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Reflected solutions of backward stochastic differential equations driven by G-Brownian motion

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Abstract In this paper, we study the reflected solutions of one-dimensional backward stochastic differential equations driven by G-Brownian motion. The reflection keeps the solution above a given stochastic process. In order to derive the uniqueness of reflected G-BSDEs, we apply a "martingale condition" instead of the Skorohod condition. Similar to the classical case, we prove the existence by approximation via penalization. We then give some applications including a generalized Feynman-Kac formula of an obstacle problem for fully nonlinear partial differential equation and option pricing of American types under volatility uncertainty.

Keywords G-expectation, reflected backward stochastic differential equations, obstacle problems for fully nonlinear PDEs

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1 Introduction

El Karoui et al. [4] introduced the problem of backward stochastic differential equation (BSDE) with reflection, which means that the solution to a BSDE is required to be above a certain given continuous boundary process, called the obstacle. For this purpose, an additional continuous increasing process should be included in the equation. Furthermore, this additional process should be chosen in a minimal way so that it satisfies the Skorohod condition. It is worth noting that the solution is the value function of an optimal stopping problem.

Due to the importance in BSDE theory and in applications, the reflected problem has attracted a great deal of attention since 1997. Many scholars tried to relax the conditions on the generator and the obstacle process. Hamadene [6] and Lepeltier and Xu [14] proposed a generalized Skorohod condition and studied the case where the obstacle process is discontinuous. Cvitanic and Karaztas [2] and Hamadene and Lepeltier [7] proved the existence and uniqueness when there are two reflecting obstacles. They also

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established the connection between this problem and Dynkin games. Matoussi [17] and Kobylanski et al. [13] extended the results to the case where the generator is not a Lipschitz function.

We should point out that the classical BSDE can only provide a probabilistic interpretation for the solution of quasilinear PDEs. In addition, this BSDE cannot be applied to pricing path-dependent contingent claims in the uncertain volatility model (UVM). Motivated by these facts, Peng [19, 20] systematically introduced a time-consistent fully nonlinear expectation theory. One of the most important cases is the G-expectation theory (see [23] and the references therein). In this framework, a new type of Brownian motion and the corresponding stochastic calculus of Itô's type were constructed. It has been widely used to study the problems of model uncertainty, nonlinear stochastic dynamical systems and fully nonlinear PDEs.

The backward stochastic differential equation driven by G-Brownian motion (i.e., G-BSDE) can be written in the following way:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\langle B\rangle_s - \int_t^T Z_s dB_s - (K_T - K_t).$$

The solution of this equation consists of a triplet of processes (Y, Z, K). The existence and uniqueness of the solution are proved in [8]. In [9], the corresponding comparison theorem, Feynman-Kac formula and related topics were established.

In this paper, we study the case where the solution of a G-BSDE is required to stay above a given stochastic process, called the lower obstacle. An increasing process should be included in this equation to push the solution above the obstacle. According to the classical case studied by [4], one may expect that the solution of a reflected G-BSDE is a quadruple of processes $\{(Y_t, Z_t, K_t, L_t), 0 \leqslant t \leqslant T\}$ such that

(1) $Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\langle B\rangle_s - \int_t^T Z_s dB_s - (K_T - K_t) + L_T - L_t$;

(2) $(Y, Z, K) \in \mathfrak{S}_G^\alpha(0, T)$ and $Y_t \geqslant S_t$, $0 \leqslant t \leqslant T$;

(3) $\{L_t\}$ is continuous and increasing, $L_0 = 0$ and $\int_0^T (Y_t - S_t)dL_t = 0$.

A shortcoming of this formulation is that, as an example provided in Remark 3.7, the solution of (1)–(3) is not unique. Our crucial observation is that, in fact, we can define a nondecreasing process $A_t = L_t - K_t$ such that A is continuous and $\{-\int_0^\cdot (Y_s - S_s)dA_s\}$ is a G-martingale. So we apply a "martingale condition" instead of the classical Skorohod condition and reformulate this problem as the following. A triplet of processes (Y, Z, A) is called a solution of a reflected G-BSDE if the following properties hold:

(a) $(Y, Z, A) \in \mathcal{S}_G^\alpha(0, T)$ and $Y_t \geqslant S_t$;

(b) $Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\langle B\rangle_s - \int_t^T Z_s dB_s + (A_T - A_t)$;

(c) $\{-\int_0^t (Y_s - S_s)dA_s\}_{t\in[0,T]}$ is a nonincreasing G-martingale.

Here, we denote by $\mathcal{S}_G^\alpha(0, T)$ the collection of processes (Y, Z, A) such that $Y \in S_G^\alpha(0, T)$, $Z \in H_G^\alpha(0, T)$, A is a continuous nondecreasing process with $A_0 = 0$ and $A \in S_G^\alpha(0, T)$. Under some appropriate assumptions, we can prove that the solution of the above reflected G-BSDE is unique. In proving the existence of this problem, we use the approximation method via penalization. This is a constructive method in the sense that the solution of the reflected G-BSDE is proved to be the limit of a sequence of penalized G-BSDEs. One of the difficulties in the proof of the existence is that the classical dominated convergence theorem cannot be applied to our G-framework. Additionally, a sequence bounded in $M_G^p(0, T)$ is no longer weakly compact. This main difficulty in carrying out this construction is to prove the convergence property in some appropriate sense. It turns out that the well-known monotonic convergence theorem (see [18]) cannot be applied and we must find a new method to overcome this difficulty. We have found a new approach involving a uniformly continuous property in $S_G^p(0, T)$ to overcome this challenge. This approach is also applicable to many other appropriate situations.

In comparison with [4], an important difference is that, if all coefficients of SDEs and reflected BSDEs are deterministic functions of the state variable, as in Section 6, then the solution Y of the corresponding reflected BSDE is a viscosity solution of an obstacle problem of a fully nonlinear parabolic PDE of (6.2). There has been tremendous interest in developing the obstacle problem for PDEs since it has wide applications to mathematical finance (see [5]) and mathematical physics (see [24]). We then obtain a new type of probabilistic interpretation for the viscosity solution of an obstacle problem of PDE via reflected BSDE.

The rest of the paper is organized as follows. In Section 2, we present some notation and results as preliminaries. The problem is formulated in detail in Section 3 and we state some a priori estimates from which we derive some integrability properties and the uniqueness of the solution. In Section 4, we apply the approximation method via penalization to prove the existence of the solution. We list some convergence properties of the solutions to the penalized G-BSDEs. Our main results are shown and proved in Section 5. Furthermore, we prove a comparison theorem similar to that in [9], specifically for nonreflected G-BSDEs. In Section 6, we give the relation between reflected G-BSDEs and the corresponding obstacle problems for fully nonlinear parabolic PDEs. Finally, we use the results of the previous sections to study the pricing problem for American contingent claims under model uncertainty in Section 7. In Appendix A, we introduce the optional stopping theorem under G-framework used for the pricing problem.

2 Preliminaries

We recall some basic notions and results of G-expectation, which are needed in the sequel. More relevant details can be found in [8, 9, 21–23].

2.1 G-expectation

Definition 2.1. Let Ω be a given set and let $\mathcal{H}$ be a vector lattice of real valued functions defined on Ω, namely $c \in \mathcal{H}$ for each constant c and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. $\mathcal{H}$ is considered as the space of random variables. A sublinear expectation $\hat{\mathrm{E}}$ on $\mathcal{H}$ is a functional $\hat{\mathrm{E}} : \mathcal{H} \to \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

(i) monotonicity: if $X \geqslant Y$, then $\hat{\mathrm{E}}[X] \geqslant \hat{\mathrm{E}}[Y]$;

(ii) constant preserving: $\hat{\mathrm{E}}[c] = c$;

(iii) sub-additivity: $\hat{\mathrm{E}}[X + Y] \leqslant \hat{\mathrm{E}}[X] + \hat{\mathrm{E}}[Y]$;

(iv) positive homogeneity: $\hat{\mathrm{E}}[\lambda X] = \lambda \hat{\mathrm{E}}[X]$ for each $\lambda \geqslant 0$.

The triple $(\Omega, \mathcal{H}, \hat{\mathrm{E}})$ is called a sublinear expectation space. $X \in \mathcal{H}$ is called a random variable in $(\Omega, \mathcal{H}, \hat{\mathrm{E}})$. We call $Y = (Y_1, \ldots, Y_d), Y_i \in \mathcal{H}$ a d-dimensional random vector in $(\Omega, \mathcal{H}, \hat{\mathrm{E}})$.

Let $\Omega_T = C_0([0, T]; \mathbb{R}^d)$, the space of $\mathbb{R}^d$-valued continuous functions on $[0, T]$ with $\omega_0 = 0$, be endowed with the supremum norm. Let $B = (B^i)_{i=1}^d$ be the canonical process. For each $T > 0$, set

$$L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, \ldots, B_{t_n}) : n \geqslant 1,\ t_1, \ldots, t_n \in [0, T],\ \varphi \in C_{b,\mathrm{Lip}}(\mathbb{R}^{d\times n})\},$$

where $C_{b,\mathrm{Lip}}(\mathbb{R}^{d\times n})$ denotes the set of bounded Lipschitz functions on $\mathbb{R}^{d\times n}$.

Denote by $\mathbb{S}_d$ the collection of all $d \times d$ symmetric matrices. For each given monotonic and sublinear function $G : \mathbb{S}_d \to \mathbb{R}$, we can construct a G-expectation $\hat{\mathrm{E}}$ as well as the conditional G-expectation $\hat{\mathrm{E}}_t$. We call $(\Omega_T, L_{ip}(\Omega_T), \hat{\mathrm{E}})$ the G-expectation space. The canonical process B is the d-dimensional G-Brownian motion under this space. In this paper, we suppose that G is non-degenerate, i.e., there exists some $\underline{\sigma}^2 > 0$ such that $G(A) - G(B) \geqslant \frac{1}{2}\underline{\sigma}^2 \mathrm{tr}[A - B]$ for any $A \geqslant B$.

Let B be the d-dimensional G-Brownian motion. For each fixed $a \in \mathbb{R}^d$, $\{B_t^a\} := \{\langle a, B_t\rangle\}$ is a 1-dimensional G_a-Brownian motion, where $G_a : \mathbb{R} \to \mathbb{R}$ satisfies

$$G_a(p) = G(aa^{\mathrm{T}})p^+ + G(-aa^{\mathrm{T}})p^-, \quad p \in \mathbb{R}.$$

Let $\pi_t^N = \{t_0^N, \ldots, t_N^N\}$, $N = 1, 2, \ldots$, be a sequence of partitions of $[0, t]$ such that $\mu(\pi_t^N) = \max\{|t_{i+1}^N - t_i^N| : i = 0, \ldots, N-1\} \to 0$, the quadratic variation process of B^a is defined by

$$\langle B^a\rangle_t = \lim_{\mu(\pi_t^N)\to 0} \sum_{j=0}^{N-1} (B^a_{t_{j+1}^N} - B^a_{t_j^N})^2.$$

For $a, \bar{a} \in \mathbb{R}^d$, we can define the mutual variation process of B^a and $B^{\bar{a}}$ by

$$\langle B^a, B^{\bar{a}} \rangle_t := \frac{1}{4}[\langle B^{a+\bar{a}} \rangle - \langle B^{a-\bar{a}} \rangle].$$

Denote by $L_G^p(\Omega_T)$ the completion of $L_{ip}(\Omega_T)$ under the norm $\|\xi\|_{L_G^p} := (\hat{\mathrm{E}}[|\xi|^p])^{1/p}$ for $p \geqslant 1$. For all $t \in [0,T]$, $\hat{\mathrm{E}}_t[\,\cdot\,]$ is a continuous mapping on $L_{ip}(\Omega_T)$ with respect to the norm $\|\cdot\|_{L_G^p}$. Therefore, it can be extended continuously to the completion $L_G^p(\Omega_T)$. Denis et al. [3] proved the following representation theorem of G-expectation on $L_G^1(\Omega_T)$.

Theorem 2.2 (See [3,10]). *There exists a weakly compact set $\mathcal{P} \subset \mathcal{M}_1(\Omega_T)$, the set of all probability measures on $(\Omega_T, \mathcal{B}(\Omega_T))$, such that*

$$\hat{\mathrm{E}}[\xi] = \sup_{P \in \mathcal{P}} E_P[\xi] \quad \text{for all} \quad \xi \in L_G^1(\Omega_T).$$

$\mathcal{P}$ is called a set that represents $\hat{\mathrm{E}}$.

Let $\mathcal{P}$ be a weakly compact set that represents $\hat{\mathrm{E}}$. For this $\mathcal{P}$, we define the capacity

$$c(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega_T).$$

Definition 2.3. A set $A \subset \mathcal{B}(\Omega_T)$ is polar if $c(A) = 0$. A property holds "*quasi-surely*" (q.s.) if it holds outside a polar set.

In the following, we do not distinguish the two random variables X and Y if $X = Y$ q.s.

For $\xi \in L_{ip}(\Omega_T)$, let $\mathcal{E}(\xi) = \hat{\mathrm{E}}[\sup_{t \in [0,T]} \hat{\mathrm{E}}_t[\xi]]$. For convenience, we call $\mathcal{E}$ the G-evaluation. For $p \geqslant 1$ and $\xi \in L_{ip}(\Omega_T)$, define $\|\xi\|_{p,\mathcal{E}} = [\mathcal{E}(|\xi|^p)]^{1/p}$ and denote by $L_{\mathcal{E}}^p(\Omega_T)$ the completion of $L_{ip}(\Omega_T)$ under $\|\cdot\|_{p,\mathcal{E}}$. The following estimate between the two norms $\|\cdot\|_{L_G^p}$ and $\|\cdot\|_{p,\mathcal{E}}$ will be frequently used in this paper.

Theorem 2.4 (See [25]). *For any $\alpha \geqslant 1$ and $\delta > 0$, $L_G^{\alpha+\delta}(\Omega_T) \subset L_{\mathcal{E}}^{\alpha}(\Omega_T)$. More precisely, for any $1 < \gamma < \beta := (\alpha+\delta)/\alpha$, $\gamma \leqslant 2$, we have*

$$\|\xi\|_{\alpha,\mathcal{E}}^{\alpha} \leqslant \gamma^* \{\|\xi\|_{L_G^{\alpha+\delta}}^{\alpha} + 14^{1/\gamma} C_{\beta/\gamma} \|\xi\|_{L_G^{\alpha+\delta}}^{(\alpha+\delta)/\gamma}\}, \quad \forall\, \xi \in L_{ip}(\Omega_T),$$

where $C_{\beta/\gamma} = \sum_{i=1}^{\infty} i^{-\beta/\gamma}$, $\gamma^* = \gamma/(\gamma - 1)$.

2.2 *G*-Itô calculus

Definition 2.5. Let $M_G^0(0,T)$ be the collection of processes in the following form: for a given partition $\{t_0, \ldots, t_N\} = \pi_T$ of $[0,T]$,

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) \mathbf{1}_{[t_j, t_{j+1})}(t),$$

where $\xi_i \in L_{ip}(\Omega_{t_i})$, $i = 0, 1, 2, \ldots, N-1$. For each $p \geqslant 1$ and $\eta \in M_G^0(0,T)$, let the norms be $\|\eta\|_{H_G^p} := \{\hat{\mathrm{E}}[(\int_0^T |\eta_s|^2 ds)^{p/2}]\}^{1/p}$, $\|\eta\|_{M_G^p} := (\hat{\mathrm{E}}[\int_0^T |\eta_s|^p ds])^{1/p}$ and denote by $H_G^p(0,T)$ and $M_G^p(0,T)$ the completions of $M_G^0(0,T)$ under the norms $\|\cdot\|_{H_G^p}$ and $\|\cdot\|_{M_G^p}$, respectively.

For two processes $\xi \in M_G^1(0,T)$ and $\eta \in M_G^2(0,T)$, the G-Itô integrals $(\int_0^t \xi_s d\langle B^i, B^j \rangle_s)_{0 \leqslant t \leqslant T}$ and $(\int_0^t \eta_s dB_s^i)_{0 \leqslant t \leqslant T}$ are well-defined (see [16,23]). Similar to the classical Burkholder-Davis-Gundy inequality, the following property holds.

Proposition 2.6 (See [9]). *If $\eta \in H_G^{\alpha}(0,T)$ with $\alpha \geqslant 1$ and $p \in (0, \alpha]$, then $\sup_{u \in [t,T]} |\int_t^u \eta_s dB_s|^p \in L_G^1(\Omega_T)$ and*

$$\underline{\sigma}^p c_p \hat{\mathrm{E}}_t\left[\left(\int_t^T |\eta_s|^2 ds\right)^{p/2}\right] \leqslant \hat{\mathrm{E}}_t\left[\sup_{u \in [t,T]} \left|\int_t^u \eta_s dB_s\right|^p\right] \leqslant \bar{\sigma}^p C_p \hat{\mathrm{E}}_t\left[\left(\int_t^T |\eta_s|^2 ds\right)^{p/2}\right],$$

where $0 < c_p < C_p < \infty$ are constants.

Let $S_G^0(0,T)=\{h(t,B_{t_1\wedge t},\ldots,B_{t_n\wedge t}) : t_1,\ldots,t_n\in[0,T], h\in C_{b,\mathrm{Lip}}(\mathbb{R}^{n+1})\}$. For $p\geqslant 1$ and $\eta\in S_G^0(0,T)$, set $\|\eta\|_{S_G^p}=\{\hat{\mathrm{E}}[\sup_{t\in[0,T]}|\eta_t|^p]\}^{1/p}$. Denote by $S_G^p(0,T)$ the completion of $S_G^0(0,T)$ under the norm $\|\cdot\|_{S_G^p}$. We have the following continuity property for any $Y\in S_G^p(0,T)$ with $p>1$.

Lemma 2.7 (See [15]). *For $Y\in S_G^p(0,T)$ with $p>1$, we have, by setting $Y_s:=Y_T$ for $s>T$,*

$$F(Y):=\limsup_{\varepsilon\to 0}\Big(\hat{\mathrm{E}}\Big[\sup_{t\in[0,T]}\sup_{s\in[t,t+\varepsilon]}|Y_t-Y_s|^p\Big]\Big)^{\frac{1}{p}}=0.$$

We now introduce some basic results of G-BSDEs. Consider the following type of G-BSDE (here we use the Einstein convention)

$$Y_t=\xi+\int_t^T f(s,Y_s,Z_s)ds+\int_t^T g_{ij}(s,Y_s,Z_s)d\langle B^i,B^j\rangle_s-\int_t^T Z_s dB_s-(K_T-K_t), \tag{2.1}$$

where $f(t,\omega,y,z)$, $g_{ij}(t,\omega,y,z):[0,T]\times\Omega_T\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ satisfying the following properties:

(H1′) there exists some $\beta>1$ such that for any y,z, $f(\cdot,\cdot,y,z), g_{ij}(\cdot,\cdot,y,z)\in M_G^\beta(0,T)$;

(H2) there exists some $L>0$ such that

$$|f(t,y,z)-f(t,y',z')|+\sum_{i,j=1}^d|g_{ij}(t,y,z)-g_{ij}(t,y',z')|\leqslant L(|y-y'|+|z-z'|).$$

For simplicity, we denote by $\mathfrak{S}_G^\alpha(0,T)$ the collection of processes (Y,Z,K) such that $Y\in S_G^\alpha(0,T)$, $Z\in H_G^\alpha(0,T;\mathbb{R}^d)$, K is a decreasing G-martingale with $K_0=0$ and $K_T\in L_G^\alpha(\Omega_T)$.

Theorem 2.8 (See [8]). *Assume that $\xi\in L_G^\beta(\Omega_T)$ and f,g_{ij} satisfy* (H1′) *and* (H2) *for some $\beta>1$. Then, for any $1<\alpha<\beta$,* (2.1) *has a unique solution $(Y,Z,K)\in\mathfrak{S}_G^\alpha(0,T)$.*

We also have the comparison theorem for G-BSDE.

Theorem 2.9 (See [9]). *Let $(Y_t^l,Z_t^l,K_t^l)_{t\leqslant T}$, $l=1,2$, be the solutions of the following G-BSDEs:*

$$Y_t^l=\xi^l+\int_t^T f^l(s,Y_s^l,Z_s^l)ds+\int_t^T g_{ij}^l(s,Y_s^l,Z_s^l)d\langle B^i,B^j\rangle_s+V_T^l-V_t^l-\int_t^T Z_s^l dB_s-(K_T^l-K_t^l),$$

where processes $\{V_t^l\}_{0\leqslant t\leqslant T}$ are assumed to be right-continuous with right limit (RCLL), quasi-surely, such that $\hat{\mathrm{E}}[\sup_{t\in[0,T]}|V_t^l|^\beta]<\infty$, f^l, g_{ij}^l satisfy (H1′) *and* (H2), *$\xi^l\in L_G^\beta(\Omega_T)$ with $\beta>1$. If $\xi^1\geqslant\xi^2$, $f^1\geqslant f^2$, $g_{ij}^1\geqslant g_{ij}^2$, for $i,j=1,\ldots,d$, $V_t^1-V_t^2$ is an increasing process, then $Y_t^1\geqslant Y_t^2$.*

3 Problem of reflected BSDE driven by *G*-Brownian motion and some a priori estimates

For simplicity, we consider the G-expectation space $(\Omega,L_G^1(\Omega_T),\hat{\mathrm{E}})$ with $\Omega_T=C_0([0,T],\mathbb{R})$ and $\bar{\sigma}^2=\hat{\mathrm{E}}[B_1^2]\geqslant-\hat{\mathrm{E}}[-B_1^2]=\underline{\sigma}^2$. Our methods and results still hold for the case $d>1$. We are given the following data: the generators f and g, the obstacle process $\{S_t\}_{t\in[0,T]}$ and the terminal value ξ, where f and g are maps $f(t,\omega,y,z)$, $g(t,\omega,y,z):[0,T]\times\Omega_T\times\mathbb{R}^2\to\mathbb{R}$.

We will make the following assumptions: there exists some $\beta>2$ such that

(H1) for any y,z, $f(\cdot,\cdot,y,z)$, $g(\cdot,\cdot,y,z)\in M_G^\beta(0,T)$;

(H2) $|f(t,\omega,y,z)-f(t,\omega,y',z')|+|g(t,\omega,y,z)-g(t,\omega,y',z')|\leqslant L(|y-y'|+|z-z'|)$ for some $L>0$;

(H3) $\xi\in L_G^\beta(\Omega_T)$ and $\xi\geqslant S_T$, q.s.;

(H4) there exists a constant c such that $\{S_t\}_{t\in[0,T]}\in S_G^\beta(0,T)$ and $S_t\leqslant c$, for each $t\in[0,T]$;

(H4′) $\{S_t\}_{t\in[0,T]}$ has the following form:

$$S_t=S_0+\int_0^t b(s)ds+\int_0^t l(s)d\langle B\rangle_s+\int_0^t\sigma(s)dB_s,$$

where $\{b(t)\}_{t\in[0,T]}$ and $\{l(t)\}_{t\in[0,T]}$ belong to $M_G^\beta(0,T)$ and $\{\sigma(t)\}_{t\in[0,T]}$ belongs to $H_G^\beta(0,T)$.

Let us now introduce our reflected G-BSDE with a lower obstacle. A triplet of processes (Y, Z, A) is called a solution of reflected G-BSDE with a lower obstacle if for some $1 < \alpha \leqslant \beta$ the following properties hold:

(a) $(Y, Z, A) \in \mathcal{S}_G^\alpha(0, T)$ and $Y_t \geqslant S_t$, $0 \leqslant t \leqslant T$;

(b) $Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\langle B\rangle_s - \int_t^T Z_s dB_s + (A_T - A_t)$;

(c) $\{-\int_0^t (Y_s - S_s)dA_s\}_{t\in[0,T]}$ is a nonincreasing G-martingale.

Here, we denote by $\mathcal{S}_G^\alpha(0, T)$ the collection of processes (Y, Z, A) such that $Y \in S_G^\alpha(0, T)$, $Z \in H_G^\alpha(0, T)$, A is a continuous nondecreasing process with $A_0 = 0$ and $A \in S_G^\alpha(0, T)$. For simplicity, we mainly consider the case where $g \equiv 0$ and $l \equiv 0$. Similar results still hold for the cases $g, l \neq 0$. Now, we give a priori estimates for the solution of the reflected G-BSDE with a lower obstacle.

Proposition 3.1. *Let f satisfy* (H1) *and* (H2). *Assume*

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s + (A_T - A_t),$$

where $(Y, Z, A) \in \mathcal{S}_G^\alpha(0, T)$ with $\alpha > 1$. Then, there exists a constant $C := C(\alpha, T, L, \underline{\sigma}) > 0$ such that for each $t \in [0, T]$,

$$\hat{\mathrm{E}}_t\bigg[\bigg(\int_t^T |Z_s|^2 ds\bigg)^{\frac{\alpha}{2}}\bigg] \leqslant C\bigg\{\hat{\mathrm{E}}_t\Big[\sup_{s\in[t,T]}|Y_s|^\alpha\Big] + \Big(\hat{\mathrm{E}}_t\Big[\sup_{s\in[t,T]}|Y_s|^\alpha\Big]\Big)^{1/2}\bigg(\hat{\mathrm{E}}_t\bigg[\bigg(\int_t^T |f(s,0,0)|ds\bigg)^\alpha\bigg]\bigg)^{1/2}\bigg\}, \tag{3.1}$$

$$\hat{\mathrm{E}}_t[|A_T - A_t|^\alpha] \leqslant C\bigg\{\hat{\mathrm{E}}_t\Big[\sup_{s\in[t,T]}|Y_s|^\alpha\Big] + \hat{\mathrm{E}}_t\bigg[\bigg(\int_t^T |f(s,0,0)|ds\bigg)^\alpha\bigg]\bigg\}. \tag{3.2}$$

Proof. The proof is similar to that of Proposition 3.5 in [8]. So we omit it. □

Proposition 3.2. *For $i = 1, 2$, let $\xi^i \in L_G^\beta(\Omega_T)$, f^i satisfy* (H1) *and* (H2) *for some $\beta > 2$. Assume*

$$Y_t^i = \xi^i + \int_t^T f^i(s, Y_s, Z_s)ds - \int_t^T Z_s^i dB_s + (A_T^i - A_t^i),$$

where $(Y^i, Z^i, A^i) \in \mathcal{S}_G^\alpha(0, T)$ for some $1 < \alpha < \beta$. Set $\hat{Y}_t = Y_t^1 - Y_t^2$, $\hat{Z}_t = Z_t^1 - Z_t^2$. Then, there exists a constant $C := C(\alpha, T, L, \underline{\sigma})$ such that

$$\begin{aligned}\hat{\mathrm{E}}\bigg[\bigg(\int_0^T |\hat{Z}|^2 ds\bigg)^{\frac{\alpha}{2}}\bigg] \leqslant C_\alpha\bigg\{&\Big(\hat{\mathrm{E}}\Big[\sup_{t\in[0,T]}|\hat{Y}_t|^\alpha\Big]\Big)^{1/2}\sum_{i=1}^2\bigg[\Big(\hat{\mathrm{E}}\Big[\sup_{t\in[0,T]}|Y_t^i|^\alpha\Big]\Big)^{1/2} \\ &+ \bigg(\hat{\mathrm{E}}\bigg[\bigg(\int_0^T |f^i(s,0,0)|ds\bigg)^\alpha\bigg]\bigg)^{1/2}\bigg] + \hat{\mathrm{E}}\Big[\sup_{t\in[0,T]}|\hat{Y}_t|^\alpha\Big]\bigg\}.\end{aligned}$$

Proof. The proof is similar to that of Proposition 3.8 in [8]. So we omit it. □

Remark 3.3. Note that in the above two propositions, we do not assume (Y, Z, A) and (Y^i, Z^i, A^i), $i = 1, 2$ to be the solutions of reflected G-BSDEs.

Proposition 3.4. *For $i = 1, 2$, let $\xi^i \in L_G^\beta(\Omega_T)$ with $\xi^i \geqslant S_T^i$, where*

$$S_t^i = S_0^i + \int_0^t b^i(s)ds + \int_0^t \sigma^i(s)dB_s.$$

Here, $\{b^i(s)\} \in M_G^\beta(0, T)$, $\{\sigma^i(s)\} \in H_G^\beta(0, T)$ for some $\beta > 2$. Let f^i satisfy (H1) *and* (H2). *Assume that $(Y^i, Z^i, A^i) \in \mathcal{S}_G^\alpha(0, T)$ for some $1 < \alpha < \beta$ are the solutions of the reflected G-BSDEs corresponding to ξ^i, f^i and S^i. Set $\tilde{Y}_t = (Y_t^1 - S_t^1) - (Y_t^2 - S_t^2)$. Then, there exists a constant $C := C(\alpha, T, L, \underline{\sigma})$ such that*

$$|Y_t^i|^\alpha \leqslant C\hat{\mathrm{E}}_t\bigg[|\xi^i|^\alpha + \sup_{s\in[t,T]}|S_s^i|^\alpha + \int_t^T |\bar{\lambda}_s^{i,0}|^\alpha ds\bigg],$$

$$|\tilde{Y}_t|^\alpha \leqslant C\hat{\mathrm{E}}_t\left[|\tilde{\xi}|^\alpha + \int_t^T (|\hat{\lambda}_s|^\alpha + |\hat{\rho}_s|^\alpha + |\hat{S}_s|^\alpha)ds\right],$$

where $\tilde{\xi} = (\xi^1 - S_T^1) - (\xi^2 - S_T^2)$, $\hat{\lambda}_s = |f^1(s, Y_s^2, Z_s^2) - f^2(s, Y_s^2, Z_s^2)|$, $\hat{\rho}_s = |b^1(s) - b^2(s)| + |\sigma^1(s) - \sigma^2(s)|$, $\hat{S}_s = S_s^1 - S_s^2$ *and* $\bar{\lambda}_s^{i,0} = |f^i(s, 0, 0)| + |b^i(s)| + |\sigma^i(s)|$.

Proof. We only show the second inequality, since the first one can be proved in a similar way.

For any $\varepsilon > 0$, set $\hat{f}_t = f^1(t, Y_t^1, Z_t^1) - f^2(t, Y_t^2, Z_t^2)$, $\hat{f}_t^1 = f^1(t, Y_t^1, Z_t^1) - f^1(t, Y_t^2, Z_t^2)$, $\hat{A}_t = A_t^1 - A_t^2$, $\tilde{Z}_t = (Z_t^1 - \sigma^1(t)) - (Z_t^2 - \sigma^2(t))$, $\varepsilon_\alpha = \varepsilon(1 - \alpha/2)^+$ and $\bar{Y}_t = |\tilde{Y}_t|^2 + \varepsilon_\alpha$. Applying Itô's formula to $\bar{Y}_t^{\frac{\alpha}{2}} \mathrm{e}^{rt}$, where $r > 0$ will be determined later, we get

$$\begin{aligned}
&\bar{Y}_t^{\alpha/2}\mathrm{e}^{rt} + \int_t^T r\mathrm{e}^{rs}\bar{Y}_s^{\alpha/2}ds + \int_t^T \frac{\alpha}{2}\mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-1}(\tilde{Z}_s)^2 d\langle B\rangle_s \\
&\quad = (\varepsilon_\alpha + |\tilde{\xi}|^2)^{\alpha/2}\mathrm{e}^{rT} + \alpha\left(1 - \frac{\alpha}{2}\right)\int_t^T \mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-2}(\tilde{Y}_s)^2(\tilde{Z}_s)^2 d\langle B\rangle_s - \int_t^T \alpha\mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-1}\tilde{Y}_s\tilde{Z}_s dB_s \\
&\qquad + \int_t^T \alpha\mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-1}\tilde{Y}_s(\hat{f}_s + b^1(s) - b^2(s))ds + \int_t^T \alpha\mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-1}\tilde{Y}_s d\hat{A}_s \\
&\quad \leqslant (\varepsilon_\alpha + |\tilde{\xi}|^2)^{\alpha/2}\mathrm{e}^{rT} + \int_t^T \alpha\mathrm{e}^{rs}\bar{Y}_s^{\frac{\alpha-1}{2}}\{|\hat{f}_s^1 + b^1(s) - b^2(s)| + \hat{\lambda}_s\}ds \\
&\qquad + \alpha\left(1 - \frac{\alpha}{2}\right)\int_t^T \mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-1}(\tilde{Z}_s)^2 d\langle B\rangle_s - (M_T - M_t),
\end{aligned} \tag{3.3}$$

where $M_t = \int_0^t \alpha\mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-1}(\tilde{Y}_s\tilde{Z}_s dB_s - (\tilde{Y}_s)^+ dA_s^1 - (\tilde{Y}_s)^- dA_s^2)$. We claim that $\{M_t\}$ is a G-martingale. Indeed, note that $\tilde{Y}_t = Y_t^1 - S_t^1 + S_t^2 - Y_t^2 \leqslant Y_t^1 - S_t^1$. Consequently, $(\tilde{Y}_t)^+ \leqslant (Y_t^1 - S_t^1)^+ = Y_t^1 - S_t^1$. Then, we obtain

$$0 \geqslant -\int_t^T (\tilde{Y}_s)^+ dA_s^1 \geqslant -\int_t^T (Y_s^1 - S_s^1)dA_s^1.$$

Thus, we can conclude that

$$0 \geqslant \hat{\mathrm{E}}_t\left[-\int_t^T (\tilde{Y}_s)^+ dA_s^1\right] \geqslant \hat{\mathrm{E}}_t\left[-\int_t^T (Y_s^1 - S_s^1)dA_s^1\right] = 0.$$

It follows that the process $\{K_t^1\} = \{-\int_0^t (\tilde{Y}_s)^+ dA_s^1\}$ is a nonincreasing G-martingale. Set

$$K_t^2 = -\int_0^t (\tilde{Y}_s)^- dA_s^2.$$

Both $\{K_t^1\}$ and $\{K_t^2\}$ are nonincreasing G-martingales, so is $\{\int_0^t \alpha\mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-1}(dK_s^1 + dK_s^2)\}$, which yields that $\{M_t\}_{t\in[0,T]}$ is a G-martingale. From the assumption of f^1, we derive that

$$\begin{aligned}
&\int_t^T \alpha\mathrm{e}^{rs}\bar{Y}_s^{\frac{\alpha-1}{2}}|\hat{f}_s^1 + b^1(s) - b^2(s)|ds \\
&\quad \leqslant \int_t^T \alpha\mathrm{e}^{rs}\bar{Y}_s^{\frac{\alpha-1}{2}}\{L(|\tilde{Y}_s| + |\tilde{Z}_s|) + (L \vee 1)(|\hat{S}_s| + |\hat{\rho}_s|)\}ds \\
&\quad \leqslant \left(\alpha L + \frac{\alpha L^2}{\underline{\sigma}^2(\alpha - 1)}\right)\int_t^T \mathrm{e}^{rs}\bar{Y}_s^{\alpha/2}ds + \frac{\alpha(\alpha - 1)}{4}\int_t^T \mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-1}(\tilde{Z}_s)^2 d\langle B\rangle_s \\
&\qquad + (L \vee 1)\int_t^T \alpha\mathrm{e}^{rs}\bar{Y}_s^{\frac{\alpha-1}{2}}\{|\hat{S}_s| + |\hat{\rho}_s|\}ds.
\end{aligned} \tag{3.4}$$

By Young's inequality, we have

$$\int_t^T \alpha\mathrm{e}^{rs}\bar{Y}_s^{\frac{\alpha-1}{2}}\{|\hat{\lambda}_s| + |\hat{S}_s| + |\hat{\rho}_s|\}ds$$

$$\leqslant 3(\alpha-1)\int_t^T \mathrm{e}^{rs}\bar{Y}_s^{\alpha/2}ds+\int_t^T \mathrm{e}^{rs}\{|\hat{\lambda}_s|^\alpha+|\hat{\rho}_s|^\alpha+|\hat{S}_s|^\alpha\}ds. \tag{3.5}$$

By (3.3)–(3.5) and setting $r=3(L\vee 1)(\alpha-1)+\alpha L+\frac{\alpha L^2}{\underline{\sigma}^2(\alpha-1)}+1$, we get

$$\bar{Y}_t^{\alpha/2}\mathrm{e}^{rt}+(M_T-M_t)\leqslant C\bigg\{(\varepsilon_\alpha+|\tilde{\xi}|^2)^{\alpha/2}\mathrm{e}^{rT}+\int_t^T \mathrm{e}^{rs}(|\hat{\lambda}_s|^\alpha+|\hat{\rho}_s|^\alpha+|\hat{S}_s|^\alpha)ds\bigg\}.$$

Taking conditional expectation on both sides and then letting $\varepsilon\downarrow 0$, we have

$$|\tilde{Y}_t|^\alpha\leqslant C\hat{\mathrm{E}}_t\bigg[|\tilde{\xi}|^\alpha+\int_t^T(|\hat{\lambda}_s|^\alpha+|\hat{\rho}_s|^\alpha+|\hat{S}_s|^\alpha)ds\bigg].$$

The proof is completed. □

Proposition 3.5. *Let (ξ,f,S) satisfy (H1)–(H4). Assume that $(Y,Z,A)\in\mathcal{S}_G^\alpha(0,T)$, for some $2\leqslant\alpha<\beta$, is a solution of the reflected G-BSDE with data (ξ,f,S). Then there exists a constant $C:=C(\alpha,T,L,\underline{\sigma},c)>0$ such that*

$$|Y_t|^\alpha\leqslant C\hat{\mathrm{E}}_t\bigg[1+|\xi|^\alpha+\int_t^T|f(s,0,0)|^\alpha ds\bigg].$$

Proof. For any $r>0$, set $\tilde{Y}_t=|Y_t-c|^2$. Applying Itô's formula to $\tilde{Y}_t^{\alpha/2}\mathrm{e}^{rt}$, noting that $S_t\leqslant c$ and A is a nondecreasing process, we have

$$\begin{aligned}
&\tilde{Y}_t^{\alpha/2}\mathrm{e}^{rt}+\int_t^T r\mathrm{e}^{rs}\tilde{Y}_s^{\alpha/2}ds+\frac{\alpha}{2}\int_t^T \mathrm{e}^{rs}\tilde{Y}_s^{\alpha/2-1}Z_s^2d\langle B\rangle_s\\
&\quad=|\xi-c|^\alpha\mathrm{e}^{rT}+\int_t^T\alpha\mathrm{e}^{rs}\tilde{Y}_s^{\alpha/2-1}(Y_s-c)f(s,Y_s,Z_s)ds+\alpha\left(1-\frac{\alpha}{2}\right)\int_t^T\mathrm{e}^{rs}\tilde{Y}_s^{\alpha/2-2}(Y_s-c)^2Z_s^2\langle B\rangle_s\\
&\qquad-\int_t^T\alpha\mathrm{e}^{rs}\tilde{Y}_s^{\alpha/2-1}(Y_s-c)Z_sdB_s+\int_t^T\alpha\mathrm{e}^{rs}\tilde{Y}_s^{\alpha/2-1}(Y_s-c)dA_s\\
&\quad\leqslant|\xi-c|^\alpha\mathrm{e}^{rT}+\int_t^T\alpha\mathrm{e}^{rs}\tilde{Y}_s^{\frac{\alpha-1}{2}}|f(s,Y_s,Z_s)|ds+\alpha\left(1-\frac{\alpha}{2}\right)\int_t^T\mathrm{e}^{rs}\tilde{Y}_s^{\alpha/2-1}Z_s^2\langle B\rangle_s-(M_T-M_t),
\end{aligned}$$

where $M_t=\int_t^T\alpha\mathrm{e}^{rs}\tilde{Y}_s^{\alpha/2-1}(Y_s-c)Z_sdB_s-\int_t^T\alpha\mathrm{e}^{rs}\tilde{Y}_s^{\alpha/2-1}(Y_s-S_s)dA_s$. By Condition (c), M is a G-martingale. By the assumption of f and Young's inequality, we get

$$\begin{aligned}
\int_t^T\alpha\mathrm{e}^{rs}\tilde{Y}_s^{\frac{\alpha-1}{2}}|f(s,Y_s,Z_s)|ds&\leqslant\int_t^T\alpha\mathrm{e}^{rs}\tilde{Y}_s^{\frac{\alpha-1}{2}}[|f(s,c,0)|+L|\tilde{Y}_s|+L|Z_s|]ds\\
&\leqslant\left(\alpha L+\frac{\alpha L^2}{\underline{\sigma}^2(\alpha-1)}\right)\int_t^T\mathrm{e}^{rs}\tilde{Y}_s^{\alpha/2}ds+(\alpha-1)\int_t^T\mathrm{e}^{rs}\tilde{Y}_s^{\alpha/2}ds\\
&\quad+\frac{\alpha(\alpha-1)}{4}\int_t^T\mathrm{e}^{rs}\tilde{Y}_s^{\alpha/2-1}Z_s^2\langle B\rangle_s+\int_t^T\mathrm{e}^{rs}|f(s,c,0)|^\alpha ds.
\end{aligned}\tag{3.6}$$

Setting $r=\alpha+\alpha L+\frac{\alpha L^2}{\underline{\sigma}^2(\alpha-1)}$ and by the above analysis, we have

$$\tilde{Y}_t^{\alpha/2}\mathrm{e}^{rt}+M_T-M_t\leqslant|\xi-c|^\alpha\mathrm{e}^{rT}+\int_t^T\mathrm{e}^{rs}|f(s,c,0)|^\alpha ds.$$

Taking conditional expectation on both sides yields that

$$|Y_t-c|^\alpha\leqslant C\hat{\mathrm{E}}_t\bigg[|\xi-c|^\alpha+\int_t^T|f(s,c,0)|^\alpha ds\bigg].$$

Noting that for $p\geqslant 1$, we have $|a+b|^p\leqslant 2^{p-1}(|a|^p+|b|^p)$. Then, the proof is completed. □

Proposition 3.6. *Let (ξ^1, f^1, S^1) and (ξ^2, f^2, S^2) be two sets of data, each one satisfying the assumptions (H1)–(H4). Let $(Y^i, Z^i, A^i) \in \mathcal{S}_G^{\alpha}(0,T)$ be the solutions of the reflected G-BSDEs with data (ξ^i, f^i, S^i), $i=1,2$, respectively, with $2 \leqslant \alpha < \beta$. Set $\hat{Y}_t = Y_t^1 - Y_t^2$, $\hat{S}_t = S_t^1 - S_t^2$, $\hat{\xi} = \xi^1 - \xi^2$. Then, there exists a constant $C := C(\alpha, T, L, \underline{\sigma}, c) > 0$ such that*

$$|\hat{Y}_t|^{\alpha} \leqslant C\left\{\hat{\mathrm{E}}_t\left[|\hat{\xi}|^{\alpha} + \int_t^T |\hat{\lambda}_s|^{\alpha} ds\right] + \left(\hat{\mathrm{E}}_t\left[\sup_{s\in[t,T]} |\hat{S}_s|^{\alpha}\right]\right)^{\frac{1}{\alpha}} \Psi_{t,T}^{\frac{\alpha-1}{\alpha}}\right\},$$

where $\hat{\lambda}_s = |f^1(s, Y_s^2, Z_s^2) - f^2(s, Y_s^2, Z_s^2)|$ and

$$\Psi_{t,T} = \sum_{i=1}^{2} \hat{\mathrm{E}}_t\left[\sup_{s\in[t,T]} \hat{\mathrm{E}}_s\left[1 + |\xi^i|^{\alpha} + \int_t^T |f^i(r,0,0)|^{\alpha} dr\right]\right].$$

Proof. Set $\hat{Z}_t = Z_t^1 - Z_t^2$, $\hat{f}_t = f^1(t, Y_t^1, Z_t^1) - f^2(t, Y_t^2, Z_t^2)$ and $\hat{f}_t^1 = f^1(t, Y_t^1, Z_t^1) - f^1(t, Y_t^2, Z_t^2)$. For any $r > 0$, by applying Itô's formula to $\bar{Y}_t^{\alpha/2} \mathrm{e}^{rt} = (|\hat{Y}_t|^2)^{\alpha/2} \mathrm{e}^{rt}$, we have

$$\begin{aligned}
&\bar{Y}_t^{\alpha/2}\mathrm{e}^{rt} + \int_t^T r\mathrm{e}^{rs}\bar{Y}_s^{\alpha/2} ds + \int_t^T \frac{\alpha}{2}\mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-1}(\hat{Z}_s)^2 d\langle B\rangle_s \\
&\quad = |\hat{\xi}|^{\alpha}\mathrm{e}^{rT} + \alpha\left(1-\frac{\alpha}{2}\right)\int_t^T \mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-2}(\hat{Y}_s)^2(\hat{Z}_s)^2 d\langle B\rangle_s - \int_t^T \alpha\mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-1}\hat{Y}_s\hat{Z}_s dB_s \\
&\qquad + \int_t^T \alpha\mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-1}\hat{Y}_s\hat{f}_s ds + \int_t^T \alpha\mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-1}\hat{Y}_s d\hat{A}_s \\
&\quad \leqslant |\hat{\xi}|^{\alpha}\mathrm{e}^{rT} + \alpha\left(1-\frac{\alpha}{2}\right)\int_t^T \mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-1}(\hat{Z}_s)^2 d\langle B\rangle_s + \int_t^T \alpha\mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-1}\hat{S}_s d\hat{A}_s \\
&\qquad + \int_t^T \alpha\mathrm{e}^{rs}\bar{Y}_s^{\frac{\alpha-1}{2}}\{|\hat{f}_s^1| + |\hat{\lambda}_s|\} ds - (M_T - M_t),
\end{aligned} \tag{3.7}$$

where $M_t = \int_0^t \alpha\mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-1}\hat{Y}_s\hat{Z}_s dB_s - \int_0^t \alpha\mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-1}(\hat{Y}_s - \hat{S}_s)^- dA_s^2 - \int_0^t \alpha\mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-1}(\hat{Y}_s - \hat{S}_s)^+ dA_s^1$. By a similar analysis as in the proof of Proposition 3.4, we conclude that $\{M_t\}_{t\in[0,T]}$ is a G-martingale. By Young's inequality and the assumption of f^1, similar to (3.4) and (3.5), we have

$$\begin{aligned}
\int_t^T \alpha\mathrm{e}^{rs}\bar{Y}_s^{\frac{\alpha-1}{2}}\{|\hat{f}_s^1| + |\hat{\lambda}_s|\} ds &\leqslant \frac{\alpha(\alpha-1)}{4}\int_t^T \mathrm{e}^{rs}\tilde{Y}_s^{\alpha/2-1}Z_s^2\langle B\rangle_s + \int_t^T \mathrm{e}^{rs}|\hat{\lambda}_s|^{\alpha} ds \\
&\quad + \left(\alpha - 1 + \alpha L + \frac{\alpha L^2}{\underline{\sigma}^2(\alpha-1)}\right)\int_t^T \mathrm{e}^{rs}\tilde{Y}_s^{\alpha/2} ds.
\end{aligned}$$

Set $r = \alpha + \alpha L + \frac{\alpha L^2}{\underline{\sigma}^2(\alpha-1)}$. Taking conditional expectation on both sides of (3.7), we obtain

$$|\hat{Y}_t|^{\alpha} \leqslant C\left\{\hat{\mathrm{E}}_t\left[|\hat{\xi}|^{\alpha} + \int_t^T |\hat{\lambda}_s|^{\alpha} ds\right] + \hat{\mathrm{E}}_t\left[\int_t^T \bar{Y}_s^{\alpha/2-1}|\hat{S}_s| d(A_s^1 + A_s^2)\right]\right\}.$$

By applying Hölder's inequality, we get

$$\begin{aligned}
\hat{\mathrm{E}}_t\left[\int_t^T \bar{Y}_s^{\alpha/2-1}|\hat{S}_s| d(A_s^1 + A_s^2)\right] &\leqslant \hat{\mathrm{E}}_t\Big[\sup_{s\in[t,T]} \bar{Y}_s^{\alpha/2-1}|\hat{S}_s|(|A_T^1 - A_t^1| + |A_T^2 - A_t^2|)\Big] \\
&\leqslant \left(\hat{\mathrm{E}}_t\Big[\sup_{s\in[0,T]} |\hat{S}_s|^{\alpha}\Big]\right)^{\frac{1}{\alpha}} \left(\hat{\mathrm{E}}_t\Big[\sup_{s\in[t,T]} \bar{Y}_s^{\alpha/2}\Big]\right)^{\frac{\alpha-2}{\alpha}} \left(\sum_{i=1}^{2} \hat{\mathrm{E}}_t[|A_T^i - A_t^i|^{\alpha}]\right)^{\frac{1}{\alpha}}.
\end{aligned}$$

From Propositions 3.1 and 3.5, we finally get the desired result. □

Remark 3.7. One may formulate a solution of reflected G-BSDE as a quadruple $\{(Y_t, Z_t, K_t, L_t), 0 \leqslant t \leqslant T\}$ satisfying Conditions (1)–(3) in the introduction. But the following example shows that the uniqueness is false in this formulation.

Let $f = -1$, $g = 0$, $\xi = 0$ and $S = 0$. It is easy to check that $(0,0,0,t)$ and $(0, 0, \frac{1}{\bar{\sigma}^2 - \underline{\sigma}^2}(\underline{\sigma}^2 t - \langle B\rangle_t), \frac{1}{\bar{\sigma}^2 - \underline{\sigma}^2}(\bar{\sigma}^2 t - \langle B\rangle_t))$ are solutions of the reflected G-BSDE with data $(0, -1, 0, 0)$ satisfying Conditions (1)–(3).

4 Penalized method and convergence properties

In order to derive the existence of the solution to the reflected G-BSDE with a lower obstacle, we apply the approximation method via penalization. In this section, we first state some convergence properties of solutions to the penalized G-BSDEs, which will be needed in the sequel.

For f and ξ satisfying (H1)–(H3), $\{S_t\}_{t\in[0,T]}$ satisfying (H4) or (H4′), we now consider the following family of G-BSDEs parameterized by $n=1,2,\dots,$

$$Y_t^n=\xi+\int_t^T f(s,Y_s^n,Z_s^n)ds+n\int_t^T (Y_s^n-S_s)^-ds-\int_t^T Z_s^n dB_s-(K_T^n-K_t^n). \tag{4.1}$$

Now, let $L_t^n=n\int_0^t(Y_s^n-S_s)^-ds$. Then, $(L_t^n)_{t\in[0,T]}$ is a nondecreasing process. By defining $L_t^n=n\int_0^t(Y_s^n-S_s)^-ds$, the G-BSDE (4.1) is written as

$$Y_t^n=\xi+\int_t^T f(s,Y_s^n,Z_s^n)ds-\int_t^T Z_s^n dB_s-(K_T^n-K_t^n)+(L_T^n-L_t^n). \tag{4.2}$$

We now establish a priori estimates on the sequences (Y^n,Z^n,K^n,L^n).

Lemma 4.1. *There exists a constant $C=C(\alpha,T,L,\underline{\sigma})$ independent of n, such that for $1<\alpha<\beta$,*

$$\hat{\mathrm{E}}\Big[\sup_{t\in[0,T]}|Y_t^n|^\alpha\Big]\leqslant C,\quad \hat{\mathrm{E}}[|K_T^n|^\alpha]\leqslant C,\quad \hat{\mathrm{E}}[|L_T^n|^\alpha]\leqslant C,\quad \hat{\mathrm{E}}\bigg[\bigg(\int_0^T|Z_t^n|^2dt\bigg)^{\frac{\alpha}{2}}\bigg]\leqslant C.$$

Proof. For simplicity, we first consider the case $S\equiv 0$. The proof of the other cases will be given in the remark. For any $r,\varepsilon>0$, set $\tilde{Y}_t=(Y_t^n)^2+\varepsilon_\alpha$, where $\varepsilon_\alpha=\varepsilon(1-\alpha/2)^+$. Note that for each $a\in\mathbb{R}$, $a\times a^-\leqslant 0$. Applying Itô's formula to $\tilde{Y}_t^{\alpha/2}\mathrm{e}^{rt}$ yields that

$$\begin{aligned}
&\tilde{Y}_t^{\alpha/2}\mathrm{e}^{rt}+\int_t^T r\mathrm{e}^{rs}\tilde{Y}_s^{\alpha/2}ds+\int_t^T\frac{\alpha}{2}\mathrm{e}^{rs}\tilde{Y}_s^{\alpha/2-1}(Z_s^n)^2d\langle B\rangle_s\\
&=(|\xi|^2+\varepsilon_\alpha)^{\frac{\alpha}{2}}\mathrm{e}^{rT}+\alpha\Big(1-\frac{\alpha}{2}\Big)\int_t^T\mathrm{e}^{rs}\tilde{Y}_s^{\alpha/2-2}(Y_s^n)^2(Z_s^n)^2d\langle B\rangle_s+\int_t^T\alpha\mathrm{e}^{rs}\tilde{Y}_s^{\alpha/2-1}Y_s^n dL_s^n\\
&\quad+\int_t^T\alpha\mathrm{e}^{rs}\tilde{Y}_s^{\alpha/2-1}Y_s^n f(s,Y_s^n,Z_s^n)ds-\int_t^T\alpha\mathrm{e}^{rs}\tilde{Y}_s^{\alpha/2-1}(Y_s^nZ_s^ndB_s+Y_s^ndK_s^n)\\
&\leqslant(|\xi|^2+\varepsilon_\alpha)^{\frac{\alpha}{2}}\mathrm{e}^{rT}+\alpha\Big(1-\frac{\alpha}{2}\Big)\int_t^T\mathrm{e}^{rs}\tilde{Y}_s^{\alpha/2-2}(Y_s^n)^2(Z_s^n)^2d\langle B\rangle_s\\
&\quad+\int_t^T\alpha\mathrm{e}^{rs}\tilde{Y}_s^{\alpha/2-1/2}|f(s,Y_s^n,Z_s^n)|ds-(M_T-M_t),
\end{aligned}$$

where $M_t=\int_t^T\alpha\mathrm{e}^{rs}\tilde{Y}_s^{\alpha/2-1}(Y_s^nZ_s^ndB_s+(Y_s^n)^+dK_s^n)$ is a G-martingale. Similar to (3.6), we have

$$\begin{aligned}
\int_t^T\alpha\mathrm{e}^{rs}\tilde{Y}_s^{\frac{\alpha-1}{2}}|f(s,Y_s^n,Z_s^n)|ds\leqslant&\int_t^T\mathrm{e}^{rs}|f(s,0,0)|^\alpha ds+\frac{\alpha(\alpha-1)}{4}\int_t^T\mathrm{e}^{rs}\tilde{Y}_s^{\alpha/2-1}(Z_s^n)^2d\langle B\rangle_s\\
&+\bigg(\alpha-1+\alpha L+\frac{\alpha L^2}{\underline{\sigma}^2(\alpha-1)}\bigg)\int_t^T\mathrm{e}^{rs}\tilde{Y}_s^{\alpha/2}ds.
\end{aligned}$$

Set $r=\alpha+\alpha L+\frac{\alpha L^2}{\underline{\sigma}^2(\alpha-1)}$. We derive that

$$\tilde{Y}_t^{\alpha/2}\mathrm{e}^{rt}+M_T-M_t\leqslant(|\xi|^2+\varepsilon_\alpha)^{\frac{\alpha}{2}}\mathrm{e}^{rT}+\int_t^T\mathrm{e}^{rs}|f(s,0,0)|^\alpha ds.$$

Taking conditional expectation on both sides and then letting $\varepsilon\to 0$, we obtain

$$|Y_t^n|^\alpha\leqslant C\hat{\mathrm{E}}_t\bigg[|\xi|^\alpha+\int_t^T|f(s,0,0)|^\alpha ds\bigg].$$

By Theorem 2.4, for $1<\alpha<\beta$, there exists a constant C independent of n such that $\hat{\mathrm{E}}[\sup_{t\in[0,T]}|Y_t^n|^\alpha] \leqslant C$. By Proposition 3.1, we have

$$\hat{\mathrm{E}}\Big[\Big(\int_0^T|Z_s^n|^2ds\Big)^{\frac{\alpha}{2}}\Big]\leqslant C_\alpha\Big\{\hat{\mathrm{E}}\Big[\sup_{t\in[0,T]}|Y_t^n|^\alpha\Big]+\Big(\hat{\mathrm{E}}\Big[\sup_{t\in[0,T]}|Y_t^n|^\alpha\Big]\Big)^{\frac{1}{2}}\Big(\hat{\mathrm{E}}\Big[\Big(\int_0^T|f(s,0,0)|ds\Big)^\alpha\Big]\Big)^{\frac{1}{2}}\Big\},$$

$$\hat{\mathrm{E}}[|L_T^n-K_T^n|^\alpha]\leqslant C_\alpha\Big\{\hat{\mathrm{E}}\Big[\sup_{t\in[0,T]}|Y_t^n|^\alpha\Big]+\hat{\mathrm{E}}\Big[\Big(\int_0^T|f(s,0,0)|ds\Big)^\alpha\Big]\Big\},$$

where the constant C_α depends on $\alpha,T,\underline{\sigma}$ and L. Thus, we conclude that there exists a constant C independent of n, such that for $1<\alpha<\beta$,

$$\hat{\mathrm{E}}\Big[\Big(\int_0^T|Z_t^n|^2dt\Big)^{\frac{\alpha}{2}}\Big]\leqslant C,\quad \hat{\mathrm{E}}[|L_T^n-K_T^n|^\alpha]\leqslant C.$$

Since L_T^n and $-K_T^n$ are non-negative, it follows that

$$\hat{\mathrm{E}}[|K_T^n|^\alpha]\leqslant C,\quad \hat{\mathrm{E}}[|L_T^n|^\alpha]=n^\alpha\hat{\mathrm{E}}\Big[\Big(\int_0^T(Y_s^n)^-s\Big)^\alpha\Big]\leqslant C.$$

This completes the proof. □

Remark 4.2. If the obstacle process $\{S_t\}_{t\in[0,T]}$ satisfies (H4), set $\tilde{Y}_t^n=Y_t^n-c$. It is simple to check that

$$\tilde{Y}_t^n=\xi-c+\int_t^T f(s,\tilde{Y}_s^n+c,Z_s^n)ds+\int_t^T n(\tilde{Y}_s^n-(S_s-c))^-ds-\int_t^T Z_s^ndB_s-(K_T^n-K_t^n).$$

By an analysis similar to the proof of Lemma 4.1, we derive that

$$|\tilde{Y}_t^n|^\alpha\leqslant C\hat{\mathrm{E}}_t\Big[|\xi-c|^\alpha+\int_t^T|f(s,c,0)|^\alpha ds\Big].$$

If S satisfies (H4′), for simplicity we suppose that $l\equiv0$. Let $\widetilde{Y}_t^n=Y_t^n-S_t$ and $\widetilde{Z}_t^n=Z_t^n-\sigma(t)$, we can rewrite (4.1) as follows:

$$\widetilde{Y}_t^n=\xi-S_T+\int_t^T[f(s,\widetilde{Y}_s^n+S_s,\widetilde{Z}_s^n+\sigma(s))+b(s)]ds+n\int_t^T(\widetilde{Y}_s^n)^-ds-\int_t^T\widetilde{Z}_s^ndB_s-(K_T^n-K_t^n).$$

Using the same method, we get $|\widetilde{Y}_t^n|^\alpha\leqslant C\hat{\mathrm{E}}_t[|\xi-S_T|^\alpha+\int_t^T|f(s,S_s,\sigma(s))+b(s)|^\alpha ds]$.

Thus, we conclude that in the above two cases, for $1<\alpha<\beta$, there exists a constant C independent of n such that $\hat{\mathrm{E}}[\sup_{t\in[0,T]}|Y_t^n|^\alpha]\leqslant C$. By Proposition 3.1, we have

$$\hat{\mathrm{E}}[|K_T^n|^\alpha]\leqslant C,\quad \hat{\mathrm{E}}[|L_T^n|^\alpha]=n^\alpha\hat{\mathrm{E}}\Big[\Big(\int_0^T(Y_s^n-S_s)^-ds\Big)^\alpha\Big]\leqslant C,\quad\text{and}\quad \hat{\mathrm{E}}\Big[\Big(\int_0^T|Z_t^n|^2dt\Big)^{\frac{\alpha}{2}}\Big]\leqslant C.$$

Lemma 4.1 implies that $(Y^n-S)^-\to0$ in $M_G^1(0,T)$. The following lemma which corresponds to [4, Lemma 6.1] shows that this convergence holds in $S_G^\alpha(0,T)$, for $1<\alpha<\beta$. This is of vital importance to prove the convergence property for (Y^n).

Lemma 4.3. *For some* $1<\alpha<\beta$, *we have* $\lim_{n\to\infty}\hat{\mathrm{E}}[\sup_{t\in[0,T]}|(Y_t^n-S_t)^-|^\alpha]=0$.

Proof. We now consider the following G-BSDEs parameterized by $n=1,2,\dots,$

$$y_t^n=\xi+\int_t^T f(s,Y_s^n,Z_s^n)ds+\int_t^T n(S_s-y_s^n)ds-\int_t^T z_s^ndB_s-(k_T^n-k_t^n).$$

By applying G-Itô's formula to $\mathrm{e}^{-nt}y_t^n$, we get

$$y_t^n=\mathrm{e}^{nt}\hat{\mathrm{E}}_t\Big[\mathrm{e}^{-nT}\xi+\int_t^T n\mathrm{e}^{-ns}S_sds+\int_t^T\mathrm{e}^{-ns}f(s,Y_s^n,Z_s^n)ds\Big].$$

By Theorem 2.9, we have for all $n \geqslant 1$, $Y_t^n \geqslant Y_t^1$ and

$$Y_t^n - S_t \geqslant y_t^n - S_t = \hat{\mathrm{E}}_t\bigg[\tilde{S}_t^n + \int_t^T \mathrm{e}^{n(t-s)} f(s, Y_s^n, Z_s^n)ds\bigg],$$

where $\tilde{S}_t^n = \mathrm{e}^{n(t-T)}(\xi - S_t) + \int_t^T n\mathrm{e}^{n(t-s)}(S_s - S_t)ds$. It follows that

$$(Y_t^n - S_t)^- \leqslant (y_t^n - S_t)^- \leqslant \hat{\mathrm{E}}_t\bigg[|\tilde{S}_t^n| + \bigg|\int_t^T \mathrm{e}^{n(t-s)} f(s, Y_s^n, Z_s^n)ds\bigg|\bigg].$$

Applying Hölder's inequality yields that

$$\begin{aligned}\bigg|\int_t^T \mathrm{e}^{n(t-s)} f(s, Y_s^n, Z_s^n)ds\bigg| &\leqslant \frac{1}{\sqrt{2n}}\bigg(\int_0^T f^2(s, Y_s^n, Z_s^n)ds\bigg)^{1/2}\\ &\leqslant \frac{C}{\sqrt{n}}\bigg(\sup_{s\in[0,T]}|Y_s^n|^2 + \int_0^T (f^2(s,0,0) + |Z_s^n|^2)ds\bigg)^{1/2}.\end{aligned}$$

By Lemma 4.1, for $1 < \alpha < \beta$, we have

$$\hat{\mathrm{E}}\bigg[\sup_{t\in[0,T]}\bigg|\int_t^T \mathrm{e}^{n(t-s)} f(s, Y_s^n, Z_s^n)ds\bigg|^\alpha\bigg] \to 0, \quad \text{as} \quad n \to \infty. \tag{4.3}$$

For $\varepsilon > 0$, it is straightforward to show that

$$\begin{aligned}|\tilde{S}_t^n| &= \bigg|\mathrm{e}^{n(t-T)}(\xi - S_t) + \int_{t+\varepsilon}^T n\mathrm{e}^{n(t-s)}(S_s - S_t)ds + \int_t^{t+\varepsilon} n\mathrm{e}^{n(t-s)}(S_s - S_t)ds\bigg|\\ &\leqslant \mathrm{e}^{n(t-T)}|\xi - S_t| + \mathrm{e}^{-n\varepsilon}\sup_{s\in[t+\varepsilon,T]}|S_t - S_s| + \sup_{s\in[t,t+\varepsilon]}|S_s - S_t|.\end{aligned}$$

For $T > \delta > 0$, from the above inequality we obtain

$$\begin{aligned}\sup_{t\in[0,T-\delta]}|\tilde{S}_t^n| &\leqslant \mathrm{e}^{-n\delta}\sup_{t\in[0,T-\delta]}|\xi - S_t| + \mathrm{e}^{-n\varepsilon}\sup_{t\in[0,T-\delta]}\sup_{s\in[t+\varepsilon,T]}|S_t - S_s| + \sup_{t\in[0,T-\delta]}\sup_{s\in[t,t+\varepsilon]}|S_s - S_t|\\ &\leqslant \mathrm{e}^{-n\delta}\Big(\sup_{t\in[0,T]}|S_t| + |\xi|\Big) + 2\mathrm{e}^{-n\varepsilon}\sup_{t\in[0,T]}|S_t| + \sup_{t\in[0,T]}\sup_{s\in[t,t+\varepsilon]}|S_s - S_t|.\end{aligned}$$

It is easy to check that for each fixed $\varepsilon, \delta > 0$,

$$\begin{aligned}\hat{\mathrm{E}}\Big[\sup_{t\in[0,T-\delta]}|\tilde{S}_t^n|^\beta\Big] &\leqslant C\Big\{(\mathrm{e}^{-n\beta\varepsilon} + \mathrm{e}^{-n\beta\delta})\hat{\mathrm{E}}\Big[\sup_{t\in[0,T]}|S_t|^\beta + |\xi|^\beta\Big] + \hat{\mathrm{E}}\Big[\sup_{t\in[0,T]}\sup_{s\in[t,t+\varepsilon]}|S_s - S_t|^\beta\Big]\Big\}\\ &\to C\hat{\mathrm{E}}\Big[\sup_{t\in[0,T]}\sup_{s\in[t,t+\varepsilon]}|S_s - S_t|^\beta\Big], \quad \text{as} \quad n \to \infty.\end{aligned} \tag{4.4}$$

For $1 < \alpha < \beta$ and $0 < \delta < T$, we have

$$\begin{aligned}&\hat{\mathrm{E}}\Big[\sup_{t\in[0,T]}|(Y_t^n - S_t)^-|^\alpha\Big]\\ &\quad\leqslant \hat{\mathrm{E}}\Big[\sup_{t\in[0,T-\delta]}|(Y_t^n - S_t)^-|^\alpha\Big] + \hat{\mathrm{E}}\Big[\sup_{t\in[T-\delta,T]}|(Y_t^n - S_t)^-|^\alpha\Big]\\ &\quad\leqslant \hat{\mathrm{E}}\bigg[\sup_{t\in[0,T-\delta]}\bigg\{\hat{\mathrm{E}}_t\bigg[|\tilde{S}_t^n| + \bigg|\int_t^T \mathrm{e}^{n(t-s)} f(s, Y_s^n, Z_s^n)ds\bigg|\bigg]\bigg\}^\alpha\bigg] + \hat{\mathrm{E}}\Big[\sup_{t\in[T-\delta,T]}|(Y_t^1 - S_t)^-|^\alpha\Big]\\ &\quad\leqslant C\bigg\{\hat{\mathrm{E}}\Big[\sup_{t\in[0,T-\delta]}\hat{\mathrm{E}}_t\Big[\sup_{u\in[0,T-\delta]}|\tilde{S}_u^n|^\alpha\Big]\Big] + \hat{\mathrm{E}}\bigg[\sup_{t\in[0,T-\delta]}\hat{\mathrm{E}}_t\bigg[\sup_{u\in[0,T]}\bigg|\int_u^T \mathrm{e}^{n(t-s)} f(s, Y_s^n, Z_s^n)ds\bigg|^\alpha\bigg]\bigg]\bigg\}\\ &\qquad + \hat{\mathrm{E}}\Big[\sup_{t\in[T-\delta,T]}|(Y_t^1 - S_t)^-|^\alpha\Big] =: I + \hat{\mathrm{E}}\Big[\sup_{t\in[T-\delta,T]}|(Y_t^1 - S_t)^-|^\alpha\Big].\end{aligned} \tag{4.5}$$

By Lemma 2.7, noting that $Y^1 - S \in S_G^\alpha(0,T)$ and $(Y_T^1 - S_T)^- = 0$, we obtain $\lim_{\delta\to 0}\hat{\mathrm{E}}[\sup_{t\in[T-\delta,T]}|(Y_t^1 - S_t)^-|^\alpha] = 0$. By Theorem 2.4, (4.3) and (4.4), we derive that

$$I \leqslant C\Big\{\hat{\mathrm{E}}\Big[\sup_{t\in[0,T]}\sup_{s\in[t,t+\varepsilon]}|S_s - S_t|^\beta\Big] + \Big(\hat{\mathrm{E}}\Big[\sup_{t\in[0,T]}\sup_{s\in[t,t+\varepsilon]}|S_s - S_t|^\beta\Big]\Big)^{\alpha/\beta}\Big\}, \quad \text{as} \quad n\to\infty.$$

Now, first let $n\to\infty$ and then let $\varepsilon,\delta\to 0$ in (4.5). By Lemma 2.7, the above analysis again proves that for $1<\alpha<\beta$, $\lim_{n\to\infty}\hat{\mathrm{E}}[\sup_{t\in[0,T]}|(Y_t^n - S_t)^-|^\alpha] = 0$. The proof is completed. □

Now, we show the convergence property of sequence $(Y^n)_{n=1}^\infty$.

Lemma 4.4. *For some $\beta > \alpha \geqslant 2$, we have* $\lim_{n,m\to\infty}\hat{\mathrm{E}}[\sup_{t\in[0,T]}|Y_t^n - Y_t^m|^\alpha] = 0$.

Proof. Without loss of generality, we may assume $S\equiv 0$ in (4.1). For any $r>0$, set $\hat{Y}_t = Y_t^n - Y_t^m$, $\hat{Z}_t = Z_t^n - Z_t^m$, $\hat{K}_t = K_t^n - K_t^m$, $\hat{L}_t = L_t^n - L_t^m$, $\bar{Y}_t = |\hat{Y}_t|^2$ and $\hat{f}_t = f(t,Y_t^n,Z_t^n) - f(t,Y_t^m,Z_t^m)$. By applying Itô's formula to $\bar{Y}_t^{\alpha/2}\mathrm{e}^{rt}$, we get

$$\begin{aligned}
&\bar{Y}_t^{\alpha/2}\mathrm{e}^{rt} + \int_t^T r\mathrm{e}^{rs}\bar{Y}_s^{\alpha/2}ds + \int_t^T \frac{\alpha}{2}\mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-1}(\hat{Z}_s)^2 d\langle B\rangle_s\\
&\quad = \alpha\Big(1-\frac{\alpha}{2}\Big)\int_t^T \mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-2}(\hat{Y}_s)^2(\hat{Z}_s)^2 d\langle B\rangle_s + \int_t^T \alpha\mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-1}\hat{Y}_s d\hat{L}_s\\
&\qquad + \int_t^T \alpha\mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-1}\hat{Y}_s\hat{f}_s ds - \int_t^T \alpha\mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-1}(\hat{Y}_s\hat{Z}_s dB_s + \hat{Y}_s d\hat{K}_s)\\
&\quad \leqslant \alpha\Big(1-\frac{\alpha}{2}\Big)\int_t^T \mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-2}(\hat{Y}_s)^2(\hat{Z}_s)^2 d\langle B\rangle_s + \int_t^T \alpha\mathrm{e}^{rs}\bar{Y}_s^{\frac{\alpha-1}{2}}|\hat{f}_s|ds\\
&\qquad - \int_t^T \alpha\mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-1}Y_s^n dL_s^m - \int_t^T \alpha\mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-1}Y_s^m dL_s^n - (M_T - M_t),
\end{aligned}$$

where $M_t = \int_0^t \alpha\mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-1}(\hat{Y}_s\hat{Z}_s dB_s + (\hat{Y}_s)^+ dK_s^m + (\hat{Y}_s)^- dK_s^n)$ is a G-martingale. Similar to (3.4), we have

$$\int_t^T \alpha\mathrm{e}^{rs}\bar{Y}_s^{\frac{\alpha-1}{2}}|\hat{f}_s|ds \leqslant \Big(\alpha L + \frac{\alpha L^2}{\underline{\sigma}^2(\alpha-1)}\Big)\int_t^T \mathrm{e}^{rs}\bar{Y}_s^{\alpha/2}ds + \frac{\alpha(\alpha-1)}{4}\int_t^T \mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-1}(\hat{Z}_s)^2 d\langle B\rangle_s.$$

Let $r = 1 + \alpha L + \frac{\alpha L^2}{\underline{\sigma}^2(\alpha-1)}$. By the above analysis, we have

$$\bar{Y}_t^{\alpha/2}\mathrm{e}^{rt} + (M_T - M_t) \leqslant -\int_t^T \alpha\mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-1}Y_s^n dL_s^m - \int_t^T \alpha\mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-1}Y_s^m dL_s^n.$$

Taking conditional expectation on both sides of the above inequality, we conclude that

$$\bar{Y}_t^{\alpha/2}\mathrm{e}^{rt} \leqslant \hat{\mathrm{E}}_t\Big[-\int_t^T \alpha\mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-1}Y_s^n dL_s^m - \int_t^T \alpha\mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-1}Y_s^m dL_s^n\Big]. \tag{4.6}$$

Observe that

$$\begin{aligned}
\hat{\mathrm{E}}_t\Big[-\int_t^T \alpha\mathrm{e}^{rs}\bar{Y}_s^{\alpha/2-1}Y_s^m dL_s^n\Big] &\leqslant \alpha\mathrm{e}^{rT}\hat{\mathrm{E}}_t\Big[\int_t^T \bar{Y}_s^{\alpha/2-1}n(Y_s^n)^-(Y_s^m)^- ds\Big]\\
&\leqslant C\hat{\mathrm{E}}_t\Big[\int_0^T n|(Y_s^n)^-|^{\alpha-1}(Y_s^m)^- ds\Big] + C\hat{\mathrm{E}}_t\Big[\int_0^T n|(Y_s^m)^-|^{\alpha-1}(Y_s^n)^- ds\Big].
\end{aligned}$$

From (4.6) and taking expectation on both sides, we deduce that

$$\begin{aligned}
\hat{\mathrm{E}}\Big[\sup_{t\in[0,T]}|Y_t^n - Y_t^m|^\alpha\Big] \leqslant C\hat{\mathrm{E}}\Big[\sup_{t\in[0,T]}\Big\{&\hat{\mathrm{E}}_t\Big[\int_0^T (n+m)|(Y_s^n)^-|^{\alpha-1}(Y_s^m)^- ds\Big]\\
&+ \hat{\mathrm{E}}_t\Big[\int_0^T (n+m)|(Y_s^m)^-|^{\alpha-1}(Y_s^n)^- ds\Big]\Big\}\Big].
\end{aligned} \tag{4.7}$$

For $2 \leqslant \alpha < \beta$, there exist α', p, q, r, p', $q' > 1$, such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, $\frac{1}{p'} + \frac{1}{q'} = 1$, $(\alpha - 2)\alpha' p < \beta$, $\alpha' q < \beta$, $\alpha' r < \beta$, $(\alpha - 1)\alpha' p' < \beta$ and $\alpha' q' < \beta$. Applying Lemmas 4.1, 4.3 and Hölder's inequality, there exists a constant C independent of m and n such that

$$\begin{aligned}
&\hat{\mathrm{E}}\bigg[\bigg(\int_0^T n|(Y_s^n)^-|^{\alpha-1}(Y_s^m)^- ds\bigg)^{\alpha'}\bigg] \\
&\quad\leqslant \hat{\mathrm{E}}\bigg[\sup_{s\in[0,T]}\{|(Y_s^n)^-|^{(\alpha-2)\alpha'}|(Y_s^m)^-|^{\alpha'}\}\bigg(\int_0^T n(Y_s^n)^- ds\bigg)^{\alpha'}\bigg] \\
&\quad\leqslant \bigg(\hat{\mathrm{E}}\bigg[\sup_{s\in[0,T]}|(Y_s^n)^-|^{(\alpha-2)\alpha' p}\bigg]\bigg)^{\frac{1}{p}}\bigg(\hat{\mathrm{E}}\bigg[\sup_{s\in[0,T]}|(Y_s^m)^-|^{\alpha' q}\bigg]\bigg)^{\frac{1}{q}}\bigg(\hat{\mathrm{E}}\bigg[\bigg(\int_0^T n(Y_s^n)^- ds\bigg)^{\alpha' r}\bigg]\bigg)^{\frac{1}{r}} \\
&\quad\leqslant C\bigg(\hat{\mathrm{E}}\bigg[\sup_{s\in[0,T]}|(Y_s^m)^-|^{\alpha' q}\bigg]\bigg)^{\frac{1}{q}},
\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}
\hat{\mathrm{E}}\bigg[\bigg(\int_0^T m|(Y_s^n)^-|^{\alpha-1}(Y_s^m)^- ds\bigg)^{\alpha'}\bigg] &\leqslant \hat{\mathrm{E}}\bigg[\sup_{s\in[0,T]}|(Y_s^n)^-|^{(\alpha-1)\alpha'}\bigg(\int_0^T m(Y_s^m)^- ds\bigg)^{\alpha'}\bigg] \\
&\leqslant \bigg(\hat{\mathrm{E}}\bigg[\sup_{s\in[0,T]}|(Y_s^n)^-|^{(\alpha-1)\alpha' p'}\bigg]\bigg)^{\frac{1}{p'}}\bigg(\hat{\mathrm{E}}\bigg[\bigg(\int_0^T m(Y_s^m)^- ds\bigg)^{\alpha' q'}\bigg]\bigg)^{\frac{1}{q'}} \\
&\leqslant C\bigg(\hat{\mathrm{E}}\bigg[\sup_{s\in[0,T]}|(Y_s^n)^-|^{(\alpha-1)\alpha' p'}\bigg]\bigg)^{\frac{1}{p'}}.
\end{aligned} \tag{4.9}$$

Then, by Theorem 2.4 and Lemma 4.3, (4.7)–(4.9) yield that

$$\lim_{n,m\to\infty}\hat{\mathrm{E}}\bigg[\sup_{t\in[0,T]}|Y_t^n - Y_t^m|^{\alpha}\bigg] = 0.$$

The proof is completed. □

5 Existence and uniqueness of reflected G-BSDE with a lower obstacle

Theorem 5.1. *Suppose that ξ, f satisfy* (H1)–(H3) *and S satisfies* (H4) *or* (H4′)*. Then, the reflected G-BSDE with data (ξ, f, S) has a unique solution (Y, Z, A). Moreover, for any $2 \leqslant \alpha < \beta$ we have $Y \in S_G^\alpha(0,T)$, $Z \in H_G^\alpha(0,T)$ and $A \in S_G^\alpha(0,T)$.*

Proof. The uniqueness of the solution is a direct consequence of the priori estimates in Propositions 3.2, 3.4 and 3.6.

To prove the existence, it suffices to prove the $S \equiv 0$ case. Recalling penalized G-BSDEs (4.1), set $\hat{Y}_t = Y_t^n - Y_t^m$, $\hat{Z}_t = Z_t^n - Z_t^m$, $\hat{K}_t = K_t^n - K_t^m$, $\hat{L}_t = L_t^n - L_t^m$ and $\hat{f}_t = f(t, Y_t^n, Z_t^n) - f(t, Y_t^m, Z_t^m)$. By Lemma 4.4, there exists $Y \in S_G^\alpha(0,T)$ satisfying $\lim_{n\to\infty}\hat{\mathrm{E}}[\sup_{t\in[0,T]}|Y_t - Y_t^n|^\alpha] = 0$. Applying Itô's formula to $|\hat{Y}_t|^2$, we get

$$\begin{aligned}
|\hat{Y}_t|^2 + \int_t^T |\hat{Z}_s|^2 d\langle B\rangle_s &= \int_t^T 2\hat{Y}_s\hat{f}_s ds - \int_t^T 2\hat{Y}_s d\hat{K}_s + \int_t^T 2\hat{Y}_s d\hat{L}_s - \int_t^T 2\hat{Y}_s\hat{Z}_s dB_s \\
&\leqslant 2L\int_t^T [|\hat{Y}_s|^2 + |\hat{Y}_s||\hat{Z}_s|]ds - \int_t^T 2\hat{Y}_s d\hat{K}_s + \int_t^T 2\hat{Y}_s d\hat{L}_s - \int_t^T 2\hat{Y}_s\hat{Z}_s dB_s.
\end{aligned}$$

Note that for each $\varepsilon > 0$,

$$2L\int_t^T |\hat{Y}_s||\hat{Z}_s|ds \leqslant L^2/\varepsilon\int_t^T |\hat{Y}_s|^2 ds + \varepsilon\int_t^T |\hat{Z}_s|^2 ds.$$

Choosing $\varepsilon < \underline{\sigma}^2$, we have

$$\begin{aligned}\int_0^T |\hat{Z}_s|^2 ds &\leqslant C\bigg(\int_0^T |\hat{Y}_s|^2 ds - \int_0^T \hat{Y}_s d\hat{K}_s + \int_0^T \hat{Y}_s d\hat{L}_s - \int_0^T \hat{Y}_s \hat{Z}_s dB_s\bigg)\\ &\leqslant C\bigg(\sup_{s\in[0,T]} |\hat{Y}_s|^2 + \sup_{s\in[0,T]} |\hat{Y}_s|(|K_T^n| + |K_T^m| + |L_T^n| + |L_T^m|) - \int_0^T \hat{Y}_s \hat{Z}_s dB_s\bigg). \end{aligned} \tag{5.1}$$

By Proposition 2.6, for any $\varepsilon' > 0$, we obtain

$$\begin{aligned}\hat{\mathrm{E}}\bigg[\bigg(\int_0^T \hat{Y}_s \hat{Z}_s dB_s\bigg)^{\frac{\alpha}{2}}\bigg] &\leqslant C\hat{\mathrm{E}}\bigg[\bigg(\int_0^T \hat{Y}_s^2 \hat{Z}_s^2 ds\bigg)^{\frac{\alpha}{4}}\bigg]\\ &\leqslant C\Big(\hat{\mathrm{E}}\Big[\sup_{t\in[0,T]} |\hat{Y}_t|^\alpha\Big]\Big)^{1/2}\bigg(\hat{\mathrm{E}}\bigg[\bigg(\int_0^T |\hat{Z}_s|^2 ds\bigg)^{\frac{\alpha}{2}}\bigg]\bigg)^{1/2}\\ &\leqslant \frac{C}{4\varepsilon'}\hat{\mathrm{E}}\Big[\sup_{t\in[0,T]} |\hat{Y}_t|^\alpha\Big] + C\varepsilon'\hat{\mathrm{E}}\bigg[\bigg(\int_0^T |\hat{Z}_s|^2 ds\bigg)^{\frac{\alpha}{2}}\bigg].\end{aligned}$$

Applying Lemma 4.1 and Hölder's inequality, choosing a small enough ε', it follows from (5.1) that

$$\hat{\mathrm{E}}\bigg[\bigg(\int_0^T |Z_s^n - Z_s^m|^2 ds\bigg)^{\frac{\alpha}{2}}\bigg] \leqslant C\Big\{\hat{\mathrm{E}}\Big[\sup_{t\in[0,T]} |\hat{Y}_t|^\alpha\Big] + \Big(\hat{\mathrm{E}}\Big[\sup_{t\in[0,T]} |\hat{Y}_t|^\alpha\Big]\Big)^{1/2}\Big\}.$$

It is straightforward to show that $\lim_{n,m\to\infty} \hat{\mathrm{E}}[(\int_0^T |Z_s^n - Z_s^m|^2 ds)^{\frac{\alpha}{2}}] = 0$. Then, there exists a process $\{Z_t\} \in H_G^\alpha(0,T)$ such that $\hat{\mathrm{E}}[(\int_0^T |Z_s - Z_s^n|^2 ds)^{\alpha/2}] \to 0$ as $n \to \infty$. Set $A_t^n = L_t^n - K_t^n$. It is easy to check that $(A_t^n)_{t\in[0,T]}$ is a nondecreasing process and

$$A_t^n - A_t^m = \hat{Y}_0 - \hat{Y}_t - \int_0^t \hat{f}_s ds + \int_0^t \hat{Z}_s dB_s.$$

By applying Proposition 2.6 and the assumption of f, it follows that

$$\begin{aligned}\hat{\mathrm{E}}\Big[\sup_{t\in[0,T]} |A_t^n - A_t^m|^\alpha\Big] &\leqslant C\hat{\mathrm{E}}\bigg[\sup_{t\in[0,T]} |\hat{Y}_t|^\alpha + \bigg(\int_0^T |\hat{f}_s| ds\bigg)^\alpha + \sup_{t\in[0,T]}\bigg|\int_0^t \hat{Z}_s dB_s\bigg|^\alpha\bigg]\\ &\leqslant C\bigg\{\hat{\mathrm{E}}\Big[\sup_{t\in[0,T]} |\hat{Y}_t|^\alpha\Big] + \hat{\mathrm{E}}\bigg[\bigg(\int_0^T |\hat{Z}_s|^2 ds\bigg)^{\alpha/2}\bigg]\bigg\} \to 0, \quad \text{as} \quad n, m \to \infty.\end{aligned}$$

Then, there exists a nondecreasing process $(A_t)_{t\in[0,T]}$ satisfying that $\lim_{n\to\infty} \hat{\mathrm{E}}[\sup_{t\in[0,T]} |A_t - A_t^n|^\alpha] = 0$.

In the following, it remains to prove that $Y_t \geqslant 0$, $t \in [0,T]$ and $\{-\int_0^t Y_s dA_s\}_{t\in[0,T]}$ is a nonincreasing G-martingale. For the first statement, it can be easily deduced from Lemma 4.3. Set $\widetilde{K}_t^n := \int_0^t Y_s dK_s^n$. Since $Y_t \geqslant 0$, for any $0 \leqslant t \leqslant T$ and K^n is a decreasing G-martingale, then $\widetilde{K}^n$ is a decreasing G-martingale. Note that

$$\begin{aligned}\sup_{t\in[0,T]}\bigg|-\int_0^t Y_s dA_s - \widetilde{K}_t^n\bigg| &\leqslant \sup_{t\in[0,T]}\bigg\{\bigg|-\int_0^t Y_s dA_s + \int_0^t Y_s dA_s^n\bigg| + \bigg|\int_0^t (Y_s^n - Y_s) dA_s^n\bigg|\\ &\quad + \bigg|\int_0^t (Y_s^n - Y_s) dK_s^n\bigg| + \bigg|\int_0^t -Y_s^n n(Y_s^n)^- ds\bigg|\bigg\}\\ &\leqslant \sup_{t\in[0,T]}\bigg\{\bigg|\int_0^t \widetilde{Y}_s^m d(A_s^n - A_s)\bigg| + \bigg|\int_0^t (Y_s - \widetilde{Y}_s^m) d(A_s^n - A_s)\bigg|\bigg\}\\ &\quad + \sup_{t\in[0,T]} |Y_s - Y_s^n|[|A_T^n| + |K_T^n|] + \sup_{t\in[0,T]} (Y_s^n)^-|L_T^n|\\ &=: I + II + III + IV,\end{aligned}$$

where $\widetilde{Y}_t^m=\sum_{i=0}^{m-1}Y_{t_i^m}I_{[t_i^m,t_{i+1}^m)}(t)$ and $t_i^m=\frac{iT}{m}$, $i=0,1,\dots,m$. By a simple calculation, we have

$$\begin{aligned}
\hat{\mathrm{E}}[I]&\leqslant\sum_{i=0}^{m-1}\hat{\mathrm{E}}\Big[\sup_{s\in[0,T]}|Y_s|(|A^n_{t^m_{i+1}}-A_{t^m_{i+1}}|+|A^n_{t^m_i}-A_{t^m_i}|)\Big]\\
&\leqslant\Big(\hat{\mathrm{E}}\Big[\sup_{s\in[0,T]}|Y_s|^2\Big]\Big)^{1/2}\sum_{i=0}^{m-1}\{(\hat{\mathrm{E}}[|A^n_{t^m_{i+1}}-A_{t^m_{i+1}}|^2])^{1/2}+(\hat{\mathrm{E}}[|A^n_{t^m_i}-A_{t^m_i}|^2])^{1/2}\},\\
\hat{\mathrm{E}}[II]&\leqslant\Big(\hat{\mathrm{E}}\Big[\sup_{s\in[0,T]}|Y_s-\widetilde{Y}^m_s|^2\Big]\Big)^{1/2}\{(\hat{\mathrm{E}}[|A^n_T|^2])^{1/2}+(\hat{\mathrm{E}}[|A_T|^2])^{1/2}\},\\
\hat{\mathrm{E}}[III]&\leqslant\Big(\hat{\mathrm{E}}\Big[\sup_{s\in[0,T]}|Y_s-Y^n_s|^2\Big]\Big)^{1/2}\{(\hat{\mathrm{E}}[|A^n_T|^2])^{1/2}+(\hat{\mathrm{E}}[|K^n_T|^2])^{1/2}\},\\
\hat{\mathrm{E}}[IV]&\leqslant\Big(\hat{\mathrm{E}}\Big[\sup_{s\in[0,T]}|(Y^n_s)^-|^2\Big]\Big)^{1/2}(\hat{\mathrm{E}}[|L^n_T|^2])^{1/2}.
\end{aligned}$$

Then, for each fixed m, letting n approach infinity, we conclude that

$$\lim_{n\to\infty}\hat{\mathrm{E}}\Big[\sup_{t\in[0,T]}\Big|-\int_0^t Y_s dA_s-\widetilde{K}^n_t\Big|\Big]\leqslant C\Big(\hat{\mathrm{E}}\Big[\sup_{t\in[0,T]}|Y_s-\widetilde{Y}^m_s|^2\Big]\Big)^{1/2}.$$

By [8, Lemma 3.2], letting m approach infinity, we get $\lim_{n\to\infty}\hat{\mathrm{E}}[\sup_{t\in[0,T]}|-\int_0^t Y_s dA_s-\widetilde{K}^n_t|]=0$. It follows that $\{-\int_0^t Y_s dA_s\}$ is a nonincreasing G-martingale. □

Furthermore, we have the following result.

Theorem 5.2. *Suppose that ξ, f and g satisfy* (H1)–(H3)*, S satisfies* (H4) *or* (H4′)*. Then, the reflected G-BSDE with data (ξ,f,g,S) has a unique solution (Y,Z,A). Moreover, for any $2\leqslant\alpha<\beta$ we have $Y\in S^\alpha_G(0,T)$, $Z\in H^\alpha_G(0,T)$ and $A\in S^\alpha_G(0,T)$.*

Proof. The proof is similar to that of Theorem 5.1. □

We next prove a comparison theorem, similar to that of [9] for non-reflected G-BSDEs. The proof is based on the approximation method via penalization.

Theorem 5.3. *Let (ξ^1,f^1,g^1,S^1) and (ξ^2,f^2,g^2,S^2) be two sets of data. Suppose S^i satisfies* (H4) *or* (H4′)*, and ξ^i, f^i and g^i satisfy* (H1)–(H3) *for $i=1,2$. We furthermore assume the following*:

(i) $\xi^1\leqslant\xi^2$, *q.s.*;

(ii) $f^1(t,y,z)\leqslant f^2(t,y,z)$, $g^1(t,y,z)\leqslant g^2(t,y,z)$, $\forall\,(y,z)\in\mathbb{R}^2$;

(iii) $S^1_t\leqslant S^2_t$, $0\leqslant t\leqslant T$, *q.s.*

Let (Y^i,Z^i,A^i) be the solutions of the reflected G-BSDE with data (ξ^i,f^i,g^i,S^i), $i=1,2$, respectively. Then

$$Y^1_t\leqslant Y^2_t,\quad 0\leqslant t\leqslant T\quad q.s.$$

Proof. We consider the following G-BSDEs parameterized by $n=1,2,\dots,$

$$y^n_t=\xi^1+\int_t^T f^1(s,y^n_s,z^n_s)ds+\int_t^T g^1(s,y^n_s,z^n_s)d\langle B\rangle_s+\int_t^T n(y^n_s-S^1_s)^-ds-\int_t^T z^n_s dB_s-(K^n_T-K^n_t).$$

By an analysis similar to the proof of Theorem 5.1, it follows that $\lim_{n\to\infty}\hat{\mathrm{E}}[\sup_{t\in[0,T]}|Y^1_t-y^n_t|^\alpha]=0$, where $2\leqslant\alpha<\beta$. Noting that (Y^2,Z^2,A^2) is the solution of the reflected G-BSDE with data (ξ^2,f^2,g^2,S^2) and $Y^2_t\geqslant S^2_t$, $0\leqslant t\leqslant T$, we have

$$Y^2_t=\xi^2+\int_t^T f^2(s,Y^2_s,Z^2_s)ds+\int_t^T g^2(s,Y^2_s,Z^2_s)d\langle B\rangle_s+\int_t^T n(Y^2_s-S^2_s)^-ds-\int_t^T Z^2_s dB_s+(A^2_T-A^2_t).$$

Applying Theorem 2.9 yields $Y^2_t\geqslant y^n_t$, for all $n\in\mathbb{N}$. Letting $n\to\infty$, we conclude that $Y^2_t\geqslant Y^1_t$. □

Remark 5.4. Actually, the process A can be represented as the sum of two nondecreasing processes A^1 and A^2 such that $\int_0^T(Y_s - S_s)dA_s^2 = 0$, and for any $0 \leqslant s \leqslant t \leqslant T$,

$$\hat{\mathrm{E}}_s\left[\int_0^t (S_r - Y_r)dA_r^1\right] = \int_0^s (S_r - Y_r)dA_r^1. \tag{5.2}$$

Indeed, set $A_t^1 = \int_0^t I_{\{Y_s > S_s\}}dA_s$, $A_t^2 = \int_0^t I_{\{Y_s = S_s\}}dA_s$. It is easy to check that $A = A^1 + A^2$ and A^2 satisfies the Skorohod condition. We now show that A^1 satisfies (5.2). Set $\widetilde{K}_t := \int_0^t (S_s - Y_s)dA_s$. By Theorem 5.1, $\widetilde{K}$ is a decreasing G-martingale and $\widetilde{K}_t \in L_G^p(\Omega_t)$ for some $1 < p < \frac{\beta}{2}$, $\forall\, t \in [0,T]$. Choose a sequence of bounded, non-negative and Lipschitz continuous functions $(\varphi_n(x))_{n=1}^\infty$ such that $\varphi_n(x) \uparrow I_{\{x>0\}}$. Set

$$\bar{K}_t^n := \int_0^t \varphi_n(Y_s - S_s)d\widetilde{K}_s = \int_0^t (S_s - Y_s)\varphi_n(Y_s - S_s)dA_s.$$

Applying [8, Lemma 3.4], we obtain that $\bar{K}^n$ is a decreasing G-martingale. Furthermore, we have

$$\int_0^t \varphi_n(Y_s - S_s)d\widetilde{K}_s \downarrow \int_0^t (S_s - Y_s)dA_s^1 \in L_G^{1^*}(\Omega_t),$$

where $L_G^{1^*}(\Omega_t)$ is defined in Appendix A. By the extended conditional G-expectation defined in [11], we derive that

$$\begin{aligned}
\hat{\mathrm{E}}_s\left[\int_0^t (S_r - Y_r)dA_r^1\right] &= \lim_{n\to\infty} \hat{\mathrm{E}}_s\left[\int_0^t (S_r - Y_r)\varphi_n(Y_r - S_r)dA_r\right] \\
&= \lim_{n\to\infty} \int_0^s (S_r - Y_r)\varphi_n(Y_r - S_r)dA_r \\
&= \int_0^s (S_r - Y_r)dA_r^1.
\end{aligned}$$

6 Relation between reflected G-BSDEs and obstacle problems for nonlinear parabolic PDEs

In this section, we give a probabilistic representation for the solution of an obstacle problem for a fully nonlinear parabolic PDE using the reflected G-BSDE mentioned in the above sections. For this purpose, we put the reflected G-BSDE in a nonlinear Markovian framework.

For each $0 \leqslant t \leqslant T$ and $\xi \in L_G^p(\Omega_t; \mathbb{R}^d)$, $p \geqslant 2$, let $\{X_s^{t,\xi}, t \leqslant s \leqslant T\}$ be the unique $\mathbb{R}^d$-valued solution of the SDE driven by G-Brownian motion (here we use the Einstein convention):

$$X_s^{t,\xi} = x + \int_t^s b(r, X_r^{t,\xi})dr + \int_t^s l_{ij}(r, X_r^{t,\xi})d\langle B^i, B^j\rangle_r + \int_t^s \sigma_i(r, X_r^{t,\xi})dB_r^i. \tag{6.1}$$

For any $(t,x) \in [0,T] \times \mathbb{R}^d$, we assume that the data $(\xi^{t,x}, f^{t,x}, g^{t,x}, S^{t,x})$ of the reflected G-BSDE takes the following form:

$$\begin{aligned}
&\xi^{t,x} = \phi(X_T^{t,x}), \quad f^{t,x}(s,y,z) = f(s, X_s^{t,x}, y, z), \\
&S_s^{t,x} = h(s, X_s^{t,x}), \quad g_{ij}^{t,x}(s,y,z) = g_{ij}(s, X_s^{t,x}, y, z),
\end{aligned}$$

where $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, $l_{ij} : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma_i : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\phi : \mathbb{R}^d \to \mathbb{R}$, $f, g_{ij} : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and $h : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ are deterministic functions and satisfy the following conditions:

(A1) $l_{ij} = l_{ji}$ and $g_{ij} = g_{ji}$ for $1 \leqslant i, j \leqslant d$;

(A2) b, l_{ij}, σ_i, f, g_{ij} and h are continuous in t;

(A3) there exist a positive integer m and a constant L such that

$$
\begin{aligned}
&|b(t,x)-b(t,x')|+\sum_{i,j=1}^{d}|l_{ij}(t,x)-l_{ij}(t,x')|+\sum_{i=1}^{d}|\sigma_i(t,x)-\sigma_i(t,x')|\leqslant L|x-x'|,\\
&|\phi(x)-\phi(x')|\leqslant L(1+|x|^m+|x'|^m)|x-x'|,\\
&|f(t,x,y,z)-f(t,x',y',z')|+\sum_{i.j=1}^{d}|g_{ij}(t,x,y,z)-g_{ij}(t,x',y',z')|\\
&\quad\leqslant L[(1+|x|^m+|x'|^m)|x-x'|+|y-y'|+|z-z'|];
\end{aligned}
$$

(A4) h is Lipschitz continuous w.r.t. x and bounded from above, $h(T,x)\leqslant\phi(x)$ for any $x\in\mathbb{R}^d$;

(A4′) h belongs to the space $C^{1,2}_{\rm Lip}([0,T]\times\mathbb{R}^d)$ and $h(T,x)\leqslant\phi(x)$ for any $x\in\mathbb{R}^d$, where $C^{1,2}_{\rm Lip}([0,T]\times\mathbb{R}^d)$ is the space of all functions of class $C^{1,2}([0,T]\times\mathbb{R}^d)$ whose partial derivatives of order less than or equal to 2 and itself are Lipschitz continuous functions with respect to x.

We have the following estimates of G-SDEs, which come from [23, Chapter V].

Proposition 6.1 (See [23]). *Let $\xi,\xi'\in L^p_G(\Omega_t;\mathbb{R}^d)$ and $p\geqslant 2$. Then we have, for each $\delta\in[0,T-t]$,*

$$
\begin{aligned}
&\hat{\mathrm{E}}_t\Big[\sup_{s\in[t,t+\delta]}|X_s^{t,\xi}-X_s^{t,\xi'}|^p\Big]\leqslant C|\xi-\xi'|^p,\\
&\hat{\mathrm{E}}_t[|X_{t+\delta}^{t,\xi}|^p]\leqslant C(1+|\xi|^p),\\
&\hat{\mathrm{E}}_t\Big[\sup_{s\in[t,t+\delta]}|X_s^{t,\xi}-\xi|^p\Big]\leqslant C(1+|\xi|^p)\delta^{p/2},
\end{aligned}
$$

where the constant C depends on L,G,p,d and T.

Proof. For the reader's convenience, we give a brief proof here. It is easy to check that $\{X_s^{t,\xi}\}_{s\in[t,T]}$, $\{X_s^{t,\xi'}\}_{s\in[t,T]}\in M_G^p(0,T;\mathbb{R}^d)$. By Proposition 2.6, we have

$$
\begin{aligned}
&\hat{\mathrm{E}}_t\Big[\sup_{s\in[t,t+\delta]}|X_s^{t,\xi}-X_s^{t,\xi'}|^p\Big]\\
&\leqslant C\hat{\mathrm{E}}_t\bigg[|\xi-\xi'|^p+\int_t^{t+\delta}|X_s^{t,\xi}-X_s^{t,\xi'}|^pds+\sup_{s\in[t,t+\delta]}\bigg|\int_t^s(\sigma(r,X_r^{t,\xi})-\sigma(r,X_r^{t,\xi'}))dB_r\bigg|^p\bigg]\\
&\leqslant C\bigg\{|\xi-\xi'|^p+\hat{\mathrm{E}}_t\bigg[\int_t^{t+\delta}|X_s^{t,\xi}-X_s^{t,\xi'}|^pds\bigg]+\hat{\mathrm{E}}_t\bigg[\bigg(\int_t^{t+\delta}|X_s^{t,\xi}-X_s^{t,\xi'}|^2ds\bigg)^{p/2}\bigg]\bigg\}\\
&\leqslant C\bigg\{|\xi-\xi'|^p+\int_t^{t+\delta}\hat{\mathrm{E}}_t\Big[\sup_{r\in[t,s]}|X_r^{t,\xi}-X_r^{t,\xi'}|^p\Big]ds\bigg\}.
\end{aligned}
$$

By Gronwall's inequality, we get the first inequality. The others can be proved similarly. □

It follows from the previous results that for each $(t,x)\in[0,T]\times\mathbb{R}^d$, there exists a unique triple $(Y_s^{t,x},Z_s^{t,x},A_s^{t,x})_{s\in[t,T]}$, which solves the following reflected G-BSDE:

(i) $Y_s^{t,x}=\phi(X_T^{t,x})+\int_s^T f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})dr+\int_s^T g_{ij}(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})d\langle B^i,B^j\rangle_r-\int_s^T Z_r^{t,x}dB_r+A_T^{t,x}-A_s^{t,x}$, $t\leqslant s\leqslant T$;

(ii) $Y_s^{t,x}\geqslant h(s,X_s^{t,x})$, $t\leqslant s\leqslant T$;

(iii) $\{A_s^{t,x}\}$ is nondecreasing and continuous, and $\{-\int_t^s(Y_r^{t,x}-h(r,X_r^{t,x}))dA_r^{t,x},t\leqslant s\leqslant T\}$ is a non-increasing G-martingale.

We now consider the following obstacle problem for a parabolic PDE:

$$
\begin{cases}
\min(-\partial_t u(t,x)-F(D_x^2u,D_xu,u,x,t),u(t,x)-h(t,x))=0, & (t,x)\in(0,T)\times\mathbb{R}^d,\\
u(T,x)=\phi(x), & x\in\mathbb{R}^d,
\end{cases}
\tag{6.2}
$$

where

$$
F(D_x^2u,D_xu,u,x,t)=G(H(D_x^2u,D_xu,u,x,t))+\langle b(t,x),D_xu\rangle
$$

$$
\begin{aligned}
&+ f(t,x,u,\langle\sigma_1(t,x),D_xu\rangle,\dots,\langle\sigma_d(t,x),D_xu\rangle),\\
H(D_x^2u,D_xu,u,x,t) = {}&\langle D_x^2u\sigma_i(t,x),\sigma_j(t,x)\rangle + 2\langle D_xu,l_{ij}(t,x)\rangle\\
&+ 2g_{ij}(t,x,u,\langle\sigma_1(t,x),D_xu\rangle,\dots,\langle\sigma_d(t,x),D_xu\rangle).
\end{aligned}
$$

We need to consider solutions of the above PDE in the viscosity sense. The best candidate to define the notion of viscosity solution is by using the language of sub- and super-jets (see [1]).

Definition 6.2. Let $u \in C((0,T)\times\mathbb{R}^d)$ and $(t,x)\in(0,T)\times\mathbb{R}^d$. We denote by $\mathcal{P}^{2,+}u(t,x)$ (the "parabolic superjet" of u at (t,x)) the set of triples $(p,q,X)\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}_d$ satisfying

$$
u(s,y)\leqslant u(t,x)+p(s-t)+\langle q,y-x\rangle+\frac{1}{2}\langle X(y-x),y-x\rangle+o(|s-t|+|y-x|^2).
$$

Similarly, we define $\mathcal{P}^{2,-}u(t,x)$ (the "parabolic subjet" of u at (t,x)) by $\mathcal{P}^{2,-}u(t,x):=-\mathcal{P}^{2,+}(-u)(t,x)$.

Then, we give the definition of the viscosity solution of the obstacle problem (6.2).

Definition 6.3. It can be said that $u\in C([0,T]\times\mathbb{R}^d)$ is a viscosity subsolution of (6.2) if $u(T,x)\leqslant\phi(x)$, $x\in\mathbb{R}^d$, and at any point $(t,x)\in(0,T)\times\mathbb{R}^d$, for any $(p,q,X)\in\mathcal{P}^{2,+}u(t,x)$,

$$
\min(u(t,x)-h(t,x),-p-F(X,q,u(t,x),x,t))\leqslant 0.
$$

It can be said that $u\in C([0,T]\times\mathbb{R}^d)$ is a viscosity supersolution of (6.2) if $u(T,x)\geqslant\phi(x)$, $x\in\mathbb{R}^d$, and at any point $(t,x)\in(0,T)\times\mathbb{R}^d$, for any $(p,q,X)\in\mathcal{P}^{2,-}u(t,x)$,

$$
\min(u(t,x)-h(t,x),-p-F(X,q,u(t,x),x,t))\geqslant 0.
$$

$u\in C([0,T]\times\mathbb{R}^d)$ is said to be a viscosity solution of (6.2) if it is both a viscosity sub- and supersolution.

We now define

$$
u(t,x):=Y_t^{t,x},\quad (t,x)\in[0,T]\times\mathbb{R}^d. \tag{6.3}
$$

It is important to note that $u(t,x)$ is a deterministic function. We claim that u is a continuous function. For simplicity, we only consider the case where $g=0$ in the next three lemmas. The results still hold for the other cases.

Lemma 6.4. *Let Assumptions* (A1)–(A3) *and* (A4′) *hold. For each* $t\in[0,T]$, $x_1,x_2\in\mathbb{R}^d$, *we have*

$$
|u(t,x_1)-u(t,x_2)|\leqslant C(1+|x_1|^{m\vee 2}+|x_2|^{m\vee 2})|x_1-x_2|.
$$

Proof. From Proposition 3.4, since $u(t,x)$ is a deterministic function, we have

$$
\begin{aligned}
|u(t,x_1)-u(t,x_2)|^2\leqslant C\bigg\{\hat{\mathrm{E}}\bigg[&|(\phi(X_T^{t,x_1})-h(T,X_T^{t,x_1}))-(\phi(X_T^{t,x_2})-h(T,X_T^{t,x_2}))|^2\\
&+\int_t^T|f(s,X_s^{t,x_1},Y_s^{t,x_1},Z_s^{t,x_1})-f(s,X_s^{t,x_2},Y_s^{t,x_1},Z_s^{t,x_1})|^2ds\\
&+\int_t^T|b^1(s)-b^2(s)|^2+|l_{ij}^1(s)-l_{ij}^2(s)|^2+|\sigma_i^1(s)-\sigma_i^2(s)|^2\\
&+|h(s,X_s^{t,x_1})-h(s,X_s^{t,x_2})|^2ds\bigg]+|h(t,x_1)-h(t,x_2)|^2\bigg\},
\end{aligned} \tag{6.4}
$$

where for $k=1,2$,

$$
\begin{aligned}
&b^k(s)=\partial_sh(s,X_s^{t,x_k})+\langle b(s,X_s^{t,x_k}),D_xh(s,X_s^{t,x_k})\rangle,\\
&l_{ij}^k(s)=\langle D_xh(s,X_s^{t,x_k}),l_{ij}(s,X_s^{t,x_k})\rangle+\frac{1}{2}\langle D_x^2h(s,X_s^{t,x_k})\sigma_i(s,X_s^{t,x_k}),\sigma_j(s,X_s^{t,x_k})\rangle,\\
&\sigma_i^k(s)=\langle\sigma_i(s,X_s^{t,x_k}),D_xh(s,X_s^{t,x_k})\rangle.
\end{aligned}
$$

Set $\hat{X}_s^t = X_s^{t,x_1} - X_s^{t,x_2}$. By Assumptions (A3), (A4′) and Proposition 6.1, we have

$$\begin{aligned}|u(t,x_1)-u(t,x_2)|^2 \leqslant C\bigg\{&\hat{\mathrm{E}}\bigg[\bigg(1+\sum_{k=1}^{2}|X_T^{t,x_k}|^m\bigg)^2|\hat{X}_T^t|^2\bigg] + \int_t^T \hat{\mathrm{E}}\bigg[\bigg(1+\sum_{k=1}^{2}|X_s^{t,x_k}|^m\bigg)^2|\hat{X}_s^t|^2\bigg]ds \\ &+ \int_t^T \hat{\mathrm{E}}\bigg[\bigg(1+\sum_{k=1}^{2}|X_s^{t,x_k}|^2\bigg)^2|\hat{X}_s^t|^2\bigg]ds + \int_t^T \hat{\mathrm{E}}[|\hat{X}_s^t|^2]ds + |x_1-x_2|^2\bigg\} \\ \leqslant{}& C(1+|x_1|^{2m\vee 4}+|x_2|^{2m\vee 4})\bigg\{\bigg(\hat{\mathrm{E}}\Big[\sup_{s\in[t,T]}|\hat{X}_s^t|^4\Big]\bigg)^{1/2} + |x_1-x_2|^2\bigg\} \\ \leqslant{}& C(1+|x_1|^{2m\vee 4}+|x_2|^{2m\vee 4})|x_1-x_2|^2.\end{aligned}$$

The proof is completed. □

Lemma 6.5. *Let Assumptions* (A1)–(A4) *hold. For each* $t\in[0,T]$, $x,x'\in\mathbb{R}^d$, *we have*

$$|u(t,x_1)-u(t,x_2)|^2 \leqslant C\{(1+|x_1|^{2m}+|x_2|^{2m})|x_1-x_2|^2 + (1+|x_1|^{m+1}+|x_2|^{m+1})|x_1-x_2|\}.$$

Proof. From Propositions 3.6 and 6.1, by an analysis similar to the proof of the above lemma, we get the desired result. □

Lemma 6.6. *The function* $u(t,x)$ *is continuous in* t.

Proof. We only need to prove the case where (A1)–(A3) and (A4′) hold. The case that (A1)–(A4) hold can be proved in a similar way. We define $X_s^{t,x}:=x$, $Y_s^{t,x}:=Y_t^{t,x}$, $Z_s^{t,x}:=0$ and $A_s^{t,x}:=0$ for $0\leqslant s\leqslant t$. Then, we define the obstacle

$$\tilde{S}_u^{t,x} = \begin{cases} h(t,x) + \displaystyle\int_t^u \tilde{b}(s,X_s^{t,x})ds + \int_t^u \tilde{l}_{ij}(s,X_s^{t,x})d\langle B^i,B^j\rangle_s + \int_t^u \tilde{\sigma}_i(s,X_s^{t,x})dB_s^i, & u\in(t,T], \\ h(t,x), & u\in[0,t], \end{cases}$$

where

$$\begin{aligned}\tilde{b}(s,X_s^{t,x}) &= \partial_s h(s,X_s^{t,x}) + \langle b(s,X_s^{t,x}), D_x h(s,X_s^{t,x})\rangle, \\ \tilde{l}_{ij}(s,X_s^{t,x}) &= \langle D_x h(s,X_s^{t,x}), l_{ij}(s,X_s^{t,x})\rangle + \frac{1}{2}\langle D_x^2 h(s,X_s^{t,x})\sigma_i(s,X_s^{t,x}), \sigma_j(s,X_s^{t,x})\rangle, \\ \tilde{\sigma}_i(s,X_s^{t,x}) &= \langle \sigma_i(s,X_s^{t,x}), D_x h(s,X_s^{t,x})\rangle.\end{aligned}$$

It is easy to check that $(Y_s^{t,x},Z_s^{t,x},A_s^{t,x})_{s\in[0,T]}$ is the solution to the reflected G-BSDE with data $(\phi(X_T^{t,x}), \tilde{f}^{t,x}, \tilde{S}^{t,x})$, where $\tilde{f}^{t,x}(s,y,z) := I_{[t,T]}(s)f(s,X_s^{t,x},y,z)$. Fixing $x\in\mathbb{R}^d$, for $0\leqslant t_1\leqslant t_2\leqslant T$, by Proposition 3.4, we have

$$\begin{aligned}|u(t_1,x)-u(t_2,x)|^2 &= |Y_0^{t_1,x}-Y_0^{t_2,x}|^2 \\ &\leqslant C\bigg\{\hat{\mathrm{E}}\bigg[|(\phi(X_T^{t_1,x})-h(T,X_T^{t_1,x})) - (\phi(X_T^{t_2,x})-h(T,X_T^{t_2,x}))|^2 \\ &\quad + |h(t_1,x)-h(t_2,x)|^2 + \int_0^T |\hat{\lambda}_{t_1,t_2}(s)|^2 + |\hat{\rho}_{t_1,t_2}(s)|^2 \\ &\quad + |h(s,X_s^{t_1,x})-h(s,X_s^{t_2,x})|^2 ds\bigg]\bigg\},\end{aligned}$$

where

$$\hat{\lambda}_{t_1,t_2}(s) = |I_{[t_1,T]}(s)f(s,X_s^{t_1,x},Y_s^{t_2,x},Z_s^{t_2,x}) - I_{[t_2,T]}(s)f(s,X_s^{t_2,x},Y_s^{t_2,x},Z_s^{t_2,x})|,$$

and

$$\hat{\rho}_{t_1,t_2}(s) = |I_{[t_1,T]}(s)\tilde{b}(s,X_s^{t_1,x}) - I_{[t_2,T]}(s)\tilde{b}(s,X_s^{t_2,x})|$$

$$+ |I_{[t_1,T]}(s)\tilde{l}_{ij}(s, X_s^{t_1,x}) - I_{[t_2,T]}(s)\tilde{l}_{ij}(s, X_s^{t_2,x})| \\ + |I_{[t_1,T]}(s)\tilde{\sigma}_i(s, X_s^{t_1,x}) - I_{[t_2,T]}(s)\tilde{\sigma}_i(s, X_s^{t_2,x})|.$$

Set $\hat{X}_s^x = X_s^{t_1,x} - X_s^{t_2,x}$. By Hölder's inequality, Assumptions (A3) and (A4′), we deduce that

$$\begin{aligned}|u(t_1,x) - u(t_2,x)|^2 \leqslant C\Bigg\{&\hat{\mathrm{E}}[(1 + |X_T^{t_1,x}|^m + |X_T^{t_2,x}|^m)^2|\hat{X}_T^x|^2] + |h(t_1,x) - h(t_2,x)|^2 \\ &+ \int_{t_2}^T \hat{\mathrm{E}}[(1 + |X_s^{t_1,x}|^{m\vee 2} + |X_s^{t_2,x}|^{m\vee 2})^2|\hat{X}_s^x|^2]ds \\ &+ \int_{t_1}^{t_2} \hat{\mathrm{E}}[1 + |X_s^{t_1,x}|^{(2m+2)\vee 6} + |Y_s^{t_2,x}|^2]ds\Bigg\}.\end{aligned}$$

Note that $\hat{X}_s^x = X_s^{t_2,X_{t_2}^{t_1,x}} - X_s^{t_2,x}$, for $s \in [t_2,T]$. Applying Proposition 6.1, it follows that

$$|u(t_1,x) - u(t_2,x)| \leqslant C\{(1 + |x|^{(m+1)\vee 3})|t_2 - t_1|^{\frac{1}{2}} + |h(t_2,x) - h(t_1,x)|\}.$$

The proof is completed. □

We will use the approximation of the reflected G-BSDE by penalization. For each $(t,x) \in [0,T] \times \mathbb{R}^d$, $n \in \mathbb{N}$, let $\{(Y_s^{n,t,x}, Z_s^{n,t,x}, K_s^{n,t,x}), t \leqslant s \leqslant T\}$ denote the solution of the G-BSDE

$$\begin{aligned}Y_s^{n,t,x} = \phi(X_T^{t,x}) &+ \int_s^T f(r, X_r^{t,x}, Y_r^{n,t,x}, Z_r^{n,t,x})dr + \int_s^T g_{ij}(r, X_r^{t,x}, Y_r^{n,t,x}, Z_r^{n,t,x})d\langle B^i, B^j\rangle_r \\ &+ n\int_s^T (Y_r^{n,t,x} - h(r, X_r^{t,x}))^- dr - \int_s^T Z_r^{n,t,x}dB_r - (K_T^{n,t,x} - K_s^{n,t,x}), \quad t \leqslant s \leqslant T.\end{aligned}$$

We define $u_n(t,x) := Y_t^{n,t,x}$, $0 \leqslant t \leqslant T, x \in \mathbb{R}^d$. By [9, Theorem 4.5], u_n is the viscosity solution of the parabolic PDE

$$\begin{cases}-\partial_t u_n(t,x) - F_n(D_x^2 u_n(t,x), D_x u_n(t,x), u_n(t,x), x, t) = 0, & (t,x) \in [0,T] \times \mathbb{R}^d, \\ u_n(T,x) = \phi(x), & x \in \mathbb{R}^d,\end{cases} \tag{6.5}$$

where $F_n(D_x^2 u, D_x u, u, x, t) = F(D_x^2 u, D_x u, u, x, t) + n(u - h(t,x))^-$.

Theorem 6.7. *The function u defined by* (6.3) *is the unique viscosity solution of the obstacle problem* (6.2).

Proof. From the results of the previous sections, for each $(t,x) \in [0,T] \times \mathbb{R}^d$, we obtain

$$u_n(t,x) \uparrow u(t,x), \quad \text{as} \quad n \to \infty.$$

By [9, Proposition 4.2 and Theorem 4.5] and Lemmas 6.4–6.6, u_n and u are continuous. Then, by applying Dini's theorem, the sequence u^n uniformly converges to u on compact sets.

We first show that u is a subsolution of (6.2). For each fixed $(t,x) \in (0,T) \times \mathbb{R}^d$, let $(p,q,X) \in \mathcal{P}^{2,+}u(t,x)$. Without loss of generality, we may assume that $u(t,x) > h(t,x)$. By [1, Lemma 6.1], there exist sequences

$$n_j \to \infty, \quad (t_j, x_j) \to (t,x), \quad (p_j, q_j, X_j) \in \mathcal{P}^{2,+}u_{n_j}(t_j, x_j),$$

such that $(p_j, q_j, X_j) \to (p,q,X)$. Since u^n is the viscosity solution to (6.5), it follows that for any j,

$$-p_j - F_{n_j}(X_j, q_j, u_{n_j}(t_j, x_j), x_j, t_j) \leqslant 0.$$

Noting the uniform convergence of u_n on compact sets, by the assumption that $u(t,x) > h(t,x)$, we derive that for a j large enough, $u_{n_j}(t_j, x_j) > h(t_j, x_j)$. Therefore, letting j approach infinity in the above inequality yields $-p - F(X, q, u(t,x), x, t) \leqslant 0$. Then, we conclude that u is a subsolution of (6.2).

It remains to prove that u is a supersolution of (6.2). For each fixed $(t,x) \in (0,T)\times\mathbb{R}^d$, and $(p,q,X) \in \mathcal{P}^{2,-}u(t,x)$. Noting that $\{Y_s^{t,x}\}_{s\in[t,T]}$ is the solution of reflected G-BSDE with data $(\xi^{t,x}, f^{t,x}, g^{t,x}, S^{t,x})$, where $S_s^{t,x} = h(s, X_s^{t,x})$, we have $u(t,x) = Y_t^{t,x} \geqslant h(t,x)$. Applying [1, Lemma 6.1] again, there exist sequences

$$n_j \to \infty, \quad (t_j, x_j) \to (t,x), \quad (p_j, q_j, X_j) \in \mathcal{P}^{2,-}u_{n_j}(t_j, x_j),$$

such that $(p_j, q_j, X_j) \to (p,q,X)$. Since u^n is the viscosity solution to (6.5), we derive that for any j,

$$-p_j - F_{n_j}(X_j, q_j, u_{n_j}(t_j, x_j), x_j, t_j) \geqslant 0.$$

Therefore,

$$-p_j - F(X_j, q_j, u_{n_j}(t_j, x_j), x_j, t_j) \geqslant 0.$$

Letting $j \to \infty$ in the above inequality, we have $-p - F(X, q, u(t,x), x, t) \geqslant 0$, which implies that u is a supersolution of (6.2). Thus, u is a viscosity solution of (6.2).

An analysis similar to the proof of Theorem 8.6 in [4] shows that there exists at most one solution of the obstacle problem (6.2) in the class of continuous functions which grow at most polynomially at infinity. The proof is completed. □

7 American options under volatility uncertainty

Now, let us consider the financial market with volatility uncertainty. The market model $\mathcal{M}$ is introduced in [26] consisting of two assets whose dynamics are given by

$$\begin{aligned} d\gamma_t &= r\gamma_t dt, \quad \gamma_0 = 1, \\ dS_t &= rS_t dt + S_t dB_t, \quad S_0 = x_0 > 0, \end{aligned}$$

where $r \geqslant 0$ is a constant interest rate. The asset $\gamma = (\gamma_t)$ represents a riskless bond. The stock price is described by a geometric G-Brownian motion. Since the deviation of the process B from its mean is unknown, this model shows the ambiguity under volatility uncertainty.

Definition 7.1 (See [26])**.** A cumulative consumption process $C = (C_t)$ is a non-negative $\mathcal{F}_t$-adapted process with values in $L_G^1(\Omega_T)$, and with nondecreasing, RCLL paths on $(0,T]$, and $C_0 = 0$, $C_T < \infty$, q.s., where $\mathcal{F}_t = \sigma\{B_s \,|\, 0 \leqslant s \leqslant t\}$. A portfolio process $\pi = (\pi_t)$ is an $\mathcal{F}_t$-adapted real valued process with values in $L_G^1(\Omega_T)$.

Definition 7.2 (See [26])**.** For a given initial capital y, a portfolio process π and a cumulative consumption process C, consider the wealth equation

$$dX_t = X_t(1-\pi_t)\frac{d\gamma_t}{\gamma_t} + X_t\pi_t\frac{dS_t}{S_t} - dC_t = rX_t dt + \pi_t X_t dB_t - dC_t$$

with initial wealth $X_0 = y$. Or, equivalently,

$$\gamma_t^{-1}X_t = y - \int_0^t \gamma_u^{-1} dC_u + \int_0^t \gamma_u^{-1} X_u \pi_u dB_u, \quad \forall\, t \leqslant T.$$

If this equation has a unique solution $X = (X_t) := X^{y,\pi,C}$, it is called the wealth process corresponding to the triple (y, π, C).

Definition 7.3 (See [26])**.** A portfolio/consumption process pair (π, C) is called admissible for an initial capital $y \in \mathbb{R}$ if

(i) the pair obeys the conditions of Definitions 7.1 and 7.2;

(ii) $(\pi_t X_t^{y,\pi,C}) \in M_G^2(0,T)$;

(iii) the solution $X_t^{y,\pi,C}$ satisfies $X_t^{y,\pi,C} \geqslant -L$, $\forall\, t \leqslant T$, q.s., where L is a non-negative random variable in $L_G^2(\Omega_T)$.

We then write $(\pi, C) \in \mathcal{A}(y)$.

We denote by $\mathcal{T}_{s,t}$ the set of all stopping times taking values in $[s,t]$, for any $0 \leqslant s \leqslant t \leqslant T$. Then, the American contingent claims may be defined by the following:

Definition 7.4 (See [12]). An American contingent claim is a financial instrument consisting of

(i) an expiration date $T \in (0,\infty)$;

(ii) the selection of an exercise time $\tau \in \mathcal{T}_{0,T}$;

(iii) a payoff H_τ at the exercise time.

We require that the payoff process $\{H_t\}_{t\in[0,T]}$ satisfies (H4) or (H4′) in Section 3. Since the financial market under volatility uncertainty is incomplete, it is natural to consider the superhedging price for the American contingent claims.

Definition 7.5. Given an American contingent claim (T,H), we define the superhedging class

$$\mathcal{U} := \{y \geqslant 0 \,|\, \exists\, (\pi, C) \in \mathcal{A}(y) : \text{for any stopping time } \tau, X_\tau^{y,\pi,C} \geqslant H_\tau, \text{q.s.}\}.$$

The superhedging price is defined as $h_{up} := \inf\{y \,|\, y \in \mathcal{U}\}$.

Theorem 7.6. *Given the financial market $\mathcal{M}$ and an American contingent claim (T,H), we have $h_{up} = Y_0$, where $Y = (Y_t)$ is the solution to the reflected G-BSDE with parameter (H_T, f, H) where $f(y) = ry$.*

Proof. Let $y \in \mathcal{U}$. By the definition of $\mathcal{U}$, there exists a pair $(\pi, C) \in \mathcal{A}(y)$ such that for any stopping time τ, $X_\tau^{y,\pi,C} \geqslant H_\tau$. Applying [16, Lemmas 3.4, 4.2 and 4.3], we derive that for any $\eta \in M_G^2(0,T)$, $\hat{\mathrm{E}}[\int_0^\tau \eta_s dB_s] = 0$. Then, we obtain

$$\begin{aligned} y &= \hat{\mathrm{E}}\bigg[y + \int_0^\tau \gamma_u^{-1} X_u^{y,\pi,C} \pi_u dB_u\bigg] \\ &\geqslant \hat{\mathrm{E}}\bigg[y + \int_0^\tau \gamma_u^{-1} X_u^{y,\pi,C} \pi_u dB_u - \int_0^\tau \gamma_u^{-1} dC_u\bigg] \\ &= \hat{\mathrm{E}}[\gamma_\tau^{-1} X_\tau^{y,\pi,C}] \geqslant \hat{\mathrm{E}}[\gamma_\tau^{-1} H_\tau]. \end{aligned}$$

It follows that $h_{up} \geqslant \sup_{\tau\in\mathcal{T}_{0,T}} \hat{\mathrm{E}}[\gamma_\tau^{-1} H_\tau]$.

Now, we turn to prove the inverse inequality. Consider the following reflected G-BSDE:

$$\begin{cases} Y_t = H_T - \displaystyle\int_t^T rY_s ds - \int_t^T Z_s dB_s + (A_T - A_t), \\ Y_t \geqslant H_t. \end{cases}$$

By Theorem 5.1, there exists a unique solution (Y, Z, A) to the above equation. Let $C = A$, $\pi = \frac{Z}{Y}$. Then $H_\tau \leqslant Y_\tau = X_\tau^{Y_0,\pi,C}$, which implies $Y_0 \in \mathcal{U}$. It follows that $h_{up} \leqslant Y_0$. Applying Itô's formula to $\widetilde{Y}_t = \gamma_t^{-1} Y_t$, we conclude that $\widetilde{Y}$ is a solution to the reflected G-BSDE with data $(\gamma_T^{-1} H_T, 0, \gamma^{-1} H)$. By the following proposition, we finally get the desired result. □

Proposition 7.7. *Let (Y, Z, A) be a solution of the reflected G-BSDE with data (ξ, f, S). Then, we have*

$$Y_0 = \sup_{\tau\in\mathcal{T}_{0,T}} \hat{\mathrm{E}}\bigg[\int_0^\tau f(s, Y_s, Z_s) ds + S_\tau I_{\{\tau<T\}} + \xi I_{\{\tau=T\}}\bigg].$$

Proof. Let $\tau \in \mathcal{T}_{0,T}$. Note the fact that $\hat{\mathrm{E}}[\int_0^\tau Z_s dB_s] = 0$. Then, we have

$$Y_0 = \hat{\mathrm{E}}\bigg[\int_0^\tau f(s, Y_s, Z_s) ds + Y_\tau + A_\tau\bigg] \geqslant \hat{\mathrm{E}}\bigg[\int_0^\tau f(s, Y_s, Z_s) ds + S_\tau I_{\{\tau<T\}} + \xi I_{\{\tau=T\}}\bigg].$$

We are now in a position to show the inverse inequality. By the definition of the solution of the reflected G-BSDE, we may define

$$K_t := -\int_0^t (Y_s - S_s) dA_s.$$

Then, K is a nonincreasing G-martingale. Let

$$D^n = \inf\left\{0 \leqslant t \leqslant T : Y_t - S_t < \frac{1}{n}\right\} \wedge T.$$

By Example A.4 in Appendix A, D^n is a $*$-stopping time for $n \geqslant 1$. It is easy to check that $D^n \to D$, where

$$D = \inf\{0 \leqslant t \leqslant T : Y_t - S_t = 0\} \wedge T.$$

Noting that A is nondecreasing, by Theorem A.5, it follows that

$$0 = \hat{\mathrm{E}}[K_{D^n}] = \hat{\mathrm{E}}\left[-\int_0^{D^n}(Y_s - S_s)dA_s\right] \leqslant \frac{1}{n}\hat{\mathrm{E}}[-A_{D^n}] \leqslant 0,$$

which yields $\hat{\mathrm{E}}[-A_{D^n}] = 0$. By the continuity property of A, we have $\hat{\mathrm{E}}[-A_D] = 0$. Then, it is easy to check that

$$Y_0 = \hat{\mathrm{E}}\left[\int_0^D f(s, Y_s, Z_s)ds + S_D I_{\{D<T\}} + \xi I_{\{D=T\}}\right].$$

Hence, the result follows. □

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References

1 Crandall M, Ishii H, Lions P L. User's guide to the viscosity solutions of second order partial differential equations. Bull Amer Math Soc (NS), 1992, 27: 1–67

2 Cvitanic J, Karatzas I. Backward stochastic differential equations with reflection and Dynkin games. Ann Probab, 1996, 24: 2024–2056

3 Denis L, Hu M, Peng S. Function spaces and capacity related to a sublinear expectation: Application to G-Brownian motion pathes. Potential Anal, 2011, 34: 139–161

4 El Karoui N, Kapoudjian C, Pardoux E, et al. Reflected solutions of backward SDE's, and related obstacle problems for PDE's. Ann Probab, 1997, 23: 702–737

5 El Karoui N, Pardoux E, Quenez M C. Reflected backward SDE's and American options. In: Numerical Methods in Finance. Cambridge: Cambridge University Press, 1997: 215–231

6 Hamadene S. Reflected BSDE's with discontinuous barrier and application. Stochastics Stochastics Rep, 2002, 74: 571–596

7 Hamadene S, Lepeltier J-P. Reflected BSDE's and mixed game problem. Stochastic Process Appl, 2000, 85: 177–188

8 Hu M, Ji S, Peng S, et al. Backward stochastic differential equations driven by G-Brownian motion. Stochastic Process Appl, 2014, 124: 759–784

9 Hu M, Ji S, Peng S, et al. Comparison theorem, Feynman-Kac formula and Girsanov transformation for BSDEs driven by G-Brownian motion. Stochastic Process Appl, 2014, 124: 1170–1195

10 Hu M, Peng S. On representation theorem of G-expectations and paths of G-Brownian motion. Acta Math Appl Sin Engl Ser, 2009, 25: 539–546

11 Hu M, Peng S. Extended conditional G-expectation and related stopping times. ArXiv:1309.3829v1, 2013

12 Karatzas I. On the pricing of American options. Appl Math Optim, 1988, 17: 37–60

13 Kobylanski M, Lepeltier J-P, Quenez M C, et al. Reflected BSDE with superlinear quadratic coefficient. Probab Math Statist, 2002, 22: 51–83

14 Lepeltier J-P, Xu M. Penalization method for reflected backward stochastic differential equations with one r.c.l.l. barrier. Statist Probab Lett, 2005, 75: 58–66

15 Li H, Peng S, Song Y. Supermartingale decomposition theorem under G-expectation. ArXiv:1703.02730, 2017

16 Li X, Peng S. Stopping times and related Itô's calculus with G-Brownian motion. Stochastic Process Appl, 2011, 121: 1492–1508

17 Matoussi A. Reflected solutions of backward stochasitc differential equations with continous coefficient. Statist Probab Lett, 1997, 34: 347–354
18 Peng S. Monotonic limit theorem of BSDE and nonlinear decomposition theorem of Doob-Meyer's type. Probab Theory Related Fields, 1999, 113: 473–499
19 Peng S. Filtration consistent nonlinear expectations and evaluations of contigent claims. Acta Math Appl Sin, 2004, 20: 1–24
20 Peng S. Nonlinear expecations and nonlinear Markov chains. Chin Ann Math Ser B, 2005, 26: 159–184
21 Peng S. G-expectation, G-Brownian motion and related stochastic calculus of Itô type. In: Stochastic Analysis and Applications. Abel Symposia, vol. 2. Berlin: Springer, 2007: 541–567
22 Peng S. Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation. Stochastic Process Appl, 2008, 118: 2223–2253
23 Peng S. Nonlinear expectations and stochastic calculus under uncertainty. ArXiv:1002.4546v1, 2010
24 Rodrigues J F. Obstacle Problems in Mathematical Physics. North-Holland Mathematics Studies, vol. 134. Notas de Matematica, vol. 114. Amsterdam: North-Holland, 1987
25 Song Y. Some properties on G-evaluation and its applications to G-martingale decomposition. Sci China Math, 2011, 54: 287–300
26 Vorbrink J. Financial markets with volatility uncertainty. J Math Econom, 2014, 53: 64–78

Appendix A

In this appendix, we introduce the extended conditional G-expectation and optional stopping theorem under G-framework. More details can be found in [11].

Let $(\Omega, L_G^1(\Omega), \hat{\mathrm{E}})$ be the G-expectation space and $\mathcal{P}$ be a weakly compact set that represents $\hat{\mathrm{E}}$. We set

$$\begin{aligned}
&L^0(\Omega) := \{X : \Omega \to [-\infty, \infty] \text{ and } X \text{ is } \mathcal{B}(\Omega)\text{-measurable}\},\\
&\mathcal{L}(\Omega) := \{X \in L^0(\Omega) : E_P[X] \text{ exists for each } P \in \mathcal{P}\}.
\end{aligned}$$

We extend G-expectation $\hat{\mathrm{E}}$ to $\mathcal{L}(\Omega)$ and still denote it by $\hat{\mathrm{E}}$. For each $X \in \mathcal{L}(\Omega)$, we define

$$\hat{\mathrm{E}}[X] = \sup_{P\in\mathcal{P}} E_P[X].$$

Then, we give some notation

$$\begin{aligned}
&\mathbb{L}^p(\Omega) := \{X \in L^0(\Omega) : \hat{\mathrm{E}}[|X|^p] < \infty\} \quad \text{for} \quad p \geqslant 1,\\
&L_G^{1^*}(\Omega) := \{X \in \mathbb{L}^1(\Omega) : \exists\, X_n \in L_G^1(\Omega) \text{ such that } X_n \downarrow X, \text{q.s.}\},\\
&L_G^{1^*_*}(\Omega) := \{X \in \mathbb{L}^1(\Omega) : \exists\, X_n \in L_G^{1^*}(\Omega) \text{ such that } X_n \uparrow X, \text{q.s.}\},\\
&\bar{L}_G^{1^*_*}(\Omega) := \{X \in \mathbb{L}^1(\Omega) : \exists\, X_n \in L_G^{1^*_*}(\Omega) \text{ such that } \hat{\mathrm{E}}[|X_n - X|] \to 0\}.
\end{aligned}$$

Set $\Omega_t = \{\omega_{\cdot\wedge t} : \omega \in \Omega\}$ for $t > 0$. Similarly, we can define $L^0(\Omega_t)$, $\mathcal{L}(\Omega_t)$, $\mathbb{L}^p(\Omega_t)$, $L_G^{1^*}(\Omega_t)$, $L_G^{1^*_*}(\Omega_t)$ and $\bar{L}_G^{1^*_*}(\Omega_t)$, respectively. Then we can extend the conditional G-expectation to space $\bar{L}_G^{1^*_*}(\Omega)$ and it satisfies the following property.

Proposition A.1 (See [11])**.** *For each $X \in \bar{L}_G^{1^*_*}(\Omega)$, we have, for each $P \in \mathcal{P}$,*

$$\hat{\mathrm{E}}_t[X] = \operatorname*{ess\,sup}_{Q\in\mathcal{P}(t,P)}{}^P E_Q[X \mid \mathcal{F}_t], \quad P\text{-}a.s.,$$

where $\mathcal{P}(t, P) = \{Q \in P : E_Q[X] = E_P[X], \forall\, X \in L_{ip}(\Omega_t)\}$.

We now give the definition of stopping times under the G-expectation framework.

Definition A.2. A random time $\tau : \Omega \to [0, \infty)$ is called a $*$-stopping time if $I_{\{\tau\geqslant t\}} \in L_G^{1^*}(\Omega_t)$ for each $t \geqslant 0$.

Definition A.3. For a given $*$-stopping time τ and $\xi \in \bar{L}_G^{1^*_*}(\Omega)$, we define $\hat{\mathrm{E}}_\tau[\xi] := M_\tau$, where $M_t = \hat{\mathrm{E}}_t[\xi]$ for $t \geqslant 0$.

We then give an example of $*$-stopping time.

Example A.4. Let $(X_t)_{t\in[0,T]}$ be a d-dimensional right continuous process such that $X_t \in L_G^1(\Omega_t)$ for $t \in [0,T]$. For each fixed closed set $F \in \mathbb{R}^d$, we define

$$\tau = \inf\{t \geqslant 0 : X_t \notin F\} \wedge T.$$

Then, τ is a $*$-stopping time.

Now, we introduce the following optional stopping theorem under the G-framework.

Theorem A.5 (See [11]). *Suppose that G is non-degenerate. Let $M_t = \hat{\mathrm{E}}_t[\xi]$ for $t \leqslant T$, $\xi \in L_G^p(\Omega_T)$ with $p > 1$ and let σ and τ be two $*$-stopping times with $\sigma \leqslant \tau \leqslant T$. Then, M_τ, $M_\sigma \in \bar{L}_G^{1^*_*}(\Omega)$ and*

$$M_\sigma = \hat{\mathrm{E}}_\sigma[M_\tau], \quad q.s.$$