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Quadratic forms and Sobolev spaces of fractional order

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Abstract

We study quadratic functionals on $L^2(\mathbb{R}^d)$ that generate seminorms in the fractional Sobolev space $H^s(\mathbb{R}^d)$ for $0 < s < 1$. The functionals under consideration appear in the study of Markov jump processes and, independently, in recent research on the Boltzmann equation. The functional measures differentiability of a function f in a similar way as the seminorm of $H^s(\mathbb{R}^d)$. The major difference is that differences $f(y) - f(x)$ are taken into account only if y lies in some double cone with apex at x or vice versa. The configuration of double cones is allowed to be inhomogeneous without any assumption on the spatial regularity. We prove that the resulting seminorm is comparable to the standard one of $H^s(\mathbb{R}^d)$. The proof follows from a similar result on discrete quadratic forms in $\mathbb{Z}^d$, which is our second main result. We establish a general scheme for discrete approximations of nonlocal quadratic forms. Applications to Markov jump processes are discussed.

1. *Introduction*

The Sobolev–Slobodeckiĭ space $H^s(\mathbb{R}^d)$, $0 < s < 1$, can be defined as the set of all functions $f \in L^2(\mathbb{R}^d)$ such that the seminorm

$$\int\limits_{\mathbb{R}^d \times \mathbb{R}^d} (f(y) - f(x))^2 |x - y|^{-d-2s} \, \mathrm{d}(x, y) \tag{1}$$

is finite, see the original work [14] or the monographs [1, 2, 12]. The normed space is complete and, together with its modifications for domains in $\mathbb{R}^d$, is of fundamental importance in the field of Partial Differential Equations. In this article, we adopt the common notation from Stochastic Analysis, where $H^s(\mathbb{R}^d)$ usually is denoted by $H^{\alpha/2}(\mathbb{R}^d)$ with $\alpha = 2s \in (0, 2)$. The corresponding stochastic process is called α-stable process, which explains the usage of α here.

We study seminorms on $L^2(\mathbb{R}^d)$, which are very similar but smaller than (1) because we consider differences $f(y) - f(x)$ only if y lies in some double cone with apex at x. Below, we explain where and why the corresponding quadratic forms appear naturally. In order to formulate our main result, let us fix some notation. By V we denote a double cone in $\mathbb{R}^d$ with apex at $0 \in \mathbb{R}^d$, symmetry axis $v \in \mathbb{R}^d$, and apex angle $\vartheta \in (0, \frac{\pi}{2}]$. Let $\mathcal{V} = (0, \frac{\pi}{2}] \times \mathbb{P}^{d-1}_{\mathbb{R}}$ denote the family of all such double cones. For $x \in \mathbb{R}^d$, we define a shifted double cone by $V[x] = V + x$. A mapping $\Gamma : \mathbb{R}^d \to \mathcal{V}$ is called a configuration. If Γ is a configuration with the property that the infimum ϑ over all apex angles of cones in $\Gamma(\mathbb{R}^d)$ is positive, then Γ is called ϑ-bounded. If in addition

$$\{\, (x, y) \in \mathbb{R}^d \times \mathbb{R}^d \mid y - x \in \Gamma(x) \,\} \text{ is a Borel set in } \mathbb{R}^d \times \mathbb{R}^d, \tag{M}$$

then Γ is called ϑ-admissible. For $x \in \mathbb{R}^d$ and Γ a configuration, we define $V^\Gamma[x] = x + \Gamma(x)$.

One of our main results is the following theorem.

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THEOREM 1.1. *Let Γ be a ϑ-admissible configuration and $\alpha \in (0,2)$. Let $k : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty]$ be a measurable function satisfying $k(x,y) = k(y,x)$ and*

$$\Lambda^{-1}\big(\mathbb{1}_{V^\Gamma[x]}(y) + \mathbb{1}_{V^\Gamma[y]}(x)\big)|x-y|^{-d-\alpha} \leqslant k(x,y) \leqslant \Lambda|x-y|^{-d-\alpha}, \tag{2}$$

for almost all x and y, where $\Lambda \geqslant 1$ is some constant. Then, there is a constant $c \geqslant 1$ such that for every ball $B \subset \mathbb{R}^d$ and for every $f \in L^2(B)$, the inequality

$$\int\limits_{B\times B} (f(x)-f(y))^2|x-y|^{-d-\alpha}\,\mathrm{d}(x,y) \leqslant c \int\limits_{B\times B} (f(x)-f(y))^2 k(x,y)\,\mathrm{d}(x,y) \tag{3}$$

holds.

The constant c depends on Λ, the dimension d, and ϑ. It is independent of k and Γ. For $0 < \alpha_0 \leqslant \alpha < 2$, the constant c depends on α_0 but not on α.

Note that the reverse inequality in (3) trivially holds true. Moreover, note that $f \in L^2(B)$ does, in general, not imply that any of the two terms in (3) is finite. The result, in particular, says that the term on the right-hand side is infinite if the term on the left-hand side is infinite.

REMARK 1.2. One strength of the theorem is that there are only two essential assumptions, namely, that the infimum of the apex angles of double cones is required to be positive and that the set

$$\{\,(x,y) \in \mathbb{R}^d \times \mathbb{R}^d \mid y - x \in \Gamma(x)\,\}$$

is a Borel set in $\mathbb{R}^d \times \mathbb{R}^d$. Other than that, the symmetry axis of the double cone and the apex angle might depend on the center in an arbitrary way. Note that the last condition (M) is nothing else but the measurability of the function $v : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}, v(x,y) = \mathbb{1}_{V^\Gamma[x]}(y)$, which is important in light of (2).

A similar result like Theorem 1.1 has recently been provided in [**9**, Lemma A.6]. One difference between the two results is that Theorem 1.1 provides comparability on every ball. This property is important for applications, for example, for regularity results, cf. [**7**, Condition (A)], and when studying function spaces over bounded sets, cf. Theorem 1.4. Another difference concerns the class of cones considered. In our setup, it is generally not true that two double cones $x + \Gamma(x)$ and $y + \Gamma(y)$ have a nonempty intersection. This is different in the framework of [**9**], cf. Lemma A.5 therein. On the other hand, we consider classical double cones and not more general union of rays.

The proof of our main result is based on discrete approximations of the quadratic forms involved. We establish a general scheme of how to approximate a given nonlocal quadratic form on $L^2(\mathbb{R}^d)$ through a sequence of discrete quadratic forms. We provide a discrete analog of Theorem 1.1 that implies Theorem 1.1. We hope the discrete result itself to be useful in different fields, for example, when studying random walks in inhomogeneous or random media. Let us formulate our main result in the discrete setup.

THEOREM 1.3. *Let Γ be a ϑ-bounded configuration and $\alpha \in (0,2)$. Let $\omega : \mathbb{Z}^d \times \mathbb{Z}^d \to [0,\infty]$ be a function satisfying $\omega(x,y) = \omega(y,x)$ and*

$$\Lambda^{-1}\big(\mathbb{1}_{V^\Gamma[x]}(y) + \mathbb{1}_{V^\Gamma[y]}(x)\big)|x-y|^{-d-\alpha} \leqslant \omega(x,y) \leqslant \Lambda|x-y|^{-d-\alpha} \tag{4}$$

for $|x-y| > R_0$, where $R_0 > 0$, $\Lambda \geqslant 1$ are some constants. There exist constants $\kappa \geqslant 1, c \geqslant 1$ such that for every $R > 0$, $x_0 \in \mathbb{R}^d$, and every function $f : (B_{\kappa R}(x_0) \cap \mathbb{Z}^d) \to \mathbb{R}$, the inequality

$$\sum_{\substack{x,y\in B_R(x_0)\cap\mathbb{Z}^d\\ |x-y|>R_0}} (f(x)-f(y))^2|x-y|^{-d-\alpha} \leqslant c \sum_{\substack{x,y\in B_{\kappa R}(x_0)\cap\mathbb{Z}^d\\ |x-y|>R_0}} (f(x)-f(y))^2\omega(x,y)$$

holds.

The constant c depends on Λ, ϑ, R_0, and on the dimension d. It does not depend on ω and Γ.

Let us present the motivation for Theorem 1.1 and provide some applications.

One motivation for our research stems from new contributions to the study of the Boltzmann equation, where recent regularity results for integro-differential operators are applied in combination with results on kinetic equations, see [**10**]. This approach allows to work without cutoff conditions on the collision kernel. The Boltzmann equation describes the evolution of the density f depending on time, space, and velocity in an ideal gas. The equation takes the form

$$\partial_t f + (v, \nabla_x f) = Q(f, f) \qquad (t \in \mathbb{R}, x \in \mathbb{R}^d, v \in \mathbb{R}^d)\,,$$

where $Q(f, f)$ is the so-called Boltzmann collision operator whose precise definition depends on assumptions on how the particles interact in the gas under consideration, in particular on the so-called cross section, cf. [**13**, **17**]. For some cases, the operator $(f, g) \mapsto Q(f, g)$ can be decomposed as $Q(f, g) = Q_1(f, g) + Q_2(f, g)$, where for given f the map $g \mapsto Q_1(f, g)$ corresponds to an operator $g \mapsto L_v g$ that satisfies

$$-\int L_v g(v) g(v) \mathrm{d}v \geqslant \lambda \|g\|^2_{\dot{H}^s} - \Lambda \|g\|^2_{L^2} \tag{5}$$

for every $g \in C^\infty(B_1)$. The constants λ and Λ depend on the dimension d and on bounds of physical quantities related to f such as mass, energy, and entropy. The exponent $s \in (0, 1)$ correlates to assumptions on the cross section that appears in the definition of the collision kernel. The operator $g \mapsto Q_2(f, g)$ turns out to be of lower order. In the aforementioned cases, the proof of (5) is well understood, see [**3**, **16**]. A direct proof making use of a geometric understanding of the collision mechanism is provided in [**9**, Proposition A.1]. A detailed analysis leads to an estimate of the form (3), cf. [**9**, Lemma A.6], where x, y are replaced by velocity variables v, v' and the kernel k is derived in a highly nontrivial way from the collision kernel. As explained above, the assumptions on the cones appearing in the kernel in this work and in [**9**] are rather different.

The first application concerns function spaces. For a domain $\Omega \subset \mathbb{R}^d$, we consider the Hilbert space

$$H_k(\Omega) \;=\; \big\{\, f \in L^2(\Omega) \;\big|\; |f|_{H_k(\Omega)} < \infty \,\big\},$$

where the seminorm $|f|_{H_k(\Omega)}$ is given by

$$|f|^2_{H_k(\Omega)} \;=\; \int\limits_{\Omega \times \Omega} (f(y) - f(x))^2 k(x, y)\, \mathrm{d}(x, y).$$

We endow $H_k(\Omega)$ with the norm $\|f\|_{H_k(\Omega)}$, $\|f\|^2_{H_k(\Omega)} = \|f\|^2_{L^2(\Omega)} + |f|^2_{H_k(\Omega)}$. The space $H^{\frac{\alpha}{2}}(\Omega)$ is defined as the Hilbert space of all $f \in L^2(\Omega)$ such that the seminorm $|f|_{H^{\frac{\alpha}{2}}(\Omega)}$ is finite. We denote the norm on $H^{\frac{\alpha}{2}}(\Omega)$ by $\|f\|_{H^{\frac{\alpha}{2}}(\Omega)}$. Note that $|\cdot|_{H^{\frac{\alpha}{2}}(\Omega)}$ dominates $|\cdot|_{H_k(\Omega)}$ because of (2). Hence, $\|\cdot\|_{H^{\frac{\alpha}{2}}(\Omega)}$ dominates $\|\cdot\|_{H_k(\Omega)}$ and we can deduce the following inclusion:

$$H^{\frac{\alpha}{2}}(\Omega) \subset H_k(\Omega)\,. \tag{6}$$

As we will show, Theorem 1.3 implies the reverse implication if Ω is a bounded Lipschitz domain. We will prove the following result in Section 3.

THEOREM 1.4. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then, $H_k(\Omega) = H^{\frac{\alpha}{2}}(\Omega)$. The seminorms $|\cdot|_{H_k(\Omega)}$ and $|\cdot|_{H^{\frac{\alpha}{2}}(\Omega)}$ as well as the corresponding norms are comparable on $H_k(\Omega)$. Moreover, the subspace $C^\infty(\overline{\Omega})$ is dense in $H_k(\Omega)$.*

In addition, $H_k(\mathbb{R}^d) = H^{\frac{\alpha}{2}}(\mathbb{R}^d)$. The seminorms $|\cdot|_{H_k(\mathbb{R}^d)}$ and $|\cdot|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)}$ as well as the corresponding norms are comparable on $H_k(\mathbb{R}^d)$. The subspace $C_c^\infty(\mathbb{R}^d)$ of smooth functions with compact support in $\mathbb{R}^d$ is dense in $H_k(\mathbb{R}^d)$.

As mentioned above, Theorem 1.1 has direct significance for the theory of Markov jump processes. Let us recall that a bilinear symmetric closed form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(\mathbb{R}^d)$ is called Dirichlet form if it is Markovian, for example, if for every $u \in \mathcal{D}(\mathcal{E})$, the function $v = (u \wedge 1) \vee 0$ belongs to $\mathcal{D}(\mathcal{E})$ and satisfies $\mathcal{E}(v, v) \leqslant \mathcal{E}(u, u)$. See [8, Section 1.1] for this definition plus comments and examples. A Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(\mathbb{R}^d)$ is called regular if $C_c(\mathbb{R}^d) \cap \mathcal{D}(\mathcal{E})$ is dense in $C_c(\mathbb{R}^d)$ with respect to the supremum norm as well as in $\mathcal{D}(\mathcal{E})$ with respect to the norm $(\mathcal{E}(u, u) + (u, u))^{\frac{1}{2}}$. A major result is that every regular Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(\mathbb{R}^d)$ corresponds to a symmetric strong Markov process on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, whose Dirichlet form is given by $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, cf. [8, Theorem 7.2.1]. Note that the rotationally symmetric α-stable process is the strong Markov process that corresponds to the regular Dirichlet form $(\mathcal{E}^\alpha, H^{\alpha/2}(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d)$, where

$$\mathcal{E}^\alpha(f, g) \;=\; \int\limits_{\mathbb{R}^d \times \mathbb{R}^d} (f(y) - f(x))(g(y) - g(x))\, |x - y|^{-d-\alpha}\, \mathrm{d}(x, y)\,.$$

Theorem 1.1 immediately implies the following result.

COROLLARY 1.5. *The Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathbb{R}^d)$ with $\mathcal{F} = H^{\alpha/2}(\mathbb{R}^d)$ and*

$$\mathcal{E}(f, g) \;=\; \int\limits_{\mathbb{R}^d \times \mathbb{R}^d} (f(y) - f(x))(g(y) - g(x))k(x, y)\, \mathrm{d}(x, y)\,,$$

is a regular Dirichlet form on $L^2(\mathbb{R}^d)$. There exists a corresponding strong Markov process.

The corresponding stochastic process is an interesting subject for further research. Presumably, it shares several properties with the related rotationally symmetric α-stable process. Establishing sharp pointwise heat kernel estimates and, if applicable, the Feller property constitute interesting but challenging tasks.

Another application concerns regularity of solutions to integro-differential equations. We can apply recent results of [7] and establish a weak Harnack inequality and Hölder *a priori* estimates to corresponding weak solutions.

COROLLARY 1.6. *Assume $\alpha \in (0, 2)$, k is as in Theorem 1.1, $\Omega \subset \mathbb{R}^d$ is open, and $f \in L^{q/\alpha}(\Omega)$ for $q > d$. Then, every weak solution $u : \mathbb{R}^d \to \mathbb{R}$ to*

$$\lim_{\varepsilon \to 0+} \int\limits_{\mathbb{R}^d \setminus B_\varepsilon(x)} (u(y) - u(x))k(x, y)\mathrm{d}y \;=\; f(x) \qquad (x \in \Omega)\,,$$

satisfies a weak Harnack inequality and is Hölder regular in the interior of Ω.

The proof uses the regularity result of [7]. It relies on Theorem 1.1, which ensures that [7, Condition (A)] is satisfied. Condition (B) is easily verified for the classical choice of Lipschitz continuous cutoff functions.

This article is organized as follows. In Section 2, we provide the technical definitions and explain the set-up in detail. In Section 3, we explain how Theorem 1.1 is derived from Theorem 1.3. To this end, we formulate a rescaled version of Theorem 1.3 on $h\mathbb{Z}^d$ for $h > 0$, Corollary 3.1, and consider the limit procedure $h \searrow 0$. We also provide the proof of Theorem 1.4. In Section 5, we provide the main tool for the proof of Theorem 1.3, which is Theorem 5.15. Since the main ideas can be better communicated when working in the Euclidean space rather than the integral lattice, we present this case separately in Section 4. Section 6 finally contains the proof of Theorem 1.3.

2. *Set-up, definitions, and preliminaries*

The aim of this section to provide the framework of Theorem 1.1 and auxiliary results needed to deduce Theorem 1.1 from Theorem 1.3.

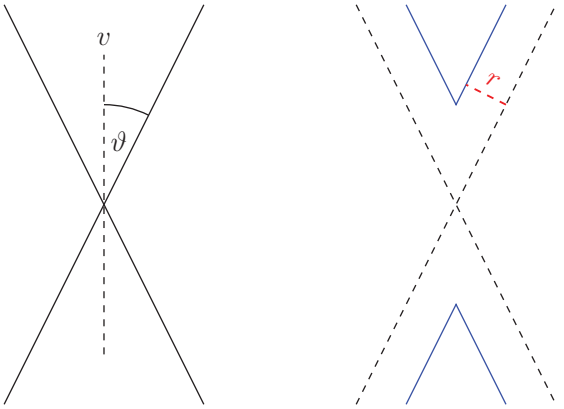

FIGURE 1 (colour online). *Example of a cone $V(v, \vartheta)$ and a double half-cone $V_r(v, \vartheta)$ for $d = 2$.*

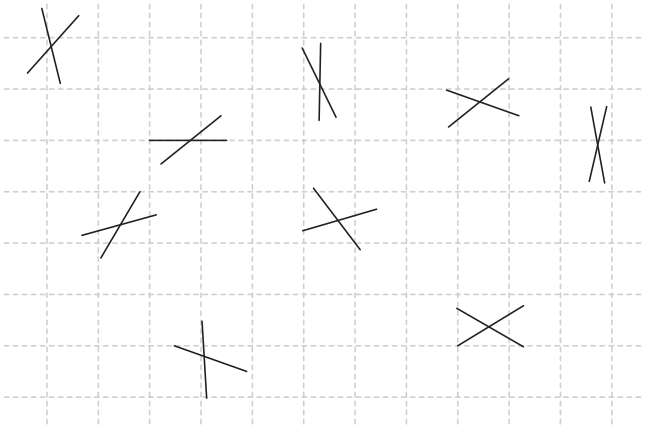

FIGURE 2. *The family $V^\Gamma[x]$ for a possible configuration Γ.*

DEFINITION 2.1. Given $v \in \mathcal{S}^{d-1}$ and $\vartheta \in (0, \frac{\pi}{2}]$, we define a *cone* by

$$\widetilde{V} \;=\; \widetilde{V}(v, \vartheta) \;=\; \left\{ h \in \mathbb{R}^d \;\middle|\; h \neq 0,\ \frac{\langle v, h\rangle}{|h|} > \cos(\vartheta) \right\}.$$

Let $\widetilde{\mathcal{V}}$ denote the family of all cones. We denote the corresponding *double cone* by V, that is,

$$V \;=\; V(v,\vartheta) \;=\; \widetilde{V} \cup (-\widetilde{V})\,.$$

The set $\mathcal{V}$ of all double cones is simply the manifold $(0,\frac{\pi}{2}] \times \mathbb{P}^{d-1}_{\mathbb{R}}$, where $\mathbb{P}^{d-1}_{\mathbb{R}}$ is the real projective space of dimension $d-1$. For $x \in \mathbb{R}^d$, we define a *shifted cone* by $\widetilde{V}[x] = \widetilde{V} + x$ and a *shifted double cone* by $V[x] = V + x$. For $r > 0$ and a given cone $\widetilde{V} = \widetilde{V}(v,\vartheta)$, we define

$$\widetilde{V}_r = \widetilde{V}_r(v,\vartheta) = \{y \in \widetilde{V} \mid \overline{B_r(y)} \subset \widetilde{V}\}.$$

For a double cone V, we define the set V_r analogously and call it a *double half-cone*. A mapping $\Gamma : \mathbb{R}^d \to \mathcal{V}$ is called *configuration*. If Γ is a configuration with the property that the infimum ϑ over all apex angles of cones in $\Gamma(\mathbb{R}^d)$ is positive, then Γ is called *ϑ-bounded*. If Γ is a ϑ-bounded configuration and

$$\big\{\,(x,y) \in \mathbb{R}^d \times \mathbb{R}^d \;\big|\; y - x \in \Gamma(x)\,\big\} \text{ is a Borel set in } \mathbb{R}^d \times \mathbb{R}^d, \tag{M}$$

then Γ is called *ϑ-admissible*, cf. Remark 1.2. For $x \in \mathbb{R}^d$ and Γ a configuration, we define $V^\Gamma[x] = x + \Gamma(x)$ and analogously for $r > 0$

$$V^\Gamma_r[x] \;=\; \Big\{\, y \in V^\Gamma[x] \;\Big|\; \overline{B_r(y)} \subset V^\Gamma[x] \,\Big\},$$

see Figure 1 and Figure 2.

One key observation of our approach is that the large, possibly uncountable, family of cones generated by a ϑ-admissible configuration Γ can be reduced to a finite family of cones.

LEMMA 2.2. *Let Γ be a ϑ-bounded configuration. There are numbers $L \in \mathbb{N}$ and $\theta \in (0,\frac{\pi}{2}]$, and double cones $V^1,\dots,V^L$ centered at 0 with apex angle θ and symmetry axis $v^1,\dots,v^L \in \mathcal{S}^{d-1}$ such that*

$$\forall x \in \mathbb{R}^d \;\; \exists m \in \{1,\dots,L\} \,:\, V^m \subset \Gamma(x)\,.$$

The constants L and θ depend on the dimension d and ϑ but not on Γ itself.

Proof. Obviously,

$$\mathcal{S}^{d-1} \;\subset \bigcup_{v \in \mathcal{S}^{d-1}} V\Big(v, \frac{\vartheta}{3}\Big),$$

where $\mathcal{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$. Since $\mathcal{S}^{d-1}$ is compact and the right-hand side is an open cover of $\mathcal{S}^{d-1}$, one can choose finite many $v^1,\dots,v^L \in \mathcal{S}^{d-1}$ such that

$$\mathcal{S}^{d-1} \;\subset \bigcup_{m=1}^{L} V\Big(v^m, \frac{\vartheta}{3}\Big).$$

Define $V^m = V(v^m, \frac{\vartheta}{3})$ for $m = 1,\dots,L$. Now the claim follows with $\theta = \vartheta/3$. □

DEFINITION 2.3. In the sequel, we write $V^m[x]$ instead of $V^m + x$. We call the set $\{V^m | 1 \leqslant i \leqslant L\}$ the *family of reference cones* associated to Γ. Each element is called a *reference cone*. Analogous to Definition 2.1, we set

$$V^m_r \;=\; \big\{\, u \in V^m \;\big|\; \overline{B_r} \subset V^m \,\big\}, \quad V^m_r[x] \;=\; V^m_r + x\,.$$

With help of Lemma 2.2, we can define a new configuration that has useful properties. The following corollary is the key tool for our reasoning in Section 4 and Section 5.

COROLLARY 2.4. *Let Γ be a ϑ-bounded configuration. Then, there exists another configuration $\widetilde{\Gamma}$ that fulfills $\#\widetilde{\Gamma}(\mathbb{R}^d) < \infty$ and for every $x \in \mathbb{R}^d$*

$$\widetilde{\Gamma}(x) \subset \Gamma(x).$$

The minimum of apex angles of cones in $\widetilde{\Gamma}(\mathbb{R}^d)$ is ϑ.

Proof. Let $V^1, \dots, V^L$ be the double cones from the preceding lemma. Define sets

$$\begin{aligned} M_1 &= \{ x \in \mathbb{R}^d \mid V^1 \subset \Gamma(x) \} \\ M_2 &= \{ x \in \mathbb{R}^d \mid V^2 \subset \Gamma(x) \} \setminus M_1 \\ &\vdots \\ M_L &= \{ x \in \mathbb{R}^d \mid V^L \subset \Gamma(x) \} \setminus M_{L-1}. \end{aligned}$$

Now,

$$\mathbb{R}^d = \bigcup_{1 \leqslant i \leqslant L} M_i$$

and this union is disjunct. Define $\widetilde{\Gamma} : \mathbb{R}^d \to \mathcal{V}, x \mapsto V^i$ for $x \in M_i$ and arrive at the assertion. □

DEFINITION 2.5. For $h > 0$ and $u = (u_1, \dots, u_d) \in \mathbb{R}^d$, let

$$A_h(u) = \left\{ x \in \mathbb{R}^d \;\middle|\; \|x - u\|_\infty < h/2 \right\}$$

be the open cube with center u. The half-closed cube with center u will be denoted by

$$\widetilde{A}_h(u) = \prod_{i=1}^{d} \Big[u_i - \frac{h}{2}, u_i + \frac{h}{2} \Big).$$

REMARK 2.6. Half-closed cubes are only needed in one proof in Section 3.

Let Γ be a ϑ-admissible configuration and $\{V^1, \dots, V^L\}$ a family of reference cones according to Lemma 2.2. Then, our assumption (M) implies for any $V \in \mathcal{V}$ that the set

$$\{ x \in \mathbb{R}^d \mid V \subset \Gamma(x) \}$$

is Lebesgue measurable. This implication is due to [4, Theorem 4.4].

Given $h > 0, u \in \mathbb{R}^d$ and $m \in \{1, \dots, L\}$, we set

$$A_h^m(u) = \{ x \in A_h(u) \mid V^m \subset \Gamma(x) \}.$$

An index $m \in \{1, \dots, L\}$ is called *h-favored by majority at u* (*or short: h-favored index at u*) if

$$\lambda_d(A_h^m(u)) = \max_{i \in \{1,\dots,L\}} \lambda_d(A_h^i(u)).$$

Here, λ_d is the Lebesgue measure on $\mathbb{R}^d$. Note that $\lambda_d(A_h^m(u)) \geqslant L^{-1} \lambda_d(A_h(u))$ for every h-favored index at u. This follows directly from

$$A_h(u) = \bigcup_{i \in \{1,\dots,L\}} A_h^i(u).$$

It is clear that the choice of an h-favored index is in general not unique.

Now we state an elementary result for the intersection of cones which will be very helpful for us.

LEMMA 2.7. *Let $\widetilde{V}$ be a cone with apex angle ϑ and let $h > 0$. Then, for each $x \in \mathbb{R}^d$ and each $\xi \in A_h(x)$*

$$\widetilde{V}_{h\sqrt{d}}[\xi] \;\subset\; \widetilde{V}_{\frac{h}{2}\sqrt{d}}[x] \;\subset\; \widetilde{V}[\xi]\,.$$

In other words,

$$\bigcup_{\xi\in A_h(x)} \widetilde{V}_{h\sqrt{d}}[\xi] \;\subset\; \widetilde{V}_{\frac{h}{2}\sqrt{d}}[x] \;\subset\; \bigcap_{\xi\in A_h(x)} \widetilde{V}[\xi]\,.$$

Proof. Let $\ell > 0$. Note

$$\begin{aligned}
\zeta \in \bigcap_{\xi\in\overline{B_\ell}} \widetilde{V}[\xi] \;&\Leftrightarrow\; \forall \xi \in \overline{B_\ell} : \zeta - \xi \in \widetilde{V} \\
&\Leftrightarrow\; \zeta - \overline{B_\ell} \subset \widetilde{V} \\
&\Leftrightarrow\; \overline{B_\ell(\zeta)} \subset \widetilde{V} \\
&\Leftrightarrow\; \zeta \in \widetilde{V}_\ell.
\end{aligned}$$

This means

$$\widetilde{V}_\ell \;=\; \bigcap_{\xi\in\overline{B_\ell}} \widetilde{V}[\xi]\,. \tag{7}$$

On the other hand, for $\zeta \in \widetilde{V}_{2\ell}$, we have $\overline{B_\ell(\zeta)} \subset \widetilde{V}_\ell$. This is equivalent to

$$\forall \zeta \in \widetilde{V}_{2\ell}\;\; \forall \xi \in \overline{B_\ell} \,:\, \zeta + \xi \in \widetilde{V}_\ell\,.$$

In other words,

$$\bigcup_{\xi\in\overline{B_\ell}} \widetilde{V}_{2\ell}[\xi] \;\subset\; \widetilde{V}_\ell\,. \tag{8}$$

From (7) and (8), we conclude for every $\xi \in \overline{B_\ell}$

$$\widetilde{V}_{2\ell}[\xi] \;\subset\; \widetilde{V}_\ell \;\subset\; \widetilde{V}[\xi]\,.$$

Translation by $x \in \mathbb{R}^d$ yields

$$\widetilde{V}_{2\ell}[\xi] \;\subset\; \widetilde{V}_\ell[x] \;\subset\; \widetilde{V}[\xi] \qquad \forall \xi \in \overline{B_\ell(x)}\,.$$

Now set $\ell = \frac{h}{2}\sqrt{d}$ and observe that $A_h(x) \subset \overline{B_\ell(x)}$. □

3. *Application of the discrete problem*

In this section, we show how to derive Theorem 1.1 from Theorem 1.3. The idea is to provide an $h\mathbb{Z}^d$-Version of Theorem 1.3, applying it to a discrete version of the kernel k from Theorem 1.1, and then pass to the limit $h \to 0$. We also prove Theorem 1.4.

By scaling, we can deduce the following $h\mathbb{Z}^d$-Version from Theorem 1.3.

COROLLARY 3.1. *Let Γ be a ϑ-bounded configuration and let $h > 0$. Let $\omega : h\mathbb{Z}^d \times h\mathbb{Z}^d \to [0,\infty]$ be a function satisfying $\omega(x,y) = \omega(y,x)$ and*

$$\Lambda^{-1}\big(\mathbb{1}_{V^\Gamma[x]}(y) + \mathbb{1}_{V^\Gamma[y]}(x)\big)|x-y|^{-d-\alpha} \;\leqslant\; \omega(x,y) \;\leqslant\; \Lambda|x-y|^{-d-\alpha} \tag{9}$$

for $|x-y| > R_0 h$, where $R_0 > 0$, $\Lambda \geqslant 1$ are some constants. There exist constants $\kappa \geqslant 1$ and $c > 0$, such that for every $R > 0$, every $x_0 \in \mathbb{R}^d$, and every function $f : (B_{\kappa R} \cap h\mathbb{Z}^d) \to \mathbb{R}$, the inequality

$$c \sum_{\substack{x,y \in B_R \cap h\mathbb{Z}^d \\ |x-y| > R_0 h}} (f(x) - f(y))^2 |x-y|^{-d-\alpha} \leqslant \sum_{\substack{x,y \in B_{\kappa R} \cap h\mathbb{Z}^d \\ |x-y| > R_0 h}} (f(x) - f(y))^2 \omega(x,y)$$

holds. The constant c depends on Λ, ϑ, R_0 and on the dimension d. It does not depend on ω, Γ, and h.

Proof. Let:

$$M = \left\{ \omega : h\mathbb{Z}^d \times h\mathbb{Z}^d \to [0,\infty] \;\middle|\; \begin{array}{l} \omega(x,y) = \omega(y,x) \text{ and} \\ (9) \text{ for some configuration } \Gamma \text{ with } \vartheta > 0 \end{array} \right\},$$

$$N = \left\{ \omega : \mathbb{Z}^d \times \mathbb{Z}^d \to [0,\infty] \;\middle|\; \begin{array}{l} \omega(x,y) = \omega(y,x) \text{ and} \\ (4) \text{ for some configuration } \Gamma \text{ with } \vartheta > 0 \end{array} \right\}.$$

Every element $\omega \in M$ is of the form $h^{-d-\alpha}\tilde{\omega}(h^{-1}x, h^{-1}y)$ for some $\tilde{\omega} \in N$. If $R > 0, x_0 \in \mathbb{R}^d$, and $f : B_{\kappa R}(x_0) \cap h\mathbb{Z}^d \to \mathbb{R}$ is some function, we define the function $g : B_{\kappa R}(x_0) \cap \mathbb{Z}^d \to \mathbb{R}$ by $g(x) = f(hx)$. Then, with use of Theorem 1.3:

$$\begin{aligned} & c \sum_{\substack{x,y \in B_R(x_0) \cap h\mathbb{Z}^d \\ |x-y| > R_0 h}} (f(x) - f(y))^2 |x-y|^{-d-\alpha} \\ = c & \sum_{\substack{x,y \in B_R(x_0) \cap \mathbb{Z}^d \\ |x-y| > R_0}} (g(x) - g(y))^2 h^{-d-\alpha} |x-y|^{-d-\alpha} \\ \leqslant & \sum_{\substack{x,y \in B_R(x_0) \cap \mathbb{Z}^d \\ |x-y| > R_0}} (g(x) - g(y))^2 h^{-d-\alpha} \tilde{\omega}(x,y) \\ = & \sum_{\substack{x,y \in B_{\kappa R}(x_0) \cap h\mathbb{Z}^d \\ |x-y| > R_0 h}} (f(x) - f(y))^2 h^{-d-\alpha} \tilde{\omega}(h^{-1}x, h^{-1}y) \\ = & \sum_{\substack{x,y \in B_{\kappa R}(x_0) \cap h\mathbb{Z}^d \\ |x-y| > R_0 h}} (f(x) - f(y))^2 \omega(x,y). \end{aligned}$$

This proves the claim. □

3.1. *The discrete version of the kernel*

In this subsection, we will always assume that Γ is a fixed ϑ-admissible configuration and $\{V^m\}_{1 \leqslant m \leqslant L}$ is the associated family of reference cones. We will always denote the symmetry axis of a reference cone V^m by v^m ($m \in \{1, \ldots, L\}$).

For $k : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty]$ a nonnegative measurable function and $h > 0$, we define $\omega_h^k : h\mathbb{Z}^d \times h\mathbb{Z}^d \to [0,\infty]$ by

$$\omega_h^k(x,y) = h^{-2d} \int\limits_{A_h(x) \times A_h(y)} k(s,t)\, \mathrm{d}(s,t).$$

Note that $\omega_h^k(x,y)$ may be infinite for x and y from neighboring cubes.

We want to apply Corollary 3.1 to $\omega = \omega_h^k$. Therefore, we need to make sure that the function ω_h^k satisfies (9). First, we show this claim for $h = 1$. The next three technical lemmas are tailor-made for this purpose.

LEMMA 3.2. *For all $x, y \in \mathbb{Z}^d$, all 1-favored indices m at x and n at y, all $t \in A_1^n(y)$, and all $s \in A_1^m(x)$, the inequality*

$$\mathbb{1}_{V^m_{\sqrt{d}/2}[x]}(t) + \mathbb{1}_{V^n_{\sqrt{d}/2}[y]}(s) \;\geqslant\; \mathbb{1}_{V^m_{\sqrt{d}}[x]}(y) + \mathbb{1}_{V^n_{\sqrt{d}}[y]}(x)$$

holds.

Proof. Let $x, y \in \mathbb{Z}^d$ and let m be a 1-favored index at x. Assume $y \in V^m_{\sqrt{d}}[x]$. Then, $B_{\sqrt{d}/2}(y) \subset V^m_{\sqrt{d}/2}[x]$. Therefore,

$$A_1^n(y) \;\subset\; A_1(y) \;\subset\; B_{\sqrt{d}/2}(y) \;\subset\; V^m_{\sqrt{d}/2}[x]. \qquad \square$$

The assertion of the following lemma is obviously true.

LEMMA 3.3. *Let $r > 0$. There is an apex angle $\theta > 0$ such that for every $m \in \{1, \dots, L\}$ there exists an axis $v(m) \in \mathbb{R}^d$ so that*

$$\big(V(v(m), \theta) \cap \mathbb{Z}^d\big) \;\subset\; \big(V_r^m \cap \mathbb{Z}^d\big).$$

LEMMA 3.4. *For every $h > 0$, all $x, y \in \mathbb{Z}^d$ with $|x - y| > \sqrt{d}h$ and all $s \in A_h(x), t \in A_h(y)$, the following holds:*

$$\frac{1}{2\sqrt{d}}|x - y| \;<\; |s - t| \;<\; 2\sqrt{d}|x - y|.$$

Proof. This is about comparing the Euclidean norm to the maximum norm on $\mathbb{R}^d$. Note that for any vector $v \in \mathbb{R}^d$, we have:

$$|v|_\infty \;\leqslant\; |v| \;\leqslant\; \sqrt{d}|v|_\infty. \tag{10}$$

Let $h = 1$ and $x, y \in \mathbb{Z}^d$ with $|x - y| > \sqrt{d}$. Since the maximum norm takes only integer values on lattice points and $|x - y| > \sqrt{d}$, it follows that $|x - y|_\infty \geqslant 2$. As a consequence of the triangle inequality, we record for $s \in A_1(x)$ and $t \in A_1(y)$:

$$\frac{1}{2}|x - y|_\infty \;\leqslant\; |x - y|_\infty - 1 \;<\; |s - t|_\infty < |x - y|_\infty + 1 \;\leqslant\; 2|x - y|_\infty.$$

Using (10), we conclude:

$$|s - t| \;\leqslant\; \sqrt{d}|s - t|_\infty \;<\; 2\sqrt{d}|x - y|_\infty \;\leqslant\; 2\sqrt{d}|x - y|,$$

$$|x - y| \;\leqslant\; \sqrt{d}|x - y|_\infty \;< 2\sqrt{d}|s - t|_\infty \;\leqslant\; 2\sqrt{d}|s - t|.$$

The general case for arbitrary $h > 0$ follows by scaling. $\square$

PROPOSITION 3.5. *Let $k : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ be a symmetric and measurable function satisfying (2) for a ϑ-admissible configuration Γ. Then, there are constants $C = C(d, \vartheta) > 0$ and $\vartheta' \in (0, \frac{\pi}{2}]$ and a ϑ'-bounded configuration Γ' such that for all $x, y \in \mathbb{Z}^d$ with $|x - y| > \sqrt{d}$:*

$$C\Lambda^{-1}\Big(\mathbb{1}_{V^{\Gamma'}[x]}(y) + \mathbb{1}_{V^{\Gamma'}[y]}(x)\Big)|x - y|^{-d-\alpha} \;\leqslant\; \omega_1^k(x, y).$$

The angle ϑ' does only depend on θ and on the infimum ϑ of the apex angles of all cones in Γ. There is no further dependence on Γ.

Proof. Note:

$$\omega_1^k(x,y) \;\geqslant\; \Lambda^{-1}\int_{A_1(x)\times A_1(y)} \big[\mathbb{1}_{V^\Gamma(s)}(t)+\mathbb{1}_{V^\Gamma(t)}(s)\big]\,|t-s|^{-d-\alpha}\,\mathrm{d}(s,t).$$

Therefore, we just need to concentrate on the integral. Let m be a 1-favored index at x and n be a 1-favored index at y. Then, with use of Lemma 2.7, Lemma 3.2, Lemma 3.3, and Lemma 3.4, we estimate

$$\begin{aligned}
&\int_{A_1(x)\times A_1(y)} \big[\mathbb{1}_{V^\Gamma[s]}(t)+\mathbb{1}_{V^\Gamma[t]}(s)\big]\,|t-s|^{-d-\alpha}\,\mathrm{d}(s,t)\\
&\geqslant \int_{A_1^m(x)\times A_1^n(y)} \big[\mathbb{1}_{V^m[s]}(t)+\mathbb{1}_{V^n[t]}(s)\big]\,|t-s|^{-d-\alpha}\,\mathrm{d}(s,t)\\
&\geqslant \int_{A_1^m(x)\times A_1^n(y)} \Big[\mathbb{1}_{V^m_{\sqrt{d}/2}[x]}(t)+\mathbb{1}_{V^n_{\sqrt{d}/2}[y]}(s)\Big]\,|t-s|^{-d-\alpha}\,\mathrm{d}(s,t)\\
&\geqslant \frac{1}{(2\sqrt{d})^{d+\alpha}}\lambda_{d\times d}(A_1^m(x)\times A_1^n(y))\Big[\mathbb{1}_{V^m_{\sqrt{d}}[x]}(y)+\mathbb{1}_{V^n_{\sqrt{d}}[y]}(x)\Big]\,|x-y|^{-d-\alpha}\\
&\geqslant \frac{1}{(2\sqrt{d})^{d+\alpha}}\lambda_{d\times d}(A_1^m(x)\times A_1^n(y))\big[\mathbb{1}_{V(v(m),\theta)[x]}(y)+\mathbb{1}_{V(v(n),\theta)[y]}(x)\big]\,|x-y|^{-d-\alpha}.
\end{aligned}$$

Now, the claim follows with $C = \frac{1}{(2\sqrt{d})^{d+2}\cdot L^2} \leqslant \frac{1}{(2\sqrt{d})^{d+\alpha}}\lambda_{d\times d}(A_1^m(x)\times A_1^n(y))$ and some appropriate choice of Γ'. □

Corollary 3.6. *Let $k:\mathbb{R}^d\times\mathbb{R}^d\to[0,\infty]$ be a symmetric and measurable function satisfying (2) for a ϑ-admissible configuration Γ. Then, there are $\vartheta'>0$ and $C>0$ so that for each $h>0$ there is a configuration Γ^h on $\mathbb{R}^d$ with the following properties:*

(i) *The infimum of the apex angles of all cones in $\Gamma^h(\mathbb{R}^d)$ equals ϑ'.*
(ii) *For all $x,y\in h\mathbb{Z}^d$ with $|x-y|>\sqrt{d}h$, the inequalities*

$$C^{-1}\Big(\mathbb{1}_{V^{\Gamma^h}[x]}(y)+\mathbb{1}_{V^{\Gamma^h}[y]}(x)\Big)|x-y|^{-d-\alpha} \;\leqslant\; \omega_h^k(x,y) \;\leqslant\; C|x-y|^{-d-\alpha} \tag{11}$$

hold.

Proof. For $h>0$, define a new configuration Γ_h on $\mathbb{R}^d$ by $\Gamma_h(x)=\Gamma(hx)$. Note that the infimum of the apex angles of all cones in $\Gamma_h(\mathbb{R}^d)$ is the same as the infimum of the apex angles of all cones in $\Gamma(\mathbb{R}^d)$. It does not depend on h. Note also that (M) holds true for Γ if and only if (M) holds true for Γ_h. Therefore, Γ_h is a ϑ-admissible configuration. Define $k_h:\mathbb{R}^d\times\mathbb{R}^d\to[0,\infty]$ via $k_h(x,y)=k(hx,hy)h^{d+\alpha}$. Since k satisfies (2), we also have for almost all $x,y\in\mathbb{R}^d$:

$$\Lambda^{-1}\big(\mathbb{1}_{V^\Gamma[hx]}(hy)+\mathbb{1}_{V^\Gamma[hy]}(hx)\big)|x-y|^{-d-\alpha} \;\leqslant\; k(hx,hy)h^{d+\alpha} \;\leqslant\; \Lambda|x-y|^{-d-\alpha}. \tag{12}$$

Fix some $h>0$. We note that for all $x,y\in\mathbb{R}^d$ the assertion $hy\in V^\Gamma[hx]$ is equivalent to $y\in V^{\Gamma_h}(x)$. This together with (12) shows that k_h and Γ_h satisfy (2). Therefore, we can apply Proposition 3.5 to $\Gamma=\Gamma_h$ and $k=k_h$. We obtain a configuration $(\Gamma_h)'$ with a positive infimum of the apex angles of all cones ϑ' and some constant $C>0$ such that for all $x,y\in\mathbb{Z}^d$ with $|x-y|>\sqrt{d}$, we have

$$C\Lambda^{-1}\Big(\mathbb{1}_{V^{(\Gamma_h)'}[x]}(y)+\mathbb{1}_{V^{(\Gamma_h)'}[y]}(x)\Big)|x-y|^{-d-\alpha} \;\leqslant\; \omega_1^{k_h}(x,y). \tag{13}$$

Note that ϑ' does only depend on the infimum of the apex angles of all cones in Γ. We define a new configuration $(\Gamma_h)'_{h^{-1}}$ via $(\Gamma_h)'_{h^{-1}}(x) = (\Gamma_h)'(h^{-1}x)$. The infimum of the apex angles of all cones in this new configuration is obviously still ϑ'. Since for all $x, y \in \mathbb{Z}^d$

$$y \in V^{(\Gamma_h)'}[x] \;\Leftrightarrow\; hy \in V^{(\Gamma_h)'_{h^{-1}}}[hx]\,,$$

inequality (13) is equivalent to

$$C\Lambda^{-1}\left(\mathbb{1}_{V^{(\Gamma_h)'_{h^{-1}}}[hx]}(hy) + \mathbb{1}_{V^{(\Gamma_h)'_{h^{-1}}}[hy]}(hx)\right)|x-y|^{-d-\alpha} \;\leqslant\; \omega_1^{k_h}(x,y) \tag{14}$$

for all $x, y \in \mathbb{Z}^d$ with $|x-y| > \sqrt{d}$.

Now, let $x, y \in h\mathbb{Z}^d$ with $|x-y| > \sqrt{d}h$. Then, $h^{-1}x, h^{-1}y \in \mathbb{Z}^d$ with $|h^{-1}x - h^{-1}y| > \sqrt{d}$. With use of (14) and the transformation formula for integrals, we obtain

$$\begin{aligned}
&C\Lambda^{-1}\left(\mathbb{1}_{V^{(\Gamma_h)'_{h^{-1}}}[x]}(y) + \mathbb{1}_{V^{(\Gamma_h)'_{h^{-1}}}[y]}(x)\right)|x-y|^{-d-\alpha}\\
&\quad= C\Lambda^{-1}\left(\mathbb{1}_{V^{(\Gamma_h)'}[h^{-1}x]}(h^{-1}y) + \mathbb{1}_{V^{(\Gamma_h)'}[h^{-1}y]}(h^{-1}x)\right)|h^{-1}x - h^{-1}y|^{-d-\alpha}h^{-d-\alpha}\\
&\quad\leqslant \omega_1^{k_h}(h^{-1}x, h^{-1}y)h^{-d-\alpha}\\
&\quad= \omega_h^{k}(x,y).
\end{aligned}$$

The upper bound in (ii) is just a consequence of Lemma 3.4. The claim follows with $\Gamma^h = (\Gamma_h)'_{h^{-1}}$. □

3.2. *Proof of the continuous version*

In this part, we prove Theorem 1.1 and Theorem 1.4.

Proof of Theorem 1.1. The inequality (3) is obviously true if the right-hand side is infinite. Hence, we can restrict ourselves to functions $f \in H_k(B)$. The following Lemma 3.7 provides comparability of the seminorms $|\cdot|_{H_k(B)}$ and $|\cdot|_{H^{\frac{\alpha}{2}}(B)}$, which implies (3). The proof is complete. □

LEMMA 3.7. *Let $B \subset \mathbb{R}^d$ be a ball. Let $\alpha \in (0,2)$ and Γ be a ϑ-admissible configuration. Let $k : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty]$ be a measurable function satisfying $k(x,y) = k(y,x)$ and (2). The spaces $H_k(B)$ and $H^{\frac{\alpha}{2}}(B)$ coincide. Furthermore,*

$$|\cdot|_{H^{\frac{\alpha}{2}}(B)} \;\leqslant\; c|\cdot|_{H_k(B)} \;\text{ on }\; H_k(B) \;=\; H^{\frac{\alpha}{2}}(B)$$

for a constant $c \geqslant 1$ independent of the ball B. For $0 < \alpha_0 \leqslant \alpha < 2$, the constant depends on α_0 but not on α.

Proof. In view of (6), we only have to show the inclusion $H_k(B) \subset H^{\frac{\alpha}{2}}(B)$. Note that comparability of the seminorms implies comparability of norms. Hence, we shall show

$$|f|_{H_k(B)} \;\asymp\; |f|_{H^{\frac{\alpha}{2}}(B)} \quad \text{for all } f \in H_k(B), \tag{15}$$

that is, the ratio of the two quantities is bounded by a constant independent of f. Let $R > 0$, $x_0 \in \mathbb{R}^d$, and κ as in Corollary 3.1. In the sequel, we use the notation $B = B_R(x_0)$ and $B^* = B_{\kappa R}(x_0)$. Let $f \in H_k(B^*)$. For $h \in (0,1)$, we consider the following piecewise constant approximation of f. We define for $x \in h\mathbb{Z}^d \cap B^*$

$$f_h(x) \;=\; h^{-d}\int_{A_h(x)\cap B^*} f(s)\,\mathrm{d}s\,.$$

Because of Proposition 3.6, there is a constant $C > 0$ and a configuration Γ_h with $\vartheta' > 0$ such that for all $x, y \in h\mathbb{Z}^d$ with $|x-y| > \sqrt{d}h$, the inequalities

$$C^{-1}\Big(\mathbb{1}_{V^{\Gamma_h}[x]}(y) + \mathbb{1}_{V^{\Gamma_h}[y]}(x)\Big)|x-y|^{-d-\alpha} \leqslant \omega_h^k(x,y) \leqslant C|x-y|^{-d-\alpha}$$

hold. Thus, $\omega = \omega_h^k$ together with $\Gamma = \Gamma_h$ fulfill (9) for $R_0 = \sqrt{d}$ and $\Lambda = C$. Corollary 3.1 implies the existence of $c > 0$, independent of f, R, α, and h, so that

$$c \sum_{\substack{x,y\in B\cap h\mathbb{Z}^d \\ |x-y|>\sqrt{d}h}} (f_h(x)-f_h(y))^2|x-y|^{-d-\alpha} \leqslant \sum_{\substack{x,y\in B^*\cap h\mathbb{Z}^d \\ |x-y|>\sqrt{d}h}} (f_h(x)-f_h(y))^2\omega(x,y).$$

Using Lemma 3.4, we obtain

$$\begin{aligned} & c \sum_{\substack{x,y\in B\cap h\mathbb{Z}^d \\ |x-y|>\sqrt{d}h}} (f_h(x)-f_h(y))^2 \int_{A_h(x)\times A_h(y)} |s-t|^{-d-\alpha}\,\mathrm{d}(s,t) \\ \leqslant & \sum_{\substack{x,y\in B^*\cap h\mathbb{Z}^d \\ |x-y|>\sqrt{d}h}} (f_h(x)-f_h(y))^2 \int_{A_h(x)\times A_h(y)} k(s,t)\,\mathrm{d}(s,t), \end{aligned} \tag{16}$$

for a constant $c > 0$ that differs from the one above by a factor only depending on the dimension d.

For technical reasons, we need the property that every x in $\mathbb{R}^d$ is contained in some cube. Therefore, we consider half-closed cubes. Given $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $h \in (0,1)$, we use the notation

$$\widetilde{A}_h(x) = \prod_{i=1}^d \Big[x_i - \frac{h}{2}, x_i + \frac{h}{2}\Big).$$

For $h \in (0,1)$, we define a function $g_h : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ via

$$g_h(s,t) = \sum_{x,y\in h\mathbb{Z}^d} \Big[(f_h(x)-f_h(y))^2 k(s,t)\,\mathbb{1}_{\widetilde{A}_h(x)\times\widetilde{A}_h(y)}(s,t)\,\mathbb{1}_{\{x,y\in B^*|\sqrt{d}h<|x-y|\}}(x,y)\Big]$$

and claim that g_h converges for $h \to 0$ almost everywhere to the function $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ with

$$g(s,t) = (f(s)-f(t))^2 k(s,t) \mathbb{1}_{B^*\times B^*}(s,t).$$

Indeed, $g_h(s,t) = (f_h(x_h) - f_h(y_h))^2 k(s,t)$ for appropriate points x_h and y_h. We conclude with help of Lemma A.2, $g_h(s,t) \to g(s,t)$ for almost every $(s,t) \in B^* \times B^*$. In the same way, we can show that the function $\widetilde{g}_h : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ with

$$\begin{aligned} \widetilde{g}_h(s,t) = \sum_{x,y\in h\mathbb{Z}^d} \Big[& (f_h(x)-f_h(y))^2|s-t|^{-d-\alpha}\,\mathbb{1}_{\widetilde{A}_h(x)\times\widetilde{A}_h(y)}(s,t) \\ & \times \mathbb{1}_{\{x,y\in B|\sqrt{d}h_n<|x-y|\}}(x,y)\Big] \end{aligned}$$

converges for $h \to 0$ pointwise almost everywhere to

$$\widetilde{g} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R},$$

$$\widetilde{g}(s,t) = (f(s)-f(t))^2|s-t|^{-d-\alpha}\mathbb{1}_{B\times B}(s,t).$$

For the right-hand side in (16), this implies with help of dominated convergence

$$\sum_{\substack{x,y\in B^*\cap h\mathbb{Z}^d\\ |x-y|>\sqrt{d}h}} (f_h(x)-f_h(y))^2 \int_{\widetilde{A}_h(x)\times\widetilde{A}_h(y)} k(s,t)\,\mathrm{d}(s,t)$$

$$= \int_{\mathbb{R}^d\times\mathbb{R}^d} g_h(s,t)\,\mathrm{d}(s,t) \overset{h\to 0}{\longrightarrow} \int_{\mathbb{R}^d\times\mathbb{R}^d} g(s,t)\,\mathrm{d}(s,t).$$

With regard to the left-hand side of (16), note that the Fatou lemma implies

$$\liminf_{h\to 0} \sum_{\substack{x,y\in B\cap h\mathbb{Z}^d\\ |x-y|>\sqrt{d}h}} (f_h(x)-f_h(y))^2 \int_{\widetilde{A}_h(x)\times\widetilde{A}_h(y)} |s-t|^{-d-\alpha}\,\mathrm{d}(s,t)$$

$$= \liminf_{h\to 0} \int_{\mathbb{R}^d\times\mathbb{R}^d} \widetilde{g}_h(s,t)\,\mathrm{d}(s,t) \geqslant \int_{\mathbb{R}^d\times\mathbb{R}^d} \widetilde{g}(s,t)\mathrm{d}(s,t).$$

In conclusion, we have shown that the discrete inequality (16) yields the continuous version

$$c|f|_{H^{\frac{\alpha}{2}}(B)} \leqslant |f|_{H_k(B^*)} \quad \text{for all } f\in H_k(B^*).$$

This is true for every ball B, since c is independent of B. Therefore, using Lemma A.1, we conclude for each ball $B\subset\mathbb{R}^d$ and each $f\in H_k(B^*)$

$$c^*|f|_{H^{\frac{\alpha}{2}}(B)} \leqslant |f|_{H_k(B)}$$

for some $c^*>0$, independent of the ball B. This proves comparability of the seminorms and $H_k(B)\subset H^{\frac{\alpha}{2}}(B)$. □

Proof of Theorem 1.4. The comparability constant c in Lemma 3.7 is independent of the radius R of the respective ball. Thus, the result for the whole space is obtained in the limit $R\to\infty$ using monotone convergence.

Now let Ω be a bounded Lipschitz domain. In view of Lemma 3.7 and Lemma A.1, we conclude

$$|\cdot|_{H^{\frac{\alpha}{2}}(\Omega)} \leqslant c|\cdot|_{H_k(\Omega)} \text{ on } H_k(\Omega)$$

for a constant $c\geqslant 1$, which leads to $H_k(\Omega)\subset H^{\frac{\alpha}{2}}(\Omega)$. Since the inclusion $H^{\frac{\alpha}{2}}(\Omega)\subset H_k(\Omega)$ is obvious by the definition of k, one obtains $H_k(\Omega)=H^{\frac{\alpha}{2}}(\Omega)$. For the assertions concerning density of smooth functions, note that $C^\infty(\overline{\Omega})$ is a dense subset of $H^{\frac{\alpha}{2}}(\Omega)$, cf. [**5**, Proposition 4.52]. Furthermore, $C_c^\infty(\mathbb{R}^d)$ is a dense subset of $H^{\frac{\alpha}{2}}(\mathbb{R}^d)$, cf. [**5**, Proposition 4.27]. □

4. *A continuous prelude*

The main tool for the proof of Theorem 1.3 is the construction of paths connecting two arbitrary points in $\mathbb{Z}^d$. In this section, we show the existence of paths connecting two arbitrary points in $\mathbb{R}^d$. This result is not needed to prove Theorem 1.3. Therefore, the reader may skip this section. However, the procedure in the continuous setting is similar to the discrete setting, but less technical. For this reason, reading this section first might provide useful intuition.

From a configuration $\Gamma\colon\mathbb{R}^d\to\mathcal{V}$, we construct a directed graph G as follows: the vertex set is $\mathbb{R}^d$ and there is a directed edge from x to y if $y\in V^\Gamma[x]$. Note that there are no loops in G as $\Gamma(x)$ is open and does not contain the tip.

We shall be concerned with the question whether G is connected as an *undirected* graph if the underlying configuration is ϑ-bounded. In this case, Corollary 2.4 allows us to assume

without loss of generality that the image of Γ contains only a finite number of elements. Thus, crucial parts of the argument can be proved by induction on the number of cones in $\Gamma(\mathbb{R}^d)$. As it often happens, one needs to strike the right balance and the statement suitable for induction is a little bit stronger (and more technical) than the primary target. We are led to consider subgraphs G_U defined by open subsets $U \subset \mathbb{R}^d$ as follows: the vertex set of G_U is still $\mathbb{R}^d$ and the rule for oriented edges is the same, however, we only put in the edges issuing from vertices in U. Note that vertices outside U still can be used in edge paths since we are interested in undirected connectivity.

In this section, we always assume that the configuration Γ is ϑ-bounded.

The main result of this part is:

THEOREM 4.1. *For any connected open set $U \subset \mathbb{R}^d$, any two points $x, y \in U$ are vertices in the same connected component of G_U.*

For the proof of Theorem 4.1, we need some auxiliary results.

DEFINITION 4.2. A point $x \in \mathbb{R}^d$ is *of cone type* V if $V = \Gamma(x)$. Two points $x, y \in \mathbb{R}^d$ *have the same type* if $\Gamma(x) = \Gamma(y)$.

LEMMA 4.3. *If two points $x, y \in U$ have the same type, then there is an edge path in G_U of length at most two connecting them.*

Proof. Let $V = \Gamma(x) = \Gamma(y)$. Then, the translated double cones $V[x]$ and $V[y]$ intersect. We pick a point of intersection (it may lie outside of U). It has an edge incoming from x and another edge incoming from y. These two edges form the desired edge path. □

DEFINITION 4.4. We call x *well connected in* U if there is an open neighborhood W of x that, considered as a set of vertices in G_U, lies entirely in a single connected component of G_U. That is, the point x is connected by edge paths in G_U to all points of an open neighborhood.

The following lemma lists *inter alia* some important features of well-connected points.

LEMMA 4.5. *The following hold.*

(1) *For $y \in U$, any point $x \in U \cap V^\Gamma[y]$ is well connected in U.*
(2) *If $U' \subset U$ is an inclusion of open sets, then any point $x \in U'$ that is well connected in U' is also well-connected in U.*
(3) *Any nonempty open set U contains a point that is well connected in U. In fact, the well-connected points are dense in U.*

Proof. For (1), we may choose $U \cap V^\Gamma[y]$ as the open neighborhood. Any two points therein are connected via an edge path of length two with y as the middle vertex. Therefore, (1) follows.

Enlarging the open set U' only adds edges to the graph. Hence, connectivity can only improve. This proves (2).

For the proof of (3), note that existence of a well-connected point follows from (1). Applying the existence statement to smaller open sets $U' \subset U$, density follows in view of (2). □

LEMMA 4.6. *Consider two points $x, y \in U$ and let $V = \Gamma(y)$ be the cone type of y. Assume that the translated double cone $V[x]$ contains a point z of cone type V. Then, x and y are connected.*

Note that we do not assume that $V = \Gamma(x)$. One may also note that in the situation of the lemma, the point x is well connected in U.

Proof. Since y and z have the same type, they are connected by an edge path of length at most two. Now, $z \in V[x]$ implies $x \in V[z] = V^\Gamma[z]$. Hence, there is an edge from z to x. □

Proof of Theorem 4.1. According to Lemma 2.2, we may assume that the image of Γ has at most L different elements since Γ is ϑ-bounded. Therefore, we can use induction on the number $\#\Gamma(U)$ of cones realized in U. If there is only a single cone type throughout U, any two points $x, y \in U$ are connected in G_U by an edge path of length at most two. This settles the base of the induction.

For $\#\Gamma(U) > 1$, we start with the following observation:

> *There is a constant $\lambda > 0$ depending only on the minimum apex angle ϑ such that for any double cone $V \in \mathcal{V}$ and any two points $x, y \in \mathbb{R}^d$ of distance $\|x - y\| < \lambda$, the intersection $V[x] \cap V[y]$ contains a point in $B_1(x)$.*

Now assume that x is well connected in U and that the r-ball $B_r(x)$ lies entirely in U. We claim that x is connected to any point $y \in B_{\lambda r}(x)$. Indeed, consider the cone type $V = \Gamma(y)$ of y. If $V[x]$ contains a point of cone type V, the points x and y are connected by Lemma 4.6.

Otherwise, within the open set $U' = U \cap V[x] \neq \emptyset$, the cone type V is not realized. We infer by induction that all points in U' are mutually connected in $G_{U'}$ and hence in G_U. However, $V[y] = V^\Gamma[y]$ intersects $U' \supset B_r(x) \cap V[x]$ by the opening observation. Hence, y is connected to a point in U' and therefore to any point in U', which contains points arbitrarily close to x. Since x is well connected in U, the points y and x are connected in G_U.

It follows that a well-connected point $x \in U$ whose r-neighborhood lies in U is actually connected to any point in its λr-neighborhood. Now, density of well-connected points in U (cf. Lemma 4.5 (3)) implies that U is covered by overlapping open well-connected subsets. □

5. *Chaining and renormalization*

In this section, we provide the chaining argument that leads to the proof of Theorem 1.3 in Section 5. The main result of this section is Theorem 5.15.

Every configuration $\Gamma\colon \mathbb{R}^d \to \mathcal{V}$ induces naturally a mapping $\Gamma|_{\mathbb{Z}^d}$ which we again call configuration and denote by Γ. As in the continuous case, a configuration Γ defines a directed graph $G = G(\Gamma)$ where the vertex set is given by $\mathbb{Z}^d$ and there is an oriented edge from x to y if $y \in V^\Gamma[x]$.

We shall be concerned with the question whether G is connected as an *undirected* graph if Γ is ϑ-bounded. Therefore, throughout this section, we assume without further notice that Γ is a ϑ-bounded configuration. In addition to the continuous version, however, we also want to keep track of how far such an edge path might take us away from the end points in question.

5.1. *Auxiliary results*

A technicality is that we always have to use lattice points. Since any closed ball of radius $\frac{\sqrt{d}}{2}$ contains a lattice point, we have:

LEMMA 5.1. *Let $\widetilde{V}$ be a cone of apex angle at least ϑ.*

(1) *Fix $r > 0$ and assume $R > \frac{r+\sqrt{d}}{\sin(\vartheta)}$. Then, $B_R(x) \cap \widetilde{V}[x]$ contains a lattice point $y \in \mathbb{Z}^d$ with $B_r(y) \subset \widetilde{V}[x]$.*

(2) *Let $x, y \in \mathbb{R}^d$. Fix $r > \|x - y\|$ and $R > \frac{r+\sqrt{d}}{\sin(\vartheta)} + r$. Then, the intersection*

$$B_R(x) \cap \widetilde{V}[x] \cap B_R(y) \cap \widetilde{V}[y]$$

contains a lattice point.

Proof. Within distance $\frac{r+\sqrt{d}/2}{\sin(\vartheta)}$ of x, we find a point z with $B_{r+\sqrt{d}/2}(z) \subset \widetilde{V}[x]$. Within the closed ball of radius $\sqrt{d}/2$ around z, we find the desired lattice point y. The second assertion can be seen as another way of looking at the same phenomenon. By (1), there is a lattice point $z \in \mathbb{Z}^d$ with $B_r(z) \subset B_{\frac{r+\sqrt{d}}{\sin(\vartheta)}}(x) \cap \widetilde{V}[x]$. Now, $z \in \widetilde{V}[y]$ since $\widetilde{V}[y]$ is obtained from $\widetilde{V}[x]$ via translation by a distance less than r. By triangle inequality, $z \in B_R(y)$. □

A quantitative version of Lemma 4.3 follows immediately.

COROLLARY 5.2. *Any two lattice points $x, y \in \mathbb{Z}^d$ with $\Gamma(x) = \Gamma(y) = V$ and of distance less than r are connected via a path of two edges of length less than $R = \frac{r+\sqrt{d}}{\sin(\vartheta)} + r$.*

DEFINITION 5.3. For $r \leqslant R$, we call a lattice point $x \in \mathbb{Z}^d$ *r-R-connected*, if any lattice point $y \in B_r(x)$ is connected in G to x via an undirected edge path not leaving $B_R(x)$.

The following lemma is the discrete version of the density of well-connected points (Lemma 4.5 (3)).

LEMMA 5.4. *For any $r \geqslant 0$, any $R > \frac{\sqrt{d}+r}{\sin(\vartheta)}$, and any lattice point $x \in \mathbb{Z}^d$, there is an r-R-connected lattice point $y \in B_R(x)$.*

Proof. The larger radius R is chosen so that $B_R(x) \cap (V^\Gamma[x])$ contains a lattice point y whose r-ball $B_r(y)$ lies within the double cone $V^\Gamma[x]$. Thus, any two points in $B_r(y)$ are connected via x, and x is within distance R from y. □

Our discrete variant of Lemma 4.6 reads as follows:

LEMMA 5.5. *Consider two lattice points $x, y \in \mathbb{Z}^d$ of distance less than r. Let $V = \Gamma(y)$ be the cone type of y and let $R > r + \frac{2r+\sqrt{d}}{\sin(\vartheta)}$. Assume that $B_r(x) \cap V[x]$ contains a lattice point z of cone type V. Then, there is an edge path from y to x not leaving $B_R(x)$.*

Proof. There is a directed edge from z to x. Note that the distance of y and z is at most $2r$. Hence, R is chosen so that the triple intersection

$$B_R(x) \cap V[z] \cap V[y] \;=\; B_R(x) \cap V^\Gamma[z] \cap V^\Gamma[y]$$

contains a lattice point. Through this point, z and y are connected. □

The assertion of the following lemma is obvious.

LEMMA 5.6. *There is a constant $\delta > 0$, depending only on ϑ and the dimension d, such that for any double cone $V \in \Gamma(\mathbb{Z}^d)$ the following condition holds:*

> *If for a lattice point $x \in V$, there is a lattice point in V closer to 0, then there is such a lattice point in $V \cap B_\delta(x)$.*

That is, we can go from x within V to a lattice point of minimum distance to the tip via a chain of jumps each bounded in length from above by δ.

5.2. *The Induction*

It is our aim to prove that every two lattice points in a given ball of radius r are connected via an edge path that does not leave a larger ball of radius R. Here, the radius R shall depend only

on r, ϑ, and d. In the following lemma, we show this for a series of values for r, respectively, R. The proof is similar to the proof of the corresponding result in the continuous setting, cf. Theorem 4.1.

LEMMA 5.7. *There are constants $r_1 \leqslant \rho_1 \leqslant R_1, r_2 \leqslant \rho_2 \leqslant R_2, \ldots$, depending only on ϑ and d, with $\delta < r_1$ and $r_i < r_{i+1}, \rho_i < \rho_{i+1}, R_i < R_{i+1}$ for every $i \in \mathbb{N}$ such that any lattice point $x \in \mathbb{Z}^d$ is r_k-R_k-connected provided at most k cone types are realized at the lattices points in $B_{\rho_k}(x)$.*

Proof. We induct on k. The case $k = 1$ follows directly from Corollary 5.2: choose $\rho_1 = r_1 > \delta$ and $R_1 > \frac{r_1+\sqrt{d}}{\sin(\vartheta)}$.

For the induction step, assume that constants up to ρ_{k-1}, r_{k-1} and R_{k-1} have already been found. Choose:

$$s \;>\; \frac{\rho_{k-1} + \sqrt{d}}{\sin(\vartheta)} \qquad \text{and} \qquad S \;>\; \frac{s + \sqrt{d}}{\sin(\vartheta)}.$$

Note that by Lemma 5.1 (1) any set $B_s(\hat{x}) \cap V[\hat{x}]$ contains a lattice point z with $B_{\rho_{k-1}}(z) \subset V[\hat{x}]$. If $\hat{x}$ is s-S-connected, there is an edge path from $\hat{x}$ to z not leaving $B_S(\hat{x})$.

So, we put $r_k = S$. By Lemma 5.4, there is an s-S-connected lattice point $\hat{x} \in B_{r_k}(x)$. Consider an arbitrary point $y \in B_{r_k}(x)$. It suffices to choose R_k and ρ_k so that we can ensure the existence of an edge path from $\hat{x}$ to y within $B_{R_k}(x)$.

Let $V = \Gamma(y)$ be the cone type of y. The distance of y and $\hat{x}$ is less than $2S$. We are interested in the double half-cone $\hat{x} + V_{\rho_{k-1}}$. Either tip of the double half-cone is within distance $s < S$ of $\hat{x}$ and thus within distance $3S$ of y. By Lemma 5.1 (2), the intersection

$$B_{\hat{s}}(\hat{x}) \;\cap\; (\hat{x} + V_{\rho_{k-1}}) \;\cap\; V[y]$$

contains a lattice point for any $\hat{s} > \frac{3S+\sqrt{d}}{\sin(\vartheta)} + 3S$.

Choosing $\rho_k > S + \hat{s} + \rho_{k-1}$, we can use the induction hypothesis as follows. If no lattice point in the region $B_{\hat{s}+\rho_{k-1}}(\hat{x}) \cap V[\hat{x}] \subset B_{\rho_k}(x)$ is of cone type V, we see that there are at most $k-1$ different cone types realized within $B_{\hat{s}+\rho_{k-1}}(\hat{x}) \cap V[\hat{x}]$. Hence, each lattice point in $B_{\hat{s}}(\hat{x}) \cap (\hat{x} + V_{\rho_{k-1}})$ is r_{k-1}-R_{k-1}-connected. Since $r_{k-1} > \delta$, all these well-connected balls overlap and are therefore connected to a lattice point z near the tip of the double half-cone. Recall that z is within distance s of $\hat{x}$ and that $\hat{x}$ is s-S-connected. Hence, all the lattice points in $B_{\hat{s}}(\hat{x}) \cap (\hat{x} + V_{\rho_{k-1}})$ are connected to $\hat{x}$.

On the other hand, one of these lattice points lies within the double cone $y + V = V^{\Gamma}[y]$ and is hence directly connected to y. Thus, y is connected to $\hat{x}$. Each edge path used will take us at most S or R_{k-1} outside of $B_{\hat{s}}(\hat{x})$. Thus, we might choose $R_k > 2S + R_{k-1} + \hat{s}$. We might need to increase this number, to ensure $\rho_k \leqslant R_k$, but the increase incurred in treating the remaining case is much worse.

It remains to deal with the possibility that there is a lattice point of cone type V in the region $B_{\hat{s}+\rho_{k-1}}(\hat{x}) \cap V[\hat{x}]$. Since $\hat{x}$ and y are of distance at most $2S < \hat{s}$, Lemma 5.5 applies and we choose $R_k > (\hat{s} + \rho_{k-1}) + \frac{2(\hat{s}+\rho_{k-1})+\sqrt{d}}{\sin(\vartheta)}$. □

COROLLARY 5.8. *For every $r > 0$, there is $R \geqslant r$, depending only on r, ϑ, and d, such that for any configuration $\Gamma\colon \mathbb{Z}^d \to \mathcal{V}$ with apex angles bounded from below by ϑ any lattice point $x \in \mathbb{Z}^d$ is r-R-connected.*

Proof. By Corollary 2.4, we can assume without loss of generality that $\#\Gamma(\mathbb{Z}^d) = L$, where L is a constant that depends only on ϑ and d. Now the claim follows from Lemma 5.7 and the following observation: If x is r-R-connected, it is r'-R-connected for any $r' \leqslant r$. □

5.3. *Renormalization: Blocks and Towns*

Since the proof of Theorem 1.3 involves a renormalization argument, it is important to restate Corollary 5.8 for structures at large scale (see Proposition 5.14). To this end, we introduce what we call blocks and towns. Recall our notation

$$A_\ell(x) \;=\; \left\{ y \in \mathbb{R}^d \,|\, \|y-x\|_\infty \leqslant \frac{\ell}{2} \right\}$$

for cubes.

LEMMA 5.9. *For any apex angle ϑ, there is a constant $\delta = \delta(\vartheta) > 0$ such that the following holds for each $\ell > 0$ and any points $x, y \in \mathbb{R}^d$ of distance at least $\delta\ell$:*

If $\widetilde{V}$ is a cone of apex angle $\frac{\vartheta}{2}$ and $y \in \widetilde{V}[x]$, then

$$A_\ell(y) \;\subset \bigcap_{z \in A_\ell(x)} \overline{V}[z]$$

for the cone $\overline{V}$ with apex angle ϑ and the same axis as $\widetilde{V}$.

Proof. Let $\widetilde{V}$ be a cone of apex angle $\frac{\vartheta}{2}$ and symmetry axis v and let $\overline{V}$ be a cone with apex angle ϑ and symmetry axis v. Let $x \in \mathbb{R}^d$ and $\ell > 0$. According to Lemma 2.7, we know

$$\bigcap_{z \in A_\ell(x)} \overline{V}[z] \;\supset\; \overline{V}_{\frac{\ell}{2}\sqrt{d}}[x].$$

It is also known that

$$B_{\frac{\ell}{2}\sqrt{d}}(y) \;\supset\; A_\ell(y)$$

for any $y \in \mathbb{R}^d$.

Therefore, we choose a point $\tilde{y} \in \partial(\widetilde{V}[x])$ with the property

$$B_{\frac{\ell}{2}\sqrt{d}}(\tilde{y}) \subset \overline{V}_{\frac{\ell}{2}\sqrt{d}}[x],$$

that is, $B_{\ell\sqrt{d}}(\tilde{y}) \subset \overline{V}[x]$ and set $\delta = \frac{|x-\tilde{y}|}{\ell} \geqslant \frac{\sqrt{d}}{\sin(\vartheta/2)}$. □

DEFINITION 5.10. A *block*

$$Q_\ell(x) \;=\; \mathbb{Z}^d \cap A_\ell(x)$$

is a collection of lattice points inside a cube. The *town at scale* (h, ℓ) is the collection

$$T(h,\ell) \;=\; \big\{ Q_\ell(x) \,|\, x \in h\mathbb{Z}^d \big\}$$

If the constant δ from Lemma 5.9 is less than $\frac{h}{\ell}$, we call the town *sparsely populated* (or ϑ-sparsely populated when we want to recall that δ depends on ϑ).

In order to employ geometric language, we implicitly may identify the block $Q_\ell(x)$ with its center x. This way, we think of the distance between two blocks as the distance of their centers. If h is large compared to ℓ, the distance between the centers is a good approximation to any distance between points from the two blocks.

DEFINITION 5.11. Let $\Gamma\colon \mathbb{Z}^d \to \mathcal{V}$ be a ϑ-bounded configuration. We call a double cone $V \in \mathcal{V}$ *favored by majority* in Q for a block $Q \subset \mathbb{Z}^d$ if the preimage $\Gamma_Q^{-1}(V) = \{x \in Q \,|\, \Gamma(x) = V\}$ has maximal size, that is,

$$\#\Gamma_Q^{-1}(V) \;\geqslant\; \#\Gamma_Q^{-1}(V') \qquad \text{for every } V' \in \mathcal{V}.$$

REMARK 5.12. Given a block Q, the choice of a cone $V \in \mathcal{V}$ that is favored by majority in Q, in general, is not unique.

DEFINITION 5.13. Given a town $T = T(h, \ell)$, we now define a directed graph as follows. The vertices are given by the blocks in T. There is an edge from a block Q to a block P if there is a cone $V \in \mathcal{V}$ favored by majority in Q with:

$$y \in V[x] \qquad \text{for all } x \in Q,\ y \in P.$$

We call the corresponding undirected graph the *favored graph*.

We derive a connectivity result for the favored graph of a sparsely populated town from Corollary 5.8.

PROPOSITION 5.14. *For any radius $r > 0$, there exists $R \geqslant r$ depending only on ϑ and d, such that in a ϑ-sparsely populated town T of scale (h, ℓ), any two blocks Q and P within distance hr of some point $z \in h\mathbb{Z}^d$ are connected by an undirected edge path in the favored graph that does not pass through blocks farther away from z than hR.*

Proof. Let $r > 0$ and $T = T(h, \ell)$ be a sparsely populated town. Let $Q, P \in T = T(h, \ell)$ be two blocks within distance hr of some point $z \in h\mathbb{Z}^d$. Denote by $W(Q) \in \mathcal{V}$ one of the cones that are favored by majority in Q. Let us show the existence of a path in the favored graph that connects Q and P and does not leave the ball $B_{hR}(z)$. In order to invoke Corollary 5.8, note that

$$\mathbb{Z}^d \longrightarrow T(h, \ell)$$

$$x \mapsto Q_\ell(hx)$$

provides an identification of the town T with the integer lattice $\mathbb{Z}^d$. Denoting by $W_{\frac{1}{2}}(Q)$ the double cone with apex $\frac{\vartheta}{2}$ and the same axis as $W(Q)$, let us consider the following configuration:

$$\mathbb{Z}^d \longrightarrow \mathcal{V}_{\frac{\vartheta}{2}}$$

$$x \mapsto W_{\frac{1}{2}}(Q_\ell(hx)).$$

If there is an edge from x to y in this configuration, then by Lemma 5.9, there is an edge from the block $Q_\ell(hx)$ to the block $Q_\ell(hy)$ in the favored graph. Choose $x, y \in \mathbb{Z}^d$ so that $P = Q_\ell(hx)$ and $Q = Q_\ell(hy)$.

Now the claim follows from Corollary 5.8. □

5.4. *Connecting Points at Scale*

From Corollary 5.8, it is clear that, for any configuration $\Gamma\colon \mathbb{Z}^d \to \mathcal{V}$ with apex angles bounded away from 0, the associated directed graph $G = G(\Gamma)$ is connected when considered as an undirected graph. Thus, there is a set of paths, which is large enough to connect any given pair x, y. The aim of this section is to prove quantitative estimates on the length of paths and the number of edges. The following contains our main result in this direction. As shown below, it implies Theorem 1.3 quite directly:

THEOREM 5.15. *Let $\Gamma\colon \mathbb{Z}^d \to \mathcal{V}$ be a configuration with apex angles bounded from below by $\vartheta > 0$. Let $R_0 > 0$. There exist positive numbers N and M and a constant $\lambda \geqslant R_0$, all independent of Γ, and a collection $(p_{xy})_{x,y\in\mathbb{Z}^d}$ of unoriented edge paths in G such that the following holds:*

(1) *The path p_{xy} starts at x and ends at y.*
(2) *Any path p_{xy} has at most N edges.*
(3) *Any edge of G is used in at most M paths p_{xy}.*
(4) *Any edge in p_{xy} has length comparable to $|x-y|$ with constant λ.*

Let us provide the setup of the proof of Theorem 5.15. We pick an even integer Δ larger than the constant $\max(\delta, R_0)$ and with the property that $\Delta/L \in \mathbb{N}$, where δ is as in Lemma 5.9 and L is as in Lemma 2.2. Hence, the towns $T_n = T(\Delta^n, \Delta^{n-1})$ are all ϑ-sparsely populated so that Proposition 5.14 applies. The distance $|x-y|$ lies in exactly one of the intervals $[\Delta^0, \Delta^1)$, $[\Delta^1, \Delta^2)$, $[\Delta^2, \Delta^3)$, etc., say: $|x-y| \in [\Delta^{n-1}, \Delta^n)$. In this case, we will consider T_n to be the appropriate town for connecting x and y. We call n the *logarithmic scale* of the town T_n. Let L denote the same constant as in Lemma 2.2. Assume $\#\Gamma(\mathbb{Z}^d) \leqslant L$. Since Δ is an even integer, each block $Q = Q_{\Delta^{n-1}}(x)$ contains at least $\frac{\Delta^{d(n-1)}}{L}$ lattice points $z \in Q$ where the associated cone $\Gamma(z)$ is favored by majority in Q.

An important step in the construction of p_{xy} is to connect x and y to blocks of T_n. The following lemma deals with this problem.

LEMMA 5.16. *There is a constant $R_1 \geqslant 1$ such that for any point $x \in \mathbb{R}^d$ and any $n \in \mathbb{N}$ there is a block $Q \in T_n$ entirely contained in $B_{\Delta^n R_1}(x) \cap x + \Gamma(x)$.*

Proof. There is a radius r such that for any cone $\widetilde{V}$ of apex at least $\frac{\vartheta}{2}$ and each point $z \in \mathbb{R}^d$, the intersection $B_r(z) \cap \widetilde{V}[z]$ contains a lattice point y. Now the claim follows by rescaling from Lemma 5.9 applied to $\Delta^n z$ and $\Delta^n y$. As we want to encircle the whole block and not just its center, we choose $R_1 > r + \sqrt{d}$. □

Now we are in the position to prove the main result of this section.

Proof of Theorem 5.15. Let R_1 be the radius from Lemma 5.16, put $r = 2\sqrt{d} + R_1$, and let R be the radius resulting with this value from Proposition 5.14. The proof consists of several steps.

Step 1: *Construction of paths in the favored graph for a fixed scale.* We fix some logarithmic scale n. For every $z \in \Delta^n \mathbb{Z}^d$, we construct a path $\mathcal{P}_z^n$ in the favored graph that traverses every block of T_n that is a subset of $B_{\Delta^n r}(z)$. By taking the union $\bigcup_{z \in \Delta^n \mathbb{Z}^d} \mathcal{P}_z^n$, we construct paths in the favored graph for a fixed scale. Let $z \in \Delta^n \mathbb{Z}^d$. Proposition 5.14 allows us to connect every block $Q \in T_n, Q \subset B_{\Delta^n r}(z)$ with every other block $P \in T_n, P \subset B_{\Delta^n r}(z)$ so that the corresponding path traverses not more than $\#(B_R \cap \mathbb{Z}^d) \asymp R^d$ blocks of T_n, which can be chosen to lie in $B_{\Delta^n R}$. If we apply Proposition 5.14 successively to all blocks of T_n, which are subsets of $B_{\Delta^n r}$, then we obtain a path

$$\mathcal{P}_z^n \;=\; Q^1 - Q^2 - \cdots - Q^t \tag{17}$$

of blocks of T_n in the favored graph with

$$r^d \;\asymp\; \#(B_{\Delta^n r} \cap \mathbb{Z}^d) \;\leqslant\; t \;\leqslant\; \#(B_{\Delta^n r} \cap \mathbb{Z}^d)(B_{\Delta^n R} \cap \mathbb{Z}^d) \;\asymp\; r^d R^d$$

such that the following holds (see Figure 3):

(1) For each $i \in \{1, \dots, t\}$, we have $Q^i \subset B_{\Delta^n R}(z)$.
(2) The blocks Q^1 and Q^t are subsets of $B_{\Delta^n r}(z)$.
(3) If Q is any block of T_n with $Q \subset B_{\Delta^n r}(z)$, then $Q = Q^i$ for some $i \in \{1, \dots, t\}$.

Finally, set $\mathcal{P}^n = \bigcup_{z \in \Delta^n \mathbb{Z}^d} \mathcal{P}_z^n$.

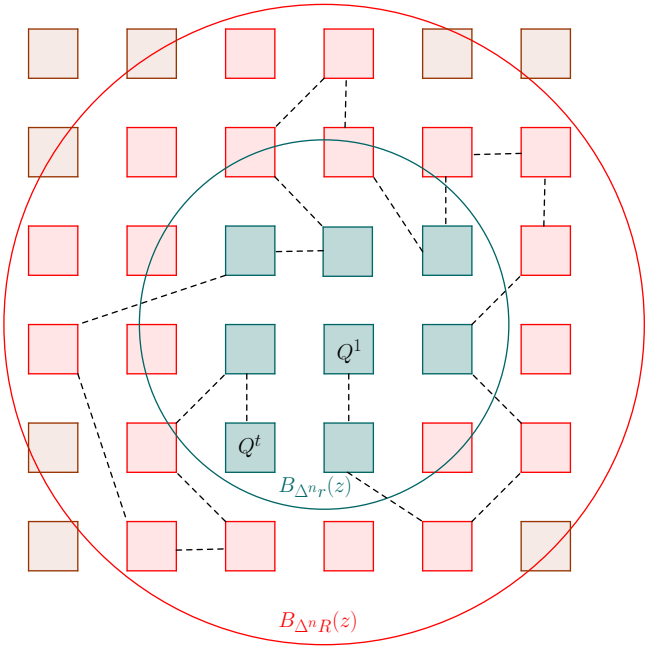

FIGURE 3 (colour online). *The path* $Q^1 - \cdots - Q^t$.

Step 2: *Construction of paths in the graph* G *for a fixed scale.* For a logarithmic scale n and $x, y \in [\Delta^{n-1}, \Delta^n)$, we construct a path in the graph G connecting x and y.

Fix a logarithmic scale n. Let $z \in \Delta^n \mathbb{Z}^d$. Choose for every block in (17) a favored cone and call the corresponding set of points in the block where this cone is associated *majority set*. Each majority set contains at least

$$a \;=\; \frac{\Delta^{d(n-1)}}{L} \;\in \mathbb{N}$$

points. Without loss of generality, we assume that every majority set contains exactly a different elements. Then, we identify a block in (17) with its majority set, that is, if Q^k is the kth block in (17), then

$$Q^k \;=\; (q_i^k)_{1 \leqslant i \leqslant a}.$$

Starting from (17), we now fix certain paths in the graph G, which then give rise to the collection (p_{xy}). Let $i \in \{1, \dots, a\}$. Without loss of generality, we assume that t is an even number (for odd t just erase the last edge in the following scheme). Let M be the set of the following paths in G:

$$\begin{array}{ccccccccccc}
q_i^1 & - & q_i^2 & - & q_i^3 & - & q_i^4 & - & \cdots & - & q_i^t \\
q_i^1 & - & q_{i+1}^2 & - & q_i^3 & - & q_{i+1}^4 & - & \cdots & - & q_{i+1}^t \\
q_i^1 & - & q_{i+2}^2 & - & q_i^3 & - & q_{i+2}^4 & - & \cdots & - & q_{i+2}^t \\
q_i^1 & - & q_{i+3}^2 & - & q_i^3 & - & q_{i+3}^4 & - & \cdots & - & q_{i+3}^t \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
q_i^1 & - & q_{i+a-1}^2 & - & q_i^3 & - & q_{i+a-1}^4 & - & \cdots & - & q_{i+a-1}^t.
\end{array} \tag{18}$$

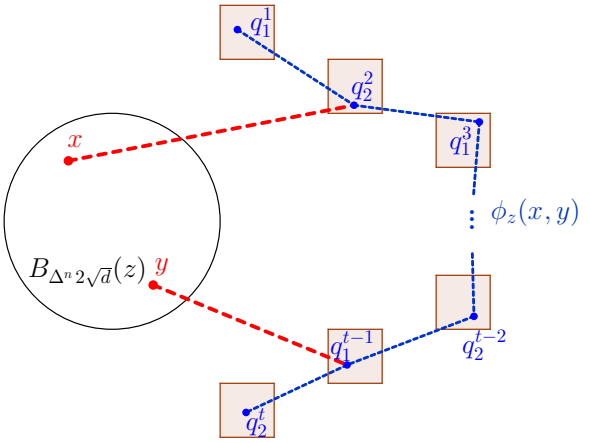

FIGURE 4 (colour online). *Construction of a path using* $\phi_z(x,y)$.

Here, the lower index is to be read modulo a, that is, $k+a=k$ for every k. Since we do this for every $i \in \{1,\dots,a\}$, the set M consists of a^2 paths. Now, we associate to every pair $(x,y) \in$

$$A \;=\; \{\,(x,y) \in B_{\Delta^n 2\sqrt{d}}(z) \times B_{\Delta^n 2\sqrt{d}}(z) \mid |x-y| \in [\Delta^{n-1},\Delta^n)\,\}$$

one path of M. Since the numbers a and $\#A$ are comparable, that is., its ratio is bounded by a number independent of n, this can be realized by a function

$$\phi_z : A \to M$$

with

$$\#\phi_z^{-1}(p) \;\leqslant\; K \quad \text{for every } p \in M$$

where $K \geqslant 1$ is independent of n and p. In order to use the path $\phi_z(x,y)$ to connect x and y, it remains to make sure that x and y are both connected in G to one element in $\phi_z(x,y)$, respectively. This follows from Lemma 5.16 which guarantees that every $x \in B_{\Delta^n 2\sqrt{d}}(z)$ is connected to every point in some block Q^k of $\mathcal{P}_z^n$. In this way, the path $\phi_z(x,y)$ induces a path in G that starts in x and ends in y (cf. Figure 4).

Using this construction scheme, we have constructed a path for each pair $(x,y) \in B_{2\sqrt{d}\Delta^n}(z) \times B_{2\sqrt{d}\Delta^n}(z)$ with $|x-y| \in [\Delta^{n-1},\Delta^n)$. Let M_z^n be the set of all these paths. This principle of construction of the paths can be carried out for every $z \in \Delta^n\mathbb{Z}^d$.

Step 3: *Construction of* p_{xy}. Note that the whole construction process of Step 2 has been performed for an arbitrary $n \in \mathbb{N}$. We define p_{xy} for $x,y \in \mathbb{Z}^d$ as follows. Choose $n \in \mathbb{N}$ such that $|x-y| \in [\Delta^{n-1},\Delta^n)$. Next, choose any $z \in \mathbb{Z}^d$ such that $\phi_z(x,y)$ represents a path connecting x and y, cf. Figure 4. In this way, $p_{xy} \in \bigcup_{z\in\Delta^n\mathbb{Z}^d} M_z^n$.

Step 4: *Bounds of the length of each edge path.* The second claim of Theorem 5.15 follows immediately from $t \leqslant \#(B_r \cap \mathbb{Z}^d) \cdot \#(B_R \cap \mathbb{Z}^d)$.

Step 5: *Bounds of the length of each edge.* By construction, all edges used in p_{xy} for some $x,y \in \mathbb{Z}^d$ with $|x-y| \in [\Delta^{n-1},\Delta^n)$ have lengths bounded from below by Δ^{n-1} and from above by $2\Delta^n R$. Ergo the fourth claim follows with $\lambda = 2R\Delta$.

Step 6: *Bounds of the multiplicity of edges.* According to step 5, it is enough to proof the third claim of Theorem 5.15 for one fixed logarithmic scale. Therefore, we fix n. Assume that e is an edge of length in $[\Delta^{n-1}, 2\Delta^n R)$. Then, there exists a point $z \in \Delta^n\mathbb{Z}^d$ so that $e \in B_{\Delta^n R}(z)$. Since the number of lattice points in B_{2R} bounds from above the number of block centers $z \in \Delta^n\mathbb{Z}^d$ for which $B_{\Delta^n R}(z)$ contains e, it is enough to bound the number of times e is used

by paths belonging to a fixed z. But now by construction (cf. step 1) for every edge in $B_{\Delta^n R}(z)$ the usage of pathst start in some point x and end in some other point y with $x, y \in B_{\Delta^n 2\sqrt{d}}(z)$ so that $|x-y| \in [\Delta^{n-1}, \Delta^n)$ is bounded by K and this number is independent of the scale. □

6. *Proof of Theorem 1.3*

We are now in the position to prove Theorem 1.3. The proof is just an easy consequence of Theorem 5.15.

Proof. Let $R > 0$ and $x_0 \in \mathbb{R}^d$. For $x, y \in B_R(x_0) \cap \mathbb{Z}^d$ denote by $(x = z_1, z_2, \dots, z_{N-1}, z_N = y)$ the path p_{xy} that satisfies properties (1)–(4) of Theorem 5.15. For simplicity, we assume here that every path in (p_{xy}) is of length N. Then, with use of the properties (1)–(4) of Theorem 5.15 and of (4) we find:

$$\begin{aligned}
&\sum_{\substack{x,y\in B_R(x_0)\cap\mathbb{Z}^d\\ |x-y|>R_0}} (f(x)-f(y))^2|x-y|^{-d-\alpha}\\
&\leqslant 2\lambda^{d+\alpha} \sum_{\substack{x,y\in B_R(x_0)\cap\mathbb{Z}^d\\ |x-y|>R_0}} \sum_{i=1}^{N-1} (f(z_{i+1})-f(z_i))^2|z_{i+1}-z_i|^{-d-\alpha}\\
&\leqslant 2\lambda^{d+\alpha} \sum_{\substack{x,y\in B_R(x_0)\cap\mathbb{Z}^d\\ |x-y|>R_0}} (N-1) \max_{i\in\{1,\dots,N-1\}} \left[(f(z_{i+1})-f(z_i))^2|z_{i+1}-z_i|^{-d-\alpha}\right]\\
&\leqslant 2\Lambda\lambda^{d+\alpha} \sum_{\substack{x,y\in B_R(x_0)\cap\mathbb{Z}^d\\ |x-y|>R_0}} (N-1) \max_{i\in\{1,\dots,N-1\}} \left[(f(z_{i+1})-f(z_i))^2\omega(z_{i+1},z_i)\right]\\
&= 2\Lambda\lambda^{d+\alpha}(N-1)M \sum_{\substack{x,y\in B_{(N-1)\lambda R}(x_0)\cap\mathbb{Z}^d\\ |x-y|>R_0}} (f(x)-f(y))^2\omega(x,y).
\end{aligned}$$

Therefore, $c = (2\Lambda\lambda^{d+\alpha}(N-1)M)^{-1}$ and $\kappa = (N-1)\lambda$. □

Appendix. *Auxiliary results*

The following lemma is a version of [7, Lemma 6.9] that matches our integral kernels. Note that [7, Lemma 6.9] is concerned with translation invariant expressions. The proof also applies to our case.

LEMMA A.1. *Let $\alpha \in (0,2)$ and $\kappa \geqslant 1$. For $B = B_R(x_0), R > 0, x_0 \in \mathbb{R}^d$, we set $B^* = B_{\kappa R}(x_0)$. Let $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a symmetric kernel that satisfies (2). Suppose that for some $c > 0$*

$$c \int_{B\times B} (f(x)-f(y))^2|x-y|^{-d-\alpha}\,\mathrm{d}(x,y) \;\leqslant\; \int_{B^*\times B^*} (f(x)-f(y))^2 k(x,y)\,\mathrm{d}(x,y)$$

for every ball $B \subset \mathbb{R}^d$ and every $f \in H_k(B^*)$. Then, for every bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ there exists a constant $\widetilde{c} = \widetilde{c}(d,\kappa,\alpha,\Omega) > 0$ such that for every $f \in H_k(\Omega)$

$$\widetilde{c}c \int_{\Omega\times\Omega} (f(x)-f(y))^2|x-y|^{-d-\alpha}\,\mathrm{d}(x,y) \;\leqslant\; \int_{\Omega\times\Omega} (f(x)-f(y))^2 k(x,y)\,\mathrm{d}(x,y).$$

The constant $\widetilde{c}$ depends on the domain Ω only up to scaling. In particular, if Ω is a ball, the constant can be chosen independently of Ω. For $0 < \alpha_0 \leqslant \alpha < 2$, the constant $\widetilde{c}$ depends on α_0 but not on α.

Proof. Let Ω be a bounded Lipschitz domain. The Whitney decomposition technique provides a family $\mathcal{B}$ of balls with the following properties.

(i) There exists a constant $c = c(d)$ such that for every $x, y \in \Omega$ with $|x - y| < c\,\mathrm{dist}(x, \partial\Omega)$, there exists a ball $B \in \mathcal{B}$ with $x, y \in B$.
(ii) For every $B \in \mathcal{B}$, $B^* \subset \Omega$.
(iii) The family $\{B^*\}_{B\in\mathcal{B}}$ has the finite overlapping property, that is, each point of Ω belongs to at most $M = M(d)$ balls B^*.

Thus, for each $f \in H_k(\Omega)$,

$$\begin{aligned}
&\int_{\Omega\times\Omega} (f(x) - f(y))^2 k(x, y)\,\mathrm{d}(x, y)\\
&\geqslant \frac{1}{M^2} \sum_{B\in\mathcal{B}} \int_{B^*\times B^*} (f(x) - f(y))^2 k(x, y)\,\mathrm{d}(x, y)\\
&\geqslant \frac{c}{M^2} \sum_{B\in\mathcal{B}} \int_{B\times B} (f(x) - f(y))^2 |x - y|^{-d-\alpha}\,\mathrm{d}(x, y)\\
&\geqslant \frac{c\widetilde{c}}{M^2} \int_{\Omega\times\Omega} (f(x) - f(y))^2 |x - y|^{-d-\alpha}\,\mathrm{d}(x, y), \qquad \text{(A.1)}
\end{aligned}$$

where we applied inequality (13) in proof of [**6**, Theorem 1] to derive the last inequality, see also [**11**, Theorem 1.6]. For a scaled version of Ω, we can scale all balls in the family $\mathcal{B}$ by the same factor and arrive at the same constant $\widetilde{c}$. The constant stays bounded when $\alpha \in [\alpha_0, 2)$ for $\alpha_0 > 0$. □

The next lemma follows from Lebesgue's differentiation theorem.

LEMMA A.2. *Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be locally integrable. The following holds for almost every $s \in \mathbb{R}^d$. If $(x_h)_{h>0}$ is a sequence in $h\mathbb{Z}^d$ such that $s \in \widetilde{A}_h(x_h)$ for every $h > 0$, then,*

$$\frac{1}{\lambda_d(A_h(x_h))} \int_{A_h(x_h)} \varphi(t)\,\mathrm{d}t \overset{h\to 0}{\longrightarrow} \varphi(s).$$

Proof. The cube $\widetilde{A}_h(x_h)$ is contained in the ball $B_{2h\sqrt{d}}(s)$ and we know $\lambda_d(\widetilde{A}_h(x_h)) = c\lambda_d(B_{2h\sqrt{d}}(s))$ for a constant c only depending on the dimension d. Thus, it follows by Lebesgue's differentiation theorem (cf [**15**, Theorem 1.4, Corollary 1.7, Chapter 3]) for almost every $s \in \mathbb{R}^d$

$$\begin{aligned}
&\frac{1}{\lambda_d(A_h(x_h))} \int_{A_h(x_h)} |\varphi(t) - \varphi(s)|\,\mathrm{d}s\\
&\leqslant \frac{1}{\lambda_d(B_{2h\sqrt{d}}(s))} \int_{B_{2h\sqrt{d}}(s)} |\varphi(t) - \varphi(s)|\,\mathrm{d}s \overset{h\to 0}{\longrightarrow} 0.
\end{aligned}$$

This implies our claim. □

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