Stochastic Heat Equations with Values in a Manifold via Dirichlet Forms*

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Abstract

In this paper, we prove the existence of martingale solutions to the stochastic heat equation taking values in a Riemannian manifold, which admits Wiener (Brownian bridge) measure on the Riemannian path (loop) space as an invariant measure using a suitable Dirichlet form. Using the Andersson-Driver approximation, we heuristically derive a form of the equation solved by the process given by the Dirichlet form.

Moreover, we establish the log-Sobolev inequality for the Dirichlet form in the path space. In addition, some characterizations for the lower bound of the Ricci curvature are presented related to the stochastic heat equation.

Keywords: Stochastic heat equation; Ricci curvature; Functional inequality; Quasi-regular Dirichlet form

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1 Introduction

This work is motivated by Tadahisa Funaki's pioneering work [26] and Martin Hairer's recent work [35]. The former had proved the existence and uniqueness of a natural evolution driven by regular noise on loop space over a Riemannian manifold (M, g) by the classical theory of stochastic differential equation, and the latter considered the singular noise case, i.e. the associated stochastic heat equation may be interpreted formally as

$$\dot{u} = -\nabla E(u) + \sum_{i=1}^{m} \sigma_i(u)\xi_i \tag{1.1}$$

for solutions $u(t, \cdot) : S^1 \to M$, i.e. the formal Langevin dynamics associated to the energy given by

$$E(u) = \int_{S^1} g_{u(s)}(\partial_s u(s), \partial_s u(s))\mathrm{d}s,$$

and $\sum_{i=1}^{m} \sigma_i(u)\xi_i$ is a suitable "white noise on loop space". By Andersson-Driver's work in [2], we know that there exists an explicit relation between the Langevin energy $E(u)$ and the Wiener (Brownian bridge) measure (see also [42, 52, 61]). In [2], the Wiener (Brownian bridge) measure μ has been interpreted as the limit of a natural approximation of the measure $\exp(-\frac{1}{2}E(u))\mathscr{D}u$, where $\mathscr{D}u$ denotes a 'Lebesgue' like measure on path space. By observing the above connection, one may think the solution to the stochastic heat equation (1.1) may have μ as an invariant (even symmetrizing) measure. In this paper, starting from the invariant measure μ we use the theory of Dirichlet forms to construct a natural evolution which admits μ as an invariant measure.

Actually, on a Riemannian path/loop space there exists another process which also admits μ as an invariant measure associated with the Dirichlet form $\mathscr{E}^{OU}$ given by the Malliavin gradient, which sometimes is called the **O-U Dirichlet process**. These processes had first been constructed by Driver and Röckner in [19] for pinned loop space and by Albeverio, Léandre and Röckner in [4] for the free loop space by using Dirichlet form theory. After that there were several follow-up papers concentrating on more general cases, see [23, 43, 57, 16]. In addition, Norris in [48, 47] obtained some similar processes based on the theory of stochastic differential equations.

Recently, Hairer [35] wrote equation (1.1) in local coordinates heuristically as

$$\partial_t u^\alpha = \partial_s^2 u^\alpha + \Gamma^\alpha_{\beta\gamma}(u)\partial_s u^\beta \partial_s u^\gamma + \sigma_i^\alpha(u)\xi_i, \tag{1.2}$$

where Einstein summation convention over repeated indices is applied and $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols for the Levi-Civita connection of (M, g). σ_i are some suitable vector fields on M. This equation may be considered as a certain kind of multi-component version of the KPZ equation. By the theory of regularity structures recently developed in [34, 8, 11], local well-posedness of (1.2) has been obtained in [35].

When $M=\mathbb{R}^d$, the process constructed by Driver and Röckner in [19] is the O-U process in the Mallivan calculus, whereas equation (1.2) corresponds to the stochastic heat equation. To explain the difference of the above two processes, let us first consider the following two stochastic equations on $L^2([0,1];\mathbb{R}^d)$:

A. Ornstein-Unlenbeck process

$$\mathrm{d}X(t)=\frac{1}{2}X(t)\mathrm{d}t+(-\Delta_D)^{-\frac{1}{2}}\mathrm{d}W(t),$$

B. Stochastic heat equation

$$\mathrm{d}X(t)=\frac{1}{2}\Delta_D X(t)\mathrm{d}t+\mathrm{d}W(t),$$

where Δ_D is the operator $\frac{\mathrm{d}^2}{\mathrm{d}s^2}$ on $L^2([0,1];\mathbb{R}^d)$ with boundary condition $h(0)=0, h(1)=0$ and W is an L^2-cylindrical Wiener process. Solutions to these two equations share the same Gaussian invariant measure $N(0,(-\Delta_D)^{-1})$ in $L^2([0,1];\mathbb{R}^d)$. It is well known that $N(0,(-\Delta_D)^{-1})$ has full (topological) support on $C([0,1];\mathbb{R}^d)$ and is the same as the law of the Brownian bridge on $C([0,1];\mathbb{R}^d)$ starting from 0.

In the first part of this paper, we construct the solutions to the stochastic heat equation taking values in a Riemannian manifold M by Dirichlet form theory (i.e. we replace in Case **B** above $\mathbb{R}^d$ by M). Unlike as in [19], we consider the closure $(\mathscr{E},\mathscr{D}(\mathscr{E}))$ of the following bilinear form with the reference measure μ = the law of Brownian motion on M (path space case)/the law of Brownian bridge on M (loop space case):

$$\mathscr{E}(F,G):=\frac{1}{2}\int_{\mathbf{E}}\langle DF,DG\rangle_{\mathbf{H}}\mathrm{d}\mu=\frac{1}{2}\sum_{k=1}^{\infty}\int_{\mathbf{E}}D_{h_k}FD_{h_k}G\mathrm{d}\mu;\quad F,G\in\mathscr{F}C_b^1,$$

where $\mathscr{F}C_b^1$ is introduced in (2.2) below, $\mathbf{H}:=L^2([0,1];\mathbb{R}^d)$, $\mathbf{E}$ is introduced in Section 2.1 and DF is the L^2-derivative defined in Section 2 with $\{h_k\}$ being an orthonormal basis in $\mathbf{H}$. In this case, we call the associated Dirichlet form **$\mathbf{L^2}$-Dirichlet form**. When $M=\mathbb{R}^d$, this Dirichlet form just corresponds to the stochastic heat equation, i.e. Case **B** above, while in [19] the considered Dirichlet form corresponds to the process from Case A above, if $M=\mathbb{R}^d$. Therefore, below we denote the Dirichlet form in [19] by $\mathscr{E}^{OU}$. By simple computations, one sees that the classical cylinder test functions $u(\gamma)=f(\gamma_t)$ considered in [19] are not in the domain of $\mathscr{D}(\mathscr{E})$, since $\mathscr{E}(u,u)$ might be infinity. Thus, we need to choose a class of suitable functions $\mathscr{F}C_b^1$ introduced in (2.2) below. As usual in Dirichlet from theory to obtain the corresponding Markov process, i.e. in our case the solution to the stochastic heat equation in the path/loop space over M, we have to prove **a) closability** and **b) quasi-regularity**, which is done in Section 2 below (see Theorem 2.2). To prove the quasi-regularity of the closure of $\mathscr{E}$, the uniform distance will be replaced by L^2-distance mentioned in Subsection 2.1.

Then we obtain martingale solutions to the stochastic heat equations, which admit μ as an invariant measure on path space and loop space, respectively.

We would like to emphasize that both the Dirichlet form approach and regularity structure theory can be used to construct rigorously the natural Markov process associated with (1.2) (see Section 4). The Markov process constructed by the Dirichlet form approach is a global martingale solution starting from quasi-every starting point (which is a path/loop), which has the law of the Brownian motion resp. Brownian bridge on M as an invariant measure. The solution obtained by regularity structure in [35] is a local strong solution to (1.2) starting from every point (=path/loop).

In this paper we consider four different cases: pinned path resp. loop spaces and free path resp. loop spaces. For a better understanding of the measure and the stochastic heat equation on these spaces, let us first look at the simplest case : $M = \mathbb{R}^d$. For the case of the path space, the reference measure is $P_x := N(x, (-\Delta_{D,N})^{-1})$, which is the law of Brownian motion starting from a fixed point $x \in \mathbb{R}^d$, where $\Delta_{D,N}$ is the operator $\frac{\mathrm{d}^2}{\mathrm{d}s^2}$ on $L^2([0,1];\mathbb{R}^d)$ with boundary condition $h(0) = 0, h'(1) = 0$, and the corresponding SPDE constructed by the L^2-Dirichlet form $\mathscr{E}$ above is the following:

$$\mathrm{d}X(t) = \frac{1}{2}\Delta_{D,N}(X(t) - x)\mathrm{d}t + \mathrm{d}W(t), \tag{1.3}$$

with W an $L^2([0,1];\mathbb{R}^d)$-cylindrical Wiener process. On the loop space, the reference measure is $P_{x,x} = N(x, (-\Delta_D)^{-1})$, which is the law of Brownian bridge starting from $x \in \mathbb{R}^d$, then the corresponding SPDE is:

$$\mathrm{d}X(t) = \frac{1}{2}\Delta_D(X(t) - x)\mathrm{d}t + \mathrm{d}W(t),$$

where Δ_D is the operator $\frac{\mathrm{d}^2}{\mathrm{d}s^2}$ on $L^2([0,1];\mathbb{R}^d)$ with boundary condition $h(0) = 0, h(1) = 0$. For the case of free path/loop space, we have the following: Let σ be a probability measure on $\mathbb{R}^d$. Then the reference measure for the free path case is given by $\int P_x\sigma(\mathrm{d}x) = \int N(x, (-\Delta_{D,N})^{-1})\sigma(\mathrm{d}x)$ and the corresponding SPDE is:

$$\mathrm{d}X(t,x) = \frac{1}{2}\Delta_{D,N}(X(t,x) - X(t,0))\mathrm{d}t + \mathrm{d}W(t).$$

Similarly, the reference measure for the free loop case is given by $\int P_{x,x}\sigma(\mathrm{d}x) = \int N(x, (-\Delta_D)^{-1})\sigma(\mathrm{d}x)$ and the corresponding SPDE is:

$$\mathrm{d}X(t,x) = \frac{1}{2}\Delta_D(X(t,x) - X(t,0))\mathrm{d}t + \mathrm{d}W(t).$$

In the second part of this paper, we use functional inequalities to study the properties of the solutions to the stochastic heat equations on path space over a Riemannian manifold M. Functional inequalities for Ornstein-Unlenbeck process on Riemannian path space have been well-studied (see [24, 3, 4, 22, 3, 38, 39, 53, 16] and references therein). Since the L^2-Dirichlet form associated with the stochastic heat equation is larger than the O-U Dirichlet form $\mathscr{E}^{OU}$ constructed in [19] (i.e., $\mathscr{E}^{OU}(u,u) \le \mathscr{E}(u,u)$ for $u \in D(\mathscr{E})$), all the functional inequalities with respect to $\mathscr{E}^{OU}$ still hold in the stochastic heat equation case, which implies that the former requires stronger Ricci curvature conditions than the latter. In fact, from recent results in [46] by Naber, we know that the Poincare inequality/log-Sobolev inequality for the twisted O-U Dirichlet form requires the uniformly bounded Ricci curvature. And for the L^2-Dirichlet form, it only needs lower bounded Ricci curvature, which had already been proved before by Gourcy-Wu in [33]. In this section we also establish the log-Sobolev inequality for $\mathscr{E}$, but our constant $C(K)$ is smaller than Gourcy-Wu's constant (see Theorem 3.1 below). In particular, when M is an Einstein manifold with constant Ricci curvature $K \in \mathbb{R}$, our constant $C(K)$ in the log-Sobolev inequality is optimal in the sense that $\lim_{K\to 0} C(K) = \frac{4}{\pi^2}$ and $\frac{4}{\pi^2}$ is the optimal constant for the log-Sobolev inequality in the flat case (see Theorem 3.3 below). Here we want to emphasize that the log-Sobolev inequality implies the L^2-ergodicity of the solution to the stochastic heat equation (see Remark 3.2 below for other consequences).

As mentioned above, the log-Sobolev inequality is a consequence of a geometric property of the manifold. It is very interesting to ask to what extent these geometric properties are also necessary for the log-Sobolev inequality to hold for $\mathscr{E}$ as above. The most interesting work is related to the Bakry-Emery criterion, which gives a characterization of the lower boundedness of the Ricci curvature in terms of the log-Sobolev inequality for the classical Dirichlet form on a Riemannian manifold (see [6]). Recently, Naber in [46] characterizes uniform boundedness of the Ricci curvature using the O-U process on path space. Wang-Wu[59] obtained a more general characterization of the Ricci curvature and the second fundamental form on the boundary of the Riemannian manifold using a new method. After that, this result has been extended to general uniform bounds of the Ricci curvature by Wu [60] and Cheng-Thalimaier [17]. In addition, Wu [60] and Wang [56] gave some characterization for the upper bound of the Ricci curvature by analysis on path space and the Weitzenböck-Bochner integration formula respectively. Similar to the above case, in Subsection 3.2 we give some (equivalent) characterizations of the lower boundedness of the Ricci curvature by using the L^2-gradient and a properly weighted decomposition of the L^2-Dirichlet form on path space.

In the last part of the paper, we discuss the form of the stochastic heat equations constructed in Section 2. Dirichlet form theory is a useful tool to construct stochastic processes on infinite dimensional spaces (see [5, 44]). In the flat case we can use Dirichlet form theory to write an SPDE which the process satisfies (see [5] for the $p(\Phi)_2$ model). This helps to obtain new properties of the Φ^4_2 field (see [50, 51]). However, in the Riemannian manifold case, the explicit form of the SPDE cannot be deduced directly since there are no linear functions on the Riemannian manifold. It will be seen from Section 2 that the martingale part is space-time white noise and thus is very rough. To define the drift part, renormalization is required (see [34]). In Section 4 we construct suitable approximation processes on the piecewise geodesic space using the approximation measures from [2] and discuss the convergence of the approximations, which gives two limiting forms of the stochastic heat equations. One is more related to the Mosco convergence of the approximating Dirichlet forms to the L^2-Dirichlet form $\mathscr{E}$ and the integration by parts formula obtained by Driver (see (4.30) and Remark 4.10 below). The other is more related to (1.2) constructed by Hairer (see (4.29) below). The different forms of the equations originate from the different choices of the diffusion coefficients. We conjecture that the Markov process constructed in Section 2 by the Dirichlet form $\mathscr{E}$ has the same law as the solution to (1.2) constructed by Hairer in [35]. To obtain the Mosco convergence of the corresponding Dirichlet forms in general Markov uniqueness of the limiting L^2-Dirichlet form is required, which is a difficult problem in Dirichlet form theory. We hope to be able to use the theory of regularity structures/ paracontrolled distribution method to make the heuristic convergence of the corresponding solutions in Section 4 rigorous in our future work.

This paper is organized as follows: In Section 2 we construct the L^2-Dirichlet form $\mathscr{E}$ on the pinned (free) path/loop space. By this we obtain existence of martingale solutions to the stochastic heat equation on path space and loop space. In Section 3.1, we derive functional inequalities for the L^2-Dirichlet form $\mathscr{E}$. The equivalent characterizations of the lower boundedness of the Ricci curvature are obtained in Section 3.2. In Section 4.2, we construct approximation processes on the piecewise geodesic space by considering Dirichlet forms with respect to the approximation measure. In Section 4.3 we discuss the convergence of the approximation processes and the form of the limiting stochastic equation heuristically.

2 Construction of Dirichlet form

2.1 Dirichlet form on pinned path Space

Throughout this article, suppose that M is a complete and stochastic complete Riemannian manifold with dimension d, and ρ is the Riemannian distance on M. In this section we assume that M is compact for simplicity and for the more general case, we refer to Remark 2.1. Fix $o \in M$ and $T > 0$, the based path space $W_o^T(M)$ over M is defined by

$$W_o^T(M) := \{\gamma \in C([0,T]; M) : \gamma(0) = o\}.$$

Then $W_o^T(M)$ is a Polish space under the uniform distance

$$d_\infty(\gamma, \sigma) := \sup_{t\in[0,T]} \rho(\gamma(t), \sigma(t)), \quad \gamma, \sigma \in W_o^T(M).$$

For convenience, we write $W_o(M) := W_o^1(M)$. In the following we consider $W_o(M)$ for simplicity. In order to construct Dirichlet forms associated to stochastic heat equations on Riemannian path space, we first need to introduce the following L^2-distance, which is a smaller distance than the above uniform distance d_∞ on $W_o(M)$:

$$\tilde{d}(\gamma,\eta)^2 := \int_0^1 \rho(\gamma_s,\eta_s)^2 \mathrm{d}s, \quad \gamma,\eta \in W_o(M).$$

The L^2-distance $\tilde{d}$ is quite crucial to prove the quasi-regularity for the Dirichlet form mentioned in Theorem 2.2. Let $\mathbf{E}$ be the closure of $W_o(M)$ in

$$\left\{\eta : [0,T] \to M; \int_0^1 \rho(o,\eta_s)^2 \mathrm{d}s < \infty\right\}$$

with respect to the distance $\tilde{d}$, then $\mathbf{E}$ is a Polish space.

Before stating our main results in this section, let us recall some basic notation and introduce the Brownian motion on M. Let ∇ be the Riemannian connection on M and the curvature tensor R of ∇ is given by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_X\nabla_Y Z - \nabla_{[X,Y]}Z$$

for all vector fields X, Y and Z on M. The Ricci tensor Ric and the scalar curvature Scal of M are traces of R and Ric respectively, i.e.,

$$\mathrm{Ric}X := \sum_{i=1}^d R(X,\bar{e}_i)\bar{e}_i, \quad \mathrm{Scal} = \sum_{i=1}^d \langle \mathrm{Ric}\bar{e}_i, \bar{e}_i\rangle,$$

where $\{\bar{e}_i\}$ is an orthonormal frame.

Let $O(M)$ be the orthonormal frame bundle over M, and let $\pi : O(M) \to M$ be the canonical projection. Furthermore, we choose a standard othornormal basis $\{H_i\}_{i=1}^d$ of horizontal vector fields on $O(M)$ and consider the following SDE:

$$\begin{cases} \mathrm{d}U_t = \sum_{i=1}^d H_i(U_t)\circ \mathrm{d}B_t^i, \quad t \geq 0 \\ U_0 = u_o, \end{cases} \tag{2.1}$$

where u_o is a fixed orthonormal basis of T_oM and $B_t^1, \cdots, B_t^d$ are independent Brownian motions on $\mathbb{R}$. Then $x_t := \pi(U_t)$, $t \geq 0$, is the Brownian motion on M with initial point o, and $U_\cdot$ is the (stochastic) horizontal lift along $x_\cdot$. Let μ be the distribution of $x_{[0,1]} := \{x(t)|t \in [0,1]\}$. Then μ is a probability measure on $W_o(M)$.

In the following we use $\langle\cdot,\cdot\rangle$ to denote the inner product in $\mathbb{R}^d$.

Let $\mathscr{F}C_b^1$ be a space of C_b^1 cylinder functions on $\mathbf{E}$, defined as follows: for every $F \in \mathscr{F}C_b^1$, there exist some $m \geq 1$, $m \in \mathbb{N}$, $f \in C_b^1(\mathbb{R}^m), g_i \in C_b^{0,1}([0,1]\times M)$, $i = 1, ..., m$, such that

$$F(\gamma) = f\left(\int_0^1 g_1(s,\gamma_s)\mathrm{d}s, \int_0^1 g_2(s,\gamma_s)\mathrm{d}s, ..., \int_0^1 g_m(s,\gamma_s)\mathrm{d}s\right), \quad \gamma \in \mathbf{E}. \tag{2.2}$$

Here $C_b^{0,1}([0,1]\times M)$ denotes the functions which are continuous w.r.t. the first variable and C^1- differentiable w.r.t. the second variable with continuous derivatives. It is easy to see that $\mathscr{F}C_b^1$ is dense in $L^2(\mathbf{E},\mu)$. For any $F\in\mathscr{F}C_b^1$ of the form (2.2) and $h\in\mathbf{H}:=L^2([0,1];\mathbb{R}^d)$, the directional derivative of F with respect to h is given by

$$D_hF(\gamma)=\sum_{j=1}^m\hat{\partial}_jf(\gamma)\int_0^1\left\langle U_s^{-1}(\gamma)\nabla g_j(s,\gamma_s),h_s\right\rangle\mathrm{d}s,\quad \gamma\in W_o(M),$$

where

$$\hat{\partial}_jf(\gamma):=\partial_jf\bigg(\int_0^1 g_1(s,\gamma_s)\mathrm{d}s,\int_0^1 g_2(s,\gamma_s)\mathrm{d}s,...,\int_0^1 g_m(s,\gamma_s)\mathrm{d}s\bigg).$$

and ∇g_j denotes the gradient w.r.t. the second variable. Without loss of generality, for $\gamma\in\mathbf{E}\backslash W_o(M)$ we take $D_hF(\gamma)=0$. By the Riesz representation theorem, there exists a gradient operator $DF(\gamma)\in\mathbf{H}$ such that $\langle DF(\gamma),h\rangle_{\mathbf{H}}=D_hF(\gamma),\gamma\in\mathbf{E},h\in\mathbf{H}$. In particular, for $\gamma\in W_o(M)$,

$$DF(\gamma)(s)=\sum_{j=1}^m\hat{\partial}_jf(\gamma)U_s^{-1}(\gamma)\nabla g_j(s,\gamma_s). \tag{2.3}$$

Remark 2.1. *In fact, for a more general Riemannian manifold the main results in this section still hold. But when we prove the quasi-regularity of $\mathscr{E}$, it is required that the function g is allowed to be the distance function, which can be approximated by C^1-functions in a suitable way. This will be considered in a forthcoming paper.*

Let $\mathbb{H}$ denote the Cameron-Martin space:

$$\mathbb{H}:=\left\{h\in C^1([0,1];\mathbb{R}^d)\bigg|h(0)=0,\|h\|_{\mathbb{H}}^2:=\int_0^1|h'(s)|^2\mathrm{d}s<\infty\right\}.$$

Taking a sequence of elements $\{h_k\}\subset\mathbb{H}$ such that it is an orthonormal basis in $\mathbf{H}$, consider the following symmetric quadratic form

$$\mathscr{E}(F,G):=\frac{1}{2}\int_{\mathbf{E}}\langle DF,DG\rangle_{\mathbf{H}}\mathrm{d}\mu=\frac{1}{2}\sum_{k=1}^\infty\int_{\mathbf{E}}D_{h_k}FD_{h_k}G\mathrm{d}\mu;\quad F,G\in\mathscr{F}C_b^1.$$

The following is the main results in this section.

Theorem 2.2. *The quadratic form $(\mathscr{E},\mathscr{F}C_b^1)$ is closable and its closure $(\mathscr{E},\mathscr{D}(\mathscr{E}))$ is a quasi-regular Dirichlet form on $L^2(\mathbf{E};\mu)=L^2(W_o(M);\mu)$.*

The proof of Theorem 2.2 will be given at the end of this subsection. Using the theory of Dirichlet forms (see [44]), we obtain the following associated diffusion process.

Theorem 2.3. *There exists a conservative (Markov) diffusion process* $M = (\Omega, \mathscr{F}, (\mathscr{M}_t), (X_t)_{t\geq 0}, (\mathbf{P}^z)_{z\in\mathbf{E}})$ *on* $\mathbf{E}$ properly associated with $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$, *i.e. for* $u \in L^2(\mathbf{E};\mu) \cap \mathscr{B}_b(\mathbf{E})$, *the transition semigroup* $P_t u(z) := \mathbf{E}^z[u(X_t)]$ *is an* $\mathscr{E}$*-quasi-continuous version of* $T_t u$ *for all* $t > 0$, *where* T_t *is the semigroup associated with* $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$.

Here $\mathscr{B}_b(\mathbf{E})$ denotes the set of the bounded Borel-measurable functions and for the notion of $\mathscr{E}$-quasi-continuity we refer to [44, ChapterIII, Definition 3.2]. Moreover, by the Fukushima decomposition we have:

Theorem 2.4. *There exists a* properly $\mathscr{E}$-exceptional set $S \subset \mathbf{E}$, *i.e.* $\mu(S) = 0$ *and* $\mathbf{P}^z[X_t \in \mathbf{E} \setminus S, \forall t \geq 0] = 1$ *for* $z \in E\backslash S$, *such that* $\forall z \in \mathbf{E}\backslash S$ *under* $\mathbf{P}^z$, *the sample paths of the associated process* $M = (\Omega, \mathscr{F}, (\mathscr{M}_t), (X_t)_{t\geq 0}, (\mathbf{P}^z)_{z\in\mathbf{E}})$ *on* $\mathbf{E}$ *satisfy the following for* $u \in \mathscr{D}(\mathscr{E})$

$$u(X_t) - u(X_0) = M_t^u + N_t^u \quad \mathbf{P}^z - a.s., \tag{2.4}$$

where M^u *is a martingale with quadratic variation process given by* $\int_0^t |Du(X_s)|^2_{\mathbf{H}} \mathrm{d}s$ *and* N_t *is a zero quadratic variation process. In particular, for* $u \in D(L)$, $N_t^u = \int_0^t Lu(X_s)ds$, *where* L *is the generator of* $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$.

Remark 2.5. *i) If we choose* $u(\gamma) = \int_{r_1}^{r_2} u^\alpha(\gamma_s)ds \in \mathscr{F}C_b^1$, $0 \leq r_1 < r_2 \leq 1$, *with* u^α *being local coordinates on* M, *then the quadratic variation process for* M^u *is the same as that for the martingale part in (1.2) (see Remark 4.10).*

ii) Although the stochastic horizontal lift $(U_t(\gamma))_{t\in[0,\infty)}$ *is applied in the definition of* $(\mathscr{E}, \mathscr{F}C_b^1)$, *the value of* $\mathscr{E}(F,F)$ *is independent of* $(U_t(\gamma))_{t\in[0,\infty)}$. *In particular, by the definition* (2.3) *of the gradient, we have*

$$\mathscr{E}(F,G) = \frac{1}{2}\int \sum_{i=1}^{m}\sum_{j=1}^{l} \hat{\partial}_i f_1(\gamma)\hat{\partial}_j f_2(\gamma) \int_0^1 \langle \nabla g_i^1(s,\gamma_s), \nabla g_j^2(s,\gamma_s)\rangle \mathrm{d}s\mathrm{d}\mu$$

for any $F, G \in \mathscr{F}C_b^1$ *with*

$$F(\gamma) = f_1\left(\int_0^1 g_1^1(s,\gamma(s))\mathrm{d}s, \int_0^1 g_2^1(s,\gamma(s))\mathrm{d}s, ..., \int_0^1 g_m^1(s,\gamma(s))\mathrm{d}s\right)$$
$$G(\gamma) = f_2\left(\int_0^1 g_1^2(s,\gamma(s))\mathrm{d}s, \int_0^1 g_2^2(s,\gamma(s))\mathrm{d}s, ..., \int_0^1 g_l^2(s,\gamma(s))\mathrm{d}s\right), \quad \gamma \in \mathbf{E}.$$

This implies the quadratic form $\mathscr{E}$ *is independent of* $(U_t(\gamma))_{t\in[0,\infty)}$. *This is different from the O-U Dirichlet form, since the latter depends on the parallel translation* $(U_t(\gamma))_{t\in[0,\infty)}$.

iii) In Section 4 we have another reference measure $\mu_0 := e^{-\frac{1}{6}\int_0^1 \mathrm{Scal}(\gamma(s))\mathrm{d}s}\mathrm{d}\mu(\gamma)$, *which is also related to* $\exp(-\frac{1}{2}E(u))\mathscr{D}u$ *mentioned in introduction. We can also construct the* L^2*-Dirichlet form* $(\mathscr{E}^0, \mathscr{D}(\mathscr{E}^0))$ *with respect to* μ_0 *and the results in Theorems 2.2-2.4 still hold in this case.*

Proof of Theorem 2.2. (a) **Closablity:** By the integration by parts formula (refer to [20], also see [36, 37]): for $h \in \mathbb{H}$,

$$\int D_h F \mathrm{d}\mu = \int F \beta_h \mathrm{d}\mu \tag{2.5}$$

for every cylinder function depending on finite times $F(\gamma) = f(\gamma_{t_1}, ..., \gamma_{t_m})$, where $f \in C_b^1(M^m)$ and $t_i \in [0,1], i = 1, ..., m$,

$$L^2(\mathbf{E}, \mu) \ni \beta_h := \int_0^1 \left\langle h_s' + \frac{1}{2}\mathrm{Ric}_{U_s}(h_s), \mathrm{d}B_s \right\rangle,$$

where

$$\langle \mathrm{Ric}_{U_s}(a_1), a_2 \rangle := \langle \mathrm{Ric}(U_s a_1), U_s a_2 \rangle_{T_{\gamma_s}M}, \quad a_1, a_2 \in \mathbb{R}^d.$$

For each $F(\gamma) = f(\int_0^1 g_1(s, \gamma_s)\mathrm{d}s, \cdots, \int_0^1 g_m(s, \gamma_s)\mathrm{d}s) \in \mathscr{F}C_b^1$, choose

$$F_n = f\left(\frac{1}{n}\sum_{i=1}^n g_1(i/n, \gamma_{i/n}), \cdots, \frac{1}{n}\sum_{i=1}^n g_m(i/n, \gamma_{i/n})\right).$$

Then F_n and $D_h F_n$ L^2-converge to F and $D_h F$ respectively. Thus, we deduce that (2.5) holds for $F \in \mathscr{F}C_b^1$.

Since $\beta_h \in L^2(\mathbf{E}, \mu)$, it is standard to prove that $(\mathscr{E}, \mathscr{F}C_b^1)$ is closable (see [19] or [43, 57, 16]). For the completeness of the proof we write it in detail. Let $\{F_n\}_{n=1}^\infty \subseteq \mathscr{F}C_b^1$ be a sequence of cylinder functions with

$$\lim_{n\to\infty} \mu\left[F_n^2\right] = 0, \quad \lim_{n,m\to\infty} \mathscr{E}\left(F_n - F_m, F_n - F_m\right) = 0. \tag{2.6}$$

Thus $\{DF_n\}_{n=1}^\infty$ is a Cauchy sequence in $L^2(\mathbf{E} \to \mathbf{H}; \mu)$ for which there exists a limit Φ. It suffices to prove that $\Phi = 0$. By (2.5), for $G \in \mathscr{F}C_b^1$ and $k \ge 1$, we have

$$\begin{aligned}
&\mu\left[\langle DF_n, h_k\rangle_{\mathbf{H}} G\right] \\
&= \mu\left[\langle D\left(F_n G\right), h_k\rangle_{\mathbf{H}}\right] - \mu\left[\langle DG, h_k\rangle_{\mathbf{H}} F_n\right] \\
&= \mu\left[F_n G \int_0^1 \left\langle h_k'(s) + \frac{1}{2}\mathrm{Ric}_{U_s} h_k(s), \mathrm{d}B_s \right\rangle\right] - \mu\left[\langle DG, h_k\rangle_{\mathbf{H}} F_n\right].
\end{aligned} \tag{2.7}$$

Since G and DG are bounded and $\int_0^1 \left\langle h_k'(s) + \frac{1}{2}\mathrm{Ric}_{U_s} h_k(s), \mathrm{d}B_s\right\rangle \in L^2(\mathbf{E}; \mu)$, F_n converges to 0 in $L^2(\mu)$, we may take the limit $n \to \infty$ under the integral in (2.7) and conclude

$$\mu\left[\langle \Phi, h_k\rangle_{\mathbf{H}} G\right] = 0, \quad \forall\, G \in \mathscr{F}C_b^1,\ k \ge 1,$$

which implies that $\Phi = 0$, a.s., and that $(\mathscr{E}, \mathscr{F}C_b^1)$ is closable. By standard methods, we show easily that its closure $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is a Dirichlet form.

(b) **Quasi-Regularity:** By the Nash embedding theorem we may assume that M is embedded isometrically into $\mathbb{R}^N$ for a large enough $N \in \mathbb{R}$:

$$\psi : p \mapsto \psi(p) = (\psi^1(p), ..., \psi^N(p)) \in \mathbb{R}^N.$$

Then the distance $\rho(p,q)$ is equivalent to $\rho_0(p,q) := |\psi(p) - \psi(q)|$ for $p, q \in M$ and ψ is smooth on M, which implies that the two distances $\tilde{d}(\gamma,\eta)^2$ and $\bar{d}(\gamma,\eta)^2 := \sum_{i=1}^N \int_0^1 (\psi^i(\gamma(s)) - \psi^i(\eta(s)))^2 \mathrm{d}s$ on the path space $\mathbf{E}$ are equivalent to each other. Since $\mathbf{E}$ is separable we can choose a fixed countable dense set $\{\xi_m | m \in \mathbb{N}\} \subset W_o(M)$ in $\mathbf{E}$. We first prove the tightness of the capacity for $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$: Let $\varphi \in C_b^\infty(\mathbb{R})$ be an increasing function satisfying

$$\varphi(t) = t, \quad \forall\, t \in [-1,1] \text{ and } \|\varphi'\|_\infty \le 1.$$

And for $m \in \mathbb{N}$, the function $v_m : \mathbf{E} \to \mathbb{R}$ is given by

$$v_m(\gamma) = \varphi(\bar{d}(\gamma, \xi_m)^2), \quad \gamma \in \mathbf{E}.$$

Suppose we can show that

$$w_n := \inf_{m \le n} v_m, n \in \mathbb{N}, \text{ converges } \mathscr{E} - \text{quasi-uniformly to zero on } \mathbf{E}, \tag{2.8}$$

then for every $k \in \mathbb{N}$ there exists a closed set F_k such that $\mathrm{Cap}(F_k^c) < \frac{1}{k}$ and $w_n \to 0$ uniformly on F_k. For every $0 < \epsilon < 1$ there exists $n \in \mathbb{N}$ such that $w_n < \epsilon$ on F_k, which implies that F_k is totally bounded, hence compact and the capacity of $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is tight. In the following we show (2.8): we fix $m \in \mathbb{N}$, consider $v_m \in D(\mathscr{E})$ and

$$D_{h_k} v_m(\gamma) = \varphi'(\bar{d}(\gamma,\xi_m)^2) \sum_{i=1}^N 2 \int_0^1 (\psi^i(\gamma(s)) - \psi^i(\xi_m(s))) \left\langle U_s^{-1} \nabla \psi^i(\gamma(s)), h_k(s) \right\rangle \mathrm{d}s.$$

Thus we obtain

(2.9)

$$\begin{aligned}
\mathscr{E}(v_m, v_m) =& \frac{1}{2} \int_{\mathbf{E}} \sum_{k=1}^\infty \left(D_{h_k} v_m(\gamma) \right)^2 \mathrm{d}\mu \\
\le& 2 \int_{\mathbf{E}} \sum_{k=1}^\infty \left(\sum_{i=1}^N \int_0^1 (\psi^i(\gamma(s)) - \psi^i(\xi_m(s))) \left\langle U_s^{-1} \nabla \psi^i(\gamma(s)), h_k(s) \right\rangle ds \right)^2 \mathrm{d}\mu \\
\le& C_N \sum_{i=1}^N \int_{\mathbf{E}} \|U_\cdot^{-1} \nabla \psi^i(\gamma)\|_{\mathbf{H}}^2 \mathrm{d}\mu \le C, \quad \forall m \in \mathbb{N}.
\end{aligned}$$

Here C is independent of m and in the last inequality we used that M is compact. Since $\{\xi_m | m \in \mathbb{N}\}$ is dense in $\mathbf{E}$, $w_n \downarrow 0$ on $\mathbf{E}$ hence in $L^2(\mathbf{E};\mu)$. By (2.9) and [44, IV.Lemma 4.1] we have

$$\mathscr{E}(w_n, w_n) \le C, \quad \forall n \in \mathbb{N}.$$

By [44, I.2.12, III.3.5] we obtain that a subsequence of the Cesaro mean of some subsequence of w_n converges to zero $\mathscr{E}$-quasi-uniformly. But since $(w_n)_{n\in\mathbb{N}}$ is decreasing, (2.8) follows.

For any $\gamma \neq \eta \in \mathbf{E}$ let $\varepsilon := \bar{d}(\gamma, \eta) > 0$. There exists a certain ξ_N such that $\bar{d}(\xi_N, \eta) < \frac{\varepsilon}{4}$ and $\bar{d}(\xi_N, \gamma) > \frac{\varepsilon}{4}$. Let $v_m(\gamma) := \varphi(\bar{d}(\gamma, \xi_m)^2), m \in \mathbb{N}$ for φ as above. Then $\{v_m\}$ separates the points of $\mathbf{E}$ and (iii) in the definition of quasi-regular Dirichlet froms (cf. [44]) follows. Now the results follow immediately. □

2.2 Dirichlet form on loop sapce

In this subsection, we construct the quasi-regular Dirichlet form on loop space. To do that, we first need the integration by parts formula with respect to the Brownian bridge measure and this formula does not only depend on bounds of the Ricci curvature, but also on the Hessian of the logarithm heat kernel on M. Fix $o \in M$, the based loop space $L_{o,o}(M)$ over M is defined by

$$L_{o,o}(M) := \{\gamma \in C([0,1]; M) : \gamma(0) = \gamma(1) = o\}.$$

Then $L_{o,o}(M)$ is a Polish space under the uniform distance d_∞.

As in the previous section, we work with the following simple but natural distance on $L_{o,o}(M)$,

$$\tilde{d}(\gamma, \eta) := \int_0^1 \rho(\gamma_s, \eta_s)^2 ds, \quad \gamma, \eta \in L_{o,o}(M).$$

Let $\mathbf{E}$ be the closure of $L_{o,o}(M)$ in $\{\eta : [0,T] \to M; \int_0^1 \rho(o, \eta_s)^2 ds < \infty\}$ with respect to the distance $\tilde{d}$. Then $\mathbf{E}$ is a Polish space.

Let $\mathbb{P}_{o,o}$ be the Brownian bridge measure on $L_{o,o}(M)$, which can be extended to a Borel measure on $\mathbf{E}$. Let $O(M)$ be the orthonormal frame bundle over M, and let $\pi : O(M) \to M$ be the canonical projection. Let $(\gamma_t)_{\{0\le t\le 1\}}$ be the coordinate process on $L_{o,o}(M)$, $(\mathscr{F}_t)_{0\le t\le 1}$ the $\mathbb{P}_{o,o}$-completed natural filtration of (γ_t). We set $\mathscr{F} = \mathscr{F}_1$. Then (γ_t) is a semimartingale on the stochastic basis $(\mathbf{E}, \mathscr{F}, \mathscr{F}_t, \mathbb{P}_{o,o})$. For a given orthonormal frame $u_0 \in \pi^{-1}(x) \subset O(M)$, there exists a unique stochastic horizontal lift (U_t) of (γ_t), determined by the Levi-Civita connection, such that $U_0 = u_0$. Let

$$\left\{ \begin{array}{l} \mathrm{d}B_t = U_t^{-1} \circ \mathrm{d}\gamma_t - U_t^{-1} \nabla \log p_{1-t}(\gamma_t, o)\mathrm{d}t, \quad t \ge 0 \\ B_0 = 0, \end{array} \right. \tag{2.10}$$

where $\circ \mathrm{d}\gamma_t$ stands for the Stratonovich differential of γ_t and $p_t(x,y)$ is the heat kernel of $\frac{1}{2}\Delta$ with $\Delta :=$ Levi-Civita Laplacian on M. $(B_t)_{0\le t\le 1}$ is an $\mathbb{R}^d$-valued standard Brownian motion.

By Driver's integration by parts formula [21] (see also [38, 13]) we have for $F \in \mathscr{F}C_b^1, h \in \mathbb{H}_0 := \{h \in \mathbb{H} | h(1) = 0\}$,

$$\int_{L_{o,o}(M)} D_h F \mathrm{d}\mathbb{P}_{o,o} = \int_{L_{o,o}(M)} F \beta_h \mathrm{d}\mathbb{P}_{o,o}, \tag{2.11}$$

with

$$L^2(\mathbf{E}, \mathbb{P}_{o,o}) \ni \beta_h := \int_0^1 \left\langle h_s' + \frac{1}{2}\mathrm{Ric}_{U_s} h_s - \mathrm{Hess}_{U_s} \log p_{1-s}(\cdot, o) h_s, \mathrm{d}B_s \right\rangle,$$

where $\mathrm{Hess}_u f a := u^{-1}\mathrm{Hess} f(\pi(u)) u a$ for $u \in O(M), a \in \mathbb{R}^d$ and smooth function f on M. Let $\{h_k\} \subset \mathbb{H}_0$ be an orthonormal basis in $\mathbf{H}$ such that $h_k \in \mathbb{H}_0, k \in \mathbb{N}$. Similarly as above we easily deduce that the form

$$\mathscr{E}(F,G) := \frac{1}{2}\int_{\mathbf{E}} \langle DF, DG\rangle_{\mathbf{H}} \mathrm{d}\mathbb{P}_{o,o} = \frac{1}{2}\sum_{k=1}^{\infty}\int_{\mathbf{E}} D_{h_k} F D_{h_k} G \mathrm{d}\mathbb{P}_{o,o}, \quad F, G \in \mathscr{F}C_b^1$$

is closable and its closure $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is a quasi-regular Dirichlet form on $L^2(\mathbf{E}; \mathbb{P}_{o,o}) = L^2(L_{o,o}(M); \mathbb{P}_{o,o})$. Moreover, Theorems 2.2-2.4 still hold in this case.

2.3 Dirichlet form on free path/loop space

Similar to the above two subsections, in this subsection, we construct a class of quasi-regular Dirichlet forms on the free path/loop space. Let σ be a probability measure on M and $\mathrm{d}\sigma(x) = v(x)\mathrm{d}x$ some C^1-function v on M, and $\mathbb{P}_\sigma$ be the distribution of the Brownian motion/Brownian bridge starting from σ up to time 1, which is then a probability measure on the free path/loop space:

$$W(M) = C([0,1]; M) \text{ or } L(M) = \cup_{y\in M} L_{y,y}(M). \tag{2.12}$$

In fact, we know that

$$\mathrm{d}\mathbb{P}_\sigma = \int_M \mathbb{P}_y \mathrm{d}\sigma(y),$$

where $\mathbb{P}_y$ is the law of Brownian motion/Brownian bridge starting at y. Similarly, we define the L^2-distance on $W(M)/L(M)$ by

$$\tilde{d}(\gamma, \eta) := \int_0^1 \rho(\gamma_s, \eta_s)^2 \mathrm{d}s, \quad \gamma, \eta \in W(M).$$

Let $\mathbf{E}$ be the closure of $W(M)/L(M)$ in $\{\eta : [0,T] \to M; \int_0^1 \rho(o,\eta_s)^2 ds < \infty\}$ with respect to the distance $\tilde{d}$. Then $\mathbf{E}$ is a Polish space. $\mathbb{P}_\sigma$ can be extended to a Borel measure on $\mathbf{E}$. Choose a sequence of $\{h_k\} \subset \mathbb{H}$ such that it is an orthonormal basis in $\mathbf{H}$. Then the quadratic form on the free path/loop space is defined by

$$\mathscr{E}(F,G) := \frac{1}{2}\int_{\mathbf{E}} \langle DF, DG\rangle_{\mathbf{H}} \mathrm{d}\mathbb{P}_\sigma = \frac{1}{2}\sum_{k=1}^{\infty}\int_{\mathbf{E}} D_{h_k}F D_{h_k}G \mathrm{d}\mathbb{P}_\sigma, \quad F,G \in \mathscr{F}C_b^1.$$

By the integration by parts formula in [28]/[15, Lemma 4.1] (and the references therein): for $F \in \mathscr{F}C_b^1, h \in \mathbb{H}$,

$$\int D_h F \mathrm{d}\mathbb{P}_\sigma = \int F\beta_h \mathrm{d}\mathbb{P}_\sigma, \tag{2.13}$$

where

$$\beta_h := \int_0^1 \left\langle h_s' + \frac{1}{2}\mathrm{Ric}_{U_s} h_s, \mathrm{d}B_s \right\rangle \text{ or } \int_0^1 \left\langle h_s' + \frac{1}{2}\mathrm{Ric}_{U_s} h_s - \mathrm{Hess}_{U_s} \log p_{1-s}(\cdot, o) h_s, \mathrm{d}B_s \right\rangle,$$

and $\beta_h \in L^2(\mathbf{E}, \mathbb{P}_\sigma)$. Here B is the corresponding Brownian motion in $\mathbb{R}^d$. This implies that the form $\mathscr{E}$ is closable, and similarly as above, we can prove that its closure $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is a quasi-regular Dirichlet form on $L^2(\mathbf{E}; \mathbb{P}_\sigma) = L^2(W(M); \mathbb{P}_\sigma)$. Moreover, Theorems 2.2-2.4 still hold in this case.

Remark 2.6. *Compared to the proof of the closability of the O-U Dirichlet form $\mathscr{E}^{OU}$ on the free path/loop space in [28], our situation is simpler now. This is because the integration by parts formula for O-U Dirichlet form depends on the initial distribution σ. The present case does not depend on the initial point since now we take the L_2-space as the intermediate space.*

3 Properties of L^2-Dirichlet form on path space

In this section, we study properties of the stochastic heat process $X_t, t \geq 0$, and L^2-Dirichlet form $\mathscr{E}$ constructed in Section 2.1. In fact, we establish some functional inequalities associated with $(\mathscr{E}, D(\mathscr{E}))$. As mentioned in Remark 2.1, the results in Section 2 also hold when M is not compact. Therefore, in this section we drop the compactness condition on M.

3.1 Log-Sobolev inequality

In this subsection, we establish log-Sobolev inequality for the L^2-Dirichlet form.

Theorem 3.1. *[Log-Sobolev inequality] Suppose that* $\mathrm{Ric} \geq -K$ *for* $K \in \mathbb{R}$*, then the log-Sobolev inequality holds*

$$\mu(F^2 \log F^2) \leq 2C(K)\mathscr{E}(F,F), \qquad F \in \mathscr{F}C_b^1,\ \mu(F^2) = 1, \tag{3.1}$$

where $C(K) := \frac{e^K - 1 - K}{K^2} \wedge C_0(K)$ *with*

$$C_0(K) = \begin{cases} \frac{4}{K^2}\left(1 - \sqrt{2e^{\frac{K}{2}} - e^K}\right), & \text{if } K < 0, \\ \frac{2}{K^2}\left(e^K - 2e^{\frac{K}{2}} + 1\right), & \text{if } K > 0. \end{cases}$$

Remark 3.2. (*i*) *In fact, Theorem 3.1 has first been proved in [33]. Compared to their results, our constant* $C(K)$ *is smaller. The constant in [33] is given by*

$$\tilde{C}(K) = \begin{cases} \frac{4}{K^2}\left(1 - \sqrt{2e^{\frac{K}{2}} - e^K}\right), & \text{if } 2e^{\frac{K}{2}} - e^K > 0, \\ \frac{2}{K^2}\left(e^K - 2e^{\frac{K}{2}} + 1\right), & \text{if } 2e^{\frac{K}{2}} - e^K < 0. \end{cases}$$

Then it is easy to see that $\tilde{C}(K) \geq C_0(K)$ *for* $K > 0$ *and* $2e^{\frac{K}{2}} - e^K > 0$*.*

Comparing the classic O-U Dirichlet form $\mathscr{E}^{OU}$ *and the* L^2*-Dirichlet form* $\mathscr{E}$*, we note that the log-Sobolev inequality associated to two Dirichlet forms are completely different. The former requires uniform bounds on the Ricci curvature, and the latter only needs lower bounds of the Ricci curvature.*

(*ii*) *According to [54], the log-Sobolev inequality implies hypercontractivity of the associated semigroup* P_t*, in particular, the* L^2*-exponential ergodicity of the process:* $\|P_t f - \int f d\mu\|_{L^2} \leq e^{-t/C(K)}\|F\|_{L^2}$*.*

(*iii*) *The log-Sobolev inequality also implies the irreducibility of the Dirichlet form* $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$*. It's obvious that the Dirichlet form* $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ *is recurrent. Combining these two results, by [27, Theorem 4.7.1], for any nearly Borel non-exceptional set* B*,*

$$\mathbf{P}^z(\sigma_B \circ \theta_n < \infty, \forall n \geq 0) = 1, \qquad \text{for q.e. } z \in \mathbf{E}.$$

Here $\sigma_B = \inf\{t > 0 : X_t \in B\}$*,* θ *is the shift operator for the Markov process* X*, and for the definition of any nearly Borel non-exceptional set we refer to [27]. Moreover by [27, Theorem 4.7.3] we obtain the following strong law of large numbers: for* $f \in L^1(\boldsymbol{E}, \mu)$

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(X_s)\mathrm{d}s = \int f \mathrm{d}\mu, \quad \mathbf{P}^z - a.s.,$$

for q.e. $z \in E$*.*

Proof of Theorem 3.1. By [33] we have the martingale representation theorem, that is, for $F \in \mathscr{F}C_b^1$,

$$F = \mathbb{E}(F) + \int_0^1 \langle H_s^F, \mathrm{d}B_s\rangle, \tag{3.2}$$

with

$$H_s^F = \mathbb{E}\left[M_s^{-1} \int_s^1 M_\tau(DF(\tau))\mathrm{d}\tau \middle| \mathscr{F}_s \right]. \tag{3.3}$$

Here and in the following $\mathbb{E}$ means the expectation w.r.t. μ, B is $\mathbb{R}^d$-valued Brownian motion under μ, $(\mathscr{F}_t)$ is the normal filtration generated by B and M_t is the solution of the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} M_t + \frac{1}{2} M_t \mathrm{Ric}_{U_t} = 0, \quad M_0 = I. \tag{3.4}$$

Let $F = G^2$ for $G \in \mathscr{F}C_b^1$ and consider the continuous version of the martingale $N_s = \mathbb{E}[F|\mathscr{F}_s]$. We have

$$N_s = \mathbb{E}F + \int_0^s \langle H_\tau^F, \mathrm{d}B_\tau\rangle.$$

Now applying Itô's formula to $N_s \log N_s$, we have

$$\mathbb{E}N_1 \log N_1 - \mathbb{E}N_0 \log N_0 = \frac{1}{2}\mathbb{E}\int_0^1 N_s^{-1}|H_s^F|^2 \mathrm{d}s. \tag{3.5}$$

Here and in the following we use $|\cdot|$ to denote the norm in $\mathbb{R}^d$. On the other hand,

$$DF = D(G^2) = 2GDG.$$

Using this relation in the explicit formula (3.3) for H^F, we have

$$H_s^F = 2\mathbb{E}\left[G M_s^{-1} \int_s^1 M_\tau DG(\tau)\mathrm{d}\tau \middle| \mathscr{F}_s \right]. \tag{3.6}$$

By the lower bound on the Ricci curvature, we have

$$\|M_s^{-1}M_\tau\| \le e^{K(\tau-s)/2}. \tag{3.7}$$

By Cauchy-Schwarz inequality in (3.6) and (3.7), we have

$$|H_s^F|^2 \le 4\mathbb{E}[G^2|\mathscr{F}_s]\mathbb{E}\left[\left(\int_s^1 e^{K(\tau-s)/2}|D_\tau G|\mathrm{d}\tau \right)^2 \middle| \mathscr{F}_s \right].$$

Here and in the following we use $D_\tau G$ to denote $DG(\tau)$ for simplicity. Thus the right hand side of (3.5) can be controlled by

$$
\begin{aligned}
2\mathbb{E}\int_0^1\left[\left(\int_s^1 e^{K(\tau-s)/2}|D_\tau G|d\tau\right)^2\right]\mathrm{d}s \leq & 2\mathbb{E}\int_0^1\left[\int_s^1 e^{K(\tau-s)}d\tau\int_s^1|D_\tau G|^2\mathrm{d}\tau\right]ds \\
\leq & \frac{2}{K}\int_0^1[e^{K(1-s)}-1]\mathrm{d}s\mathscr{E}(G,G) \\
\leq & 2\frac{e^K-1-K}{K^2}\mathscr{E}(G,G).
\end{aligned} \tag{3.8}
$$

Now we use another way to control the left hand side of (3.8). We have the following estimate, which follows essentially from [29]: Hölder's inequality implies that

$$
\left(\int_s^1 e^{K(\tau-s)/2}|D_\tau G|\mathrm{d}\tau\right)^2 \leq \int_s^1 e^{K(\tau-s)/2}\mathrm{d}\tau\int_s^1 e^{K(\tau-s)/2}|D_\tau G|^2\mathrm{d}\tau.
$$

Then changing the order of integration we obtain

$$
\mathbb{E}\int_0^1\left[\left(\int_s^1 e^{K(\tau-s)/2}|D_\tau G|\mathrm{d}\tau\right)^2\right]\mathrm{d}s \leq \mathbb{E}\int_0^1 J_1(s)|D_sG|^2\mathrm{d}s,
$$

where

$$
\begin{aligned}
J_1(s) =& -\int_0^s\frac{2}{K}\big(1-e^{K(1-t)/2}\big)e^{K(s-t)/2}\mathrm{d}t \\
=& \frac{2}{K^2}\left[2(1-e^{\frac{Ks}{2}})-e^{\frac{K(1-s)}{2}}+e^{\frac{K(1+s)}{2}}\right].
\end{aligned}
$$

Taking the derivative of $t\to J_1(t)$ gives

$$
J_1'(t) = -\frac{2}{K}\left[e^{\frac{Kt}{2}}-\frac{1}{2}e^{\frac{K(1-t)}{2}}-\frac{1}{2}e^{\frac{K(1+t)}{2}}\right].
$$

In addition, we have $J_1(0)=0, J_1(1)=\frac{2}{K^2}\big(1-e^{\frac{K}{2}}\big)^2$. Moreover,

$$
J_1'(0) = -\frac{2}{K}\left[1-e^{\frac{K}{2}}\right]>0, \quad J_1'(1)=\frac{1}{K}\left[e^{\frac{K}{2}}-1\right]^2.
$$

For $J_1'(t)=0$ we only have one solution $e^{-Kt}=2e^{-\frac{K}{2}}-1$, which implies that when $K>0$, $J_1'(s)\geq 0$ for all $s\geq 0$. Then for $K>0$, $J(s)$ is increasing, which implies that

$$
\mathbb{E}\int_0^1\left[\left(\int_s^1 e^{K(\tau-s)/2}|D_\tau G|\mathrm{d}\tau\right)^2\right]\mathrm{d}s \leq J_1(1)\mathbb{E}\int_0^1|D_sG|^2\mathrm{d}s \leq C_0(K)\mathbb{E}\int_0^1|D_sG|^2\mathrm{d}s.
$$

For $K<0$, we suppose $t_0\in(0,1)$ satisfying $J_1'(t_0)=0$, which is the maximum point of J_1. Then for $K<0$,

$$\mathbb{E}\int_0^1\left[\left(\int_s^1 e^{K(\tau-s)/2}|D_\tau G|\mathrm{d}\tau\right)^2\right]\mathrm{d}s\le J_1(t_0)\int_0^1|D_sG|^2\mathrm{d}s\le C_0(K)\int_0^1|D_sG|^2\mathrm{d}s.$$

Combining all the above, we complete the proof. $\square$

In the following Theorem 3.3, we obtain a new constant for the log-Sobolev inequality for Einstein manifolds. In this case the constant $C(K)$ tends to the optimal constant in the flat case as $K\to 0$ (see [9]).

Theorem 3.3. *Suppose that M is an Einstein manifold with constant Ricci curvature $-K\in\mathbb{R}$. Then the log-Sobolev inequality for $(\mathscr{E},\mathscr{D}(\mathscr{E}))$ holds:*

$$\mu(F^2\log F^2)\le 2C(K)\mathscr{E}(F,F),\qquad F\in\mathscr{F}C_b^1,\ \mu(F^2)=1, \tag{3.9}$$

where

$$C(K):=\left\{\left[4d\left(\sum_k|A_k|\right)^2\frac{e^K-1}{K}+2A_0^2\right]^{1/2}+\frac{2\pi}{(K^2+\pi^2)}\vee\frac{1}{\pi}\right\}^2$$

and

$$A_k:=\left[\frac{K}{2}+\frac{2\pi^2}{K}\left(k+\frac{1}{2}\right)^2\right]^{-1},\quad k\in\mathbb{N}\cup\{0\}.$$

Remark 3.4. *In fact, we have*

$$\lim_{K\to 0}C(K)=\frac{4}{\pi^2},$$

and $\frac{4}{\pi^2}$ is the optimal constant in the $\mathbb{R}^d$ case (see [9, 18] and the references therein).

Proof. Let $h_{\alpha,k}:=\sqrt{2}\sin\left[\left(k+\frac{1}{2}\right)\pi\tau\right]e_\alpha$ for $\alpha=1,...,d,k\in\mathbb{N}\cup\{0\}$. Here $\{e_\alpha\}$ is the usual orthonormal basis for $\mathbb{R}^d$ given by $e_\alpha=(0,...,1,..,0)$. It is easy to see that $\{h_{\alpha,k}\}$ is an orthonormal basis of $\mathbf{H}$. We start with the following computation:

$$\int_s^1 e^{K(\tau-s)/2}h_{\alpha,k}\mathrm{d}\tau=e_\alpha e^{-Ks/2}\int_s^1 e^{K\tau/2}\sqrt{2}\sin\left[\left(k+\frac{1}{2}\right)\pi\tau\right]\mathrm{d}\tau:=\sqrt{2}e_\alpha e^{-Ks/2}B(s,k,K),$$

with

$$\begin{aligned}B(s,k,K)=&\left[\frac{K}{2}+\frac{2\pi^2}{K}\left(k+\frac{1}{2}\right)^2\right]^{-1}\\&\times\left\{(-1)^ke^{K/2}-e^{Ks/2}\sin\left[\left(k+\frac{1}{2}\right)\pi s\right]+\frac{2\pi}{K}\left(k+\frac{1}{2}\right)e^{Ks/2}\cos\left[\left(k+\frac{1}{2}\right)\pi s\right]\right\}.\end{aligned}$$

Thus, we have

$$\int_s^1 e^{\frac{K(\tau-s)}{2}} h_{\alpha,k}\mathrm{d}\tau = A_k(-1)^k e^{K(1-s)/2}\sqrt{2}e_\alpha - A_k h_{\alpha,k} + B_k \bar{h}_{\alpha,k}, \tag{3.10}$$

where

$$A_k = \left[\frac{K}{2} + \frac{2\pi^2}{K}\left(k+\frac{1}{2}\right)^2\right]^{-1}, \quad B_k = \left[\frac{K}{2} + \frac{2\pi^2}{K}\left(k+\frac{1}{2}\right)^2\right]^{-1}\frac{2\pi}{K}\left(k+\frac{1}{2}\right),$$
$$\bar{h}_{\alpha,k}(\tau) := \sqrt{2}\cos\left[\left(k+\frac{1}{2}\right)\pi\tau\right]e_\alpha.$$

It is easy to see that $B_k \le \frac{2\pi}{K^2+\pi^2} \vee \frac{1}{\pi}$. Indeed, if $K^2 < \pi^2$ then B_k is decreasing with respect to k and if $K^2 \ge \pi^2$ then $B_k \le \frac{1}{|K|} \le \frac{1}{\pi}$. Now since $\mathrm{Ric} = -K$ we have $M_s^{-1}M_\tau = e^{\frac{K(\tau-s)}{2}}I$. A similar argument as in the proof of Theorem 3.1 implies the left hand side of (3.8) can be controlled by

$$\begin{aligned}
&2\mathbb{E}\Big[\int_0^1\Big|\sum_{\alpha=1}^d\sum_{k=0}^\infty \langle DG, h_{\alpha,k}\rangle_{\mathbf{H}}\int_s^1 e^{\frac{K(\tau-s)}{2}}h_{\alpha,k}\mathrm{d}\tau\Big|^2\mathrm{d}s\Big]\\
=&2\mathbb{E}\Big[\int_0^1\Big|\sum_{\alpha=1}^d\sum_{k=0}^\infty \langle DG, h_{\alpha,k}\rangle_{\mathbf{H}} A_k(-1)^k\sqrt{2}e^{K(1-s)/2}e_\alpha\\
&-\sum_{\alpha=1}^d\sum_{k=0}^\infty \langle DG, h_{\alpha,k}\rangle_{\mathbf{H}} A_k h_{\alpha,k} + \sum_{\alpha=1}^d\sum_{k=0}^\infty \langle DG, h_{\alpha,k}\rangle_{\mathbf{H}} B_k\bar{h}_{\alpha,k}\Big|^2\mathrm{d}s\Big]\\
=&2\Bigg(\mathbb{E}\int_0^1\Big|\sum_{\alpha=1}^d\sum_{k=0}^\infty \langle DG, h_{\alpha,k}\rangle_{\mathbf{H}} A_k(-1)^k\sqrt{2}e^{K(1-s)/2}e_\alpha - \sum_{\alpha=1}^d\sum_{k=0}^\infty \langle DG, h_{\alpha,k}\rangle_{\mathbf{H}} A_k h_{\alpha,k}\Big|^2\mathrm{d}s\\
&+2\mathbb{E}\int_0^1\Big\langle\sum_{\alpha=1}^d\sum_{k=0}^\infty \langle DG, h_{\alpha,k}\rangle_{\mathbf{H}} A_k(-1)^k\sqrt{2}e^{K(1-s)/2}e_\alpha - \sum_{\alpha=1}^d\sum_{k=0}^\infty \langle DG, h_{\alpha,k}\rangle_{\mathbf{H}} A_k h_{\alpha,k},\\
&\sum_{\alpha=1}^d\sum_{k=0}^\infty \langle DG, h_{\alpha,k}\rangle_{\mathbf{H}} B_k\bar{h}_{\alpha,k}\Big\rangle\mathrm{d}s + \mathbb{E}\int_0^1\Big|\sum_{\alpha=1}^d\sum_{k=0}^\infty \langle DG, h_{\alpha,k}\rangle_{\mathbf{H}} B_k\bar{h}_{\alpha,k}\Big|^2\mathrm{d}s\Bigg)\\
=&:2(I_1+I_2+I_3),
\end{aligned}$$

where we used (3.10) in the first equality. Then we have for I_1

$$\begin{aligned}
I_1 \le& 4d\mathbb{E}\int_0^1 |DG|_{\mathbf{H}}^2\Big(\sum_k |A_k|\Big)^2 e^{K(1-s)}\mathrm{d}s + 2\mathbb{E}|DG|_{\mathbf{H}}^2 A_0^2\\
=&\mathbb{E}|DG|_{\mathbf{H}}^2\left[4d(\sum_k|A_k|)^2\frac{e^K-1}{K} + 2|A_0|^2\right] =: C_1(K)\mathscr{E}(G,G),
\end{aligned}$$

where the first inequality is due to that A_k^2 is decreasing w.r.t. k and $\{h_{\alpha,k}\}$ is an orthonormal basis of $\mathbf{H}$. For I_3 we have

$$I_3 = \sum_{\alpha=1}^{d}\sum_{k=0}^{\infty} \langle DG, h_{\alpha,k}\rangle_{\mathbf{H}}^2 B_k^2 \le \left(\frac{4\pi^2}{(K^2+\pi^2)^2} \vee \frac{1}{\pi^2}\right) \mathscr{E}(G,G) =: C_2(K)\mathscr{E}(G,G).$$

Using Hölder's inequality we obtain

$$I_2 \le 2I_1^{1/2} I_3^{1/2} \le 2C_1(K)^{1/2} C_2(K)^{1/2} \mathscr{E}(G,G).$$

Combining the above estimates we obtain

$$\mu(F^2 \log F^2) \le 2(C_1(K)^{1/2} + C_2(K)^{1/2})^2 \mathscr{E}(G,G).$$

□

3.2 Characterization of the lower bound of the Ricci curvature

The upper and lower bounds for the Ricci curvature on a Riemannian manifold were well characterized in terms of the twisted Malliavin gradient-Dirichlet form $\mathscr{E}^{OU}$ for the O-U process on the path space (see **A** in the introduction) in [46, 59, 60, 17]. If the Malliavin gradient is replaced by the L^2-gradient DF, then we obtain characterizations for the lower boundedness of the Ricci curvature in terms of a properly decomposition of the L^2 gradient -Dirichlet form. This subsection is devoted to prove such characterizations.

In fact, all the results in Section 2 and Theorem 3.1 also hold when we change 1 to $T > 0$. To state our results, let us first introduce some notations: For any point $y \in M$ and $T > 0$, let $x_{y,[0,T]}$ be the Brownian motion starting from $y \in M$ up to T, and $\mu_{T,y}$ be the distribution of $x_{y,[0,T]}$. Define $\mathscr{F}C_b^T$ as in (2.2) with 1 replaced by T. For any $n \ge 1$ and $G \in \mathscr{F}C_b^T$, define the following quadratic form

$$\begin{aligned}\mathscr{E}_{T,n,y}^{K}(G,G) = (1+n)C_1(K)\int_{W_y^T(M)}\int_0^{T-\frac{1}{n}} |DG(\gamma)(s)|_{\mathbb{R}^d}^2 \mathrm{d}s\mathrm{d}\mu_{T,y}(\gamma) \\ + \left(\frac{1}{n}+\frac{1}{n^2}\right) C_{2,n}(K)\int_{W_y^T(M)}\int_{T-\frac{1}{n}}^{T} |DG(\gamma)(s)|_{\mathbb{R}^d}^2 \mathrm{d}s\mathrm{d}\mu_{T,y}(\gamma).\end{aligned}$$

where

$$C_1(K) = \left[\frac{1}{K^2}\left(TKe^{KT} - e^{KT} + 1\right)\right] \bigvee \frac{T^2}{2}, \quad C_{2,n}(K) = \frac{e^{KT}-1}{K}\left(1 \vee e^{-\frac{K}{n}}\right).$$

Similarly as in the proof of Theorem 2.2 we see that $(\mathscr{E}_{T,n,y}^{K}, \mathscr{F}C_b^T)$ is closable and its closure is a Dirichlet form. Let p_t be the Markov semigroup of the Brownian motion x_y starting from $y \in M$, i.e. given by $p_t f(y) = \mathbb{E}[f(x_{y,t})], y \in M, f \in \mathscr{B}_b(M), t \ge 0$. Let $C_0^\infty(M)$ denote the set of all smooth functions with compact supports on M.

Theorem 3.5. *For $K \in \mathbb{R}$, the following statements are equivalent:*

(1) $\mathrm{Ric} \geq -K$.

(2) *For any $f \in C_0^\infty(M), T_1 > T_2 \geq 0$ and $y \in M$, we have*

$$\left|\int_{T_2}^{T_1} \nabla p_s f(y)\mathrm{d}s\right| \leq \int_{T_2}^{T_1} \mathrm{e}^{\frac{Ks}{2}} p_s|\nabla f|(y)\mathrm{d}s.$$

(3) *For any $F \in \left\{\sum_{i=1}^n a_i \int_{s_i}^{t_i} f_i(\gamma_s)\mathrm{d}s, n \in \mathbb{N}, f_i \in C_0^\infty(M), s_i, t_i \in [0,T], a_i \in \mathbb{R}\right\}$ and $y \in M, T > 0$*

$$\left|\nabla_y \mathbb{E}F(x_{y,[0,T]})\right| \leq \int_0^T e^{\frac{Ks}{2}} \mathbb{E}|DF(x_{y,[0,T]})|(s)\mathrm{d}s,$$

where ∇_y denotes the gradient w.r.t. y and $\mathbb{E}$ means expectation w.r.t. $\mu_{T,y}$.

(4) *For any $y \in M, T > 0$, the following log-Sobolev inequality holds for any $n \in \mathbb{N}$:*

$$\mu_{T,y}(F^2 \log F^2) \leq 2\mathscr{E}_{T,n,y}^K(F,F), \quad F \in \mathscr{F}C_b^T,\ \mu_{T,y}(F^2) = 1.$$

(5) *For any $y \in M, T > 0$, the following Poincaré-inequality holds for any $n \in \mathbb{N}$:*

$$\mu_{T,y}(F^2) \leq \mathscr{E}_{T,n,y}^K(F,F), \quad F \in \mathscr{F}C_b^T,\ \mu_{T,y}(F) = 0.$$

Proof. (1) $\Rightarrow$ (3) By the gradient formula in [38] (see also [28, 55, 59]), for $F(\gamma) := f(\gamma_t)$ with $f \in C_0^\infty(M)$ we have

$$\nabla_y \mathbb{E}[F(x_{y,[0,T]})] = U_0^y \mathbb{E}\Big[M_t^y (U_t^y)^{-1} \nabla f(x_{y,t})\Big],$$

where U_t^y is the solution to (2.1) with o replaced by y and M_t^y is the solution to (3.4) with U_t replaced by U_t^y. Applying the above formula to $F(\gamma) = \sum_{i=1}^n a_i \int_{s_i}^{t_i} f_i(\gamma_s)\mathrm{d}s$, we have

$$\begin{aligned}
\nabla_y \mathbb{E}[F(x_{y,[0,T]})] &= \sum_{i=1}^n a_i \int_{s_i}^{t_i} \nabla_y \mathbb{E}[f_i(x_{y,s})]\mathrm{d}s \\
&= U_0^y \sum_{i=1}^n a_i \int_{s_i}^{t_i} \mathbb{E}\Big[M_s^y (U_s^y)^{-1} \nabla f_i(x_{y,s})\Big]\mathrm{d}s \\
&= U_0^y \int_0^T \mathbb{E}\big[M_s^y DF(x_{y,[0,T]})(s)\big]\mathrm{d}s.
\end{aligned} \tag{3.11}$$

Combining this with $\mathrm{Ric} \geq -K$, we have

$$|\nabla_y \mathbb{E} F(x_{y,[0,T]})| \leq \int_0^T e^{\frac{Ks}{2}} \mathbb{E}|DF(x_{y,[0,T]})|(s)\mathrm{d}s. \tag{3.12}$$

(3) $\Rightarrow$ (2) Taking $F(\gamma) := \int_{T_2}^{T_1} f(\gamma_s)ds$ for $f \in C_0^1(M)$ and $T_1 > T_2 \geq 0$, by (3) we have

$$\begin{aligned}\left|\int_{T_2}^{T_1} \nabla p_s f(y)ds\right| &= \left|\int_{T_2}^{T_1} \nabla_y \mathbb{E}[f(x_{y,s})]\mathrm{d}s\right| = \left|\nabla_y \mathbb{E}[F]\right| \\ &\leq \int_0^T e^{\frac{Ks}{2}} \mathbb{E}|DF(x_{y,[0,T]})|(s)\mathrm{d}s \\ &= \int_{T_2}^{T_1} e^{\frac{Ks}{2}} p_s|\nabla f|(y)\mathrm{d}s.\end{aligned} \tag{3.13}$$

(2) $\Rightarrow$ (1) Let $f \in C_0^\infty(M)$ with $|\nabla f(y)| = 1$ and $\mathrm{Hess}_f(y) = 0$. For any $T > 0$ and $\varepsilon > 0$, according to (2), we obtain

$$\left|\int_T^{T+\varepsilon} \nabla p_s f(y)ds\right| \leq \int_T^{T+\varepsilon} \mathrm{e}^{\frac{Ks}{2}} p_s|\nabla f|(y)\mathrm{d}s. \tag{3.14}$$

Dividing the two sides of the above equation by ε and letting ε go to zero, we get

$$\left|\nabla p_T f\right|(y) \leq \mathrm{e}^{\frac{KT}{2}} p_T|\nabla f|(y). \tag{3.15}$$

Then by the classical result (or refer to [54] and references therein), (1) follows. Note that $p_{2T} = \tilde{p}_T$, where $\tilde{p}_T$ is the semigroup associated with the generator Δ. Thus we complete the proof of this step.

(5) $\Rightarrow$ (1) Let $f \in C_0^\infty(M)$ with $|\nabla f(y)| = 1$ and $\mathrm{Hess}_f(y) = 0$. Taking $F(\gamma) = n\int_{T-1/n}^T f(\gamma_s)ds$, then $DF(\gamma)(s) = nU_s^{-1}\nabla f(\gamma_s)1_{s\in[T-1/n,T]}$. By (5) we have that

$$\begin{aligned}&\mathbb{E}\left[n\int_{T-1/n}^T f(\gamma_s)\mathrm{d}s\right]^2 - \left[\mathbb{E}n\int_{T-1/n}^T f(\gamma_s)\mathrm{d}s\right]^2 \\ \leq & C_{2,n}(K)(n+1)\int_{T-1/n}^T \mathbb{E}|\nabla f(\gamma_s)|^2\mathrm{d}s.\end{aligned}$$

Letting $n \to \infty$ we obtain

$$p_T f^2 - (p_T f)^2 \leq C_2(K) p_T|\nabla f|^2, \tag{3.16}$$

with $C_2(K) = \frac{e^{KT}-1}{K}$. According to [55, Theorems 3.2.3], we have

$$\begin{aligned}\mathrm{Ric}(\nabla f, \nabla f)(y) &= \lim_{t\downarrow 0}\frac{1}{T}\left(\frac{\tilde{p}_T f^2(y) - (\tilde{p}_T f)^2(y)}{2T} - |\nabla\tilde{p}_T f(y)|^2\right) \\ &= \lim_{T\downarrow 0}\frac{1}{T}\left(\frac{p_{2T} f^2(y) - (p_{2T} f)^2(y)}{2T} - |\nabla p_{2T} f(y)|^2\right).\end{aligned} \tag{3.17}$$

Combining the above inequality with (3.16), we get

$$\begin{aligned}
\mathrm{Ric}(\nabla f, \nabla f)(y) &\le \lim_{T\downarrow 0} \frac{1}{T}\left(\frac{e^{2KT}-1}{2KT} p_{2T}|\nabla f|^2(y) - |\nabla p_{2T} f(y)|^2\right)\\
&= \lim_{T\downarrow 0} \frac{1}{T}\left(\left(\frac{e^{2KT}-1}{2KT}-1\right) p_{2T}|\nabla f|^2(y)\right) + \lim_{T\downarrow 0} \frac{1}{T}\left(p_{2T}|\nabla f|^2(y) - |\nabla p_{2T} f(y)|^2\right)\\
&= K + \lim_{T\downarrow 0} \frac{1}{T}\left(\tilde{p}_T|\nabla f|^2 - |\nabla \tilde{p}_T f(y)|^2\right)\\
&= K + 2\mathrm{Ric}(\nabla f, \nabla f)(y),
\end{aligned}$$

where the last inequality follows due to the formula in [55, Theorems 3.2.3]:

$$\mathrm{Ric}(\nabla f, \nabla f)(y) = \lim_{T\downarrow 0} \frac{\tilde{p}_T|\nabla f|^2(y) - |\nabla \tilde{p}_T f|^2(y)}{2T}.$$

Therefore, we complete the proof of this step.

(1) $\Rightarrow$ (4) According to the proof of Theorem 3.1, we only need to prove the following: for any $n \in \mathbb{N}$

$$\mathbb{E}\int_0^T \left[\int_t^T e^{\frac{(s-t)K}{2}}|DF|\mathrm{d}s\right]^2 \mathrm{d}t \le \tilde{\mathscr{E}}^K_{T,n,y}(F,F).$$

In fact, we know that

$$\begin{aligned}
&\mathbb{E}\int_0^T \left[\int_t^T e^{\frac{(s-t)K}{2}}|DF|\mathrm{d}s\right]^2 \mathrm{d}t\\
\le&\mathbb{E}\int_0^T \left[\int_{t\vee(T-1/n)}^{T-1/n} e^{\frac{(s-t)K}{2}}|DF|\mathrm{d}s + \int_{T-1/n}^T e^{\frac{(s-t)K}{2}}|DF|\mathrm{d}s\right]^2 \mathrm{d}t\\
\le&(1+n)\int_0^T (T-t)\mathbb{E}\int_{t\vee(T-1/n)}^{T-1/n} e^{(s-t)K}|DF|^2\mathrm{d}s\mathrm{d}t\\
&+ (1/n+1/n^2)\int_0^T \mathbb{E}\int_{T-1/n}^T e^{(s-t)K}|DF|^2\mathrm{d}s\mathrm{d}t\\
\le&(1+n)\int_0^T (T-t)(1\vee e^{(T-t)K})\mathrm{d}t\mathbb{E}\int_0^{T-1/n} |DF|^2\mathrm{d}s\\
&+ (1/n+1/n^2)\int_0^T e^{(T-t)K}\mathrm{d}t(1\vee e^{-K/n})\mathbb{E}\int_{T-1/n}^T |DF|^2\mathrm{d}s\\
=&(1+n)C_1(K)\mathbb{E}\int_0^{T-1/n}|DF|^2\mathrm{d}s + (1/n+1/n^2)C_2(K,n)\mathbb{E}\int_{T-1/n}^T |DF|^2\mathrm{d}s,
\end{aligned}$$

where we used Hölder's inequality and Young's inequality in the second inequality. Thus we obtain the result. □

4 Stochastic heat equation

Based on the Andersson-Driver approximation of the Wiener measure, we now present a heuristic derivation of the equation for the process (constructed in Section 2) on path space. When M is Euclidean space, we may choose some suitable linear functions, which are in the domain of the generator, through which we can deduce the associated stochastic heat equation. However, when M is a Riemannian manifold, in general, it is not easy to find suitable test functions on $\mathbf{E}$ belonging to the domain of the generator and derive the associated equation. Instead, we will use a suitable approximation to give some intuitive idea how to deduce the equation. As mentioned in Section 1, it is proved in [2] that natural approximations of $\exp(-\frac{1}{2}E(\gamma))\mathscr{D}\gamma$ do indeed converge to Wiener measure on M. For the sake of simplicity, in this section, we suppose that M is compact. First we write the equations associated with the approximation measures.

4.1 Preliminary

Before going on, we need to introduce some notations from [2]. We will also use $\langle\cdot,\cdot\rangle$ to denote the Riemmanian metric. Let $\mathscr{T}$ be the set of all partitions of $[0,1]$ and

$$E(\gamma) := \int_0^1 \langle\gamma'(s),\gamma'(s)\rangle \mathrm{d}s \tag{4.1}$$

for all absolutely continuous curves $\gamma \in W_o(M)$, where $\gamma'(s) := \frac{\mathrm{d}}{\mathrm{d}s}\gamma(s)$. Otherwise, set $E(\gamma) = \infty$. Define the space of finite energy paths:

$$H(M) := \{\gamma \in W_o(M) : \gamma \text{ is an absolutely continuous curve and } E(\gamma) < \infty\}. \tag{4.2}$$

For each $\gamma \in H(M)$, the tangent space $T_\gamma H(M)$ of $H(M)$ at γ may be naturally identified with the space of all absolutely continuous vector fields $X : [0,1] \to TM$ along γ with $X(0) = 0$ and $G^1(X,X) < \infty$, where

$$G^1(X,X) := \int_0^1 \left\langle \frac{\nabla X(s)}{\mathrm{d}s}, \frac{\nabla X(s)}{\mathrm{d}s} \right\rangle \mathrm{d}s, \tag{4.3}$$

$$\frac{\nabla X(s)}{\mathrm{d}s} := /\!/_s(\gamma)\frac{\mathrm{d}}{\mathrm{d}s}(/\!/_s^{-1}(\gamma)X(s)) \tag{4.4}$$

and $/\!/_s(\gamma) : T_oM \to T_{\gamma(s)}M$ is parallel translation along γ relative to the Levi-Civita covariant derivative ∇. As mentioned in [2], on the tangent space $TH(M)$ there exists a natural metric given by

$$G^0(X,X) := \int_0^1 \langle X(s), X(s)\rangle \, \mathrm{d}s, \tag{4.5}$$

for any $X \in TH(M)$.

Now we introduce finite dimensional approximations to $(H(M), G^0)$: for every $\mathscr{P} := \{0 = s_0 < s_1 < s_2 < ... < s_n = 1\} \in \mathscr{T}$ with $\Delta_i s = s_i - s_{i-1}$, define

$$H_{\mathscr{P}}(M) := \{\gamma \in H(M) \cap C^2([0,1]/\mathscr{P}) : \nabla\gamma'(s)/ds = 0, s \notin \mathscr{P}\}. \tag{4.6}$$

These are the piecewise geodesics paths in $H(M)$, which change directions only at the partition points. For $\gamma \in H_{\mathscr{P}}(M)$ the tangent space $T_\gamma H_{\mathscr{P}}(M)$ can be identified with elements $X \in T_\gamma H_{\mathscr{P}}(M)$ satisfying the Jacobi equations on $[0,1]\backslash\mathscr{P}$, see [2, Prop. 4.4] for more details.

By induction, we may easily get the metric on $TH_{\mathscr{P}}(M)$ for the partition $\mathscr{P} \in \mathscr{T}$,

$$G^0_{\mathscr{P}}(X,Y) := \sum_{i=1}^n \langle X(s_i), Y(s_i)\rangle \, \Delta_i s, \tag{4.7}$$

for all $X, Y \in T_\gamma H_{\mathscr{P}}(M)$ and $\gamma \in H_{\mathscr{P}}(M)$. Let $\mathrm{Vol}_{G^0_{\mathscr{P}}}$ be the volume form on $H_{\mathscr{P}}(M)$ determined by $G^0_{\mathscr{P}}$. By the arguments in [2], $\mathrm{Vol}_{G^0_{\mathscr{P}}}$ may be interpreted as a suitable approximation to $\mathscr{D}\gamma$ mentioned in introduction.

Denote by $\nu^0_{\mathscr{P}}$ the measure on $H_{\mathscr{P}}(M)$ given by

$$\nu^0_{\mathscr{P}} := \frac{1}{Z_{\mathscr{P}}} e^{-\frac{1}{2}E}\mathrm{Vol}_{G^0_{\mathscr{P}}},$$

where $E : H(M) \to [0,\infty)$ is the energy functional defined in (4.1) and $Z_{\mathscr{P}}$ is a normalization constant given by

$$Z_{\mathscr{P}} := \Pi_{i=1}^n (\sqrt{2\pi}\Delta_i s)^d.$$

The following is one of the main results from [2].

Theorem 4.1. *[2, Theorem 1.8] Suppose that $f : W_o(M) \to \mathbb{R}$ is bounded and continuous,*

$$\lim_{|\mathscr{P}|\to 0} \int_{H_{\mathscr{P}}(M)} f(\gamma)\mathrm{d}\nu^0_{\mathscr{P}}(\gamma) = \int_{W_o(M)} f(\gamma) e^{-\frac{1}{6}\int_0^1 \mathrm{Scal}(\gamma(s))\mathrm{d}s}\mathrm{d}\mu(\gamma),$$

where Scal *is the scalar curvature of M and μ is the law of Brownian motion on M introduced in Section 2.1.*

For technical reasons, we need to introduce the following subspace $H^\delta_{\mathscr{P}}(M)$ of $H_{\mathscr{P}}(M)$ such that every element $\gamma \in H^\delta_{\mathscr{P}}(M)$ is a piecewise geodesic and each part $\gamma([s_{i-1}, s_i])$ is the unique geodesic linking $\gamma(s_{i-1})$ and $\gamma(s_i)$ (see [2, Sec. 5]). In fact, for any partition $\mathscr{T} \ni \mathscr{P} := \{0 = s_0 < s_1 < s_2 < ... < s_n = 1\}$ with $\Delta s_i = \varepsilon$ for $i = 1, .., n$ and each $\delta > 0$ less than the injectivity radius of M, define

$$H^\delta_{\mathscr{P}}(M) := \left\{\gamma \in H_{\mathscr{P}}(M) : \int_{s_{i-1}}^{s_i} |\gamma'(s)|\mathrm{d}s < \delta \text{ for } i = 1, 2, ..., n\right\},$$

where $s_0 = 0$. In the following we always suppose that $\delta > 0$ is less than the injectivity radius of M. Then we can easily check that $H^\delta_{\mathscr{P}}(M)$ is a locally compact separable metric space with the distance given by

$$d^{\mathscr{P}}(\gamma, \eta) = \varepsilon \sum_{i=1}^{n} \rho(\gamma_{s_i}, \eta_{s_i}), \quad \forall\, \gamma, \eta \in H^\delta_{\mathscr{P}}(M).$$

Moreover, it is easy to show that each $\gamma \in H^\delta_{\mathscr{P}}(M)$ is determined uniquely by finite points $o, \gamma(s_1), \gamma(s_2), ..., \gamma(s_n)$ (see e.g. [2, Section 5]). By Theorem 4.1 and [2, (6.1), Prop. 5.13] we have the following convergence result.

Theorem 4.2. *Suppose that $f : W(M) \to \mathbb{R}$ is bounded and continuous. Then*

$$\lim_{|\mathscr{P}|\to 0} \int_{H^\delta_{\mathscr{P}}(M)} f(\gamma) \mathrm{d}\nu^0_{\mathscr{P}}(\gamma) = \int_{W_o(M)} f(\gamma) e^{-\frac{1}{6}\int_0^1 \mathrm{Scal}(\gamma(s))\mathrm{d}s} \mathrm{d}\mu(\gamma),$$

where Scal *is the scalar curvature of M and μ is the law of Brownian motion on M introduced in Section 2.1.*

Next, we will recall some basic geometrical concepts of a Riemannian manifold M. As in Section 2, let $O(M)$ be the orthonormal frame bundle over M and let $\pi : O(M) \to M$ denote the bundle of orthogonal frames on M. Let $\mathscr{X}(M)$ be the set of all smooth vector fields and let ∇ be the Riemannian connection on M. The curvature tensor is given in terms of the Riemannian connection ∇ by the following formula:

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_X \nabla_Y Z - \nabla_{[X,Y]} Z,$$

for any vector fields $X, Y, Z \in \mathscr{X}(M)$ on M, where $[X, Y]$ is the Lie bracket of vector fields X and Y.

The Ricci curvature may be interpreted as the trace of the curvature tensor and the scalar curvature may be considered as the trace of the Ricci curvature tensor on M, that is to say,

$$\mathrm{Ric}(X) := \sum_{i=1}^{d} R(X, \bar{e}_i)\bar{e}_i, \quad \forall\, X \in T(M),$$

$$\mathrm{Scal}(x) := \sum_{i=1}^{d} \langle \mathrm{Ric}(\bar{e}_i), \bar{e}_i \rangle_{T_x M}, x \in M,$$

where $\{\bar{e}_i\}$ is an orthonormal frame. Denote the curvature form by

$$\Omega(\eta_1, \eta_2) = u^{-1} R(\pi_* \eta_1, \pi_* \eta_2) u,$$

for all $u \in O(M)$ and $\eta_1, \eta_2 \in T_u O(M)$, and for $a, b \in \mathbb{R}^d$, let

$$\Omega_u(a, b) := u^{-1} R(ua, ub) u.$$

Define

$$R_u(v,w) = u^{-1}R(v,w)u, \quad u \in O(M),\ v,w \in T_{\pi(u)}M.$$

Fix $\gamma \in H(M)$ and $X \in T_\gamma H(M)$, define $q_s(X)$ by

$$q_s(X) = \int_0^s R_{u(r)}(\gamma'(r), X(r))\mathrm{d}r, \tag{4.8}$$

where $u = /\!/(\gamma)$ is the horizontal lift of γ.

The development map $\phi : H(\mathbb{R}^d) \to H(M)$ is defined by $\phi(b) := \gamma \in H(M)$ for $b \in H(\mathbb{R}^d)$, where γ solves the functional differential equation,

$$\gamma'(s) = /\!/_s(\gamma)b'(s), \quad \gamma(0) = o.$$

The anti-development map $\phi^{-1} : H(M) \to H(\mathbb{R}^d)$ is given by $b = \phi^{-1}(\gamma)$, where

$$b(s) = \int_0^s /\!/_r^{-1}(\gamma)\gamma'(r)\mathrm{d}r.$$

For each $h \in C^\infty(H(M) \to H(\mathbb{R}^d))$ and $\gamma \in H(M)$, let $X^h(\gamma) \in T_\gamma H(M)$ be given by

$$X_s^h(\gamma) := /\!/_s(\gamma)h_s(\gamma) \quad \text{for all } s \in [0,1],$$

where $h_s(\gamma) := h(\gamma)(s)$. Given $\gamma \in H_{\mathscr{P}}(M)$, let $H_{\mathscr{P},\gamma}$ be the subspace of $H(\mathbb{R}^d)$ given by

$$H_{\mathscr{P},\gamma} := \{v \in H(\mathbb{R}^d) : v''(s) = \Omega_{u(s)}(b'(s), v(s))b'(s), \forall s \notin \mathscr{P}\},$$

where $u = /\!/(\gamma)$ and $b = \phi^{-1}(\gamma)$. By [2] we know that $v \in H_{\mathscr{P},\gamma}$ if and only if $X^v(\gamma) := /\!/(\gamma)v \in T_\gamma H_{\mathscr{P}}(M)$.

4.2 The approximation Dirichlet form $\mathscr{E}^{\mathscr{P}}$

In this subsection we will mainly derive the Dirichlet form associated with the approximation measures $\nu_{\mathscr{P}}^0$. To do that, we need to construct a family of special basis on $TH_{\mathscr{P}}(M)$.

For any $\varepsilon > 0$, take $\mathscr{T} \ni \mathscr{P} = \{0 = s_0 < s_1 < ... < s_n = 1\}$ with $\Delta s_i = s_i - s_{i-1} = \varepsilon$ for $i = 1, ..., n$. Let $\{e_a\}$ be an orthonormal basis for $\mathbb{R}^d$ given by $e_a = (0, .., 1, ...0)$. Consider the space $l^2(\mathscr{P};\mathbb{R}^d) := \left\{\hat{h} : \mathscr{P} \mapsto \mathbb{R}^d \middle| \|\hat{h}\|^2_{l^2(\mathscr{P};\mathbb{R}^d)} < \infty\right\}$ under the norm given by

$$\left\langle \hat{h}_1, \hat{h}_2 \right\rangle_{l^2(\mathscr{P};\mathbb{R}^d)} = \varepsilon \sum_{i=1}^n \langle \hat{h}_1(s_i), \hat{h}_2(s_i)\rangle_{\mathbb{R}^d}.$$

Choose an orthonormal basis $\hat{h}_{a,i} \in l^2(\mathscr{P};\mathbb{R}^d), i = 1, ..., n, a = 1, ..., d$, be given by

$$\hat{h}_{a,i}(s_j) = \begin{cases} 0, & j \neq i \\ \frac{1}{\sqrt{\varepsilon}} e_a, & j = i. \end{cases}$$

For fixed $i = 1, ..., n, a = 1, ..., d$, define $h_{a,i} : H_{\mathscr{P}}(M) \to H(\mathbb{R}^d)$ by requiring $h_{a,i}(\gamma) \in H_{\mathscr{P},\gamma}$ for all $\gamma \in H_{\mathscr{P}}(M)$ and for $s \in \mathscr{P}$, $h_{a,i}(\gamma)(s) = \hat{h}_{a,i}(s)$ for all $\gamma \in H_{\mathscr{P}}(M)$. For $\gamma \in H^{\delta}_{\mathscr{P}}(M)$, $h_{a,i}(\gamma)$ is uniquely determined by the above properties (see the proof of Lemma A.1 below). The following lemma is used to prove the quasi-regularity of the approximation Dirichlet form $\mathscr{E}^{\mathscr{P}}$.

Lemma 4.3. $\sup_{r\in[0,1]} |h_{a,i}(\gamma)(r)| \in L^p(H^{\delta}_{\mathscr{P}}(M), \nu^0_{\mathscr{P}}), p > 1$, *with* $\delta > 0$ *satisfying* $\cosh(\sqrt{\kappa_0}\delta)\kappa_0\delta^2 < 1$. *Here* κ_0 *is an upper bound for the norms of the curvature tensor* R *(or equivalently* Ω*).*

Proof. We only consider $h_{a,1}(r)$ on $[0, \varepsilon]$. The other cases can be handled similarly. We use the following notations: $\gamma \in H^{\delta}_{\mathscr{P}}(M)$, $b := \phi^{-1}(\gamma)$ the anti-development map, $u := /\!/(\gamma)$ and $A(s) := \Omega_{u(s)}(b'(s), \cdot)b'(s)$, $b'(s) = \Delta_i b/\varepsilon$ for $s \in (s_{i-1}, s_i]$ with $\Delta_i b = b(s_i) - b(s_{i-1})$. A similar argument as in the proof of Lemma 8.2 in [2] implies that for $r \in [0, \varepsilon]$,

$$|h_{a,1}(r)| \leq |h'_{a,1}(0)|\frac{\sinh\sqrt{K}r}{\sqrt{K}}, \tag{4.9}$$

where

$$K := \sup_{s\in[0,\varepsilon]} \|A(s)\| \leq \kappa_0\frac{|\Delta_1 b|^2}{\varepsilon^2} \leq \kappa_0\frac{\delta^2}{\varepsilon^2} \tag{4.10}$$

and $\|\cdot\|$ is the norm of the matrix. In fact, by Taylor's theorem we have for $s \in [0, \varepsilon]$

$$\begin{aligned} h(s) =& h(0) + sh'(0) + \int_0^s h''(u)(s-u)\mathrm{d}u \\ =& sh'(0) + \int_0^s A(u)h(u)(s-u)\mathrm{d}u. \end{aligned} \tag{4.11}$$

Here and in the following we omit the subindex of h if there's no confusion. Then for $s \in [0, \varepsilon]$

$$|h(s)| \leq s|h'(0)| + K\int_0^s |h(u)|(s-u)\mathrm{d}u =: f(s).$$

Note that $f(0) = 0, f'(s) = |h'(0)| + K\int_0^s |h(u)|\mathrm{d}u$ and

$$f''(s) = K|h(s)| \leq Kf(s),$$

that is

$$f''(s) = Kf(s) + \eta(s), \quad f(0) = 0, \quad f'(0) = |h'(0)|,$$

where $\eta(s) := f''(s) - Kf(s) \leq 0$. Then by the variation of parameter (cf. the proof of [2, Lemma 8.2]) we have

$$f(s) =|h'(0)|\frac{\sinh\sqrt{K}s}{\sqrt{K}} + \int_0^s \frac{\sinh\sqrt{K}(s-r)}{\sqrt{K}}\eta(r)\mathrm{d}r \leq |h'(0)|\frac{\sinh\sqrt{K}s}{\sqrt{K}},$$

which implies (4.9). Also (4.11) implies that

$$h'(0) = \frac{1}{\varepsilon}\left[h(\varepsilon) - \int_0^\varepsilon (\varepsilon - u)A(u)h(u)\mathrm{d}u\right].$$

Then by (4.9) we have

$$\begin{aligned}|h(s)| \leq & \frac{\sinh\sqrt{K}s}{\sqrt{K}}\frac{1}{\varepsilon}\left[\frac{1}{\sqrt{\varepsilon}} + K\int_0^\varepsilon (\varepsilon - u)|h(u)|\mathrm{d}u\right] \\ \leq & \frac{\sinh\sqrt{K}\varepsilon}{\sqrt{K}}\frac{1}{\varepsilon\sqrt{\varepsilon}} + \frac{\sqrt{K}\varepsilon}{2}\sinh\sqrt{K}\varepsilon \sup_{u\in[0,\varepsilon]}|h(u)|\end{aligned}$$

By (4.10) we have

$$\sqrt{K}\varepsilon \sinh\sqrt{K}\varepsilon \leq \sqrt{\kappa_0}|\Delta_1 b|\sinh(\sqrt{\kappa_0}|\Delta_1 b|) \leq \kappa_0\delta^2\cosh(\sqrt{\kappa_0}\delta) \leq 1.$$

Thus we know that

$$\sup_{s\in[0,\varepsilon]}|h(s)| \leq \frac{2}{\varepsilon\sqrt{\varepsilon}}\frac{\sinh\sqrt{K}\varepsilon}{\sqrt{K}} \leq \frac{2}{\sqrt{\varepsilon}}\cosh\sqrt{K}\varepsilon \leq \frac{2}{\sqrt{\varepsilon}}\cosh\kappa_0\delta,$$

which implies the result. Here we used the elementary inequality $\sinh(a)/a \leq \cosh(a)$. $\square$

In the following we fix a δ as in Lemma 4.3 and we consider $H^\delta_{\mathscr{P}}(M)$ as the state space for the approximation Dirichlet form. Let

$$\mathscr{F}C_0^{\mathscr{P}} := \left\{H^\delta_{\mathscr{P}}(M) \ni \gamma \mapsto F(\gamma) := f(\gamma_{s_1}, \gamma_{s_2}, ..., \gamma_{s_n}), f \in C_b^1(M^n)\right\} \cap C_0^1(H^\delta_{\mathscr{P}}(M)),$$

with $C_0^1(H^\delta_{\mathscr{P}}(M))$ being continuous, differentiable functions from $H^\delta_{\mathscr{P}}(M)$ to $\mathbb{R}$ with compact support. Since $\gamma \in H^\delta_{\mathscr{P}}(M)$, γ is determined by $\gamma(s_1), ..., \gamma(s_n)$. This implies that every $u = f(\gamma_{t_1}, ..., \gamma_{t_m})$ with $f \in C_b^1(M^m), 0 < t_1 < t_2 < ... < t_m \leq 1$, can be expressed as $g(\gamma_{s_1}, \gamma_{s_2}, ..., \gamma_{s_n}) \in \mathscr{F}C_0^{\mathscr{P}}$. By this we can easily conclude that $\mathscr{F}C_0^{\mathscr{P}}$ is dense in $L^2(H^\delta_{\mathscr{P}}(M), \nu^0_{\mathscr{P}})$.

For each $F \in \mathscr{F}C_0^{\mathscr{P}}$, the directional derivative of F with respect to $h_{a,i}$ is given by

$$D_{h_{a,i}}F(\gamma) = \langle\nabla_i f(\gamma), /\!/_{s_i}(\gamma)h_{a,i}(\gamma)(s_i)\rangle_{T_{\gamma_{s_i}}M}, \quad \gamma \in H^\delta_{\mathscr{P}}(M), \tag{4.12}$$

where $\nabla_i f(\gamma) = \nabla_i f(\gamma_{s_1}, ..., \gamma_{s_n})$. Define for $\gamma \in H^\delta_{\mathscr{P}}(M)$,

$$DF(\gamma) := \sum_{i=1}^n \sum_{a=1}^d D_{h_{a,i}}F(\gamma)\hat{h}_{a,i} \in l^2(\mathscr{P}; \mathbb{R}^d).$$

In this section we also use the notation DF as in Section 2 for simplicity.

Remark 4.4. *By the definition of h and $\hat{h}$ we know that $DF(\gamma)=\sum_{i=1}^n\sum_{a=1}^d D_{h_{a,i}}F(\gamma)$ $h_{a,i}|_{\mathscr{P}}$. For $F\in\mathscr{F}C_0^{\mathscr{P}}$ the directional derivative should be along $h_{a,i}\in H_{\mathscr{P},\gamma}$, which does not form a basis for $L^2([0,1];\mathbb{R}^d)$. Therefore, we replace $L^2([0,1];\mathbb{R}^d)$ in $\mathscr{E}$ by $l^2(\mathscr{P};\mathbb{R}^d)$ in the Dirichlet form $\mathscr{E}^{\mathscr{P}}$ and consider $DF(\gamma)\in l^2(\mathscr{P};\mathbb{R}^d)$. For $F\in\mathscr{F}C_b^1,\gamma\in H^\delta_{\mathscr{P}}$, we can find $F_\varepsilon\in\mathscr{F}C_0^{\mathscr{P}}$ such that $\|DF_\varepsilon(\gamma)\|_{l^2(\mathscr{P},\mathbb{R}^d)}\to\|DF(\gamma)\|_{L^2([0,1];\mathbb{R}^d)}$, as $\varepsilon\to 0$, where the second DF is the L^2-gradient in Section 2.*

Next, we will introduce the quadratic form on $H^\delta_{\mathscr{P}}(M)$. For any $u,v\in\mathscr{F}C_0^{\mathscr{P}}$, define

$$\mathscr{E}^{\mathscr{P}}(u,v)=\frac{1}{2}\int_{H^\delta_{\mathscr{P}}(M)}\langle Du,Dv\rangle_{l^2(\mathscr{P},\mathbb{R}^d)}\mathrm{d}\nu^0_{\mathscr{P}}=\sum_{i=1}^n\sum_{a=1}^d\frac{1}{2}\int_{H^\delta_{\mathscr{P}}(M)}D_{h_{a,i}}u(\gamma)D_{h_{a,i}}v(\gamma)\mathrm{d}\nu^0_{\mathscr{P}}.$$

To prove the closability of the form $\mathscr{E}^{\mathscr{P}}$, we need to establish the following integration by parts formula for $\nu^0_{\mathscr{P}}$.

Lemma 4.5 (Integration by parts formula)**.** *For every $h_{a,j}$, $a=1,2,...,d$, $j=1,2,...,n$, we have the following integration by parts formula*

$$\int_{H^\delta_{\mathscr{P}}(M)}X^{h_{a,j}}f\mathrm{d}\nu^0_{\mathscr{P}}=\int_{H^\delta_{\mathscr{P}}(M)}f\beta_{\mathscr{P}}(h_{a,j})\mathrm{d}\nu^0_{\mathscr{P}}\tag{4.13}$$

for all $f\in C_0^1(H^\delta_{\mathscr{P}}(M))$ with δ as in Lemma 4.3, where for $p>1$

$$L^p(H^\delta_{\mathscr{P}}(M),\nu^0_{\mathscr{P}})\ni\beta_{\mathscr{P}}(h_{a,j})=\frac{1}{\varepsilon}\langle\Delta_j b-\Delta_{j+1}b,h_{a,j}(s_j)\rangle+\varepsilon\sum_{a_1=1}^d\left\langle q(X^{h_{a_1,j}})h_{a_1,j},h_{a,j}\right\rangle(s_j).$$

Here $b=\phi^{-1}(\gamma)$ with $\Delta_j b=b(s_j)-b(s_{j-1})$ for $j=1,...,n$, $\Delta_{n+1}b=0$ and q is defined in (4.8)*.*

Proof. By Stoke's theorem we have for $f\in C_0^1(H^\delta_{\mathscr{P}}(M))$

$$0=\int_{H^\delta_{\mathscr{P}}(M)}\left[(X^{h_{a,j}}f)\nu^0_{\mathscr{P}}+fL_{X^{h_{a,j}}}\nu^0_{\mathscr{P}}\right],$$

where we recall

$$\nu^0_{\mathscr{P}}=\frac{1}{Z^0_{\mathscr{P}}}e^{-\frac{1}{2}E}\mathrm{Vol}_{G^0_{\mathscr{P}}}.$$

By the same arguments as in [2, Lemma 7.3] and (4.7), we know that $\{X^{h_{a,i}},i=1,...,n,a=1,...,d\}$ is a globally defined orthonormal frame for $(H_{\mathscr{P}}(M),G^0_{\mathscr{P}})$. Then we have for $a=1,...,d, j=1,...,n$,

$$\begin{aligned}L_{X^{h_{a,j}}}\nu^0_{\mathscr{P}} &= -\frac{1}{2}(X^{h_{a,j}}E)(\gamma)\cdot\nu^0_{\mathscr{P}} + L_{X^{h_{a,j}}}\mathrm{Vol}_{G^0_{\mathscr{P}}}\\ &= -\frac{1}{2}(X^{h_{a,j}}E)(\gamma)\cdot\nu^0_{\mathscr{P}} + \frac{1}{Z^0_{\mathscr{P}}}e^{-\frac{1}{2}E}\sum_{i=1}^n\sum_{a_1=1}^d G^0_{\mathscr{P}}([X^{h_{a_1,i}},X^{h_{a,j}}],X^{h_{a_1,i}})\cdot\mathrm{Vol}_{G^0_{\mathscr{P}}}\\ &= -\frac{1}{2}(X^{h_{a,j}}E)(\gamma)\cdot\nu^0_{\mathscr{P}} + \sum_{i=1}^n\sum_{a_1=1}^d G^0_{\mathscr{P}}([X^{h_{a_1,i}},X^{h_{a,j}}],X^{h_{a_1,i}})\cdot\nu^0_{\mathscr{P}}.\end{aligned}$$

By the Cartan development map and [2, Lemma 7.1], we know

$$\begin{aligned}(X^{h_{a,j}}E)(\gamma) &= 2\int_0^1\left\langle\gamma'(s),\frac{\nabla X^{h_{a,j}}(\gamma)(s)}{\mathrm{d}s}\right\rangle_{T_{\gamma_s}M}\mathrm{d}s = 2\int_0^1\left\langle /\!/_s(\gamma)b'(s), /\!/_s(\gamma)h'_{a,j}(s)\right\rangle_{T_{\gamma_s}M}\mathrm{d}s\\ &= 2\int_0^1\left\langle b'(s),h'_{a,j}(s)\right\rangle\mathrm{d}s = \frac{2}{\varepsilon}\left\langle\Delta_j b-\Delta_{j+1}b, h_{a,j}(s_j)\right\rangle,\end{aligned}$$

where we used $b'(s_i)=b'(r)=\Delta_i b/\varepsilon$ for $r\in(s_{i-1},s_i], i=1,...,n$, in the last equality.

Furthermore, by [2, Theorem 3.5] we have

$$\begin{aligned}&\sum_{i=1}^n\sum_{a_1=1}^d G^0_{\mathscr{P}}([X^{h_{a_1,i}},X^{h_{a,j}}],X^{h_{a_1,i}})\\ =&\varepsilon\sum_{i=1}^n\sum_{a_1=1}^d\sum_{k=1}^n\left\langle X^{h_{a_1,i}}h_{a,j}-X^{h_{a,j}}h_{a_1,i},h_{a_1,i}\right\rangle(s_k)\\ &-\varepsilon\sum_{i=1}^n\sum_{a_1=1}^d\sum_{k=1}^n\left\langle q(X^{h_{a,j}})h_{a_1,i}-q(X^{h_{a_1,i}})h_{a,j},h_{a_1,i}\right\rangle(s_k)\\ =&\varepsilon\sum_{a_1=1}^d\left\langle q(X^{h_{a_1,j}})h_{a,j},h_{a_1,j}\right\rangle(s_j).\end{aligned}$$

Here we used $X^{h_{a_1,i}}h_{a,j}(s_k)=0$, since $h_{a,j}(s_k)$ is independent of γ and we also used $\left\langle q(X^{h_{a_1,i}})h_{a,j},h_{a_1,i}\right\rangle(s_k)\neq 0$ only for $i=j=k$ and the skew symmetry of $q(X^{h_{a_1,j}})$ to deduce $\left\langle q(X^{h_{a,j}})h_{a_1,i},h_{a_1,i}\right\rangle=0$. Thus, by Stoke's theorem we know that (4.13) holds. □

Based on the above integration by parts formula, we obtain the closablity of the following quadratic form $(\mathscr{E}^{\mathscr{P}},\mathscr{F}C_0^{\mathscr{P}})$ on $L^2(H^\delta_{\mathscr{P}}(M);\nu^0_{\mathscr{P}})$. Now we prove:

Theorem 4.6. *The quadratic form $(\mathscr{E}^{\mathscr{P}},\mathscr{F}C_0^{\mathscr{P}})$ is closable in $L^2(H^\delta_{\mathscr{P}}(M);\nu^0_{\mathscr{P}})$ and its closure $(\mathscr{E}^{\mathscr{P}},\mathscr{D}(\mathscr{E}^{\mathscr{P}}))$ is a quasi-regular Dirichlet form.*

Proof. (a) **Dirichlet form:** First we prove that $(\mathscr{E}^{\mathscr{P}}, \mathscr{F}C_0^{\mathscr{P}})$ is closable. Let $\{F_k\}_{k=1}^{\infty} \subseteq \mathscr{F}C_0^{\mathscr{P}}$ with

$$\lim_{k\to\infty} \nu_{\mathscr{P}}^0\left[F_k^2\right] = 0, \quad \lim_{k,m\to\infty} \mathscr{E}^{\mathscr{P}}(F_k - F_m, F_k - F_m) = 0. \tag{4.14}$$

By (4.14), we know that $\{DF_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $L^2\left(H_{\mathscr{P}}^{\delta}(M) \to l^2(\mathscr{P};\mathbb{R}^d);\nu_{\mathscr{P}}^0\right)$, for which there exists a limit Φ. It suffices to prove that $\Phi = 0$. Taking $F \in \mathscr{F}C_0^{\mathscr{P}}$, we have for $a = 1, ..., d, i = 1, ..., n$,

$$X^{h_{a,i}}F = D_{h_{a,i}}F = \langle DF, \hat{h}_{a,i}\rangle_{l^2(\mathscr{P};\mathbb{R}^d)}. \tag{4.15}$$

Thus, by the above integration by parts formula (4.13), we have for $G \in \mathscr{F}C_0^{\mathscr{P}}$

$$\begin{aligned} \nu_{\mathscr{P}}^0\left[G\langle DF_k, \hat{h}_{a,i}\rangle_{l^2(\mathscr{P};\mathbb{R}^d)}\right] &= \nu_{\mathscr{P}}^0\left[GX^{h_{a,i}}F_k\right] \\ &= \nu_{\mathscr{P}}^0\left[F_kG\beta_{\mathscr{P}}(h_{a,i})\right] - \nu_{\mathscr{P}}^0\left[F_kX^{h_{a,i}}G\right]. \end{aligned} \tag{4.16}$$

Since G and DG are bounded and $\beta_{\mathscr{P}}(h_{a,i}) \in L^2(H_{\mathscr{P}}^{\delta}(M);\nu_{\mathscr{P}}^0)$, F_k converges to 0 in $L^2(\nu_{\mathscr{P}}^0)$. By the dominated convergence theorem, taking the limit in (4.16), we obtain

$$\nu_{\mathscr{P}}^0\left[G\langle \Phi, \hat{h}_{a,i}\rangle_{l^2(\mathscr{P};\mathbb{R}^d)}\right] = 0, \quad \forall\, G \in \mathscr{F}C_0^{\mathscr{P}}.$$

Therefore, there exists a $\nu_{\mathscr{P}}^0$-null set Ω_0, such that

$$\langle \Phi(\gamma), \hat{h}_{a,i}\rangle_{l^2(\mathscr{P};\mathbb{R}^d)} = 0, \gamma \notin \Omega_0.$$

Since $\{\hat{h}_{a,i}\}$ is an orthonormal basis in $l^2(\mathscr{P};\mathbb{R}^d)$, we conclude that $\Phi = 0$, a.s., and hence $(\mathscr{E}^{\mathscr{P}}, \mathscr{F}C_0^{\mathscr{P}})$ is closable. Moreover, it is standard that the closure $(\mathscr{E}^{\mathscr{P}}, D(\mathscr{E}^{\mathscr{P}}))$ is a Dirichlet form.

(b) **Quasi-regularity:** Since $\gamma \in H_{\mathscr{P}}^{\delta}(M)$ is uniquely determined by $(\gamma(s_1), \gamma(s_2), ..., \gamma(s_n))$, we can easily find a countable dense subset in $\mathscr{F}C_0^{\mathscr{P}}$ to separate the points in $H_{\mathscr{P}}^{\delta}(M)$. In fact, similarly as in the proof of Theorem 2.2, we use ψ to denote the Nash embedding map. For $k \in \mathbb{N}$, choose $\chi_k \in \mathscr{F}C_0^{\mathscr{P}}$ satisfying $\chi_k(\gamma) = 1$ if $d(\gamma_{s_{i-1}}, \gamma_{s_i}) \leq \delta - \frac{1}{k}$ for every $i = 1, ..., n$. Since $H_{\mathscr{P}}^{\delta}(M)$ is separable we can choose a fixed countable dense set $\{\xi^m | m \in \mathbb{N}\}$ in $H_{\mathscr{P}}^{\delta}(M)$. Take $\{v_{mki}(\gamma) = |\psi(\gamma_{s_i}) - \psi(\xi_{s_i}^m)|^2\chi_k(\gamma), k, m \in \mathbb{N}, i = 1, ..., n\}$, which is a countable dense subset in $\mathscr{F}C_0^{\mathscr{P}}$ and separate the points in $H_{\mathscr{P}}^{\delta}(M)$. Since $H_{\mathscr{P}}^{\delta}(M)$ is locally compact, the tightness of the corresponding capacity follows immediately. Now the quasi-regularity of the Dirichlet form follows. $\square$

Similarly as in Section 2, we can construct a Markov process associated with the above Dirichlet form. We consider $H_{\mathscr{P},\Delta}^{\delta}(M)$ as the one point compactification of $H_{\mathscr{P}}^{\delta}(M)$ (c.f. [44, P88]). Any function $f : H_{\mathscr{P}}^{\delta}(M) \to \mathbb{R}$ is considered as a function on $H_{\mathscr{P},\Delta}^{\delta}(M)$ by setting $f(\Delta) = 0$. By the above proof for quasi-regularity and [44, Chap. V Corollary 2.16] we obtain:

Theorem 4.7. *There exists a Markov (Hunt) diffusion process* $M^{\mathscr{P}} = \big(\Omega, \mathscr{F}, \mathscr{M}_t, (x_t^{\mathscr{P}})_{t\geq 0}, (P^z)_{z\in H^\delta_{\mathscr{P},\Delta}(M)}\big)$ *with state space* $H^\delta_{\mathscr{P}}(M)$ properly associated with $(\mathscr{E}^{\mathscr{P}}, \mathscr{D}(\mathscr{E}^{\mathscr{P}}))$, *i.e. for* $u \in L^2(H^\delta_{\mathscr{P}}(M); \nu^0_{\mathscr{P}}) \cap \mathscr{B}_b(H^\delta_{\mathscr{P}}(M))$, *the transition semigroup* $P_t^{\mathscr{P}} u(z) := E^z[u(x_t^{\mathscr{P}})]$ *is an* $\mathscr{E}^{\mathscr{P}}$*-quasi-continuous version of* $T_t^{\mathscr{P}} u$ *for all* $t > 0$, *where* $T_t^{\mathscr{P}}$ *is the semigroup associated with* $(\mathscr{E}^{\mathscr{P}}, \mathscr{D}(\mathscr{E}^{\mathscr{P}}))$.

By the integration by parts formula in Lemma 4.5 we can write the explicit martingale solution to the Markov process constructed for $\nu^0_{\mathscr{P}}$.

Theorem 4.8. *There exists a* properly $\mathscr{E}^{\mathscr{P}}$-exceptional set $S \subset E$, *i.e.* $\nu^0_{\mathscr{P}}(S) = 0$ *and* $P^z[x^{\mathscr{P}}(t) \in H^\delta_{\mathscr{P},\Delta}(M) \setminus S, \forall t \geq 0] = 1$ *for* $z \in H^\delta_{\mathscr{P},\Delta}(M) \setminus S$, *such that* $\forall z \in H^\delta_{\mathscr{P}}(M) \setminus S$ *under* P^z, *the sample paths of the associated process* $M^{\mathscr{P}}$ *satisfy the following for* $u(\gamma) = f(\gamma_{s_1}, .., \gamma_{s_n}) \in \mathscr{F}C_0^{\mathscr{P}}$ *with* $f \in C^\infty(M^n)$,

$$
\begin{aligned}
u(x_t^{\mathscr{P}}) - u(x_0^{\mathscr{P}}) &= \frac{1}{2}\sum_{i=1}^n \sum_{a=1}^d \int_0^t X^{h_{a,i}} X^{h_{a,i}} u(x_l^{\mathscr{P}}) \mathrm{d}l \\
&\quad - \frac{1}{2}\sum_{i=1}^n \sum_{a=1}^d \int_0^t D_{h_{a,i}} u(x_l^{\mathscr{P}}) \beta_{\mathscr{P}}(h_{a,i})(x_l^{\mathscr{P}}) \mathrm{d}l + M_t^u,
\end{aligned}
\tag{4.17}
$$

where $\beta_{\mathscr{P}}(h_{a,j})$ *is given in Lemma 4.5 and* M_t^u *is a martingale with the quadratic variation process given by*

$$
\begin{aligned}
\langle M_t^u \rangle &= \sum_{i=1}^n \sum_{a=1}^d \int_0^t \left\langle X^{h_{a,i}} u(x_r^{\mathscr{P}}), X^{h_{a,i}} u(x_r^{\mathscr{P}}) \right\rangle \mathrm{d}r \\
&= \frac{1}{\varepsilon} \sum_{i=1}^n \int_0^t \left\langle /\!/_{s_i}^{-1} \nabla_i f(x_r^{\mathscr{P}}), /\!/_{s_i}^{-1} \nabla_i f(x_r^{\mathscr{P}}) \right\rangle \mathrm{d}r \\
&= \frac{1}{\varepsilon} \sum_{i=1}^n \int_0^t \left\langle \nabla_i f(x_r^{\mathscr{P}}), \nabla_i f(x_r^{\mathscr{P}}) \right\rangle \mathrm{d}r,
\end{aligned}
\tag{4.18}
$$

with $\nabla_i f(\gamma) = \nabla_i f(\gamma_{s_1}, ..., \gamma_{s_n})$.

Proof. By (4.12) and applying the integration by parts formula (4.13) we have for

$v \in \mathscr{F}C_0^{\mathscr{P}}$,

$$
\begin{aligned}
&\mathscr{E}^{\mathscr{P}}(u,v)\\
&=\frac{1}{2}\sum_{i=1}^{n}\sum_{a=1}^{d}\int_{H^{\delta}_{\mathscr{P}}(M)}\left\langle Du,\hat{h}_{a,i}\right\rangle_{l^2(\mathscr{P};\mathbb{R}^d)}\left\langle Dv,\hat{h}_{a,i}\right\rangle_{l^2(\mathscr{P};\mathbb{R}^d)}\mathrm{d}\nu^0_{\mathscr{P}}\\
&=\frac{1}{2}\sum_{i=1}^{n}\sum_{a=1}^{d}\int_{H^{\delta}_{\mathscr{P}}(M)}X^{h_{a,i}}uX^{h_{a,i}}v\mathrm{d}\nu^0_{\mathscr{P}}\\
&=\frac{1}{2}\sum_{i=1}^{n}\sum_{a=1}^{d}\int_{H^{\delta}_{\mathscr{P}}(M)}v\left[(X^{h_{a,i}}u)\beta_{\mathscr{P}}(h_{a,i})-X^{h_{a,i}}(X^{h_{a,i}}u)\right]\mathrm{d}\nu^0_{\mathscr{P}}\\
&=-\int_{H^{\delta}_{\mathscr{P}}(M)}v\left\{\frac{1}{2}\sum_{i=1}^{n}\sum_{a=1}^{d}X^{h_{a,i}}(X^{h_{a,i}}u)-\frac{1}{2}\sum_{i=1}^{n}\sum_{a=1}^{d}\beta_{\mathscr{P}}(h_{a,i})D_{h_{a,i}}u\right\}\mathrm{d}\nu^0_{\mathscr{P}}\\
&=-\int_{H^{\delta}_{\mathscr{P}}(M)}vL^{\mathscr{P}}u\mathrm{d}\nu^0_{\mathscr{P}},
\end{aligned}
$$

where $L^{\mathscr{P}}$ is the generator of $\mathscr{E}^{\mathscr{P}}$ (see [44, Chap. 1]) and in the third equality we apply (4.13) to $vX^{h_{a,j}}u$, which is also a smooth function on $H^{\delta}_{\mathscr{P}}(M)$. Then by the Fukushima decomposition we have under P^z

$$
u(x_t^{\mathscr{P}})-u(x_0^{\mathscr{P}})=\int_0^t L^{\mathscr{P}}u(x_r^{\mathscr{P}})\mathrm{d}r+M_t^u,
$$

where M_t^u is a martingale with the quadratic variation process given by (4.18) (cf. [27, Thm. 5.2.3]. Thus the result follows. □

Remark 4.9. *We can also write $\mathscr{E}^{\mathscr{P}}$ in the following way. In fact, by [2, Sec. 5] we know that for any $u=f(\gamma_{s_1},...,\gamma_{s_n})\in\mathscr{F}C_0^{\mathscr{P}}$,*

$$
\begin{aligned}
\mathscr{E}^{\mathscr{P}}(u,u)=&\frac{1}{2}\int_{H^{\delta}_{\mathscr{P}}(M)}\langle Du,Du\rangle_{l^2(\mathscr{P},\mathbb{R}^d)}d\nu^0_{\mathscr{P}}\\
=&\sum_{i=1}^{n}\sum_{a=1}^{d}\frac{1}{2}\int_{H^{\delta}_{\mathscr{P}}(M)}D_{h_{a,i}}u(\gamma)D_{h_{a,i}}u(\gamma)d\nu^0_{\mathscr{P}}\\
=&\frac{1}{\varepsilon}\sum_{i=1}^{n}\frac{1}{2}\int_{H^{\delta}_{\mathscr{P}}(M)}\langle\nabla_i f,\nabla_i f\rangle_{T_{\gamma_{s_i}}M}d\nu^0_{\mathscr{P}}\\
=&\frac{1}{\varepsilon}\sum_{i=1}^{n}\frac{1}{2}\int_{M^{\mathscr{P}}_{\delta}}\langle\nabla_i f,\nabla_i f\rangle_{T_{\gamma_{s_i}}M}d\mu_{\mathscr{P}},
\end{aligned}
\tag{4.19}
$$

with

$$
M^{\mathscr{P}}_{\delta}:=\left\{\mathbf{x}=(x_{s_1},...,x_{s_n})\in M^n:\rho(x_{s_{i-1}},x_{s_i})<\delta\ \textit{for}\ i=1,2,...,n\right\},
$$

and $\mu_{\mathscr{P}} := \frac{1}{Z_{\mathscr{P}}}\exp(-\frac{1}{2}E_{\mathscr{P}})\mathrm{Vol}_{g_{\mathscr{P}}}$, *where the energy form* $E_{\mathscr{P}}(\mathbf{x})$ *is defined by*

$$E_{\mathscr{P}}(\mathbf{x}) := \sum_{i=1}^{n} \frac{\rho^2(x_{s_{i-1}}, x_{s_i})}{\varepsilon},$$

and $\mathrm{Vol}_{g_{\mathscr{P}}}$ *denotes the volume measure on* M^n *with respect to the metric* $g_\varepsilon^{\mathscr{P}} := \varepsilon g \times \varepsilon g \times ... \times \varepsilon g$. *As a result, we can also view* $\mathscr{E}^{\mathscr{P}}$ *as a quasi-regular Dirichlet form in* $L^2(M_\delta^{\mathscr{P}}, g_\varepsilon^{\mathscr{P}})$.

4.3 Derivation of the limiting process

In order to present a better understand of the stochastic heat equation in Section 4.2. We have two ways to write the limiting equation. The first one is invoking the stochastic parallel translation U:

4.3.1 Limiting equation invoking the stochastic parallel translation U:

In the following we choose $\delta > 0$ satisfying the conditions in Lemma A.1 below. Then the associated finite dimension geodesic space $(H_{\mathscr{P}}^\delta(M), G_{\mathscr{P}}^0)$ is a smooth manifold with nd dimensions. We know that $X^{h_{a,j}}$ is a standard orthonormal frame fields in $(H_{\mathscr{P}}^\delta(M), G_{\mathscr{P}}^0)$ and the associated Laplace operator $\Delta_{\mathscr{P}}$ is defined by

$$\Delta_{\mathscr{P}} u = \sum_{i=1}^{n}\sum_{a=1}^{d} (X^{h_{a,i}})^2 u, \quad u \in C^\infty(H_{\mathscr{P}}^\delta(M)). \tag{4.20}$$

Set

$$\beta_{\mathscr{P}} = \frac{1}{2}\sum_{i=1}^{n}\sum_{a=1}^{d} \beta_{\mathscr{P}}(h_{a,i})h_{a,i}. \tag{4.21}$$

Then, the generator associated to the Dirichlet form $\mathscr{E}^{\mathscr{P}}$ can be written as

$$L_{\mathscr{P}} = \frac{1}{2}\Delta_{\mathscr{P}} - X^{\beta_{\mathscr{P}}} \cdot \nabla_{\mathscr{P}} \tag{4.22}$$

where $\nabla_{\mathscr{P}}$ is the unique gradient associated to the metric $G_{\mathscr{P}}^0$. Thus, the associated diffusion process satisfies the following equation under P^z, for q.-e. $z \in H_{\mathscr{P}}^\delta(M)$: for $i = 1, ..., n$,

$$\mathrm{d}x_t^{\mathscr{P}}(s_i) = \sum_{a=1}^{d} X^{h_{a,i}}(x_t^{\mathscr{P}})(s_i) \circ \mathrm{d}W_t^{a,i} - X^{\beta_{\mathscr{P}}}(x_t^{\mathscr{P}})(s_i)\mathrm{d}t, \tag{4.23}$$

where $\{W^{a,i}\}$ is a sequence of independent Brownian motions, $\circ$ means the Stratonovich integral and

$$
\begin{aligned}
X^{\beta_{\mathscr{P}}}(\gamma)(s_i) &= /\!/_{s_i}(\gamma)\beta_{\mathscr{P}}(s_i) = \frac{1}{2}\sum_{a=1}^{d}\beta_{\mathscr{P}}(h_{a,i})/\!/_{s_i}(\gamma)h_{a,i}(s_i) \\
&= \frac{1}{2\sqrt{\varepsilon}}\sum_{a=1}^{d}\beta_{\mathscr{P}}(h_{a,i})/\!/_{s_i}(\gamma)e_a.
\end{aligned} \tag{4.24}
$$

As mentioned in Remark 2.5 (iii) we can also construct the L^2-Dirichlet form $(\mathscr{E}^0, \mathscr{D}(\mathscr{E}^0))$ with respect to the reference measure $\mu_0 := e^{-\frac{1}{6}\int_0^1 \mathrm{Scal}(\gamma(s))\mathrm{d}s}\mathrm{d}\mu(\gamma)$. We conjecture for which heuristic proofs are included in the appendix.

Conjecture I (with limiting equation in terms of stochastic parallel translation U): $(\mathscr{E}^{\mathscr{P}}, \mathscr{D}(\mathscr{E}^{\mathscr{P}}))$ Mosco converges to $(\mathscr{E}^0, \mathscr{D}(\mathscr{E}^0))$, as $\varepsilon \to 0$. The Markov process Φ given by $(\mathscr{E}^0, \mathscr{D}(\mathscr{E}^0))$ satisfies the following heuristic equation

$$
\mathrm{d}\Phi_{t,s} = \frac{1}{2}\frac{\nabla}{ds}\partial_s\Phi_{t,s}\mathrm{d}t - \frac{1}{4}\mathrm{Ric}(\partial_s\Phi_{t,s})\mathrm{d}t + \frac{1}{12}\nabla\mathrm{Scal}(\Phi_{t,s})\mathrm{d}t + U_{t,s}(\Phi)\circ\mathrm{d}W_t, \tag{4.25}
$$

where $U_{t,\cdot}$ is the stochastic parallel translation for Φ_t introduced in Section 2.1, W is an $L^2([0,1];\mathbb{R}^d)$-cylindrical Wiener process and $\circ$ means renormalization (see Remark 4.10).

4.3.2 Limiting equation invoking vector fields σ:

Above we used the Laplace operator $\Delta_{\mathscr{P}}$. Now we derive the diffusion equation associated to the Dirichelt form $\mathscr{E}^{\mathscr{P}}$ by using the Laplacian on finite dimensional manifolds and the vector fields σ in (1.2). In this case, $\sum_{a=1}^m \sigma_a^2$ is equal to the Laplace-Beltrami operator. Thus by using (4.22) and [42, Lemma 5.23], it is easy to prove that for $u = f(\gamma_{s_1}, ..., \gamma_{s_n}) \in \mathscr{F}C_0^{\mathscr{P}}$,

$$
L_{\mathscr{P}}u = \sum_{i=1}^{n}[\frac{1}{2\varepsilon}\Delta^{(i)}f - X^{\beta^0_{\mathscr{P}}}(\gamma)(s_i)\cdot\nabla_i f], \tag{4.26}
$$

where $\Delta^{(i)}, \nabla_i$ mean the Laplace-Beltrami operator and the gradient with respect to the i-th variable.

$$
X^{\beta^0_{\mathscr{P}}}(\gamma)(s_i) = \frac{1}{2}/\!/_{s_i}(\gamma)\frac{1}{\varepsilon}(\Delta_j b - \Delta_{j+1}b) = \frac{1}{2\varepsilon}(\gamma'(s_i-) - \gamma'(s_i+)). \tag{4.27}
$$

Therefore, the associated diffusion process satisfies the following equation under P^z q.-e. $z \in H^{\delta}_{\mathscr{P}}(M)$: for $i = 1, ..., n$,

$$
\mathrm{d}x_t^{\mathscr{P}}(s_i) = \frac{1}{\sqrt{\varepsilon}}\sum_{a=1}^{m}\sigma_a(x_t^{\mathscr{P}})(s_i)\circ\mathrm{d}W_t^{a,i} - X^{\beta^0_{\mathscr{P}}}(x_t^{\mathscr{P}})(s_i)\mathrm{d}t, \tag{4.28}
$$

where $\{W^{a,i}\}$ is a sequence of independent Brownian motions, $\circ$ means the Stratonovich integral. Now we conjecture:

Conjecture II in terms of vector fields σ: $x^{\mathscr{P}}$ converge to Φ, as $\varepsilon \to 0$, with Φ satisfying the following heuristic equation

$$\mathrm{d}\Phi = \frac{1}{2}\frac{\nabla}{ds}\partial_s \Phi \mathrm{d}t + \sigma_a(\Phi) \circ \mathrm{d}W_t^a, \tag{4.29}$$

where W is an $L^2([0,1];\mathbb{R}^d)$-cylindrical Wiener process and $\circ$ means renormalization (see Remark 4.10).

Remark 4.10. (i) *We only present (so far) heuristic proofs in the appendix. For the flat case, the convergence can be made rigorous by classical argument (see e.g. [62]). For Conjecture II we believe that the convergence above can be made rigorous by using the theory of regularity structures introduced by Hairer in [34] or by using the paracontrolled distribution method proposed in [30]. In fact,*

$$\frac{1}{2}\frac{\nabla}{\mathrm{d}s}\partial_s\gamma + \frac{1}{4}\mathrm{Ric}(\partial_s\gamma) \quad \textit{and} \quad U \circ \mathrm{d}W, \sigma \circ \mathrm{d}W$$

are not well-defined in the classical sense and we need to multiply two distributions. To make the proof rigorous, renormalization techniques should be involved. As there are more than 40 terms required for the renormalization for the equation (1.2)*, the BPHZ theorem in the regularity structure theory developed in [11] has been used in [35]. To prove the convergence rigorously in Conjecture II, the discrete version of the BPHZ theorem is required. However, there is no useful version of the discrete BPHZ theorem until now. This is one reason we do not prove Conjecture II in this paper. We hope to be able to prove the convergence rigorously in our future work.*

(ii) *We have two ways to write the limiting equations, which give us two different equations with different diffusion coefficients. Since the different approximated processes have the same law, the solutions to two different equations* (4.25) *and* (4.29) *should have the same law. For* (4.25) *this is more related to the integration by parts formula (see (iii) below). For* (4.29) *this requires by regularity structure theory.*

(iii) *By Conjecture I, we expect that the process given by the Dirichlet form* $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ *in Section 2.1 can be interpreted as a solution to the following heuristic stochastic heat equation*

$$\mathrm{d}X_{t,s} = \frac{1}{2}\frac{\nabla}{ds}\partial_s X_{t,s}\mathrm{d}t - \frac{1}{4}\mathrm{Ric}(\partial_s X_{t,s})\mathrm{d}t + U_{t,s} \circ \mathrm{d}W_t. \tag{4.30}$$

By the integration by parts formula by Driver in [20], we can also derive (4.30) heuristically: We have the following relations:

$$\frac{\nabla}{\mathrm{d}s}\partial_s X_{t,s} \longleftrightarrow \int_0^1 \langle h'(s), \mathrm{d}B_s\rangle, \quad \mathrm{Ric}(\partial_s X_{t,s}) \longleftrightarrow \left\langle \int_0^1 \mathrm{Ric}_{U_s} h_s, \mathrm{d}B_s \right\rangle,$$

for $h \in \mathbb{H}$. Remark 4.4 gives part of the proof of the Mosco convergence of Dirichlet forms $\mathscr{E}^{\mathscr{P}}$, which is equivalent to the convergence of the associated semigroups. However, the Mosco convergence in Conjecture I still requires Markov uniqueness of the limiting Dirichlet form $(\mathscr{E}^0, \mathscr{D}(\mathscr{E}^0))$, which is a very difficult problem in this case.

(iii) If we write (4.29) *in local coordinates it is the same as equation (1.2) considered in [35]. To use the theory of regularity structures in [34] or the paracontrolled distribution method in [30] for equation* (4.29)*, we may embed the manifold M into a high dimensional Euclidean space $\mathbb{R}^N$. In this case, equation (4.29) can be written as*

$$\mathrm{d}X^i = \frac{1}{2}[\partial_{ss}^2 X^i - S^i_{jl}(X)\partial_s X^j \partial_s X^l]\mathrm{d}t + \sigma^i \circ dW, \tag{4.31}$$

where $X^i = \langle X, e_i\rangle$ with $\{e_i\}$ a basis in $\mathbb{R}^N$, S is the second fundamental form and π_p is the projection map from $\mathbb{R}^N$ to T_pM for $p \in M$ (see also [7]). Here we used that

$$\frac{\nabla}{ds}\partial_s X = \partial_{ss}^2 X - S_X(\partial_s X, \partial_s X),$$

for the second fundamental form S (see [49]). By using the recent results for the theory of regularity structures in [8] and [11], the local well-posedness of the equation (4.31) follows. Moreover, by the results in [7] for the smooth noise case, the solution should stay in the Riemannian manifold M.

Remark 4.11. *In [2] another Riemannian metric $G^1_{\mathscr{P}}$ has also been introduced and the corresponding measures $\nu^1_{\mathscr{P}}$ converge to the Wiener measure μ. By [2, Corollary 7.7] we can also consider the Dirichlet form associated with $\nu^1_{\mathscr{P}}$ and obtain that it is a quasi-regular Dirichlet form. However, it seems not easy to derive the equation for the approximation processes as in* (4.23)*.*

A Appendix

In the appendix we give some heuristic calculations leading to proofs of Conjectures I and II. Before this, we prove the following results for the basis $h_{a,j}, a = 1, 2, ..., d, j = 1, ..., n$.

Lemma A.1. *Fix κ_0, δ as in Lemma 4.3 satisfying $\kappa_0\delta^2 < \frac{1}{3}$ and for each $\gamma \in H^\delta_{\mathscr{P}}(M)$, let $b := \phi^{-1}(\gamma)$ be the associated anti-development map and let $u := /\!/(\gamma)$ be the parallel translate of γ. Let $h : [0, \varepsilon] \to \mathbb{R}^d$ be the solution of the equation*

$$h''(s) = \Omega_{u(s)}(b'(s), h(s))b'(s), \quad s \in (0, \varepsilon) \tag{A.1}$$

with boundary conditions $h(0) = 0, h(\varepsilon) = \frac{1}{\sqrt{\varepsilon}}e_a$, where $b'(s) = \frac{\Delta_i b}{\varepsilon}$ for $\Delta_i b = b(s_i) - b(s_{i-1})$. Then for $r \in [0, \varepsilon]$

$$h(r) = \varepsilon^{-3/2}\left(rI + \frac{[\Delta_i b]^2 r^3}{6\varepsilon^2} + O(|\Delta_i b|^4)\right)\left(I - \frac{[\Delta_i b]^2}{6} + O(|\Delta_i b|^4)\right)e_a + g(r),$$

where $[\Delta_i b]^2 := \Omega_{u(0)}(\Delta_i b, \cdot)\Delta_i b$, $O(|\Delta_i b|^4)$ *denotes a matrix (term) with its norm bounded by* $C|\Delta_i b|^4$ *and for* $r \in [0, \varepsilon]$

$$|g(r)| \le C|\Delta_i b|^3 \varepsilon^{-1/2}$$

for some constant C*, which is independent of* γ *and* r*.*

Proof. For convenience, let $A(s) := \Omega_{u(s)}(b'(s), \cdot)b'(s)$. By the definition of the derivative, we know that $b'(s) = \Delta_i b/\varepsilon$. It is easy to see that

$$A(u) = A(0) + \int_0^u A'(r)dr. \tag{A.2}$$

Let $\hat{h}$ be the solution to the equation (A.1) with $A(s)$ replaced by $A(0)$. Then it is not difficult to obtain that $\hat{h}$ satisfies the following (see [12, Page72]):

$$\hat{h}(r) = \left(\sum_{n=0}^{\infty} \frac{A(0)^n r^{2n+1}}{(2n+1)!}\right)\left(\sum_{n=0}^{\infty} \frac{A(0)^n \varepsilon^{2n+1}}{(2n+1)!}\right)^{-1} \frac{1}{\sqrt{\varepsilon}} e_a := BD_0^{-1}\frac{1}{\sqrt{\varepsilon}} e_a. \tag{A.3}$$

Here D_0 is invertible, since $D_0 = \varepsilon\left(I + \sum_{n=1}^{\infty} \frac{A(0)^n \varepsilon^{2n}}{(2n+1)!}\right) := \varepsilon(I + D)$ and

$$\|D\| \le \sum_{n=1}^{\infty} \frac{\kappa_0^n |\Delta_i b|^{2n}}{(2n+1)!} \le \frac{\kappa_0 |\Delta_i b|^2}{1 - \kappa_0|\Delta_i b|^2} < \frac{1}{2} \quad (\text{ since } \kappa_0 |\Delta_i b|^2 \le \kappa_0 \delta^2 < 1/3),$$

where we used $\|A(0)\| \le \kappa_0 |b'|^2$ in the first inequality. Moreover, we have

$$D_0^{-1} = \varepsilon^{-1} \sum_{n=0}^{\infty} (-1)^n D^n = \varepsilon^{-1}\left(I - \frac{[\Delta_i b]^2}{6} + O(|\Delta_i b|^4)\right),$$

where we used $\|\sum_{n=2}^{\infty}(-1)^n D^n\| \le \sum_{n=2}^{\infty} \|D\|^n = \frac{\|D\|^2}{1-\|D\|} \lesssim |\Delta_i b|^4$. In addition,

$$B = \sum_{n=0}^{\infty} \frac{A(0)^n r^{2n+1}}{(2n+1)!} = rI + \frac{[\Delta_i b]^2 r^3}{6\varepsilon^2} + O(|\Delta_i b|^4).$$

Combining this with (A.3) we obtain that

$$\hat{h}(r) = \varepsilon^{-3/2}\left(rI + \frac{[\Delta_i b]^2 r^3}{6\varepsilon^2} + O(|\Delta_i b|^4)\right)\left(I - \frac{[\Delta_i b]^2}{6} + O(|\Delta_i b|^4)\right) e_a.$$

Now we give an estimate of $|h - \hat{h}|$: By Taylor's theorem we have

$$h(r) - \hat{h}(r) = (h'(0) - \hat{h}'(0))r + \int_0^r (r-u)[A(u)h(u) - A(0)\hat{h}(u)]\mathrm{d}u, \tag{A.4}$$

which implies that

$$
\begin{aligned}
|h(r)-\hat{h}(r)| \leq & |h'(0)-\hat{h}'(0)|r + K\int_0^r (r-u)|h(u)-\hat{h}(u)|\mathrm{d}u \\
& + \int_0^r (r-u)|A(u)-A(0)||\hat{h}(u)|\mathrm{d}u \\
\leq & |h'(0)-\hat{h}'(0)|r + K\int_0^r (r-u)|h(u)-\hat{h}(u)|\mathrm{d}u + Cr\varepsilon^{-3/2}|\Delta_i b|^3 \\
:= & f(r),
\end{aligned}
\tag{A.5}
$$

where in the last inequality we used (A.2), that $|\hat{h}(r)| \leq C\varepsilon^{-1/2}$, and that $||A'(r)|| \leq C|\Delta_i b|^3/\varepsilon^3$ from the proof of [2, Prop. 6.2]. Here $K := \max ||A(r)||$. A similar argument as in the proof of Lemma 4.3 implies that

$$
\begin{aligned}
|h(r)-\hat{h}(r)| \leq & f(r) \leq \left(|h'(0)-\hat{h}'(0)| + C|\Delta_i b|^3\varepsilon^{-3/2}\right)\frac{\sinh\sqrt{K}r}{\sqrt{K}} \\
& \leq \cosh(\sqrt{\kappa_0}\delta)\varepsilon\left(|h'(0)-\hat{h}'(0)| + C|\Delta_i b|^3\varepsilon^{-3/2}\right),
\end{aligned}
\tag{A.6}
$$

where we used the elementary inequality $\frac{\sinh a}{a} \leq \cosh(a)$ for $a \in \mathbb{R}$ and

$$\sqrt{K}r \leq \sqrt{\kappa_0}|\Delta_i b|r/\varepsilon \leq \sqrt{\kappa_0}|\Delta_i b| \leq \sqrt{\kappa_0}\delta.$$

Also by (A.4) and a similar argument as in (A.5) we have

$$
\begin{aligned}
|h'(0)-\hat{h}'(0)|\varepsilon \leq & \int_0^\varepsilon (\varepsilon-u)|A(u)h(u)-A(0)\hat{h}(u)|\mathrm{d}u \\
\leq & K\int_0^\varepsilon (\varepsilon-u)|h(u)-\hat{h}(u)|\mathrm{d}u + C\varepsilon^{-7/2}\int_0^\varepsilon (\varepsilon-u)u|\Delta_i b|^3\mathrm{d}u \\
\leq & \cosh(\sqrt{\kappa_0}\delta)K\varepsilon\int_0^\varepsilon (\varepsilon-u)\mathrm{d}u|h'(0)-\hat{h}'(0)| + CK\varepsilon^{3/2}|\Delta_i b|^3 + C\varepsilon^{-1/2}|\Delta_i b|^3 \\
\leq & \frac{\cosh(\sqrt{\kappa_0}\delta)\kappa_0\varepsilon|\Delta_i b|^2}{2}|h'(0)-\hat{h}'(0)| + C\varepsilon^{-1/2}|\Delta_i b|^3,
\end{aligned}
$$

where in the second inequality we used that $|\hat{h}(r)| \leq C\varepsilon^{-1/2}$, and that $||A'(r)|| \leq C|\Delta_i b|^3/\varepsilon^3$. In the third inequality we used (A.6) and in the last inequality we used that $K \leq \kappa_0|\Delta_i b|^2/\varepsilon^2$. Since δ satisfies $\kappa_0\cosh(\sqrt{\kappa_0}\delta)\delta^2 < 1$, we obtain

$$|h'(0)-\hat{h}'(0)| \leq C\varepsilon^{-3/2}|\Delta_i b|^3.$$

Therefore, combining this with (A.6) we have

$$|h(r)-\hat{h}(r)| \leq C\varepsilon^{-1/2}|\Delta_i b|^3,$$

which implies the result. □

Heuristic proof of Conjectures I and II: We derive the convergence by heuristically analyzing the limit of the corresponding diffusion and drift parts. The convergence of the diffusion part follows from the definition of $X^{h_{a,i}}$. In fact, for Conjecture I the diffusion part is $/\!/_{s_i}(x_t^{\mathscr{P}}) \circ \mathrm{d}W^{\mathscr{P}}$ with $W^{\mathscr{P}}$ being an $l^2(\mathscr{P}, \mathbb{R}^d)$-cylindrical Wiener process, which converges to the diffusion part heuristically. We emphasize that the diffusion part is not well-defined in the classical sense and it requires renormalization (see Remark 4.10). For Conjecture II it is easy to see the convergence heuristically.

In the following we consider the drift part. For Conjecture I, we analyze $\frac{1}{\sqrt{\varepsilon}}\beta_{\mathscr{P}}(h_{a,j})$: By (4.13), (4.8) we know that

(A.7)

$$\begin{aligned}
&\frac{1}{\sqrt{\varepsilon}}\beta_{\mathscr{P}}(h_{a,j})(\gamma) = \frac{1}{\varepsilon\sqrt{\varepsilon}}\left\langle \Delta_j b - \Delta_{j+1} b, h_{a,j}(s_j)\right\rangle + \sqrt{\varepsilon}\sum_{a_1=1}^{d}\left\langle q(X^{h_{a_1,j}})h_{a_1,j}, h_{a,j}\right\rangle(s_j) \\
&= \frac{1}{\varepsilon^2}\langle \Delta_j b - \Delta_{j+1} b, e_a\rangle + \sqrt{\varepsilon}\sum_{a_1=1}^{d}\left\langle h_{a,j}(s_j), \int_{s_{j-1}}^{s_j} R_{u(r)}\big(\gamma'(r), /\!/_r h_{a_1,j}(r)\big)\mathrm{d}r h_{a_1,j}(s_j)\right\rangle \\
&:= I_1(\varepsilon) + I_2(\varepsilon),
\end{aligned}$$

where we used that $h_{a,j}(r) = 0$ for $r \in [0, s_{j-1}]$. Here and in the following we use $/\!/_r$ to denote $/\!/_r(\gamma)$ for simplicity. Now we consider $I_2(\varepsilon)$. Since

$$h_{a_1,j}(r) = \frac{1}{2}h_{a_1,j}(s_j) + \frac{1}{2}[2h_{a_1,j}(r) - h_{a_1,j}(s_j)] := \psi_1(r) + \psi_2(r), \tag{A.8}$$

we get

$$\begin{aligned}
I_2(\varepsilon) &= \frac{\sqrt{\varepsilon}}{2}\sum_{a_1=1}^{d}\left\langle h_{a,j}(s_j), \int_{s_{j-1}}^{s_j} R_{u(r)}\big(\gamma'(r), /\!/_r h_{a_1,j}(s_j)\big)\mathrm{d}r h_{a_1,j}(s_j)\right\rangle \\
&\quad + \sqrt{\varepsilon}\sum_{a_1=1}^{d}\left\langle h_{a,j}(s_j), \int_{s_{j-1}}^{s_j} R_{u(r)}\big(\gamma'(r), X_r^{\psi_2}\big)\mathrm{d}r h_{a_1,j}(s_j)\right\rangle \\
&=: I_{21}(\varepsilon) + I_{22}(\varepsilon),
\end{aligned} \tag{A.9}$$

where $X_r^{\psi_2} = /\!\!/_r \psi_2(r)$. For $I_{21}(\varepsilon)$, we deduce that
(A.10)

$$\begin{aligned}
I_{21}(\varepsilon) =& \frac{1}{2\varepsilon} \sum_{a_1=1}^d \left\langle e_a, \int_{s_{j-1}}^{s_j} /\!\!/_r^{-1} R(\gamma'(r), /\!\!/_r e_{a_1}) /\!\!/_r e_{a_1} \mathrm{d}r \right\rangle \\
=& \frac{1}{2\varepsilon} \left\langle e_a, \int_{s_{j-1}}^{s_j} /\!\!/_r^{-1} \mathrm{Ric}(\gamma'(r)) \mathrm{d}r \right\rangle = \frac{1}{2\varepsilon} \left\langle e_a, \int_{s_{j-1}}^{s_j} \mathrm{Ric}_{u(r)}(b'(r)) \mathrm{d}r \right\rangle \\
=& \frac{1}{2} \left\langle e_a, /\!\!/_{s_j}^{-1} \mathrm{Ric}(\gamma'(s_j-)) \right\rangle + \frac{1}{2\varepsilon} \left\langle e_a, \int_{s_{j-1}}^{s_j} \mathrm{Ric}_{u(r)}(b'(r)) - \mathrm{Ric}_{u(s_j)}(b'(r)) \mathrm{d}r \right\rangle \\
=& \frac{1}{2} \left\langle /\!\!/_{s_j} e_a, \mathrm{Ric}(\gamma'(s_j-)) \right\rangle - \frac{1}{2\varepsilon} \left\langle e_a, \int_{s_{j-1}}^{s_j} \int_r^{s_j} (D\mathrm{Ric})_{u_s}(b'(r), b'(r)) \mathrm{d}s \mathrm{d}r \right\rangle \\
:=& I_{211} + I_{212},
\end{aligned}$$

where $\gamma'(s_j-) = /\!\!/_{s_j} b'(r)$, $\mathrm{Ric}_{u(r)} = /\!\!/_r^{-1} \mathrm{Ric} /\!\!/_r$, $(D\mathrm{Ric})_{u_s}(b'(s), \cdot) := (\mathrm{d}/\mathrm{d}s) \mathrm{Ric}_{u(s)}$ and we used that $\{/\!\!/_r e_{a_1}\}$ is an orthonormal frame in $T_{\gamma_r}M$ in the second equality. For I_{212} we have

$$I_{212} = -\frac{1}{4\varepsilon} \left\langle e_a, (D\mathrm{Ric})_{u_{s_j}}(\Delta_j b, \Delta_j b) \right\rangle + O\left(\frac{|\Delta_j b|^3}{\varepsilon} \right).$$

For I_{22} we have
(A.11)

$$\begin{aligned}
I_{22}(\varepsilon) =& \frac{1}{\sqrt{\varepsilon}} \sum_{a_1=1}^d \left\langle e_a, \int_{s_{j-1}}^{s_j} /\!\!/_r^{-1} R(\gamma'(r), /\!\!/_r \psi_2(r)) /\!\!/_r e_{a_1} \mathrm{d}r \right\rangle \\
=& \frac{1}{\sqrt{\varepsilon}} \sum_{a_1=1}^d \left\langle e_a, \int_{s_{j-1}}^{s_j} \Omega_{u(r)}(b'(r), \psi_2(r)) e_{a_1} \mathrm{d}r \right\rangle \\
=& \frac{1}{\sqrt{\varepsilon}} \sum_{a_1=1}^d \left\langle e_a, \int_{s_{j-1}}^{s_j} \Omega_{u(s_{j-1})}(b'(r), \psi_2(r)) e_{a_1} \mathrm{d}r \right\rangle \\
&+ \frac{1}{\sqrt{\varepsilon}} \sum_{a_1=1}^d \left\langle e_a, \int_{s_{j-1}}^{s_j} \left(\Omega_{u(r)}(b'(r), \psi_2(r)) e_{a_1} - \Omega_{u(s_{j-1})}(b'(r), \psi_2(r)) e_{a_1} \right) \mathrm{d}r \right\rangle \\
:=& I_{221}(\varepsilon) + I_{222}(\varepsilon).
\end{aligned}$$

According to Lemma A.1, we know that

$$
\begin{aligned}
2\psi_2(r) =&2h_{a_1,j}(r) - h_{a_1,j}(s_j) = \Bigg[2\varepsilon^{-\frac{3}{2}}\Bigg((r-s_{j-1})I + \frac{[\Delta_j b]^2(r-s_{j-1})^3}{6\varepsilon^2} \\
&+ O(|\Delta_j b|^4)\Bigg)\Bigg(I - \frac{[\Delta_j b]^2}{6} + O(|\Delta_j b|^4)\Bigg) - \frac{1}{\sqrt{\varepsilon}}\Bigg]e_{a_1} + g(r-s_{j-1}) \\
=&:\theta(r,\varepsilon)e_{a_1} + O(\frac{|\Delta_j b|^2}{\varepsilon^{\frac{1}{2}}}) + O(\frac{|\Delta_j b|^4}{\varepsilon^{\frac{3}{2}}}),
\end{aligned}
\tag{A.12}
$$

where $\theta(r,\varepsilon) = 2\varepsilon^{-\frac{3}{2}}(r-s_{j-1}) - \frac{1}{\sqrt{\varepsilon}}$. This implies that

$$
\begin{aligned}
I_{221}(\varepsilon) =&\frac{1}{\varepsilon^{\frac{3}{2}}}\sum_{a_1=1}^{d}\Bigg\langle e_a, \int_{s_{j-1}}^{s_j}\Omega_{u(s_{j-1})}(\Delta_j b, \psi_2(r))e_{a_1}\mathrm{d}r\Bigg\rangle \\
=&\frac{1}{2\varepsilon^{\frac{3}{2}}}\sum_{a_1=1}^{d}\Bigg\langle e_a, \int_{s_{j-1}}^{s_j}\Omega_{u(s_{j-1})}\left(\Delta_j b, \theta(r,\varepsilon)e_{a_1}\right)e_{a_1}\mathrm{d}r\Bigg\rangle \\
&+ \frac{1}{\varepsilon^{\frac{3}{2}}}\Big(O(|\Delta_j b|^3\varepsilon^{\frac{1}{2}}) + O(|\Delta_j b|^5\varepsilon^{-\frac{1}{2}})\Big) \\
=&O(|\Delta_j b|^3\varepsilon^{-1}) + O(|\Delta_j b|^5\varepsilon^{-2}),
\end{aligned}
$$

where we used that $\int_{s_{j-1}}^{s_j}\theta(r,\varepsilon)\mathrm{d}r = 0$ in the third equality. In the following, we use (A.12) to estimate $I_{222}(\varepsilon)$:

(A.13)

$$
\begin{aligned}
I_{222}(\varepsilon) =&\frac{1}{\sqrt{\varepsilon}}\sum_{a_1=1}^{d}\Bigg\langle e_a, \int_{s_{j-1}}^{s_j}\left(\Omega_{u(r)}(b'(r), \psi_2(r))e_{a_1} - \Omega_{u(s_{j-1})}(b'(r), \psi_2(r))e_{a_1}\right)\mathrm{d}r\Bigg\rangle \\
=&\frac{1}{2\sqrt{\varepsilon}}\Bigg\langle e_a, \int_{s_{j-1}}^{s_j}\theta(r,\varepsilon)\left[\mathrm{Ric}_{u(r)}(b'(r)) - \mathrm{Ric}_{u(s_{j-1})}(b'(r))\right]\mathrm{d}r\Bigg\rangle + O_{b,\varepsilon} \\
=&\frac{1}{2\sqrt{\varepsilon}}\Bigg\langle e_a, \int_{s_{j-1}}^{s_j}\theta(r,\varepsilon)\int_{s_{j-1}}^{r}(D\mathrm{Ric})_{u(s)}\left(b'(s), b'(s)\right)\mathrm{d}s\mathrm{d}r\Bigg\rangle + O_{b,\varepsilon} \\
=&\frac{1}{2\varepsilon^{\frac{5}{2}}}\Big\langle e_a, (D\mathrm{Ric})_{u_{s_j}}\left(\Delta_j b, \Delta_j b\right)\Big\rangle\int_{s_{j-1}}^{s_j}\left[\frac{2(r-s_{j-1})}{\varepsilon^{3/2}} - \frac{1}{\sqrt{\varepsilon}}\right](r-s_{j-1})\mathrm{d}r + O_{b,\varepsilon} \\
=&\frac{1}{12\varepsilon}\Big\langle e_a, (D\mathrm{Ric})_{u_{s_j}}\left(\Delta_j b, \Delta_j b\right)\Big\rangle + O_{b,\varepsilon},
\end{aligned}
$$

with $O_{b,\varepsilon} := O\left(\frac{|\Delta_j b|^3}{\varepsilon}\right) + O\left(\frac{|\Delta_j b|^5}{\varepsilon^2}\right)$. Combining the computation for I_{212} and I_{221}, I_{222} we have

$$
I_{212} + I_{22} = -\frac{1}{6\varepsilon}\Big\langle e_a, (D\mathrm{Ric})_{u_{s_j}}\left(\Delta_j b, \Delta_j b\right)\Big\rangle + O_{b,\varepsilon},
$$

which combined with (A.7), (A.9), (A.10) implies that

$$\begin{aligned}\frac{1}{\sqrt{\varepsilon}}\beta_{\mathscr{P}}(h_{a,j})(\gamma) =&\frac{1}{\varepsilon^2}\langle \Delta_j b-\Delta_{j+1}b, e_a\rangle+\frac{1}{2}\left\langle /\!/_{s_j}e_a, \mathrm{Ric}(\gamma'(s_j-))\right\rangle\\ &-\frac{1}{6\varepsilon}\langle (D\mathrm{Ric})_{u_{s_j}}(\Delta_j b,\Delta_j b), e_a\rangle+O_{b,\varepsilon}.\end{aligned}$$

Then we obtain

$$\begin{aligned}X^{\beta_{\mathscr{P}}}(\gamma)(s_i)=&\frac{1}{2\varepsilon^2}\sum_{a=1}^d\langle \Delta_j b-\Delta_{j+1}b, e_a\rangle /\!/_{s_j}e_a+\frac{1}{4}\sum_{a=1}^d\left\langle /\!/_{s_j}e_a, \mathrm{Ric}(\gamma'(s_j-))\right\rangle /\!/_{s_j}e_a\\ &-\frac{1}{12\varepsilon}\sum_{a=1}^d\langle (D\mathrm{Ric})_{u_{s_j}}(\Delta_j b,\Delta_j b), e_a\rangle /\!/_{s_j}e_a+O_{b,\varepsilon}\\ =&\frac{1}{2\varepsilon^2}/\!/_{s_j}(\Delta_j b-\Delta_{j+1}b)+\frac{1}{4}\mathrm{Ric}(\gamma'(s_j-))-\frac{1}{12\varepsilon}/\!/_{s_j}(D\mathrm{Ric})_{u_{s_j}}(\Delta_j b,\Delta_j b)+O_{b,\varepsilon},\end{aligned}$$

where we used that $\{/\!/_{s_j}e_a\}$ is an orthonormal frame in $T_{\gamma_{s_j}}M$. Heuristically, we have $\frac{1}{\varepsilon^2}/\!/_{s_j}(\Delta_j b-\Delta_{j+1}b)\to /\!/_s\partial^2_{ss}b$ and $O_{b,\varepsilon}\to 0$ as $\varepsilon\to 0$, since $|\Delta_j b|\simeq\varepsilon^{\frac{1}{2}-}$. We also have $\sum_{i=1}^d /\!/_{s_j}(D\mathrm{Ric})_{u_{s_j}}(e_i,e_i)=\nabla\mathrm{Scal}$, which suggests the third term above converges to $\frac{1}{12}\nabla\mathrm{Scal}(\gamma)$ by a similar argument as in [2, Section 6]. Now we have

$$\begin{aligned}X^{\beta_{\mathscr{P}}}(\gamma)\to&-\frac{1}{2}/\!/_s\partial^2_{ss}b+\frac{1}{4}\mathrm{Ric}(\partial_s\gamma)-\frac{1}{12}\nabla\mathrm{Scal}(\gamma)\\ =&-\frac{1}{2}\frac{\nabla}{ds}\partial_s\gamma+\frac{1}{4}\mathrm{Ric}(\partial_s\gamma)-\frac{1}{12}\nabla\mathrm{Scal}(\gamma),\end{aligned}\tag{A.14}$$

where we used $/\!/_s\partial^2_{ss}b=/\!/_s\frac{\mathrm{d}}{\mathrm{d}s}/\!/_s^{-1}\partial_s\gamma=\frac{\nabla}{\mathrm{d}s}\partial_s\gamma$.

For Conjecture II it is easy to see that $\frac{1}{\varepsilon^2}/\!/_{s_j}(\Delta_j b-\Delta_{j+1}b)\to /\!/_s\partial^2_{ss}b=\frac{\nabla}{\mathrm{d}s}\partial_s\gamma$.

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