1}\right)\big), \quad p \in \big[1, \frac{2\overline{m}}{1+\overline{m}}\big),$$

all $\phi \in C_c^\infty(\mathbb{R}_v)$, $\gamma \in [0,1)$ large enough and $\mathcal{O} \subset\subset \mathbb{R}^d$ there is a constant $C \geq 0$ such that

$$\|\int f\phi\,dv\|_{L^p([0,T];W^{s,p}(\mathcal{O}))} \leq C\left(\|u_0\|_{L^1_x} + \|u_0\|_{L^{2-\gamma}_x}^{2-\gamma} + \|S\|_{L^1_{t,x}} + \|S\|_{L^{2-\gamma}_{t,x}}^{2-\gamma} + 1\right). \tag{2.27}$$

As a special case, for $m_j = n_j = m$, $j = 1,\dots,d$, we obtain (2.27) for all

$$s \in \big[0, \frac{2}{m}\big), \quad p \in \big[1, 2\frac{m}{m+1}\big).$$

Proof. We have

$$\begin{aligned}\mathcal{L}(i\tau, i\xi, v) &= i\tau + i\sum_{j=1}^d n_j v^{n_j-1}\xi_j + \sum_{j=1}^d m_j|v|^{m_j-1}|\xi_j|^2\\ &=: \mathcal{L}_{hyp}(i\tau, i\xi, v) + \mathcal{L}_{par}(\xi, v).\end{aligned}$$

Let $I \subseteq \mathbb{R}$ be a bounded set. Then, for $|\xi| \sim J$,

$$\begin{aligned}\Omega_{\mathcal{L}}(\tau,\xi;\delta) &= \{v \in I : |\mathcal{L}(i\tau, i\xi, v)| \leq \delta\}\\ &\subseteq \Omega_{\mathcal{L}_{par}}(\xi;\delta) = \{v \in I : \sum_{j=1}^d m_j|v|^{m_j-1}|\xi_j|^2 \leq \delta\}\\ &\subseteq \{v \in I : |v|^{\overline{m}-1}J^2 \lesssim \delta\}.\end{aligned} \tag{2.28}$$

Thus,

$$|\Omega_{\mathcal{L}}(\tau,\xi;\delta)| \lesssim \left(\frac{\delta}{J^2}\right)^{\frac{1}{\overline{m}-1}},$$

i.e. (2.5) is satisfied with $\beta = 2$, $\alpha = \frac{1}{\overline{m}-1}$. Moreover, due to (2.28), for $|\xi| \sim J$, $v \in \Omega_{\mathcal{L}}(\tau,\xi;\delta) \setminus \{0\}$,

$$\begin{aligned}|\partial_v \mathcal{L}(i\tau, i\xi, v)||v|^\gamma &= \Big| i \sum_{j=1}^d n_j(n_j-1)v^{n_j-2}\xi_j + \sum_{j=1}^d m_j(m_j-1)v^{[m_j-2]}|\xi_j|^2 \Big| |v|^\gamma \\ &\lesssim |v|^{\underline{n}-2+\gamma} J + |v|^{\underline{m}-2+\gamma} J^2 \\ &\lesssim \delta^{\frac{\underline{n}-2+\gamma}{\overline{m}-1}} J^{-\frac{2(\underline{n}-2+\gamma)}{\overline{m}-1}+1} + \delta^{\frac{\underline{m}-2+\gamma}{\overline{m}-1}} J^{-\frac{2(\underline{m}-2+\gamma)}{\overline{m}-1}+2}.\end{aligned}$$

Using $\delta, J \geq 1$ we get

$$|\partial_v \mathcal{L}(i\tau, i\xi, v)||v|^\gamma \lesssim \delta^{\frac{\underline{m}\vee\underline{n}-2+\gamma}{\overline{m}-1}} J^{2-2\frac{\underline{m}\wedge\underline{n}-2+\gamma}{\overline{m}-1}}, \tag{2.29}$$

i.e. (2.6) is satisfied with $\lambda = 2 - 2\frac{\underline{m}\wedge\underline{n}-2+\gamma}{\overline{m}-1}$, $\mu = \frac{\underline{m}\vee\underline{n}-2+\gamma}{\overline{m}-1}$. An application of Corollary 2.5 with γ close to one implies for all

$$s < s^* = \frac{2}{\overline{m}} \left(\frac{\underline{m}\wedge\underline{n}-1}{\overline{m}-1} \right),$$

all $p < p^* = \frac{2\overline{m}}{1+\overline{m}}$, all $\phi \in C_c^\infty(\mathbb{R}_v)$, $\gamma \in [0,1)$ large enough, $\mathcal{O} \subset\subset \mathbb{R}^d$ that there is a constant $C \geq 0$ and

$$\|\int f\phi\, dv\|_{L^p([0,T];W^{s,p}(\mathcal{O}))} \leq C(\|u_0\|_{L^1_x} + \|u_0\|^{2-\gamma}_{L^{2-\gamma}_x} + \|S\|_{L^1_{t,x}} + \|S\|^{2-\gamma}_{L^{2-\gamma}_{t,x}} + 1).$$

□

Remark 2.7. In Example 2.6 only the regularizing effect of the parabolic part is used. It may be possible that in cases $n_j << m_j$ the hyperbolic regularizing effect would dominate. Since we are mostly interested in the parabolic regularization we do not consider this point here. For related work on hyperbolic averaging we refer to [19].

3. Isotropic case

In this section we consider parabolic-hyperbolic PDE with isotropic parabolic part, that is,

$$\begin{aligned}\partial_t f(t,x,v) + a(v)\cdot\nabla_x f(t,x,v) - b(v)\Delta_x f(t,x,v) &=: \mathcal{L}(\partial_t, \nabla_x, v) f(t,x,v) \\ &= g_0(t,x,v) + \partial_v g_1(t,x,v),\end{aligned} \tag{3.1}$$

where $a : \mathbb{R} \to \mathbb{R}^d$, $b : \mathbb{R} \to \mathbb{R}_+ \cup \{0\}$ are twice continuously differentiable. The operator $\mathcal{L}$ is given by its symbol

$$\begin{aligned}\mathcal{L}(i\tau, i\xi, v) &:= \mathcal{L}_{hyp}(i\tau, i\xi, v) + \mathcal{L}_{par}(\xi, v) \\ &:= i\tau + ia(v)\cdot\xi + b(v)|\xi|^2.\end{aligned}$$

In this isotropic case we may work with a more restrictive non-degeneracy condition, which will allow to improve the order of integrability obtained in Example 2.6.

Definition 3.1 (Isotropic truncation property).

i. We say that a function $m : \mathbb{R}^d_\xi \to \mathbb{C}$ is isotropic if m is radial, that is, it depends only on $|\xi|^2$.

ii. Let $m : \mathbb{R}^d_\xi \times \mathbb{R}_v \to \mathbb{C}$ be a Caratheodory function such that $m(\cdot, v)$ is isotropic for all $v \in \mathbb{R}$. Then m is said to satisfy the isotropic truncation property if for every bump function ψ supported on a ball in $\mathbb{C}$, every bump function φ supported in $\{\xi \in \mathbb{C} : 1 \le |\xi| \le 4\}$ and every $1 < p < \infty$

$$M_{\psi,J} f(x,v) := \mathcal{F}_x^{-1} \varphi\left(\frac{|\xi|^2}{J^2}\right) \psi\left(\frac{m(\xi,v)}{\delta}\right) \mathcal{F}_x f(x)$$

is an L^p_x-multiplier for all $v \in \mathbb{R}$, $J = 2^j$, $j \in \mathbb{N}$ and, for all $r \ge 1$,

$$\Big\| \|M_{\psi,J}\|_{\mathcal{M}^p} \Big\|_{L^r_v} \lesssim |\Omega_m(J,\delta)|^{\frac{1}{r}},$$

where

$$\Omega_m(J,\delta) := \{v \in \mathbb{R} : |\frac{m(J,v)}{\delta}| \in \operatorname{supp} \psi\}.$$

Example 3.2. Consider

$$\mathcal{L}(\xi, v) = |\xi|^2 b(v),$$

for $b : \mathbb{R} \to \mathbb{R}_+ \cup \{0\}$ being measurable. Then $\mathcal{L}$ satisfies the isotropic truncation property.

Proof. Let φ, ψ be as in the definition of the isotropic truncation property. In order to prove that $M_{\psi,J}$ is an L^p-multiplier we will invoke the Hörmander–Mihlin Multiplier Theorem [21, Theorem 5.2.7]. We note that

$$\sup_{\xi \in \mathbb{R}^d} \varphi\left(\frac{|\xi|^2}{J^2}\right) \psi\left(\frac{\mathcal{L}(\xi,v)}{\delta}\right) < \infty$$

and

$$\begin{aligned}
&\partial_{\xi_i} \varphi\left(\frac{|\xi|^2}{J^2}\right) \psi\left(\frac{\mathcal{L}(\xi,v)}{\delta}\right) \\
&= \varphi'\left(\frac{|\xi|^2}{J^2}\right) \frac{|\xi|^2}{J^2} \frac{2\xi_i}{|\xi|^2} \psi\left(\frac{\mathcal{L}(\xi,v)}{\delta}\right) + \varphi\left(\frac{|\xi|^2}{J^2}\right) \psi'\left(\frac{\mathcal{L}(\xi,v)}{\delta}\right) \frac{\mathcal{L}(\xi,v)}{\delta} \frac{2\xi_i}{|\xi|^2} \\
&= \left[\varphi'\left(\frac{|\xi|^2}{J^2}\right) \frac{|\xi|^2}{J^2} \psi\left(\frac{\mathcal{L}(\xi,v)}{\delta}\right) + \varphi\left(\frac{|\xi|^2}{J^2}\right) \psi'\left(\frac{\mathcal{L}(\xi,v)}{\delta}\right) \frac{\mathcal{L}(\xi,v)}{\delta}\right] \frac{2\xi_i}{|\xi|^2} \\
&= \tilde{\varphi}\left(\frac{|\xi|^2}{J^2}\right) \tilde{\psi}\left(\frac{\mathcal{L}(\xi,v)}{\delta}\right) \frac{2\xi_i}{|\xi|^2},
\end{aligned}$$

where $\tilde{\varphi}$, $\tilde{\psi}$ are bump functions with the same support properties as φ, ψ. Hence, induction yields

$$|\partial^\alpha_\xi \varphi\left(\frac{|\xi|^2}{J^2}\right) \psi\left(\frac{\mathcal{L}(\xi,v)}{\delta}\right)| \le \tilde{\varphi^\alpha}\left(\frac{|\xi|^2}{J^2}\right) \tilde{\psi^\alpha}\left(\frac{\mathcal{L}(\xi,v)}{\delta}\right) \frac{C_\alpha}{|\xi|^{|\alpha|}},$$

for all multi-indices α with $|\alpha| \le [\frac{d}{2}] + 1$, where where $\tilde{\varphi}^\alpha$, $\tilde{\psi^\alpha}$ are bump functions with the same support properties as φ, ψ. The Hörmander–Mihlin Multiplier Theorem thus implies that

$$\varphi\left(\frac{|\xi|^2}{J^2}\right) \psi\left(\frac{\mathcal{L}(\xi,v)}{\delta}\right) \in \mathcal{M}^p$$

for all $1 < p < \infty$ with

$$\|\varphi\left(\frac{|\xi|^2}{J^2}\right)\psi\left(\frac{\mathcal{L}(\xi,v)}{\delta}\right)\|_{\mathcal{M}^p} \le C_{d,p}\sup_{\xi\in\mathbb{R}^d}\tilde{\varphi}\left(\frac{|\xi|^2}{J^2}\right)\tilde{\psi}\left(\frac{\mathcal{L}(\xi,v)}{\delta}\right),$$

where $\tilde{\varphi}$, $\tilde{\psi}$ are bump functions as above. Hence,

$$\|\varphi\left(\frac{|\xi|^2}{J^2}\right)\psi\left(\frac{\mathcal{L}(\xi,v)}{\delta}\right)\|_{\mathcal{M}^p} \le C_{d,p}\sup_{J\le|\xi|\le 2J}\tilde{\psi}\left(\frac{\mathcal{L}(\xi,v)}{\delta}\right).$$

Hence,

$$\begin{aligned}\Big\|\|\varphi\left(\frac{|\xi|^2}{J^2}\right)\psi\left(\frac{\mathcal{L}(\xi,v)}{\delta}\right)\|_{\mathcal{M}^p}\Big\|_{L^r_v} &\lesssim \left(\int\sup_{J\le|\xi|\le 2J}\tilde{\psi}\left(\frac{\mathcal{L}(\xi,v)}{\delta}\right)dv\right)^{\frac{1}{r}}\\ &\lesssim \left(\int\sup_{J\le|\xi|\le 2J}1_{\frac{|\xi|^2b(v)}{\delta}\in\,\mathrm{supp}\,\tilde{\psi}}\,dv\right)^{\frac{1}{r}} \lesssim \left(\int 1_{\frac{|J|^2b(v)}{\delta}\in\,\mathrm{supp}\,\tilde{\psi}}\,dv\right)^{\frac{1}{r}}\\ &\lesssim \left(|\{v\in\mathbb{R}:\ \frac{|J|^2b(v)}{\delta}\in\ \mathrm{supp}\,\tilde{\psi}|\}\right)^{\frac{1}{r}} = |\Omega_{\mathcal{L}}(J,\delta)|^{\frac{1}{r}}.\end{aligned}$$

□

3.1. **Averaging Lemma.** Working with the isotropic truncation property allows to prove a similar statement to Lemma 2.1, but without the restriction to $p \le 2$. This leads to an improved estimate on the integrability of the solution.

Lemma 3.3. *Let $f \in L^{r'}_v(L^p_{t,x})$ for $1 < p < \infty$, $r' \in [1,\infty]$ solve, in the sense of distributions,*

$$\mathcal{L}(\partial_t,\nabla_x,v)f(t,x,v) = \Delta_x^{\frac{\eta}{2}}g_0(t,x,v) + \partial_v\Delta_x^{\frac{\eta}{2}}g_1(t,x,v)\ \text{ on } \mathbb{R}_t\times\mathbb{R}^d_x\times\mathbb{R}_v \tag{3.2}$$

with g_i being Radon measures satisfying

$$|g_0|(t,x,v) + |g_1|(t,x,v)|v|^{-\gamma} \in \begin{cases} L^q(\mathbb{R}_t\times\mathbb{R}^d_x\times\mathbb{R}_v), & 1<q\le 2\\ \mathcal{M}_{TV}(\mathbb{R}_t\times\mathbb{R}^d_x\times\mathbb{R}_v), & q=1,\end{cases} \tag{3.3}$$

for some $\gamma\ge 0$, $\eta\ge 0$, $1\le q\le p$ and $\mathcal{L}(\partial_t,\nabla_x,v)$ as in (3.1) with corresponding symbol $\mathcal{L}(i\tau,i\xi,v) = \mathcal{L}_{hyp}(i\tau,i\xi,v)+\mathcal{L}_{par}(\xi,v)$. Assume that $\mathcal{L}(i\tau,i\xi,v)$ satisfies the truncation property uniformly in $v\in\mathbb{R}$. Let $I\subseteq\mathbb{R}$ be a not necessarily bounded interval, set

$$\omega_{\mathcal{L}}(J;\delta) := \sup_{\tau\in\mathbb{R},\,\xi\in\mathbb{R}^d,|\xi|\sim J}|\Omega_{\mathcal{L}}(\tau,\xi;\delta)|,\quad \Omega_{\mathcal{L}}(\tau,\xi;\delta) = \{v\in I:\ |\mathcal{L}(i\tau,i\xi,v)|\le\delta\},$$

and suppose that the following non-degeneracy condition holds: There exist $\alpha,\beta>0$ such that

$$\omega_{\mathcal{L}}(J;\delta)\lesssim(\frac{\delta}{J^\beta})^\alpha\quad \forall\delta\ge 1,\ J\ge 1. \tag{3.4}$$

Moreover, assume that there exist $\lambda\ge 0$ and $\mu\in[0,1]$ such that, for all $\delta\ge 1$, $J\ge 1$,

$$\sup_{\tau,|\xi|\sim J}\ \sup_{v\in\Omega_{\mathcal{L}}(\tau,\xi;\delta)}|\partial_v\mathcal{L}(i\tau,i\xi,v)||v|^\gamma \lesssim J^\lambda\delta^\mu \tag{3.5}$$

and $\frac{\alpha\beta}{q'} \le \lambda + \eta$. *Assume that* $\mathcal{L}_{par}$ *satisfies the isotropic truncation property with*

$$|\Omega_{\mathcal{L}_{par}}(J,\delta)| \lesssim (\frac{\delta}{J^\beta})^\alpha \quad \forall \delta \ge 1,\ J \ge 1. \tag{3.6}$$

Then, for all $\phi \in C_b^\infty(I)$, $s \in [0,s^*)$, $\tilde{p} \in [1,p^*)$, $T \ge 0$, $\mathcal{O} \subset\subset \mathbb{R}^d$, *there is a constant* $C \ge 0$ *such that*

$$\begin{aligned} \|\int f(t,x,v)\phi(v)\,dv\|_{L^{\tilde{p}}([0,T];W^{s,\tilde{p}}(\mathcal{O}))} \le C\big(&\|g_0\phi\|_{L^q_{t,x,v}} + \||v|^{-\gamma}g_1\phi\|_{L^q_{t,x,v}} \\ &+ \|g_1\phi'\|_{L^q_{t,x,v}} + \|f\phi\|_{L^{r'}_v(L^p_{t,x})} + \|f\phi\|_{L^q_{t,x}L^1_v} + \|f\phi\|_{L^{\tilde{p}}_{t,x}L^1_v}\big) \end{aligned} \tag{3.7}$$

with $s^* := (1-\theta)\frac{\alpha\beta}{r} + \theta(\frac{\alpha\beta}{q'} - \lambda - \eta)$, *where* $\theta = \theta_\alpha$ *and* p^* *are given by*

$$\theta := \frac{\frac{\alpha}{r}}{\alpha(\frac{1}{r} - \frac{1}{q'}) + 1} \in (0,1), \quad \frac{1}{p^*} := \frac{1-\theta}{p} + \frac{\theta}{q},\ \frac{1}{r} + \frac{1}{r'} = 1.$$

Proof. The proof proceeds analogously to the one of Lemma 2.1. The only change appears in the estimation of f^0. We may assume that ψ_0 is of the form $\psi_0(ia+b) = \psi_0^1(a)\psi_0^2(b)$ with ψ_0^i being locally supported bump functions. Hence,

$$\psi_0\left(\frac{\mathcal{L}(i\tau,i\xi,v)}{\delta}\right) = \psi_0^1\left(\frac{\mathcal{L}_{hyp}(i\tau,i\xi,v)}{\delta}\right)\psi_0^2\left(\frac{\mathcal{L}_{par}(\xi,v)}{\delta}\right)$$

and

$$\|\varphi_1(\frac{\xi}{2^j})\psi_0\left(\frac{\mathcal{L}(i\tau,i\xi,v)}{\delta}\right)\|_{\mathcal{M}^p} \lesssim \|\varphi_1(\frac{\xi}{2^j})\psi_0^2\left(\frac{\mathcal{L}_{par}(\xi,v)}{\delta}\right)\|_{\mathcal{M}^p}.$$

The isotropic truncation property and (3.6) then imply

$$\Big\|\|\varphi_1(\frac{\xi}{2^j})\psi_0\left(\frac{\mathcal{L}(i\tau,i\xi,v)}{\delta}\right)\|_{\mathcal{M}^p}\Big\|_{L^r_v} \lesssim |\Omega_{\mathcal{L}_{par}}(2^j,\delta)|^{\frac{1}{r}} \lesssim (\frac{\delta}{2^{j\beta}})^{\frac{\alpha}{r}}.$$

Hence,

$$\begin{aligned} \|\int f_j^0\phi\,dv\|_{L^p_{t,x}} &= \|\int \mathcal{F}^{-1}_{t,x}\varphi_1(\frac{\xi}{2^j})\psi_0\left(\frac{\mathcal{L}(i\tau,i\xi,v)}{\delta}\right)\mathcal{F}_{t,x}f^0\phi\,dv\|_{L^p_{t,x}} \\ &\le \int \|\mathcal{F}^{-1}_{t,x}\varphi_1(\frac{\xi}{2^j})\psi_0\left(\frac{\mathcal{L}(i\tau,i\xi,v)}{\delta}\right)\mathcal{F}_{t,x}f^0\phi\|_{L^p_{t,x}}\,dv \\ &\lesssim \int \|\mathcal{F}^{-1}_{t,x}\varphi_1(\frac{\xi}{2^j})\psi_0\left(\frac{\mathcal{L}(i\tau,i\xi,v)}{\delta}\right)\mathcal{F}_{t,x}\|_{\mathcal{M}^p}\|f^0\phi\|_{L^p_{t,x}}\,dv \\ &\le \Big\|\|\varphi_1(\frac{\xi}{2^j})\psi_0\left(\frac{\mathcal{L}(i\tau,i\xi,v)}{\delta}\right)\|_{\mathcal{M}^p}\Big\|_{L^r_v}\|f^0\phi\|_{L^{r'}_vL^p_{t,x}} \\ &\lesssim (\frac{\delta}{2^{j\beta}})^{\frac{\alpha}{r}}\|f^0\phi\|_{L^{r'}_vL^p_{t,x}}. \end{aligned}$$

The proof then proceeds as before, the only difference being that we do not have to restrict to $1 < p \le 2$ and the modified definition of r, r'. □

3.2. **Porous media equations.** In this section we consider porous media equations with a source of the type

$$\begin{aligned}\partial_t u - \Delta u^{[m]} &= S(t,x) \text{ on } (0,T)\times\mathbb{R}^d_x, \\ u(0) &= u_0,\end{aligned} \tag{3.8}$$

where $u_0 \in L^1(\mathbb{R}^d_x)$, $S \in L^1([0,T]\times\mathbb{R}^d_x)$, $T \geq 0$ and $m > 1$.

In [11] the kinetic form of (3.8) with $S \equiv 0$ was introduced. Analogously, the kinetic form to (3.8) reads, with $f = \chi(u(t,x),v)$, $q \in \mathcal{M}^+$,

$$\partial_t f + m|v|^{m-1}\Delta f = \partial_v q + S(t,x)\delta_{u(t,x)}(v) \text{ on } (0,T)\times\mathbb{R}^d_x\times\mathbb{R}_v. \tag{3.9}$$

For the notion and well-posedness of entropy solutions to (3.8) see Appendix B. As before let $\mathcal{L}(\partial_t,\nabla_x,v)f = \partial_t f + m|v|^{m-1}\Delta f$ with symbol

$$\begin{aligned}\mathcal{L}(i\tau,\xi,v) :=& \mathcal{L}_{hyp}(i\tau) + \mathcal{L}_{par}(\xi,v) \\ :=& i\tau + m|v|^{m-1}|\xi|^2.\end{aligned}$$

Example 3.4. Let $u_0 \in L^1(\mathbb{R}^d_x)$, $S \in L^1([0,T]\times\mathbb{R}^d_x)$ and let u be the unique entropy solution to (3.8). Then, for all

$$s \in [0,\frac{2}{m}), \quad p \in [1,m),$$

all $\gamma \in (0,1)$ large enough, $\mathcal{O} \subset\subset \mathbb{R}^d$, there is a constant $C \geq 0$ such that

$$\|u\|_{L^p([0,T];W^{s,p}(\mathcal{O}))} \leq C\left(\|u_0\|^2_{(L^1\cap L^{2-\gamma}\cap L^p)(\mathbb{R}^d_x)} + \|S\|^2_{(L^1\cap L^{2-\gamma}\cap L^p)([0,T]\times\mathbb{R}^d_x)} + 1\right).$$

Proof. Let $s \in [0,\frac{2}{m})$, $p \in [1,m)$. We have $f \in L^1_{t,x,v}\cap L^\infty_{t,x,v}$ and thus $f \in L^{\tilde p}_v(L^{\tilde p}_{t,x})$ for all $\tilde p \geq 1$ with

$$\|f\|^{\tilde p}_{L^{\tilde p}_v(L^{\tilde p}_{t,x})} \leq \|f\|_{L^1_v(L^1_{t,x})}. \tag{3.10}$$

This bound will replace the property $f \in L^2_{t,x;loc}(H^{\sigma,2}_v)$ used in the proof of Corollary 2.5, which is possible due to Lemma 3.3. As a consequence, the localization of f performed in Corollary 2.5 is not required here. In order to apply (3.3) we need to extend (3.9) to all time $t \in \mathbb{R}$, which can be done by multiplication with a smooth cut-off function $\varphi \in C^\infty_c(0,T)$. Let $\eta = 0$, $\alpha = \frac{1}{m-1}$, $\beta = 2$ and choose $\gamma \in [0,1)$ large enough and $r \geq 1$ small enough such that $\lambda = 2 - 2\frac{m-2+\gamma}{m-1} = 2(\frac{1-\gamma}{m-1})$ is such that

$$\begin{aligned}(1-\theta)\beta\frac{\alpha}{r} - \theta(\lambda+\eta) &= \theta(\frac{\beta}{r} - \lambda) \\ &= \frac{2}{m}\left(\frac{1}{r} - (\frac{1-\gamma}{m-1})\right) \\ &> s,\end{aligned}$$

where $\theta = \frac{1}{m}$. Next, choose $\tilde p$ large enough, such that $p^* = m\left(\frac{\tilde p}{m-1+\tilde p}\right) > p$ and note $\frac{1-\theta}{\tilde p} + \theta = \frac{1}{p^*}$. We can choose $\tilde p$, r such that $\tilde p = r'$. Let $g_0 = \delta_{v=u(t,x)}S + f\partial_t\varphi$, $g_1 = q$. In order to treat the possible singularity of $\partial_v\mathcal{L}$ at $v = 0$ we proceed as in Corollary 2.5, i.e. first cutting out the singularity, then controlling the respective error uniformly by

Lemma 2.4. Note that $\mathcal{L}$ satisfies (3.4), (3.5) on $\mathbb{R}\setminus\{0\}$ for all $\gamma\in[0,1)$ and $\mathcal{L}_{par}$ satisfies the isotropic truncation property with (3.6). With these choices, Lemma 3.3 with $p=\tilde p$, $q=1$ and $\phi\equiv 1$ yields

$$\begin{aligned}\|u\|_{L^p([0,T];W^{s,p}(\mathcal{O}))} &\lesssim \|\delta_{v=u(t,x)}S\|_{\mathcal{M}_{TV}} + \|f_0\|_{L^1_xL^1_v} + \||v|^{-\gamma}q\|_{\mathcal{M}_{TV}}\\ &\quad + \|f\|_{L^{\tilde p}_v(L^{\tilde p}_{t,x})} + \|f\|_{L^1_{t,x}L^1_v} + \|f\|_{L^p_{t,x}L^1_v}\\ &\lesssim \|S\|_{L^1_{t,x}} + \|u_0\|_{L^1_x} + \||v|^{-\gamma}q\|_{\mathcal{M}_{TV}} + \|f\|_{L^1_{t,x}L^1_v} + \|f\|_{L^p_{t,x}L^1_v} + 1.\end{aligned}$$

The fact that, for all $\eta\in[1,\infty)$,

$$\|f\|_{L^\eta_{t,x}L^1_v} = \|u\|_{L^\eta_{t,x}} \lesssim \|u_0\|_{L^\eta_x} + \|S\|_{L^\eta_{t,x}}$$

and Lemma 2.3 thus imply

$$\|u\|_{L^p([0,T];W^{s,p}(\mathcal{O}))} \lesssim \|u_0\|^2_{L^1_x\cap L^{2-\gamma}_x\cap L^p_x} + \|S\|^2_{L^1_{t,x}\cap L^{2-\gamma}_{t,x}\cap L^p_{t,x}} + 1.$$

Since $p^*>p$ this yields the claim. $\square$

Remark 3.5. We note that for $u_0\in L^1_x$ or $S\in L^1_{t,x}$ the kinetic measure q does not necessarily have finite mass (cf. e.g. [26]). Therefore, in the literature the cut-off $\phi\in C^\infty_c(\mathbb{R})$ in (3.7) is required to be compactly supported, which prevents to deduce regularity estimates for u itself, unless u is bounded. Our arguments allow to avoid this restriction since we work with the singular moments $|v|^{-\gamma}q$ only, which are shown to be finite in Lemma 2.3, provided $u\in L^{2-\gamma}_x$, $S\in L^{2-\gamma}_{t,x}$.

Remark 3.6. As it has been pointed out in the introduction, the results obtained in [16] are restricted to fractional differentiability of an order less than one. This restriction is inherent to the method used in [16]. More precisely, the estimates obtained in [16] are (informally) based on testing (3.8) with $\int_0^t \Delta u^{[m]}\,dr$, integrating in space and time and using Hölder's inequality, which leads to the energy inequality (neglecting constants)

$$\int_0^T\int(\nabla u^{[\frac{m+1}{2}]})^2dxdr \le \int u^2(0)dx. \tag{3.11}$$

The regularity estimates are then deduced from (3.11) alone. In [16] these formal computations are made rigorous, a careful treatment of boundary conditions is given and the bound on $\int_0^T\int(\nabla u^{[\frac{m+1}{2}]})^2dxdr$ is used to prove (1.2). Since (3.11) only involves derivatives of first order, it does not seem possible to deduce higher than first order differentiability from this.

4. Degenerate parabolic Anderson model

We consider the degenerate parabolic Anderson model

$$\begin{aligned}\partial_t u &= \partial_{xx}u^{[m]} + u\,S \text{ on } (0,T)\times I,\\ u^\varepsilon &= 0 \text{ on } \partial I,\end{aligned} \tag{4.1}$$

with $m\in(1,2)$, $I\subseteq\mathbb{R}$ a bounded, open interval and S being a distribution only. As for the parabolic Anderson model (cf [17,18]), the particular example we have in mind is

$S=\xi$ being spatial white noise. Accordingly, we assume that, locally on $\mathbb{R}$,

$$S\in B^{-\frac{1}{2}-\varepsilon}_{\infty,\infty} \text{ for all } \varepsilon>0. \tag{4.2}$$

The choice of zero Dirichlet boundary data in (4.1) is for simplicity only and the arguments of this section can easily be adapted to the Cauchy problem. We will prove the following regularity estimate for a weak (i.e. distributional) solution to (4.1).

Proposition 4.1. *Let $u_0\in L^{m+1}(I)$. Then there exists a weak solution u to (4.1) satisfying, for all $p\in[1,m)$, $s\in[0,\frac{3}{2}\frac{1}{m})$,*

$$u\in L^p([0,T];W^{s,p}_{loc}(I)),$$

with, for all $T\geq 0$, $\mathcal{O}\subset\subset I$,

$$\|u\|_{L^p([0,T];W^{s,p}(\mathcal{O}))}\lesssim \|u_0\|^{m+1}_{L^{m+1}(I)}+\|S\|^{\tau}_{B^{-\eta}_{\infty,\infty}}+1,$$

for some $\tau\geq 2$ and $\eta\in(\frac{1}{2},1]$ small enough.

The proof of the above Proposition is a consequence of establishing according uniform regularity estimates (see Theorem 4.5 below) for the approximating problem

$$\begin{aligned}\partial_t u^\varepsilon &= \partial_{xx}(u^\varepsilon)^{[m]}+u^\varepsilon S^\varepsilon(x) \text{ on } (0,T)\times I,\\ u^\varepsilon &= 0 \text{ on } \partial I,\end{aligned} \tag{4.3}$$

where $S^\varepsilon\in C^\infty(\mathbb{R})$ with $\|S^\varepsilon\|_{B^{-\frac{1}{2}-\varepsilon}_{\infty,\infty}}\leq\|S\|_{B^{-\frac{1}{2}-\varepsilon}_{\infty,\infty}}$ and $S^\varepsilon\to S$ locally in $B^{-\frac{1}{2}-\varepsilon}_{\infty,\infty}$ for all $\varepsilon>0$. These estimates will be derived from the kinetic formulation of (4.3). Informally, with $\chi^\varepsilon:=\chi(u^\varepsilon)$ the kinetic form reads, in the sense of distributions,

$$\begin{aligned}\partial_t\chi^\varepsilon &= m|v|^{m-1}\partial_{xx}\chi^\varepsilon+\delta_{u^\varepsilon(t,x)=v}u^\varepsilon S^\varepsilon+\partial_v q^\varepsilon\\ &= m|v|^{m-1}\partial_{xx}\chi^\varepsilon+\chi^\varepsilon S^\varepsilon+\partial_v q^\varepsilon-\partial_v(\chi^\varepsilon v S^\varepsilon) \text{ on } (0,T)\times I\times\mathbb{R}.\end{aligned} \tag{4.4}$$

Definition 4.2. We say that $u^\varepsilon\in L^1([0,T]\times I)$ is an entropy solution to (4.3) if

(i) for every $\alpha\in(0,m]$ there is a constant $K_1\geq 0$ such that

$$\|\partial_x(u^\varepsilon)^{[\frac{m+\alpha}{2}]}\|_{L^2([0,T]\times I)}\leq K_1. \tag{4.5}$$

(ii) $\chi^\varepsilon=\chi(u^\varepsilon)$ satisfies (4.4), in the sense of distributions on $(0,T)\times I\times\mathbb{R}$, for some non-negative, finite measure q^ε such that,

$$q^\varepsilon=m^\varepsilon+n^\varepsilon$$

with m^ε being a non-negative measure and n^ε given by

$$n^\varepsilon=\delta_{v=u^\varepsilon}(\partial_x(u^\varepsilon)^{[\frac{m+1}{2}]})^2$$

and satisfying, for every $\alpha\in(0,m]$ with K_1 as in (i),

$$\int_{[0,T]\times\mathbb{R}^d\times\mathbb{R}}|v|^{\alpha-1}q^\varepsilon\,dtdxdv\leq K_1. \tag{4.6}$$

Following the arguments of [11] it is not difficult to see that there is a unique entropy solution u^ε to (4.3), see also Appendix B. The additional complication in the proof of well-posedness due to the forcing has been resolved in [19] in the case of scalar conservation laws. The same arguments may be applied here. Comparing to [11, Definition 2.2] it only remains to show that the constant K_1 in (4.5) and (4.6) can be chosen uniformly in ε.

Lemma 4.3. *Let $\alpha > 0$ and $\tau = \frac{2\alpha+2}{2\alpha+3-m} \in (1,2]$. Then, for some constant $C = C(\alpha, m, T)$,*

$$\sup_{t\in[0,T]} \int_I |u^\varepsilon(t)|^{\alpha+1}dx + \int_0^T \int_I (\partial_x (u^\varepsilon)^{[\frac{m+\alpha}{2}]})^2 dxdr \le C \int_I |u_0|^{\alpha+1}dx + C\|S\|^{\tau'}_{W^{-1,\tau'}}.$$

and

$$\int_{[0,T]\times \bar I\times\mathbb{R}} |v|^{\alpha-1} q^\varepsilon \, drdxdv \le C \int_I |u_0|^{\alpha+1}dx + C\|S\|^{\tau'}_{W^{-1,\tau'}}. \tag{4.7}$$

Proof. In the following we present an informal derivation of the claimed energy estimates. These arguments can be made rigorous by considering a vanishing viscosity approximation

$$\partial_t u^{\varepsilon,\delta} = \delta\partial_{xx} u^{\varepsilon,\delta} + \partial_{xx}(u^{\varepsilon,\delta})^{[m]} + u^{\varepsilon,\delta} S^\varepsilon(x) \text{ on } (0,T)\times I.$$

For simplicity we drop the ε in the notation. Testing (4.3) with $u^{[\alpha]}$ yields

$$\begin{aligned}\partial_t \int_I |u|^{\alpha+1}dx &= (\alpha+1)\int_I u^{[\alpha]}(\partial_{xx}u^{[m]} + uS)dx \\ &= -\frac{4(\alpha+1)\alpha m}{(m+\alpha)^2}\int_I (\partial_x u^{[\frac{m+\alpha}{2}]})^2 dx + \int_I |u|^{\alpha+1} S dx\end{aligned}$$

We further have, for $\tau \in [1,2)$ to be chosen later,

$$\int_I |u|^{\alpha+1} S dx \lesssim \| |u|^{\alpha+1}\|^\tau_{W^{1,\tau}} + \|S\|^{\tau'}_{W^{-1,\tau'}} \tag{4.8}$$

and, for every $\eta > 0$ and some $C_\eta \ge 0$,

$$\begin{aligned}\| |u|^{\alpha+1}\|^\tau_{W^{1,\tau}} &\lesssim \int_I |\partial_x |u|^{\alpha+1}|^\tau \, dx = (\alpha+1)^\tau \int_I |u^{[\alpha]}\partial_x u|^\tau \, dx \\ &= (\alpha+1)^\tau \int_I \left| u^{[\alpha - \frac{m+\alpha-2}{2}]} |u|^{\frac{m+\alpha-2}{2}} \partial_x u\right|^\tau dx \\ &= \frac{4(\alpha+1)^\tau}{(m+\alpha)^2}\int_I |u^{\frac{\alpha-m+2}{2}}|^\tau |\partial_x u^{[\frac{m+\alpha}{2}]}|^\tau \, dx \\ &\le \frac{4(\alpha+1)^\tau}{(m+\alpha)^2}(\int_I C_\eta |u^{\frac{\alpha-m+2}{2}}|^{\frac{2\tau}{2-\tau}} + \eta |\partial_x u^{\frac{m+\alpha}{2}}|^2) \, dx.\end{aligned} \tag{4.9}$$

Thus, since $\tau < 2$ and choosing η small enough,

$$\begin{aligned}\partial_t \int_I |u|^{\alpha+1}dx \lesssim &-\frac{2(\alpha+1)\alpha m}{(m+\alpha)^2}\int_I (\partial_x u^{[\frac{m+\alpha}{2}]})^2 dx \\ &+ C\frac{4(\alpha+1)^\tau}{(m+\alpha)^2}\int_I |u|^{(\frac{\alpha-m+2}{2})(\frac{2\tau}{2-\tau})}dx + \|S\|^{\tau'}_{W^{-1,\tau'}}.\end{aligned}$$

Now we choose τ such that $(\frac{\alpha-m+2}{2})(\frac{2\tau}{2-\tau})=\alpha+1$, i.e. since $m-2<\alpha$,

$$\tau = \frac{2\alpha+2}{2\alpha+3-m} \in (1,2].$$

In conclusion,

$$\begin{aligned}\partial_t \int_I |u|^{\alpha+1}dx \lesssim & -\frac{2(\alpha+1)\alpha m}{(m+\alpha)^2}\int_I (\partial_x u^{[\frac{m+\alpha}{2}]})^2 dx \\ & + C\frac{4(\alpha+1)^\tau}{(m+\alpha)^2}\int_I |u|^{\alpha+1}dx + \|S\|^{\tau'}_{W^{-1,\tau'}}.\end{aligned}$$

Gronwall's inequality implies

$$\int_I |u(t)|^{\alpha+1}dx + \int_0^t \int_I (\partial_x u^{[\frac{m+\alpha}{2}]})^2 dxdr \lesssim \int_I |u_0|^{\alpha+1}dx + \|S\|^{\tau'}_{W^{-1,\tau'}}.$$

In order to establish (4.7) we note that on the approximative level $u^{\varepsilon,\delta}$ the kinetic form is satisfied with $q^{\varepsilon,\delta}=\delta_{v=u^{\varepsilon,\delta}}(\partial_x (u^{\varepsilon,\delta})^{[\frac{m+1}{2}]})^2$. Thus,

$$\begin{aligned}\int_{[0,T]\times\bar I\times\mathbb{R}} |v|^{\alpha-1}q^{\varepsilon,\delta}drdxdv &= \int_I (\partial_x (u^{\varepsilon,\delta})^{[\frac{m+\alpha}{2}]})^2 dtdx \\ &\lesssim \int_I |u_0|^{\alpha+1}dx + \|S^\varepsilon\|^{\tau'}_{W^{-1,\tau'}}.\end{aligned}$$

Passing to the limit $\delta\to 0$ yields (4.7). □

Corollary 4.4. *Let $u_0\in L^{m+1}(I)$. Then, there is a unique entropy solution u^ε to (4.3) and u^ε satisfies Definition 4.2 with*

$$K_1 \lesssim \|u_0\|^{m+1}_{L^{m+1}} + \|S\|^\tau_{B^{-\eta}_{\infty,\infty}} + 1$$

for some $\tau\ge 2$ and some $\eta\in(\frac{1}{2},1)$. In particular, the constants K_1 in Definition 4.2 can be chosen uniformly in ε and

$$\|u^\varepsilon\|^2_{L^2([0,T];H^1_0(I))} \le K_1.$$

Proof. We apply Lemma 4.3 with $\alpha\in(0,m]$. □

Theorem 4.5. *Assume (4.2) and let u^ε be the entropy solution to (4.3). Then, for all $p\in[1,m)$, $s\in[0,\frac{3}{2}\frac{1}{m})$ we have*

$$u^\varepsilon \in L^p([0,T];W^{s,p}_{loc}(I))$$

with, for all $T\ge 0$, $\mathcal{O}\subset\subset I$,

$$\|u^\varepsilon\|_{L^p([0,T];W^{s,p}(\mathcal{O}))} \le C(\|u_0\|^{m+1}_{L^{m+1}(I)} + \|S\|^\tau_{B^{-\eta}_{\infty,\infty}} + 1),$$

for some $\tau\ge 2$, C independent of $\varepsilon>0$ and $\eta\in(\frac{1}{2},1)$ small enough.

Proof. Let $p \in [1,m)$, $s \in [0, \frac{3}{2}\frac{1}{m})$. For simplicity we drop the ε in the notation. Rewriting (4.4) we obtain, for $\eta \in (\frac{1}{2}, 1)$,

$$\begin{aligned}\partial_t \chi &= m|v|^{m-1}\partial_{xx}\chi + \Delta_x^{\frac{\eta}{2}} \underbrace{\Delta_x^{-\frac{\eta}{2}}\chi S}_{:=g_0} + \Delta_x^{\frac{\eta}{2}}\partial_v \underbrace{\Delta_x^{-\frac{\eta}{2}} q}_{=:g_1} - \Delta_x^{\frac{\eta}{2}}\partial_v \underbrace{\Delta_x^{-\frac{\eta}{2}}\chi v S}_{=:g_2} \\ &= m|v|^{m-1}\partial_{xx}\chi + \Delta_x^{\frac{\eta}{2}} g_0 + \Delta_x^{\frac{\eta}{2}}\partial_v g_1 - \Delta_x^{\frac{\eta}{2}}\partial_v g_2 \text{ on } (0,T)\times I \times \mathbb{R}.\end{aligned} \tag{4.10}$$

An elementary computation shows $\|\chi\|_{L^1_{t,v}W^{\eta,1}_x} \lesssim \|u\|_{L^1_t W^{\eta,1}_x}$. We next use embedding results for Besov spaces [3, Proposition 2.78], estimates for the paraproduct of functions and distributions [28, Section 4.4.3, Theorem 1] and Corollary 4.4 to obtain, for $\delta > 0$ small enough,

$$\begin{aligned}\|g_0\|_{L^1_{t,x,v}} &= \|\Delta_x^{-\frac{\eta}{2}}\chi S\|_{L^1_{t,x,v}} \lesssim \|\chi S\|_{L^1_{t,v}B^{-\eta}_{1,1}} \lesssim \|\chi\|_{L^1_{t,v}B^{\eta+\delta}_{1,1}}\|S\|_{B^{-\eta}_{\infty,\infty}} \\ &\lesssim \|u\|_{L^1_t(W^{\eta+2\delta,1}_x)}\|S\|_{B^{-\eta}_{\infty,\infty}} \lesssim \|u\|^2_{L^2_t(H^1_0)} + \|S\|^2_{B^{-\eta}_{\infty,\infty}} \le K_1 + \|S\|^2_{B^{-\eta}_{\infty,\infty}}.\end{aligned} \tag{4.11}$$

Moreover, using the same reasoning we obtain

$$\||v|^{-1}g_2\|_{L^1_{t,x,v}} = \||v|^{-1}\Delta_x^{-\frac{\eta}{2}}\chi v S\|_{L^1_{t,x,v}} = \|\Delta_x^{-\frac{\eta}{2}}|\chi| S\|_{L^1_{t,x,v}} \lesssim K_1 + \|S\|^2_{B^{-\eta}_{\infty,\infty}}. \tag{4.12}$$

We choose a cut-off function and localize (4.10) as in the proof of Corollary 2.5. Hence, using (3.10), we may apply Lemma 3.3, with η sufficiently close to $\frac{1}{2}$, $\alpha = \frac{1}{m-1}$, $\beta = 2$, $\lambda = 2 - 2\frac{m-2+\gamma}{m-1}$ small enough by choosing γ close to one, $r > 1$ small enough, $p = r'$, $q = 1$, $\theta = \frac{1}{m}$, such that

$$\begin{aligned}(1-\theta)\beta\frac{\alpha}{r} - \theta(\lambda+\eta) &= \theta(\frac{\beta}{r} - \lambda - \eta) \\ &= \frac{1}{m}\left(\frac{3}{2} + (\frac{2}{r} - 2) + (2\frac{\gamma-1}{m-1}) + (\frac{1}{2} - \eta)\right) > s.\end{aligned}$$

This yields, for all $\mathcal{O} \subset\subset I$,

$$\begin{aligned}\|u\|_{L^p([0,T];W^{s,p}(\mathcal{O}))} &\lesssim \|\Delta_x^{-\frac{\eta}{2}}\chi S\|_{\mathcal{M}_{t,x,v}} + \|\Delta_x^{-\frac{\eta}{2}}|v|^{-\gamma}q\|_{\mathcal{M}_{t,x,v}} + \||v|^{-1}\Delta_x^{-\frac{\eta}{2}}\chi v S\|_{\mathcal{M}_{t,x,v}} \\ &\quad + \|f\|_{L^{r'}_{t,x,v}} + \|f\|_{L^1_{t,x,v}} + \|f\|_{L^p_{t,x}L^1_v} + 1.\end{aligned}$$

Hence, since

$$\|f\|_{L^{r'}_{t,x,v}} \lesssim \|f\|_{L^1_{t,x,v}} + 1, \quad \|f\|_{L^1_{t,x,v}} = \|u\|_{L^1_{t,x}}, \quad \|f\|_{L^p_{t,x}L^1_v} = \|u\|_{L^p_{t,x}}$$

we have, using (4.11), (4.12),

$$\|u\|_{L^p([0,T];W^{s,p}(\mathcal{O}))} \lesssim K_1 + \|S\|^2_{B^{-\eta}_{\infty,\infty}} + \|u\|_{L^1_{t,x}} + \|u\|_{L^p_{t,x}} + 1.$$

In fact, (4.10) is not exactly of the form (3.1), since g_1, g_2 allow singular moments of different order, i.e. $\gamma \in (0,1)$ for g_1, $\gamma = 1$ for g_2. However, in the proof of Lemma 3.3, the terms involving g_2 only lead to better behaved terms than g_1 and thus may be absorbed. We next note that by the arguments of Lemma 4.3

$$\|u\|_{L^1_{t,x}} \lesssim \|u_0\|_{L^1_x} + \|S\|^{\tau}_{W^{-1,\tau}} + 1, \quad \|u\|_{L^p_{t,x}} \lesssim \|u_0\|_{L^{m+1}_x} + \|S\|^{\tau}_{W^{-1,\tau}} + 1$$

for some $\tau \geq 2$. Hence, by Corollary 4.4 we obtain

$$\begin{aligned}\|u\|_{L^p([0,T];W^{s,p}(\mathcal{O}))} \lesssim& \|u_0\|_{L^{m+1}}^{m+1} + \|S\|_{B^{-\eta}_{\infty,\infty}}^{\tau} + \|u_0\|_{L^1_x} + \|u_0\|_{L_x^{m+1}} + \|S\|_{W^{-1,\tau}}^{\tau} + 1\\ \lesssim& \|u_0\|_{L^{m+1}}^{m+1} + \|S\|_{B^{-\eta}_{\infty,\infty}}^{\tau} + 1,\end{aligned}$$

for some $\tau \geq 2$. □

Proof of Proposition 4.1. By Lemma 4.3 we have

$$\|u^\varepsilon\|_{L^2([0,T];H_0^1)}^2 + \|\partial_x (u^\varepsilon)^{[m]}\|_{L^2([0,T];L^2)}^2 \leq C.$$

Hence, we also have $\|u^\varepsilon S^\varepsilon\|_{W^{-1,2}}^2 \lesssim \|u^\varepsilon\|_{W^{1,2}}^2 \|S^\varepsilon\|_{W^{-1,2}}^2 \leq C$. By (4.3) we obtain

$$\|\partial_t u^\varepsilon\|_{L^2([0,T];W^{-1,2})}^2 \leq C.$$

The Aubin-Lions compactness Lemma yields (for a subsequence)

$$u^\varepsilon \to u \quad \text{in } L^2([0,T]; L^2(I)).$$

This allows to pass to the limit in the weak form of (4.3). Hence, Theorem 4.5 finishes the proof. □

Appendix A. Truncation property and basic estimates

From [31, Definition 2.1] we recall the following definition.

Definition A.1. Let m be a complex-valued Fourier multiplier. We say that m has the truncation property if, for any locally supported bump function ψ on $\mathbb{C}$ and any $1 \leq p < \infty$, the multiplier with symbol $\psi(\frac{m(\xi)}{\delta})$ is an L^p-multiplier as well as an $\mathcal{M}_{TV}$-multiplier uniformly in $\delta > 0$, that is, its L^p-multiplier norm ($\mathcal{M}_{TV}$-multiplier norm resp.) depends only on the support and C^l size of ψ (for some large l that may depend on m) but otherwise is independent of δ.

We slightly deviate from the definition of the truncation property given in [31, Definition 2.1] since we require it to hold also for $p = 1$ and on $\mathcal{M}_{TV}$. In [31, Section 2.4] it was shown that multipliers corresponding to parabolic-hyperbolic PDE satisfy the truncation property for $p > 1$. Accordingly we extend this property to our Definition in the following example.

Example A.2. Let

$$m(\tau, \xi, v) = i\tau + ia(v) \cdot \xi + (\xi, b(v)\xi)$$

for some measurable $a : \mathbb{R} \to \mathbb{R}^d$, $b : \mathbb{R} \to \mathcal{S}_+^{d\times d}$. Then, m satisfies the truncation property uniformly in v.

Proof. Following [31, Section 2.4] it remains to consider the cases $p = 1$ and $\mathcal{M}_{TV}$. Arguing as in [31, Section 2.4] we can consider the cases $m(\tau, \xi, v) = i\tau + ia(v) \cdot \xi$ and $m(\tau, \xi, v) = (\xi, b(v)\xi)$ separately. By invariance under linear transformations, arguing again as in [31, Section 2.4] it is enough to consider $\psi(i\xi_1)$, $\psi(|\xi|^2)$. Due to [21, Theorem 2.5.8] in order to prove that these are L^1-multipliers, we need to show that their inverse Fourier transforms have finite L^1 norm, which is true since ψ is a bump function. Again by [21, Theorem 2.5.8] an operator is an L^1-multiplier if and only if it is given by the convolution with a finite Borel measure. As such, it can be extended to a multiplier on $\mathcal{M}_{TV}$ with the same norm. $\square$

We next provide a basic L^p estimate for symbols satisfying the truncation property uniformly. The following estimate is an extension of [31, Lemma 2.2] by making use of regularity in the v component of f. As pointed out in the introduction, this allows to avoid bootstrapping arguments in the applications, which is crucial, since these bootstrapping arguments do not allow to conclude a regularity of order more than one.

Lemma A.3. *Assume that $m(\xi,v)$ satisfies the truncation property uniformly in v. Let φ,ϕ be bounded, smooth functions, ψ be a smooth cut-off function and M_ψ be the Fourier multiplier with symbol $\varphi(\xi)\psi\left(\frac{m(\xi,v)}{\delta}\right)$. Then, for all $1<p\le 2$, $\sigma\ge 0$, $r\in(\frac{p'}{1+\sigma p'},p']\cap(1,\infty)$,*

$$\|\int M_\psi f\phi\,dv\|_{L^p_x}\lesssim \|f\phi\|_{L^p_x(H^{\sigma,p}_v)}\sup_{\xi\in\ supp\ \varphi}|\Omega_m(\xi,\delta)|^{\frac{1}{r}},$$

where $\Omega_m(\xi,\delta)=\{v\in\ supp\ \phi:\ |m(\xi,v)|\le\delta\}$. Moreover,

$$\|\int M_\psi f\phi\,dv\|_{\mathcal{M}_{TV;x}}\lesssim\|f\phi\|_{\mathcal{M}_{TV;x}}.$$

Proof. We first consider the case $p=2$. Then

$$\begin{aligned}
\|\int M_\psi f\phi\,dv\|_{L^2_x}&\lesssim\|\int \mathcal{F}_x^{-1}\varphi(\xi)\psi\left(\frac{m(\xi,v)}{\delta}\right)\hat f\phi\,dv\|_{L^2_x}\\
&=\|\int\varphi(\xi)\psi\left(\frac{m(\xi,v)}{\delta}\right)\hat f\phi\,dv\|_{L^2_\xi}\lesssim\Big\||\varphi(\xi)|\|\psi\left(\frac{m(\xi,v)}{\delta}\right)\|_{H^{-\sigma,2}_v}\|\hat f\phi\|_{H^{\sigma,2}_v}\Big\|_{L^2_\xi}\\
&\lesssim\sup_{\xi\in\text{ supp }\varphi}\|\psi\left(\frac{m(\xi,v)}{\delta}\right)\|_{H^{-\sigma,2}_v}\|\hat f\phi\|_{L^2_\xi(H^{\sigma,2}_v)}.
\end{aligned}$$

Note

$$\begin{aligned}
\|\hat f\phi\|^2_{L^2_\xi(H^{\sigma,2}_v)}&=\int\|\hat f\phi\|^2_{H^{\sigma,2}_v}d\xi=\int|(1+\Delta_v)^{\frac{\sigma}{2}}\hat f\phi|^2\,dvd\xi\\
&=\int|\mathcal{F}_x(1+\Delta_v)^{\frac{\sigma}{2}}f\phi|^2\,d\xi dv=\int|(1+\Delta_v)^{\frac{\sigma}{2}}f\phi|^2\,dxdv\\
&=\int\|f\phi\|^2_{H^{\sigma,2}_v}\,dx=\|f\phi\|^2_{L^2_xH^{\sigma,2}_v}.
\end{aligned}$$

By Sobolev embeddings (cf. e.g. [3, Theorem 1.66]) we have $H^{\sigma,2}_v\hookrightarrow L^{r'}_v$ for all $r'\in[2,\frac{2}{1-2\sigma}]\cap\mathbb{R}$. Hence, for $r\in[\frac{2}{1+2\sigma},2]\cap(1,\infty)$ we have $L^r_v\hookrightarrow H^{-\sigma,2}_v$. Fix $r\in[\frac{2}{1+2\sigma},2]\cap(1,\infty)$ arbitrary. Then

$$\begin{aligned}
\|\int M_\psi f\phi\,dv\|_{L^2_x}&\lesssim\sup_{\xi\in\text{ supp }\varphi}\|\psi\left(\frac{m(\xi,v)}{\delta}\right)\|_{L^r_v}\|f\phi\|_{L^2_x(H^{\sigma,2}_v)}\\
&\lesssim\sup_{\xi\in\text{ supp }\varphi}|\Omega_m(\xi,\delta)|^{\frac{1}{r}}\|f\phi\|_{L^2_x(H^{\sigma,2}_v)}.
\end{aligned}$$

This finishes the proof in case of $p=2$.

Due to the truncation property (on L^1 and $\mathcal{M}_{TV}$) uniform in v, we have, for all $\eta \geq 1$,

$$\|\int M_\psi f\phi\, dv\|_{L^\eta_x} \lesssim \|f\phi\|_{L^\eta_{x,v}}$$

and

$$\|\int M_\psi f\phi\, dv\|_{\mathcal{M}_{TV}} \lesssim \|f\phi\|_{\mathcal{M}_{TV}}.$$

We now conclude by interpolation: From the above we have that $\overline{M}_\psi f := \int M_\psi f\phi\, dv$ is a bounded linear operator in $L(L^2_x(H^{\sigma,2}_v); L^2_x) \cap L(L^\eta_{x,v}; L^\eta_x)$. By complex interpolation, for $\theta \in (0,1)$, $\overline{M}_\psi$ is a bounded linear operator in $L([L^2_x(H^{\sigma,2}_v), L^\eta_{x,v}]_\theta; [L^2_x, L^\eta_x]_\theta)$. Interpolation of Banach space valued L^p-spaces yields

$$[L^2_x(H^{\sigma,2}_v), L^\eta_{x,v}]_\theta = L_x^{\frac{2}{1+\theta(\frac{2}{\eta}-1)}}([H^{\sigma,2}_v, L^\eta_v]_\theta).$$

Next we note that, for $\eta > 1$,

$$[H^{\sigma,2}_v, L^\eta_v]_\theta = H_v^{(1-\theta)\sigma, \frac{2}{1+\theta(\frac{2}{\eta}-1)}}$$

Hence,

$$\begin{aligned}
[L^2_x(H^{\sigma,2}_v), L^\eta_{x,v}]_\theta &\supseteq L_x^{\frac{2}{1+\theta(\frac{2}{\eta}-1)}}(H_v^{(1-\theta)\sigma, \frac{2}{1+\theta(\frac{2}{\eta}-1)}}) \\
[L^2_x, L^\eta_x]_\theta &= L^{\frac{2}{1+\theta(\frac{2}{\eta}-1)}}.
\end{aligned}$$

Let now $p \in (1,2)$. Let $\eta > 1$ be such that $\theta = \frac{2-p}{p}\frac{\eta}{2-\eta} \in (0,1)$, i.e. $p = \frac{2}{1+\theta(\frac{2}{\eta}-1)}$. Then, in conclusion, for all $\sigma > 0$ and all $r \in [\frac{2}{1+2\sigma}, 2] \cap (1,\infty)$,

$$\begin{aligned}
\|\int M_\psi f\phi\, dv\|_{L^p} &= \|\int M_\psi f\phi\, dv\|_{L^{\frac{2}{1+\theta(\frac{2}{\eta}-1)}}} \\
&\lesssim \|\overline{M}_\psi\|^{1-\theta}_{L(L^2_x(H^{\sigma,2}_v);L^2_x)} \|\overline{M}_\psi\|^\theta_{L(L^\eta_{x,v};L^\eta_x)} \|f\phi\|_{L_x^{\frac{2}{1+\theta(\frac{2}{\eta}-1)}}(H_v^{(1-\theta)\sigma, \frac{2}{1+\theta(\frac{2}{\eta}-1)}})} \\
&\lesssim \sup_\xi |\Omega_m(\xi,\delta)|^{\frac{2}{rp'}} \|f\phi\|_{L^p_x(H_v^{2\sigma\frac{p-\eta}{p(2-\eta)},p})}.
\end{aligned}$$

Now given $\sigma > 0$ we apply the above with σ replaced by $\sigma' := \frac{p(2-\eta)}{2(p-\eta)}\sigma > 0$ and $\eta > 1$ small enough. Again choosing $\eta > 1$ small enough, this yields the claim for all $r \in (\frac{p'}{1+\sigma p'}, p'] \cap (1,\infty)$. □

Appendix B. Entropy solutions for parabolic-hyperbolic PDE with a source

In this section we recall some details on the concept of entropy/kinetic solutions and their well-posedness for PDE of the type

$$\partial_t u + \mathrm{div} A(u) - \mathrm{div}\,(b(u)\nabla u) = S(t,x) \quad \text{on } (0,T)\times\mathbb{R}^d \tag{B.1}$$

with

$$
\begin{aligned}
&u_0 \in L^1(\mathbb{R}^d),\ S \in L^1([0,T]\times\mathbb{R}^d)\\
&a = A' \in L^\infty_{loc}(\mathbb{R};\mathbb{R}^d)\\
&b_{ij}(\cdot) = \sum_{k=1}^d \sigma_{ik}(\cdot)\sigma_{kj}(\cdot), \quad \sigma_{ik} \in L^\infty_{loc}(\mathbb{R};\mathbb{R}^d).
\end{aligned} \tag{B.2}
$$

We will use the terms kinetic and entropy solution synonymously. From [11] we recall

Definition B.1. We say that $u \in C([0,T];L^1(\mathbb{R}^d))$ is an entropy solution to (B.1) if $f = \chi(u)$ satisfies

i. for any non-negative $\psi \in \mathcal{D}(\mathbb{R})$, $k = 1,\dots,d$,

$$\sum_{i=1}^d \partial_{x_i}\beta^\psi_{ik}(u) \in L^2([0,T]\times\mathbb{R}^d)$$

ii. for any two non-negative functions $\psi_1,\psi_2 \in \mathcal{D}(\mathbb{R})$,

$$\sqrt{\psi_1(u(t,x))}\sum_{i=1}^d \partial_{x_i}\beta^{\psi_2}_{ik}(u(t,x)) = \sum_{i=1}^d \partial_{x_i}\beta^{\psi_1\psi_2}_{ik}(u(t,x)) \quad \text{a.e.}$$

iii. there are non-negative measures $m, n \in \mathcal{M}^+$ such that, in the sense of distributions,

$$\partial_t f + a(v)\cdot\nabla_x f - \operatorname{div}(b(v)\nabla_x f) = \partial_v(m+n) + \delta_{v=u(t,x)}S \quad \text{on } (0,T)\times\mathbb{R}^d_x\times\mathbb{R}_v$$

where n is defined by

$$\int \psi(v)n(t,x,v)\,dv = \sum_{k=1}^d\left(\sum_{i=1}^d \partial_{x_i}\beta^\psi_{ik}(u(t,x))\right)^2$$

for any $\psi \in \mathcal{D}(\mathbb{R})$ with $\psi \geq 0$

iv. we have

$$\int (m+n)\,dxdt \leq \mu(v) \in L^\infty_0(\mathbb{R}),$$

where L^∞_0 is the space of L^∞-functions vanishing for $|v| \to \infty$.

The proof of well-posedness of entropy solutions to (B.1) follows along the same lines of [11]. The additional difficulty of the force S in (B.1) can be resolved as in [19, Theorem 10]. The construction of solutions relies on a smooth approximation of u_0, S. This yields

Theorem B.2. *Let $u_0 \in L^1(\mathbb{R}^n)$, $S \in L^1([0,T]\times\mathbb{R}^n)$. Then there is a unique entropy solution u to* (B.1) *satisfying $u \in C([0,T];L^1(\mathbb{R}^n))$. For two entropy solutions u^1, u^2 with initial conditions u^1_0, u^2_0 we have*

$$\sup_{t\in[0,T]} \|u^1(t) - u^2(t)\|_{L^1(\mathbb{R}^n)} \leq \|u^1_0 - u^2_0\|_{L^1(\mathbb{R}^n)} + \|S\|_{L^1([0,T]\times\mathbb{R}^n)}.$$

Appendix C. The case $m \geq 2$

In this section we present a slight improvement on the results obtained in [16]. We consider

$$\partial_t u + \mathrm{div} A(u) - \Delta u^{[m]} = S(t,x) \quad \text{on } (0,T) \times \mathbb{R}^d_x \tag{C.1}$$

where

$$\begin{gathered} u_0 \in L^1(\mathbb{R}^d_x),\ S \in L^1([0,T]\times\mathbb{R}^d_x) \\ a = A' \in L^\infty_{loc}(\mathbb{R};\mathbb{R}^d), \\ u^{[m]} = |u|^{m-1}u \text{ with } m \geq 2. \end{gathered} \tag{C.2}$$

By [11] and Appendix B there is a unique entropy solution to (C.1).

Lemma C.1. *For each $\gamma > 0$ there are $c_\gamma, C_\gamma > 0$ such that*

$$\sup_{t\in[0,T]} \|u(t)\|_{1+\gamma}^{1+\gamma} + c_{\gamma,m}\int_0^T \int_{\mathbb{R}^d_x} (\nabla u^{[\frac{\gamma+m}{2}]})^2 dx \leq C_\gamma(\|u_0\|^{1+\gamma}_{L^{1+\gamma}_x} + \|S\|^{1+\gamma}_{L^{1+\gamma}_{t,x}}).$$

Proof. We present an informal derivation of the stated energy estimate. The rigorous justification of these calculations is a simple consequence of first considering a vanishing viscosity approximation and then using weak lower semicontinuity of the right hand side of the inequality. We have, with $A^\gamma(u) := \int_0^u A'(v)v^{[\gamma]}\,dv$,

$$\begin{aligned}
&\partial_t \int_{\mathbb{R}^d_x} |u|^{1+\gamma} dx \\
&= (1+\gamma)\int_{\mathbb{R}^d_x} u^{[\gamma]}(-\mathrm{div}A(u) + \Delta u^{[m]} + S(t,x))\,dx \\
&= (1+\gamma)\int_{\mathbb{R}^d_x} -u^{[\gamma]}\mathrm{div}A(u) + u^{[\gamma]}\Delta u^{[m]} + u^{[\gamma]}S(t,x)\,dx \\
&\leq (1+\gamma)\int_{\mathbb{R}^d_x} \mathrm{div}A^\gamma(u) - \frac{4\gamma m}{(\gamma+m)^2}(\nabla u^{[\frac{\gamma+m}{2}]})^2 + \frac{\gamma}{1+\gamma}|u^{[\gamma]}|^{\frac{1+\gamma}{\gamma}} + \frac{1}{1+\gamma}|S(t,x)|^{1+\gamma}\,dx \\
&= -\frac{4\gamma m(1+\gamma)}{(\gamma+m)^2}\int_{\mathbb{R}^d_x} (\nabla u^{[\frac{\gamma+m}{2}]})^2\,dx + \int_{\mathbb{R}^d_x} \gamma|u|^{1+\gamma} + |S(t,x)|^{1+\gamma}\,dx.
\end{aligned}$$

Gronwall's inequality yields

$$\begin{aligned}
&\int_{\mathbb{R}^d_x} |u(t)|^{1+\gamma}dx + \frac{4\gamma m(1+\gamma)}{(\gamma+m)^2}\int_0^t e^{\gamma(t-s)}\int_{\mathbb{R}^d_x}(\nabla u^{[\frac{\gamma+m}{2}]})^2 dx \\
&\leq e^{\gamma t}\int_{\mathbb{R}^d_x}|u_0|^{1+\gamma}dx + \int_0^t e^{\gamma(t-s)}\int_{\mathbb{R}^d_x}|S(s,x)|^{1+\gamma}\,dxds.
\end{aligned}$$

In conclusion, for $\gamma > 0$ we obtain that

$$\sup_{t\in[0,T]} \|u(t)\|_{1+\gamma}^{1+\gamma} + c_{\gamma,m}\int_0^T \int_{\mathbb{R}^d_x} (\nabla u^{[\frac{\gamma+m}{2}]})^2 dx \leq C_\gamma(\|u_0\|^{1+\gamma}_{L^{1+\gamma}_x} + \|S\|^{1+\gamma}_{L^{1+\gamma}_{t,x}}).$$

$\square$

For $p \in [1,\infty)$, $s \in (0,1)$ we recall

$$|f|^p_{\mathcal{N}^{s,p}} := \sup_{\delta>0} \sup_{0<|z|<\delta} \int_{\mathbb{R}^d_x} \frac{|f(x+z)-f(x)|}{|z|^s}\, dx$$

and

$$\|f\|^p_{\mathcal{N}^{s,p}} = \|f\|^p_{L^p} + |f|^p_{\mathcal{N}^{s,p}}.$$

Theorem C.2. *Let $\gamma > 0$, $m \geq 2$ and $u_0 \in L^{1+\gamma}(\mathbb{R}^d_x)$, $S \in L^{1+\gamma}([0,T]\times\mathbb{R}^d_x)$. Then*

$$\int_0^T |u(t,\cdot)|^{m+\gamma}_{\mathcal{N}^{\frac{2}{m+\gamma},m+\gamma}(\mathbb{R}^d_x)}\, dt \leq C_{\gamma,m}(\|u_0\|^{1+\gamma}_{L^{1+\gamma}_x} + \|S\|^{1+\gamma}_{L^{1+\gamma}_{t,x}}).$$

If, in addition, $u_0 \in L^{m+\gamma}(\mathbb{R}^d_x)$, $S \in L^{m+\gamma}([0,T]\times\mathbb{R}^d_x)$ then $u \in L^{m+\gamma}([0,T];\mathcal{N}^{\frac{2}{m+\gamma},m+\gamma}(\mathbb{R}^d_x))$ with

$$\|u\|^{m+\gamma}_{L^{m+\gamma}([0,T];\mathcal{N}^{\frac{2}{m+\gamma},m+\gamma}(\mathbb{R}^d_x))} \leq C_{\gamma,m}(\|u_0\|^{1+\gamma}_{L^{1+\gamma}_x} + \|S\|^{1+\gamma}_{L^{1+\gamma}_{t,x}} + \|u_0\|^{m+\gamma}_{L^{m+\gamma}_x} + \|S\|^{m+\gamma}_{L^{m+\gamma}_{t,x}}). \tag{C.3}$$

Proof. We again restrict to giving the informal derivation, the rigorous justification is standard by considering a vanishing viscosity approximation first, then using lower semi-continuity. From [16, Lemma 4.1] we recall the elementary inequality, for $m \geq 2$,

$$|r-s|^m \leq c|r^{[\frac{m}{2}]} - s^{[\frac{m}{2}]}|^2 \quad \forall r,s \in \mathbb{R},$$

for some $c > 0$. Hence,

$$\begin{aligned}|\Delta^h_e u(x)|^m &= |u(x+he)-u(x)|^m \leq c|u(x+he)^{[\frac{m}{2}]} - u(x)^{[\frac{m}{2}]}|^2\\ &= c|\Delta^h_e u^{[\frac{m}{2}]}(x)|^2\end{aligned}$$

and thus, using Lemma C.1,

$$\begin{aligned}&\int_0^T \sup_{e\in\mathbb{R}^N,|e|=1} \sup_{h>0} \int_{\mathbb{R}^d} \left|\frac{\Delta^h_e u(t,x)}{h^{\frac{2}{m+\gamma}}}\right|^{m+\gamma} dxdt\\ &= \int_0^T \sup_{e\in\mathbb{R}^N,|e|=1} \sup_{h>0} \int_{\mathbb{R}^d} h^{-2} \left|\Delta^h_e u(t,x)\right|^{m+\gamma} dxdt\\ &\leq c\int_0^T \sup_{e\in\mathbb{R}^N,|e|=1} \sup_{h>0} \int_{\mathbb{R}^d} h^{-2}|\Delta^h_e u^{[\frac{m+\gamma}{2}]}(t,x)|^2 dxdt\\ &\leq c\int_0^T \int_{\mathbb{R}^d} |\nabla u^{[\frac{m+\gamma}{2}]}(t,x)|^2 dxdt\\ &\leq C_{\gamma,m}(\|u_0\|^{1+\gamma}_{L^{1+\gamma}_x} + \|S\|^{1+\gamma}_{L^{1+\gamma}_{t,x}}).\end{aligned}$$

This implies

$$\int_0^T |u(t,\cdot)|^{m+\gamma}_{\mathcal{N}^{\frac{2}{m+\gamma},m+\gamma}(\mathbb{R}^d_x)}\, dt \leq C_{\gamma,m}(\|u_0\|^{1+\gamma}_{L^{1+\gamma}_x} + \|S\|^{1+\gamma}_{L^{1+\gamma}_{t,x}}).$$

Using Lemma C.1 with γ replaced by $m-1+\gamma$ yields

$$\|u\|^{m+\gamma}_{L^\infty([0,T];L^{m+\gamma}(\mathbb{R}^d_x))} \leq C_{m,\gamma}(\|u_0\|^{m+\gamma}_{L^{m+\gamma}_x} + \|S\|^{m+\gamma}_{L^{m+\gamma}_{t,x}}).$$

This implies that

$$\|u\|^{m+\gamma}_{L^{m+\gamma}([0,T];\mathcal{N}^{\frac{2}{m+\gamma},m+\gamma}(\mathbb{R}^d))} \leq C_\gamma(\|u_0\|^{1+\gamma}_{L^{1+\gamma}_x} + \|S\|^{1+\gamma}_{L^{1+\gamma}_{t,x}}) + C_{m,\gamma}(\|u_0\|^{m+\gamma}_{L^{m+\gamma}_x} + \|S\|^{m+\gamma}_{L^{m+\gamma}_{t,x}}).$$

□

Remark C.3 (Optimality of Theorem C.2). For simplicity we consider the elliptic problem

$$0 = \Delta u^{[m]} + S \quad \text{on } \mathbb{R}^d \tag{C.4}$$

and note that the arguments leading to Theorem C.2 can be applied here without essential change. Assume that for some $\alpha \in (0,1)$, $p, p_1, p_2 \in [1,\infty)$ we have

$$|u(\cdot)|^p_{\mathcal{N}^{\alpha,p}(\mathbb{R}^d_x)} \leq C(\|S\|^{p_1}_{L^{p_1}(\mathbb{R}^d_x)} + \|S\|^{p_2}_{L^{p_2}(\mathbb{R}^d_x)}) \tag{C.5}$$

for some constant $C \geq 0$, all $S \in (L^{p_1} \cap L^{p_2})(\mathbb{R}^d_x)$ and u being the corresponding solution to (C.4).

Rescaling $\tilde{u}(x) := \eta^{-\frac{2}{m}} u(\eta x)$, $\tilde{S}(x) = S(\eta x)$ yields (C.4) for $\tilde{u}$, $\tilde{S}$. Moreover,

$$\begin{aligned} |\tilde{u}(\cdot)|^p_{\mathcal{N}^{\alpha,p}(\mathbb{R}^d_x)} &= \eta^{-\frac{2}{m}p+\alpha p-1}|u(\cdot)|^p_{\mathcal{N}^{\alpha,p}(\mathbb{R}^d_x)} \\ \|\tilde{S}\|^{p_i}_{L^{p_i}(\mathbb{R}^d_x)} &= \eta^{-1}\|S\|^{p_i}_{L^{p_i}(\mathbb{R}^d_x)}. \end{aligned}$$

Hence, (C.5) applied to $\tilde{u}$ implies

$$|u(\cdot)|^p_{\mathcal{N}^{\alpha,p}(\mathbb{R}^d_x)} \leq C\eta^{(\frac{2}{m}-\alpha)p}(\|S\|^{p_1}_{L^{p_1}(\mathbb{R}^d_x)} + \|S\|^{p_2}_{L^{p_2}(\mathbb{R}^d_x)}). \tag{C.6}$$

Thus, if $\alpha > \frac{2}{m}$ we may let $\eta \to \infty$ and get $|u(\cdot)|^p_{\mathcal{N}^{\alpha,p}(\mathbb{R}^d_x)} = 0$. Together with $u \in L^1(\mathbb{R}^d_x)$ this implies $u \equiv 0$. Hence, $\alpha = \frac{2}{m}$ is the optimal regularity exponent for (C.5).

References

[1] R. A. Adams. *Sobolev spaces*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[2] D. G. Aronson and J. L. Vázquez. Eventual C^∞-regularity and concavity for flows in one-dimensional porous media. *Arch. Rational Mech. Anal.*, 99(4):329–348, 1987.

[3] H. Bahouri, J.-Y. Chemin, and R. Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011.

[4] J. Bergh and J. Löfström. *Interpolation spaces. An introduction*. Springer-Verlag, Berlin-New York, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[5] V. Bögelein, F. Duzaar, and U. Gianazza. Porous medium type equations with measure data and potential estimates. *SIAM J. Math. Anal.*, 45(6):3283–3330, 2013.

[6] V. Bögelein, F. Duzaar, and U. Gianazza. Continuity estimates for porous medium type equations with measure data. *J. Funct. Anal.*, 267(9):3351–3396, 2014.

[7] L. A. Caffarelli and L. C. Evans. Continuity of the temperature in the two-phase Stefan problem. *Arch. Rational Mech. Anal.*, 81(3):199–220, 1983.

[8] L. A. Caffarelli and A. Friedman. Continuity of the density of a gas flow in a porous medium. *Trans. Amer. Math. Soc.*, 252:99–113, 1979.

[9] L. A. Caffarelli and A. Friedman. Regularity of the free boundary of a gas flow in an n-dimensional porous medium. *Indiana Univ. Math. J.*, 29(3):361–391, 1980.

[10] L. A. Caffarelli, J. L. Vázquez, and N. I. Wolanski. Lipschitz continuity of solutions and interfaces of the N-dimensional porous medium equation. *Indiana Univ. Math. J.*, 36(2):373–401, 1987.

[11] G.-Q. Chen and B. Perthame. Well-posedness for non-isotropic degenerate parabolic-hyperbolic equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 20(4):645–668, 2003.

[12] C. De Lellis, F. Otto, and M. Westdickenberg. Structure of entropy solutions for multi-dimensional scalar conservation laws. *Arch. Ration. Mech. Anal.*, 170(2):137–184, 2003.

[13] C. De Lellis and M. Westdickenberg. On the optimality of velocity averaging lemmas. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 20(6):1075–1085, 2003.

[14] E. DiBenedetto. Continuity of weak solutions to a general porous medium equation. *Indiana Univ. Math. J.*, 32(1):83–118, 1983.

[15] E. DiBenedetto and A. Friedman. Hölder estimates for nonlinear degenerate parabolic systems. *J. Reine Angew. Math.*, 357:1–22, 1985.

[16] C. Ebmeyer. Regularity in Sobolev spaces for the fast diffusion and the porous medium equation. *J. Math. Anal. Appl.*, 307(1), 2005.

[17] J. Gärtner and S. A. Molchanov. Parabolic problems for the Anderson model. I. Intermittency and related topics. *Comm. Math. Phys.*, 132(3):613–655, 1990.

[18] J. Gärtner and S. A. Molchanov. Parabolic problems for the Anderson model. II. Second-order asymptotics and structure of high peaks. *Probab. Theory Related Fields*, 111(1):17–55, 1998.

[19] B. Gess and X. Lamy. On averaging lemmas for scalar conservation laws. *preprint*, 2017.

[20] B. Gess and P. E. Souganidis. Stochastic non-isotropic degenerate parabolic-hyperbolic equations. *to appear in: Stochastic Process. Appl.*, 2017.

[21] L. Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2014.

[22] L. Grafakos. *Modern Fourier analysis*, volume 250 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2014.

[23] C. Kienzler, H. Koch, and J. L. Vazquez. Flatness implies smoothness for solutions of the porous medium equation. *arXiv:1609.09048*, 2016.

[24] N. V. Krylov. An analytic approach to SPDEs. In *Stochastic partial differential equations: six perspectives*, volume 64 of *Math. Surveys Monogr.*, pages 185–242. Amer. Math. Soc., Providence, RI, 1999.

[25] M. Mizuno. Hölder estimates for solutions of the Cauchy problem for the porous medium equation with external forces. *Manuscripta Math.*, 141(1-2):273–313, 2013.

[26] B. Perthame. *Kinetic formulation of conservation laws*, volume 21 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2002.

[27] L. s. Pick, A. Kufner, O. r. John, and S. Fu ˇ cík. *Function spaces. Vol. 1*, volume 14 of *De Gruyter Series in Nonlinear Analysis and Applications*. Walter de Gruyter & Co., Berlin, extended edition, 2013.

[28] T. Runst and W. Sickel. *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, volume 3 of *De Gruyter Series in Nonlinear Analysis and Applications*. Walter de Gruyter & Co., Berlin, 1996.

[29] P. E. Sacks. Continuity of solutions of a singular parabolic equation. *Nonlinear Anal.*, 7(4):387–409, 1983.

[30] P. E. Sacks. The inital and boundary value problem for a class of degenerate parabolic equations. *Comm. Partial Differential Equations*, 8(7):693–733, 1983.

[31] E. Tadmor and T. Tao. Velocity averaging, kinetic formulations, and regularizing effects in quasi-linear PDEs. *Comm. Pure Appl. Math.*, 60(10):1488–1521, 2007.

[32] J. L. Vázquez. *The porous medium equation*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, Oxford, 2007. Mathematical theory.

[33] M. Wiegner. On C_α-regularity of the gradient of solutions of degenerate parabolic systems. *Ann. Mat. Pura Appl. (4)*, 145:385–405, 1986.

[34] W. P. Ziemer. Interior and boundary continuity of weak solutions of degenerate parabolic equations. *Trans. Amer. Math. Soc.*, 271(2):733–748, 1982.