Quick or Persistent? On the Feedback Effects between First and Second-Mover Advantages in a Stochastic Investment Game*

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Abstract

We analyse a dynamic model of investment under uncertainty in a duopoly where firms have an option to switch from one market to another. We construct a subgame perfect equilibrium in Markovian mixed strategies and show that both preemption and attrition can occur along typical equilibrium paths. The presence of both effects implies different tradeoffs than in existing strategic real option models that focus on only one of them, and accordingly it changes the nature of stopping problems to be solved. This is in contrast to the model's deterministic version, where the standard behaviour in each regime can simply be pasted.

Keywords: Stochastic timing games, preemption, war of attrition, real options, Markov perfect equilibrium, mixed strategies, two-dimensional optimal stopping.

JEL subject classification: C61, C73, D21, D43, L13

1 Introduction

In this paper we study the interaction of first and second-mover advantages in a strategic real option exercise model. So far, the literature has only focussed on each of these in isolation. Therefore, existing theory cannot deal with realistic economic situations that can dynamically generate both first and second-mover advantages and that actually induce option exercise in any regime in equilibrium. We consider a seemingly simple variant of standard real option settings, where firms can typically either invest into, or abandon some given market. Our

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two firms are initially operating in the same market and thus face duopolistic competition. Both have the option to switch to a new market at some sunk cost, such that they become differentiated monopolists if one firm switches. The relative profitability of the markets evolves randomly, and the relative values of (i) making the investment or (ii) becoming monopolist in the present market, evolve accordingly. Thus our model generates both first and second-mover advantages randomly over time and this random evolution creates feedback effects between them. This is in contrast to the deterministic version of the model, where one can simply paste together the behaviours known from studies of first or second-mover advantages in isolation. We therefore show that uncertainty induces qualitatively different behaviour, a feature that is not always present for real options models.

In the literature, the impact of competition on the valuation and exercise timing of real options has been shown for the following two classes of situations, which focus on first or second-mover advantages, respectively. First, if at least two firms have the option to choose the best time for an irreversible investment in some market, the possibility to preempt the other firm(s) typically generates a first-mover advantage. Competition for that advantage drastically reduces the classical option value of waiting. For instance, Fudenberg and Tirole (1985) obtain rent-equalization in equilibrium, meaning that no firm can realize a first-mover advantage in a deterministic technology-adoption model. Back and Paulsen (2009) show that the zero-NPV investment rule constitutes a subgame-perfect equilibrium in a stochastic capital-accumulation model, completely eliminating any option value.[1]

The second, strategically very different situation is that of two firms considering to exit from some duopoly that yields insufficient revenues to cover running costs. If the market is still profitable for a monopolist, then there is a second-mover advantage and the strategic timing problem takes the form of a war of attrition. It can only be won by credibly threatening not to use one's option. In equilibrium, the losing firm quits as soon as any delay becomes indeed costly and thus behaves as if it was the only one having an exit option.[2] In this sense

[1]Fudenberg and Tirole (1985) use the model of Reinganum (1981), who only considered precommitment or "open loop" strategies that are not subgame perfect. Similarly Back and Paulsen (2009) study the effect of "closed loop" strategies and subgame perfectness in the model of Grenadier (2002). With open loop strategies, the effect of competition in the respective first papers rather resembles that of Cournot as opposed to Bertrand competition.

Some game-theoretic analyses of models with uncertainty and a single investment opportunity for each firm are Smets (1991), Huisman and Kort (1999), Weeds (2002), Boyer et al. (2004), Pawlina and Kort (2006) and Thijssen et al. (2012). A recent contribution that combines preemption with capacity choice is Huisman and Kort (2015). Extensive surveys can be found in Chevalier-Roignant and Trigeorgis (2011) and Azevedo and Paxson (2014).

[2]This is the case in the deterministic model of Ghemawat and Nalebuff (1985) and the stochastic models of Lambrecht (2001) and Murto (2004). The winning firm in subgame-perfect equilibrium is the one who can sustain monopoly longer. A second-mover advantage may also arise in the investment model of Hoppe (2000) by informational spillovers, since the profitability of investment is only revealed by making it. Her equilibria are of the same qualitative type as earlier mentioned. Hendricks et al. (1988) derive additional mixed strategy equilibria in a deterministic war of attrition, where possibly both agents obtain the payoff that each would get if the respective other never quitted.

competition affects only the equilibrium payoff of the winning firm.

Our model, despite its simple structure, cannot only generate both of these effects, but actually induces more complex behaviour that has not been studied before.[3]

Instead of running losses in the initial duopoly we consider a particular outside option for each firm that makes leaving the duopoly worthwhile when this outside option is in the money. The profitability of the new, emerging, market that any firm may switch to by a related investment evolves randomly, as does the profitability of the current market for a potential monopolist. While both firms share the interest to separate, it depends on the evolution of the random state which market is more attractive, i.e., whether there is a current first or second-mover advantage. The uncertainty induces qualitatively different behaviour from previously considered models and also from the deterministic version of the model as follows.

Our state space is two-dimensional, corresponding to the profitabilities of the two markets. In subgame-perfect equilibrium this state space effectively will be split into three regions: a *preemption region* where both players try to preempt each other because the new market is sufficiently profitable, an *attrition region* where the new market is profitable, but less than monopoly in the current market (so that each firm prefers its competitor to switch), and a *continuation region* where neither firm acts because the new market is currently so unprofitable that there is no opportunity cost of waiting. Along a sample path the following may then happen. If the state hits the preemption region, then at least one firm invests instantly. If the state moves through the attrition region, each firm invests at a certain hazard rate (the *attrition rate*), which may result in no investment until either the preemption region is hit, or the attrition region is exited and the continuation region is re-entered. Inside the latter region nothing happens until either the preemption region is hit directly or the attrition region is entered again. Our mixed strategies represent explicitly the strategic tradeoffs that the firms face.

The distinctive feature of our model, compared to the existing literature and to its deterministic version, is that it is not clear ex ante in which sequence the different regions will be passed. That induces several important effects.

First, the uncertain evolution of the state creates an additional risk of getting stuck in preemption when waiting for an optimal time to switch. In previous models (having a one-dimensional structure), as well as in the deterministic version of our model, preemption occurs only when the payoff from exercising the option is sufficiently high. As a consequence, there

[3] In the models of Kwon et al. (2016) or Hoppe (2000), either preemption or attrition can occur, depending on the values of underlying parameters. Once parameters are fixed, however, the respective model falls into one of the two classes. In our model, different parts of the state-space exhibit strategically relevant first or second-mover advantages, even once the parameters are fixed. Typical preemption models like those listed above may also have phases with a second-mover advantage, but no action will be taken there, in contrast to our model.

is a relatively simple tradeoff and the common solution is either to wait until reaching the preemption constraint, or the latter is not even binding and the option is exercised where this is optimal, *independently* of the threat of being preempted later on. In our model, a firm determining an optimal time to switch has to consider the risk of both markets deteriorating, with the present one deteriorating relatively more, so that a first-mover advantage will arise. This triggers preemptive option exercise at less favourable prospects. Indeed, for this reason waiting does become costly while there is still a second-mover advantage, but before a firm that was not threatened by preemption would exercise its option. So, there truly is a *feedback effect* between the regimes. We show that the cost of waiting is the opportunity cost of foregone (net) revenue in the new market and that, thus, the main question is the characterization of when this cost applies, i.e., when actual attrition happens.

Second, additionally to the location of the attrition region, the outcomes of our mixed strategies in that region also differ qualitatively from those in simpler models. Again, the deterministic version resembles the standard war of attrition and the switch occurs definitely in the attrition region if it is ever entered. In our stochastic model, there is *always* a positive probability of passing the attrition region without a switch, which will then occur due to preemption.

Despite the standard setting of our model, there are some technical challenges. Firstly, in regimes with a second-mover advantage, payoffs of pure strategy equilibria are generally asymmetric and each firm prefers to become follower. Determining the roles is then a (commitment) problem left open. By considering mixed strategies we can make the firms indifferent between the two roles. With uncertainty, however, the attrition region (which is where the mixing occurs), can be entered and left repeatedly at random. Despite this complication, we are still able to obtain an equilibrium with *Markovian* stopping probabilities per small time interval (the aforementioned attrition rate). Note that deterministic models typically make some simplifying smoothness or monotonicity assumption on the payoff functions, which we do not need to make.

Another important issue relates to the value functions of the leader and follower roles. Determining the attrition region in terms of the problem of when to switch up to hitting the preemption region will require solving a genuinely two-dimensional constrained optimal stopping problem. There are no known general techniques to solve such problems. While our problem *cannot* be transformed into a one-dimensional problem (like some models considered in the optimal stopping literature), we are still able to show that, quite remarkably, our model has enough structure to fully characterize the attrition and preemption regions of the state space. The fact that we can obtain such sharp results is partly due to the use of continuous time, for which a well-developed stochastic calculus and theory of semi-martingales with continuous sample paths exists.

Finally, we propose a simulation-based approach to numerically analyse the model. This simulation shows, for a particular starting point in the continuation region, that many sample paths actually reach the preemption region via the attrition region, illustrating that our results on stochastic versus deterministic models are economically important.

The paper is organized as follows. In Section 2 we present the basic model. In Section 3 we describe our results in an informal way, before formally defining, analysing, and discussing the timing game in detail in Section 4. In particular, we construct a subgame perfect equilibrium in Markovian mixed strategies in Section 4.4. An important part of this construction consists of solving a constrained optimal stopping problem in Section 4.3. Section 5 presents a numerical example that we use to discuss the economic content of our results. In Section 6 we contrast our stochastic model with its deterministic version and in Section 7 we conclude and briefly discuss how strategic behaviour here influences the exposure of firms to market risk factors. All proofs are relegated to an appendix.

2 A Model with First and Second-Mover Advantages: Product Differentiation in a Duopoly

Consider two firms that are initially operating in the same market A, e.g., producing a homogeneous good. Suppose for simplicity that the firms make no profits while they are in duopoly, like under Bertrand competition. Each firm has an option to engage in product differentiation and switch to a new market B. This switch involves some sunk costs $I > 0$ (of adjusting production, licensing fees, advertising etc.). If one firm switches, both can be seen as monopolists in their respective markets and they will earn some profit streams. The monopoly profits in markets A and B are given by the processes $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$, respectively. For simplicity, there are also no profits when both firms are active in market B. For completeness we allow some additional fixed running costs before and after switching denoted by c_0, c_A and c_B, respectively, but these have no qualitative impact and may also be ignored. Depending on the net present value of the two monopoly profits X and Y and the total costs of switching, it may at any time be more profitable to switch to the new market, to become monopolist in the current market, or to stay in duopoly.

There is a fixed probability space $(\Omega, \mathscr{F}, P)$ to capture uncertainty about the state of the world, in particular concerning future profits. Furthermore, there is a dynamic revelation of information about the state, represented by a filtration $\mathbf{F} = (\mathscr{F}_t)_{t\geq 0}$. We assume that $\mathbf{F}$ satisfies the usual conditions of right-continuity and completeness and that X and Y are adapted to $\mathbf{F}$.

To obtain some explicit results, we assume that X and Y are geometric Brownian motions,[4]

[4]This is the standard model in real options theory. It allows us to evaluate the feedback effect between first and second-mover advantages very explicitly. Our equilibrium construction will work in much more general settings, however.

i.e., that they are the unique solutions to the system of stochastic differential equations

$$\frac{dX}{X} = \mu_X dt + \sigma_X dB^X \quad \text{and} \quad \frac{dY}{Y} = \mu_Y dt + \sigma_Y dB^Y, \tag{2.1}$$

with given initial values $(X_0, Y_0) = (x, y) \in \mathbb{R}^2_+$. B^X and B^Y are Brownian motions that are correlated with coefficient ρ, and μ_X, μ_Y, σ_X and σ_Y are some constants representing the drift and volatility of the growth rates of the profit processes X and Y, respectively. We assume that $\sigma_X^2, \sigma_Y^2 > 0$ and $|\rho| < 1$ to exclude potential degeneracies that require separate treatment. The case where $\sigma_X = \sigma_Y = 0$ is considered in Section 6, however, whereas the case where $\rho = 1$ essentially gets us to the standard preemption model as analysed in the existing literature.

We also assume that profits are discounted at a common rate $r > \max(0, \mu_X, \mu_Y)$ to ensure finite values of the following payoff processes and the subsequent stopping problems.[5]

Indeed, the basic payoffs can now be formulated by processes L, F and M, which represent, respectively, the expected payoffs at the time of the first switch to *(i)* a firm that switches solely and becomes the *leader*, *(ii)* a firm that remains and becomes the *follower* and *(iii)* a firm that switches simultaneously with the other. At any time $t \geq 0$, these processes take the values

$$\begin{aligned}
L_t :=& -\int_0^t e^{-rs} c_0 \, ds + E\Big[\int_t^\infty e^{-rs}(Y_s - c_B)\, ds \Big| \mathscr{F}_t\Big] - e^{-rt} I \\
=& -\frac{c_0}{r} + e^{-rt}\Big(\frac{Y_t}{r - \mu_Y} - \frac{c_B - c_0}{r} - I\Big) := L(t, Y_t), \\
F_t :=& -\int_0^t e^{-rs} c_0 \, ds + E\Big[\int_t^\infty e^{-rs}(X_s - c_A)\, ds \Big| \mathscr{F}_t\Big] \\
=& -\frac{c_0}{r} + e^{-rt}\Big(\frac{X_t}{r - \mu_X} - \frac{c_A - c_0}{r}\Big) := F(t, X_t)
\end{aligned} \tag{2.2}$$

and

$$M_t := -\int_0^\infty e^{-rs} c_0 \, ds - e^{-rt} I = -\frac{c_0}{r} - e^{-rt} I.$$

Note that, unlike the standard literature, these payoffs are measured in time 0 units, not current time t units. As all processes are continuous functions of the profit processes X and Y and time t, they, too, have continuous paths and are adapted to $\mathbf{F}$ like the latter. Thanks to their Markovian representations, we may furthermore evaluate the three payoff processes also at any stopping time at which the switch might occur to obtain the correct payoffs.[6]

[5]Then $(e^{-rt}X_t)$ is bounded by an integrable random variable. Indeed, for $\sigma_X > 0$ we have $\sup_t e^{-rt}X_t = X_0 e^{\sigma_X Z}$ with $Z = \sup_t B_t^X - t(r - \mu_X + \sigma_X^2/2)/\sigma_X$, which is exponentially distributed with rate $2(r - \mu_X)/\sigma_X + \sigma_X$ (see, e.g., Revuz and Yor (1999), Exercise (3.12) 4°). Thus, $E[\sup_t e^{-rt}X_t] = X_0(1 + \sigma_X^2/2(r - \mu_X)) \in \mathbb{R}_+$, implying that $(e^{-rt}X_t)$ is of class (D); analogously for $\sigma_X < 0$ and Y.

[6]If the switch occurs at a stopping time τ, one has to take conditional expectations w.r.t. $\mathscr{F}_\tau$ to determine

Finally note that all payoff processes are bounded by integrable random variables, cf. fn. 5, and that they converge (in $L^1(P)$) to $-c_0/r =: L_\infty$ as $t \to \infty$.

To keep the model as simple as possible, we assume that it is optimal for at most one firm to switch. Specifically, we assume that $(c_0 - c_A)/r + I \geq 0$, which ensures that $F \geq M$, and that the relative capitalized total cost of a single firm that switches is nonnegative, i.e., $(c_B - c_A)/r + I \geq 0$.

As is well known from the literature, the firms will try to preempt each other in switching to the market B whenever $L > F$. In our model, this is the case iff

$$(X, Y) \in \mathcal{P} := \Big\{(x, y) \in \mathbb{R}^2_+ \;\Big|\; y > (r - \mu_Y)\Big(\frac{x}{r - \mu_X} + \frac{c_B - c_A}{r} + I\Big)\Big\}.$$

We accordingly call $\mathcal{P}$ the *preemption region* of the state space $\mathbb{R}^2_+$ of our process (X, Y). When $F > L$, each firm prefers to be the one who stays if there is a switch, which potentially induces a war of attrition.[7] We will formally model this strategic conflict as a timing game in Section 4.

3 An Informal Preview of the Results

At this stage it is instructive to analyse the model informally and anticipate the equilibrium construction that follows. To begin, in line with the literature on preemption games, our game has to end as soon as the set $\mathcal{P}$ is hit. However, to model this preemption outcome it is not enough to consider distribution functions over time as (mixed) strategies.[8] As there is no "next period" in continuous time, one has to enable the firms both to try to invest immediately but avoiding simultaneous investment (at least partially), in particular on the boundary of $\mathcal{P}$, where they are still indifferent. We follow the endogenous approach of Fudenberg and Tirole (1985), augmenting strategies by "intensities" $\alpha_i \in [0, 1]$ that determine the outcome when both firms try to invest simultaneously, like in an infinitely repeated grab-the-dollar game. A firm that grabs first invests. If firm j grabs with stationary probability $\alpha_j > 0$, then firm i can obtain the follower payoff F by never grabbing, and $\alpha_j M + (1 - \alpha_j)L$ by grabbing with probability 1. Hence the firms are just indifferent between both actions if

$$\alpha_1 = \alpha_2 = \frac{L - F}{L - M} \equiv \alpha \in (0, 1],$$

implying that the expected local payoffs are F (note that we have $F \geq M$).

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the appropriate payoff L_τ for example, which in general need not be consistent at all with a family of conditional expectations indexed by deterministic times t and a pointwise construction via $t = \tau(\omega)$.

[7] In the commonly used case where $\rho = 1$, $F > L$ will never induce switching. In our model, with $|\rho| < 1$, this is no longer the case.

[8] See Hendricks and Wilson (1992) for a proof of equilibrium non-existence.

With symmetric intensities the probability that either firm becomes leader is then

$$\alpha(1-\alpha)+(1-\alpha)^2\alpha(1-\alpha)+\cdots=\frac{1-\alpha}{2-\alpha}.$$

Taking limits, that probability becomes $\frac{1}{2}$ on the boundary of $\mathcal{P}$ where α vanishes (see Remark A.2 in the Appendix on the limit outcome). In the interior of $\mathcal{P}$, however, there is a positive probability $\alpha/(2-\alpha)$ of simultaneous investment representing the cost of preemption, in contrast to more ad hoc coordination devices like coin tosses.

Now consider states outside $\mathcal{P}$. Suppose first that some firm tries to determine when it would be optimal to make the switch and obtain the leader value L, which directly depends only on Y. If the competitor could not preempt the firm, standard techniques would yield that there is a threshold y^* that separates the *continuation region*, where the firm remains inactive, and the *stopping region* where the firm makes the switch. The optimal stopping time is thus the first time the process Y enters the set $[y^*,\infty)$, i.e., the option to wait is worthless if and only if the market B is in a sufficiently profitable state. The process X plays no role in this problem at all.

The firm should realize, however, that as soon as $\mathcal{P}$ is hit the game is over and it receives, in expectation, the follower value, which *does* depend on X. Thus the firm should actually solve the *constrained* optimal stopping problem of becoming leader up to the first hitting time of $\mathcal{P}$. Intuitively, the solution to this problem should again divide the (genuinely two-dimensional) state space into two sets: a continuation set $\mathcal{C}$ and a stopping set $\mathcal{A}$. The boundary between these regions turns out to be given by a mapping $x\mapsto b(x)$, see Figure 1. The function b increases to y^* for $x\to\infty$, since then the probability of reaching $\mathcal{P}$ while waiting to get closer to the unconstrained threshold y^* is small.

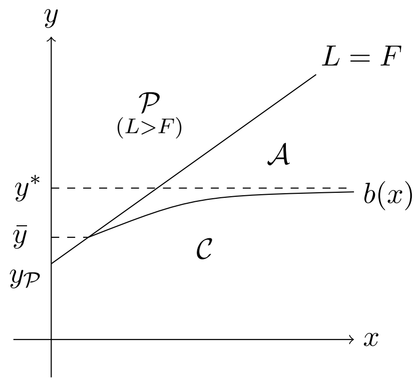


Figure 1: Continuation, Attrition and Preemption regions.
– The definitions of $\bar{y}$ and $y_{\mathcal{P}}$ are given in Section 4.3. –

For smaller values of x, there is a higher risk to get trapped in preemption if $\mathcal{P}$ is hit with $Y<y^*$. Thus it may be better to secure the current leader value before $\mathcal{P}$ is hit at a possibly

even lower value. In Section 4.3, we formally show that it is optimal to stop strictly before $\mathcal{P} \cup \mathbb{R}_+ \times [y^*, \infty)$ is hit in the constrained stopping problem, i.e., the stopping boundary $b(x)$ lies below that region (in particular when preemption is "near"). In the interior of the stopping set $\mathcal{A}$ we now have a situation alike a war of attrition: waiting without a switch occuring is costly (the constrained leader value decreases in expectation), but becoming follower in this region would yield a higher payoff $F > L$.

We characterize the attrition region $\mathcal{A}$ below in terms of the stopping boundary $b(x)$ for the constrained leader's stopping problem and identify the cost of waiting in this region as the *drift* of L. In equilibrium, the firms are only willing to continue (and bear the cost of waiting) in $\mathcal{A}$ because there is a chance that the opponent drops out, thereby handing them the follower's payoff. We will propose symmetric Markovian equilibrium hazard rates of investing – called *attrition rates* – that make both firms exactly indifferent to continue in $\mathcal{A}$. If we denote those rates here by λ_t to give a heuristic argument, the probability that a firm switches in a small time interval $[t, t + dt]$ is $\lambda_t \, dt$. Since each firm is supposed to be indifferent to wait, the equilibrium value (i.e., expected payoff) V_t should satisfy

$$\begin{aligned} & V_t = F_t \lambda_t \, dt + (1 - \lambda_t \, dt) E[V_{t+dt} \,|\, \mathscr{F}_t] = F_t \lambda_t \, dt + (1 - \lambda_t \, dt) E[V_t + dV_t \,|\, \mathscr{F}_t] \\ \Leftrightarrow \quad & \lambda_t \, dt = \frac{-E[dV_t \,|\, \mathscr{F}_t]}{F_t - V_t - E[dV_t \,|\, \mathscr{F}_t]}. \end{aligned}$$

On the other hand, we should have $V{=}L$ in $\mathcal{A}$ by indifference and because the probability of simultaneous switching is zero if the other firm switches at a rate. Hence, $E[dV_t \,|\, \mathscr{F}_t]$ is the drift of L_t (which is here negative) of order $o(dt)$, implying that

$$\lambda_t \, dt = \frac{-E[dL_t \,|\, \mathscr{F}_t]}{F_t - L_t}.$$

We will model strategies more generally as distribution functions $G_i(t)$ over time, such that the hazard rates are actually $\lambda_t \, dt = dG_i(t)/(1 - G_i(t))$. Note that in contrast to deterministic models that develop linearly (with time t the state variable), we have to account for the possibility that the state enters and leaves the attrition region frequently and randomly, which makes indifference much more complex to verify. A distinctive feature of the uncertainty in the evolution of (X, Y) is that there is always a positive probability of reaching the preemption region with no firm having switched before, even though the hazard rates λ_t grow unboundedly as $F - L \to 0$ near $\mathcal{P}$. In the deterministic analogue of our model, the unbounded growth of λ_t near $\mathcal{P}$ enforces switching in $\mathcal{A}$, so that $\mathcal{P}$ is never actually reached.

4 Formal Analysis of Subgame Perfect Equilibria

In this section we present and discuss in more detail the formal results that have been introduced informally in Section 3. We start by formalizing the timing game (in particular strategies and the equilibrium concept) in Section 4.1, followed by establishing equilibria for subgames starting in the preemption region $\mathcal{P}$ in Section 4.2. The constrained optimal stopping problem that identifies the attrition region $\mathcal{A}$ is analysed in Section 4.3. In Section 4.4 we establish (Markovian) equilibrium strategies for arbitrary subgames.

4.1 Timing Games: Strategies and Equilibrium

As continuous time is not well ordered, one cannot model a timing game in terms of elementary actions like "wait" and "stop". We thus adopt the common approach of letting players choose the time $t \in \mathbb{R}_+$ of their first (and only) move, called a "plan of action" to be followed as long as no other player has moved, see Fudenberg and Tirole (1985) or Laraki et al. (2005). Hence, players whose plans are minimal move at their planned time, i.e., potentially all. The remaining players are allowed to react conditionally at the time of the first move. Here no firm can have an interest to switch anymore after the other has switched, so all non-minimal plans will simply be abandoned at the minimum.

In our model, these plans can depend on the states of the world. Recall that the exogenous dynamic information about the true state is represented by the filtration $\mathbf{F} = (\mathscr{F}_t)_{t \geq 0}$. The state-dependent dates τ that can be identified via $\mathbf{F}$ are the *stopping times*, i.e., all random variables τ that satisfy $\{\tau \leq t\} \in \mathscr{F}_t$ for all $t \in \mathbb{R}_+$. These stopping times act as pure strategies, the set of which we denote by $\mathscr{T}$.

As we are interested in subgame perfect equilibria, we will consider behaviour at all identifiable random dates – also off the path of intended play – by following the framework of Riedel and Steg (2017). Specifically, any stopping time $\vartheta \in \mathscr{T}$ is considered as the start of a potential subgame in which no firm has switched, yet, even if it is past an initially planned date to switch.[9] In every such subgame, the firms are further allowed to randomize by choosing distribution functions over time that still may be state-dependent through the available information, i.e., adapted to $\mathbf{F}$. The (pure) plan to switch at $\tau \geq \vartheta$, e.g., corresponds to the distribution function $\mathbf{1}_{t \geq \tau}$, $t \geq 0$. Finally, additional strategy extensions are needed to model preemption appropriately in continuous time.

Definition 4.1. An *extended mixed strategy* for firm $i \in \{1, 2\}$ in the subgame starting at $\vartheta \in \mathscr{T}$ (with no firm having switched, yet), also called *ϑ-strategy*, is a pair of processes $(G_i^\vartheta, \alpha_i^\vartheta)$ taking values in $[0, 1]$, respectively, with the following properties.

[9] The random variables (in particular, plans) that are measurable w.r.t. the information $\mathscr{F}_\vartheta$ for some $\vartheta \in \mathscr{T}$ are generally richer than those that can be constructed from a family of $\mathscr{F}_t$-measurable random variables, $t \geq 0$, so it is not enough to consider deterministic starting dates for subgames.

(i) G_i^ϑ is adapted. It is right-continuous and nondecreasing with $G_i^\vartheta(t) = 0$ for all $t < \vartheta$, a.s.

(ii) α_i^ϑ is progressively measurable.[10] It is right-continuous where $\alpha_i^\vartheta < 1$, a.s.[11]

(iii)
$$\alpha_i^\vartheta(t) > 0 \Rightarrow G_i^\vartheta(t) = 1 \qquad \text{for all } t \geq 0, \text{ a.s.}$$

We further define $G_i^\vartheta(0-) \equiv 0$, $G_i^\vartheta(\infty) \equiv 1$ and $\alpha_i^\vartheta(\infty) \equiv 1$ for every extended mixed strategy.

The extensions α_i^ϑ were introduced as an endogenous coordination device by Fudenberg and Tirole (1985) to resolve equilibrium existence issues. They enable richer outcome probabilities where the G_i^ϑ jump to 1 simultaneously, so that simultaneous stopping (here: switching) can be avoided to a certain extent. The extensions are meant to capture limit behaviour from discrete time, but they can also be thought of as measuring a "preemption intensity" of firms. The resulting outcome probabilities $\lambda_{L,i}^\vartheta$, $\lambda_{L,j}^\vartheta$ and λ_M^ϑ at $\hat{\tau}^\vartheta := \inf\{t \geq \vartheta \mid \alpha_1^\vartheta(t) + \alpha_2^\vartheta(t) > 0\}$ as defined in Riedel and Steg (2017) for the stochastic generalization are provided in Appendix A.2 for completeness. Here, $\lambda_{L,i}^\vartheta$ and $\lambda_{L,j}^\vartheta$ denote the probabilities that firm i or firm j becomes the leader, respectively, and λ_M^ϑ the probability that both firms switch simultaneously. Since the latter outcome is never desirable (in the sense that firms would rather be the follower than end up in a situation of simultaneous switching), this outcome has traditionally been referred to as a "coordination failure" (see, for example, Huisman and Kort, 1999).

If no firm has switched prior to $\vartheta \in \mathscr{T}$, the conditional expected payoffs are then the following.

Definition 4.2. Given two extended mixed strategies $(G_i^\vartheta, \alpha_i^\vartheta)$, $(G_j^\vartheta, \alpha_j^\vartheta)$, $i, j \in \{1, 2\}$, $i \neq j$, the *payoff* of firm i in the subgame starting at $\vartheta \in \mathscr{T}$ is

$$\begin{aligned} V_i^\vartheta(G_i^\vartheta, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta) := E\Big[\int_{[0,\hat{\tau}^\vartheta)} (1 - G_j^\vartheta(s)) L_s \, dG_i^\vartheta(s) + \int_{[0,\hat{\tau}^\vartheta)} (1 - G_i^\vartheta(s)) F_s \, dG_j^\vartheta(s) \\ + \sum_{s \in [0,\hat{\tau}^\vartheta)} \Delta G_i^\vartheta(s) \Delta G_j^\vartheta(s) M_s + \lambda_{L,i}^\vartheta L_{\hat{\tau}^\vartheta} + \lambda_{L,j}^\vartheta F_{\hat{\tau}^\vartheta} + \lambda_M^\vartheta M_{\hat{\tau}^\vartheta} \,\Big|\, \mathscr{F}_\vartheta\Big]. \end{aligned}$$

A full strategy for the timing game is a system of plans that is time consistent in the sense of satisfying Bayes' law, i.e., the conditional probability of switching at any time has to be independent of when the plan has been made.

[10] Formally, the mapping $\alpha_i^\vartheta : \Omega \times [0, t] \to \mathbb{R}$, $(\omega, s) \mapsto \alpha_i^\vartheta(\omega, s)$ must be $\mathscr{F}_t \otimes \mathscr{B}([0, t])$-measurable for any $t \in \mathbb{R}_+$. It is a stronger condition than adaptedness, but weaker than optionality, which we automatically have for G_i^ϑ by right-continuity. Progressive measurability implies that $\alpha_i^\vartheta(\tau)$ will be $\mathscr{F}_\tau$-measurable for any $\tau \in \mathscr{T}$.

[11] This means that with probability 1, $\alpha_i^\vartheta(\cdot)$ is right-continuous at all $t \in [0, \infty)$ for which $\alpha_i^\vartheta(t) < 1$. Since we are here only interested in *symmetric* games, we may demand the extensions $\alpha_i^\vartheta(\cdot)$ to be right-continuous also where they take the value 0, which simplifies the definition of outcomes. See Section 3 of Riedel and Steg (2017) for issues with asymmetric games and corresponding weaker regularity restrictions.

Definition 4.3. An *extended mixed strategy* for firm $i \in \{1,2\}$ in the timing game is a family

$$(G_i, \alpha_i) := (G_i^\vartheta, \alpha_i^\vartheta)_{\vartheta \in \mathscr{T}}$$

of extended mixed strategies for all subgames $\vartheta \in \mathscr{T}$.

An extended mixed strategy (G_i, α_i) is *time-consistent* if for all $\vartheta \leq \vartheta' \in \mathscr{T}$

$$\vartheta' \leq t \in \mathbb{R}_+ \Rightarrow \; G_i^\vartheta(t) = G_i^\vartheta(\vartheta'-) + \big(1 - G_i^\vartheta(\vartheta'-)\big) G_i^{\vartheta'}(t) \quad \text{a.s.}$$

and

$$\vartheta' \leq \tau \in \mathscr{T} \Rightarrow \; \alpha_i^\vartheta(\tau) = \alpha_i^{\vartheta'}(\tau) \quad \text{a.s.}$$

Now the equilibrium concept is standard.

Definition 4.4. A *subgame perfect equilibrium* for the timing game is a pair (G_1, α_1), (G_2, α_2) of time-consistent extended mixed strategies, such that for all $\vartheta \in \mathscr{T}$, $i, j \in \{1,2\}$, $i \neq j$, and extended mixed strategies $(G_a^\vartheta, \alpha_a^\vartheta)$, it holds that

$$V_i^\vartheta(G_i^\vartheta, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta) \geq V_i^\vartheta(G_a^\vartheta, \alpha_a^\vartheta, G_j^\vartheta, \alpha_j^\vartheta) \quad \text{a.s.}$$

That is, every pair $(G_1^\vartheta, \alpha_1^\vartheta)$, $(G_2^\vartheta, \alpha_2^\vartheta)$ is an *equilibrium* in the subgame at $\vartheta \in \mathscr{T}$.

4.2 Preemption Equilibria

We start our construction of a subgame perfect equilibrium by considering subgames with a first-mover advantage $L_\vartheta > F_\vartheta$, i.e., starting in the preemption region $\mathcal{P}$, where we can refer to the following established equilibria in which at least one firm switches immediately.

Proposition 4.5 (Riedel and Steg (2017), Proposition 3.1)**.** *Fix $\vartheta \in \mathscr{T}$ and suppose $\vartheta = \inf\{t \geq \vartheta \,|\, L_t > F_t\}$ a.s. Then $(G_1^\vartheta, \alpha_1^\vartheta)$, $(G_2^\vartheta, \alpha_2^\vartheta)$ defined by*

$$\alpha_i^\vartheta(t) = \mathbf{1}_{L_t > F_t} \frac{L_t - F_t}{L_t - M_t}$$

for any $t \in [\vartheta, \infty)$ and $G_i^\vartheta = \mathbf{1}_{t \geq \vartheta}$, $i = 1, 2$, are an equilibrium in the subgame at ϑ. The resulting payoffs are $V_i^\vartheta(G_i^\vartheta, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta) = F_\vartheta$.

In these equilibria, the firms are indifferent between stopping and waiting. The latter would mean becoming follower instantaneously, as then the opponent switches with certainty. If $L_\vartheta > F_\vartheta$, then there is a positive probability of simultaneous switching, which is the "cost of preemption", driving the payoffs down to F_ϑ. Of particular interest are however subgames ϑ with $L_\vartheta = F_\vartheta$, where each firm becomes leader or follower with probability $\frac{1}{2}$ in the equilibrium of Proposition 4.5 (given the outcome probabilities defined in Section A.2). This is the case in "continuation" equilibria at $\tau_{\mathcal{P}}(\vartheta) := \inf\{t \geq \vartheta \,|\, L_t > F_t\}$ (where the game will thus end with probability 1) for subgames which begin with $L_\vartheta \leq F_\vartheta$. By these continuation equilibria the firms know that *if* the preemption region is ever reached, then each firm can expect to earn the follower value *at that time*. This observation will turn out to be important for constructing the equilibrium in the case of attrition.

4.3 Constrained Leader's Stopping Problem

We now turn to equilibria for subgames beginning outside the preemption region where we will observe a war of attrition, in contrast to typical strategic real option models in the literature, in which only preemption is considered. Our equilibria are closely related to a particular stopping problem that we here discuss in detail. Its value (function) will also turn out to be the firm's continuation value in equilibrium and its solution is needed to characterize and understand equilibrium strategies. This stopping problem is also of interest on its own as it is two-dimensional and therefore not at all standard in the optimal stopping literature.

In accordance with Proposition 4.5 we fix the (equilibrium) payoff F whenever the state hits the preemption region $\mathcal{P}$. Suppose now that only one firm, say i, can make the switch before $\mathcal{P}$ is reached. Then i can determine when to become optimally the leader up to hitting $\mathcal{P}$, where the game ends with the current value of F as the expected payoff.

Letting $\tau_{\mathcal{P}} := \inf\{t \geq 0 \,|\, (X_t, Y_t) \in \mathcal{P}\} = \inf\{t \geq 0 \,|\, L_t > F_t\}$ denote the first hitting time of the preemption region $\mathcal{P}$, firm i now faces the problem of optimally stopping the auxiliary payoff process

$$\tilde{L} := L\mathbf{1}_{t<\tau_{\mathcal{P}}} + F_{\tau_{\mathcal{P}}}\mathbf{1}_{t\geq\tau_{\mathcal{P}}}.$$

It obviously suffices to consider stopping times $\tau \leq \tau_{\mathcal{P}}$. We are also only interested in initial states $(X_0, Y_0) = (x, y) \in \mathcal{P}^c$, which implies $F_{\tau_{\mathcal{P}}} = L_{\tau_{\mathcal{P}}}$ by continuity. Then the value of our stopping problem is

$$V_{\tilde{L}}(x,y) := \operatorname*{ess\,sup}_{\tau\geq 0} E\big[\tilde{L}_\tau\big] = \operatorname*{ess\,sup}_{\tau\in[0,\tau_{\mathcal{P}}]} E\big[L_\tau\big]$$

for $(X_0, Y_0) = (x, y) \in \mathcal{P}^c$ (and $V_{\tilde{L}}(x,y) := F_0$ for $(X_0, Y_0) = (x, y) \in \mathcal{P}$). Thanks to the strong Markov property, the solution of this problem can be characterized (see Krylov (1980)) by identifying the *stopping region* of the state space $\{(x,y) \in \mathbb{R}^2_+ \,|\, V_{\tilde{L}}(x,y) = \tilde{L}_0 \text{ for } (X_0,Y_0) = (x,y)\} = \mathcal{P} \cup \{(x,y) \in \mathcal{P}^c \,|\, V_{\tilde{L}}(x,y) = L(0,y)\}$ and the *continuation region*

$$\mathcal{C} := \{(x,y) \in \mathcal{P}^c \,|\, V_{\tilde{L}}(x,y) > L(0,y)\} \subset \mathcal{P}^c.$$

By the continuity of $\tilde{L}$ (resp. L) it is indeed optimal to stop as soon as (X,Y) hits $\mathcal{C}^c$.

The stopping and continuation regions are related to those for the unconstrained problem $\sup_{\tau\geq 0} E\big[L_\tau\big]$, which only depends on Y and its initial value $Y_0 = y$. This is a standard problem of the real options literature, which is uniquely solved[12] by stopping the first time Y exceeds the threshold

$$y^* = \frac{\beta_1}{\beta_1 - 1}(r - \mu_Y)\Big(I + \frac{c_B - c_0}{r}\Big),$$

with $\beta_1 > 1$ the positive root of the quadratic equation

$$\mathcal{Q}(\beta) \equiv \frac{1}{2}\sigma_Y^2\beta(\beta - 1) + \mu_Y\beta - r = 0. \tag{4.1}$$

At y^*, the net present value of investing in B as a monopolist is just high enough to offset the option value of waiting.[13]

The constraint is binding iff the stopping region for the unconstrained problem does *not* completely contain the preemption region $\mathcal{P}$, which is the situation depicted in Figure 1 above with

$$y^* > y_{\mathcal{P}} := (r - \mu_Y)\Big(\frac{c_B - c_A}{r} + I\Big).^{14}$$

Indeed, as stopping L is optimal for $Y_t \geq y^*$ in the unconstrained case, it is too under the constraint $\tau \leq \tau_{\mathcal{P}}$. Hence, if $y^* \leq y_{\mathcal{P}}$, the continuation regions for the constrained and unconstrained problems agree.

In the displayed case $y^* > y_{\mathcal{P}}$, however, we have a much more complicated, truly two-dimensional stopping problem.[15] Then the stopping region for $\tilde{L}$ extends below $\mathcal{P} \cup \{(x,y) \in \mathbb{R}_+^2 \mid y \geq y^*\}$ due to the risk of hitting $\mathcal{P}$ at a lower value of Y when (X_0, Y_0) is close to $\mathcal{P}$ but below y^*. Stopping is dominated, however, below the value

$$\bar{y} := c_B - c_0 + rI,$$

where the drift of L is positive, whence also $\bar{y} < y^*$ if either is positive.[16] Furthermore we show in the proof of the following proposition that it is always worthwhile to wait until Y exceeds at least $y_{\mathcal{P}}$ (if this is less than y^*). The continuation region $\mathcal{C}$ and the stopping region

[12] The solution also holds in the degenerate case $Y_0 = y^* = 0$, iff L is constant, but then it is not unique, of course.

[13] Above y^*, the forgone revenue from any delay is so high that investment is strictly optimal. Below y^*, the depreciation effect on the investment cost is dominant to make waiting strictly optimal.

[14] $y_{\mathcal{P}} \geq 0$ by our assumption on the parameters.

[15] Genuinely two-dimensional problems and their free boundaries are rarely studied in the literature due to their general complexity. Sometimes problems that appear two-dimensional are considered, e.g. involving a one-dimensional diffusion and its running supremum, which are then reduced to a one-dimensional, more standard problem. See, e.g., Peskir and Shiryaev (2006).

[16] The drift of L, $-e^{-rt}(Y_t - \bar{y})\,dt$, can be derived by applying Itō's formula to $L(t, Y_t)$ in (2.2). With $\bar{y}$ we also have

$$L_t = -\frac{c_0}{r} + E\bigg[\int_t^\infty e^{-rs}(Y_s - \bar{y})\,ds\,\bigg|\,\mathscr{F}_t\bigg].$$

It holds that $\bar{y} = 0 \Leftrightarrow y^* = 0$ and otherwise $y^*/\bar{y} \in (0,1)$, since $\beta_1\mu_Y < r$ (obviously if $\mu_Y \leq 0$ and due to $\mathcal{Q}(r/\mu_Y) > 0$ in (4.1) if $\mu_Y > 0$); furthermore $\bar{y} > y_{\mathcal{P}}$ iff $(c_0 - c_A)/(c_B - c_A + rI) < \mu_Y/r$, e.g., if c_0 is sufficiently small or if I or c_B are sufficiently large (while $\mu_Y > 0$).

$\mathcal{C}^c$ for $\tilde{L}$ are then indeed separated by a function $b(x)$ as shown.

Proposition 4.6. *There exists a function $b : \mathbb{R}_+ \to [\min(y_\mathcal{P}, y^*), y^*]$ which is nondecreasing and continuous, such that, up to the origin,*

$$\mathcal{C} = \{(x,y) \in \mathbb{R}_+^2 \mid y < b(x)\} \subset \mathcal{P}^c$$

and $\tau_{\mathcal{C}^c} := \inf\{t \geq 0 \,|\, (X_t, Y_t) \in \mathcal{C}^c\}$ attains $V_{\tilde{L}}(x,y) = E[\tilde{L}_{\tau_{\mathcal{C}^c}}]$ for $(X_0, Y_0) = (x,y) \in \mathbb{R}_+^2$. The origin belongs to $\mathcal{C}$ iff $y^* > 0$, i.e., iff $\bar{y} > 0$. In that case b further satisfies $b(x) \geq \min(\bar{y}, y_\mathcal{P} + x(r-\mu_Y)/(r-\mu_X))$ (and otherwise $b \equiv y^*$ and $\mathcal{C} = \emptyset$).

We show in Lemma 4.8 below that indeed $b < y^*$ if $y_\mathcal{P} < y^*$; nevertheless $b(x) \nearrow y^*$ since the value of the constrained problem converges to that of the unconstrained problem as $X_0 \to \infty$.

By the shape of $\mathcal{C}$ identified in Proposition 4.6, a firm that was sure never to become follower outside $\mathcal{P}$ would switch as soon as the state leaves $\mathcal{C}$: any delay on $\{Y > b(X)\} \cap \{F > L\}$ would induce a running expected loss given by the drift of L, which is $-e^{-rt}(Y_t - \bar{y})\,dt < 0$ there. Switching would yield the other, staying firm the prize $F > L$, however. Since both firms face the same situation, we obtain a war of attrition in the stopping region of the constrained stopping problem: there is a running cost of waiting for a prize that is obtained if the opponent gives in and switches. Therefore we call it the *attrition region*

$$\mathcal{A} := \{(x,y) \in \mathbb{R}_+^2 \,|\, y \geq b(x)\} \setminus \mathcal{P}.^{17}$$

In order to derive equilibria outside the preemption region $\mathcal{P}$, we need to know exactly the expected cost of abstaining to stop in $\mathcal{A}$. In general this need not be just the (negative of the) drift of L, $e^{-rt}(Y_t - \bar{y})\,dt \geq 0$ in $\mathcal{A}$, since the state can transit very frequently between $\mathcal{A}$ and $\mathcal{C}$.[18] Our equilibrium verification uses the following characterization of the constrained stopping problem and the cost of stopping too late. By the general theory of optimal stopping, the value *process* of the stopping problem $V_{\tilde{L}}(X,Y) := U_{\tilde{L}}$ is the smallest supermartingale dominating the payoff process $\tilde{L}$, called the Snell envelope. As a supermartingale it has a decomposition $U_{\tilde{L}} = M_{\tilde{L}} - D_{\tilde{L}}$ with a martingale $M_{\tilde{L}}$ and a nondecreasing process $D_{\tilde{L}}$ called the *compensator* starting from $D_{\tilde{L}}(0) = 0$. Hence $U_{\tilde{L}}(0) - E[U_{\tilde{L}}(\tau)] = E[D_{\tilde{L}}(\tau)] \geq 0$ is the expected cost if one considers only stopping after τ.

[17] Formally we only have $\mathcal{A} = \mathcal{C}^c \cap \mathcal{P}^c$ up to the origin by Proposition 4.6, but we prefer to work with the boundary representation. Precisely $\mathcal{A} \supseteq \mathcal{C}^c \cap \mathcal{P}^c$, and as $b(x)$ lies below the preemption boundary $y_\mathcal{P} + x(r - \mu_Y)/(r-\mu_X)$, in fact $b(0) = \min(y_\mathcal{P}, y^*)$, so we have $(0,0) \in \mathcal{C} \cap \mathcal{A}$ iff $y^* > 0 \geq y_\mathcal{P}$, resp. $\mathcal{A} = \mathcal{C}^c \cap \mathcal{P}^c$ iff $y^* \leq 0$ or $y_\mathcal{P} > 0$.

[18] This "switching on and off" of the waiting cost could lead to non-trivial behaviour, depending on the *local time* our process (X,Y) spends on the boundary between the two regions given by $y = b(x)$. See Jacka (1993) for an example based on Brownian motion where that local time is non-trivial.

Given the geometry of the boundary between $\mathcal{A}$ and $\mathcal{C}$ that we have identified in Proposition 4.6, the increments of $D_{\tilde{L}}$ are indeed given by the (absolute value of the) drift of L where the state is in $\mathcal{A}$.

Proposition 4.7. *With $b(x)$ as in Proposition 4.6 and $(X_0, Y_0) \neq (0,0)$ we have*

$$dD_{\tilde{L}}(t) = \mathbf{1}_{t<\tau_{\mathcal{P}}, Y_t \geq b(X_t)} e^{-rt}(Y_t - \bar{y})\, dt \tag{4.2}$$

for all $t \in \mathbb{R}_+$ a.s. If $(X_0, Y_0) = (0,0)$, (4.2) still holds if $y^* \leq 0$ or $y_{\mathcal{P}} > 0$; otherwise $dD_{\tilde{L}} \equiv 0$.

Using the Snell envelope $U_{\tilde{L}}$, we can now show that b is not trivial, i.e., not just the constraint applied to the unconstrained solution.

Lemma 4.8. *If $y_{\mathcal{P}} < y^*$, then also $b < y^*$.*

Remark 4.9. It is a difficult problem to characterize b more explicitly. However, by similar arguments as in the proof of Lemma 4.8 one can obtain a scheme to approximate b (see also our numerical study in Section 5). The Snell envelope $U_{\tilde{L}}$, being a supermartingale of class (D), converges in $L^1(P)$ to $U_{\tilde{L}}(\infty) = \tilde{L}_\infty = L_{\tau_{\mathcal{P}}}$. Also its martingale and monotone parts both converge in $L^1(P)$ as $t \to \infty$. If $Y_0 \geq b(X_0)$, we further have $U_{\tilde{L}}(0) = \tilde{L}_0 = L_0$. Therefore,

$$\begin{aligned} U_{\tilde{L}}(0) - E[U_{\tilde{L}}(\infty)] = L_0 - E[L_{\tau_{\mathcal{P}}}] &= E\Big[\int_0^{\tau_{\mathcal{P}}} e^{-rt}(Y_t - \bar{y})\, dt\Big] \\ &= E[D_{\tilde{L}}(\infty)] = E\Big[\int_0^{\tau_{\mathcal{P}}} \mathbf{1}_{Y_t \geq b(X_t)} e^{-rt}(Y_t - \bar{y})\, dt\Big] \end{aligned}$$

and hence

$$Y_0 \geq b(X_0) \quad \Rightarrow \quad E\Big[\int_0^{\tau_{\mathcal{P}}} \mathbf{1}_{Y_t < b(X_t)} e^{-rt}(Y_t - \bar{y})\, dt\Big] = 0. \tag{4.3}$$

Since $y^* \geq b$, one can also use $\tau_{\mathcal{P}} \wedge \inf\{t \geq 0 \,|\, Y_t \geq y^*\}$ in (4.3). Any candidate $\hat{b}$ for b should satisfy $\hat{b} \geq \bar{y}$ like b itself. Therefore, if $\hat{b}$ is (locally) too high (low), the expectation in (4.3) will be positive (negative) for $Y_0 = \hat{b}(X_0)$.

4.4 Subgame Perfect Equilibria

We are now ready to combine the previous results to construct equilibria for arbitrary subgames. If the state reaches $\mathcal{P}$, there is preemption with immediate switching by some firm. In the continuation region $\mathcal{C}$, the firms just wait. In the attrition region $\mathcal{A}$, the firms switch at a specific rate such that the resulting probability for the other to become (more profitable) follower exactly compensates for the negative drift of the leader's payoff, making each firm indifferent between waiting and immediate switching.

Proposition 4.10. *Suppose* $(X_0, Y_0) \neq (0,0)$ *or* $y_{\mathcal{P}} > 0$*. Fix* $\vartheta \in \mathscr{T}$ *and set* $\tau_{\mathcal{P}} := \inf\{t \geq \vartheta \,|\, L_t > F_t\}$*. Then* $(G_1^\vartheta, \alpha_1^\vartheta)$*,* $(G_2^\vartheta, \alpha_2^\vartheta)$ *defined by*

$$\frac{dG_i^\vartheta(t)}{1 - G_i^\vartheta(t)} = \frac{\mathbf{1}_{Y_t \geq b(X_t)}(Y_t - \bar{y})\,dt}{X_t/(r - \mu_X) - (Y_t - y_{\mathcal{P}})/(r - \mu_Y)} \tag{4.4}$$

on $[\vartheta, \tau_{\mathcal{P}})$ *and* $G_i^\vartheta(t) = 1$ *on* $[\tau_{\mathcal{P}}, \infty]$*, and*

$$\alpha_i^\vartheta(t) = \mathbf{1}_{L_t > F_t} \frac{L_t - F_t}{L_t - M_t}$$

on $[\vartheta, \infty]$*,* $i = 1, 2$*, are an equilibrium in the subgame at* ϑ*.*

The resulting payoffs are

$$V_i^\vartheta(G_i^\vartheta, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta) = V_{\tilde{L}}(X_\vartheta, Y_\vartheta) = \operatorname*{ess\,sup}_{\tau \geq \vartheta} E\big[\tilde{L}_\tau \,\big|\, \mathscr{F}_\vartheta\big]$$

with $\tilde{L} := L\mathbf{1}_{t < \tau_{\mathcal{P}}} + F_{\tau_{\mathcal{P}}}\mathbf{1}_{t \geq \tau_{\mathcal{P}}}$*.*

The Markovian equilibrium stopping rate (4.4) is the (absolute value of the) drift of L in $\mathcal{A}$, divided by the difference $F - L$, ensuring the exact compensation that makes each firm indifferent between the two roles, as discussed before. Waiting is strictly optimal if $(X, Y) \in \mathcal{C}$, where the equilibrium payoff is $U_{\tilde{L}} > L$.

Although the stopping rate (4.4) becomes unboundedly large as the state approaches the preemption region – because the denominator $F - L$ vanishes while the numerator is bounded away from zero – the cumulative stopping probability does *not* converge to 1. There will indeed be some mass left when reaching the preemption boundary. This is a distinctive feature of our stochastic model[19] that is not observed in deterministic versions, cf. Section 6.

Proposition 4.11. *The strategies* G_i^ϑ *specified in Proposition 4.10 satisfy* $\Delta G_i^\vartheta(\tau_{\mathcal{P}}) > 0$ *on* $\{\tau_{\mathcal{P}} < \infty\}$ *a.s.*

Concerning the regularity of α_i, note that we can only have $L \leq M$ on $\{Y \leq y_{\mathcal{P}}\}$ by our assumption $(c_0 - c_A)/r + I \geq 0$, whence $L > M$ on $\{(X, Y) \in \mathcal{P} \cup \partial\mathcal{P}\}$ a.s. if $X_0 > 0$; so α_i will be continuous and vanish on $\{(X, Y) \in \partial\mathcal{P}\}$.[20]

[19] The mathematical question underlying Proposition 4.11 is of interest in its own right. With the same arguments used in the proof (which then become much simpler) one can show that for a Brownian motion B started at $a > 0$ one has

$$\int_0^{\tau_0} \frac{1}{B_t}\,dt < \infty$$

a.s., where $\tau_0 = \inf\{t \geq 0 \,|\, B_t \leq 0\}$. We actually consider the reciprocal of the process $Z_t = X_t - Y_t + a$ with our geometric Brownian motions X, Y.

[20] In the particular case $X_0 = 0$, $X \equiv 0$ and

$$\alpha_i(t) = \frac{L_t - F_t}{L_t - M_t}\mathbf{1}_{L_t > F_t} = \frac{(Y_t - y_{\mathcal{P}})\mathbf{1}_{Y_t > y_{\mathcal{P}}}}{Y_t - y_{\mathcal{P}} + (r - \mu_Y)(I - (c_A - c_0)/r)}.$$

We can easily aggregate the equilibria to a subgame perfect equilibrium since all quantities are Markovian.

Theorem 4.12. *The strategies* $(G_i^\vartheta, \alpha_i^\vartheta)_{\vartheta\in\mathscr{T}}$, $i = 1,2$ *of Proposition 4.10 form a subgame perfect equilibrium.*

Remark 4.13. If $X_0 = 0 < Y_0$, the game is played "on the y-axis" and the derived equilibria are as follows. There is preemption as soon as Y exceeds the preemption point $y_{\mathcal{P}}$. Any y less than $b(0) = \min(y_{\mathcal{P}}, y^*)$ is in the continuation region, so there is attrition iff $(y^*)^+ < y_{\mathcal{P}}$, between those points.

If $Y_0 = 0$, the game is played "on the x-axis" and is actually deterministic since we then have $\tau_{\mathcal{P}} = \infty$ by our assumption $y_{\mathcal{P}} \geq 0$. In fact, now $F_t > L_t \,\forall t \in \mathbb{R}_+$ if $X_0 \vee y_{\mathcal{P}} > 0$, and $dL_t = e^{-rt}\bar{y}\,dt$. Thus, if $\bar{y} \geq 0$, stopping is dominated for $t \in \mathbb{R}_+$, strictly if $\bar{y} > 0$ or against any strategy that charges $[0,\infty)$, while the proposed equilibrium strategies only charge $[\infty]$.[21] If $\bar{y} < 0$, also $b \equiv y^* < 0$, and there is permanent attrition with rate $-dL/(F-L)$.

In the completely degenerate case that further $X_0 = y_{\mathcal{P}} = 0$, the stopping rate of Proposition 4.10 is not well defined. Now, however, $L \equiv F$ and there are the following equilibria: for $\bar{y} > 0$, both firms wait indefinitely, for $\bar{y} = 0$, any strategy pair with no joint mass points is an equilibrium, and for $\bar{y} < 0$, one firm switches immediately and the other abstains.

5 A Numerical Example

As an illustration of the model we present a numerical example to show the consequences of the results presented so far. The parameter values of our base case are given in Table 1.

$c_0 = 3.5$	$c_A = 4.5$	$c_B = 5$	$I = 100$	$r = .1$
$\mu_Y = .04$	$\mu_X = .04$	$\sigma_Y = .25$	$\sigma_X = .25$	$\rho = .4$

Table 1: Parameter values for a numerical example

For this case it holds that $y^* = 17.45$. The preemption region is depicted in Figure 2a. Finding the boundary $x \mapsto b(x)$ as characterized in Proposition 4.6 is not trivial as can be seen from Remark 4.9. However, it is known from the literature on numerical methods for pricing American options (see, for example, Glasserman, 2004 for an overview) that a piecewise linear or exponential approximation often works well. In our case, we hypothesize an exponential boundary of the form

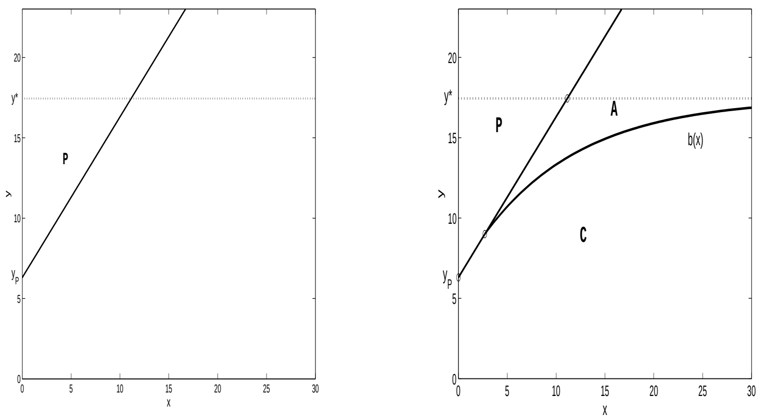
$$b(x) = y^* - (y^* - \bar{y})e^{-\gamma(x-\bar{x})},$$

Then α_i will still be continuous for $I > (c_A - c_0)/r$ and unity on $\{Y \geq y_{\mathcal{P}}\}$ for $I = (c_A - c_0)/r$.

[21]Indeed, if $\bar{y} = y^* = 0$, $b \equiv y^* = 0$, while if $\bar{y} > 0 \Leftrightarrow y^* > 0$, then $b(X_t) > 0$ for $X_0 + y_{\mathcal{P}} > 0$ by $b(x) \geq \min(\bar{y}, y_{\mathcal{P}} + x(r-\mu_Y)/(r-\mu_X))$.

where $\bar{x}$ is the unique value for x such that $(\bar{x}, \bar{y})$ is on the boundary of $\mathcal{P}$, and γ is a free parameter.

A rough implementation now consists of hypothesizing a value for γ, followed by simulating 3,000 sample paths for several starting values close to this boundary. The optimal boundary should be such that simulated continuation values are uniformly slightly larger than the leader value. Using a manual search across several values for γ suggests a value of .0984 and a boundary as depicted in Figure 2b.



(a) Preemption region

(b) Optimal stopping boundary for constrained leader value

Figure 2: Preemption, attrition and continuation regions for numerical example.

Once the boundary is established we can simulate equilibrium sample paths using the equilibrium strategies from Theorem 4.12. Two sample paths, together with their realizations of the attrition rate (4.4) are given in Figures 3 and 4. In parallel to the paths we sample the investment decisions from the respective attrition rates. The displayed paths end where investment takes place along that particular sample path.

Note that, even though the two sample paths start at exactly the same point $(X_0, Y_0) = (6, 8)$, their equilibrium realizations are very different. In the sample path $t \mapsto (X_t(\omega_1), Y_t(\omega_1))$ in Figure 3, the attrition region is entered and exited several times before, finally, the sample path hits the preemption boundary. At that time the realized attrition rate has gone up to about $G_j^0(\tau_{\mathcal{P}}(\omega_1)-) \approx .045$, $j = 1, 2$. However, no firm has invested up to time $\tau_{\mathcal{P}}(\omega_1)$. At time $\tau_{\mathcal{P}}(\omega_1)$ we have $(G_j^{\tau_{\mathcal{P}}(\omega_1)}(\tau_{\mathcal{P}}(\omega_1)), \alpha_j^{\tau_{\mathcal{P}}(\omega_1)}(\tau_{\mathcal{P}}(\omega_1))) = (1, 0)$, $j = 1, 2$, so that (see Appendix A.2) each firm is the first to invest at time $\tau_{\mathcal{P}}(\omega_1)$ with probability $\lambda_{L,i}^{\tau_{\mathcal{P}}(\omega_1)} = .5$. So, the ω_1 sample path ends with investment taking place when there is, just, a first-mover advantage. In the sample path $t \mapsto (X_t(\omega_2), Y_t(\omega_2))$ in Figure 4 the attrition region is again entered and exited several times. Note that the first entry of the attrition region is at roughly the same time as along the ω_1 sample path. At time 2.5 (approximately), when the attrition

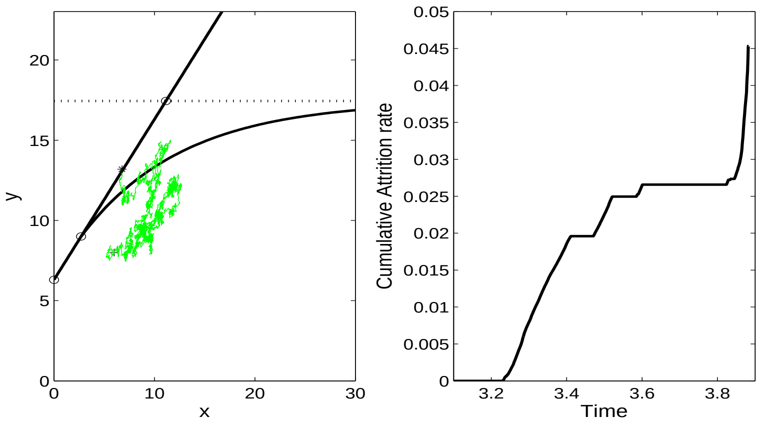


Figure 3: Sample path ending in preemption and its realized attrition rate.

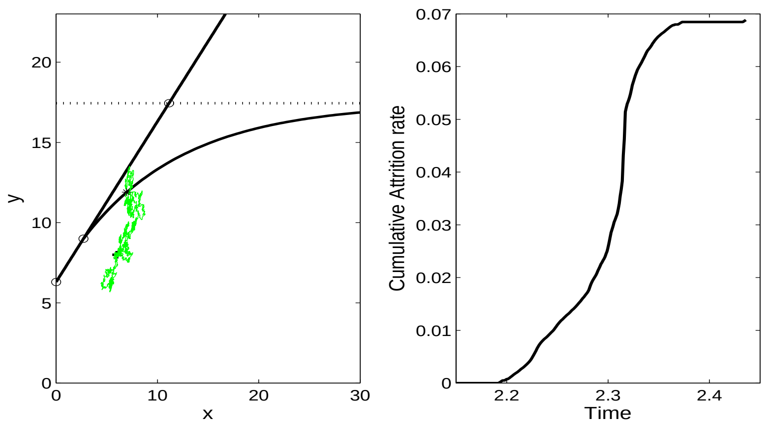


Figure 4: Sample path ending in attrition region and its realized attrition rate.

rate realization has gone up to $G_j^0(1.4) \approx .07$, $j = 1, 2$, one of the two firms actually invests, thereby handing the other firm the second-mover advantage.

Finally, we run a simulation of 3,000 sample paths, all starting at $(X_0, Y_0) = (6, 8)$, and record whether the game stops in the attrition or preemption region. We find that 80% of sample paths end with preemption, whereas 20% of sample paths end in the attrition region. A scatter diagram of values $(X_{\tau(\omega)}, Y_{\tau(\omega)})$ at the time of first investment is given in Figure 5. The average fraction of time spent in the attrition region is 6.76% and all of sample paths experience attrition before the time of first investment. The value of immediate investment is -16.67, whereas the (simulated) value of the equilibrium strategies is 12.02.

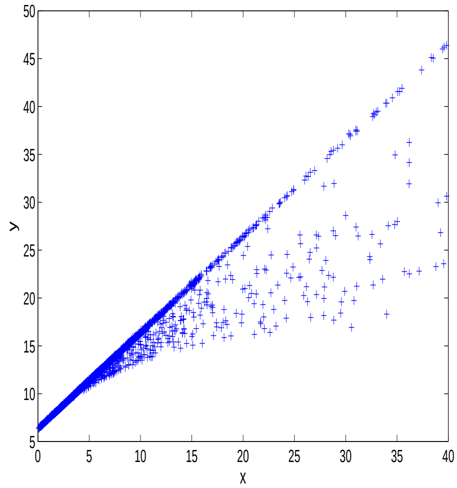


Figure 5: Scatter plot of realizations of (X_{τ^*}, Y_{τ^*}).
– This figure excludes outliers in both dimensions. –

6 Differences between Deterministic and Stochastic Models

For comparative purposes, we now briefly analyse a deterministic version of our model, i.e., setting $\sigma_X = \sigma_Y = 0$. The expressions for L_t, F_t and M_t provided in (2.2) then remain the same. In order to avoid the distinction of many different cases, we concentrate on the most "typical" one, representing the economic story that we have in mind. Therefore, assume that the market B is *growing* at the rate $\mu_Y > 0$ from the initial value $Y_0 > 0$. Further assume that $\mu_Y \geq \mu_X$, so that it is only market B that can possibly overtake the other, A, in profitability.[22] The same holds then for the functions L and F: Now we have preemption due to $L_t > F_t$ iff $(X_t, Y_t) \in \mathcal{P}$, i.e., iff

$$e^{\mu_Y t}\left(\frac{Y_0}{r-\mu_Y} - \frac{X_0}{r-\mu_X}e^{(\mu_X-\mu_Y)t}\right) > \frac{c_B - c_A}{r} + I. \tag{6.2}$$

[22]If $\mu_X > \mu_Y$ and $X_0 > 0$, the behaviour differs from that discussed in the main text as follows. Now the profitability of market A will eventually outgrow that of market B and preemption occurs during a bounded time interval (if at all). Indeed, since

$$\frac{\partial(L_t - F_t)}{\partial t} = e^{\mu_X t}\left(\frac{\mu_Y Y_0}{r-\mu_Y}e^{(\mu_Y-\mu_X)t} - \frac{\mu_X X_0}{r-\mu_X}\right) \tag{6.1}$$

has at most one zero and is negative for t sufficiently large, $(L_t - F_t)$ grows until its maximum and then declines, so $\{t \mid L_t - F_t > 0\}$ forms an interval (if nonempty). Arranging $(L_t - F_t)$ similarly as (6.1) shows that $(L_t - F_t)$ is negative for t sufficiently large, which bounds the previous interval.

Supposing that the preemption interval is nonempty and that $\mu_Y > 0$, the unique maximum of L_t attained at $t = \bar{t}$ may occur before, during or after preemption. We can observe attrition followed by preemption only in the first case, i.e., if $L_{\bar{t}} < F_{\bar{t}}$ and $\partial(L_{\bar{t}} - F_{\bar{t}}) > 0$, which is if $X_{\bar{t}}/(r-\mu_X)$ (see the right of (6.3)) lies in $\big(\bar{y}/(r-\mu_Y) - (c_B - c_A)/r - I, \mu_Y\bar{y}/(\mu_X(r-\mu_Y))\big)$. This interval is nonempty, e.g., if μ_X and μ_Y are sufficiently close.

In subgames after the preemption interval there will be indefinite attrition for all $t \geq \bar{t}$.

The left-hand side is the product of two nondecreasing functions and the right-hand side is nonnegative by our assumption made in Section 2. Hence $L_t > F_t$ for all

$$t > \inf\{t \geq 0 \,|\, L_t > F_t\} =: t_{\mathcal{P}}$$

and preemption will indeed occur iff the LHS of (6.2) ever becomes positive, i.e., $t_{\mathcal{P}} < \infty \Leftrightarrow \mu_Y > \mu_X$ or $Y_0/(r-\mu_Y) > X_0/(r-\mu_X)$. Preemption does not start immediately at $t = 0$ iff $Y_0/(r-\mu_Y) - X_0/(r-\mu_X) < (c_B - c_A)/r + I$, i.e., $t_{\mathcal{P}} > 0 \Leftrightarrow (X_0, Y_0) \notin \overline{\mathcal{P}}$.

Now the constrained stopping problem to maximize L_t over $t \leq t_{\mathcal{P}}$ is very easy. As L_t is increasing iff $Y_t < \bar{y}$, i.e., iff $Y_0 < \bar{y}$ and

$$t < \frac{1}{\mu_Y} \ln\left(\frac{\bar{y}}{Y_0}\right) =: \bar{t},$$

the solution is to stop at the unconstrained optimum $t = \bar{t}$ (resp. at $t = \bar{t} := 0$ if $Y_0 \geq \bar{y}$) or at the constraint $t = t_{\mathcal{P}}$ if it is binding. The solution is also unique, since L_t is strictly increasing for $Y_t < \bar{y}$ and strictly decreasing for $Y_t \geq \bar{y}$.

As a consequence, stopping is strictly dominated before the constrained optimum $\min(\bar{t}, t_{\mathcal{P}})$ is reached and we can only observe attrition in equilibrium if the *un*constrained maximum of L is reached *before* preemption starts. It indeed holds that $L_{\bar{t}} < F_{\bar{t}}$ (given $Y_0 < \bar{y}$) iff

$$\begin{aligned} & \frac{Y_{\bar{t}}}{r-\mu_Y} - \frac{c_B - c_A}{r} - I < \frac{X_{\bar{t}}}{r-\mu_X} \\ \Leftrightarrow \qquad & \frac{\bar{y}}{r-\mu_Y} - \frac{c_B - c_A}{r} - I < \frac{X_0 e^{\mu_X \bar{t}}}{r-\mu_X} = \frac{X_0}{r-\mu_X}\left(\frac{\bar{y}}{Y_0}\right)^{\frac{\mu_X}{\mu_Y}} \end{aligned} \tag{6.3}$$

(and $L_0 < F_0 \Leftrightarrow (X_0, Y_0) \notin \overline{\mathcal{P}}$ for $Y_0 \geq \bar{y}$), so that there will be attrition iff the initial profitability X_0 of market A is large enough to let Y_t exceed $\bar{y}$ before $(X_t, Y_t) \in \overline{\mathcal{P}}$.[23] Once $Y_t > \bar{y}$, resp. $t > \bar{t}$, L_t strictly decreases and we observe attrition until preemption starts at $t_{\mathcal{P}}$.

The equilibrium can now be represented as in Proposition 4.10, resp. Theorem 4.12, with the fixed preemption start date $t_{\mathcal{P}}$ replacing $\tau_{\mathcal{P}}(\vartheta)$ throughout and $t \geq \bar{t}$ replacing the dynamic attrition boundary $Y_t \geq b(X_t)$. Note that if there is attrition before the preemption region is reached, stopping will now occur with probability 1 due to attrition.[24]

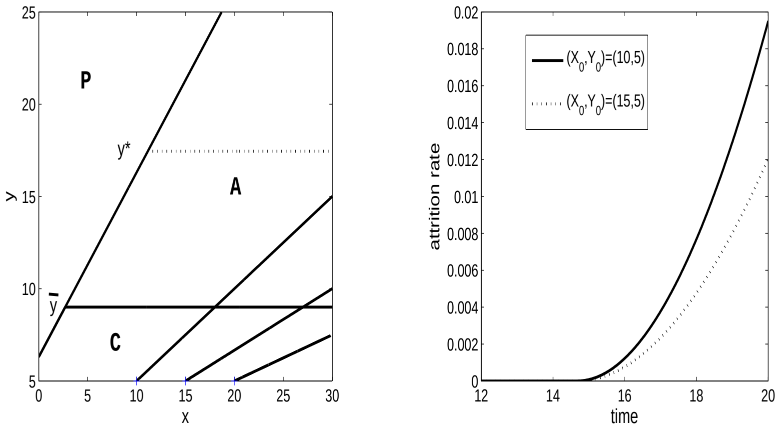
[23] If $\mu_X \neq 0$, the path that (X_t, Y_t) takes in the state space R_+^2 is given by the relation $y = Y_0(x/X_0)^{(\mu_Y/\mu_X)}$. For $\mu_X = 0$, it is of course just $\{(X_0, y) \,|\, y \geq Y_0\}$.

[24] In analogy (in fact contrast) to Proposition 4.11 and its proof it suffices to verify that

$$\int_0^{\tau_{\mathcal{P}}} \frac{dt}{X_t - Y_t + a} = \infty$$

when in the simplified notation $\tau_{\mathcal{P}} = \inf\{t \geq 0 \,|\, X_t - Y_t \leq a\} \in (0, \infty)$, since we are only interested in the case $Y_{\tau_{\mathcal{P}}} > \bar{y}$ so that Y_t is bounded away from $\bar{y}$ near $\tau_{\mathcal{P}}$, where the above integral explodes. Indeed, with

As a numerical example, we use the same parameter values as before, but set $\sigma_X = \sigma_Y = 0$. For easy reference, the left-panel of Figure 6 includes the approximate boundary for the attrition region as derived in Section 5. Note that for each sample path it is clear *a priori* that the game ends in attrition and that, once the attrition region is entered, it will never be exited again. As a result, the equilibrium attrition rate is a smooth function of time as can be seen in the right-panel of Figure 6. In this example none of the sample paths will ever end in preemption.



(a) Sample path (b) Attrition rate

Figure 6: Paths and attrition rates for deterministic case.

If, however, we take the same example, but with $\mu_Y = .06$ and $\mu_X = .02$, then the situation is very different. Figure 7 shows three sample paths that all end in preemption. Consequently, the attrition rate increases smoothly to 1 at the time the preemption region is hit. This is, again, very different from the typical sample path of the attrition rate as discussed in Section 5.

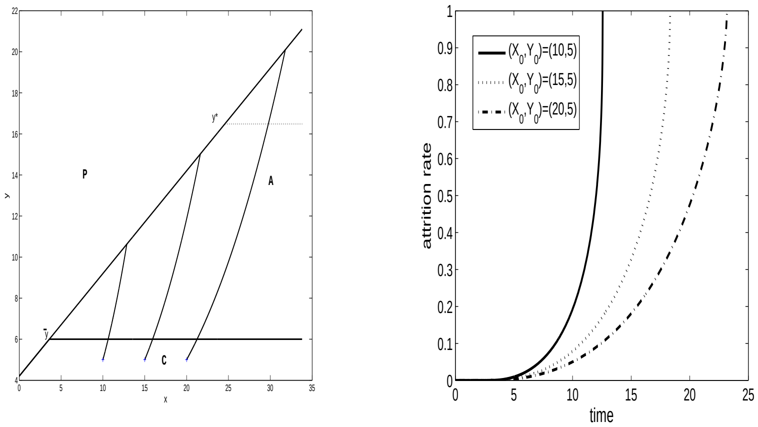
7 Concluding Remarks

In this paper we have built a continuous time stochastic model capturing a natural strategic investment problem that induces a sequence of local first and second-mover advantages depending on the random evolution of the environment. Due to that randomness, in equilibrium the firms exercise their options in qualitatively different stages than in models that focus on

$Z_t = X_t - Y_t + a$ we have $\lim_{t\to\tau_{\mathcal{P}}} Z_t = 0$ and $dZ_t = (\mu_X X_0 e^{\mu_X t} - \mu_Y Y_0 e^{\mu_Y t})\,dt$ and hence

$$\infty = \lim_{t\to\tau_{\mathcal{P}}} -\ln(Z_t) = -\ln(Z_0) - \int_0^{\tau_{\mathcal{P}}} \frac{\mu_X X_0 e^{\mu_X t} - \mu_Y Y_0 e^{\mu_Y t}}{X_t - Y_t + a}\,dt.$$

As the numerator in the integral is bounded on the assumed finite interval $[0, \tau_{\mathcal{P}}]$, the claim follows.



(a) Sample path (b) Attrition rate

Figure 7: Paths and attrition rates for deterministic case.

either preemption or a war of attrition only. Hence, we demonstrate a clear strategic feedback effect between different regimes of the model, so that it will generally be worthwhile to study more complex option exercise models.

It is quite clear that the principles underlying our equilibrium construction, in particular the identification of the central stopping problems, generalize considerably. However, by having chosen a rather explicit model close to the literature, we have been able to quantify the equilibrium incentives in terms of basic factors, as apparent from our Markovian mixed strategies. This quantification generates deeper economic insights than the common derivation of threshold-type equilibrium strategies in the literature. Moreover, having shown how to construct an equilibrium in mixed strategies for a timing game without any monotonicity assumptions is also a methodological contribution.

An economically important avenue for future research is to enrich the economic environment by introducing a continuation value for the follower. This could even be extended to a model where firms can switch as often as they wish. Such an extension, however, does not fall into the framework of timing games employed here, but will require a richer model of subgames and, particularly, histories (of multiple actions).

Our equilibria with Markovian hazard rates of stopping also have a novel effect that presents an interesting avenue of future research in the finance literature. Strategic behaviour of firms, as modeled in this paper, endogenously determines firm betas – here in the quite drastic way that firms completely "compete away" exposure to one of the risk factors, which generally would not be the case with pure strategies. To see why this is the case, recall that firm value is typically composed of the value of assets in place and the present value of growth options (PVGO). In our setting, the PVGO is determined entirely by the option to switch. Contrary to the existing literature on firm value and asset betas, our model has two risk factors, i.e., the processes X and Y. If one assumes, following Carlson et al. (2004) and Aguerrevere (2009), that both factors are traded in the market, then a firm's beta with respect to each factor is the elasticity of the firm's value, $V(x,y)$, with respect to the two factors. So, when $(X_\cdot, Y_\cdot) = (x,y)$ it holds that

$$\beta_X = \frac{\partial V(x,y)}{\partial x} \frac{x}{V(x,y)} \quad \text{and} \quad \beta_Y = \frac{\partial V(x,y)}{\partial y} \frac{y}{V(x,y)}.$$

In our subgame perfect equilibrium, the value of the firm for initial states $(x,y) \in \mathcal{A} \cup \mathcal{P}$ is $V(x,y) = \min(L_0, F_0)$, while for $(x,y) \in \mathcal{C}$ it holds that

$$V(x,y) = E\big[\min(L_{\tau_{\mathcal{C}^c}}, F_{\tau_{\mathcal{C}^c}})\big], \quad \text{where} \quad \tau_{\mathcal{C}^c} = \inf\{t \geq 0 \,|\, (X_t, Y_t) \notin \mathcal{C}\}.$$

This implies that $\beta_X, \beta_Y > 0$ in the continuation region $\mathcal{C}$, but that in $\mathcal{A}$, strategic considerations lead firms to manipulate factor exposures in such a way that $\beta_X = 0$. Also note that, *ex post*, for a game that starts in $\mathcal{C}$, there is always exactly one firm that switches, so that, *ex post*, for one firm it holds that $\beta_X > 0$ and $\beta_Y = 0$, whereas for the other firm it holds that $\beta_X = 0$ and $\beta_Y > 0$. However, *ex ante* it is not clear which firm will take which role. This endogeneity of firm betas and their time-varying nature are, to our knowledge, not explored as equilibrium phenomena in the asset pricing literature. Our results suggest that a (regime-specific) proxy for competition could be added to factor models when firms are exposed to several risk factors. This points to a potential new avenue of research in the literature on time-varying expected stock returns as summarized, for example, in Eugene Fama's Nobel lecture (Fama, 2014).

A Appendix

A.1 Proofs

Proof of Proposition 4.6. First consider $Y_0 = 0$, which is absorbing. By our assumptions $y_{\mathcal{P}} \geq 0$, so $(\mathbb{R}_+, 0) \subset \mathcal{P}^c$. For $Y \equiv 0$ it holds that $\tau_{\mathcal{P}} = \infty$ (so we face an unconstrained, deterministic problem) and thus $(x,0) \in \mathcal{C} \Leftrightarrow V_{\tilde{L}}(x,0) > L(0,0) \Leftrightarrow 0 < y^*$ by the solution of the unconstrained problem, or directly from $dL_t = e^{-rt}\bar{y}\,dt$ for $Y_0 = 0$ and $y^* > 0 \Leftrightarrow \bar{y} > 0$. In this case $\tau_{\mathcal{C}^c} = \infty$ is optimal, and $\tau_{\mathcal{C}^c} = 0$ if $y^* \leq 0$.

We now establish the boundary between the stopping and continuation regions for $\tilde{L}$ in the whole state space by strong path comparisons. Therefore, denote the solution to (2.1) for given initial condition $(X_0, Y_0) = (x,y) \in \mathbb{R}^2_+$ by $(X^x, Y^y) = (xX^1, yY^1)$. Further, write the continuation region as $\mathcal{C} = \{(x,y) \in \mathbb{R}^2_+ \,|\, V_{\tilde{L}}(x,y) > \tilde{L}(x,y)\} \subset \mathcal{P}^c$, where $\tilde{L}(x,y) := L(0,y)\mathbf{1}_{(x,y)\in\mathcal{P}^c} + F(0,x)\mathbf{1}_{(x,y)\in\mathcal{P}}$; cf. (2.2). This means that if we fix any $(x_0, y_0) \in \mathcal{C}$, then there exists a stopping time $\tau^* \in (0, \tau_{\mathcal{P}}]$ a.s. with $V_{\tilde{L}}(x_0, y_0) \geq E\big[\tilde{L}_{\tau^*}\big] = E\big[L(\tau^*, Y^{y_0}_{\tau^*})\big] > \tilde{L}(x_0, y_0) = L(0, y_0)$. Now fix an arbitrary $\varepsilon \in (0, y_0)$ implying that $(X^{x_0}, Y^{y_0-\varepsilon}) = (X^{x_0}, Y^{y_0} - \varepsilon Y^1)$ starts at $(x_0, y_0 - \varepsilon) \in \mathcal{P}^c$ and $\tau^* \leq \tau_{\mathcal{P}} \leq \inf\{t \geq 0 \,|\, (X^{x_0}_t, Y^{y_0-\varepsilon}_t) \in \mathcal{P}\}$.

Hence, $V_{\tilde{L}}(x_0, y_0-\varepsilon) \geq E\big[L(\tau^*, Y^{y_0-\varepsilon}_{\tau^*})\big] = E\big[L(\tau^*, Y^{y_0}_{\tau^*}) - e^{-r\tau^*}\varepsilon Y^1_{\tau^*}/(r-\mu_Y)\big] > L(0, y_0) - E\big[e^{-r\tau^*}\varepsilon Y^1_{\tau^*}/(r-\mu_Y)\big] \geq L(0, y_0) - \varepsilon/(r-\mu_Y) = L(0, y_0-\varepsilon) = \tilde{L}(x_0, y_0-\varepsilon)$. The last inequality is due to $(e^{-rt}Y^1)$ being a supermartingale by $r > \mu_Y$.

As ε was arbitrary we can define $b(x) := \sup\{y \geq 0 \,|\, V_{\tilde{L}}(x, y) > \tilde{L}(x, y)\}$ for any $x \in \mathbb{R}_+$ where that section of the continuation region is nonempty and conclude $V_{\tilde{L}}(x, y) > \tilde{L}(x, y)$ for all $y \in (0, b(x))$. Then we have $b(x) \leq y^*$ because immediate stopping is optimal in the unconstrained problem for $Y_0 \geq y^*$. Formally, $L(0, y) = \tilde{L}(x, y) \leq V_{\tilde{L}}(x, y) \leq U_L(0) = L(0, y)$ for any $(X_0, Y_0) = (x, y) \in \mathcal{P}^c$ with $y \geq y^*$.[25] The same argument shows that the continuation region is empty if $y^* \leq 0$. In this case we can set $b(x) := y^*$.

The section of the continuation region for arbitrary $x \geq 0$ is in fact *only* empty if $y^* \leq 0$, which completes the definition of b. Indeed, we have seen at the beginning of the proof that $(\mathbb{R}_+, 0) \subset \mathcal{C}$ if $y^* > 0$ (and therefore actually $(x, y) \in \mathcal{C}$ for all $y \in [0, b(x))$). We could also have applied the following important estimate for $x > 0$. If $y^* > 0$, i.e., if $\bar{y} > 0$, then $b(x) \geq \min(\bar{y}, y_{\mathcal{P}} + x(r-\mu_Y)/(r-\mu_X))$, which follows from the fact that L is a continuous semimartingale with finite variation part

$$\int_0^t -e^{-rs}(Y_s - \bar{y})\, ds,$$

as can be seen from applying Itō's Lemma. This drift is strictly positive on $\{Y < \bar{y}\}$, where stopping L is therefore suboptimal and so it is too for $\tilde{L}$ (up to $\tau_{\mathcal{P}}$).

For the monotonicity of b, pick any $(X_0, Y_0) = (x_0, y_0) \in \mathcal{C}$ and $\tau^* \leq \tau_{\mathcal{P}}$ as before and fix an arbitrary $\varepsilon > 0$. Then $(X^{x_0+\varepsilon}, Y^{y_0}) = (X^{x_0} + \varepsilon X^1, Y^{y_0})$ starts at $(x_0 + \varepsilon, y_0) \in \mathcal{P}^c$ and $\tau^* \leq \tau_{\mathcal{P}} \leq \inf\{t \geq 0 \,|\, (X^{x_0+\varepsilon}_t, Y^{y_0}_t) \in \mathcal{P}\}$. Now $V_{\tilde{L}}(x_0 + \varepsilon, y_0) \geq E\big[L(\tau^*, Y^{y_0}_{\tau^*})\big] > L(0, y_0) = \tilde{L}(x_0 + \varepsilon, y_0)$, whence $(x_0 + \varepsilon, y_0 \in \mathcal{C}$. Therefore, $b(x_0 + \varepsilon) \geq b(x_0)$.

The continuity of b in $x_0 = 0$ holds by the following estimates. Below we show that $b(x) \geq \min(y_{\mathcal{P}}, y^*)$, which together with monotonicity yields a right-hand limit and $b(0), b(0+) \geq \min(y_{\mathcal{P}}, y^*)$. On the other hand, by definition $b(x) \leq y_{\mathcal{P}} + x(r-\mu_Y)/(r-\mu_X)$ and $b(x) \leq y^*$ as shown above. Hence $b(0) = b(0+) = \min(y_{\mathcal{P}}, y^*)$. To verify the continuity of b in $x_0 > 0$, we show that if $(x_0, y_0) \in \mathcal{C}$, then also the line $\{(x, y) \in \mathbb{R}^2_+ \,|\, x \in (0, x_0], y = xy_0/x_0\} \subset \mathcal{C}$.[26] As the first step, we check that given $(X_0, Y_0) = (x_0, y_0) \in \mathcal{P}^c$, $\tau_{\mathcal{P}} \leq \inf\{t \geq 0 \,|\, (X^x_t, Y^y_t) \in \mathcal{P}\}$

[25] Here U_L is the Snell envelope of L , i.e., the value process of the unconstrained stopping problem. See also the corresponding discussion for $\tilde{L}$ following Proposition 4.6.

[26] A graphical illustration helps to convey the continuity argument for $b(x)$.

for any point (x, y) on the specified line. Indeed, for any (X^x, Y^y) we can write

$$L_t \le F_t \Leftrightarrow \frac{Y_t^y}{r-\mu_Y} - \frac{X_t^x}{r-\mu_X} \le \frac{c_B - c_A}{r} + I = \frac{y_{\mathcal{P}}}{r-\mu_Y}$$
$$\Leftrightarrow \frac{y}{r-\mu_Y} Y_t^1 - \frac{x}{r-\mu_X} X_t^1 \le \frac{y_{\mathcal{P}}}{r-\mu_Y}.$$

If $y = xy_0/x_0$ this becomes

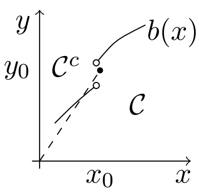
$$\frac{y}{r-\mu_Y} Y_t^1 - \frac{x}{r-\mu_X} X_t^1 = \frac{x}{x_0}\left(\frac{y_0}{r-\mu_Y} Y_t^1 - \frac{x_0}{r-\mu_X} X_t^1\right) \le \frac{y_{\mathcal{P}}}{r-\mu_Y}.$$

As $y_{\mathcal{P}}/(r-\mu_Y)$ is nonnegative, the condition for (x, y) is implied by the one for (x_0, y_0) if $x \in [0, x_0]$. Therefore, if we fix any point on the line $\{(x, y) \in \mathbb{R}_+^2 \,|\, x \in (0, x_0], y = xy_0/x_0\}$, we have $\tau^* \le \inf\{t \ge 0 \,|\, (X_t^x, Y_t^y) \in \mathcal{P}\}$, where τ^* as before satisfies $E\big[L(\tau^*, Y_{\tau^*}^{y_0})\big] > L(0, y_0)$ for the given $(x_0, y_0) \in \mathcal{C}$.

In particular, $(x, y) \in \mathcal{P}^c$. Now suppose that $(x, y) \in \mathcal{C}^c$. Then $L(0, y) = V_{\tilde{L}}(x, y) \ge E\big[L(\tau^*, Y_{\tau^*}^y)\big] = E\big[L(\tau^*, Y_{\tau^*}^{y_0}) - e^{-r\tau^*}(y_0 - y)/(r-\mu_Y)Y_{\tau^*}^1\big] > L(0, y_0) - E\big[e^{-r\tau^*}(y_0 - y)/(r-\mu_Y)Y_{\tau^*}^1\big] \ge L(0, y_0) - (y_0 - y)/(r-\mu_Y) = L(0, y)$, a contradiction (where we again used the fact that $(e^{-rt}Y^1)$ is a supermartingale). Thus, $(x, y) \in \mathcal{C}$.

Now we argue that $\partial\mathcal{C} \subset \mathcal{C}^c$, except possibly for the origin. Suppose $y^* > 0$, i.e., $\bar{y} > 0$ (otherwise $\mathcal{C} = \emptyset$). By $b(x) \ge \min(\bar{y}, y_{\mathcal{P}} + x(r-\mu_Y)/(r-\mu_X))$, the only point in $\partial\mathcal{C}$ with $y = b(x) = 0$ can be the origin (and that only if $y_{\mathcal{P}} = 0$). If $y = Y_0 > 0$ and $(X_0, Y_0) \in \partial\mathcal{C}$, then (X, Y) will enter the interior of $\mathcal{C}^c$ immediately with probability 1. Indeed we have shown that $b(x+h) \le b(x) + hb(x)/x$ for any $x, h > 0$, which implies with the monotonicity of b that if $Y_0 = b(X_0) > 0$ and $X_0 > 0$, then $\{Y_t > b(X_t)\} \supset \{Y_t > Y_0\} \cap \{Y_t > Y_0 X_t/X_0\} = \{(\mu_Y - \sigma_Y^2/2)t + \sigma_Y B_t^Y > 0\} \cap \{(\mu_Y - \sigma_Y^2/2)t + \sigma_Y B_t^Y - (\mu_X - \sigma_X^2/2)t - \sigma_X B_t^X > 0\}$. After normalization we see that the latter two sets are both of the form $\{B_t + \mu t > 0\}$ for some Brownian motion B with drift $\mu \in \mathbb{R}$ (we normalize by σ_Y in the first and $(\sigma_Y^2 - 2\rho\sigma_X\sigma_Y + \sigma_X^2)^{1/2}$ in the second case; recall $\sigma_Y B^Y$ and $\sigma_Y B^Y - \sigma_X B^X$ are not degenerate). Therefore, the hitting times of either set, which we denote by τ_1 and τ_2 are 0 a.s., and $P[\inf\{t \ge 0 \,|\, Y_t > b(X_t)\} = 0] \ge 1 - P[\tau_1 > 0] - P[\tau_2 > 0] = 1$, i.e., the correlation ρ between B^X and B^Y is not important. A simpler version of the argument applies to the case $X_0 = 0$ and $Y_0 = b(0) > 0$, when $\{Y_t > b(X_t)\} = \{Y_t > Y_0\}$.

It remains to prove that $b(x) \ge \min(y_{\mathcal{P}}, y^*)$. If $y^* \le y_{\mathcal{P}}$ then $\tau_{\mathcal{P}} \ge \tau^*(y) := \inf\{t \ge$



$0 \,|\, Y_t \geq y^*\}$ for $(X_0, Y_0) = (x, y) \in \mathcal{P}^c$ and thus $E[L_{\tau^*(y)}] = U_L(0) \geq V_{\tilde{L}}(x, y) \geq E[L_{\tau^*(y)}]$. Hence, $U_L(0) = V_{\tilde{L}}(x, y) > L(0, y) = \tilde{L}(x, y)$ for $y < y^*$, implying $b(x) \geq y^*$.

For $y^* > y_{\mathcal{P}}$ we will show that $\tilde{L}(x, y) = L(0, y) < E[L_{\tau(y_{\mathcal{P}})}] \leq V_{\tilde{L}}(x, y)$ for any (x, y) with $y < y_{\mathcal{P}}$, where $\tau(y_{\mathcal{P}}) := \inf\{t \geq 0 \,|\, Y_t \geq y_{\mathcal{P}}\} \leq \tau_{\mathcal{P}}$, which implies that $b(x) \geq y_{\mathcal{P}}$. On $\{Y_0 = y < y_{\mathcal{P}}\}$ we obtain by the definition of $y_{\mathcal{P}}$ that $E[L_{\tau(y_{\mathcal{P}})}] = -c_0/r + (y/y_{\mathcal{P}})^{\beta_1}(c_0 - c_A)/r$,[27] which is a continuous function of y that extends to $E[L_{\tau(y_{\mathcal{P}})}] = L(0, y_{\mathcal{P}})$ for $Y_0 = y = y_{\mathcal{P}}$ and to $E[L_{\tau(y_{\mathcal{P}})}] = -c_0/r$ for $Y_0 = y = 0$. Hence, $E[L_{\tau(y_{\mathcal{P}})}] - L(0, y)$ vanishes at $y = y_{\mathcal{P}}$, while at $y = 0$ it attains $I + (c_B - c_0)/r$, which is strictly positive if $y^* > 0$. Now $\partial_y E[L_{\tau(y_{\mathcal{P}})} - L(0, y)] < 0$ for all $y \in (0, y_{\mathcal{P}})$ iff $\beta_1(c_0 - c_A)(y/y_{\mathcal{P}})^{\beta_1 - 1} < c_B - c_A + rI$ for all $y \in (0, y_{\mathcal{P}})$, iff $\beta_1(c_0 - c_A) \leq c_B - c_A + rI$ (since $\beta_1 > 1$), i.e., iff $y^* \geq y_{\mathcal{P}}$. $\square$

Proof of Proposition 4.7. As in the proof of Proposition 4.6 we begin with the deterministic case $Y_0 = 0$, in which case $\tau_{\mathcal{P}} = \infty$ and $dL_t = e^{-rt}\bar{y}\, dt$. In this case the Snell envelope of $\tilde{L} = L$ is the smallest nonincreasing function dominating L. For $\bar{y} \geq 0 \Leftrightarrow y^* \geq 0$, it is just the constant $L_\infty = -c_0/r$ and thus $dD_{\tilde{L}} = 0$, while for $\bar{y} < 0 \Leftrightarrow y^* < 0$ it is L itself and thus $dD_{\tilde{L}} = -dL$. Both correspond to the claim, cf. also Remark 4.13.

For $Y_0 > 0 = X_0$, the solution is by Proposition 4.6 to stop the first time Y exceeds $b(0) = \min(y_{\mathcal{P}}, y^*)$. Then the Snell envelope is explicitly known for $t \leq \tau_{\mathcal{P}}$, as in footnote 27 with $\hat{y} = \min(y_{\mathcal{P}}, y^*)$, inserting Y_t and discounting the brackets by e^{-rt} (and $U_{\tilde{L}} = \tilde{L}$ for $Y \geq \hat{y}$). In this case $dD_{\tilde{L}}$ can be determined as the drift of $U_{\tilde{L}}$ from Itō's formula. One can also reduce the following argument to the 1-dimensional case.

For $X_0, Y_0 > 0$ first note that the Snell envelope $U_{\tilde{L}} = V_{\tilde{L}}(X, Y)$ is continuous because $V_{\tilde{L}}(\cdot)$ is a continuous function; see, e.g., Krylov (1980). The question when $D_{\tilde{L}}$ is exactly the decreasing component of the continuous semimartingale $\tilde{L}$ in the stopping region is answered by Jacka (1993): it is the case if the local time of the non-negative semimartingale $U_{\tilde{L}} - \tilde{L}$ at 0 is trivial, i.e., if $L^0(U_{\tilde{L}} - \tilde{L}) \equiv 0$ (a.s.).

We now show that $L^0(U_{\tilde{L}} - \tilde{L}) \equiv 0$ (a.s.) by applying the argument of Jacka's Theorem 6. Specifically, Itō's Lemma shows that $\tilde{L}$ is a continuous semimartingale with finite variation part $A_t := \int_0^t -\mathbf{1}_{s<\tau_{\mathcal{P}}} e^{-rs}(Y_s - c_B + c_0 - rI)\, ds$. Denote its decreasing part by A^-, which here satisfies

$$dA_t^- = \mathbf{1}_{Y>\bar{y}}\, e^{-rt}(Y_t - c_B + c_0 - rI)\, dt$$

for $t < \tau_{\mathcal{P}}$. By Theorem 3 of Jacka (1993), $dL_t^0(U_{\tilde{L}} - \tilde{L}) \leq \mathbf{1}_{U_{\tilde{L}}=\tilde{L}, t<\tau_{\mathcal{P}}} 2\, dA_t^-$, and as in his

[27] For any threshold $\hat{y} > 0$ and its hitting time $\tau(\hat{y}) := \inf\{t \geq 0 \,|\, Y_t \geq \hat{y}\}$ by the geometric Brownian motion Y we have by standard results

$$E[L_{\tau(\hat{y})}] = -\frac{c_0}{r} + \left(\frac{Y_0}{\hat{y}}\right)^{\beta_1} \left(\frac{\hat{y}}{r - \mu_Y} - \frac{c_B - c_0}{r} - I\right) \text{ if } Y_0 < \hat{y}.$$

Theorem 6, $dL^0(U_{\tilde{L}} - \tilde{L})$ is supported by $\{(X,Y) \in \partial\mathcal{C}\}$. For all $t \in \mathbb{R}_+$, it holds that

$$E\big[L^0_t(U_{\tilde{L}} - \tilde{L})\big] \leq E\bigg[2\int_0^t \mathbf{1}_{(X,Y)\in\partial\mathcal{C}, s<\tau_{\mathcal{P}}}\, dA^-_s\bigg] \leq E\bigg[2\int_0^t \mathbf{1}_{(X,Y)\in\partial\mathcal{C}}\, dA^-_s\bigg].$$

Note that dA^- has a Markovian density with respect to Lebesgue measure on $\mathbb{R}_+$, and that our underlying diffusion (X,Y) has a log-normal transition distribution, which thus has a density with respect to Lebesgue measure on $\mathbb{R}^2_+$. As $\partial\mathcal{C} = \{(x,y) \in \mathbb{R}^2_+ \,|\, y = b(x)\}$ has Lebesgue measure 0 in $\mathbb{R}^2_+$, we conclude, like in the proof of (Jacka, 1993, Theorem 6), that $L^0(U_{\tilde{L}} - \tilde{L}) \equiv 0$ (a.s.). □

Proof of Lemma 4.8. Suppose $b(\hat{x}) = y^* > y_{\mathcal{P}}$, hence $y^* > 0$. Then in particular for all $X_0 \geq \hat{x}$ and $Y_0 = y^*$, $U_{\tilde{L}}(0) = \tilde{L}_0 = L_0 = U_L(0)$ with U_L the (unconstrained) Snell envelope of L. For any stopping time τ,

$$U_{\tilde{L}}(0) - E[U_{\tilde{L}}(\tau)] = E[D_{\tilde{L}}(\tau)] = E\bigg[\int_0^{\tau\wedge\tau_{\mathcal{P}}} \mathbf{1}_{Y_t \geq b(X_t)} e^{-rt}(Y_t - \bar{y})\, dt\bigg]$$

and similarly for the unconstrained problem

$$U_L(0) - E[U_L(\tau)] = E[D_L(\tau)] = E\bigg[\int_0^{\tau} \mathbf{1}_{Y_t \geq y^*} e^{-rt}(Y_t - \bar{y})\, dt\bigg].$$

Let now $X_0 > \hat{x}$, $Y_0 = y^*$ and $\tau = \inf\{t \geq 0 \,|\, X_t \leq \hat{x}\} \wedge \tau_{\mathcal{P}} > 0$. Then $U_{\tilde{L}}(\tau) < U_L(\tau)$ on $\{Y_\tau < y^*\}$, which has positive probability by our non-degeneracy assumption, so $E[U_{\tilde{L}}(\tau)] < E[U_L(\tau)]$.[28] This contradicts all other terms in the previous two displays being equal, respectively. □

Proof of Proposition 4.10. Fix $i \in \{1,2\}$ and let j denote the other firm. The strategies satisfy the conditions of Definition 4.1 (presupposing $G^\vartheta_i(t) = \alpha^\vartheta_i(t) = 0$ on $[0,\vartheta)$). Note in particular that by our non-degeneracy assumption, $\tau_{\mathcal{P}} = \inf\{t \geq \vartheta \,|\, L_t \geq F_t\}$ unless $(X_\vartheta, Y_\vartheta) = (X_0, Y_0) = (0,0)$ and $y_{\mathcal{P}} = 0$, and thus except for that excluded case the denominator in (4.4) will not vanish before $\tau_{\mathcal{P}}$. Now $\bar{y} > Y_t \geq b(X_t)$ implies $t \geq \tau_{\mathcal{P}}$ by $b(x) \geq \min(\bar{y}, y_{\mathcal{P}} + x(r-\mu_Y)/(r-\mu_X))$ and hence the proposed stopping rate is also nonnegative.[29]

The strategies are mutual best replies at $\tau_{\mathcal{P}}$ by Proposition 4.5, i.e., given any value of $G^\vartheta_i(\tau_{\mathcal{P}}-)$, $G^\vartheta_i(t) = 1$ on $[\tau_{\mathcal{P}}, \infty]$ and $\alpha^\vartheta_i(t)$ as proposed are optimal against $(G^\vartheta_j, \alpha^\vartheta_j)$, and the related payoff to firm i is $E\big[(1 - G^\vartheta_i(\tau_{\mathcal{P}}-))(1 - G^\vartheta_j(\tau_{\mathcal{P}}-))F_{\tau_{\mathcal{P}}} \,\big|\, \mathscr{F}_\vartheta\big]$. It remains to show optimality of G^ϑ_i on $[\vartheta, \tau_{\mathcal{P}})$. Since G^ϑ_j is continuous on $[\vartheta, \tau_{\mathcal{P}})$ it follows from Riedel and Steg

[28] For $Y_0 > 0$, the optimal stopping time for L, i.e., the first time Y exceeds y^*, is unique (up to null sets), but not admissible in the constrained problem for $Y_0, y_{\mathcal{P}} < y^*$, so analogously $U_{\tilde{L}}(\tau) < U_L(\tau)$ on $\{Y_\tau < y^*\}$.

[29] Given the proposed rate, $G^\vartheta_i(t) = 1 - \exp\{-\int_\vartheta^t (1 - G^\vartheta_i(s))^{-1}\, dG^\vartheta_i(s)\}$ on $[\vartheta, \tau_{\mathcal{P}})$.

(2017) that it is not necessary to consider the possibility that $\alpha_i^\vartheta(t) > 0$.

Given the proposed α_i^ϑ, note that time-consistency requires that $G_i^\vartheta(t) = 1$ on $[\tau_\mathcal{P}, \infty]$. Because of the continuity of G_j^ϑ up to $\tau_\mathcal{P}$, we can write the payoff from any G_i^ϑ on $[\vartheta, \tau_\mathcal{P})$ as

$$V_i^\vartheta(G_i^\vartheta, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta) = E\Big[\int_{[0,\tau_\mathcal{P}]} \Big(\int_{[0,s)} F_t \, dG_j^\vartheta(t) + (1 - G_j^\vartheta(s))L_s\Big) \, dG_i^\vartheta(s) \\ + (1 - G_i^\vartheta(\tau_\mathcal{P}-))(1 - G_j^\vartheta(\tau_\mathcal{P}-))F_{\tau_\mathcal{P}} \Big| \mathscr{F}_\vartheta\Big].$$

Furthermore, by noting that $(1 - G_i^\vartheta(\tau_\mathcal{P}-))(1 - G_j^\vartheta(\tau_\mathcal{P}-))F_{\tau_\mathcal{P}} = \Delta G_i^\vartheta(\tau_\mathcal{P})\Delta G_j^\vartheta(\tau_\mathcal{P})F_{\tau_\mathcal{P}}$ and $\Delta G_j^\vartheta(t) = 0$ on $[0, \tau_\mathcal{P})$, we obtain that

$$V_i^\vartheta(G_i^\vartheta, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta) = E\Big[\int_{[0,\tau_\mathcal{P}]} S_i(s) \, dG_i^\vartheta(s) \Big| \mathscr{F}_\vartheta\Big]$$

where $S_i(s) = \int_{[0,s]} F_t \, dG_j^\vartheta(t) + \big(1 - G_j^\vartheta(s)\big)L_s$. Thus G_i^ϑ is optimal iff it increases when it is optimal to stop S_i. We now show that this is the case anywhere in the attrition region $\mathcal{A}$, or at $\tau_\mathcal{P}$.

The process S_i is continuous on $\{\vartheta < \tau_\mathcal{P}\}$ because there $L_{\tau_\mathcal{P}} = F_{\tau_\mathcal{P}}$ implies $\Delta S_i(\tau_\mathcal{P}) = \Delta G_j^\vartheta(\tau_\mathcal{P})\big(F_{\tau_\mathcal{P}} - L_{\tau_\mathcal{P}}\big) = 0$. Since L is a continuous semimartingale and G_j^ϑ is monotone and continuous on $[0, \tau_\mathcal{P})$, S_i satisfies

$$dS_i(s) = (F_s - L_s) \, dG_j^\vartheta(s) + \big(1 - G_j^\vartheta(s)\big) \, dL_s.$$

Using $(F_s - L_s)dG_j^\vartheta(s) = \big(1 - G_j^\vartheta(s)\big)\mathbf{1}_{Y_s \geq b(X_s)}e^{-rs}(Y_s - \bar{y}) \, ds$ on $[\vartheta, \tau_\mathcal{P})$ yields

$$dS_i(s) = \big(1 - G_j^\vartheta(s)\big)\big(dL_s + \mathbf{1}_{Y_s \geq b(X_s)}e^{-rs}(Y_s - \bar{y}) \, ds\big).$$

Recalling $\tilde{L}$ from Subsection 4.3 (with $\tau_\mathcal{P}$ adjusted for ϑ), and its Snell envelope $U_{\tilde{L}}$ with compensator $D_{\tilde{L}}$, identified in Proposition 4.7, we have in fact

$$dS_i(s) = \big(1 - G_j^\vartheta(s)\big)\big(d\tilde{L}_s + dD_{\tilde{L}}(s)\big)$$

on $[\vartheta, \tau_\mathcal{P}]$ where $\vartheta < \tau_\mathcal{P}$. We first ignore $(1 - G_j^\vartheta)$ and argue that it is optimal to stop $\tilde{L} + D_{\tilde{L}}$ anywhere in the attrition region $\mathcal{A}$ or at $\tau_\mathcal{P}$. Indeed, as $U_{\tilde{L}} = M_{\tilde{L}} - D_{\tilde{L}}$ is the smallest supermartingale dominating $\tilde{L}$, $U_{\tilde{L}} + D_{\tilde{L}} = M_{\tilde{L}}$ must be the smallest supermartingale dominating $\tilde{L} + D_{\tilde{L}}$, i.e., the Snell envelope of $\tilde{L} + D_{\tilde{L}}$. As $M_{\tilde{L}}$ is a martingale, any stopping time τ with $M_{\tilde{L}}(\tau) = \tilde{L}_\tau + D_{\tilde{L}}(\tau) \Leftrightarrow U_{\tilde{L}}(\tau) = \tilde{L}_\tau$ a.s., is optimal for $\tilde{L} + D_{\tilde{L}}$, which is satisfied where the proposed G_i^ϑ increases (i.e., in $\mathcal{A}$ or at $\tau_\mathcal{P}$).[30] Instead of stopping $\tilde{L} + D_{\tilde{L}}$, we

[30] It is possible, by Proposition 4.6, that $\mathcal{A} \cap \mathcal{C}$ is only reached at the origin, i.e., by $(X_\tau, Y_\tau) = (0, 0) \Leftrightarrow (X_0, Y_0) = (0, 0)$ and if $y_\mathcal{P} = 0$ (cf. footnote 17), which we have excluded. $U_{\tilde{L}}(\tau_\mathcal{P}) = \tilde{L}_{\tau_\mathcal{P}}$ holds by definition.

actually have to consider, for any $\tau \in [\vartheta, \tau_{\mathcal{P}}]$,

$$\begin{aligned} S_i(\tau) - S_i(\vartheta) &= \int_{[\vartheta,\tau]} (1 - G_j^\vartheta(s))\big(d\tilde{L}_s + dD_{\tilde{L}}(s)\big) \\ &= (1 - G_j^\vartheta(\tau))\big(\tilde{L}_\tau + D_{\tilde{L}}(\tau)\big) - \tilde{L}_\vartheta - D_{\tilde{L}}(\vartheta) \\ &+ \int_{[\vartheta,\tau]} \big(\tilde{L}_s + D_{\tilde{L}}(s)\big)\, dG_j^\vartheta(s). \end{aligned}$$

The presence of $(1 - G_j^\vartheta)$ thus acts like a constraint compared to stopping $\tilde{L} + D_{\tilde{L}}$. However, G_j^ϑ places its entire mass favorably, as it also only increases when it is optimal to stop $\tilde{L} + D_{\tilde{L}}$. Therefore the payoff from optimally stopping S_i is indeed that from $\tilde{L} + D_{\tilde{L}}$, i.e., $M_{\tilde{L}}(\vartheta) = U_{\tilde{L}}(\vartheta)$ (with $D_{\tilde{L}}(\vartheta) = 0$), and the same stopping times are optimal. The proof is now complete. □

Proof of Proposition 4.11. We need to show that

$$\int_0^{\tau_{\mathcal{P}}} \frac{dG_i^\vartheta(t)}{1 - G_i^\vartheta(t)} = \int_0^{\tau_{\mathcal{P}}} \frac{\mathbf{1}_{Y_t \geq b(X_t)}(Y_t - \bar{y})\, dt}{X_t/(r - \mu_X) - (Y_t - y_{\mathcal{P}})/(r - \mu_Y)} < \infty \tag{A.1}$$

on $\{\tau_{\mathcal{P}} < \infty\}$ a.s. Note that $\tau_{\mathcal{P}} = \inf\{t \geq \vartheta \,|\, L_t \geq F_t\}$ if $(X_0, Y_0) \neq (0,0)$ or $y_{\mathcal{P}} > 0$ (see the proof of Proposition 4.10). As Y is continuous, it is bounded on $[0, \tau_{\mathcal{P}}]$ where this interval is finite. Hence we may just use dt as the numerator in (A.1). We will also ignore $(r - \mu_X)$ and $(r - \mu_Y)$, which may be encoded in the initial position (X_0, Y_0). By the strong Markov property we set $\vartheta = 0$. Hence, we will prove that

$$\int_0^{\tau_0} \frac{dt}{X_t - Y_t + a} < \infty \tag{A.2}$$

on $\{\tau_0 < \infty\}$ a.s. for nonnegative geometric Brownian motions X and Y (under our non-degeneracy assumption $\sigma_X^2, \sigma_Y^2 > 0$, $|\rho| < 1$), some fixed level $a > 0$ and the stopping time $\tau_\varepsilon := \inf\{t \geq 0 \,|\, X_t - Y_t + a \leq \varepsilon\}$ with $\varepsilon = 0$. We will treat the special case $y_{\mathcal{P}} = a = 0$ at the very end of the proof. Define the process $Z := X - Y + a$ to simplify notation.

As a first step and tool, we derive the weaker result $E[\int_0^{\tau_0 \wedge T} \ln(Z_t)\, dt] \in \mathbb{R}$ for any time $T > 0$ (which implies also $\int_0^{\tau_0} |\ln(Z_t)|\, dt < \infty$ on $\{\tau_0 < \infty\}$ a.s. by the arguments towards the end of the proof).

Define the function $f\colon \mathbb{R}_+ \to \mathbb{R}$ by

$$f(x) = x\ln(x) - x \in [-1, x^2]$$

and the function $F\colon \mathbb{R}_+ \to \mathbb{R}$ by $F(x) = \int_0^x f(y)\, dy \in [-x, x^3]$, such that $F''(x) = \ln(x)$ for all $x > 0$. For localization purposes fix an $\varepsilon > 0$ and a time $T > 0$. By Itō's formula we have

$$\begin{aligned}
F(Z_{\tau_\varepsilon\wedge T}) &= F(Z_0) + \int_0^{\tau_\varepsilon\wedge T} f(Z_t)\,dZ_t + \frac{1}{2}\int_0^{\tau_\varepsilon\wedge T} \ln(Z_t)\,d[Z]_t \\
&= F(Z_0) + \int_0^{\tau_\varepsilon\wedge T} f(Z_t)(\mu_X X_t - \mu_Y Y_t)\,dt \\
&\quad + \int_0^{\tau_\varepsilon\wedge T} f(Z_t)\Big(\sigma_X X_t\,dB_t^X - \sigma_Y Y_t\,dB_t^Y\Big) \\
&\quad + \frac{1}{2}\int_0^{\tau_\varepsilon\wedge T} \ln(Z_t)\big(\underbrace{\sigma_X^2 X_t^2 + \sigma_Y^2 Y_t^2 - 2\rho\sigma_X\sigma_Y X_t Y_t}_{=:\sigma^2(X_t,Y_t)}\big)\,dt.
\end{aligned}$$

We want to establish $\lim_{\varepsilon\searrow 0} E[F(Z_{\tau_\varepsilon\wedge T})]$, first in terms of the integrals, for which we need some estimates.

In order to eliminate the second, stochastic integral by taking expectations, it is sufficient to verify that $\mathbf{1}_{t<\tau_0} f(Z_t)X_t$ and $\mathbf{1}_{t<\tau_0} f(Z_t)Y_t$ are square $P\otimes dt$-integrable on $\Omega\times[0,T]$. We have $|f(Z_t)| \le 1 + Z_t^2$. For $t \le \tau_0$ further $0 \le Z_t \le X_t + a$ by $Y_t \ge 0$ and hence $Z_t^2 \le (X_t + a)^2 \le 2X_t^2 + 2a^2$. The sought square-integrability with X follows now from the $P\otimes dt$-integrability of X_t^n on $\Omega\times[0,T]$ for any $n\in\mathbb{N}$ and analogously that for Y thanks to $0 \le Y_t \le X_t + a$ for $t \le \tau_0$.

The same estimates guarantee that the expectation of the first integral converges to the finite expectation at τ_0 as $\varepsilon \searrow 0$. For the third integral we have $\ln(Z_t) \le Z_t \le X_t + a$. The second term $\sigma^2(X_t,Y_t)$ is bounded from below by $(\sigma_X X_t - \sigma_Y Y_t)^2 \ge 0$ and from above by $(\sigma_X X_t + \sigma_Y Y_t)^2 \le ((\sigma_X + \sigma_Y)X_t + \sigma_Y a)^2 \le 2(\sigma_X+\sigma_Y)^2 X_t^2 + 2\sigma_Y^2 a^2$ for $t \le \tau_0$ (supposing $\sigma_X, \sigma_Y > 0$ wlog.). Hence the positive part of the integrand is bounded by a $P\otimes dt$-integrable process on $\Omega\times[0,T]$, while the negative part converges monotonically, and we may take the limit of the expectation of the whole integral.

That the latter is finite follows from analysing the limit of the LHS, $\lim_{\varepsilon\searrow 0} E[F(Z_{\tau_\varepsilon\wedge T})]$, directly. We have $F(Z_{\tau_\varepsilon\wedge T}) = \mathbf{1}_{\tau_\varepsilon\le T}F(\varepsilon) + \mathbf{1}_{T<\tau_\varepsilon}F(Z_T)$, and it is continuous in $\varepsilon \searrow 0$. For $T < \tau_0$ again $0 \le Z_T \le X_T + a$ and thus $|F(Z_T)| \le |Z_T| + |Z_T|^3 \le (X_T + a) + (X_T + a)^3$. As also $|F(\varepsilon)| \le 1$ for all $\varepsilon \le 1$, $|F(Z_{\tau_\varepsilon\wedge T})|$ is bounded by an integrable random variable as $\varepsilon \searrow 0$. Consequently, $\lim_{\varepsilon\searrow 0} E[F(Z_{\tau_\varepsilon\wedge T})] = E[F(Z_{\tau_0\wedge T})] \in \mathbb{R}$ and also $E[\int_0^{\tau_0\wedge T} \ln(Z_t)\sigma^2(X_t,Y_t)\,dt] \in \mathbb{R}$ on the RHS. In the integral we may ignore the term $\sigma^2(X_t,Y_t)$, which completes the first step. Indeed, with $|\rho| < 1$ we can only have $\sigma^2(X_t,Y_t) = 0$ if $X_t = Y_t = 0$, i.e., if $Z_t = a$. Therefore $\inf\{\sigma^2(X_t,Y_t)\,|\,Z_t \le \varepsilon\} > 0$ for any fixed $\varepsilon \in (0,a)$, so $\sigma^2(X_t,Y_t)$ does not "kill" the downside of $\ln(Z_t)$ and

$$E\Big[\int_0^{\tau_0\wedge T} \ln(Z_t)\,dt\Big] \in \mathbb{R}.$$

In the following we further need $E[\int_0^{\tau_0\wedge T} \ln(Z_t)X_t\,dt] \in \mathbb{R}$, which obtains as follows. With $|\rho| < 1$, $\inf\{\sigma^2(X_t,Y_t)\,|\,Z_t \le \varepsilon\}$ is attained only with the constraint binding, $Y_t = X_t + a - \varepsilon$.

Thus, for $Z_t \leq \varepsilon$, $\sigma^2(X_t, Y_t) \geq \sigma^2(X_t, X_t + a - \varepsilon)$. The latter is a quadratic function of X_t, with X_t^2 having coefficient $\sigma^2(1,1) > 0$ given $|\rho| < 1$. The quadratic function hence exceeds X for all X sufficiently large, i.e., we can pick $K > 0$ such that $\sigma^2(X_t, Y_t) \geq X_t$ on $\{Z_t \leq \varepsilon\} \cap \{X_t \geq K\}$. X_t thus does not "blow up" the downside of $\ln(Z_t)$ more than $\sigma^2(X_t, Y_t)$.

We are now ready to analyse

$$
\begin{aligned}
f(Z_{\tau_\varepsilon \wedge T}) = f(Z_0) &+ \int_0^{\tau_\varepsilon \wedge T} \ln(Z_t)\, dZ_t + \frac{1}{2}\int_0^{\tau_\varepsilon \wedge T} \frac{1}{Z_t}\, d[Z]_t \\
= f(Z_0) &+ \int_0^{\tau_\varepsilon \wedge T} \ln(Z_t)\big(\mu_X X_t - \mu_Y Y_t\big)\, dt \\
&+ \int_0^{\tau_\varepsilon \wedge T} \ln(Z_t)\Big(\sigma_X X_t\, dB_t^X - \sigma_Y Y_t\, dB_t^Y\Big) \\
&+ \frac{1}{2}\int_0^{\tau_\varepsilon \wedge T} \frac{1}{Z_t}\sigma^2(X_t, Y_t)\, dt
\end{aligned}
$$

when taking the limit $\varepsilon \searrow 0$ under expectations as before. By our final observations of step one the first integral converges (note again $Y_t \leq X_t + a$ for $t \leq \tau_0$). In the second, stochastic integral we now have $|\ln(Z_t)| \leq |\ln(\varepsilon)| + |Z_t|$ for $t \leq \tau_\varepsilon$, an even smaller bound than above, making the expectation vanish. In the third integral, $\sigma^2(X_t, Y_t)/Z_t \geq 0$ for $t \leq \tau_0$, so monotone convergence holds.

On the LHS, $|f(Z_{\tau_\varepsilon \wedge T})|$ is bounded by an integrable random variable for all $\varepsilon \leq 1$ analogously to the first step, implying $\lim_{\varepsilon \searrow 0} E[f(Z_{\tau_\varepsilon \wedge T})] = E[f(Z_{\tau_0 \wedge T})] \in \mathbb{R}$ and thus $E[\int_0^{\tau_0 \wedge T} \sigma^2(X_t, Y_t)/Z_t\, dt] < \infty$. By the same arguments brought forward at the end of the first step we can again ignore $\sigma^2(X_t, Y_t)$.

With $E[\int_0^{\tau_0 \wedge T} 1/Z_t\, dt] < \infty$, $P[\{\tau_0 \leq T\} \cap \{\int_0^{\tau_0} 1/Z_t\, dt = \infty\}] = 0$. As T was arbitrary, we may take the union over all integer T to conclude $P[\{\tau_0 < \infty\} \cap \{\int_0^{\tau_0} 1/Z_t\, dt = \infty\}] = 0$. This completes the proof of (A.2) for the case $a > 0$ (and that of (A.1) for $y_{\mathcal{P}} > 0$).

Some modification is due for the case $y_{\mathcal{P}} = a = 0$ and if $(X_0, Y_0) \neq (0,0)$ (otherwise $\tau_0 = 0$ and (A.2) is trivial). Then $\sigma^2(X_t, Y_t)$ is not bounded away from 0 for Z_t small, it may be an important factor in the integrability of $\ln(Z_t)\sigma^2(X_t, Y_t)$ when (X_t, Y_t) is close to the origin. In order to infer the required well-behaved limit of $E[\int_0^{\tau_\varepsilon \wedge T} \ln(Z_t)\big(\mu_X X_t - \mu_Y Y_t\big)\, dt]$ and to finally remove $\sigma^2(X_t, Y_t)$ from $\int_0^{\tau_0} \sigma^2(X_t, Y_t)/Z_t\, dt$, one can employ another localization procedure: Fix a small $\delta > 0$ and use the minimum of $\sigma_\delta := \inf\{t \geq 0 \,|\, X_t + Y_t < \delta\}$ and $\tau_\varepsilon \wedge T$ everywhere above. Then $\sigma^2(X_t, Y_t)$ is bounded away from 0 on $[0, \sigma_\delta \wedge \tau_0]$ for $|\rho| < 1$ and again bounded below by a quadratic function with X_t^2 having coefficient $\sigma^2(1,1) > 0$. The result now obtains as above for all paths with $X_t + Y_t \geq \delta$ on $[0, \tau_0]$. As $\delta > 0$ is indeed arbitrary and any path with $(X_0, Y_0) \neq (0,0)$ is bounded away from the origin on $[0, \tau_0]$ finite by continuity, the claim follows. $\square$

A.2 Outcome Probabilities

The following definition is a simplification of that in Riedel and Steg (2017), resulting from right-continuity of any $\alpha_i^\vartheta(\cdot)$ also where it takes the value 0.

Define the functions μ_L and μ_M from $[0,1]^2 \setminus (0,0)$ to $[0,1]$ by

$$\mu_L(x,y) := \frac{x(1-y)}{x+y-xy} \qquad \text{and} \qquad \mu_M(x,y) := \frac{xy}{x+y-xy}.$$

$\mu_L(a_i, a_j)$ is the probability that firm i stops first in an infinitely repeated stopping game where i plays constant stage stopping probabilities a_i and firm j plays constant stage probabilities a_j. $\mu_M(a_i, a_j)$ is the probability of simultaneous stopping and $1 - \mu_L(a_i, a_j) - \mu_M(a_i, a_j) = \mu_L(a_j, a_i)$ that of firm j stopping first.

Definition A.1. Given $\vartheta \in \mathscr{T}$ and a pair of extended mixed strategies $(G_1^\vartheta, \alpha_1^\vartheta)$ and $(G_2^\vartheta, \alpha_2^\vartheta)$, the *outcome probabilities* $\lambda_{L,1}^\vartheta$, $\lambda_{L,2}^\vartheta$ and λ_M^ϑ at $\hat{\tau}^\vartheta := \inf\{t \geq \vartheta \,|\, \alpha_1^\vartheta(t) + \alpha_2^\vartheta(t) > 0\}$ are defined as follows. Let $i, j \in \{1,2\}$, $i \neq j$.
If $\hat{\tau}^\vartheta < \hat{\tau}_j^\vartheta := \inf\{t \geq \vartheta \,|\, \alpha_j^\vartheta(t) > 0\}$, then

$$\begin{aligned}
\lambda_{L,i}^\vartheta &:= \big(1 - G_i^\vartheta(\hat{\tau}^\vartheta-)\big)\big(1 - G_j^\vartheta(\hat{\tau}^\vartheta)\big), \\
\lambda_M^\vartheta &:= \big(1 - G_i^\vartheta(\hat{\tau}^\vartheta-)\big)\alpha_i^\vartheta(\hat{\tau}^\vartheta)\Delta G_j^\vartheta(\hat{\tau}^\vartheta).
\end{aligned}$$

If $\hat{\tau}^\vartheta < \hat{\tau}_i^\vartheta := \inf\{t \geq \vartheta \,|\, \alpha_i^\vartheta(t) > 0\}$, then

$$\begin{aligned}
\lambda_{L,i}^\vartheta &:= \big(1 - G_j^\vartheta(\hat{\tau}^\vartheta-)\big)\big(1 - \alpha_j(\hat{\tau}^\vartheta)\big)\Delta G_i^\vartheta(\hat{\tau}^\vartheta), \\
\lambda_M^\vartheta &:= \big(1 - G_j^\vartheta(\hat{\tau}^\vartheta-)\big)\alpha_j^\vartheta(\hat{\tau}^\vartheta)\Delta G_i^\vartheta(\hat{\tau}^\vartheta).
\end{aligned}$$

If $\hat{\tau}^\vartheta = \hat{\tau}_1^\vartheta = \hat{\tau}_2^\vartheta$ and $\alpha_1^\vartheta(\hat{\tau}^\vartheta) + \alpha_2^\vartheta(\hat{\tau}^\vartheta) > 0$, then

$$\begin{aligned}
\lambda_{L,i}^\vartheta &:= \big(1 - G_i^\vartheta(\hat{\tau}^\vartheta-)\big)\big(1 - G_j^\vartheta(\hat{\tau}^\vartheta-)\big)\mu_L\big(\alpha_i^\vartheta(\hat{\tau}^\vartheta), \alpha_j^\vartheta(\hat{\tau}^\vartheta)\big), \\
\lambda_M^\vartheta &:= \big(1 - G_i^\vartheta(\hat{\tau}^\vartheta-)\big)\big(1 - G_j^\vartheta(\hat{\tau}^\vartheta-)\big)\mu_M\big(\alpha_1^\vartheta(\hat{\tau}^\vartheta), \alpha_2^\vartheta(\hat{\tau}^\vartheta)\big).
\end{aligned}$$

If $\hat\tau^\vartheta = \hat\tau^\vartheta_1 = \hat\tau^\vartheta_2$ and $\alpha^\vartheta_1(\hat\tau^\vartheta) + \alpha^\vartheta_2(\hat\tau^\vartheta) = 0$, then

$$\begin{aligned}\lambda^\vartheta_{L,i} := \bigl(1 - G^\vartheta_i(\hat\tau^\vartheta-)\bigr)\bigl(1 - G^\vartheta_j(\hat\tau^\vartheta-)\bigr)\,\frac{1}{2}\Biggl\{ &\liminf_{\substack{t\searrow\hat\tau^\vartheta\\ \alpha^\vartheta_i(t)+\alpha^\vartheta_j(t)>0}} \mu_L\bigl(\alpha^\vartheta_i(t), \alpha^\vartheta_j(t)\bigr)\\ &+ \limsup_{\substack{t\searrow\hat\tau^\vartheta\\ \alpha^\vartheta_i(t)+\alpha^\vartheta_j(t)>0}} \mu_L\bigl(\alpha^\vartheta_i(t), \alpha^\vartheta_j(t)\bigr)\Biggr\},\\ \lambda^\vartheta_M := 0.&\end{aligned}$$

Remark A.2.

(i) λ^ϑ_M is the probability of simultaneous stopping at $\hat\tau^\vartheta$, while $\lambda^\vartheta_{L,i}$ is the probability of firm i becoming the leader, i.e., that of firm j becoming follower. It holds that $\lambda^\vartheta_M + \lambda^\vartheta_{L,i} + \lambda^\vartheta_{L,j} = \bigl(1 - G^\vartheta_i(\hat\tau^\vartheta-)\bigr)\bigl(1 - G^\vartheta_j(\hat\tau^\vartheta-)\bigr)$. Dividing by $\bigl(1 - G^\vartheta_i(\hat\tau^\vartheta-)\bigr)\bigl(1 - G^\vartheta_j(\hat\tau^\vartheta-)\bigr)$ where feasible yields the corresponding conditional probabilities.

(ii) If any $\alpha^\vartheta_\cdot = 1$, then no limit argument is needed. Otherwise both $\alpha^\vartheta_\cdot$ are right-continuous and the corresponding limit of μ_M exists. μ_L, however, has no continuous extension at the origin, whence we use the symmetric combination of lim inf and lim sup, ensuring consistency whenever the limit does exist. If the limit in a potential equilibrium does not exist, both firms will be indifferent about the roles; see Lemma A.5 in Riedel and Steg (2017).

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