

Semipolar sets and intrinsic Hausdorff measure

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Abstract

Given a “Green function” G on a locally compact space X with countable base, a Borel set A in X is called G -semipolar, if there is no measure $\nu \neq 0$ supported by A such that $G\nu := \int G(\cdot, y) d\nu(y)$ is a continuous real function on X . Introducing an *intrinsic Hausdorff measure* m_G using G -balls $B(x, \rho) := \{y \in X : G(x, y) > 1/\rho\}$, it is shown that every set A in X with $m_G(A) < \infty$ is contained in a G -semipolar Borel set. This is of interest, since G -semipolar sets are semipolar in the potential-theoretic sense (countable unions of totally thin sets, hit by a corresponding process at most countably many times), if G is a genuine Green function.

The result has immediate consequences for classical potential theory, Riesz potentials and the heat equation (where it solves an open problem). More generally, it is applied to metric measure spaces (X, d, μ) , where a continuous heat kernel with upper and lower bounds of the form $t^{-\alpha/\beta} \Phi_j(d(x, y)t^{-1/\beta})$, $j = 1, 2$, is given. Then the intrinsic Hausdorff measure on X is equivalent to an ordinary Hausdorff measure $m_{\alpha-\beta}$. For the corresponding space-time structure on $X \times \mathbb{R}$, the intrinsic Hausdorff measure turns out to be equivalent to an anisotropic Hausdorff measure $m_{\alpha, \beta}$.

Keywords: Heat equation; metric measure space; heat kernel; balayage space; Green function; Hausdorff measure; semipolar set; space-time process

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1 Introduction

In 1985, S.J. Taylor and N.A. Watson published a paper on a Hausdorff measure classification of polar sets for the heat equation

$$\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} - \frac{\partial u}{\partial t} = 0$$

on \mathbb{R}^{n+1} ; see [30]. To that end they introduced an anisotropic (parabolic) measure m_P of Hausdorff type (their notation is $\mathcal{P} - \Lambda^n - m$) defined as follows. For $\rho > 0$, $x' \in \mathbb{R}^{n+1}$, $A \subset \mathbb{R}^{n+1}$ and $\delta > 0$, let

$$(1.1) \quad P(0, \rho) := [-\rho/2, \rho/2]^n \times [-\rho^2/2, \rho^2/2], \quad P(x', \rho) := x' + P(0, \rho),$$

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$$m_P^\delta(A) := \inf \left\{ \sum_{j=1}^{\infty} (\text{diam } P(x'_j, \rho_j))^n : A \subset \bigcup_{j=1}^{\infty} P(x'_j, \rho_j), \text{diam } P(x'_j, \rho_j) < \delta \right\},$$

$$m_P(A) := \lim_{\delta \rightarrow 0} m_P^\delta(A).$$

They showed that A is polar if $m_P(A) = 0$ ([30, Theorem 1]), and noted that Borel sets $A \subset \mathbb{R}^n \times \{0\}$ of strictly positive finite n -dimensional Lebesgue measure are nonpolar, but satisfy $m_P(A) < \infty$. Since such sets are semipolar, they raised the problem, if *every* set A in \mathbb{R}^{n+1} satisfying $m_P(A) < \infty$ is semipolar.

In this paper, we shall give an affirmative answer to this question. In fact, we shall prove such a result in the abstract setting of locally compact space X with countable base, where we consider function kernels G having simple regularity properties which are satisfied by Green functions for a wide class of elliptic or parabolic second order partial differential operators (leading to harmonic spaces; see, for example, [13, Section 7]) as well as for rather general jump processes (leading to balayage spaces; see [5]). Special attention is given to an application on metric measure spaces; see Section 7.

The reader who is not familiar with or not interested in this generality may suppose that we mainly consider the case of the heat equation, classical potential theory, and Riesz potentials.

The clue will be the introduction of an intrinsic measure m_G of Hausdorff type using G -balls

$$B(x, \rho) := \{y \in X : G(x, y) > \rho^{-1}\}, \quad x \in X, \rho > 0,$$

and defining, for sets A in X and $\delta > 0$,

$$m_G^\delta(A) := \inf \left\{ \sum_{j=1}^{\infty} \rho_j : A \subset \bigcup_{j=1}^{\infty} B(x_j, \rho_j), 0 < \rho_j < \delta \right\}, \quad m_G(A) := \lim_{\delta \rightarrow 0} m_G^\delta(A).$$

A Borel set A in X will be called G -semipolar if there is no measure $\mu \neq 0$ on X such that $\mu(X \setminus A) = 0$ and $G\mu := \int G(\cdot, y) d\mu(y)$ is a continuous real function.

Under simple assumptions on the regularity of G we shall prove the following; see properties (i) – (iii) in Section 3 and Theorem 3.3.

THEOREM 1.1. *If a set A in X satisfies $m_G(A) < \infty$, then it is contained in a G -semipolar Borel set.*

In fairly general potential-theoretic settings, semipolar sets (countable unions of totally sets, sets which a corresponding process hits (almost surely) at most countably many times) can be characterized by not supporting continuous real potentials. Therefore, assuming that G is a genuine Green function, we have the following consequence; see properties (i')– (iv) in Section 4 and Corollary 4.2.

COROLLARY 1.2. *If A is a set in X such that $m_G(A) < \infty$, then A is semipolar.*

Whereas the proof of Theorem 1.1 is rather involved and the characterization of semipolar sets is very subtle, there are many situations, where it is easy to see that every set A in X satisfying $m_G(A) = 0$ is contained in a (G -)polar Borel set; see Corollaries 3.6 and 4.4.

In classical potential theory on \mathbb{R}^n , every semipolar set is polar. In the case $n \geq 3$, Theorem 1.1 reduces to the well known fact that sets A in \mathbb{R}^n having finite $(n - 2)$ -dimensional Hausdorff measure are polar: It suffices to take

$$G(x, y) := |x - y|^{2-n}, \quad x, y \in \mathbb{R}^n,$$

and to observe that G -balls $B(x, \rho)$ are Euclidean balls with center x and radius ρ^{n-2} . In the case $n = 2$, Theorem 1.1 implies that sets A in \mathbb{R}^2 having finite ϕ -Hausdorff measure, where $\phi(t) := \log^+ t^{-1}$, are polar; cf., for example, [1, Theorem 5.9.4].

Taking, for the heat equation,

$$(1.2) \quad G(x', y') := G_0(x' - y'), \quad x', y' \in \mathbb{R}^{n+1},$$

where

$$(1.3) \quad G_0(x') := 1_{(0, \infty)}(t) \cdot t^{-n/2} \exp\left(-\frac{|x|^2}{4t}\right), \quad x' = (x, t) \in \mathbb{R}^n \times \mathbb{R},$$

we shall prove (Corollary 5.4) that,

$$(1.4) \quad (2n)^{-n} m_P \leq m_G \leq (2/n)^{n/2} m_P.$$

So Corollaries 1.2 and 4.4 show that sets A in \mathbb{R}^{n+1} are semipolar, polar respectively if $m_P(A) < \infty$, $m_P(A) = 0$ respectively.

In the space-time setting given by general heat kernels $(x, y, t) \mapsto p_t(x, y)$ on metric measure spaces (X, d, μ) the intrinsic Hausdorff measure will be equivalent to an anisotropic Hausdorff measure $m_{\alpha, \beta}$ on $X' := X \times \mathbb{R}$; see Section 7.

In detail, our paper is organized as follows. We start with a short Section 2 on comparison of measures of Hausdorff type on arbitrary sets. In Section 3, we prove Theorem 3.3 and show that sets A with $m_G(A) = 0$ are contained in G -polar Borel sets. In Section 4, we apply our general results to harmonic spaces and balayage spaces (diffusions and jump processes). In Section 5, we briefly discuss the results in the standard settings of classical potential theory, heat equation and Riesz potentials. In Section 6, we prove results for rather general space-time settings. In Section 7, we finally apply our results to heat kernels on metric measure spaces covering diffusions and jump processes on manifolds and fractals.

2 Comparison of measures of Hausdorff type

To prove (1.4) and similar estimates it will be helpful to have a general comparison result for measures of Hausdorff type. To that end let us consider an arbitrary set X and let \mathcal{F} denote the set of all mappings F which associate to all $x \in X$ and $\rho > 0$ some subset $F(x, \rho)$ of X . For all $F \in \mathcal{F}$, $\eta > 0$ and $A \subset X$, we define

$$m_{\eta, F}^\delta(A) := \inf \left\{ \sum_{j=1}^{\infty} \rho_j^\eta : A \subset \bigcup_{j=1}^{\infty} F(x_j, \rho_j), x_j \in X, 0 < \rho_j < \delta \right\}, \quad \delta > 0,$$

and

$$m_{\eta, F}(A) := \lim_{\delta \rightarrow 0} m_{\eta, F}^\delta(A).$$

PROPOSITION 2.1. *Let $\eta, \tilde{\eta} \in (0, \infty)$, $F, \tilde{F} \in \mathcal{F}$ and $\kappa > 0$ such that, for all $x \in X$ and $\rho > 0$, there exists $z \in X$ with $F(x, \rho) \subset \tilde{F}(z, \kappa\rho^{\eta/\tilde{\eta}})$. Then*

$$m_{\tilde{\eta}, \tilde{F}} \leq \kappa^{\tilde{\eta}} m_{\eta, F}.$$

Proof. Let $A \subset X$, $x_j \in X$ and $0 < \rho_j < \delta$, $j \in \mathbb{N}$, such that $A \subset \bigcup_{j=1}^{\infty} F(x_j, \rho_j)$. Then there are $z_j \in X$ such that $A \subset \bigcup_{j=1}^{\infty} \tilde{F}(z_j, \kappa\rho_j^{\eta/\tilde{\eta}})$. Taking $\tilde{\delta} := \kappa\delta^{\tilde{\eta}/\eta}$ we conclude that

$$m_{\tilde{\eta}, \tilde{F}}^{\tilde{\delta}}(A) \leq \sum_{j=1}^{\infty} (\kappa\rho_j^{\eta/\tilde{\eta}})^{\tilde{\eta}} = \kappa^{\tilde{\eta}} \sum_{j=1}^{\infty} \rho_j^{\eta}.$$

This implies that

$$m_{\tilde{\eta}, \tilde{F}}^{\tilde{\delta}}(A) \leq \kappa^{\tilde{\eta}} m_{\eta, F}^{\delta}(A).$$

The proof is completed letting δ tend to 0. □

3 Hausdorff measure with respect to a kernel

Let X be a locally compact space with countable base and let G be a positive numerical function on $X \times X$ having the following properties:

- (i) For every $y \in X$, $G(\cdot, y)$ is lower semicontinuous and $\limsup_{x \rightarrow y} G(x, y) = \infty$.
- (ii) G is continuous outside the diagonal Δ and Borel measurable on Δ .
- (iii) For every compact K in X , there exists a compact L in X such that G is bounded on $(X \setminus L) \times K$.

Let $\mathcal{M}(X)$ denote the set of all (positive Radon) measures on X . For $\mu \in \mathcal{M}(X)$, we define

$$G\mu(x) := \int G(x, y) d\mu(y), \quad x \in X.$$

By (i) and Fatou's lemma, these functions $G\mu$ are lower semicontinuous. This implies that, for $\mu, \nu \in \mathcal{M}(X)$ such that $\nu \leq \mu$ and $G\mu$ is continuous and real, the function $G\nu$ is continuous and real.

The following definitions are justified by characterizations of polar and semipolar sets related to the Laplace equation, the heat equation, harmonic spaces and balayage spaces (associated with diffusions and jump processes), where positive constants are superharmonic and there exists a Green function.

DEFINITIONS 3.1. *Let us say that a Borel set A in X is G -polar if there is no measure $\mu \neq 0$ such that $\mu(X \setminus A) = 0$ and $G\mu$ is bounded.*

Let us say that a Borel set A in X is G -semipolar if there is no measure $\mu \neq 0$ such that $\mu(X \setminus A) = 0$ and $G\mu$ is continuous and real.

REMARKS 3.2. 1. A Borel set A is G -semipolar if there is no measure $\mu \neq 0$ such that $\mu(X \setminus A) = 0$ and $G\mu$ is continuous and *bounded*, which implies that every G -polar set is G -semipolar.

Indeed, suppose that A is not G -semipolar. Then there exists a measure $\mu \neq 0$ such that $\mu(X \setminus A) = 0$ and $G\mu$ is continuous and real. Of course, we may choose

a compact K in A such that $\mu(K) > 0$. Let $\nu = 1_K\mu$. Then $G\nu$ is continuous and real. In particular, $G\nu$ is bounded on every compact in X . Choosing a compact neighborhood L of K such that G is bounded by some constant a on $(X \setminus L) \times K$, we obtain that $G\nu(x) = \int G(x, y) d\nu(y) \leq a\nu(K) < \infty$ for every $x \in X \setminus L$.

2. If $A_j, j \in \mathbb{N}$, are Borel sets in X , then the union $A := \bigcup_{j=1}^{\infty} A_j$ is G -semipolar if and only if every A_j is G -semipolar.

Indeed, suppose that the sets $A_j, j \in \mathbb{N}$, are G -semipolar and consider $\mu \in \mathcal{M}(X)$ such that $\mu(X \setminus A) = 0$ and $G\mu$ is continuous and real. Defining $\mu_j := 1_{A_j}\mu$ we know that $\mu_j(X \setminus A_j) = 0$ and $G\mu_j$ is continuous and real, hence $\mu_j = 0$ for $j \in \mathbb{N}$. So $\mu = 0$, and we see that A is G -semipolar. The converse is trivial.

We define G -balls $B(x, \rho)$ by

$$B(x, \rho) := \{y \in X : G(x, y) > \rho^{-1}\}, \quad x \in X, \rho > 0.$$

Let us observe that obviously, for all such balls $B(x, \rho)$ (which, by (ii), are Borel sets) and measures ν on X ,

$$(3.1) \quad \nu(B(x, \rho)) \leq \rho \int_{B(x, \rho)} G(x, y) d\nu(y) \leq \rho G\nu(x).$$

Moreover, we define an *intrinsic Hausdorff measure* m_G as follows. For every subset A of X , let

$$m_G^\delta(A) := \inf \left\{ \sum_{j=1}^{\infty} \rho_j : A \subset \bigcup_{j=1}^{\infty} B(x_j, \rho_j), x_j \in X, 0 < \rho_j < \delta \right\}, \quad \delta > 0,$$

$$m_G(A) := \lim_{\delta \rightarrow 0} m_G^\delta(A).$$

Let us note right away that, for every subset A of X ,

$$(3.2) \quad m_G(A) = m_G(\tilde{A}) \quad \text{for some Borel set } \tilde{A} \text{ containing } A.$$

If $k \in \mathbb{N}$ and $\alpha_k := m_G^{1/k}(A) < \infty$, we choose a covering of A by G -balls $B(x_{jk}, \rho_{jk})$, $j \in \mathbb{N}$, with $\rho_{jk} < 1/k$ and $\sum_{j=1}^{\infty} \rho_{jk} < \alpha_k + 1/k$, and define $A_k := \bigcup_{j=1}^{\infty} B(x_{jk}, \rho_{jk})$. Taking $A_k := X$ otherwise, the set $\tilde{A} := \bigcap_{k=1}^{\infty} A_k$ has the desired property.

Our main result is the following.

THEOREM 3.3. *If a set A in X satisfies $m_G(A) < \infty$, then it is contained in a G -semipolar Borel set.*

In its proof we shall use the following simple generalization of the Lebesgue dominated convergence theorem (straightforward application of Fatou's lemma both to the sequence (f_n) and the sequence $(g_n - f_n)1_{\{g_n < \infty\}}$).

LEMMA 3.4. *Let $(Y, \mathcal{A}, \lambda)$ be a measure space and let f_n, g_n be \mathcal{A} -measurable functions on Y such that $0 \leq f_n \leq g_n$, the sequence (f_n) converges pointwise to f , the sequence (g_n) converges pointwise to g , and $\lim_{n \rightarrow \infty} \int g_n d\lambda = \int g d\lambda < \infty$. Then $\lim_{n \rightarrow \infty} \int f_n d\lambda = \int f d\lambda$.*

Proof of Theorem 3.3. By (3.2), it suffices to consider a Borel set A in X .

a) Suppose that A is not G -semipolar. Then we may choose $\mu_0 \in \mathcal{M}(X)$ such that $\mu_0(A) > 0$, $\mu_0(X \setminus A) = 0$, and $G\mu_0$ is continuous and real. Let K be a compact in A such that $\mu_0(K) > 0$, and define $\mu := 1_K\mu_0$. By (i), $\limsup_{x \rightarrow y} G(x, y) = \infty$, and hence

$$(3.3) \quad \mu(\{y\}) = 0 \quad \text{for every } y \in X.$$

b) Now let us fix a decreasing continuous function φ on $[0, \infty]$ such that $\varphi = 1$ on $[0, 1]$ and $\varphi = 0$ on $[2, \infty]$. For $x \in X$ and $\rho > 0$, let

$$\mu_{x,\rho} := \varphi\left(\frac{1}{\rho G(x, \cdot)}\right)\mu \quad \text{and} \quad p_\rho(x) := G\mu_{x,\rho}(x).$$

Then

$$(3.4) \quad 1_{B(x,\rho)}\mu \leq \mu_{x,\rho} \leq 1_{B(x,2\rho)}\mu \quad \text{and} \quad p_\rho(x) \leq p_\sigma(x) \text{ if } \rho \leq \sigma.$$

Since $\mu(\{x\}) = 0$, by (3.3), and since $\bigcap_{\rho>0} B(x, 2\rho)$ is either empty or $\{x\}$, we see, in particular, that $p_\rho(x) \downarrow 0$ as $\rho \downarrow 0$.

Next we claim that all functions p_ρ are continuous. To that end let us fix $\rho > 0$, $x_0 \in X$, and $\varepsilon > 0$. Since $\mu(\{x_0\}) = 0$, we may choose a continuous function $0 \leq \psi \leq 1$ which equals 1 on a neighborhood V_0 of x_0 and satisfies $G(\psi\mu)(x_0) < \varepsilon$. Then $G(\psi\mu)$ is continuous and real. So there exists a compact neighborhood K_0 of x_0 in V_0 such that $G(\psi\mu) < \varepsilon$ on K_0 . For every $x \in K_0$, let

$$\varphi_x := (1 - \psi)\varphi\left(\frac{1}{\rho G(x, \cdot)}\right), \quad \mu_x := \varphi_x\mu, \quad q(x) := G\mu_x(x).$$

Then $0 \leq \varphi_x \leq 1$ and, for every $x \in K_0$,

$$(3.5) \quad p_\rho(x) - q(x) = G(\psi\mu_{x,\rho})(x) \leq G(\psi\mu)(x) < \varepsilon.$$

To finish the proof of our claim let us finally fix a sequence (x_n) in K_0 converging to x_0 . If $y \in X \setminus V_0$, then

$$\lim_{n \rightarrow \infty} G(x_n, y) = G(x_0, y) \quad \text{and} \quad \lim_{n \rightarrow \infty} \varphi_{x_n}(y) = \varphi_{x_0}(y).$$

If $y \in V_0$, then $\varphi_{x_n}(y) = 0 = \varphi_{x_0}(y)$ for every $n \in \mathbb{N}$. Hence the sequence $(\varphi_{x_n}G(x_n, \cdot))$ converges pointwise to $\varphi_{x_0}G(x_0, \cdot)$. Since $\lim_{n \rightarrow \infty} G\mu(x_n) = G\mu(x_0)$ and $G\mu(x_0)$ is finite, we thus conclude, by Lemma 3.4, that

$$\lim_{n \rightarrow \infty} q(x_n) = q(x_0).$$

If n is large enough, then $|q(x_n) - q(x_0)| < \varepsilon$, and $|p_\rho(x_n) - p_\rho(x_0)| < 3\varepsilon$, by (3.5).

c) By (iii), there exist a compact neighborhood L of K and $a > 0$ such that $G \leq a$ on $(X \setminus L) \times K$. By Dini's theorem, there exists a decreasing sequence (τ_n) such that $0 < \tau_n < 1/n$ and

$$(3.6) \quad G\mu_{x,\tau_n}(x) = p_{\tau_n}(x) < 2^{-n}, \quad x \in L,$$

for every $n \in \mathbb{N}$. For the moment, let us fix $k \in \mathbb{N}$, $x \in L$, $0 < \rho < \tau_k$ and define

$$\nu := \sum_{n=1}^k \mu_{x, \tau_n}.$$

By (3.1) and (3.6), $\nu(B(x, \rho)) \leq \rho G\nu(x) < \rho$. By (3.4), $\mu_{x, \tau_n}(B(x, \tau_n)) = \mu(B(x, \tau_n))$ for every $1 \leq n \leq k$. Thus

$$(3.7) \quad k\mu(B(x, \rho)) = \sum_{n=1}^k \mu_{x, \tau_n}(B(x, \rho)) = \nu(B(x, \rho)) < \rho.$$

d) Again let $k \in \mathbb{N}$ and let us fix $0 < \delta < \min\{\tau_k, a^{-1}\}$. Let us consider $x_j \in X$ and $0 < \rho_j < \delta$, $j \in \mathbb{N}$, such that the sets $B(x_j, \rho_j)$ cover K . If $j \in \mathbb{N}$ such that $x_j \in X \setminus L$, then $G(x_j, y) \leq a < \delta^{-1} < \rho_j^{-1}$ for all $y \in K$, and thus $B(x_j, \rho_j) \cap K = \emptyset$. So we may assume that $x_j \in L$ for every $j \in \mathbb{N}$. Then, by (3.7),

$$k\mu(K) \leq k \sum_{j=1}^{\infty} \mu(B(x_j, \rho_j)) < \sum_{j=1}^{\infty} \rho_j.$$

Thus $m_G^\delta(K) \geq k\mu(K)$. This shows that $m_G(K) = \infty$, and hence $m_G(A) = \infty$. \square

To deal with G -polar sets, we define, for sets A in X ,

$$c(A) := \sup\{\nu^*(A) : \nu \in \mathcal{M}(X), G\nu \leq 1\}$$

($\nu^*(A)$ denoting the outer measure of A). If A is universally measurable, then

$$c(A) := \sup\{\nu(A) : \nu \in \mathcal{M}(X), \nu(X \setminus A) = 0, G\nu \leq 1\}$$

(it suffices to note that whenever $\nu \in \mathcal{M}(X)$ such that $G\nu \leq 1$, then $\nu' := 1_A\nu$ satisfies $G\nu' \leq 1$ and $\nu'(A) = \nu(A)$). In particular, a Borel set A in X is G -polar if and only if $c(A) = 0$.

PROPOSITION 3.5. *For every set A in X , $c(A) \leq m_G(A)$.*

Proof. Let $\nu \in \mathcal{M}(X)$ such that $G\nu \leq 1$, and let $x_j \in X$ and $\rho_j > 0$, $j \in \mathbb{N}$, such that $A \subset \bigcup_{j=1}^{\infty} B(x_j, \rho_j)$. Then, by (3.1),

$$\nu^*(A) \leq \sum_{j=1}^{\infty} \nu(B(x_j, \rho_j)) \leq \sum_{j=1}^{\infty} \rho_j G\nu(x_j) \leq \sum_{j=1}^{\infty} \rho_j.$$

This clearly implies that $c(A) \leq m_G(A)$. \square

COROLLARY 3.6. *If a set A in X satisfies $m_G(A) = 0$, then it is contained in a G -polar Borel set.*

4 Application to harmonic spaces and balayage spaces

Let (X, \mathcal{H}) be a \mathcal{P} -harmonic space or, more generally, let (X, \mathcal{W}) be a balayage space; see [4, 5, 22, 8]. Let us assume that the constant function 1 is superharmonic and that we have a (Green) function $G: X \times X \rightarrow [0, \infty]$ having the following properties (note that (i') implies (i) from Section 3):

- (i') For every $y \in X$, the function $G(\cdot, y)$ is a potential on X which has superharmonic support $\{y\}$ and satisfies $\limsup_{x \rightarrow y} G(x, y) = \infty$.
- (ii) G is continuous outside the diagonal Δ and Borel measurable on Δ .
- (iii) For every compact K in X , there exists a compact L in X such that G is bounded on $(X \setminus L) \times K$.
- (iv) For every continuous real potential p on X having compact superharmonic support there exists a measure $\mu \in \mathcal{M}(X)$ such that

$$p = G\mu := \int G(\cdot, y) d\mu(y).$$

In the case of a harmonic space (X, \mathcal{H}) , property (iii) follows from (i') and (ii). Indeed, let K be a compact in X and let L be any compact neighborhood of X . Then G is bounded on $\partial L \times K$ by some $a > 0$, and hence $G \leq a$ on $(X \setminus L) \times K$, by the minimum principle; see [5, III.6.6].

Let us recall that, for all subsets A of X and superharmonic functions $u \geq 0$, the reduced function R_u^A , defined to be the infimum of all positive superharmonic functions majorizing u on A , is harmonic on $X \setminus \bar{A}$, and that its greatest lower semicontinuous minorant \hat{R}_u^A (the balayage of u on A) is superharmonic on X .

A subset A of X is called *polar* if $\hat{R}_1^A = 0$ (or, equivalently, if $\hat{R}_u^A = 0$ for every superharmonic function $u \geq 0$ on X). It is called *semipolar* if it is a countable union of sets T_j , $j \in \mathbb{N}$, which are *totally thin*, that is, such that, for every $x \in X$, there exists a superharmonic function $u \geq 0$ on X with $\hat{R}_u^{T_j}(x) < u(x)$. Of course, every polar set is totally thin and hence semipolar. Moreover, it is immediately seen that countable unions of polar (semipolar) sets are polar (semipolar) and that every subset of a polar (totally thin, semipolar) set is polar (totally thin, semipolar).

PROPOSITION 4.1. *Let A be a Borel set in X . Then A is semipolar if and only if A is G -semipolar.*

Proof. Immediate consequence of (iv) and [5, VI.8.8] (using Remark 3.2,2). □

COROLLARY 4.2. *If $A \subset X$ satisfies $m_G(A) < \infty$, then A is semipolar.*

Proof. Theorem 3.3 and Proposition 4.1. □

To get a corresponding result for polar sets let us suppose from now on in this section that the following holds:

- (*) For every compact K in X , the potential \hat{R}_1^K is harmonic on $X \setminus K$.

We observe that (*) does not hold in general; see [5, V.9.1] for an example of a balayage space and a compact K such that \hat{R}_1^K is not harmonic on $X \setminus K$.

However, (*) is satisfied in the case of a harmonic space, since R_1^K is harmonic on $X \setminus K$, and hence $\hat{R}_1^K = R_1^K$ on $X \setminus K$. It also holds in the case of a balayage space, where every semipolar set is polar (since then the semipolar sets $\{\hat{R}_1^K < R_1^K\}$ are polar and harmonic measures do not charge polar sets; see [5, VI.5.11 and VI.5.6]).

Moreover, we assume that G has the following property:

(iv') For every bounded potential p with compact superharmonic support there exists a measure $\mu \in \mathcal{M}(X)$ such that $p = G\mu$.

Clearly, (iv') implies (iv), since, by the minimum principle, every continuous real potential p with compact superharmonic support K is bounded by its maximum on K ; see [5, III.6.6].

Conversely, by [24, Theorem 4.1], (iv') is a consequence of (iv) if, for every $x \in X$, the function $G(x, \cdot)$ is lower semicontinuous at x and, if x is finely isolated (and not isolated), continuous at x .

PROPOSITION 4.3. *Every G -polar Borel set in X is polar.*

Proof. Let A be a Borel set in X which is not polar. By [5, VI.5.5], there exists a compact K in A such that K is not polar, that is, $\hat{R}_1^K \neq 0$. Of course, $\hat{R}_1^K \leq 1$. By (iv'), $\hat{R}_1^K = G\mu$ for some $\mu \in \mathcal{M}(X)$. By (*), \hat{R}_1^K is harmonic on $X \setminus K$, and hence $\mu(X \setminus K) = 0$, by (i'). Therefore A is not G -polar. \square

COROLLARY 4.4. *If $A \subset X$ satisfies $m_G(A) = 0$, then A is polar.*

Proof. Corollary 3.6 and Proposition 4.3. \square

5 Application to standard examples

5.1 Classical potential theory and Riesz potentials

Let $X = \mathbb{R}^n$, $0 < \beta < n$, $0 < \beta \leq 2$. If $\beta = 2$ (and $n \geq 3$), let $(\mathbb{R}^n, \mathcal{H})$ be the harmonic space associated with the Laplace operator. If $\beta < 2$, let $(\mathbb{R}^n, \mathcal{W})$ be the balayage space, where every superharmonic function is an increasing limit of Riesz potentials; see [5, Section V.4]. In both cases, the function G defined by

$$G(x, y) := |x - y|^{\beta-n}, \quad x, y \in \mathbb{R}^n,$$

has the properties (i'), (ii), (iii) and (iv') and every semipolar set is polar. Of course, for all $x \in \mathbb{R}^n$ and $\rho > 0$,

$$\{y \in \mathbb{R}^n : G(x, y) > \rho^{-1}\} = \{y \in \mathbb{R}^n : |x - y| < \rho^{n-\beta}\}.$$

Hence m_G is the usual $(n - \beta)$ -dimensional Hausdorff measure $m_{t^{n-\beta}}$ on \mathbb{R}^n . Therefore Corollary 4.2 yields the following,

THEOREM 5.1. *If a set A in \mathbb{R}^n satisfies $m_{t^{n-\beta}}(A) < \infty$, then A is polar.*

Applying Corollary 4.2 to classical Green functions G_D on discs D in \mathbb{R}^2 , we obtain the following.

THEOREM 5.2. *Let $A \subset \mathbb{R}^2$ and $\phi(t) := \log^+(1/t)$, $t > 0$. If $m_\phi(A) < \infty$, then A is polar (with respect to classical potential theory).*

In particular, we have recovered known results dealing with classical or Riesz potential theory; see [1, Theorem 5.9.4], [6, Theorem IV.1], [26, Theorem 3.14].

5.2 Heat equation

In this subsection we consider the harmonic space $(\mathbb{R}^{n+1}, \mathcal{H})$ associated with the heat equation with Green function G defined by (1.2) and (1.3). Then G obviously has the properties (i) – (iii) from Section 3, and property (iv') follows from [31, Corollary 6.39 and Theorem 6.34].

Let $x' \in \mathbb{R}^{n+1}$, $\rho > 0$. Then $B(x', \rho) = x' + B(0, \rho)$, where $B(0, \rho)$ denotes the *heat ball with center 0 and radius $\rho^{2/n}$* , that is,

$$(5.1) \quad B(0, \rho) := \left\{ (y, -s) : y \in \mathbb{R}^n, 0 < s < \rho^{2/n}, |y|^2 < 2ns \log \frac{\rho^{2/n}}{s} \right\};$$

in particular, $B(0, \rho)$ is convex and contained in the cylinder

$$\{(y, -s) : y \in \mathbb{R}^n, 0 < s < \rho^{2/n}, |y| < n^{1/2} \rho^{1/n}\};$$

see [31, page 2]. Hence, by our definition (1.1),

$$(5.2) \quad B(x', \rho) \subset P(x', 2n^{1/2} \rho^{1/n}).$$

Further, we claim that, defining $z' := (0, \rho^2)$,

$$(5.3) \quad P(-z', \rho) \subset B(0, 2^{n/2} \rho^n)$$

and hence, by translation invariance,

$$(5.4) \quad P(x', \rho) \subset B(x' + z', 2^{n/2} \rho^n).$$

To prove (5.3) we only have to show that every vertex of $P(-z', \rho)$ is contained in the convex set $B(0, 2^{n/2} \rho^n)$ which, by (5.1), is the set of all $(y, -s) \in \mathbb{R}^{n+1}$ such that

$$(5.5) \quad 0 < s < 2\rho^2 \quad \text{and} \quad |y|^2 < 2ns \log \frac{2\rho^2}{s}.$$

Such a vertex has the form $(y, -s)$ with $s = (k/2)\rho^2$, $k \in \{1, 3\}$ and $|y|^2 = n(\rho/2)^2$, hence it is contained in $B(0, 2^{n/2} \rho^n)$ if

$$\frac{1}{4} < 2 \frac{k}{2} \log \frac{4}{k}, \quad k \in \{1, 3\}.$$

This holds, of course, for $k = 1$. It also holds for $k = 3$ since already $4 \log(4/3) = \log(256/81) > 1$. So (5.4) holds.

PROPOSITION 5.3. *For every subset A of \mathbb{R}^{n+1} ,*

$$m_{n,P}(A) \leq 2^n n^{n/2} m_G(A) \quad \text{and} \quad m_G(A) \leq 2^{n/2} m_{n,P}(A).$$

Proof. By our definitions, $m_G = m_{1,B}$; see Section 2. Hence the first inequality follows from (5.2) and Proposition 2.1 with $F = B$ and $\eta = 1$, $\tilde{F} = P$ and $\tilde{\eta} = n$, $\kappa = 2n^{1/2}$.

The second inequality follows from (5.4) and Proposition 2.1 with $F = P$ and $\eta = n$, $\tilde{F} = B$ and $\tilde{\eta} = 1$, $\kappa = 2^{n/2}$. \square

Let us next observe that, for m_P from Section 1 and $m_{n,P}$ we have

$$(5.6) \quad m_P = n^{n/2} m_{n,P}.$$

Indeed, if $\delta > 0$, then $\sqrt{n}\rho \leq \text{diam } P(x', \rho) \leq \sqrt{n}\rho(1 + \delta)$ for all $x' \in \mathbb{R}^{n+1}$ and $0 < \rho < \delta$, and hence

$$n^{n/2} m_{n,P}^\delta \leq m_P^\delta \leq (1 + \delta)^n n^{n/2} m_{n,P}^\delta,$$

which clearly implies (5.6). So we may state Proposition 5.3 in the following way.

COROLLARY 5.4. *For every subset A of \mathbb{R}^{n+1} ,*

$$(2n)^{-n} m_P(A) \leq m_G(A) \leq (2/n)^{n/2} m_P(A).$$

In particular, $(1/2)m_P \leq m_G \leq \sqrt{2}m_P$, if $n = 1$.

So Corollaries 4.2 and 4.4 yield the following, where the first statement is the positive answer to the question raised in [30, page 330] and the second one is [30, Theorem 1].

THEOREM 5.5. *Let A be a subset of X .*

1. *If $m_P(A) < \infty$, then A is semipolar.*
2. *If $m_P(A) = 0$, then A is polar.*

Let us note that an analogous result to that in Theorem 5.5.2 was established in [28, Theorem 3], where a modified heat kernel is investigated.

Now let us suppose that $n = 1$ and show that subsets of vertical lines in \mathbb{R}^2 are polar if they are semipolar (as we already noticed, this is not true for subsets of horizontal lines).

PROPOSITION 5.6. *Every semipolar set A in $\{0\} \times \mathbb{R}$ is polar.*

Proof. Let A be a semipolar set in $\{0\} \times \mathbb{R}$. By [5, VI.5.7.3], we may suppose that A is a Borel set. We claim that every compact K in A is polar, and hence A is polar, by [5, VI.5.5].

To that end suppose that A contains a compact K which is not polar, and let ν be the thermal capacity distribution of K ; see [31, Definition 7.33]. Then $\nu \neq 0$, ν is supported by K , and $G\nu \leq 1$. By Lusin's theorem, there exists a compact \tilde{K} in K such that $\tilde{\nu} := 1_{\tilde{K}}\nu \neq 0$ and the restriction of $G\nu$ to \tilde{K} is continuous. Since both $G\tilde{\nu}$ and $G(\nu - \tilde{\nu})$ are lower semicontinuous, we see that also the restriction of $G\nu$ to \tilde{K} is continuous. By [9, Theorem 5], this implies that $G\tilde{\nu}$ is continuous on \mathbb{R}^2 . So \tilde{K} is not semipolar, a contradiction. \square

Let us remark that a set in $\{0\} \times \mathbb{R}$ is polar if and only if it has zero Riesz $\frac{1}{2}$ -capacity; see [25, Theorem 2].

6 Application to space-time processes

In this section we shall consider a space-time setting which is more general than for the heat equation and which will be discussed in the situation of heat semigroups on metric measure spaces in Section 7.

As in [23, Sections 7 and 8] we assume the following. Let $X \neq \emptyset$ be a locally compact space with countable base, $X' := X \times \mathbb{R}$, and let $\mathcal{B}^+(X)$, $\mathcal{B}^+(X')$ denote the set of all Borel measurable positive numerical functions on X , X' respectively. We suppose that we have a measure μ on X , not charging points, and a strictly positive continuous real function $(x, y, t) \mapsto p_t(x, y)$ on $X \times X \times (0, \infty)$ satisfying the Chapman-Kolmogorov equations:

(CK) For all $s, t \in (0, \infty)$ and $x, y \in X$,

$$p_{s+t}(x, y) = \int p_s(x, z)p_t(z, y) d\mu(z).$$

For $t > 0$, $f \in \mathcal{B}^+(X)$ and $x \in X$, let

$$P_t f(x) := \int p_t(x, y)f(y) d\mu(y).$$

Then $\mathbb{P} := (P_t)_{t>0}$ is a semigroup on X . Let $E_{\mathbb{P}}$ denote the corresponding cone of excessive functions, that is,

$$E_{\mathbb{P}} := \{u \in \mathcal{B}^+(X) : \sup_{t>0} P_t u = u\}.$$

We suppose that, in addition, the following holds.

(E) $1 \in E_{\mathbb{P}}$.

(C) For all $x_0, y_0 \in X$,

$$\limsup_{(x,t) \rightarrow (x_0,0)} p_t(x, x_0) = \infty \quad \text{and} \quad \lim_{(x,y,t) \rightarrow (x_0,y_0,0)} p_t(x, y) = 0, \quad \text{if } x_0 \neq y_0.$$

(KL) For all compacts K in X , $T > 0$ and $\varepsilon > 0$, there exists a compact L in X such that

$$\|1_{X \setminus L} p_t(\cdot, y)\|_{L^1(\mu)} + \|1_{X \setminus L} p_t(\cdot, y)\|_{\infty} < \varepsilon \quad \text{for all } (y, t) \in K \times (0, T].$$

In [23] we were supposing that also the dual function $(x, y, t) \mapsto p_t(y, x)$ has the properties above. However, since this played a role only starting with [23, Lemma 8.5], we shall not need that here.

Let $\mathbb{T} := (T_t)_{t>0}$ denote the semigroup of uniform translation to the left, that is, $T_t(r, \cdot) := \varepsilon_{r-t}$. We define

$$\mathbb{P}' := \mathbb{P} \otimes \mathbb{T}$$

which means that $\mathbb{P}' = (P'_t)_{t>0}$, where, for all $t > 0$, $f \in \mathcal{B}^+(X')$ and $(x, r) \in X'$,

$$P'_t f(x, r) := P_t f(\cdot, r - t)(x).$$

Clearly, \mathbb{P}' is a semigroup on X' . Let $E_{\mathbb{P}'}$ denote its cone of excessive functions. Then, by [23, Proposition 8.1] and (E), the following holds; cf. also [5, Section V.5].

PROPOSITION 6.1. $(X', E_{\mathbb{P}'})$ is a balayage space, $1 \in E_{\mathbb{P}'}$.

We define $G' : X' \times X' \rightarrow [0, \infty)$ by

$$G'((x, r), (y, s)) := \begin{cases} p_{r-s}(x, y), & \text{if } r > s, \\ 0, & \text{if } r \leq s. \end{cases}$$

THEOREM 6.2. The function G' has the properties (i'), (ii), (iii) and (iv') of Section 4.

Proof. By (C), the function G' is continuous outside the diagonal in $X' \times X'$ and lower semicontinuous on $X' \times X'$. Hence G' satisfies (ii). Moreover, (i') holds, by [23, Proposition 8.4].

Property (iii) is a consequence of (KL), (CK) and (E). Indeed, let K' be a compact in X' . Then there exist a compact K in X , $t_0 \in \mathbb{R}$, and $T > 0$, such that K' is contained in $K \times [t_0, t_0 + T]$. By (KL), there exists a compact L in X such that

$$(6.1) \quad p_t(x, y) \leq 1 \quad \text{for all } x \in X \setminus L, y \in K \text{ and } t \in (0, T].$$

Let $a \geq 1$ be such that $p_T \leq a$ on the compact $L \times K$. By (6.1), $p_T \leq 1$ on $(X \setminus L) \times K$. Therefore $p_T \leq a$ on $X \times K$ and hence

$$(6.2) \quad p_s(x, y) = \int p_{s-T}(x, z) p_T(z, y) d\mu(z) \leq a P_{s-T} 1(x) \leq a$$

for all $x \in X$, $y \in K$ and $s > T$. Defining $L' := L \times [t_0, t_0 + T]$ and combining (6.1) and (6.2), we obtain that $G' \leq a$ on $(X' \setminus L') \times K'$.

To see that G' has the property (iv') we recall that the potential kernel V' of \mathbb{P}' is proper and, defining $m := \mu \otimes \lambda_{\mathbb{R}}$,

$$\begin{aligned} V'f(x, r) &= \int_0^\infty P'_t f(x, r) dt = \int_0^\infty P_t f(\cdot, r-t)(x) dt \\ &= \int_0^\infty \left(\int_X p_t(x, y) f(y, r-t) d\mu(y) \right) dt \\ &= \int G'((x, r), (y, s)) f(y, s) dm(y, s) \end{aligned}$$

for every $f \in \mathcal{B}^+(X')$; see [23, page 674].

Let q be a potential for the balayage space $(X', E_{\mathbb{P}'})$. By [5, II.3.11], there exist bounded functions $f_n \in \mathcal{B}^+(X')$ such that $V'f_n \uparrow q$ as $n \rightarrow \infty$. Defining $\mu_n := f_n m$ we hence know that $G'\mu_n \uparrow q$. Moreover, we note that every balayage space contains only countably many finely isolated points; see [5, III.7.2]. Hence there is no point $(x, r) \in X \times \mathbb{R}$ which is finely isolated, since otherwise *all* points (x, t) , $t \in \mathbb{R}$, would be finely isolated. By [24, Theorem 1.1], we finally conclude that there exists $\mu' \in \mathcal{M}(X')$ such that $q = G'\mu'$; cf. also [23, Remark 8.7] and the references therein. \square

Since $(X', E_{\mathbb{P}'})$ is a balayage space, there exists a Hunt process \mathfrak{X}' on X' with transition semigroup \mathbb{P}' , the *space-time process* associated with $(x, y, t) \mapsto p_t(x, y)$ and μ ; see [5, Theorem IV.8.1] and its proof. So, by Corollary 4.2, we have the following result.

COROLLARY 6.3. *If A is a subset of X' such that $m_{G'}(A) < \infty$, then there exists a Borel set \tilde{A} containing A which is semipolar, that is, the process \mathfrak{X}' hits the set \tilde{A} at most countably many times.*

7 Application to heat kernels on metric measure spaces

Let (X, d) be a separable metric space, $X \neq \emptyset$, where balls $D(x, r) := \{d(\cdot, x) < r\}$ are relatively compact, let μ be a positive Radon measure on X with full support, and suppose that we have a continuous positive real function

$$(7.1) \quad (x, y, t) \mapsto p_t(x, y) \quad \text{on } X \times X \times (0, \infty)$$

such that the following holds:

(CK) *Chapman-Kolmogorov equations:* For all $s, t \in (0, \infty)$ and $x, y \in X$,

$$p_{s+t}(x, y) = \int p_s(x, z)p_t(z, y) d\mu(z).$$

(E) For every $x \in X$,

$$\sup_{t>0} \int p_t(x, y) d\mu(y) = 1.$$

(B) There exist constants $\alpha, \beta > 0$ and positive decreasing real functions Φ_1, Φ_2 on $[0, \infty)$ such that $\Phi_1(1) > 0$, $\int_0^\infty \sigma^{\alpha-1}\Phi_2(\sigma) d\sigma < \infty$ and

$$(7.2) \quad \frac{1}{t^{\alpha/\beta}}\Phi_1\left(\frac{d(x, y)}{t^{1/\beta}}\right) \leq p_t(x, y) \leq \frac{1}{t^{\alpha/\beta}}\Phi_2\left(\frac{d(x, y)}{t^{1/\beta}}\right)$$

for all $x, y \in X$ and $t > 0$.

REMARKS 7.1. 1. See [17, (1.5) and (H₀) on p. 2067] for assumption (B) and [21] as well as [16, Theorem 2.10] for conditions implying the continuity of (7.1). For the definition of an abstract heat kernel, the relation to Dirichlet forms and a discussion of various examples see [18]. For manifolds, (E) follows from [12, (7.50) and (7.53) in Theorem 7.13].

2. A striking fact is the following dichotomy; see [20, Theorem 4.1]. Suppose that we have an abstract heat kernel satisfying (7.2) with functions of the form $\Phi_j(\sigma) = C_j\Phi(c_j\sigma)$, where $C_j, c_j \in (0, \infty)$ and $\Phi: [0, \infty) \rightarrow [0, \infty)$ is decreasing with $\Phi(\sigma_0) > 0$ for some $\sigma_0 > 0$. Then, under mild additional assumptions on X and $p_t(x, y)$, either $\beta \geq 2$ and (7.2) holds with

$$(7.3) \quad \Phi(\sigma) = \exp\left(-\sigma^{\frac{\beta}{\beta-1}}\right),$$

leading to *sub-Gaussian* lower and upper bounds, or (7.2) holds with

$$(7.4) \quad \Phi(\sigma) = (1 + \sigma)^{-(\alpha+\beta)},$$

leading to *stable-like* lower and upper bounds. In the case (7.3) the corresponding process will be a diffusion, in the case (7.4) it will be a jump process.

3. A special case for sub-Gaussian bounds is, of course, the classical Gauss-Weierstrass kernel on $X = \mathbb{R}^n$ (with $d(x, y) = |x - y|$ and Lebesgue measure μ), where $\alpha = n$, $\beta = 2$, $C_1 = C_2 = (4\pi)^{-n/2}$ and $c_1 = c_2 = 1/4$. See also [27] and [10] for two-sided Gaussian bounds (sub-Gaussian bounds with $c_1 = c_2$) for heat kernels on manifolds. Many further examples for sub-Gaussian bounds (with $\beta > 2$) are given by fractal spaces like Sierpinski gaskets and carpets; see [2] and [3]. For properties which in the setting of regular local Dirichlet forms are equivalent to two-sided sub-Gaussian bounds we refer the reader to [21], [14] and [19].

4. A special case with stable-like bounds is the β -stable heat kernel given by the fractional Laplacian $(-\Delta)^{\beta/2}$ on \mathbb{R}^n with $\alpha = n$ and $0 < \beta < \min\{2, n\}$. More generally, subordination applied to heat kernels with Gaussian bounds leads to heat kernels with stable-like bounds; see [29] and [11] for further examples obtained by subordination. For a direct approach on d -sets see [7]. Moreover, see [15] for properties characterizing the existence of two-sided stable like bounds in the setting of regular Dirichlet forms having a jumping part (but no killing part).

Let us now verify the properties (C) and (KL) introduced in the previous section. To that end we define

$$(7.5) \quad \varphi_2(\sigma) := \sigma^\alpha \Phi_2(\sigma), \quad \sigma > 0,$$

and observe that taking $\sigma := d(x, y)t^{-1/\beta}$ the upper bound $q_t(x, y)$ in (7.2) can be written as

$$(7.6) \quad q_t(x, y) = t^{-\alpha/\beta} \Phi_2(\sigma) = d(x, y)^{-\alpha} \varphi_2(\sigma).$$

Moreover, we note that $\lim_{\sigma \rightarrow \infty} \varphi_2(\sigma) = 0$ and hence, in particular,

$$M := \sup\{\varphi_2(\sigma) : 0 \leq \sigma < \infty\} < \infty.$$

Indeed, let $I_k := [2^k, 2^{k+1}]$ and $\gamma_k := \inf \varphi_2(I_k)$, $k \in \mathbb{N}$. Clearly, $\int_{I_k} \varphi_2(\sigma) \sigma^{-1} d\sigma \geq \gamma_k/2$ for every $k \in \mathbb{N}$, and hence the integrability of $\sigma^{\alpha-1} \Phi_2$ implies that $\lim_{k \rightarrow \infty} \gamma_k = 0$. Since Φ_2 is decreasing and $\sigma^\alpha \leq 4^\alpha \tau^\alpha$ for all $\sigma, \tau \in I_k \cup I_{k+1}$, we obtain that $\varphi_2 \leq 4^\alpha \gamma_k$ on I_{k+1} , $k \in \mathbb{N}$. Thus $\lim_{\sigma \rightarrow \infty} \varphi_2(\sigma) = 0$.

LEMMA 7.2. *Property (C) of Section 6 holds.*

Proof. The lower estimate in (7.2) yields that $\lim_{t \rightarrow 0} p_t(x_0, x_0) = \infty$ for all $x_0 \in X$. Since $\lim_{\sigma \rightarrow \infty} \varphi_2(\sigma) = 0$, (7.2) and (7.6) imply that $\lim_{(x, y, t) \rightarrow (x_0, y_0, 0)} p_t(x, y) = 0$, whenever x_0, y_0 are different points in X . \square

LEMMA 7.3. *Let K be a compact in X and $\varepsilon > 0$. Then there exist $T > 0$ and a compact L in X such that, for $t > 0$, $x \in X$ and $y \in K$,*

$$(7.7) \quad p_t(x, y) < \varepsilon, \quad \text{whenever } t > T \text{ or } x \in X \setminus L.$$

Proof. There exist $y_0 \in X$ and $R > 0$ such that $K \subset D(y_0, R)$. Let us choose $T, N \in (0, \infty)$ such that $T^{-\alpha/\beta} \Phi_2(0) < \varepsilon$ and $N^{-\alpha} M < \varepsilon$. Let L be the closure of $D(x_0, R + N)$ and $t > 0$, $x \in X$, $y \in K$.

If $t > T$, then $p_t(x, y) < \varepsilon$, by (7.2) and our choice of T . If $x \in X \setminus L$, then $d(x, y) > N$, and hence $p_t(x, y) < \varepsilon$, by (7.6) and our choice of N . \square

Having (E) and the lower bound in (B), we obtain that, taking $c_\mu := \Phi_1(1)^{-1}$,

$$(7.8) \quad \mu(D(x, r)) \leq c_\mu r^\alpha \quad \text{for all } x \in X \text{ and } r > 0.$$

In particular, μ does not charge points and no point in X is isolated. Indeed, it suffices to observe that, by the first inequality in (7.2),

$$1 \geq \int p_{r^\beta}(x, y) d\mu(y) \geq \mu(D(x, r)) \inf_{y \in D(x, r)} p_{r^\beta}(x, y) \geq \mu(D(x, r)) r^{-\alpha} \Phi_1(1)$$

(cf. the first lines of the proof for [17, Theorem 3.2]).

LEMMA 7.4. *Property (KL) of Section 6 holds.*

Proof. Let K be a compact in X , $T > 0$ and $\varepsilon > 0$. We fix $y_0 \in X$ and $R \geq 1$ such that $K \subset D(y_0, R)$ and define

$$D_j := D(y_0, 2^j R), \quad j \in \mathbb{N}.$$

Since $\sigma^{\alpha-1} \Phi_2$ is integrable, there exists $k \in \mathbb{N}$ such that, defining $\sigma_k := 2^{k-2} R / T^{1/\beta}$,

$$(7.9) \quad 2^{3\alpha+1} c_\mu \int_{\sigma_k}^{\infty} \varphi_2(\sigma) \frac{d\sigma}{\sigma} < \varepsilon/2.$$

By Lemma 7.3, we may assume that $p_t(x, y) < \varepsilon/2$ whenever $y \in K$, $x \in X \setminus D_k$ and $t > 0$. We claim that the closure L of D_k has the desired properties.

So let us fix $0 < t \leq T$ and $y \in K$. If $j \in \mathbb{N}$ and $x \in D_{j+1} \setminus D_j$, then $d(x, y) \geq 2^{j-1} R$, and hence, by the monotonicity of Φ_2 and (7.8),

$$\int_{D_{j+1} \setminus D_j} p_t(x, y) d\mu(x) \leq \frac{1}{t^{\alpha/\beta}} \Phi_2 \left(\frac{2^{j-1} R}{t^{1/\beta}} \right) \cdot c_\mu (2^{j+1} R)^\alpha = 4^\alpha c_\mu \varphi_2 \left(\frac{2^{j-1} R}{t^{1/\beta}} \right).$$

If $b \in (0, \infty)$, then $\varphi_2(2b) = (2b)^\alpha \Phi_2(2b) \leq 2^{\alpha+1} \sigma^\alpha \Phi_2(\sigma) \cdot (b/\sigma)$ for every $\sigma \in [b, 2b]$, and therefore, integrating on $[b, 2b]$,

$$\varphi_2(2b) \leq 2^{\alpha+1} \int_b^{2b} \varphi_2(\sigma) \frac{d\sigma}{\sigma}.$$

Since $2^{k-2} R / t^{1/\beta} \geq \sigma_k$, we conclude that

$$\int_{X \setminus D_k} p_t(x, y) d\mu(x) \leq 4^\alpha c_\mu \sum_{j=k}^{\infty} \varphi_2 \left(\frac{2^{j-1} R}{t^{1/\beta}} \right) \leq 2^{3\alpha+1} c_\mu \int_{\sigma_k}^{\infty} \varphi_2(\sigma) \frac{d\sigma}{\sigma} < \varepsilon/2.$$

Thus (KL) holds. □

7.1 Semipolar sets in $X' = X \times \mathbb{R}$

Having verified the properties (C) and (KL) we may apply the results from Section 6. Let us first recall the definitions

$$P'_t f(x, r) := \int p_t(x, y) f(y, r - t) d\mu(y), \quad t > 0, (x, r) \in X', f \in \mathcal{B}^+(X'),$$

and

$$G'((x, r), (y, s)) := \begin{cases} p_{r-s}(x, y), & \text{if } r > s, \\ 0, & \text{if } r \leq s, \end{cases} \quad (x, r), (y, s) \in X'.$$

Then we know the following; see Proposition 6.1, Theorem 6.2 and Corollary 6.3.

THEOREM 7.5. 1. $\mathbb{P}' = (P'_t)_{t>0}$ is a sub-Markov semigroup on X' such that $(X', E_{\mathbb{P}'})$ is a balayage space with $1 \in E_{\mathbb{P}'}$.

2. There exists a Hunt process \mathfrak{X}' on X' with transition semigroup \mathbb{P}' .
3. G' is a Green function for $(X', E_{\mathbb{P}'})$ satisfying (i'), (ii), (iii), (iv') of Section 4.
4. Every set A in X' with $m_{G'}(A) < \infty$ is contained in a Borel set which is semipolar, that is, which the process \mathfrak{X}' hits at most countably many times.

Moreover, we shall see that $m_{G'}$ is equivalent to an anisotropic Hausdorff measure $m_{\alpha, \beta}$. To that end we first recall that in our setting, for $x' \in X'$ and $\rho > 0$,

$$B(x', \rho) = \{y' \in X' : G'(x', y') > 1/\rho\}$$

and that $m_{G'} = m_{1, B}$; see Section 2. That is, for every set A in X' ,

$$m_{G'}(A) = \liminf_{\delta \rightarrow 0} \left\{ \sum_{j=1}^{\infty} \rho_j : A \subset \bigcup_{j=1}^{\infty} B(x'_j, \rho_j), x'_j \in X', 0 < \rho_j < \delta \right\}.$$

We now consider cylinders $Z_\beta(x', \rho)$ in X' given, for $x' = (x, r) \in X'$ and $\rho > 0$, by

$$Z_\beta(x', \rho) := \{(y, s) \in X' : d(x, y) < \rho \text{ and } |r - s| < \rho^\beta\}$$

and define, for subsets A of X' and $\delta > 0$,

$$m_{\alpha, \beta}^\delta(A) := \inf \left\{ \sum_{j=1}^{\infty} \rho_j^\alpha : A \subset \bigcup_{j=1}^{\infty} Z_\beta(x'_j, \rho_j), x'_j \in X', 0 < \rho_j < \delta \right\},$$

,

$$m_{\alpha, \beta}(A) := \lim_{\delta \rightarrow 0} m_{\alpha, \beta}^\delta(A).$$

PROPOSITION 7.6. There exists $C > 0$ such that, for all $x' \in X'$ and $\rho > 0$,

$$(7.10) \quad B(x', \rho) \subset Z_\beta(x', (C\rho)^{1/\alpha}).$$

In particular, $m_{\alpha, \beta} \leq C m_{G'}$.

Proof. Let $x' = (x, r) \in X'$, $\rho > 0$ and $y' = (y, s)$ in $B(x', \rho)$. Then $t = r - s > 0$ and, taking $\sigma := d(x, y)t^{-1/\beta}$,

$$1/\rho < G'(x', y') = p_t(x, y) \leq t^{-\alpha/\beta} \Phi_2(\sigma) = d(x, y)^{-\alpha} \varphi_2(\sigma).$$

So $t < (\Phi_2(0)\rho)^{\beta/\alpha}$ and $d(x, y) < (M\rho)^{1/\alpha}$. That is, (7.10) holds with $C := \max\{\Phi_2(0), M\}$. An application of Proposition 2.1 completes the proof. \square

PROPOSITION 7.7. *There exists $\kappa > 0$ such that, for all $x' \in X'$ and $\rho > 0$, there is a point $z' \in X'$ such that*

$$(7.11) \quad Z_\beta(x', (\kappa\rho)^{1/\alpha}) \subset B(z', \rho).$$

In particular, $\kappa m_{G'} \leq m_{\alpha, \beta}$.

Proof. We choose $\eta \in (0, 1)$ such that $(3\eta)^{\alpha/\beta} < \Phi_1(0)$ and define $\kappa := \eta^{\alpha/\beta}$. Let $x' = (x, r) \in X'$, $\rho > 0$, $z' := (x, r + 2\eta\rho^{\beta/\alpha})$ and $y' = (y, s) \in Z_\beta(x', (\kappa\rho)^{1/\alpha})$. Then $|r - s| < (\kappa\rho^{1/\alpha})^\beta = \eta\rho^{\beta/\alpha}$ and $t := (r + 2\eta\rho^{\beta/\alpha}) - s \in (\eta\rho^{\beta/\alpha}, 3\eta\rho^{\beta/\alpha})$. Hence

$$\sigma := \frac{d(x, y)}{t^{1/\beta}} < \frac{(\kappa\rho)^{1/\alpha}}{(\eta\rho^{\beta/\alpha})^{1/\beta}} = 1$$

and

$$G'(x', y') = p_t(x, y) \geq t^{-\alpha/\beta} \Phi_1(\sigma) \geq (3\eta)^{-\alpha/\beta} \rho^{-1} \Phi_1(1) > \rho^{-1}.$$

So $y' \in B(x', \rho)$ proving (7.11). Again an application of Proposition 2.1 completes the proof. \square

In particular, Theorem 7.5 and Proposition 7.7 yield the following.

COROLLARY 7.8. *Every set A in X' with $m_{\alpha, \beta}(A) < \infty$, is contained in a Borel set which is semipolar, that is, which the process \mathfrak{X}' hits at most countably many times.*

7.2 Semipolar sets in X

To complete the paper let us show that we may also generalize our results for classical potential theory and Riesz potentials to our setting of heat semigroups on metric measure spaces. To that end we assume in this section that $\beta < \alpha$, define

$$G(x, y) := \int_0^\infty p_t(x, y) dt, \quad x, y \in X,$$

and introduce constants $c, C \in (0, \infty)$ by

$$c := \frac{\beta}{\alpha - \beta} \Phi_1(1) \quad \text{and} \quad C := \beta \int_0^\infty \sigma^{\alpha - \beta} \Phi_2(\sigma) \frac{d\sigma}{\sigma}.$$

PROPOSITION 7.9. *G has the properties (i) – (iii) of Section 3 and*

$$(7.12) \quad cd(x, y)^{-(\alpha - \beta)} \leq G(x, y) \leq Cd(x, y)^{-(\alpha - \beta)}, \quad x, y \in X.$$

Proof. Since $\Phi_1(0) \geq \Phi_1(1) > 0$, we know that $G = \infty$ on the diagonal of $X \times X$. Considering $x, y \in X$, $x \neq y$, and taking $\sigma := d(x, y)t^{-1/\beta}$ we have

$$\frac{d\sigma}{dt} = -\frac{1}{\beta} t^{-1} \sigma = -\frac{1}{\beta} d(x, y)^{-\beta} \sigma^{\beta+1},$$

and hence, using (7.6),

$$\int_0^\infty q_t(x, y) dt = \beta d(x, y)^{-(\alpha - \beta)} \int_0^\infty \sigma^{\alpha - \beta} \Phi_2(\sigma) \frac{d\sigma}{\sigma} = Cd(x, y)^{-(\alpha - \beta)}.$$

By (7.2), we thus conclude that $G(x, y) \leq Cd(x, y)^{-(\alpha-\beta)}$ and

$$G(x, y) \geq \beta d(x, y)^{-(\alpha-\beta)} \int_0^1 \sigma^{\alpha-\beta} \Phi_1(\sigma) \frac{d\sigma}{\sigma} \geq cd(x, y)^{-(\alpha-\beta)}.$$

In particular, (i) and (iii) hold. Moreover, the continuity of G outside the diagonal of $X \times X$ follows immediately, by Lemma 3.4. \square

Defining $m_{\alpha-\beta} := m_{\alpha-\beta, F}$ with $F(x, \rho) := D(x, \rho)$, Theorem 3.3 and Corollary 3.6 already yield the following.

THEOREM 7.10. *Let A be a subset X .*

1. *If $m_{\alpha-\beta}(A) < \infty$, then A is contained in a G -semipolar Borel set.*
2. *If $m_{\alpha-\beta}(A) = 0$, then A is contained in a G -polar Borel set.*

Further, an easy consequence of (7.8) and (7.12) is the following estimate.

PROPOSITION 7.11. *Let $C_\mu := 2^\alpha(2^\beta - 1)^{-1}c_\mu C$. Then, for all $x \in X$ and $R > 0$,*

$$\int_{D(x, R)} G(x, y) d\mu(y) \leq C_\mu R^\beta.$$

Proof. Let $x \in X$. For every $r > 0$,

$$\int_{D(x, r) \setminus D(x, r/2)} d(x, y)^{-(\alpha-\beta)} d\mu(y) \leq \left(\frac{r}{2}\right)^{-(\alpha-\beta)} \mu(D(x, r)) \leq 2^{\alpha-\beta} c_\mu r^\beta,$$

and hence, for every $R > 0$,

$$\int_{D(x, R)} G(x, y) d\mu(y) \leq 2^{\alpha-\beta} c_\mu C \sum_{j=0}^{\infty} (2^{-j} R)^\beta = C_\mu R^\beta.$$

\square

For every $t > 0$, let

$$P_t f(x) := \int p_t(x, y) f(y) d\mu(y), \quad f \in \mathcal{B}^+(X), x \in X.$$

By [23, Lemma 8.2], $\mathbb{P} := (P_t)_{t>0}$ is a strong Feller sub-Markov semigroup on X mapping $\mathcal{C}_0(X)$ into $C_0(X)$ and satisfying $\lim_{t \rightarrow 0} P_t f = f$ for every $f \in \mathcal{C}_0(X)$.

Let V denote the potential kernel of \mathbb{P} , that is, for every $f \in \mathcal{B}^+(X)$,

$$Vf := \int_0^\infty P_t f dt = G(f\mu).$$

PROPOSITION 7.12. *If f is a bounded function in $\mathcal{B}^+(X)$ with compact support, then $Vf \in \mathcal{C}_0(X)$.*

Proof. Given such a function f , there exist $x_0 \in X$ and $R > 0$ such that f is bounded by a multiple $a f_0$ of $f_0 := 1_{D(x_0, R)}$. Since both Vf and $V(a f_0 - f)$ are lower semicontinuous, it hence suffices to show that $Vf_0 \in \mathcal{C}_0(X)$.

For all $N > 0$ and $x \in X \setminus D(x_0, R + N)$,

$$G(f_0 \mu)(x) \leq C_\mu \int_{D(x_0, R)} d(x, y)^{-(\alpha-\beta)} d\mu(y) \leq C_\mu N^{-(\alpha-\beta)} \mu(D(x_0, R)).$$

So $Vf_0 = G(f_0 \mu)$ vanishes at infinity.

Next let $x \in X$ and $\varepsilon > 0$. By Proposition 7.11, there exists $r > 0$ such that $V1_{D(z, 2r)}(z) < \varepsilon/3$ for every $z \in X$. Let

$$g := 1_{D(x_0, R) \setminus D(x, r)}.$$

By the continuity of G outside the diagonal, the function Vg is continuous and real on $D(x, r)$. So there exists $0 < \delta < r$ such that $|Vg(z) - Vg(x)| < \varepsilon/3$ for every $z \in D(x, \delta)$. Finally, let us fix $z \in D(x, \delta)$. Then $D(x, r) \subset D(z, 2r)$, and hence

$$|Vf_0(z) - Vf_0(x)| < \frac{\varepsilon}{3} + |Vg(z) - Vg(x)| + \frac{\varepsilon}{3} < \varepsilon.$$

Thus Vf is continuous at x . □

Let us choose a sequence (g_k) in $\mathcal{C}_0(X)$ such that $\bigcup_{k=1}^{\infty} \{g_k > 0\} = X$. There exist $a_k > 0$ such that $g_k + Vg_k \leq a_k$, $k \in \mathbb{N}$. Then $g := \sum_{k=1}^{\infty} 2^{-k} a_k^{-1} g_k \in \mathcal{C}_0(X)$, $g > 0$ and $u := Vg \in E_{\mathbb{P}} \cap \mathcal{C}_0(X)$, $u > 0$. So the kernel V is proper and we obtain the following; see, for example, [22, Corollary 2.3.8, 2].

COROLLARY 7.13. *$(X, E_{\mathbb{P}})$ is a balayage space.*

Moreover, we know that there exists a Hunt process \mathfrak{X} on X with transition semigroup \mathbb{P} ; see [5, IV.8.1].

PROPOSITION 7.14. *For every $y \in X$, the function $G(\cdot, y)$ is a potential on X with superharmonic support $\{y\}$.*

Proof. Let $y \in X$. It is straightforward to show that $G(\cdot, y) \in E_{\mathbb{P}}$. Indeed, it suffices to note that, by Fubini's theorem and (CK),

$$P_s G(\cdot, y)(x) = \int_0^{\infty} P_s p_t(\cdot, y)(x) dt = \int_0^{\infty} p_{s+t}(x, y) dt = \int_s^{\infty} p_t(x, y) dt$$

for all $x \in X$ and $s > 0$. Moreover, clearly

$$\inf\{v \in E_{\mathbb{P}} : v \geq G(\cdot, y) \text{ outside a compact in } X\} = 0$$

that is, $G(\cdot, y)$ is a potential. To show that $G(\cdot, y)$ is harmonic on $X \setminus \{y\}$ let us fix $x \in X$, $x \neq y$, and a relatively compact open neighborhood U of x such that $y \notin \bar{U}$. Let μ_x^U be the harmonic measure for U and x , that is, for every $v \in E_{\mathbb{P}}$,

$$\int v d\mu_x^U = R_v^{X \setminus U}(x) := \inf\{w(x) : w \in E_{\mathbb{P}}, w \geq v \text{ on } X \setminus U\},$$

and μ_x^U is supported by $X \setminus U$. Clearly, $\int G(\cdot, y) d\mu_x^U \leq G(x, y)$.

Suppose that $\int G(\cdot, y) d\mu_x^U < G(x, y)$. Then, by continuity of G outside the diagonal, there exists $r > 0$ such that $D(y, r) \cap U = \emptyset$ and $\int G(\cdot, z) d\mu_x^U < G(x, z)$ for every $z \in D(y, r)$. Since $\mu(D(y, r)) > 0$, integration with respect to μ on $D(y, r)$ and Fubini's theorem yield that the function $v := V1_{D(y,r)} \in E_{\mathbb{P}}$ satisfies

$$(7.13) \quad \int v d\mu_x^U < v(x).$$

However, by [5, II.7.1],

$$\int v d\mu_x^U = R_v^{X \setminus U}(x) \geq R_v^{D(y,r)}(x) = v(x)$$

contradicting (7.13). Thus $\int G(\cdot, y) d\mu_x^U = G(x, y)$ completing the proof. \square

We claim that no point is finely isolated. So let $y \in X$. Clearly, $R_1^{\{y\}} \leq aG(\cdot, y)$ for every $a > 0$, and hence $R_1^{\{y\}} = 1_{\{y\}}$. Knowing already that y is not isolated we obtain that $\hat{R}_1^{\{y\}}(y) = \liminf_{z \rightarrow y} R_1^{\{y\}}(z) = 0$. This implies that $\{y\}$ is not finely isolated, since $\hat{R}_1^{\{y\}}(y)$ is also the fine lower limit of $R_1^{\{y\}}$ at y ; see [5, III.5.9].

By [24, Theorem 1.1], we hence obtain that G has property (iv) from Section 4. Proposition 7.14 and (7.12) show that also (i'), (ii) and (iii) hold. By Theorem 7.10 and Proposition 4.1, we therefore conclude the following.

COROLLARY 7.15. *Every set A in X with $m_{\alpha-\beta}(A) < \infty$, is contained in a Borel set which is semipolar, that is, which the process \mathfrak{X} hits at most countably many times.*

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