Non–Implementability of Arrow–Debreu Equilibria by Continuous Trading under Volatility Uncertainty

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Abstract

In diffusion models, few suitably chosen financial securities allow to complete the market. As a consequence, the efficient allocations of static Arrow–Debreu equilibria can be attained in Radner equilibria by dynamic trading. We show that this celebrated result generically fails if there is Knightian uncertainty about volatility. A Radner equilibrium with the same efficient allocation as in an Arrow–Debreu equilibrium exists if and only if the discounted net trades of the equilibrium allocation display no ambiguity in the mean. This property is violated generically in endowments, and thus Arrow–Debreu equilibrium allocations are generically unattainable by dynamically trading few long–lived assets.

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1 Introduction

A fundamental result of financial economics establishes equivalence between Arrow–Debreu and Radner equilibria if asset markets are dynamically complete. When information is generated by a d–dimensional Brownian motion, d suitably chosen assets suffice to span a dynamically complete market. In such a setting, the rather heroic Arrow–Debreu equilibria, where all trade takes place on a perfect market for contingent claims at time zero, and no trade ever takes place afterwards, and the more realistic Radner equilibria, where agents dynamically trade long–lived financial assets, lead to the same allocation. Such equivalence of static and dynamic equilibria for diffusion models has been established in different settings and at different levels of generality. For continuous–time models, see, e.g., Duffie and Huang (1985), Duffie and Zame (1989), Karatzas, Lehoczky, and Shreve (1990), Dana and Pontier (1992), Zitkovic (2006), Anderson and Raimondo (2008), Riedel and Herzberg (2013), Hugonnier, Malamud, and Trubowitz (2012), Kramkov (2015) and Ehling and Heyerdahl-Larsen (2015). The basic ideas in discrete time date back to Arrow's papers from 1953 and 1971; Magill and Quinzii (1998) and Duffie (1992) are good textbook treatments of the subject.

In this paper, we show that the celebrated equivalence generically fails under Knightian uncertainty about volatility. We place ourselves in a framework in which market spanning is as easy as possible. Even then, Arrow–Debreu equilibria will usually not be implementable by a dynamic market if there is Knightian uncertainty in individual endowments. Asset markets thus remain substantially dynamically incomplete.

In which sense do we make it easy for the market to be complete? First, we consider a model in which a d–dimensional Brownian motion with ambiguous volatility generates the economy's information flow. Brownian motion is the basic diffusion; one cannot expect to obtain the result for more general diffusion processes if it fails for Brownian motion. Second, as in the Duffie–Huang–approach, we consider nominal asset markets. The nominal asset structure allows for an exogenously chosen asset structure. The market is thus free to choose the financial asset structure. If there is no spanning in this setting, one cannot expect spanning in the more demanding real asset setting where security prices are endogenously determined in equilibrium and linked to consumption prices via the real dividend structure. Third, we consider a setting where aggregate endowment is ambiguity–free. This is the archetypical starting point for an economic analysis of insurance properties of competitive markets. In this setting, an efficient economic institution should lead to an ambiguity–free allocation for ambiguity–averse individuals. Indeed, we show that efficient (and thus, Arrow–Debreu equilibrium) allocations in this Knightian economy provide full insurance against uncertainty. We thus extend analogous results of Billot, Chateauneuf, Gilboa, and Tallon (2000), Dana (2002), Tallon (1998), and de Castro and Chateauneuf (2011) to the continuous–time setting with non–dominated sets of priors.

The paper proceeds as follows. In the next section, we set up the continuous–time model with volatility uncertainty by using the concepts developed mainly by Shige Peng (2010 for an overview) and discuss some issues related to models with non–dominated priors. Section 3 discusses existence of Arrow–Debreu equilibria; without existence, the question of implementation would be void. In our framework with uncertainty–free aggregate endowments, every Arrow–Debreu equilibrium in a corresponding expected utility economy where all agents use the same prior P is also an equilibrium under Knightian uncertainty. As a by–product, we obtain indeterminacy of equilibria, as in other Knightian settings, such as Tallon (1998), Dana (2002), Rigotti and Shannon (2005), or Dana and Riedel (2013)[1]. Section 4 studies the possibility of implementation in the so–called Bachelier model where the risky (or, in this Knightian setting, maybe better: uncertain) asset is given by the Brownian motion itself because this case is particularly transparent. Indeed, in the classic case, the martingale representation theorem yields directly the portfolio strategies that finance the Arrow–Debreu (net) consumption plans. We study under what condition this result holds in an uncertain world. It turns out that implementation is possible if and only if

[1]Existence is not a trivial application of the well–known results on existence of general equilibrium for Banach lattices. Under volatility uncertainty, the natural commodity space consists of bounded and *quasi–continuous* functions. A mapping is quasi–continuous if it is continuous in nearly all its domain. The property of quasi–continuity comes for free in the probabilistic setting: Lusin's theorem establishes the fact that any measurable function on a nice topological space is quasi–continuous. Under volatility uncertainty, this equivalence between measurability and quasi–continuity no longer holds true. We are thus led to study a new commodity space which has not been studied so far in general equilibrium theory. Compare also the discussion of this space in the recent contributions Epstein and Ji (2013), Vorbrink (2014), and Beissner (2014). For this commodity space, the available existence theorems do not immediately apply. The abstract question of existence must thus be dealt with separately, but we leave the general question of existence for the future as it is not the main concern of this paper.

the value of net trades is mean ambiguity–free, i.e. if the expected value of net trades is the same for all priors. A crucial role in the proof plays the recent martingale representation theorem for G–Brownian motion derived in Soner, Touzi, and Zhang (2011). Finally, we show that generically, implementation will be impossible under Knightian uncertainty about volatility when there is no aggregate uncertainty in the economy. The set of economies for which implementation fails is prevalent in the set of economies parametrized by initial endowments. The notion of prevalence is an extension of the idea of full Lebesgue measure in finite–dimensional contexts. It has been developed in Hunt, Sauer, and Yorke (1992) and Anderson and Zame (2001). In contrast to topological notions of large and small sets, it has the advantage to coincide with the notion of full Lebesgue measure in finite dimensions. Section 5 proves the main result and Section 6 concludes. Appendix A collects the proofs of Section 3. Appendix B discusses more general asset dynamics.

2 The Economy under Knightian Uncertainty

2.1 Setting

We consider an economy over the time interval $[0,T]$ with finitely many agents $i \in \mathbb{I} = \{1,\dots,I\}$. The state space $\Omega = C[0,T]^d$ consists of all continuous functions on $[0,T]$ with values in $\mathbb{R}^d$ for $d \in \mathbb{N}$. Ω is endowed with the usual topology of uniform convergence. We denote by $\mathcal{F}$ the Borel σ–field on Ω. Let $B_t(\omega) = \omega(t)$ denote the canonical process for $\omega \in \Omega$, and denote by $\mathbb{F} = (\mathcal{F}_t)_{t\leq T}$ the canonical filtration induced by B.
Let $0 < \underline{\sigma}^k \leq \overline{\sigma}^k, k = 1,\dots,d$ and $\Sigma = \prod_{k=1}^d [\underline{\sigma}^k, \overline{\sigma}^k]$. Let $\mathcal{P}$ be the set of probability measures P on $(\Omega,\mathcal{F})$ such that B is a martingale under P with respect to $\mathbb{F}$, the covariation between B^k and B^l is zero for $k \neq l$, and we have for the quadratic variation $\langle B^k \rangle$ of B the inequality $\left(\underline{\sigma}^k\right)^2 t \leq \langle B^k\rangle_t \leq \left(\overline{\sigma}^k\right)^2 t$ for all $t \in [0,T]$ P–a.s. The process B is then a d–dimensional G–Brownian motion. The components $B^1,\dots,B^d$ are independent with unknown volatility in the bounds $[\underline{\sigma}^k, \overline{\sigma}^k]$, $k = 1,\dots,d$. We refer to Peng (2006), Denis, Hu, and Peng (2011) and Section 2 of Soner, Touzi, and Zhang (2011) for further details on G–Brownian motion.
The set of priors $\mathcal{P}$ is not dominated by a single probability measure. In such a context, sets that are conceived as null by the agents cannot be identified

with null sets under a single probability measure. We say an event holds $\mathcal{P}$ quasi–surely (q.s.) if it holds almost surely under all $P \in \mathcal{P}$. Let $\mathcal{N} := \{X : \mathcal{F}\text{-}measurable\ and\ X = 0\ \mathcal{P}\text{-}q.s.\}$ be the set of (negligible) payoffs with respect to $\mathcal{P}$ that do not charge any $P \in \mathcal{P}$. Let L_0 denote the space of $\mathcal{N}$-equivalence classes of $\mathcal{F}_T$–measurable payoffs. We say that X has a $\mathcal{P}$-*q.c. version* if there is a quasi–continuous[2] function $Y : \Omega \to \mathbb{R}$ with $X = Y$ $\mathcal{P}$–q.s. We denote by $\|X\|_\infty = \inf\{M \geq 0 : |X| \leq M\ \mathcal{P}\text{-q.s.}\}$ the quasi–sure–supremum norm. The commodity space consists of all quasi–surely bounded, quasi–continuous (q.c.), $\mathcal{F}_T$–measurable functions:

$$\mathbb{H} = L^\infty_{\mathcal{P}} = \{X \in L_0 : X \text{ has a } \mathcal{P}\text{-q.c. version and } \|X\|_\infty < \infty\}. \tag{2.1}$$

As shown in Theorem 27 of Denis, Hu, and Peng (2011), the space $L^\infty_{\mathcal{P}}$ is obtained by closing the space $C_b(\Omega)$ of bounded and continuous functions on Ω under the norm $\|X\|_\infty$. The consumption set $\mathbb{H}_+$ consists of quasi-surely nonnegative functions in $\mathbb{H}$.

For $X \in \mathbb{H}$, we denote by $E^P[X]$ the linear expectation under $P \in \mathcal{P}$. We introduce the sublinear expectation

$$\mathbb{E}[X] = \sup_{P \in \mathcal{P}} E^P[X] \tag{2.2}$$

and the superlinear expectation $\underline{\mathbb{E}}[X] = \inf_{P\in\mathcal{P}} E^P[X] = -\mathbb{E}[-X]$.

2.2 Martingales and their Representation

Define $\mathcal{P}_{t,P} = \{P' \in \mathcal{P} : P' = P \text{ on } \mathcal{F}_t\}$ for every $t \in [0,T]$ and $P \in \mathcal{P}$. Let $\mathbb{H}_t \subset \mathbb{H}$ denote the subspace of $\mathcal{F}_t$–measurable payoffs. For all $X \in \mathbb{H}$ there exists $\mathbb{E}_t[X] \in \mathbb{H}_t$ such that

$$\mathbb{E}_t[X] = \operatorname{esssup}^P_{P' \in \mathcal{P}_{t,P}} E^{P'}[X|\mathcal{F}_t], \quad P\text{-a.s. for every } P \in \mathcal{P}. \tag{2.3}$$

The conditional expectations $(\mathbb{E}_t)$ satisfy the law of iterated expectations $\mathbb{E}_s \circ \mathbb{E}_t = \mathbb{E}_s$, $s \leq t$. We also have the constants preserving property in the

[2] Knightian uncertainty requires a reconsideration of some measure theoretic results. Under risk, a measurable function is "almost" continuous in the sense that for every $\epsilon > 0$ there is an open set O with probability at least $1 - \epsilon$ such that the function is continuous on O; this is Lusin's theorem. Under non–dominated Knightian uncertainty, this Lusin property, or quasi–continuity, does not come for free from measurability, and one needs to impose it. We refer to Epstein and Ji (2013) and Denis, Hu, and Peng (2011) for the financial and measure–theoretic background.

conditional sense, i.e. $\mathbb{E}_t X = X$ for every $X \in \mathbb{H}_t$, see again Denis, Hu, and Peng (2011).
Following Denis, Hu, and Peng (2011), we introduce the norm $\|\cdot\|_2 = (\mathbb{E}|\cdot|^2)^{\frac{1}{2}}$ and the Banach space $L^2_{\mathcal{P}}$ as the closure of $C_b(\Omega)$ under $\|\cdot\|_2$.

Definition 2.1 *An $\mathbb{F}$-adapted process $X = (X_t) \in \mathbb{L}^2_{\mathcal{P}}$ is an $\mathbb{E}$-martingale if $X_s = \mathbb{E}_s[X_t]$ $\mathcal{P}$-q.s. for all $s \leq t$. We call X a symmetric $\mathbb{E}$-martingale if X and $-X$ are both $\mathbb{E}$-martingales.*

Denote by $\mathcal{M}$ the closure of piecewise constant processes $\sum_{k\geq 0} \eta_{t_k} 1_{[t_k, t_{k+1})}$ with $\eta_{t_k} \in \mathbb{H}^d$, under the norm

$$\|\eta\|_{\mathcal{M}} = \mathbb{E}[\int_0^T \mathrm{d}\langle B\rangle_t \eta_t \cdot \eta_t]^{1/2} = \mathbb{E}[\int_0^T \sum_{k=1}^d (\eta_t^k)^2 \mathrm{d}\langle B^k\rangle_t]^{1/2} .$$

For $\mathbb{F}$–progressively measurable processes (X_t), we introduce the norm $||X||_{\mathbb{S}} = \left(\mathbb{E}[\sup_{t\in[0,T]} X_t^2]\right)^{1/2}$ and let

$$\mathbb{S} = \Big\{(K_t)\colon \mathbb{F}\text{-prog. measurable with cont. paths } \mathcal{P}\text{-q.s., } ||K||_{\mathbb{S}} < \infty\Big\}. \quad (2.4)$$

The following result, Theorem 5.1 in Soner, Touzi, and Zhang (2011), is crucial for the proof of Theorem 4.2.

Theorem 2.2 (Martingale Representation Theorem under $\mathbb{E}$) *For every $X \in L^2_{\mathcal{P}}$ there exists a unique pair $(\eta, K) \in \mathcal{M} \times \mathbb{S}$ such that $(-K_t)$ is an increasing $\mathbb{E}$–martingale, $K_0 = 0$, and for all $t \in [0, T]$*

$$\mathbb{E}_t[X] = \mathbb{E}_0[X] + \int_0^t \eta_s \mathrm{d}B_s - K_t \quad \mathcal{P}\text{-q.s.} \quad (2.5)$$

2.3 Standing Assumptions on the Primitives

Agents' preferences are given by ambiguity–averse utility functionals $U^i : \mathbb{H}_+ \to \mathbb{R}$ of the form

$$U^i(c) = \underline{\mathbb{E}}[u^i(c)]$$

for a Bernoulli utility function $u^i : [0, \infty) \to \mathbb{R}$. The endowment of agent $i \in \mathbb{I}$ is denoted by $e^i \in \mathbb{H}_+$. Aggregate endowment is $e = \sum_{i\in\mathbb{I}} e^i$.

We call $\mathcal{E} = \left((\Omega, \mathcal{F}, \mathcal{P}), \mathbb{H}, (U^i, e^i)_{i\in\mathbb{I}}\right)$ the *Knightian economy.* For a fixed probability measure $P \in \mathcal{P}$, we denote by

$$U^{i,P}(c) = E^P[u^i(c)] \qquad (c \in L^\infty(P)_+)$$

the expected utility under P. In an expected utility economy, the commodity space is the space $L^\infty(P)$ of all measurable, P–a.s. bounded functions. We call $\mathcal{E}^P = \left((\Omega, \mathcal{F}, P), L^\infty(P), \left(U^{i,P}, e^i\right)_{i\in\mathbb{I}}\right)$ the expected utility economy.

Assumption 1 *The Bernoulli utility functions* $u^i : [0, \infty) \to \mathbb{R}$*,* $i \in \mathbb{I}$ *are strictly concave, strictly increasing, twice continuously differentiable on* $(0, \infty)$*, and satisfy the Inada condition* $\lim_{x\downarrow 0} \frac{du^i}{dx}(x) = \infty$.
Individual endowments e^i *are quasi–surely bounded away from zero.*

In the following, we consider a situation where the market can potentially insure individuals against their individual uncertainty because (Knightian) uncertainty disappears in the aggregate.

Definition 2.3 $X \in \mathbb{H}$ *is (*$\mathcal{P}$*–)ambiguity–free if* X *has the same probability distribution under all priors* $P \in \mathcal{P}$*.* X *is (*$\mathcal{P}$*–)mean ambiguity–free if for all* $P, Q \in \mathcal{P}$ *we have* $E^P[X] = E^Q[X]$*.*

We will use either of the following assumptions in the sequel.

Assumption 2 *The economy* $\mathcal{E} = \left((\Omega, \mathcal{F}, \mathcal{P}), \mathbb{H}, \left(U^i, e^i\right)_{i\in\mathbb{I}}\right)$ *has no aggregate risk, that is, aggregate endowment* $e = \sum_{i\in\mathbb{I}} e^i$ *is* $\mathcal{P}$*–ambiguity–free.*

Assumption 3 *The economy* $\mathcal{E}$ *has no aggregate uncertainty, that is, aggregate endowment* $e = \sum_{i\in\mathbb{I}} e^i$ *is* $\mathcal{P}$*–quasi–surely constant.*

Obviously, Assumption 3 implies Assumption 2. Note that we do allow for Knightian uncertainty at the individual level. We do not assume that *individual* endowments are risk– or uncertainty–free.
In the economic literature, economies with no aggregate uncertainty are the benchmark case to study welfare properties of markets and economic institutions, see Billot, Chateauneuf, Gilboa, and Tallon (2000) for a case study under Knightian uncertainty. When there is no uncertainty in the aggregate, but uncertainty at the individual level, a good social institution should allow ambiguity–averse agents to achieve full insurance. We will see in the next section that full insurance holds true in Arrow–Debreu equilibrium.

3 Existence and Structure of Arrow–Debreu Equilibria

In a first step, we study existence and structure of Arrow–Debreu equilibria. We will see that existence can be reduced to existence of Arrow–Debreu equilibria in expected utility economies if endowments are ambiguity–free. Existence of an Arrow–Debreu equilibrium for $\mathcal{E}^P$ has been shown by Bewley (1969) and Bewley (1972) for general utility functions; for expected utility, see Dana (1993) or Mas-Colell and Zame (1991).

An *Arrow–Debreu equilibrium* for $\mathcal{E}$ (for $\mathcal{E}^P$ for some $P \in \mathcal{P}$) consists of a positive, continuous linear functional $\Psi : \mathbb{H} \to \mathbb{R}$ (a positive linear functional on $L^\infty(P)$) and an allocation $c = (c^i)_{i\in\mathbb{I}} \in \mathbb{H}_+^{\mathbb{I}}$ such that markets clear, $\sum_{i\in\mathbb{I}} c^i = e$ $\mathcal{P}$–quasi surely (P–almost surely), and agents maximize utility subject to their budget constraint, i.e. for $i \in \mathbb{I}$ and $d \in \mathbb{H}_+$ with $U^i(d) > U^i(c^i)$ we have $\Psi(d) > \Psi(e^i)$.

Theorem 3.1 *Under Assumption 1 and 2, Arrow–Debreu equilibria exist.*

The proofs of this section are deferred to Appendix A.

It is well known that equilibria need not be unique. For our purpose, this is not important as we are going to characterize all equilibria in the following. By the first welfare theorem, every Arrow–Debreu equilibrium allocation is efficient. Recall that an allocation $c = (c^i)_{i\in\mathbb{I}} \in \mathbb{H}_+^{\mathbb{I}}$ is *efficient* if it is feasible, $\sum_{i\in\mathbb{I}} c^i \le e$ $\mathcal{P}$-q.s., and there is no other feasible allocation $d = (d^i)_{i\in\mathbb{I}} \in \mathbb{H}_+^{\mathbb{I}}$ with $U^i(d^i) > U^i(c^i)$ for all $i \in \mathbb{I}$. An allocation $c = (c^i)_{i\in\mathbb{I}}$ is called *full insurance* if all c^i are quasi–surely constant.

Theorem 3.2 *Let Assumption 1 hold true. Let c be an Arrow–Debreu equilibrium allocation of $\mathcal{E}$.*

1. *Under Assumption 2,*

   (a) *c is ambiguity–free;*

   (b) *there is an Arrow–Debreu equilibrium price functional Ψ for c that can be represented as*

   $$\Psi(d) = E^P[\psi d]$$

   for some probability measure $P \in \mathcal{P}$ and a function $\psi \in \mathbb{H}_+$ which is ambiguity–free, quasi–surely bounded and bounded away from zero.

2. *Under Assumption 3,*
   (a) *c is full insurance;*
   (b) *there is an Arrow–Debreu equilibrium price functional Ψ for c that can be represented as*

$$\Psi(d) = E^P[d]$$

   for some probability measure $P \in \mathcal{P}$.

4 (Non–)Implementability

We tackle the main question of this paper: can Arrow–Debreu equilibria be implemented by trading a few long–lived assets dynamically over time? Under risk, the answer is affirmative: Duffie and Huang (1985) show how to construct dynamically complete financial markets that support the equilibrium allocation. Subsequently, the literature has provided positive answers for the more complex case of asset structures where the asset prices are determined endogenously in equilibrium[3]. The aim of this section is to show that generically, the answer is negative under Knightian uncertainty about volatility.

Let us describe the financial market. There is a riskless asset $S_t^0 = 1$. Moreover, the price of the other d assets is given by our d-dimensional G–Brownian motion $S_t = B_t$. This is the asset price model of Bachelier, the natural candidate for a dynamically complete market. We discuss more general specifications in Appendix B.

A trading strategy consists of a process $(\theta^1, \ldots, \theta^d) \in \mathcal{M}$, the space of admissible integrands for G–Brownian motion (see Peng (2006) or Denis, Hu, and Peng (2011) for details on stochastic integration for G–Brownian motion).

[3] In that case, the question of Radner implementability is much more complex and was only recently solved by Anderson and Raimondo (2008), Hugonnier, Malamud, and Trubowitz (2012), Riedel and Herzberg (2013) and Kramkov (2015). If the asset market is *potentially complete* in the sense that sufficiently many independent dividend streams are traded, then one can obtain endogenously dynamically complete asset markets in sufficiently smooth Markovian economies. For non–smooth economies and non–Markovian state variables, the question is still open. As we focus on the *limits* of implementability under Knightian uncertainty, we consider the case of an exogenous asset structure as in Duffie and Huang (1985). If one cannot even implement the Arrow–Debreu equilibrium in this case, one cannot expect to implement in the more complex situations either.

The gains from trade are

$$G_t^\theta = \int_0^t \theta_u \mathrm{d}S_u = \sum_{1\le k\le d} \int_0^t \theta_u^k \mathrm{d}S_u^k \quad q.s. \tag{4.1}$$

We call θ admissible if the gains from trade G_T^θ are quasi–surely bounded from below. A spot consumption price ψ is a quasi–surely bounded, nonnegative $\mathcal{F}_T$–measurable function. A consumption plan is an element $c^i \in \mathbb{H}_+$. A budget–feasible plan (c^i, θ^i) is a pair of a consumption plan c^i and an admissible trading strategy θ^i with

$$(c^i - e^i)\psi = G_T^{\theta^i} \quad q.s. \tag{4.2}$$

Definition 4.1 *1. A Radner equilibrium $(\psi, (c^i, \theta^i)_{i\in\mathbb{I}})$ consists of a spot consumption price ψ and budget–feasible plans $(c^i, \theta^i)_{i\in\mathbb{I}}$ such that markets clear, i.e.*

$$\sum_{i\in\mathbb{I}} c^i = e, \quad \sum_{i\in\mathbb{I}} \theta^i = 0 \quad q.s.,$$

and for each agent $i \in \mathbb{I}$ there is no budget–feasible plan (d, η) with $U^i(c^i) < U^i(d)$.

2. Let (Ψ, c) be an Arrow–Debreu equilibrium for the economy $\mathcal{E}$ with $\Psi(\cdot) = E^P[\psi\cdot]$ for some $P \in \mathcal{P}$ and a nonnegative $\mathcal{F}_T$–measurable function ψ. We say that (Ψ, c) is implementable (in the Bachelier model) if there exists a Radner equilibrium of the form $(\psi, (c^i, \theta^i)_{i\in\mathbb{I}})$.

The following theorem characterizes implementability.

Theorem 4.2 *1. Suppose Assumption 2 holds true. Let (Ψ, c) be an Arrow–Debreu equilibrium for the economy $\mathcal{E}$ with $\Psi(d) = E^P[\psi d]$ for a quasi–surely bounded positive function $\psi \in \mathbb{H}_+$ and some $P \in \mathcal{P}$.*

(Ψ, c) is implementable if and only if the values of net trades $\xi^i = \psi(c^i - e^i)$ are mean–ambiguity–free for each $i \in \mathbb{I}$.

2. Suppose Assumption 3 holds true. Let (Ψ, c) be an Arrow–Debreu equilibrium for the economy $\mathcal{E}$ with $\Psi(d) = E^P d$ for some $P \in \mathcal{P}$.

(Ψ, c) is implementable if and only if net trades $\xi^i = c^i - e^i$ are mean–ambiguity–free for each $i \in \mathbb{I}$.

The proof is given in Section 5. Let us consider a concrete example.

Example 4.3 *Let $e(\omega) \equiv 1$, $I = 2$, $d = 1$, $T = 1$, and $u^i = \log$ for $i = 1, 2$. Assume $e^1 = (\exp(B_1) \wedge \frac{3}{4}) \vee \frac{1}{2}$ and $e^2 = 1 - e^1$. By Theorem 3.2, Arrow–Debreu equilibrium allocations are full insurance, i.e. constant, and the price functional is given by $d \mapsto \Psi(d) = E^P[d]$ for some $P \in \mathcal{P}$. In this case, the expected value of net trades $\xi^i = c^i - e^i$ depends on the particular measure $P \in \mathcal{P}$. For example, let $P^\sigma \in \mathcal{P}$ denote the measure where B has constant volatility σ. Then*

$$E^{P^\sigma} e^1 = \frac{1}{2}\Phi\left(-\frac{\log 2}{\sigma}\right) + e^{\sigma^2/2}\left(\Phi\left(\frac{\log(3/4)}{\sigma} - \sigma\right) - \Phi\left(-\sigma - \frac{\log 2}{\sigma}\right)\right) + \frac{3}{4}\Phi\left(-\frac{\log(3/4)}{\sigma}\right)$$

where Φ is the standard normal distribution. This expression depends on σ and thus, the net trades are not mean–ambiguity–free. Radner implementation is therefore impossible.

The previous example suggests that the Radner implementation of Arrow–Debreu equilibria might be the exception rather than the rule, in general. In the next step, we clarify this question under Assumption 3. We know from our analysis that all Arrow–Debreu equilibria fully insure all agents in such a setting. We claim that "for almost all" economies, or "generically", Radner implementation is impossible.

The notion of null set has a natural meaning in the finite–dimensional context because one can define negligible sets as Lebesgue null sets. A generalization to infinite–dimensional spaces has been provided by Hunt, Sauer, and Yorke (1992) and Anderson and Zame (2001). Let $C \subset \mathbb{H}$ be convex and completely metrizable in the relative topology. A universally measurable subset $E \subset C$ is (finitely) shy in C if there is a finite–dimensional subspace V of $\mathbb{H}$ such that for Lebesgue measure λ_V on V we have

1. $\lambda_V(C + h) > 0$ for some $h \in \mathbb{H}$
2. $\lambda_V(E + h) = 0$ for all $h \in \mathbb{H}$.

The set $C \setminus E$ is called prevalent in C. The concept of shyness coincides with the usual notion of zero Lebesgue measure in finite–dimensional contexts and

is thus an appropriate generalization to infinite–dimensional settings where no Lebesgue measure exists.
In the following, we fix aggregate endowment $e = 1$ and consider the class of economies parametrized by individual endowments in the set

$$\mathcal{A} = \left\{ (e^i)_{i=1,\dots,I-1} \in \mathbb{H}_+^{I-1} : \delta \leq \min_{j=1,\dots,I-1} e^j, \delta \leq 1 - \sum_{j=1}^{I-1} e^j \text{ for some } \delta > 0 \right\} .$$

We thus keep the uncertainty structure $(\Omega, \mathcal{F}, \mathcal{P})$, the space $\mathbb{H}$, and the utility functions $U^i, i \in \mathbb{I}$ fixed. As aggregate endowment is kept fixed to 1, we vary the initial endowments of the first $I-1$ agents since the endowment $e^I = 1 - \sum_{j=1}^{I-1}$ of the last agent is then fixed. Note that $\mathcal{A}$ is the countable union of closed convex sets and thus a completely metrizable Borel set.

Definition 4.4 *Let Assumption 3 hold true. We say that an economy $\mathcal{E}$ with endowments $(e^i)_{i=1,\dots,I-1} \in \mathcal{A}$ does not allow for implementation if there is no implementable Arrow–Debreu equilibrium (Ψ, c). Otherwise, we say that $\mathcal{E}$ allows for implementation.*

Theorem 4.5 *Under Assumption 3, Arrow–Debreu equilibria are generically not implementable. More precisely: the set*

$$\mathcal{R} = \{(e^i)_{i=1,\dots,I-1} \in \mathcal{A} : \text{ the economy } \mathcal{E} \text{ with endowments } (e^i)_{i=1,\dots,I-1} \text{ allows for implementation}\}$$

is shy in $\mathcal{A}$.

The proof relies on the intuitively plausible fact that the subspace of all mean–ambiguity–free functions $X \in \mathbb{H}$ is "small" when the set of priors $\mathcal{P}$ is non–trivial (see Lemma 5.2 below).

5 Proofs of the Main Theorems

As a preparation, the following corollary of Theorem 2.2 characterizes mean ambiguity free random variables.

Corollary 5.1 *The space $\mathbb{M}$ of mean ambiguous–free random variables is a $\|\cdot\|_\infty$–closed subspace of $\mathbb{H}$. We have*

$$\mathbb{M} = \Big\{ X \in \mathbb{H} : \ X = \mathbb{E}[X] + \int_0^T \eta_s \mathrm{d}B_s \ \mathcal{P}\text{-}q.s. \quad \text{for some } \eta \in \mathcal{M} \Big\}.$$

For $X \in \mathbb{M}$, the process $(\mathbb{E}_t[X])$ is a symmetric $\mathbb{E}$–martingale.

Proof: The sublinear expectation $\mathbb{E}$ is 1–Lipschitz–continuous with respect to $\|\cdot\|_\infty$. For a sequence $(X_n) \subset \mathbb{M}$ and $X \in \mathbb{H}$ with $\|X_n - X\|_\infty \to 0$, we thus have $\mathbb{E}[X_n] \to \mathbb{E}[X]$ and $\mathbb{E}[-X_n] \to \mathbb{E}[-X]$. Hence, X is also mean ambiguity–free, and $\mathbb{M}$ is $\|\cdot\|_\infty$–closed.
Now let $X \in \mathbb{M}$ and set $Y = -X$. As $\mathbb{E}_t$ is sublinear, we have $\mathbb{E}_t[Y] \geq -\mathbb{E}_t[X]$ q.s. Hence $Z := \mathbb{E}_t[Y] + \mathbb{E}_t[X] \geq 0$ q.s. Since $X \in \mathbb{M}$, we conclude with the help of sublinearity and the law of iterated expectations that

$$\mathbb{E}[Z] = \mathbb{E}\left[\mathbb{E}_t[Y] + \mathbb{E}_t[X]\right] \leq \mathbb{E}\left[\mathbb{E}_t[Y]\right] + \mathbb{E}[\mathbb{E}_t[X]] = \mathbb{E}[Y] + \mathbb{E}[X] = 0\,.$$

Hence, $Z = 0$ q.s., and so $(\mathbb{E}_t[X])$ is a symmetric $\mathbb{E}$–martingale. Theorem 5.5 of Soner, Touzi, and Zhang (2011) establishes the representation $X = \mathbb{E}[X] + \int_0^T \eta_s \mathrm{d}B_s$ for some $\eta \in \mathcal{M}$. On the other hand, if $X \in \mathbb{M}$ has such a representation as a stochastic integral, it is mean ambiguity–free. □

5.1 Proof of Theorem 4

We only need to prove part 1. Part 2. follows by exactly the same arguments by setting $\psi = 1$.
We start with the implication $\Rightarrow$. Let $(c, \psi \cdot P)$ be an Arrow-Debreu equilibrium. Suppose we have an implementation in the Bachelier model with trading strategies θ^i. The Radner budget constraint gives

$$\xi^i = G_T^{\theta^i} = \int_0^T \theta_t^i \mathrm{d}B_t \quad \mathcal{P}\text{-}q.s.$$

By Corollary 5.1, ξ^i is mean ambiguity–free.
We come to the converse implication $\Leftarrow$. Let (Ψ, c) be an Arrow–Debreu equilibrium for the economy $\mathcal{E}$ with $\Psi(d) = E^P[\psi d]$ for a quasi–surely bounded positive function $\psi \in \mathbb{H}_+$ and some $P \in \mathcal{P}$. We need to show that implementation is possible if each value of net trade ξ_i is mean ambiguity–free.

We proceed in three steps. In step 1 we introduce the candidate trading strategies for agent $i \in \mathbb{I}$; we show market clearing in step 2. Finally, step 3 shows that these strategies are optimal given the dynamic budget constraint.

1. Let the value of the net trade ξ^i be mean ambiguity–free for all agents $i \in \mathbb{I}$. The Arrow–Debreu budget constraint gives $\mathbb{E}[\xi^i] = E^P \xi^i = 0$. By Corollary 5.1, the process $t \mapsto \mathbb{E}_t[\xi^i]$ is a symmetric $\mathbb{E}$-martingale and we have
$$\mathbb{E}_t[\xi^i] = \int_0^t \theta_r^i \mathrm{d}S_r, \quad \mathcal{P}\text{-}q.s.$$
and
$$\xi^i = \int_0^T \theta_t^i \mathrm{d}S_t \quad \mathcal{P}\text{-}q.s$$
for some $\theta^i \in \mathcal{M}$. The processes θ^i are candidates for trading strategies in a Radner equilibrium with spot consumption price ψ.

2. By market–clearing in an Arrow–Debreu equilibrium, we have
$$0 = \sum_{i \in \mathbb{I}} \xi^i = \int_0^T \sum_{i \in \mathbb{I}} \theta_t^i \mathrm{d}B_t \quad \mathcal{P}\text{-}q.s\,.$$
As stochastic integrals that are zero have a q.s. zero integrand (confer, e.g., Proposition 3.3 in Soner, Touzi, and Zhang (2011)), we conclude that $\sum_{i \in \mathbb{I}} \theta^i = 0$ $\mathcal{P}$-quasi surely.

3. It remains to check that the consumption–portfolio strategy (c^i, θ^i) is optimal for agent i under the Radner–budget constraint. Suppose there is a trading strategy (d, η) with
$$\psi(d - e^i) = \int_0^T \eta_t \mathrm{d}S_t \quad \mathcal{P}\text{-}q.s.$$
By Corollary 5.1, we have $\psi(d - e^i) \in \mathbb{M}$ with $\mathbb{E}[\psi(d - e^i)] = 0$ and we then get
$$\Psi(d - e^i) = E^P[\psi(d - e^i)] = \mathbb{E}\left[\int_0^T \eta_t \mathrm{d}S_t\right] = 0\,,$$
and d is thus budget–feasible in the Arrow–Debreu model. We conclude that $U^i(d) \leq U^i(c^i)$.

This completes the proof.

5.2 Proof of Theorem 4.5

We start with a crucial preliminary lemma.

Lemma 5.2 *Let* $C = \{Y \in \mathbb{H} : \delta \leq Y \leq 1 - \delta \text{ for some } \delta > 0\}$. *Then* $M_C = \mathbb{M} \cap C$ *is a shy set in* C.

PROOF: C is a convex subset of $\mathbb{H}$ which is a countable union of closed sets, and thus a completely metrizable Borel set. By Corollary 5.1, $\mathbb{M}$ is a closed subspace of $\mathbb{H}$; thus M_C is convex and the countable union of closed sets, so universally measurable. By Corollary 5.1, each $X \in M_C$ has the form

$$X = x + \int_0^T \eta_t \mathrm{d}B_t$$

for some $x \in \mathbb{R}$ and some $\eta \in \mathcal{M}$.
We proceed by checking the two properties for (finite) shyness (Definition 2.3 in Anderson and Zame (2001)). Let

$$K_t = \langle B^1 \rangle_t - t \left(\overline{\sigma}^1\right)^2 . \tag{5.1}$$

The process K is a decreasing $\mathbb{E}$–martingale **?**Chapter IV, Example 1.3]peng2007g2. We have $-T\left(\left(\overline{\sigma}^1\right)^2 - \left(\underline{\sigma}^1\right)^2\right) \leq K_T \leq 0$ q.s. Note that K_T is not mean ambiguity–free because we have $\mathbb{E}[K_T] = 0$ and $\mathbb{E}[-K_T] = -T\left(\left(\overline{\sigma}^1\right)^2 - \left(\underline{\sigma}^1\right)^2\right)$. Our "test space" V is the one–dimensional subspace of $\mathbb{H}$ generated by K. The Lebesgue measure on V is denoted by λ_V. Lebesgue measure on the real numbers is denoted by λ.

1. There is a $X \in \mathbb{H}$ with $\lambda_V\Big(C + X\Big) > 0$.

   We take $X = 0$. Let $L := \left(T\left(\left(\overline{\sigma}^1\right)^2 - \left(\underline{\sigma}^1\right)^2\right)\right)^{-1}$. Then

   $$C \cap V = \{aK : -L < a < 0\} .$$

   Hence,

   $$\lambda_V(C) = \lambda((-L, 0)) > 0 .$$

2. For all $X \in \mathbb{H}$ we have $\lambda_V\Big(M_C + X\Big) = 0$.

   Let $X \in \mathbb{H}$. By Theorem 2.2, we have

$$X = \mathbb{E}[X] + \int_0^T \eta_s^X \mathrm{d}B_s - K_T^X$$

   for some $\eta^X \in \mathcal{M}$ and some $K^X \in \mathbb{S}$. For $Z \in V \cap (M_C + X)$, we have then $Z = aK_T$ for some $a \in \mathbb{R}$ and, using Corollary 5.1 for Z,

$$Z = z + \int_0^T \eta_s^Z dB_s + X = z + \mathbb{E}[X] + \int_0^T (\eta_s^X + \eta_s^Z) dB_s - K_T^X$$

   for some $\eta^Z \in \mathcal{M}$. Hence, we obtain

$$aK_T = z + \mathbb{E}[X] + \int_0^T (\eta_s^X + \eta_s^Z) dB_s - K_T^X \,.$$

   As the decomposition in the martingale representation theorem (Theorem 2.2) is unique, this entails $z + \mathbb{E}X = 0$, $\eta^X + \eta^Z = 0$, and $aK_T = -K_T^X$. In particular, there can be at most one real number a with this property. We conclude that $\lambda_V(M_C + X) = 0$.

□

PROOF OF THEOREM 4.5: Consider an economy with endowments $(e^i)_{i=1,\dots,I-1} \in \mathcal{A}$ (and recall that $e^I = 1 - \sum_{j=1}^{I-1} e^j$). In view of Theorem 4.2.2, an Arrow–Debreu equilibrium of the economy is given by an allocation c_α for some $\alpha \in \Delta$ and a price functional of the form $\Psi(d) = E^P d$ for some $P \in \mathcal{P}$.
By part 2.(a) of Theorem 3.2, c_α^i is constant. By Theorem 4.2, implementability thus fails if there is a $i \in \mathbb{I}$ with $e^i \in \mathbb{M}^c = \mathbb{H} \setminus \mathbb{M}$. Therefore,

$$\mathcal{R} \subseteq M_{\mathcal{A}} := \{(e^i)_{i=1,\dots,I-1} \in \mathcal{A} : e^i \in \mathbb{M} \text{ for all } i = 1, \dots, I-1\}\,.$$

It is thus sufficient to show that $M_{\mathcal{A}}$ is shy in $\mathcal{A}$ (by Fact 1 in Anderson and Zame (2001), subsets of shy sets are shy).
In the case $I = 2$, we can directly apply Lemma 5.2 since then $\mathcal{A} = C$ and $M_{\mathcal{A}} = M_C$. For $I > 2$, we repeat the argument of Lemma 5.2 with the test space V^{I-1} where V is the one–dimensional subspace of $\mathbb{H}$ generated by K_T as in Lemma 5.2. □

6 Concluding Remarks

This paper establishes a crucial difference of risk and Knightian uncertainty. Under risk, dynamic trading of few long–lived assets suffices to implement the efficient allocations of Arrow–Debreu equilibria as dynamic Radner equilibria if the number of traded assets is equal to the number of sources of uncertainty. This result generically fails under Knightian volatility uncertainty even if we allow the market to choose the asset structure and without aggregate uncertainty.

This shows a crucial difference of volatility uncertainty to other cases of uncertainty. For stochastic volatility models in which the volatility is modeled as an additional diffusion process as, e.g., in the Heston (1993) model, a suitably chosen option on volatility usually allows to complete the market (compare Davis (2004), Romano and Touzi (1997), Davis and Obłój (2008), and Schwarz (2017)).

With Knightian uncertainty and ambiguity–averse agents, such a completion is usually impossible. Knightian uncertainty about volatility is also very different from Knightian uncertainty about the drift process as in Chen and Epstein (2002). With drift uncertainty, the class of priors is dominated by one single reference measure and one would be able to replicate the result of Duffie and Huang.[4]

Our result has important economic implications. To give an example, with heterogeneous agents, one cannot expect to obtain an efficient allocation in financial market equilibria. The consumption–based capital asset pricing model as derived in Epstein and Ji (2013) which is based on the existence of a representative agent would thus need to be amended suitably in the case of heterogeneous agents.

In Appendix B, we also discuss possible extensions of our results beyond the Bachelier model.

A Proofs of Section 3

We recall underlying results for expected utility economies from Dana (1993). For weights $\alpha^i \geq 0$ and $\omega \in \Omega$, denote by $c_\alpha(\omega)$ the maximizer of $\sum_{i\in\mathbb{I}} \alpha^i u^i(c^i)$ over $\{c \in \mathbb{R}^I_+ : \sum_{i\in\mathbb{I}} c^i = e(\omega)\}$. The so–called α–efficient allocation c_α is

[4] Although this task has not been carried out formally, as far as we know.

characterized by the first–order conditions

$$\alpha^i \frac{du^i}{dx}(c^i_\alpha(\omega)) = \alpha^j \frac{du^j}{dx}(c^j_\alpha(\omega)) =: \psi_\alpha(\omega) \tag{A.1}$$

for agents with strictly positive weights $\alpha^i, \alpha^j > 0$; agents i with weight $\alpha_i = 0$ have $c^i_\alpha(\omega) = 0$, of course. As an implicit function, efficient allocations $c_\alpha = (c^1_\alpha, \ldots, c^I_\alpha)$ and the corresponding state price ψ_α are continuous in e. Set $\Delta = \{\alpha \in \mathbb{R}^I_+ : \sum_i \alpha_i = 1\}$ and denote by $\mathbb{O} = (c_\alpha)_{\alpha\in\Delta}$ the set of efficient allocations in $\mathcal{E}^P$. The definition of $\mathbb{O}$ does not depend on the particular $P \in \mathcal{P}$ as the α–efficient allocations are defined pointwise.

Our strategy of proof is to pick an equilibrium in $\mathcal{E}^P$ for an arbitrary $P \in \mathcal{P}$ and to show that this equilibrium is an equilibrium in $\mathcal{E}$. In general, equilibrium allocations in $\mathcal{E}^P$ are determined only P–almost surely. For $\mathcal{E}$, we need however that market clearing occurs quasi–surely. The allocations c_α in (A.1) are defined pointwise for all $\omega \in \Omega$. In particular, we have $\sum_{i\in\mathbb{I}} c^i_\alpha(\omega) = e(\omega)$ for all ω, hence also quasi–surely.

Proof of Theorem 3.1: We first show that the allocations $c_\alpha \in \mathbb{O}$ belong to our commodity space $\mathbb{H}$. From $0 \le c^i_\alpha \le e$, we see that c^i_α is quasi–surely bounded. As c^i_α can be written as a continuous function of aggregate endowment e, c^i_α is also quasi–continuous. Under Assumption 2, e is ambiguity–free; as a continuous function of e, c^i_α is also ambiguity–free.

Pick any $P \in \mathcal{P}$. Due to our Assumption 1, the assumptions (i) to (iv) in **?**p.954]Dana93 are satisfied. For Assumptions (i) (strict concavity and monotonicity), (ii) (twice continuous differentiability), and (iv) (Inada condition), this is immediate. For Assumption (ii), note that our Bernoulli utility functions are independent of the state ω; by concavity, they are bounded by some linear function. Hence, Assumption (ii) of Dana is also satisfied. Since endowments are bounded away from zero by Assumption 1, Assumption (E) in **?**p.954]Dana93 is also satisfied. We can thus apply Theorem 2.5 of Dana (1993): there exists an $\alpha \in \Delta$ and an equilibrium (Ψ, c) in $\mathcal{E}^P$ with $c = c_\alpha$ P-a.s. and $\Psi(X) = E^P[\psi_\alpha X]$ for $X \in L^\infty(P)$. The state price ψ_α is a continuous function of e, and hence bounded q.s. By (A.1), ψ_α is strictly positive. Due to Assumption 1, individual endowments e^i are bounded away from zero quasi–surely. Hence, we have $\Psi(e^i) > 0$ for all $i \in \mathbb{I}$. As a consequence, $\alpha^i > 0$ since otherwise $c^i_\alpha = 0$ would be dominated by some strictly positive consumption plan (e.g., choose $x > 0$ with $E^P[\psi_\alpha x] = \Psi(e^i)$. By Assumption 1, $U^i(x) = u^i(x) > u^i(0)$).

We claim that (Ψ, c) is an equilibrium in $\mathcal{E}$.

Note that Ψ is well–defined on $\mathbb{H} \subset L^\infty(P)$. Since $\sum_{i\in\mathbb{I}} c^i_\alpha(\omega) = e(\omega)$ for all ω; c_α clears the market for every $\omega \in \Omega$, hence quasi–surely. The budget constraint $\Psi(c^i) = \Psi(e^i)$ is satisfied because (Ψ, c) is an Arrow–Debreu equilibrium in $\mathcal{E}^P$.

It remains to show that c^i_α maximizes utility in $\mathcal{E}$ subject to the budget constraint. Let d be budget–feasible for agent i. As c_α is an Arrow–Debreu equilibrium in the expected utility economy $\mathcal{E}^P$, we have $E^P[u^i(c^i_\alpha)] \geq E^P[u^i(d)]$. As c^i_α is ambiguity–free, we have $U^i(c^i_\alpha) = E^P[u^i(c^i_\alpha)]$. Therefore,

$$U^i(d) \leq E^P[u^i(d)] \leq E^P[u^i(c^i_\alpha)] = U^i(c^i_\alpha)\,.$$

□ As a preparation for the proof of Theorem 3.2, we now show that the allocations in $\mathbb{O}$ are also efficient in the Knightian economy $\mathcal{E}$. This is not obvious as, in general, different measures in $\mathcal{P}$ could be the "worst case" measure for different agents and efficient allocations would thus depend on those worst–case measures. Our result hinges on Assumption 2.

Proposition A.1 *1. Under Assumption 2, the efficient allocations in the Knightian economy $\mathcal{E}$ coincide with the allocations in $\mathbb{O} = (c_\alpha)_{\alpha\in\Delta}$. Each c_α is ambiguity–free.*

2. Under Assumption 3, each c_α is full insurance.

PROOF: Let $d \in \mathbb{O}$ be an efficient allocation. By applying the separation theorem, one can show that d maximizes the weighted sum of utilities $\sum_{i\in\mathbb{I}} \alpha^i U^i(c^i)$ for some $\alpha \in \Delta$. Set $\Gamma = \{\omega \in \Omega : \exists i \in \mathbb{I} \text{ with } d^i(\omega) \neq c^i_\alpha(\omega)\}$. As the Bernoulli utility functions u^i are strictly concave by Assumption 1 and by definition of c_α, we have

$$\sum_{i\in\mathbb{I}} \alpha^i u^i(d^i(\omega)) < \sum_{i\in\mathbb{I}} \alpha^i u^i(c^i_\alpha(\omega))$$

for all $\omega \in \Gamma$. Assume that Γ is not a polar set. Then there is $P \in \mathcal{P}$ with $P(\Gamma) > 0$. Therefore, we have

$$E^P\Big[\sum_{i\in\mathbb{I}} \alpha^i u^i(d^i)\Big] < E^P\Big[\sum_{i\in\mathbb{I}} \alpha^i u^i(c^i_\alpha)\Big]\,.$$

Since c_α is ambiguity–free, $E^P[\sum_{i\in\mathbb{I}} \alpha^i u^i(c^i_\alpha)] = \sum_{i\in\mathbb{I}} \alpha^i U^i(c^i_\alpha)$. On the other hand, by ambiguity–aversion,

$$\sum_{i\in\mathbb{I}} \alpha^i U^i(c^i_\alpha) \leq E^P\Big[\sum_{i\in\mathbb{I}} \alpha^i u^i(d^i)\Big]\,,$$

and we obtain a contradiction. We thus conclude that Γ is a polar set and thus $d = c_\alpha$ quasi–surely.
From Proposition 2.2. of Dana (1993) (see also our discussion above), we know that c_α is a continuous function of aggregate endowment e; under Assumption 2, c_α is thus ambiguity–free, and under Assumption 3, c_α is quasi–surely constant, or full insurance. □ □

Proof of Theorem 3.2: Let (Ψ, c) be an Arrow–Debreu equilibrium of $\mathcal{E}$. By the first welfare theorem, c is efficient. By Proposition A.1, there exist $\alpha \in \Delta$ with $c = c_\alpha$ and c_α is ambiguity–free. This proves 1 (a). If we impose even Assumption 3, Proposition A.1 shows that c_α is full insurance, proving 2 (a).

For part 1 (b), note that we have $\alpha^i > 0$ for all $i \in \mathbb{I}$, as individual endowments are strictly positive. Otherwise, $c_\alpha^i = 0$ which is dominated by the strictly positive individual endowment e^i (Assumption 1). Due to the first oder condition of individual utility maximization, any equilibrium price functional Ψ is colinear with some supergradient of U^i at c_α^i. For any $i \in \mathbb{I}$, the set of supergradients contains all linear functionals of the form $\frac{d}{dx}u^i(c_\alpha^i) \cdot P$, where P is a minimizer in the set of priors. Since $u^i(c_\alpha^i)$ is ambiguity–free, $E^P[u^i(c_\alpha^i)]$ is constant on $\mathcal{P}$ and hence every element in $\mathcal{P}$ is a minimizer of the multiple prior expected utility. From (A.1), $\alpha^i \frac{d}{dx}u^i(c_\alpha^i) = \psi_\alpha$ which is independent of i and ambiguity–free by 1(a).

By Assumption 1, marginal utilities $\frac{d}{dx}u^i$ are continuous and decreasing. Since c_α is a continuous function of aggregate endowment e, and since e is bounded away from zero, ψ_α is quasicontinuous and bounded. Since e is also bounded and marginal utilities $\frac{d}{dx}u^i$ are decreasing, ψ_α is also bounded away from zero. We thus obtain a price functional of the form

$$\Psi(d) = E^P[\psi_\alpha d]$$

such that (c, Ψ) is an Arrow–Debreu equilibrium.

For part 2(b), note that ψ_α is constant, since c_α is full insurance. Hence, without loss of generality, we can replace the price functional $\Psi(d) = E^P[\psi_\alpha d]$ by $\Psi(d) = E^P[d]$. □

B Beyond the Bachelier Model

The Bachelier model we presented allows for negative values of the price process. Theorem 4.2 is still valid, when our G-Brownian motion $B = B^+ + B^-$

of the Bachelier model is decomposed into the positive B^+ and negative part B^-. The trading strategies are then given by $\theta_t^{k,+} = \theta_t^k 1_{\{B_t^k \geq 0\}}$ and $\theta_t^{k,-} = -\theta_t^k 1_{\{B_t^k < 0\}}$ where θ^k denotes the fractions invested in the k-th uncertain assets of Theorem 4.2. In the same fashion, as mentioned in Section 5 of Duffie and Huang (1985), the number of assets becomes $2 \cdot d + 1$.

Theorem 4.2 is still valid if we replace the process B with a symmetric $\mathbb{E}$–martingale of the form $M_t = M_0 + \int_0^t V_t \mathrm{d}B_t$, such that $V_t \in \mathbb{H}_+^{d \times d}$ with $V_t^{ij} = 0$ and V^{ii} q.s. bounded away from zero.

It suffices to show that every stochastic integral of the form $\int_0^T \theta_s \mathrm{d}B_s$ for some $\theta \in \mathcal{M}$ can be written as $\int_0^T \theta_s \mathrm{d}B_s = \int_0^T \theta_s^M \mathrm{d}M_s$ for some suitable $\theta^M \in \mathcal{M}$. The proof then follows the same lines as the proof of Theorem 4.2, by substituting B with M. The obvious candidate is $\theta_t^{M,k} = \theta_t^k / V_t^{kk}$. We need to show that the stochastic integral $\int_0^T \theta_t^M dM_t$ is well–defined. Note that $\theta^M \in \mathcal{M}$ because V^{kk} is bounded away from zero q.s.

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