

Dynamically Consistent α -Maxmin Expected Utility*

Patrick Beissner[†] Qian Lin[‡] Frank Riedel[§]

December 1, 2017

Abstract

The α -maxmin model is a prominent example of preferences under Knightian uncertainty as it allows to distinguish ambiguity and ambiguity attitude. These preferences are dynamically inconsistent for nontrivial versions of α . In this paper, we derive a recursive, dynamically consistent version of the α -maxmin model. In the continuous-time limit, the resulting dynamic utility function can be represented as a convex mixture between worst and best case, but now at the local, infinitesimal level.

We study the properties of the utility function and provide an Arrow-Pratt approximation of the static and dynamic certainty equivalent. We derive a consumption-based capital asset pricing formula and study the implications for derivative valuation under indifference pricing.

Key words: Dynamic consistency, α -maxmin expected utility, Knightian uncertainty, ambiguity attitude

JEL subject classification: C60, D81, D90

*We thank Shige Peng for fruitful discussions. Financial support through CRC 1283 “Taming Uncertainty” by the German Research Foundation DFG is gratefully acknowledged.

[†]Center for Mathematical Economics, Bielefeld University, Germany. Email: patrick.beissner@uni-bielefeld.de

[‡]School of Economics and Management, Wuhan University, Wuhan 430072, China. Email: linqian@whu.edu.cn.

[§]Center for Mathematical Economics, Bielefeld University, Germany. Email: frank.riedel@uni-bielefeld.de and Department of Economic and Financial Sciences, University of Johannesburg

1 Introduction

In an effort to differentiate conceptually ambiguity from ambiguity attitude, Ghirardato, Maccheroni, and Marinacci (2004) introduce the α -maxmin model of preferences under Knightian uncertainty. These preferences can be represented by a utility function of the form

$$U(X) = \alpha \min_{P \in \mathcal{P}} E^P[u(X)] + (1 - \alpha) \max_{P \in \mathcal{P}} E^P[u(X)] \quad (1)$$

for a Bernoulli utility function u , a class of priors \mathcal{P} , and an index of ambiguity attitude $\alpha \in [0, 1]$. Such preferences generalize the well-known α -maxmin rule of Hurwicz to settings of Knightian uncertainty where the subjective perception of ambiguity can be described by a set of probability measures and the attitude towards ambiguity by a parameter α which describes the relative weight put on pessimism versus optimism¹.

In this paper, we discuss α -maxmin utility in a dynamic framework. For the purely pessimistic case ($\alpha = 1$), Epstein and Schneider (2003) have shown that the multiple priors model of Gilboa and Schmeidler (1989) is dynamically consistent if and only if the set of priors is rectangular, i.e. stable under pasting marginal and conditional distributions.

Our starting point is the following fact: *Even if* the set of priors is rectangular, α -maxmin utility is not time-consistent for non-trivial values of ambiguity attitude α .

We thus set out to define a *recursive* version of α -maxmin utility where we apply the logic of α -maxmin utility conditionally upon the available information in every discrete time step. Such a recursive construction leads to a time-consistent overall utility function.

In discrete time, tractable representations of the resulting utility function are usually not available. The continuous-time limit of our recursive construction, however, admits a nice representation. The dynamic utility form satisfies the backward stochastic differential equation

$$dU_t(X) = - \left(\alpha \min_{\theta \in \Theta} \theta \sigma_t + (1 - \alpha) \max_{\theta \in \Theta} \theta \sigma_t \right) dt + \sigma_t dB_t$$

where now the set Θ describes the perceived ambiguity. We thus obtain again an α -maxmin representation, but now *locally*, at the infinitesimal (one-step ahead) level².

The representation of the utility functional as backward stochastic differential equation allows a more detailed study of its properties and the economic

¹The neo-additive capacities of Chateauneuf, Eichberger, and Grant (2007) are another instance of such preferences.

²This representation generalizes the representation for time-consistent pessimistic multiple prior preferences of Chen and Epstein (2002).

consequences for agents with such preferences. We discuss the properties of the utility functional and derive a representation for the certainty equivalent.

As an application, we show the implications for consumption-based asset pricing models. Ambiguity leads to an additional premium for uncertain assets, similar to Chen and Epstein (2002). The ambiguity premium is reduced by optimism. The short interest rate increases usually with pessimism.

The paper is set up as follows. The next section shows that the naive version of α -maxmin utility is not dynamically consistent. Section 3 derives the recursive, dynamically consistent version and its continuous-time limit. Section 4 discusses the properties of the resulting preferences. Section 5 discusses the implications for equilibrium asset prices in the framework of the consumption-based capital asset pricing model and the implications for derivative valuation if one uses the method of indifference pricing.

2 Dynamic Inconsistency of α -maxmin Utility

Gilboa-Schmeidler utility functions are dynamically consistent if (and only if) the set of priors is rectangular (Epstein and Schneider (2003)). This result carries over to optimistic, ambiguity-loving agents ($\alpha = 0$). For intermediate values of α , rectangularity is not sufficient for dynamic consistency as we show in this section with the help of two examples, in discrete and continuous time.

Consider the two period binomial tree of Figure 1. The transition probabilities of moving up in the tree are given by $p, q, r \in [\frac{1}{4}, \frac{3}{4}]$. By construction, the resulting set of priors

$$\mathcal{P} = \left\{ \left(pq, p(1-q), (1-p)r, (1-p)(1-r) \right) \in \Delta(\Omega) : r, p, q \in \left[\frac{1}{4}, \frac{3}{4} \right] \right\}$$

is rectangular. We write $\underline{p} = \underline{q} = \underline{r} = \frac{1}{4}$ and $\bar{p} = \bar{q} = \bar{r} = \frac{3}{4}$.

Consider the two payoffs X and Y depicted in the figure. Note that Y has a payoff which is known at time 1 but uncertain at time 0. For simplicity³, we take $u(x) = x$ and $\alpha = 1/2$.

We first show that Y is uniformly preferred to X at time 1. Indeed, in the upper node, the utility of X is

$$U_1[X] = \frac{1}{2} \min_{q \in [\frac{1}{4}, \frac{3}{4}]} q \cdot 0 + (1-q) \cdot 8 + \frac{1}{2} \max_{q \in [\frac{1}{4}, \frac{3}{4}]} q \cdot 0 + (1-q) \cdot 8 = 4,$$

which is strictly smaller than the utility of Y which is 4.1 (recall that Y is known at time 1). Similarly, in the lower node we obtain $U_1(X) = 2 < U_1(Y) = 2.2$.

³The result carries over easily to arbitrary isotone u and values of $\alpha \in (0, 1)$.

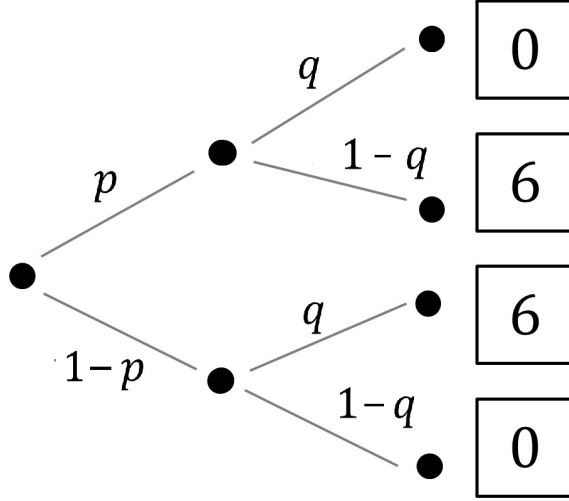


Figure 1: Two period binomial model and rectangular set of priors. For $\alpha = 1/2$, the agent prefers X to Y ex ante and reverses the ranking in all nodes at time 1.

Now we show that the ranking is reversed at time 0. We compute

$$\begin{aligned}
 U_0[X] &= \frac{1}{2} \min_{P \in \mathcal{P}} E^P[X] + \frac{1}{2} \max_{P \in \mathcal{P}} E^P[X] \\
 &= \frac{1}{2} \left(8\underline{p}(1 - \bar{q}) + 4(1 - \underline{p})\underline{r} \right) + \frac{1}{2} \left(8\bar{p}(1 - \underline{q}) + 4(1 - \bar{p})\bar{r} \right) \\
 &= \frac{1}{2} \left(\frac{1}{16} \cdot 8 + \frac{3}{16} \cdot 4 \right) + \frac{1}{2} \left(\frac{9}{16} \cdot 8 + \frac{3}{16} \cdot 4 \right) = 3\frac{1}{4}
 \end{aligned}$$

and $U_0[Y] = \frac{1}{2} \left(\frac{1}{4} \cdot 4.1 + \frac{3}{4} \cdot 2.2 \right) + \frac{1}{2} \left(\frac{3}{4} \cdot 4.1 + \frac{1}{4} \cdot 2.2 \right) = 3.15$. At time $t = 0$, X is preferred to Y .

Dynamic consistency is closely related to recursivity, or the dynamic programming principle. The iterated α -maxmin expected utility is $U_0[U_1[X]]$. We plug $U_1[X] = (4, 2)$ into U_0 and get

$$U_0[U_1[X]] = \frac{1}{2} \left[\left(\frac{1}{4} \cdot 4 + \frac{3}{4} \cdot 2 \right) + \left(\frac{3}{4} \cdot 4 + \frac{1}{4} \cdot 2 \right) \right] = 3 \neq U_0[X].$$

The inequality $U_0[U_1[X]] \neq U_0[X]$ shows that recursivity fails.

For our continuous-time example, we choose the drift ambiguity model of Chen and Epstein (2002). Fix a finite time interval $[0, T]$. Let B be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}_t)_{t \in [0, T]}$ the filtration generated by B , and completed by null sets. The set of priors consists of all probability measures P^θ such that B has drift θ under P^θ . More specifically, we denote by Θ the set of all adapted processes $\theta = (\theta_t)_{t \in [0, T]}$ with values in the interval $[-1, 0]$. The martingale

$$\left. \frac{dP^\theta}{d\mathbb{P}} \right|_{\mathcal{F}_t} = \exp \left(-\frac{1}{2} \int_0^t |\theta_s|^2 ds + \int_0^t \theta_s dB_s \right) \tag{2}$$

then defines a measure P^θ under which B has drift θ by Girsanov's theorem.

Now let us consider the three dates $t = 0$, $t = 1$ and $T = 2$. Again, we take $\alpha = \frac{1}{2}$. Consider the payoffs $X = e^{B_T}$ and $Y = \frac{1}{2}(C + \varepsilon)e^{B_t}$ for $C = e^{\frac{1}{2}} + e^{-\frac{1}{2}}$ and $\varepsilon > 0$ sufficiently small.

As we deal with drift ambiguity, and X is a monotone function of B_T , the worst-case prior \underline{P} assigns drift -1 and the best case prior drift 0 , i.e. the reference measure \mathbb{P} is the best prior.

At time $t = 1$, we thus get

$$\begin{aligned} U_1[e^{B_T}] &= \frac{1}{2} \left(E_t^{\mathbb{P}}[e^{B_T}] + E_t^{\underline{P}}[e^{B_T}] \right) \\ &= \frac{1}{2} \left(e^{\frac{1}{2}T} E_t^{\mathbb{P}}[e^{B_T - \frac{1}{2}T}] + e^{B_t + \frac{1}{2}t} E_t^{\underline{P}}[e^{-B_T - \frac{1}{2}T + B_T}] \right) \\ &= \frac{1}{2} e^{B_t} \left(e^{\frac{1}{2}(T-t)} + e^{-\frac{1}{2}(T-t)} \right) = \frac{1}{2} C e^{B_t}. \end{aligned}$$

Note that this is strictly smaller than $U_1(Y) = Y$.

At time 0 , we have

$$U_0[e^{B_T}] = \frac{1}{2} \left(E^{\mathbb{P}}[e^{B_T}] + E^{\underline{P}}[e^{B_T}] \right) = \frac{1}{2} \left(e^1 + E^{\mathbb{P}}[e^{-B_T - 1/2T + B_T}] \right) \approx 1.54,$$

whereas the utility of Y is

$$U_0[Y] = \frac{1}{2}(C + \varepsilon) \left(\frac{1}{2} E^{\mathbb{P}} e^{B_t} + \frac{1}{2} E^{\underline{P}} e^{B_t} \right) \approx \frac{C^2}{4} \approx 1.27.$$

Again, the ranking is reversed at time 0 .

3 Recursive α -maxmin utility and its Continuous-Time Limit

This section introduces a dynamically consistent version of α -maxmin utility. We start with a recursive formulation in discrete time. Then we introduce the counterpart in continuous time. We show (Theorem 2) that the discrete-time version converges to the continuous-time version.

3.1 General Setup in Continuous Time

Fix a finite time interval $[0, T]$. Let B be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(\mathcal{F}_t)_{t \in [0, T]}$ be the filtration generated by B , completed by \mathbb{P} -null sets. Set $L_t = L^2(\Omega, \mathcal{F}_t, \mathbb{P})$, for every $t \in [0, T]$, the space of square integrable and \mathcal{F}_t -measurable random variables.

Let $\Theta : \Omega \times [0, T] \Rightarrow \mathbb{R}$ be an adapted set-valued process and assume for every (ω, t) the set $\Theta_t(\omega)$ is a convex and closed subset of some compact set

$K \subset \mathbb{R}$. For a real-valued process $\theta = (\theta_t)$ with $\theta_t \in \Theta_t$ we define the density process

$$z_t^\theta \equiv \exp\left(-\frac{1}{2} \int_0^t |\theta_s|^2 ds + \int_0^t \theta_s dB_s\right).$$

By the Girsanov theorem, z^θ determines a probability measure P^θ . Given Θ , we thus obtain a corresponding set of priors:

$$\mathcal{P} = \{P^\theta : \theta \in \Theta, P^\theta \text{ is defined by (2)}\}. \quad (3)$$

The induced set of priors \mathcal{P} is weakly compact, convex and rectangular (see Chen and Epstein (2002)).

3.2 Recursive α -maxmin utility in discrete time

On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with set of priors \mathcal{P} introduced in (3), we construct a recursive α -maxmin utility in discrete time. For an integer N , we let $\Delta = \frac{T}{N}$. The collection of all adapted processes (a_t) taking values in the unit interval $[0, 1]$ is denoted by $\llbracket 0, 1 \rrbracket$.

We define the (naive and time inconsistent) nonlinear expectation, for $X \in L_T$ and $t \in [0, T]$.

$$I_t[X] = a_t \min_{P \in \mathcal{P}} E_t^P[X] + (1 - a_t) \max_{P \in \mathcal{P}} E_t^P[X].$$

Let u be a concave and increasing function. We construct a recursive utility in discrete time as follows. For the terminal time $t = T$, we define

$$U_T^N[X] = I_T[u(X)] = u(X).$$

For $t \in [i\Delta, (i+1)\Delta)$, where $i = 0, 1, \dots, N-1$, we define

$$U_t^N[X] = I_t[U_{(i+1)\Delta}^N[X]]. \quad (4)$$

By construction the family $(U_{i\Delta})_{i=0, \dots, N}$ is a family of recursive utilities.

Theorem 1 *The family $(U_{i\Delta}^N)_{i=0, \dots, N}$ is dynamically consistent in the following sense: for all $X, Y \in L_T$ and all $i < j$, if $U_{j\Delta}^N(X) \geq U_{j\Delta}^N(Y)$, a.s., then $U_{i\Delta}^N(X) \geq U_{i\Delta}^N(Y)$, a.s.*

3.3 Continuous-Time Limit of α -maxmin utility

The continuous-time setup allows to describe the recursive relation of nonlinear conditional expectations in differential terms. This differential formulation in (5) is the continuous-time counterpart of (4).

For every $a \in \llbracket 0, 1 \rrbracket$ and every $X \in L_T$, there exists a unique solution $(\mathcal{E}_t[X], \sigma_t)$ of the backward stochastic differential equation (BSDE)

$$d\mathcal{E}_t[X] = \underbrace{a_t \max_{\theta \in \Theta} \theta_t \sigma_t + (1 - a_t) \min_{\theta \in \Theta} \theta_t \sigma_t}_{=e(t, \sigma_t)} dt + \sigma_t dB_t, \quad (5)$$

with⁴ $\mathcal{E}_T[X] = X$.

Example 1 1. If $\Theta = \{\theta\}$ is a singleton, then a_t is irrelevant and we obtain the usual linear expectation $\mathcal{E}_t[X] = E_t^{P^\theta}[X]$, where the subjective prior P^θ is again given by (2).

2. If $a \equiv 1$, i.e., a form of maximal pessimism in beliefs, then $\underline{\mathcal{E}}_t[X] = \min_{P \in \mathcal{P}} E_t^P[X]$ reduces to the continuous-time analog of Gilboa–Schmeidler preferences of Chen and Epstein (2002). The case of an extremely optimistic expectation $\bar{\mathcal{E}}_t[X]$ is obtained⁵ with $a \equiv 0$.

Definition 1 Let $a \in \llbracket 0, 1 \rrbracket$. For every $X \in L_T$, the α -maxmin conditional expectation $(\mathcal{E}_t[X])_{t \in [0, T]}$ is the unique solution of (5).

The BSDE formulation of \mathcal{E}_t implies a dynamic stability of the functional form. In the notation of Example 1, the comparison principle for BSDEs yields $\mathcal{E}_t[X] \in [\underline{\mathcal{E}}_t[X], \bar{\mathcal{E}}_t[X]]$ for any X and time t . Consequently, there is a process $(\alpha_t^X) \in \llbracket 0, 1 \rrbracket$, depending on (a_t) and X , that allows for a global representation

$$\mathcal{E}_t[X] = \alpha_t^X \underline{\mathcal{E}}_t[X] + (1 - \alpha_t^X) \bar{\mathcal{E}}_t[X]. \quad (6)$$

The dynamically inconsistent expectation of Section 2 employed a constant weight $\alpha \equiv \alpha_t^X$. A stochastic and X -dependent α_t^X provides the dynamic consistency of \mathcal{E}_t . For perspective, Proposition 5 in Appendix B collects a list of further properties.

The following theorem establishes the announced connection between the discrete- and continuous-time formulation.

Theorem 2 Let u be concave and increasing. For every $X \in L_T$ and every t , $U_t^N[X]$ from (4) converges to $\mathcal{E}_t[u(X)]$, in the norm of L_T :

$$\lim_{N \rightarrow \infty} \left\| U_t^N[X] - \mathcal{E}_t[u(X)] \right\| = 0.$$

⁴ Equation (5) is a special BSDE. For more details, see Appendix A and Peng (1997).

⁵Note that the operators max and min are interchanged in the differential formulation; the maximum corresponds to the pessimistic part.

4 Properties of Recursive α -maxmin Expected Utility and the Certainty Equivalent

In this section, we study the properties of the continuous-time α -maxmin utility function as given by

$$U(X) = \mathcal{E}[u(X)]$$

for the time 0 utility and

$$U_t(X) = \mathcal{E}_t[u(X)]$$

for the dynamic utility process. We start with the basic continuity properties and dynamic consistency.

Proposition 1 *Let u be continuous and strictly increasing. Then the nonlinear expected-utility functional $U : L_T \rightarrow \mathbb{R}$ is*

- (i) norm continuous: *if $X_n \rightarrow X$ in L_T , then $\lim_{n \rightarrow \infty} U(X_n) = U(X)$.*
- (ii) order continuous: *if $X_n \searrow X$, \mathbb{P} -a.s., then $U(X_n) \searrow U(X)$.*
- (iii) (strictly) monotone: *if $X \geq Y$ then $U_t(X) \geq U_t(Y)$, for all $t \in [0, T]$. If also $\mathbb{P}(X > Y) > 0$ and u is strictly increasing, then $U(X) > U(Y)$.*
- (iv) dynamically consistent: *let $t \geq s$, if $U_t(X) \geq U_t(Y)$ then $U_s(X) \geq U_s(Y)$.*

Let us now come to risk aversion. As the utility functional U is *not* concave, one might wonder if U displays risk aversion. We will show that for a natural extension of the concept of risk aversion to Knightian uncertainty, risk aversion is still equivalent to the concavity of the Bernoulli utility function u .

Definition 2 *An agent is conditionally \mathcal{E} -risk averse on L_T if*

$$U_t(X) \leq u(\mathcal{E}_t[X]), \quad \text{for all } X \in L_T, t \in [0, T].$$

Proposition 2 *Let $u \in C^2(\mathbb{R})$ be increasing. The agent is conditionally \mathcal{E} -risk averse if and only if u is concave.*

We continue with a discussion of the certainty equivalent and extend the Arrow-Pratt analysis.

Definition 3 $\mathcal{C}^X \in \mathbb{R}$ *is called certainty equivalent of $X \in L_T$ if*

$$u(\mathcal{C}^X) = \mathcal{E}[u(X)]$$

holds true.

For derivation of the second-order approximation of the certainty equivalent consider a given wealth $w \in \mathbb{R}$ and denote the absolute risk aversion by $A(x) = -\frac{u''(x)}{u'(x)}$. For a Taylor expansion, we need some further terminology: The expression $\mathbf{var}(X) = \mathcal{E}[(X - \mathcal{E}[X])^2]$ denotes the variance under \mathcal{E} and $\mathbf{co}(X, Y) = \mathcal{E}[X - Y] - \mathcal{E}[X] + \mathcal{E}[Y]$ refers to the so-called *coexpectation* and quantifies the compensation for the nonlinearity of \mathcal{E} .⁶ The case $\alpha = 0$ in Example 1, yields a sub-linear expectation, hence $\mathbf{co}(\cdot, \cdot) \geq 0$.

Theorem 3 *Let u be concave and twice differentiable and \mathcal{E} be an α -maxmin conditional expectation. Then,*

$$\mathcal{C}^{w+X} - w \approx \frac{1}{2}A(w)\mathbf{var}(X) + \mathbf{co}\left(X, \frac{1}{2}A(w)X^2\right), \quad (7)$$

where X with $\mathcal{E}[X] = 0$ denotes a centered distortion.

For perspective, we state two examples in a static setup, that investigate the role of the coexpectation and the resulting uncertainty premium.

Example 2 *Let the normally distributed distortion X be ambiguous in the volatility parameter, i.e. $\text{Law}^{P_\sigma}(X) = N(0, \sigma)$. With $\mathcal{P} = \{P_\sigma : \sigma \in [\underline{\sigma}, \bar{\sigma}]\}$, each $\sigma \in [\underline{\sigma}, \bar{\sigma}]$ induces a law P_σ for X . Let u be of CARA type such that $A(w) = 2$. To calculate the \mathbf{co} -part in (7) for arbitrary $\alpha \in [0, 1]$, we begin with*

$$\begin{aligned} \mathcal{E}^\alpha[X - X^2] &= \alpha \min_{\sigma \in [\underline{\sigma}, \bar{\sigma}]} E^{P_\sigma}[X - X^2] + (1 - \alpha) \max_{\sigma \in [\underline{\sigma}, \bar{\sigma}]} E^{P_\sigma}[X - X^2] \\ &= \alpha \bar{\sigma} + (1 - \alpha)\underline{\sigma} \end{aligned}$$

and similarly $\mathcal{E}^\alpha[X^2] = \alpha\underline{\sigma} + (1 - \alpha)\bar{\sigma}$. Since $\mathcal{E}^\alpha[X] = 0$, we have

$$\mathbf{co}\left(X, \frac{1}{2}A(w)X^2\right) = (1 - 2\alpha)(\bar{\sigma} - \underline{\sigma}).$$

Every $\alpha \leq \frac{1}{2}$ yields a positive coexpectation.

The following example discusses the quantitative differences of risk premia when comparing with the standard expected utility model.

Example 3 *Let there be two states of the world $\Omega = \{\text{good}, \text{bad}\}$. The nonlinear expectation given by $\mathcal{P} = \{P = (p, 1-p) \in \Delta(\Omega) : p \in [\frac{2}{5}, \frac{3}{5}]\}$ and $\alpha = (\frac{1}{3}, \frac{2}{3})$. We compare the quantitative effect with a linear expectation $P = (\frac{1}{2}, \frac{1}{2}) \in \Delta(\Omega)$.*

⁶In comparison to the approximation of the (second order) smooth ambiguity certainty equivalent in Maccheroni, Marinacci, and Ruffino (2013), the present certainty equivalent reveals the nonlinear structure of the expectation. In our approximation this is exposed by the coexpectation.

For expected utilities, take $u(x) = \sqrt{x}$ as the utility index. Consider the gamble $X = (1, 4)$. A direct calculation yields

$$\mathcal{E}[X] = \frac{1}{3} \min_{P \in \mathcal{P}} E^P[X] + \frac{2}{3} \max_{P \in \mathcal{P}} E^P[X] \approx 2.53 > 2.5 = E^P[X]$$

and similarly for the expected utilities we derive $\mathcal{E}(\sqrt{X}) \approx 1.53 > 1.5 = E^P[\sqrt{X}]$. Since the inverse of \sqrt{x} is x^2 , we then have for the certainty equivalents $\mathcal{C}^X \approx 2.34 > 2.25 = C^X$. In contrast to the risk premium $R(X)$ under P , the uncertainty premium $\mathcal{R}(X) = \mathcal{C}^X - \mathcal{E}[X]$ contains a nonlinear component. The second term in the decomposition $R(X) + (\mathcal{R}(X) - R(X))$ results in an ambiguity premium.

For a small distortion, the example points out that under a nonlinear expectation the uncertainty premium may vary considerably in comparison to the linear case. This is consistent with the derivations in (7), where the coexpectation $\mathbf{co} : L_T \times L_T \rightarrow \mathbb{R}$ controls this issue. The possibly negative ambiguity premium, caused by preferences for ambiguity, is manifested in the nonlinear behavior of the risk premium $\mathcal{R}(X)$.

We extend the concept of certainty equivalent now to the dynamic case and begin with the complete description of the *conditional certainty equivalent* $\mathcal{C}_t^X = u^{-1}(U_t(X))$.

Proposition 3 *Let $u \in C^2(\mathbb{R})$ be strictly increasing and concave and $X \in L_T$. The conditional certainty equivalent \mathcal{C}_t^X satisfies the following:*

1. $\mathcal{C}_t^X \leq \mathcal{E}_t[X]$.
2. Let $(U_t(X), \sigma_t^u)$ be the unique solution of $dU_t(X) = \mathbf{e}(t, \sigma_t^u)dt + \sigma_t^u dB_t$, $U_T(X) = u(X)$, where $\mathbf{e}(t, \sigma_t^u) = a_t \max_{\theta \in \Theta} \theta_t \sigma_t^u + (1 - a_t) \min_{\theta \in \Theta} \theta_t \sigma_t^u$ and denote $\sigma_t^c = \frac{\sigma_t^u}{u'(\mathcal{C}_t^X)}$. Then, the pair $(\mathcal{C}_t^X, \sigma_t^c)$ solves

$$d\mathcal{C}_t^X = \left(\frac{1}{2} A(\mathcal{C}_t^X) \sigma_t^c \cdot \sigma_t^c + \frac{\mathbf{e}(t, \sigma_t^u)}{\sigma_t^u} \sigma_t^c \right) dt + \sigma_t^c dB_t, \quad \mathcal{C}_T^X = X. \quad (8)$$

Since $\sigma_t^c \cdot \sigma_t^c$ is the derivative of the quadratic variation $\langle \mathcal{C}^X \rangle$ of \mathcal{C}^X , the variance multiplier of (7) appears again in the conditional version (8). From this perspective, (8) describes the local decomposition of the conditional certainty equivalent. The residual compensation $\frac{\mathbf{e}(t, \sigma_t^u)}{\sigma_t^u} \sigma_t^c$ stems from the nonlinearity of the expectation and corresponds to the coexpectation in the static approximation of Theorem 3.

So far, we have fixed the time T of the payoff. To discuss aspects about time consistency of the certainty equivalent, we need to vary the terminal time. For fixed $0 \leq t < \infty$, define as in (5):

$$d\mathcal{E}_{s,t}[u(X)] = \mathbf{e}(s, \sigma_s) ds + \sigma_s dB_s, \quad s \in [0, t], \quad \mathcal{E}_{t,t}[u(X)] = u(X). \quad (9)$$

For given $X \in L_t$, (9) has a unique solution by the same arguments as before. The two time parameters in (9) corresponds to (5) via $\mathcal{E}_{t,T} = \mathcal{E}_t$.

Definition 4 Let $u \in C^2(\mathbb{R})$ be strictly increasing and concave and $X \in L_t$. The dynamic certainty equivalent $\mathcal{C}_{s,t} : L_t \rightarrow L_s$ at X , with $s \in [0, t]$, is defined by

$$u(\mathcal{C}_{s,t}(X)) = \mathcal{E}_{s,t}[u(X)],$$

where $(\mathcal{E}_{s,t}[u(X)], \sigma_s)_{s \leq t}$ is the unique solution of (9).

The conditional certainty equivalent of Proposition 3 considers a fixed $t = T$. The dynamic certainty equivalent has the following properties.

Proposition 4 For $0 \leq r \leq s \leq t < \infty$, $A \in \mathcal{F}_s$ and $X, Y \in L_t$, the following properties hold:

- (i) Constant-preserving: $\mathcal{C}_{t,t}(X) = X$.
- (ii) Recursivity: $\mathcal{C}_{r,t}(X) = \mathcal{C}_{r,s}(\mathcal{C}_{s,t}(X))$.
- (iii) Dynamic consistency: $\mathcal{C}_{r,t}(X) \leq \mathcal{C}_{r,t}(Y)$, if $\mathcal{C}_{s,t}(X) \leq \mathcal{C}_{s,t}(Y)$.
- (iv) Monotonicity: If $X \leq Y$, then $\mathcal{C}_{r,t}(X) \leq \mathcal{C}_{r,t}(Y)$.
- (v) Zero-one law: $\mathcal{C}_{s,t}(X \mathbf{1}_A) = \mathcal{C}_{s,t}(X) \mathbf{1}_A$ and

$$\mathcal{C}_{s,t}(X \mathbf{1}_A + Y \mathbf{1}_{A^c}) = \mathcal{C}_{s,t}(X) \mathbf{1}_A + \mathcal{C}_{s,t}(Y) \mathbf{1}_{A^c}.$$

- (vi) Dominance: $\mathcal{C}_{r,t}(X) \leq \mathcal{E}_{r,s}[\mathcal{C}_{s,t}(X)]$. In particular, $\mathcal{C}_{r,t}(X) \leq \mathcal{E}_{r,t}[X]$.

The type of recursivity for the dynamic certainty equivalent is illustrated in Figure 2. The certainty equivalent of X on $[r, t]$ can be obtained directly.

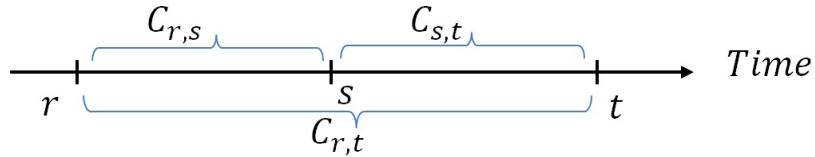


Figure 2: Time consistency of dynamic certainty equivalent

Another way determines $\mathcal{C}_{s,t}$ for X in a first step and then evaluate $\mathcal{C}_{s,t}(X)$ under the dynamic certainty equivalent on $[r, s]$.

5 Applications to Dynamic Asset Pricing

5.1 A Consumption-Based CAPM with Mild Optimism

Consider a single agent economy with aggregate endowment $de_t = \mu_t^e dt + \sigma_t^e dB_t$ and cumulative dividend process $dD_t = \mu_t^D dt + \sigma_t^D dB_t$ of a long-lived asset with initial conditions $e_0, D_0 \in \mathbb{R}_{++}$ and adapted integrable processes μ_t^e, μ_t^D and σ_t^e, σ_t^D . Assume a variant of the martingale generator condition (see Section 10 D in Duffie (1996) for a detailed account): $\sigma_t^D > 0$ almost everywhere.

At time t , the α -maxmin expected utility of the representative agent is

$$U_t(e) = \mathcal{E}_t \left[\int_t^T u(e_s) ds \right],$$

where u is a concave and three-times differentiable. If a_t is sufficiently close to 1, as discussed in Example 4 below, $\sigma \mapsto \mathbf{e}(t, \sigma)$ is sub-linear, the generator of \mathcal{E}_t in (5). Then, $\mathcal{E}_t[X] = \min_{P \in \mathcal{P}^\Theta} E_t^P[X]$ results in a super-linear expectation.

Example 4 Consider only a constant weighting process $a_t = a \in [0, 1]$ and the case of κ -ignorance, i.e., $\Theta_t = [-\kappa, \kappa]$ and $\kappa \in \mathbb{R}_+$. The generator \mathbf{e} of the α -maximin expectation in (5) simplifies considerably:

$$\mathbf{e}(t, \sigma_t) = (1 - a) \min_{\theta \in [-\kappa, \kappa]} \theta_t \sigma_t + a \max_{\theta \in [-\kappa, \kappa]} \theta_t \sigma_t = (2a - 1) \kappa \cdot |\sigma_t|. \quad (10)$$

If $a > \frac{1}{2}$, \mathbf{e} is sub-linear and yields a super-linear expectation given by

$$\mathcal{E}_t[X] = \min_{P \in \mathcal{P}^\Theta} E_t^P X, \quad \text{where } \underline{\Theta}_t = [-\underline{\theta}_t, \underline{\theta}_t] = (2a - 1) \cdot [-\kappa, \kappa].$$

Consequently, optimism appears as a shrinkage of the size of ambiguity in \mathcal{E}_t .

Departing from Example 4, we restrict the subsequent analysis to a weight process a_t that is sufficiently close to 1 in a way the nonlinear expectation remains super-linear. In view of Example 4, the residual ambiguity is denoted by $\underline{\Theta}_t = [-\underline{\theta}_t, \underline{\theta}_t]$. The resulting concavity of $c \mapsto U_0(c)$ allows to follow Section 2.4 of Beißner (2015) for the single-agent case and parts of Section 5 in Chen and Epstein (2002).

By the assumptions on Θ and the linearity of $P \mapsto E^P X$, there is a minimizing density process $\theta^* \in \underline{\Theta}$ such that⁷

$$U_t(e) = \min_{P \in \mathcal{P}^\Theta} E_t^P \left[\int_t^T u(e_s) ds \right] = E_t^{P^{\theta^*}} \left[\int_t^T u(e_s) ds \right],$$

⁷There is a prior in \mathcal{P} that is a minimizer of the super-linear expectation. By construction, there is an associated drift process θ^* , such that (2) yields the density z^{θ^*} .

for every $t \in [0, T]$. Under $P^{\theta^*} \in \mathcal{P}$, the state-price density at time t is given by $\psi_t = u'(e_t)$.⁸ Assuming a complete market, standard arguments yield a description of the risky asset by a stochastic Euler equation

$$S_t = \frac{1}{z_t^{\theta^*} \cdot u'(e_t)} E_t^{\mathbb{P}} \left[\int_t^T z_s^{\theta^*} u'(e_s) dD_s \right], \quad t \in [0, T]. \quad (11)$$

The process $z_t^{\theta^*} = \frac{dP^{\theta^*}}{d\mathbb{P}} | \mathcal{F}_t$ is given by (2) and, by virtue of (10), solves $dz_t^{\theta^*} = z_t^{\theta^*} \theta_t^* dB_t$ with $\theta_t^* = (2a_t - 1)\theta_t \cdot \text{sgn}(\sigma_t^U)$.⁹ The process σ_t^U is the second component in the BSDE formulation of $U_t(e)$

$$dU_t(e) = (2a_t - 1)\theta_t \cdot |\sigma_t^U| + u(e_t)dt + \sigma_t^U dB_t, \quad U_T(e) = 0. \quad (12)$$

With the Euler equation in (11) we follow the arguments in Section 10 H of Duffie (1996) to derive a consumption-based CAPM relation. Furthermore, the asset price can be rewritten as the cumulative return $\frac{dS_t}{S_t} = dR_t = \mu_t^R dt + \sigma_t^R dB_t$, for details see Section 6D in Duffie (1996). $P(x) = \frac{u''(x)}{u'(x)}$ denotes the degree of absolute prudence. The measure of absolute risk aversion is again denoted by $A(x)$.

Theorem 4 (CCAPM) *Assume that drift ambiguity is symmetric, i.e., $\Theta_t = [-\theta_t, \theta_t]$, with $\theta_t > 0$, and suppose optimism is mild, i.e., $a_t \in [\frac{1}{2}, 1]$.*

Then there exists a security spot market in which, at any time t , the excess return of the security satisfies

$$\mu_t^R - r_t = A(e_t) \cdot \sigma_t^R \sigma_t^e + (2a_t - 1)\theta_t \cdot \text{sgn}(\sigma_t^U) \cdot \sigma_t^R. \quad (13)$$

The equilibrium interest rate satisfies

$$r_t = A(e_t) \left[\mu_t^e + \sigma_t^e (2a_t - 1)\theta_t \cdot \text{sgn}(\sigma_t^U) - \frac{1}{2} \sigma_t^e \sigma_t^e P(e_t) \right]. \quad (14)$$

The second term of the right hand side of (13) refers to the *ambiguity premium under mild optimism* and yields a refined explanation of the equity premium. Specifically, the ambiguity premium becomes a function of a_t . The comparative statics are as follows: an increase in optimism, that is a decrease of a_t , yields a smaller ambiguity premium. This functional dependency has an intuitive appeal, as preferences for ambiguity, encoded in \mathcal{E}_t and given by a_t , directly quantifies the size of the ambiguity premium via the *optimism factor* $(2a_t - 1)$.

The boundary case $a_t = \frac{1}{2}$ let the ambiguity premium vanish. In the case of no optimism, $a_t = 1$, we get the CCAPM formula of Chen and Epstein (2002).

In several cases, the process σ_t^U in (13) can be written explicitly. This is to some extent of importance, as the sign of σ_t^U determines the form of the ambiguity premium.

⁸For details on the necessary and sufficient first order conditions of the resulting equilibrium we refer to Duffie (1996).

⁹Here, $\text{sgn}(x) = 1$ if $x > 0$, $= -1$ for $-x$ and $= 0$ for $x = 0$. The form of θ^* follows from $x \cdot \text{sgn}(x) = |x|$. If $\sigma^U > 0$, (12) is a linear BSDE. Example 5 relies on this aspect.

Example 5 Suppose the aggregate endowment follows a geometric Brownian motion $de_t = e_t(\mu^e dt + \sigma^e dB_t)$ starting in $e_0 = 1$. The degree of ambiguity is given by $\Theta \equiv [-\kappa, \kappa]$. Consider $u(x) = (x^\beta - 1)/\beta$, with $\beta \in (-\infty, 1] \setminus \{0\}$ and let $a \in [\frac{1}{2}, 1]$. These additional assumptions on the primitives allows for an explicit formulation¹⁰ of

$$\sigma_t^U = \frac{1 - \exp(\rho(a)(t - T))}{\rho(a)} e_t^\beta \sigma^e,$$

where $\rho(a) = -\beta \left(\mu^e - \frac{1-\beta}{2} (\sigma^e)^2 - (1 - 2a)\kappa|\sigma^e| \right)$ is linear and decreasing in a .

Following ? about the stylized facts on aggregate consumption, set $\sigma^e = \mu^e = 2\%$. A moderate relative risk aversion of 2 with $\beta = -1$ yields a positive $\rho(a) \approx \frac{\kappa}{25}a$ and consequently $\text{sgn}(\sigma_t^U) = 1$ almost everywhere. The ambiguity premium in (13) takes now the simple form $(2a - 1)\kappa\sigma_t^R$.

5.2 Application to Indifference Pricing

The dynamic certainty equivalent studied in Section 4 enables us to price contingent claims also via indifference pricing. This yields an alternative time consistent pricing principle. The novelty of the present modeling rests on the non-concave utility specification $X \mapsto \mathcal{E}[u(X)]$.

Hodges and Neuberger (1989) first use certainty equivalents to price claims in a static setting, i.e., from the seller point of view the indifference price is the smallest amount money $\pi \in \mathbb{R}$ that the seller would willingly sell the claims X :

$$u(\pi) = E^P[u(X)].$$

Indifference pricing under E^P can be extended to our dynamically consistent version of α -maxmin expected utility. Let the utility index be twice differentiable, strictly increasing and concave. From Definition 4, for fixed $\tau > s$, the dynamic certainty equivalent of a claim $X \in L_\tau$ at time s , $\pi_s(X) \in L_s$, satisfies $u(\pi_s(X)) = \mathcal{E}_{s,\tau}[u(X)]$. $(\mathcal{E}_{s,\tau}[u(X)], \sigma_s)_{s \leq \tau}$ is the unique solution of (9), with terminal condition $\mathcal{E}_{\tau,\tau}[u(X)] = u(X)$. Thus we define the pricing rule as the certainty equivalent:

$$\pi_s(X) = u^{-1}(\mathcal{E}_{s,\tau}[u(X)]).$$

$\pi_s(X)$ is the amount of money that the decision maker would pay at time s for the claim X with maturity at time τ . By virtue of Proposition 4, the indifference pricing rule $\pi_s : L_\tau \rightarrow L_s$ is time-consistent, monotone increasing and satisfies the zero-one law. Furthermore, by the Jensen inequality (see Appendix B) for $\mathcal{E}_{s,\tau}$ with fixed τ the price of X satisfies $\pi_s(X) \leq \mathcal{E}_{s,\tau}[X]$.

We now consider the special case of extreme pessimism. In view of Example 1, we set $a = 1$. This particular case captures a form of robust utility indifference pricing by incorporating risk aversion and model uncertainty.

¹⁰ The volatility of the utility σ_t^U is the second part in the solution of the BSDE (12). Since a_t is constant the argument follows the same line as Section 2.4 of ?.

Example 6 For fixed $t > 0$, let $(U_s)_{s \in [0, \tau]}$ be a dynamic worst case expected utility defined by

$$U_s(X) = \min_{P \in \mathcal{P}} E_s^P[u(X)], \quad s \in [0, \tau].$$

For $s < \tau$, we define the dynamic certainty equivalent of a $X \in L_t$ as above. The pricing rule becomes

$$\pi_s(X) = u^{-1} \left(\min_{P \in \mathcal{P}} E_s^P[u(X)] \right).$$

From this expression, it is apparent that preferences for risk and ambiguity are again disentangled. As argued above, $\pi_s : L_\tau \rightarrow L_s$ is time-consistent, monotone and satisfies the zero-one law. Moreover, the price of X defined by the certainty equivalent is less than $\min_{P \in \mathcal{P}} E_s^P[X]$.

6 Conclusion

We have derived a dynamically consistent extension of the α -maxmin model. In continuous time, the time-consistent version retains the α -maxmin structure and thus allows to distinguish ambiguity and ambiguity attitude, as the static model does.

We characterize risk aversion through the concavity of Bernoulli utility functions. The Arrow-Pratt approximation of the certainty equivalent contains an additional ambiguity premium that depends on the nonlinearity of the expectation and therefore on local ambiguity attitudes.

We present a consumption-based CAPM formula that allows to explain how the interplay of optimism and pessimism affects the excess return in terms of an ambiguity premium. Optimism can decrease the ambiguity premium.

We finally characterize the dynamic certainty equivalent and use it to discuss the consequences for indifference pricing.

A Backward stochastic differential equations

For the convenience of the reader, we gather some results on backward stochastic differential equations (BSDE) here. Pardoux and Peng (1990) introduced the following equation:

$$dy_t = f(t, y_t, \sigma_t)dt + \sigma_t dB_t, \quad t \in [0, T], \quad y_T = X, \quad (15)$$

where the terminal condition $X \in L_T$ and $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, so that the generator $f(\cdot, y, z)$ of the BSDE is an adapted process for every $y, z \in \mathbb{R}$.

A pair of adapted real-valued processes (y, σ) is called a solution of the above BSDE, if $E^{\mathbb{P}}[\sup_t |y_t|^2] < \infty$, $E^{\mathbb{P}}[\int_0^T |\sigma_t|^2 dt] < \infty$ and (y, σ) satisfies (15). Pardoux and Peng (1990) obtained the following existence and uniqueness of the solution of (15).

Lemma 1 *If $E^{\mathbb{P}}[\int_0^T |f(t, 0, 0)|^2 dt] < \infty$ and $f(t, \cdot, \cdot)$ is Lipschitz continuous on $\mathbb{R} \times \mathbb{R}$, then the above BSDE has a unique adapted solution (y, σ) .*

B Properties of Equation (5)

In view of (15), the BSDE in (5) considers the following generator

$$f(t, y_t, \sigma_t) = a_t \max_{\theta \in \Theta} \theta_t \sigma_t + (1 - a_t) \min_{\theta \in \Theta} \theta_t \sigma_t,$$

where Θ captures the multiple prior uncertainty \mathcal{P} .

Proposition 5 *Let $a \in \llbracket 0, 1 \rrbracket$ and \mathcal{P} be an arbitrary specification of an α -maxmin conditional expectation \mathcal{E}_t . For every $X \in L_T$, there exists a unique solution $(\mathcal{E}_t[X], \sigma_t)$ of equation (5). Moreover, the following properties hold true for every $s, t \in [0, T]$, $X, Y \in L_T$:*

- (i) (strict) Monotonicity: *If $X \geq Y$, then $\mathcal{E}_t[X] \geq \mathcal{E}_t[Y]$.
If also $\mathbb{P}(X > Y) > 0$, then $\mathcal{E}_0[X] > \mathcal{E}_0[Y]$.*
- (ii) Constant-preserving: *$\mathcal{E}_t[\eta] = \eta$, if $\eta \in L_t$ and $\mathcal{E}_t[c] = c$, for all $c \in \mathbb{R}$.*
- (iii) Tower property: *$\mathcal{E}_s[X] = \mathcal{E}_s[\mathcal{E}_t[X]]$, for all $s \leq t$.*
- (iv) Conditional linearity: *$\mathcal{E}_t[X + \eta] = \mathcal{E}_t[X] + \eta$, for every $\eta \in L_t$.*
- (v) Zero-one law: *For any $A \in \mathcal{F}_t$, we have $\mathcal{E}_t[X \mathbf{1}_A] = \mathcal{E}_t[X] \mathbf{1}_A$.*
- (vi) Positive homogeneity: *$\mathcal{E}_t[\eta X] = \eta \mathcal{E}_t[X]$, for all $\eta \geq 0$.*
- (vii) Jensen inequality: *If $u \in C^2(\mathbb{R})$ is increasing and concave, then*

$$\mathcal{E}_t[u(X)] \leq u(\mathcal{E}_t[X]).$$

By $\mathbf{1}_A$, for some $A \in \mathcal{F}$, we denote the usual indicator function, being 1 on A and 0 on $A^c = \Omega \setminus A$.

PROOF: We start with the uniqueness and existence of the solution of the BSDE. For all $x, y \in \mathbb{R}$, we have

$$\begin{aligned} & |\mathbf{e}(t, x) - \mathbf{e}(t, y)| \\ &= |a_t \max_{\theta \in \Theta} \theta_t x + (1 - a_t) \min_{\theta \in \Theta} \theta_t x - a_t \max_{\theta \in \Theta} \theta_t y - (1 - a_t) \min_{\theta \in \Theta} \theta_t y| \\ &\leq a_t |\max_{\theta \in \Theta} \theta_t x - \max_{\theta \in \Theta} \theta_t y| + (1 - a_t) |\min_{\theta \in \Theta} \theta_t x - \min_{\theta \in \Theta} \theta_t y| \\ &\leq \max_{\theta \in \Theta} |\theta_t(x - y)| + \max_{\theta \in \Theta} |\theta_t(x - y)|. \end{aligned}$$

Since Θ is compact, then there exists a positive constant C such that

$$|\mathbf{e}(t, x) - \mathbf{e}(t, y)| \leq C|x - y|.$$

Therefore, $\mathbf{e}(t, \cdot)$ is uniformly Lipschitz and $\mathbf{e}(t, 0) = 0$, then from Lemma 1 in Appendix A, equation (5) has a unique solution,

Properties (i) to (v) directly follow from Lemma 36.6 and Theorem 37.3 in Peng (1997).

To show (vi), note that for all $x \in \mathbb{R}, \beta > 0$, we have positive homogeneity of $\mathbf{e}(t, x)$

$$\begin{aligned}\mathbf{e}(t, \beta x) &= a_t \max_{\theta \in \Theta} \theta_t(\beta x) + (1 - a_t) \min_{\theta \in \Theta} \theta_t(\beta x) \\ &= \beta \mathbf{e}(t, x),\end{aligned}$$

Application of Lemma 36.9 (see also Example 10 therein) in Peng (1997) gives us (vi).

Since $u \in C^2(\mathbb{R})$ is increasing and concave, we can get (vii) from Theorem 1 in Jia and Peng (2010). \square

C Proofs

Proof of Theorem 1 *In order to prove that the family $(U_{i\Delta})_{i=0, \dots, N}$ is dynamically-consistent, we only need to show that for $i = 0, 1, \dots, N - 1$, and $X, Y \in L_T$, if $U_{(i+1)\Delta}^N[X] \geq U_{(i+1)\Delta}^N[Y]$ then $U_{i\Delta}^N[X] \geq U_{i\Delta}^N[Y]$. From the definition of $I_t[X], t \in [0, T]$, we know that $I_t[X]$ is increasing in X . Therefore,*

$$U_{i\Delta}^N[X] = I_{i\Delta} \left[U_{(i+1)\Delta}^N[X] \right] \geq I_{i\Delta} \left[U_{(i+1)\Delta}^N[Y] \right] = U_{i\Delta}^N[Y],$$

from which we complete the proof.

Proof of Theorem 2 *For $t \in [t_{N-1}^N, t_N^N] := [(N-1)\Delta, N\Delta)$, we have*

$$U_t^N[X] = I_t[U_{t_N^N}^N[X]] = I_t[X].$$

Let $(\bar{\mathcal{E}}_t[u(X)], \bar{\sigma}_t)$ and $(\underline{\mathcal{E}}_t[u(X)], \underline{\sigma}_t)$ be the solutions of the following BSDEs, respectively,

$$d\bar{\mathcal{E}}_t[u(X)] = \min_{\theta \in \Theta} \theta_t \bar{\sigma}_t dt + \bar{\sigma}_t dB_t, \quad \bar{\mathcal{E}}_T[u(X)] = u(X), \quad (16)$$

and

$$d\underline{\mathcal{E}}_t[u(X)] = \max_{\theta \in \Theta} \theta_t \underline{\sigma}_t dt + \underline{\sigma}_t dB_t, \quad \underline{\mathcal{E}}_T[u(X)] = u(X). \quad (17)$$

This implies

$$\underline{\mathcal{E}}_t[u(X)] = \min_{P \in \mathcal{P}} E_t^P[u(X)], \quad \text{and} \quad \bar{\mathcal{E}}_t[u(X)] = \max_{P \in \mathcal{P}} E_t^P[u(X)].$$

Let $(\mathcal{E}_t[u(X)], \sigma_t)$ be the solutions of the following BSDEs.

$$d\mathcal{E}_t[u(X)] = e(t, \sigma_t)dt + \sigma_t dB_t, \mathcal{E}_T[u(X)] = u(X). \quad (18)$$

Then, using the standard estimates of BSDEs (16) and (18), there exists a constant C (C is independent of Δ and can be different from line to line) such that

$$\begin{aligned} E^{\mathbb{P}}\left[\sup_{s \in [t, T]} |\underline{\mathcal{E}}_s[u(X)] - \mathcal{E}_t[u(X)]|^2\right] &\leq CE^{\mathbb{P}}\left[\int_t^T |e(r, \sigma_r) - \max_{\theta \in \Theta} \theta_r \sigma_r| dr\right]^2 \\ &\leq CE^{\mathbb{P}}\left[\int_t^T |\max_{\theta \in \Theta} \theta_r \sigma_r - \min_{\theta \in \Theta} \theta_r \sigma_r| dr\right]^2 \\ &\leq C(T-t)E^{\mathbb{P}}\left[\int_t^T |\sigma_r|^2 dr\right] \\ &\leq C\Delta E^{\mathbb{P}}\left[\int_0^T |\sigma_r|^2 dr\right] = C\Delta. \end{aligned}$$

In a similar way, we have the following estimate of BSDEs (17) and (18)

$$E^{\mathbb{P}}\left[\sup_{s \in [t, T]} |\bar{\mathcal{E}}_s[u(X)] - \mathcal{E}_t[u(X)]|^2\right] \leq C\Delta.$$

Therefore,

$$\begin{aligned} &E^{\mathbb{P}}[|U_t^N[X] - \mathcal{E}_t[u(X)]|^2] \\ &\leq 2E^{\mathbb{P}}[|\underline{\mathcal{E}}_t[u(X)] - \mathcal{E}_t[u(X)]|^2] + 2E^{\mathbb{P}}[|\bar{\mathcal{E}}_t[u(X)] - \mathcal{E}_t[u(X)]|^2] \\ &\leq C\Delta. \end{aligned} \quad (19)$$

For $t \in [t_{N-2}^N, t_{N-1}^N)$, we have

$$U_t^N[X] = I_t[U_{t_{N-1}^N}^N[X]].$$

Let $(\bar{\mathcal{E}}'_t[X], \bar{\sigma}'_t)$ and $(\underline{\mathcal{E}}'_t[X], \underline{\sigma}'_t)$ be the solutions of the following BSDEs, respectively,

$$d\bar{\mathcal{E}}'_t[X] = \min_{\theta \in \Theta} \theta_t \bar{\sigma}'_t dt + \sigma_t dB_t, \bar{\mathcal{E}}'_{t_{N-1}^N}[X] = U_{t_{N-1}^N}^N[X], \quad (20)$$

and

$$d\underline{\mathcal{E}}'_t[X] = \max_{\theta \in \Theta} \theta_t \underline{\sigma}'_t dt + \underline{\sigma}'_t dB_t, \underline{\mathcal{E}}'_{t_{N-1}^N}[X] = U_{t_{N-1}^N}^N[X]. \quad (21)$$

Then

$$\underline{\mathcal{E}}'_t[X] = \min_{P \in \mathcal{P}} E_t^P[U_{t_{N-1}^N}^N[X]], \quad \bar{\mathcal{E}}'_t[X] = \max_{P \in \mathcal{P}} E_t^P[U_{t_{N-1}^N}^N[X]].$$

Therefore, using the standard estimates of BSDEs (18) and (21), there exists a constant C (C is independent of Δ and can be different from line to line) such that

$$\begin{aligned}
& E^{\mathbb{P}} \left[\sup_{s \in [t, T]} |\underline{\mathcal{E}}'_s[X] - \mathcal{E}_s[u(X)]|^2 \right] \\
& \leq CE^{\mathbb{P}} \left[\int_t^{t_{N-1}^N} |e(r, \sigma_r) - \max_{\theta \in \Theta} \theta_r \sigma_r|^2 dr \right] + E^{\mathbb{P}} [|U_{t_{N-1}^N}^N[X] - \mathcal{E}_{t_{N-1}^N}[u(X)]|^2] \\
& \leq CE^{\mathbb{P}} \left[\int_t^{t_{N-1}^N} |\max_{\theta \in \Theta} \theta_r \sigma_r - \min_{\theta \in \Theta} \theta_r \sigma_r|^2 dr \right] + E^{\mathbb{P}} [|U_{t_{N-1}^N}^N[X] - \mathcal{E}_{t_{N-1}^N}[u(X)]|^2] \\
& \leq C(T-t)E^{\mathbb{P}} \left[\int_t^{t_{N-1}^N} |\sigma_r|^2 dr \right] + E^{\mathbb{P}} [|U_{t_{N-1}^N}^N[X] - \mathcal{E}_{t_{N-1}^N}[u(X)]|^2] \\
& \leq C\Delta E^{\mathbb{P}} \left[\int_0^T |\sigma_r|^2 dr \right] + E^{\mathbb{P}} [|U_{t_{N-1}^N}^N[X] - \mathcal{E}_{t_{N-1}^N}[u(X)]|^2].
\end{aligned}$$

From (19) it follows that

$$E^{\mathbb{P}} \left[\sup_{s \in [t, T]} |\underline{\mathcal{E}}'_s[X] - \mathcal{E}_s[u(X)]|^2 \right] \leq C\Delta.$$

In a similar way, we have the following estimate of BSDEs (5) and (20)

$$E^{\mathbb{P}} \left[\sup_{s \in [t, T]} |\overline{\mathcal{E}}'_s[X] - \mathcal{E}_s[u(X)]|^2 \right] \leq C\Delta.$$

Therefore,

$$\begin{aligned}
E^{\mathbb{P}} [|U_t^N[X] - \mathcal{E}_t[u(X)]|^2] & \leq 2E^{\mathbb{P}} [|\underline{\mathcal{E}}'_t[X] - \mathcal{E}_t[u(X)]|^2] + 2E^{\mathbb{P}} [|\overline{\mathcal{E}}'_t[X] - \mathcal{E}_t[u(X)]|^2] \\
& \leq C\Delta.
\end{aligned}$$

Using the above approach, we can prove that, for all $t \in [t_i^N, t_{i+1}^N)$, $i = 0, 1, \dots, N-2$,

$$E^{\mathbb{P}} [|U_t^N[X] - \mathcal{E}_t[u(X)]|^2] \leq C\Delta,$$

and the result follows by letting $N \rightarrow \infty$.

Proof of Proposition 1 (i) The continuity follows from the presence of a dominating sublinear expectation, which implies norm-continuity.

(ii) We just give the proof when $\{X_n\}_{n \geq 1}$ is decreasing. From the monotonicity of the nonlinear expectation \mathcal{E} , we know that $\{U(X_n)\}_{n \geq 1}$ is decreasing.

Since $\{X_n\}_{n \geq 1}$ is decreasing and $\lim_{n \rightarrow \infty} X_n = X$, \mathbb{P} -a.s., we get that $|u(X_n) - u(X)| \leq |u(X_n)| + |u(X)| \in L_T$, and $\lim_{n \rightarrow \infty} |u(X_n) - u(X)| = 0$, \mathbb{P} -a.s. Then by

virtue of the dominated convergence theorem we have, $\lim_{n \rightarrow \infty} E^{\mathbb{P}} |u(X_n) - u(X)|^2 = 0$. From (vii) in Proposition 5 we know that, there is a constant $C > 0$ such that,

$$|U(X_n) - U(X)|^2 \leq CE^{\mathbb{P}} [|u(X_n) - u(X)|^2],$$

from which we can get $\lim_{n \rightarrow \infty} U(X_n) = U(X)$.

(iii) Since $X \geq Y$, \mathbb{P} -a.s., and u is increasing, we have $u(X) \geq u(Y)$, \mathbb{P} -a.s. From (i) in Proposition 5 it follows that

$$U_t(X) = \mathcal{E}_t[u(X)] \geq \mathcal{E}_t[u(Y)] = U_t(Y).$$

Moreover, if $\mathbb{P}(X > Y) > 0$ and u is strictly increasing, then $\mathbb{P}(u(X) > u(Y)) > 0$. Using (i) in Proposition 5 again

$$U(X) = \mathcal{E}[u(X)] > \mathcal{E}[u(Y)] = U(Y).$$

(iv) By (i) and (iii) in Proposition 5, it is easily to get this.

Proof of Proposition 2 Since $e(t, \sigma)$ is a convex combination of an inf and sup operation, $e(t, \sigma)$ is positive homogeneous in σ . By an application of Theorem 3.2 in Jia and Peng (2010) to $e(t, \sigma)$, which is independent of $\mathcal{E}_t[X]$, the conditional \mathcal{E} -concavity, i.e., $u(\mathcal{E}_t[X]) \geq \mathcal{E}_t[u(X)]$, can be characterized as follow

$$\frac{1}{2}u''(x)|\sigma|^2 + e(t, u'(x)\sigma) - u'(x)e(t, \sigma) \leq 0$$

By the positive homogeneity of $e(t, \cdot)$ this is equivalent to $u''(x) \leq 0$, being equivalent to concavity.

Proof of Theorem 3 We consider the second-order Taylor expansion around w for $u(w)$

$$\begin{aligned} \mathcal{E}[u(X + w)] &\approx u(w) + \mathcal{E} \left[u'(w)X + \frac{1}{2}u''(w)X^2 \right] \\ &= u(w) + u'(w)\mathcal{E} \left[X - \frac{1}{2}A(w)X^2 \right] \\ &= u(w) + u'(w) \left(\frac{1}{2}A(w)\mathcal{E} [X^2] + \mathbf{co}(X, \frac{1}{2}A(w)X^2) \right), \end{aligned}$$

where we applied the concavity of u via $u' \geq 0, u'' \leq 0$ and Proposition 5 (iv) and (iv). Using the first-order Taylor expansion for $u(\mathcal{C}^{w+X})$ around w :

$$u(\mathcal{C}^{w+X}) \approx u(w) + u'(w)(\mathcal{C}^{w+X} - w).$$

Combining both approximations establishes the desired result.

Proof of Proposition 3 1. From Proposition 5 (vii) we know that

$$\mathcal{E}_t[u(X)] \leq u(\mathcal{E}_t[X]).$$

Since u is strictly increasing, we have $\mathcal{C}_t(X) = u^{-1}(\mathcal{E}_t[u(X)]) \leq \mathcal{E}_t[X]$.

2. Let $(\mathcal{E}_t[u(X)], \sigma_t^u)_{t \in [0, T]}$ be the unique solution of the following equation:

$$\begin{aligned} d\mathcal{E}_t[u(X)] &= \mathbf{e}(t, \sigma_t^u) dt + \sigma_t^u dB_t, \quad t \in [0, T], \\ \mathcal{E}_T[u(X)] &= u(X). \end{aligned}$$

Then from Itô Lemma with respect to $u^{-1}(\mathcal{E}_t[u(X)])$ it follows that

$$d\mathcal{C}_t(X) = \left(\frac{\mathbf{e}(t, \sigma_t^X)}{u'(\mathcal{C}_t(X))} - \frac{1}{2} \frac{u''(\mathcal{C}_t(X))}{u'(\mathcal{C}_t(X))^3} (\sigma_t^u)^2 \right) dt + \frac{\sigma_t^u}{u'(\mathcal{C}_t(X))} dB_t.$$

We denote $\sigma_t^{\mathcal{C}} = \frac{\sigma_t^u}{u'(\mathcal{C}_t(X))}$, then

$$\begin{aligned} d\mathcal{C}_t(X) &= \left(\frac{\sigma_t^{\mathcal{C}}}{\sigma_t^u} \mathbf{e}(t, \sigma_t^X) - \frac{1}{2} \frac{u''(\mathcal{C}_t(X))}{u'(\mathcal{C}_t(X))} (\sigma_t^{\mathcal{C}})^2 \right) dt + \sigma_t^{\mathcal{C}} dB_t \\ &= \frac{\sigma_t^{\mathcal{C}}}{\sigma_t^u} \mathbf{e}(t, \sigma_t^u) + \frac{1}{2} (\sigma_t^{\mathcal{C}})^2 A(\mathcal{C}_t^X) dt + \sigma_t^{\mathcal{C}} dB_t. \end{aligned}$$

Proof of Proposition 4 (i) By (vi) in Proposition 5 and the definition of the dynamic certainty equivalent, we have

$$\mathcal{C}_t(X) = u^{-1}(\mathcal{E}_{t,t}[u(X)]) = u^{-1}(u(X)) = X.$$

(ii) By (iv) in Proposition 5 and the definition of the dynamic certainty equivalent, we have

$$\begin{aligned} \mathcal{C}_{r,t}(X) &= u^{-1}(\mathcal{E}_{r,t}[u(X)]) \\ &= u^{-1}(\mathcal{E}_{r,s}[\mathcal{E}_{s,t}[u(X)]]) \\ &= u^{-1}(\mathcal{E}_{r,s}[u(\mathcal{C}_{s,t}(X))]) \\ &= u^{-1}(u(\mathcal{C}_{r,s}(\mathcal{C}_{s,t}(X)))) \\ &= \mathcal{C}_{r,s}(\mathcal{C}_{s,t}(X)). \end{aligned}$$

(iii) From (ii) it follows that

$$\mathcal{C}_{r,t}(X) = \mathcal{C}_{r,s}(\mathcal{C}_{s,t}(X)) \leq \mathcal{C}_{r,s}(\mathcal{C}_{s,t}(Y)) = \mathcal{C}_{r,t}(Y).$$

(iv) We take $v = t$ in (iii) and from (i) it follows that, if $X \leq Y$, then $\mathcal{C}_{r,t}(X) \leq \mathcal{C}_{r,t}(Y)$.

(v) Since

$$\begin{aligned} u(X\mathbf{1}_A + Y\mathbf{1}_{A^c}) &= u(X\mathbf{1}_A + Y\mathbf{1}_{A^c})\mathbf{1}_A + u(X\mathbf{1}_A + Y\mathbf{1}_{A^c})\mathbf{1}_{A^c} \\ &= u(X)\mathbf{1}_A + u(Y)\mathbf{1}_{A^c}, \end{aligned}$$

we have

$$\begin{aligned} \mathcal{C}_{s,t}(X\mathbf{1}_A + Y\mathbf{1}_{A^c}) &= u^{-1}(\mathcal{E}_{s,t}[u(X\mathbf{1}_A + Y\mathbf{1}_{A^c})]) \\ &= u^{-1}(\mathcal{E}_{s,t}[u(X)\mathbf{1}_A + u(Y)\mathbf{1}_{A^c}]). \end{aligned} \quad (22)$$

Let us consider the following two BSDEs

$$d\mathcal{E}_{s,t}[u(X)] = \hat{e}(\sigma_s^1)ds + \sigma_s^1dB_s, \quad \mathcal{E}_{t,t}[u(X)] = u(X), \quad (23)$$

and

$$d\mathcal{E}_{s,t}[u(Y)] = \hat{e}(\sigma_s^2)ds + \sigma_s^2dB_s, \quad \mathcal{E}_{t,t}[u(Y)] = u(Y). \quad (24)$$

Then (23) $\times \mathbf{1}_A + (24) \times \mathbf{1}_{A^c}$ yields

$$\begin{aligned} &d(\mathcal{E}_{s,t}[u(X)]\mathbf{1}_A + \mathcal{E}_{s,t}[u(Y)]\mathbf{1}_{A^c}) \\ &= [\hat{e}(\sigma_s^1)\mathbf{1}_A + \hat{e}(\sigma_s^2)\mathbf{1}_{A^c}]ds + (\sigma_s^1\mathbf{1}_A + \sigma_s^2\mathbf{1}_{A^c})dB_s \\ &= \hat{e}(\sigma_s^1\mathbf{1}_A + \sigma_s^2\mathbf{1}_{A^c})ds + (\sigma_s^1\mathbf{1}_A + \sigma_s^2\mathbf{1}_{A^c})dB_s, \end{aligned}$$

with the terminal condition

$$\mathcal{E}_{t,t}[u(X)]\mathbf{1}_A + \mathcal{E}_{t,t}[u(Y)]\mathbf{1}_{A^c} = u(X)\mathbf{1}_A + u(Y)\mathbf{1}_{A^c}.$$

Recall the following BSDE

$$\begin{aligned} d\mathcal{E}_{s,t}[u(X)\mathbf{1}_A + u(Y)\mathbf{1}_{A^c}] &= \hat{e}(\sigma_s)ds + \sigma_s dB_s, \\ \mathcal{E}_{t,t}[u(X)\mathbf{1}_A + u(Y)\mathbf{1}_{A^c}] &= u(X)\mathbf{1}_A + u(Y)\mathbf{1}_{A^c}. \end{aligned}$$

From the uniqueness of the solution of the above equations, we have

$$\mathcal{E}_{s,t}[u(X)]\mathbf{1}_A + \mathcal{E}_{s,t}[u(Y)]\mathbf{1}_{A^c} = \mathcal{E}_{s,t}[u(X)]\mathbf{1}_A + \mathcal{E}_{s,t}[u(Y)]\mathbf{1}_{A^c}.$$

Therefore, from (22) we have

$$\begin{aligned} \mathcal{C}_{s,t}(X\mathbf{1}_A + Y\mathbf{1}_{A^c}) &= u^{-1}(\mathcal{E}_{s,t}[u(X)]\mathbf{1}_A + \mathcal{E}_{s,t}[u(Y)]\mathbf{1}_{A^c}) \\ &= u^{-1}(\mathcal{E}_{s,t}[u(X)]\mathbf{1}_A + u^{-1}(\mathcal{E}_{s,t}[u(Y)]\mathbf{1}_{A^c}))\mathbf{1}_{A^c} \\ &= \mathcal{C}_{s,t}(X)\mathbf{1}_A + \mathcal{C}_{s,t}(Y)\mathbf{1}_{A^c}. \end{aligned}$$

Let $Y = 0$, then by $\mathcal{E}_{s,t}[0] = 0$ we have $\mathcal{C}_{s,t}(X\mathbf{1}_A) = \mathcal{C}_{s,t}(X)\mathbf{1}_A$.

(vi) From (ii) we have

$$\mathcal{C}_{r,t}(X) = \mathcal{C}_{r,s}(\mathcal{C}_{s,t}(X)) = u^{-1}(\mathcal{E}_{r,s}[u(\mathcal{C}_{s,t}(X))]).$$

Therefore, by Jensen inequality in Proposition 5 we get

$$\mathcal{E}_{r,s}[u(\mathcal{C}_{s,t}(X))] \leq u(\mathcal{E}_{r,s}[\mathcal{C}_{s,t}(X)]).$$

From the above inequalities it follows that

$$\mathcal{C}_{r,t}(X) \leq \mathcal{E}_{r,s}[\mathcal{C}_{s,t}(X)].$$

In particular, by using (i) we get $\mathcal{C}_{r,t}(X) \leq \mathcal{E}_{r,t}[X]$.

Proof of Theorem 4 A usual application of Itô's formula to $u'(e_t)$ and $\psi_t = u'(e_t) \cdot z_t^{\theta^*}$ (under \mathbb{P}) allows to formulate the dynamics of the state-price density as

$$\frac{d\psi_t}{z_t^{\theta^*}} = \left(u''(e_t)\mu_t^e + \sigma_t^e\theta_t^* + \frac{1}{2}\sigma_t^e\sigma_t^e u'''(e_t) \right) dt + u''(e_t)\sigma_t^e - u'(e_t)\theta_t^* dB_t. \quad (25)$$

The equilibrium–short rate must satisfy $r_t = -\frac{\mu_t^\psi}{\psi_t}$, where μ_t^ψ denotes the drift part in (25). By virtue of (25), the excess return turns out to be

$$\begin{aligned} \mu_t^R - r_t &= -\frac{\sigma_t^\psi}{\psi_t} \cdot \sigma_t^R \\ &= A(e_t) \cdot \sigma_t^R \sigma_t^e + (2a_t - 1)\theta_t \cdot \text{sgn}(\sigma_t^U) \cdot \sigma_t^R, \end{aligned}$$

where $\sigma_t^\psi = z_t^{\theta^*} \cdot (u''(e_t)\sigma_t^e - u'(e_t)\theta_t^*)$ is the volatility part in (25). As in Theorem 3, $A(e_t)$ denotes the Arrow–Pratt measure of absolute risk aversion at e_t .

References

- BEISSNER, P. (2015): “Brownian equilibria under Knightian uncertainty,” *Mathematics and Financial Economics*, 9(1), 39–56.
- CHATEAUNEUF, A., J. EICHBERGER, AND S. GRANT (2007): “Choice under uncertainty with the best and worst in mind: Neo-additive capacities,” *Journal of Economic Theory*, 137(1), 538–567.
- CHEN, Z., AND L. EPSTEIN (2002): “Ambiguity, risk, and asset returns in continuous time,” *Econometrica*, 70(4), 1403–1443.
- DUFFIE, D. (1996): “Dynamic asset pricing theory,” .
- EPSTEIN, L. G., AND M. SCHNEIDER (2003): “Recursive multiple-priors,” *Journal of Economic Theory*, 113(1), 1–31.
- GHIRARDATO, P., F. MACCHERONI, AND M. MARINACCI (2004): “Differentiating ambiguity and ambiguity attitude,” *Journal of Economic Theory*, 118(2), 133–173.

- GILBOA, I., AND D. SCHMEIDLER (1989): “Maxmin expected utility with non-unique prior,” *Journal of Mathematical Economics*, 18(2), 141–153.
- HODGES, S. D., AND A. NEUBERGER (1989): “Optimal replication of contingent claims under transaction costs,” *Review of futures markets*, 8(2), 222–239.
- JIA, G., AND S. PENG (2010): “Jensen’s inequality for g -convex function under g -expectation,” *Probability Theory and related Fields*, 147(1-2), 217–239.
- MACCHERONI, F., M. MARINACCI, AND D. RUFFINO (2013): “Alpha as ambiguity: robust mean-variance portfolio analysis,” *Econometrica*, 81(3), 1075–1113.
- PARDOUX, E., AND S. PENG (1990): “Adapted solution of a backward stochastic differential equation,” *Systems and Control Letters*, 14(1), 55–61.
- PENG, S. (1997): “Backward SDE and related g -expectation,” *Pitman research notes in mathematics series*, pp. 141–160.