

Probabilistic representation for solutions to nonlinear Fokker–Planck equations

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Abstract

One obtains a probabilistic representation for the entropic generalized solutions to a nonlinear Fokker–Planck equation in \mathbb{R}^d with multivalued nonlinear diffusion term as density probabilities of solutions to a nonlinear stochastic differential equation. The case of a nonlinear Fokker–Planck equation with linear space dependent drift is also studied.

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1 Introduction

Consider here the nonlinear Fokker–Planck equation (NFPE)

$$\begin{aligned} u_t(t, x) + \operatorname{div}(b(u)u) - \Delta\beta(u) &= 0 \quad \text{in } (0, \infty) \times \mathbb{R}^d, \\ u(0, x) &= u_0(x), \quad x \in \mathbb{R}^d, \quad d \geq 1, \end{aligned} \tag{1.1}$$

under the following assumptions

- (i) $\beta : \mathbb{R} \rightarrow 2^{\mathbb{R}}$ is a maximal monotone (multivalued) function $\beta(0) = 0$.
- (ii) $b \in C_b(\mathbb{R}; \mathbb{R}^d)$.

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Assumption (i) means that $(\eta_1 - \eta_2)(u_1 - u_2) \geq 0$ for all $(u_i, \eta_i) \in \mathbb{R} \times \mathbb{R}$, such that $\eta_i \in \beta(u_i)$, $i = 1, 2$, and the range of the mapping $r \rightarrow r + \beta(r)$ is all of \mathbb{R} . In particular, this holds if β is continuous and monotonically nondecreasing. If β is a discontinuous, single-valued, monotonically increasing function on \mathbb{R} , by filling the jumps r_j , that is, redefining β as $\beta(r_j) = \beta(r_j + 0) - \beta(r_j - 0)$, one gets a maximal monotone $\beta : \mathbb{R} \rightarrow 2^{\mathbb{R}}$.

In the mean field theory and statistical mechanics, equation (1.1) describes the particle transport dynamics in disordered media and $u(t)$ is the probability density.

In general, a Fokker-Planck equation of the form (1.1) is associated with a certain entropic functional. For instance, in $1-D$, equation (1.1) is derived from the entropic functional

$$S[u] = \int_{\mathbb{R}} G(u(x)) dx,$$

where $G \in C^2(0, \infty)$, $G'(0) = \infty$, $G''(u) < 0$, $\forall u > 0$. Then the NFPE associated with processes with entropy S is (see, e.g., [11], [12])

$$P_t + (b(P)P - \gamma(G(P) - PG'(P)))_x = 0,$$

where γ is some positive constant. This is an equation of the form (1.1), where $\beta(u) \equiv G(u) - uG'(u)$. A notorious example is $b = 1$ and $\beta(u) = \gamma \ln(1 + u)$ (equation of classical bosons).

Another important example is $\beta(t) \equiv rH(r - \rho_c)$, where H is the Heaviside function and $r_\varepsilon > 0$. In this case, equation (1.1) describes the self-organized criticality with the entropic functional (see [4])

$$g(r) = \begin{cases} \frac{r}{\rho_c} \ln \left(\frac{r}{\rho_c} \right) & \text{for } r > \rho_c, \\ 0 & \text{for } r < \rho_c. \end{cases}$$

In general, NFPE (1.1) has not a classical solution, but under assumptions (i)–(ii) it has an *entropic generalized solutions* in the sense of S. Kruřkov (see [2]). (See Section 3.)

The main result of this paper, Theorem 4.1, amounts to saying that, for the single valued β and b as above, the generalized entropic solution to NFPE (1.1) can be represented as the law density of a stochastic process Y which is (in the probabilistic sense) a weak solution to a stochastic differential

equation with the drift b and the diffusion term $\Phi(u) = (2\beta(u)/u\mathbf{1}_{[0,\infty)}(u))^{1/2}$, that is,

$$dY(t) = b(u(t), Y(t))dt + \Phi(u(t), Y(t))dW(t), \quad (1.2)$$

with a suitable modification for multivalued functions β . Equation (1.2) can be viewed as a stochastic model of NFPE (1.1). Previously, such a $1-D$ result was obtained in [8], [6], [7] for $b \equiv 0$, $\beta(u) = u^m$ and $\beta(u) = H(u - u_c)u$, respectively. In the special case $\beta(u) \equiv u$, one obtains the representation of solutions to the linear Fokker-Planck equation as probability density of diffusion processes with the Boltzmann-Gibbs entropy

$$S[u] = - \int_{\mathbb{R}} u \log u dx.$$

In [15], the existence of a solution to a McKean type SDE of the form (1.2) is studied and it is proved that its law density is a distributional solution to a corresponding Fokker-Planck equation.

The principal problem encountered with the probabilistic representation of entropic solutions to (1.1) is their weak regularity. (See Definition 3.1 below.) To circumventing this, we developed here a new approach which, in a few words, can be described as follows; one approximates (1.1) by a family of smooth parabolic problems for which one has such a representation and gets the result by passing to the limit in the corresponding stochastic equation.

The structure of the paper is as follows. In Section 2, we briefly sketch the standard technique to obtain probabilistic representations for solutions of nonlinear Fokker-Planck equations. In Section 3, we recall the notion of entropic solutions and present a proof, via approximation for existence of a solution. In Section 4, we derive a probabilistic representation for the latter. In Section 5, we study the case when b is independent of u , but depending on the spatial variable. In Section 6, an application to $1-D$ degenerate parabolic equation is given.

Notation. For $1 \leq p \leq \infty$, we denote by L^p the space of Lebesgue p integrable functions $u : \mathbb{R}^d \rightarrow \mathbb{R}$ with norm denoted by $|\cdot|_p$, by H^1 we denote the Sobolev space, $H^1(\mathbb{R}^d) = \left\{ u \in L^2; \frac{\partial u}{\partial x_i} \text{ in } L^2(\mathbb{R}^d) \right\}$, and by $W_{loc}^{1,p}$, H_{loc}^1 , the corresponding local spaces; H^{-1} is the dual of H^1 . $C^k(\mathbb{R}^d)$, $k = 0, 1, 2$, is the space of k -times continuously differentiable functions. We denote by $\text{Lip}(\mathbb{R})$ the space of Lipschitz functions on \mathbb{R} and by $\text{Lip}_p^{\text{loc}}(\mathbb{R})$ the space

of locally Lipschitz functions on \mathbb{R} . By $C^k(\mathbb{R}^d; \mathbb{R}^m)$, we denote the corresponding \mathbb{R}^m -valued function space. We set $C(\mathbb{R}^d; \mathbb{R}^m) = C^0(\mathbb{R}^d; \mathbb{R}^m)$ and let $C_b(\mathbb{R}^d; \mathbb{R}^m)$ denote the set of all bounded functions in $C(\mathbb{R}^d; \mathbb{R}^m)$. Furthermore, $C_0^k(\mathbb{R}^d; \mathbb{R}^m)$ denotes the space of all the functions in $C^k(\mathbb{R}^d; \mathbb{R}^m)$ with compact support. We shall use also the notations: $x = (x_1, \dots, x_d)$, $b = (b_1, b_2, \dots, b_d)$, $\text{sign}_r = \frac{r}{|r|}$ for $r \neq 0$, $\text{sign } 0 = 0$, $\text{div } a(u) = \sum_{i=1}^d \frac{\partial}{\partial x_i} a_i(u)$, $\nabla u = \left\{ \frac{\partial u}{\partial x_i} \right\}_{i=1}^d$, $\Delta u = \sum_{i=1}^d \frac{\partial^2 u}{\partial x_i^2}$, $u_t = \frac{\partial u}{\partial t}$. By $|\cdot|_d$, we denote the Euclidean norm of \mathbb{R}^d , $d \geq 1$. Given a Hilbert space X , $C([0, T]; X)$ denote the space of X -valued continuous functions on $[0, T]$.

2 Probabilistic representation of solutions to NFPE

Let $T \in (0, \infty)$ and $a^{ij} : [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$, $b^i : [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ be Borel-measurable maps such that $(a^{ij}(t, r, x))_{1 \leq i, j \leq d}$ is a symmetric nonnegative matrix for all $(t, r, x) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$. Consider the following nonlinear Fokker-Planck equation

$$\begin{aligned} \partial_t u &= \partial_i \partial_j (a^{ij}(u)u) - \partial_j (b^j(u)u), \\ u(0, \cdot) &= u_0, \end{aligned} \tag{2.1}$$

where u_0 is a probability density (with respect to the Lebesgue measure on \mathbb{R}^d) and we used Einstein's summation convention. Equation (2.1) is to be understood in the weak sense, i.e.,

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^d} [\partial_t \varphi(t, x) + a^{ij}(t, u(t, x), x) \partial_i \partial_j \varphi(t, x) \\ + b^i(t, u(t, x), x) \partial_i \varphi(t, x)] u(t, x) dx dt \\ = - \int_{\mathbb{R}^d} \varphi(0, x) u_0(x) dx, \end{aligned} \tag{2.2}$$

for all φ of the form

$$\varphi(t, x) = f(t)g(x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d,$$

with $f \in C^1([0, T]; \mathbb{R})$, $f(T) = 0$, $g \in C_0^2(\mathbb{R}^d)$, where, in (2.2), it is assumed that

$$\int_0^T \int_{\mathbb{R}^d} (|a^{ij}(u)| + |b^i(u)|) |u| dx dt < \infty, \quad 1 \leq i, j \leq d, \quad (2.3)$$

(see [19]).

Assume that a solution $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ exists such that

$$\int_{\mathbb{R}^d} u(t, x) dx = 1 \text{ for all } t \in [0, T],$$

and $t \mapsto \int_{\mathbb{R}^d} u(t, x) dx$ is weakly continuous on $[0, T]$. Then, as a consequence of the so-called *superposition principle*, more precisely Theorem 2.5 in [19] (see also [10]), there exists a probability measure P_{u_0} on $C([0, T]; \mathbb{R}^d)$ equipped with its Borel σ -algebra and its natural filtration generated by the evaluation maps π_t , $t \in [0, T]$, defined by

$$\pi_t(w) := w(t). \quad w \in C([0, T], \mathbb{R}^d),$$

solving the martingale problem in the sense of [19] for the Kolmogorov operator

$$L = \partial_t + a^{ij}(u) \partial_i \partial_j + b^i(u) \partial_i$$

with the initial distribution $P_{u_0} \circ \pi_0^{-1} = u_0 dx$ and, moreover, with marginals $P_{u_0} \circ \pi_t^{-1} = u(t, x) dx$. Then, by a standard result (see, e.g., [17], Theorem 2.6), there exists a d -dimensional (\mathcal{F}_t) -Brownian motion $W(t)$, $t \in [0, T]$, on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, Q)$ and a continuous (\mathcal{F}_t) -progressively measurable map $Y : [0, T] \times \Omega \rightarrow \mathbb{R}^d$ satisfying the following distribution dependent stochastic differential equation (SDDDE) (see [20])

$$dY(t) = b(t, u(t, Y(t)), Y(t)) dt + \sqrt{2} \sigma(t, u(t, Y(t)), Y(t)) dW(t) \quad (2.4)$$

and the law

$$Q \circ Y^{-1} = P_{u_0}, \quad (2.5)$$

where $\sigma = ((a^{ij})_{1 \leq i, j \leq d})^{\frac{1}{2}}$. In particular, we have, for the marginals,

$$Q \circ (Y(t))^{-1} = u(t, x) dx, \quad t \in [0, T].$$

So, we have obtained a probabilistic representation of the solution u of (2.1) (in the sense of (2.2)), i.e., $u(t, \cdot)$ is the law density of a stochastic process Y which is a weak solution to (2.5).

It is much harder to prove that the solution to (2.4) is unique in law, provided its initial distribution is $u_0 dx$, which would of course be very desirable. For this, one has to prove that the solutions to the linear Fokker-Planck equation

$$\begin{aligned}\partial_t v &= \partial_i \partial_j (a^{ij}(u)v) - \partial_j (b^j(u)v) \\ v(0, \cdot) &= u_0\end{aligned}\tag{2.6}$$

in the sense of (2.2) with $u(t, x) dx dt$ replaced by $v(t, x) dx dt$. This was, however, achieved in certain cases (see [6], [7] and also [16]).

As explained in the introduction of this paper, we look at generalized (= entropic) solutions for a special case of (2.1). In this case, the above approach applies directly, because the entropic solution to (1.1) in the sense of (3.1) below is also a distribution solution and has the properties required above. However, the idea to approximate entropic solutions by solutions of more regular equations satisfying (2.2) was necessary in order to get the necessary continuity properties. This procedure will be implemented in the following two sections.

3 Generalized entropic solutions to NFPE

Let b as in (ii) above and set $a(r) := b(r)r$, $\forall r \in \mathbb{R}$.

Definition 3.1. A function $u \in L^1((0, T) \times \mathbb{R}^d) \cap C([0, T]; L^1_{\text{loc}})$ is said to be an entropic solution to NFPE (1.1) if there is $\eta \in L^2(0, T; H^1_{\text{loc}})$ such that $\eta(t, x) \in \beta(u(t, x))$, a.e. $(t, x) \in (0, T) \times \mathbb{R}^d$ and, for all $k \in \mathbb{R}$,

$$\begin{aligned}\frac{\partial}{\partial t} |u - k| + \operatorname{div}_x(\operatorname{sign}(u - k)(a(u) - a(k))) \\ + \operatorname{div}_x(\nabla_x \eta \operatorname{sign}(u - k)) \leq 0 \text{ in } \mathcal{D}'((0, T) \times \mathbb{R}^d), \\ u(0, x) = u_0(x), \quad x \in \mathbb{R}^d.\end{aligned}\tag{3.1}$$

Equivalently, the initial condition holds and

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} (|u(t, x) - k| \varphi_t(t, x) + \operatorname{sign}(u(t, x) - k) \\ a(u(t, x) - a(k)) \cdot \nabla_x \varphi(t, x) - \operatorname{sign}(u(t, x) - k) \\ \nabla_x \eta(t, x) \cdot \nabla_x \varphi(t, x)) dt \geq 0\end{aligned}\tag{3.2}$$

for all $\varphi \in C_0^\infty((0, T) \times \mathbb{R}^d)$, $\varphi \geq 0$ and all $k \in \mathbb{R}$.

(In the following, we shall simply write ∇ instead of ∇_x .)

If $\beta = 0$, this is just Kruzkov's entropic solution to the conservation law equation (see [9], [14]).

The function η associated with u in Definition 3.1 will be called an entropic co-solution to (1.1) and will be denoted by η^u .

We note that, if the entropic solution u is in $L^\infty((0, T) \times \mathbb{R}^d)$, then it is also a solution in sense of distributions to (1.1), that is,

$$\int_0^T \int_{\mathbb{R}^d} (u\varphi_t + a(u) \cdot \nabla\varphi - \nabla_x \eta^u \cdot \nabla\varphi) dt dx = 0, \quad \forall \varphi \in C_0^\infty((0, T) \times \mathbb{R}^d).$$

This follows by taking in (4.1) $k = |u|_\infty + 1$ and, respectively, $k = -|u|_\infty - 1$.

In [2], the existence of an entropic solution $u \in C([0, T]; L^1)$ was proven via the Crandall and Liggett generation theorem in L^1 . Here we shall give a direct constructive proof based on an appropriate smooth approximating equation. However, as seen in Remark 3.3 below, the construction leads to the same concept of solution.

Namely, for $\varepsilon \in (0, 1]$, consider the equation

$$\begin{aligned} (u_\varepsilon)_t + \operatorname{div}(a_\varepsilon(u_\varepsilon)) - \Delta(\beta_\varepsilon(u_\varepsilon) + \varepsilon u_\varepsilon) &= 0 \text{ in } (0, \infty) \times \mathbb{R}^d, \\ u_\varepsilon(0, x) &= u_0(x), \quad x \in \mathbb{R}^d. \end{aligned} \quad (3.3)$$

Here $a_\varepsilon(u) \equiv b_\varepsilon(u)u$ and the functions $\beta_\varepsilon, b_\varepsilon$ are defined as follows.

$$\beta_\varepsilon(r) = \frac{1}{\varepsilon} (r - (1 + \varepsilon\beta)^{-1}r) \in \beta((1 + \varepsilon\beta)^{-1}r), \quad \forall r \in \mathbb{R}, \quad (3.4)$$

$$b_\varepsilon(r) = \frac{1}{\varepsilon} \int_{\mathbb{R}} \frac{b(\theta)}{1 + \varepsilon\theta^2} \rho\left(\frac{r - \theta}{\varepsilon}\right) d\theta, \quad \forall r \in \mathbb{R}, \quad \varepsilon > 0, \quad (3.5)$$

where $\rho \in C_0^\infty(\mathbb{R})$, support $\rho \subset [-1, 1]$, $\rho \geq 0$, $\int_{\mathbb{R}} \rho dx = 1$.

By (3.4)-(3.5), we see that $\beta_\varepsilon \in \operatorname{Lip}(\mathbb{R})$, $\beta'_\varepsilon \geq 0$, $b_\varepsilon \in C^\infty(\mathbb{R})$ and

$$\sup\{|rb_\varepsilon(r)|; r \in \mathbb{R}\} < \infty. \quad (3.6)$$

Moreover, $b_\varepsilon \rightarrow b$ uniformly on compacts as $\varepsilon \rightarrow 0$. We note that, by (3.4), it follows that

$$\beta_\varepsilon(0) = 0, \quad |\beta_\varepsilon(r)| \leq \inf\{|z|; z \in \beta(r)\}, \quad \forall \varepsilon > 0, \quad r \in \mathbb{R}. \quad (3.7)$$

Theorem 3.2. *Let $u_0 \in L^1 \cap L^\infty$. Then there is a unique solution $u_\varepsilon = u_\varepsilon(t, u_0)$ to (3.3) which satisfies, for all $T > 0$,*

$$u_\varepsilon \in C([0, T]; L^1) \cap C(0, T; H^1) \cap L^\infty((0, T) \times \mathbb{R}^d), \quad (3.8)$$

$$\beta_\varepsilon(u_\varepsilon), \gamma_\varepsilon(u_\varepsilon) \in L^2(0, T; H^1), \quad (3.9)$$

$$(u_\varepsilon)_t \in L^2(0, T; H^{-1}), \quad (3.10)$$

$$|u_\varepsilon(t, u_0) - u_\varepsilon(t, \bar{u}_0)|_1 \leq |u_0 - \bar{u}_0|_1, \quad \forall u_0, \bar{u}_0 \in L^1 \cap L^\infty \cap H^1, \quad (3.11)$$

$$|u_\varepsilon(t)|_p \leq |u_0|_p, \quad 1 \leq p \leq \infty. \quad (3.12)$$

$u_\varepsilon : [0, T) \rightarrow H^1$ is continuous from the right and, if $u_0 \geq 0$, a.e. in \mathbb{R}^d , then $u_\varepsilon \geq 0$, a.e. in $(0, T) \times \mathbb{R}^d$. Moreover, on a subsequence $\{\varepsilon\} \rightarrow 0$, we have

$$u_\varepsilon \rightarrow u \text{ in } C([0, T]; L^1_{\text{loc}}), \quad (3.13)$$

$$\beta_\varepsilon(u_\varepsilon) \rightarrow \eta^u \text{ weakly in } L^2(0, T; H^1), \quad (3.14)$$

and weak-star in $L^\infty(0, T; L^\infty \cap L^2)$, where u is an entropic solution to NFPE (1.1) with co-solution η^u . We have $u(0, x) = u_0(x)$, $\forall x \in \mathbb{R}^d$ and

$$u \in C([0, T]; L^1_{\text{loc}}), \quad \eta_u \in L^\infty((0, T) \times \mathbb{R}^d),$$

$$|u(t)|_p \leq |u_0|_p, \quad \forall p \in [1, \infty], \quad \forall t \in [0, T],$$

$$|u(t, u_0) - u(t, \bar{u}_0)|_1 \leq |u_0 - \bar{u}_0|_1, \quad \forall u_0, \bar{u}_0 \in L^1, \quad (3.15)$$

$$u \geq 0, \text{ a.e. in } (0, T) \times \mathbb{R}^d, \text{ if } u_0 \geq 0, \text{ a.e. in } \mathbb{R}^d.$$

$$\int_{\mathbb{R}^d} u(t, x) dx = \int_{\mathbb{R}^d} u_0(t, x) dx, \quad \forall u_0 \in L^1, \quad u_0 \geq 0, \quad \forall t \in [0, T]. \quad (3.16)$$

Proof. To prove the existence for (3.3), consider the operator A_ε in H^{-1} defined by

$$A_\varepsilon u = \text{div}(a_\varepsilon(u)) - \Delta \beta_\varepsilon(u) - \varepsilon \Delta u, \quad \forall u \in D(A_\varepsilon) = H^1.$$

Arguing as in [5] (see Lemma 3.1), it follows that A_ε is quasi- m -accretive in H^{-1} and so, there is a unique solution $u_\varepsilon \in W^{1,\infty}([0, T]; H^{-1})$, which is H^1 -valued continuous from the right on $[0, T]$.

On the other hand, the operator A_ε with the domain

$$\{u \in L^1 \cap H^1; \Delta(\beta_\varepsilon(u) + \varepsilon u) \in L^1\}$$

is accretive in L^1 , its range contains $L^1 \cap H^{-1}$, and so, its closure $\overline{A_\varepsilon}$ in $L^1 \times L^1$ is m -accretive in L^1 .

Here is the argument. As seen above for $f \in L^1 \cap H^{-1}$ and λ sufficiently large, the equation

$$\lambda v_\varepsilon + A_\varepsilon v_\varepsilon = f \quad (3.17)$$

has a unique solution $v_\varepsilon \in H^1$. Let $\mathcal{X}_\delta \in \text{Lip}(\mathbb{R})$, $\delta \in (0, 1)$, be the following approximation of the signum function

$$\mathcal{X}_\delta(r) = \begin{cases} 1 & \text{if } r > \delta, \\ \frac{r}{\delta} & \text{if } |r| \leq \delta, \\ -1 & \text{if } r < -\delta, \end{cases} \quad (3.18)$$

where $\delta > 0$. The function \mathcal{X}_δ is a Lipschitzian approximation of the signum function and, clearly, $\mathcal{X}_\delta(v_\varepsilon) \in H^1$. By (3.17), we get that

$$\begin{aligned} \lambda \int_{\mathbb{R}^d} v_\varepsilon \mathcal{X}_\delta(v_\varepsilon) dx + \int_{\mathbb{R}^d} (\beta'_\varepsilon(v_\varepsilon) + \varepsilon) |\nabla v_\varepsilon|^2 \mathcal{X}'_\delta(v_\varepsilon) dx \\ - \int_{\mathbb{R}^d} (a_\varepsilon(v_\varepsilon) \cdot \nabla v_\varepsilon) \mathcal{X}'_\delta(v_\varepsilon) dx = \int_{\mathbb{R}^d} f \mathcal{X}_\delta(v_\varepsilon) dx. \end{aligned} \quad (3.19)$$

On the other hand, we have

$$\lim_{\delta \rightarrow 0} \left| \int_{\mathbb{R}^d} (a_\varepsilon(v_\varepsilon) \cdot \nabla v_\varepsilon) \mathcal{X}'_\delta(v_\varepsilon) dx \right| = \lim_{\delta \rightarrow 0} \left| \frac{1}{\tau} \int_{|u_\varepsilon| \leq \delta} a_\varepsilon(v_\varepsilon) \cdot \nabla v_\varepsilon dx \right| = 0,$$

because $\nabla v_\varepsilon = 0$ on $\{x; v_\varepsilon(x) = 0\}$. Then, letting $\delta \rightarrow 0$ in (3.19), we get

$$|v_\varepsilon|_1 \leq \lambda^{-1} |f|_1, \quad \forall \varepsilon > 0,$$

and so $v_\varepsilon \in L^1$. Similarly, if v_ε and \bar{v}_ε are solutions to (3.13) corresponding to f and \bar{f} in $L^1 \cap H^{-1}$, we get as above

$$|v_\varepsilon - \bar{v}_\varepsilon|_1 \leq \frac{1}{\lambda} |f - \bar{f}|_1, \quad \forall \lambda > 0. \quad (3.20)$$

This means that the operator A_ε is accretive in L^1 and $L^1 \cap H^{-1} \subset R(\lambda I + A_\varepsilon)$, $\forall \lambda > 0$.

Moreover, if $f \in L^1 \cap L^\infty \cap H^{-1}$, then, as easily follows by (3.17), we have

$$|(\lambda I + A_\varepsilon)^{-1} f|_\infty \leq \lambda^{-1} |f|_\infty, \quad \forall \lambda > 0, \quad (3.21)$$

$$|(\lambda I + A_\varepsilon)^{-1} f|_1 \leq \lambda^{-1} |f|_1, \quad \forall \lambda > 0. \quad (3.22)$$

We note also that, if $f > 0$, a.e. in \mathbb{R}^d , then $(\lambda I + A_\varepsilon)^{-1}f \geq 0$, a.e. in \mathbb{R}^d . (The latter follows by multiplying (3.17) by $\text{sign } v_\varepsilon^-$ or, more exactly, by $\mathcal{X}_\delta(v_\varepsilon^-)$ and integrating over \mathbb{R}^d .) In particular, this implies that $u_\varepsilon \geq 0$.

Now, let us come back to equation (3.3).

Multiplying by u_ε , $\beta(u_\varepsilon)$ and integrating on $\delta, y) \times \mathbb{R}^d$, we get

$$\begin{aligned} \frac{1}{2} |u_\varepsilon(t)|_2^2 + \int_0^t \int_{\mathbb{R}^d} \left(\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + |\nabla \beta_\varepsilon(u_\varepsilon)|^2 \right) dt dx \\ \leq \frac{1}{2} |u_0|_2^2 + \int_{\mathbb{R}^d} j(u_0) dx \leq C, \end{aligned} \quad (3.23)$$

because

$$\int_{\mathbb{R}^d} \text{div } a_\varepsilon(v_\varepsilon) \beta_\varepsilon(u_\varepsilon) dx = \int_{\mathbb{R}^d} a'_\varepsilon(u_\varepsilon) \beta_\varepsilon(u_\varepsilon) \cdot \nabla u_\varepsilon dx = \int_{\mathbb{R}^d} \text{div } g_\varepsilon(u_\varepsilon) dx = 0,$$

and similarly for $\gamma_\varepsilon(u_\varepsilon)$. Here $g_\varepsilon(r) = \int_0^r a'_\varepsilon(s) \beta_\varepsilon(s) ds$, $j(r) = \int_0^r \beta_\varepsilon(s) ds$, $\forall r \in \mathbb{R}$.

We also note that

$$\int_{\mathbb{R}^d} u_\varepsilon(t, x) dx = \int_{\mathbb{R}^d} u_0(x) dx, \quad \forall t \in [0, T]. \quad (3.24)$$

By the Crandall & Liggett exponential formula (see [1])

$$u_\varepsilon(t) = e^{-tA_\varepsilon} u_0 = \lim_{n \rightarrow \infty} \left(I + \frac{t}{n} A_\varepsilon \right)^{-1} u_0 \text{ in } H^{-1} \cap L^1, \quad (3.25)$$

it suffices to show that the solution $v \in H^{-1} \cap L^1$ to the equation

$$v + \lambda A_\varepsilon v = f, \quad f \in H^{-1} \cap L^1, \quad \lambda > 0, \quad (3.26)$$

satisfies the conservation of the mass equality

$$\int_{\mathbb{R}^d} v(x) dx = \int_{\mathbb{R}^d} f(x) dx. \quad (3.27)$$

By (3.27), we have

$$\int_{\mathbb{R}^d} v(x) \varphi(x) dx = \int_{\mathbb{R}^d} f(x) \varphi(x) + \int_{\mathbb{R}^d} (a_\varepsilon(v) - \nabla \beta_\varepsilon(v) - \varepsilon \nabla v) \cdot \nabla \varphi dx,$$

for all $\varphi \in C_0^\infty(\mathbb{R}^d)$. Let $\mathbb{B}_N = \{x \in \mathbb{R}^N; |x| \leq N\}$ and $\varphi = 1$ on \mathbb{B}_N , $\varphi = 0$ on \mathbb{B}_N^c . Taking into account that $a_\varepsilon(v) \in L^1$, $\nabla\beta_\varepsilon(v), \nabla v \in L^2$, we get, for $N \rightarrow \infty$, that (3.26) holds.

Note also by (3.20)–(3.22) that it follows that (3.11) and (3.12) hold.

We consider the finite difference scheme associated with equation (3.3), that is,

$$\begin{aligned} \frac{1}{h} (u_\varepsilon^{i+1} - u_\varepsilon^i) + \operatorname{div} a_\varepsilon(u_\varepsilon^{i+1}) - \Delta\beta_\varepsilon(u_\varepsilon^{i+1}) + \varepsilon\Delta u_\varepsilon^{i+1} &= 0, \\ i = 0, 1, \dots, N = \lceil \frac{T}{h} \rceil, \end{aligned} \quad (3.28)$$

where $u_\varepsilon^0 = u_0$. We set

$$u_h^\varepsilon(t) = u_\varepsilon^i \text{ for } t \in [ih, (i+1)h). \quad (3.29)$$

By the Crandall & Liggett formula (3.25), we know that, for $h \rightarrow 0$,

$$u_h^\varepsilon \rightarrow u_\varepsilon \text{ in } C([0, T]; H^{-1} \cap L^1). \quad (3.30)$$

On the other hand, since $u_\varepsilon^0 = u_0 \in L^1 \cap L^\infty$, we have by (3.20)–(3.21) and (3.29) that $u_h^\varepsilon \in L^\infty(0, T; L^1 \cap L^\infty)$ and

$$|u_h^\varepsilon|_1 + |u_h^\varepsilon(t)|_\infty \leq C, \quad \forall t \in [0, T], \quad (3.31)$$

where C is independent of ε and h .

We are going to prove that, for all ε and h ,

$$|u_h^\varepsilon(t + \bar{t}) - u_h^\varepsilon(t)|_1 \leq C|\bar{t}|; \quad t, t + \bar{t} \in [0, T]. \quad (3.32)$$

To this purpose, we fix $\bar{t} = \ell h$, $0 < \ell \leq N$, and note that, by (3.28) and (3.29), we have

$$|u_\varepsilon^{i+1+\ell} - u_\varepsilon^{i+1}|_1 \leq |u_\varepsilon^{i+\ell} - u_\varepsilon^i|_1 \leq |u_\varepsilon^\ell - u_0|_1.$$

Then (3.29) follows by (3.32), as claimed.

We note also that, by (3.29), it follows that

$$\int_0^T \|u_\varepsilon^h(t)\|_{H^1}^2 dt \leq C,$$

where C is independent of h .

Since $H^1(\mathbb{B}_R)$ is compact in $L^1(\mathbb{B}_R)$ for all balls \mathbb{B}_R of center 0 and radius R , we conclude by (3.32) that along a subsequence $\{h\} \rightarrow 0$ we have

$$u_\varepsilon^h(t) \rightarrow u_\varepsilon(t) \text{ strongly in } L^1_{\text{loc}}, \forall \varepsilon > 0,$$

uniformly on $[0, T]$.

Hence $u_\varepsilon \in C([0, T]; L^1_{\text{loc}})$ and $u_\varepsilon \in L^\infty(0, T; L^1)$. By (3.29) and the accretivity of A_ε in L^1 , (3.11) and (3.12) follow.

Now, by (3.13) and (3.23), it follows that along a subsequence converging to zero, again denoted by ε , we have

$$u_\varepsilon \rightarrow u \text{ weak-star in } L^\infty(0, T; L^2 \cap L^\infty), \quad (3.33)$$

$$\beta_\varepsilon(u_\varepsilon) \rightarrow \eta \text{ weakly in } L^2(0, T; H^1) \quad (3.34)$$

$$\sqrt{\varepsilon} u_\varepsilon \rightarrow 0 \text{ in } L^2(0, T; H^{-1}). \quad (3.35)$$

Moreover, since $\{u_\varepsilon\}$ is bounded in $L^\infty((0, T) \times \mathbb{R}^d)$, we have also by (3.4) that, for a subsequence $\{\varepsilon\} \rightarrow 0$,

$$\eta_\varepsilon = \beta_\varepsilon(u_\varepsilon) \rightarrow \eta^u \text{ weak-star in } L^\infty((0, T) \times \mathbb{R}^d). \quad (3.36)$$

Note also that, by (3.32), we get for $h \rightarrow 0$

$$|u_\varepsilon(t + \bar{t}) - u_\varepsilon(t)|_1 \leq C|\bar{t}|, \forall t, t + \bar{t} \in [0, T], \quad (3.37)$$

and so (3.8) follows. Coming back to (3.29), we see that, for each $\nu \in \mathbb{R}^d$, we have

$$\frac{1}{h} (|u_\varepsilon^{i+1})_\nu - (u_\varepsilon^i)_\nu) + \text{div}((a_\varepsilon(u_\varepsilon^{i+1}))_\nu - \Delta(\beta(u_\varepsilon^{i+1}))_\nu) + \varepsilon(u_\varepsilon^{i+1})_\nu = 0,$$

where $v_\nu(x) = v(x + \nu)$, $x \in \mathbb{R}^d$. Multiplying the latter by $\text{sign}((u_\varepsilon)^{i+1})_\nu$ (or, more exactly, by $\mathcal{X}_\delta((u_\varepsilon^{i+1})_\nu)$), we get, as above,

$$|(u_\varepsilon^{i+1})_\nu|_1 \leq |(u_0)_\nu|_1, \forall \nu \in \mathbb{R}^d, i = 0, 1, \dots, N.$$

Hence

$$|(u_\varepsilon)_\nu(t)|_1 \leq |(u_0)_\nu|_1, \forall t \in [0, T], \nu \in \mathbb{R}^d, \quad (3.38)$$

for all $\nu \in \mathbb{R}^d$. By (3.37) and (3.38), it follows by the Kolmogorov compactness criterium that $\{u_\varepsilon\}_{\varepsilon > 0}$ is compact in each space $L^1(\mathbb{B}_R)$. Hence, for $\varepsilon \rightarrow 0$,

$$u_\varepsilon \rightarrow u \text{ in } L^1((0, T) \times L^1_{\text{loc}}). \quad (3.39)$$

In particular, it follows via Egorov's theorem that, for each $\delta > 0$, there is a measurable subset $\Sigma_\delta \subset (0, T) \times \mathbb{B}_R$ whose complement in $(0, T) \times \mathbb{B}_R$ has Lebesgue measure less than δ such that $u_\varepsilon \rightarrow u$ in $L^\infty(\Sigma_\delta)$. Since $\beta : L^2(\Sigma_\delta) \rightarrow L^2(\Sigma_\delta)$ is maximal monotone, strongly-weakly closed, and so it follows by (3.34) and (3.36) that $\eta^u \in \beta(u)$ a.e. in Σ_δ and, since δ is arbitrary, it follows that $\eta^u \in \beta(u)$, a.e. in $(0, T) \times \mathbb{R}^d$. Moreover, by (3.34), it follows that $\nabla\beta_\varepsilon(u_\varepsilon) \rightarrow \nabla\eta^u$ weakly in $L^2(0, T; L^2)$, as $\varepsilon \rightarrow 0$.

Note also that, letting $\varepsilon \rightarrow 0$ in (3.11)–(3.12), it follows by (3.39) via Fatou's lemma that (3.15) holds.

Since by (3.37), $\{u_\varepsilon\}$ is equi-uniformly continuous in $C([0, T]; L^1)$, by (3.39) it follows that $u_\varepsilon \rightarrow u$ in $C([0, T]; L^1_{\text{loc}})$ as $\varepsilon \rightarrow 0$. In particular, this implies that $u \in C([0, T]; L^1_{\text{loc}})$. Letting $\varepsilon \rightarrow 0$ in (3.24), we get also (3.16).

Let us prove now that u is an entropic solution for equation (1.1), that is, (3.2) holds. By (3.3), we have

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^d} \mathcal{X}_\delta(u_\varepsilon - k) [(u_\varepsilon)_t \varphi + \operatorname{div}(a_\varepsilon(u_\varepsilon)) \varphi + \varepsilon u_\varepsilon \varphi] dx dt \\ & \quad + \int_0^T \int_{\mathbb{R}^d} (\nabla\beta_\varepsilon(u_\varepsilon)) \cdot \nabla(\mathcal{X}_\delta(u_\varepsilon - k) \varphi) dx dt = 0. \end{aligned} \quad (3.40)$$

Now, letting $\varepsilon \rightarrow 0$ in (3.24), we get (3.16), as claimed.

We have, for $j_\delta(r) \equiv \int_0^r \mathcal{X}_\delta(s) ds$,

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^d} \mathcal{X}_\delta(u_\varepsilon - k) (u_\varepsilon)_t \varphi dx dt \\ & = - \int_0^T \int_{\mathbb{R}^d} \mathcal{X}_\delta(u_\varepsilon - k) (u_\varepsilon - k)_t \varphi dx dt \\ & = \int_0^T \int_{\mathbb{R}^d} j_\delta(u_\varepsilon - k) \varphi_t dx dt \xrightarrow{\delta \rightarrow 0} \int_0^T \int_{\mathbb{R}^d} |u_\varepsilon - k| \varphi_t dx dt. \end{aligned} \quad (3.41)$$

We also have

$$\left| \int_0^T \int_{\mathbb{R}^d} \mathcal{X}_\delta(u_\varepsilon - k) (\varepsilon u_\varepsilon \varphi) dx dt \right| \leq C \sqrt{\varepsilon} \|\varphi\|_{L^1((0, T) \times \mathbb{R}^d)}, \quad (3.42)$$

and

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}^d} \nabla \beta_\varepsilon(u_\varepsilon) \cdot \nabla (\mathcal{X}_\delta(u_\varepsilon - k)) \varphi dx dt \\
&= \int_0^T \int_{\mathbb{R}^d} \beta'_\varepsilon(u_\varepsilon) |\nabla u_\varepsilon|^2 \mathcal{X}'_\delta(u_\varepsilon - k) \varphi dx dt \\
&+ \int_0^T \int_{\mathbb{R}^d} (\nabla \beta_\varepsilon(u_\varepsilon) \cdot \nabla \varphi) \mathcal{X}_\delta(u_\varepsilon - k) dx dt \\
&\geq \int_0^T \int_{\mathbb{R}^d} (\nabla \beta_\varepsilon(u_\varepsilon) \cdot \nabla \varphi) \mathcal{X}_\delta(u_\varepsilon - k) dx dt,
\end{aligned} \tag{3.43}$$

because $\varphi \geq 0$. Furthermore,

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}^d} \mathcal{X}_\delta(u_\varepsilon - k) \operatorname{div}(a_\varepsilon(u_\varepsilon)) \varphi dx dt \\
&= \int_0^T \int_{\mathbb{R}^d} \mathcal{X}_\delta(u_\varepsilon - k) \operatorname{div}(a_\varepsilon(u_\varepsilon) - a_\varepsilon(k)) \varphi dx dt \\
&= \int_0^T \int_{\mathbb{R}^d} \mathcal{X}_\delta(u_\varepsilon - k) \operatorname{div}(a_\varepsilon(u_\varepsilon) - a_\varepsilon(k)) \varphi dx dt \\
&+ \int_0^T \int_{\mathbb{R}^d} \mathcal{X}_\delta(u_\varepsilon - k) (a_\varepsilon(u_\varepsilon) - a_\varepsilon(k)) \cdot \nabla \varphi dx dt = I_{\varepsilon, \delta}^1 + I_{\varepsilon, \delta}^2.
\end{aligned} \tag{3.44}$$

We note that

$$\begin{aligned}
& \mathcal{X}_\delta(u_\varepsilon - k) \operatorname{div}(a_\varepsilon(u_\varepsilon) - a_\varepsilon(k)) = \operatorname{div}(\mathcal{X}_\delta(u_\varepsilon - k) (a_\varepsilon(u_\varepsilon) - a_\varepsilon(k))) \\
&\quad - \int_k^{u_\varepsilon} \mathcal{X}'_\delta(s) (a_\varepsilon(u_\varepsilon(s)) - a_\varepsilon(k)) ds.
\end{aligned}$$

This yields

$$\begin{aligned}
I_{\varepsilon, \delta}^1 &= - \int_0^T \int_{\mathbb{R}^d} \nabla \varphi \cdot (\mathcal{X}_\delta(u_\varepsilon - k) (a_\varepsilon(u_\varepsilon) - a_\varepsilon(k))) dx dt \\
&\quad - \int_k^{u_\varepsilon} \mathcal{X}'_\delta(s) (a_\varepsilon(u_\varepsilon(s)) - a_\varepsilon(k)) ds dx dt
\end{aligned}$$

and so, we have

$$\lim_{\varepsilon, \delta \rightarrow 0} I_{\varepsilon, \delta}^1 = 0. \tag{3.45}$$

We have also, by Lebesgue's dominated convergence theorem, that

$$\lim_{\varepsilon, \delta \rightarrow 0} I_{\varepsilon, \delta}^2 = \int_0^T \int_{\mathbb{R}^d} \text{sign}(u - k)(a(u) - a(k)) \cdot \nabla \varphi \, dx \, dt. \quad (3.46)$$

Now, combining (3.40)–(3.46) and letting $\varepsilon, \delta \rightarrow 0$, we obtain that

$$\begin{aligned} - \int_0^T \int_{\mathbb{R}^d} (u - k) \varphi_t \, dx \, dt - \int_0^T \int_{\mathbb{R}^d} \text{sign}(u - k)(a(u) - a(k)) \cdot \nabla \varphi \, dx \, dt \\ + \int_0^T \int_{\mathbb{R}^d} \nabla \eta^u \cdot \nabla \varphi \, \text{sign}(u - k) \, dx \, dt \leq 0, \end{aligned}$$

and so (3.2) follows.

Then it follows that u is an entropic solution to (1.1) with co-solution η^u . This completes the proof.

Remark 3.3. It should be mentioned that, if a is not Lipschitz, then the entropic solution u to (1.1) in general is not unique (this happens also for the special case of the conservation law equation, that is, for $\beta \equiv 0$). On the other hand, the solution $u = \lim_{\varepsilon \rightarrow 0} u_\varepsilon$ given by Theorem 3.2 is just the mild solution (see [1], p. 128) to the Cauchy problem

$$\begin{aligned} \frac{du}{dt} + Au \ni 0, \quad t \in (0, T), \\ u(0) = u_0, \end{aligned} \quad (3.47)$$

where A is the closure in $L^1 \times L^1$ of the operator $A_0 : D(A_0) \subset L^1 \rightarrow L^1$ defined by $v \in A_0 u$ if $u, v \in L^1$, $a(u) \in L^1$, $\eta \in H_{\text{loc}}^1$, $\eta \in \beta(u)$, a.e. in \mathbb{R}^d , and

$$\int_{\mathbb{R}^d} |\text{sign}(u - k)((a(u) - a(k)) \cdot \nabla \varphi - \nabla(\eta \cdot \nabla \varphi) + v \varphi) \, dx \geq 0,$$

for all $k \in \mathbb{R}$ and $\varphi \in C_0^\infty(\mathbb{R}^d)$, $\varphi \geq 0$.

It turns out (see [2]) that A is m -accretive in L^1 and so, by the Crandall & Liggett theorem, (3.47) has a unique mild solution $y = e^{-tA} y_0 = \lim_{n \rightarrow \infty} (I + \frac{t}{n} A)^{-n} y_0$ in L^1 . On the other hand, arguing as in the proof of Theorem 3.2, it follows that, for each $f \in L^1 \cap L^\infty$, the solution u^ε to the

equation $\lambda u^\varepsilon + A_\varepsilon u^\varepsilon = f$ is strongly convergent in L^1_{loc} for $\varepsilon \rightarrow 0$ and $\lambda > 0$ to $v = (\lambda I + A_0)^{-1}f$. This implies by an argument similar to that used in the proof of the Trotter–Kato theorem for nonlinear semigroups (see, e.g., [1], p.169) that, for $u_0 \in L^1 \cap L^\infty$, the solutions u_ε to equation (3.3) are convergent in $C([0, T]; L^1)$ to $y = e^{-tA}y_0$. Hence, we can identify in the class of entropic solutions to (1.1) the function u given by Theorem 3.2 as the evolution generated by the operator A in L^1 . Such a solution will be called in the following the *generalized entropic solution* to FPNE (1.1).

4 The probabilistic representation of generalized entropic solution

We set

$$\Phi(r) = \left(\frac{\beta(r)}{r} \mathbf{1}_{[0, \infty)}(r) \right)^{\frac{1}{2}}, \quad \forall r \in \mathbb{R}, \quad (4.1)$$

and assume that, besides (i), the following conditions hold

$$(iii) \sup\{z; z \in \Phi(r)\} \leq \rho < +\infty, \quad \forall r \in [0, \infty).$$

Consider the Wiener process

$$W = \text{col}(W_j)_{j=1}^d, \quad (4.2)$$

where W_j are independent Brownian motions on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. We note that, by virtue of assumptions (ii) and (iii), the function $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and bounded.

Let $u = u(t, x)$ be the generalized entropic solution to (1.1) with the corresponding co-solution η^u . We set

$$\mathcal{X}^u(t, x) = \begin{cases} \left(\frac{\eta^u(t, x)}{u(t, x)} \right)^{\frac{1}{2}} & \text{on } [(t, x); u(t, x) > 0], \\ 0 & \text{on } [(t, x); u(t, x) = 0], \end{cases}$$

and note that \mathcal{X}^u is measurable and bounded on $(0, T) \times \mathbb{R}^d$.

Consider the stochastic differential equation

$$\begin{aligned} dY(t) &= b(u(t, Y(t)))dt + \sqrt{2} \mathcal{X}^u(t, u(t, Y(t)))dW(t), \quad t \geq 0, \\ Y(0) &= Y_0. \end{aligned} \quad (4.3)$$

A probabilistically weak solution $Y : [0, T] \times \Omega \rightarrow \mathbb{R}^+$ to equation (4.3) is a progressively measurable and pathwise continuous process which satisfies (4.3) in integral form for some \mathbb{R}^d -valued Wiener process on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_t, P)$ as above. Theorem 4.1 below is the main result of this work.

Theorem 4.1. *Let assumptions (i)-(iii) hold and let u be the generalized entropic solution to NFPE (1.1) given by Theorem 3.2 with $u_0 \geq 0$ in \mathbb{R}^d , $u_0 \in L^1 \cap L^\infty$, $\int_{\mathbb{R}^d} u_0 dx = 1$. Then there is a probabilistically weak solution Y to (4.3) such that*

$$P \circ Y(t)^{-1}(dx) = u(t, x)dx \text{ for every } t \in [0, T]. \quad (4.4)$$

Proof. We recall that u is also a distributional solution and also that the function $t \rightarrow u(t, x)$ is L^1_{loc} continuous on $[0, T]$, $u \in L^1((0, T) \times \mathbb{R}^d) \cap L^\infty((0, T) \times \mathbb{R}^d)$ and $\int_{\mathbb{R}^d} u(t, x)dx = 1, \forall t \in [0, T]$. Moreover, by (iii) we have also

$$\int_0^T \int_{\mathbb{R}^d} (|b(u(t, x))| + |\eta^u(t, x)|)|u(t, x)|dt dx < \infty.$$

Then, as explained in Section 2, by [19], Theorem 2.5, and [17], Theorem 2.6, there exists a d -dimensional $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion $W(H), t \in [0, T]$, on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and a continuous (\mathcal{F}_t) -progressively measurable map $Y : [0, T] \times \Omega \rightarrow \mathbb{R}^d$ which satisfies equation (4.3), and $P_{u_0} \circ Y(t)^{-1}(dx) = u(t, x)dx$ for all $t \in [0, T]$. This completes the proof.

5 NFPE with linear drift; distributional solutions

Consider the equation

$$\begin{aligned} u_t + \operatorname{div}(D(x)u) - \Delta\beta(u) &= 0 \text{ in } (0, \infty) \times \mathbb{R}^d, \\ u(0, x) &= u_0(x), \end{aligned} \quad (5.1)$$

where $D : \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfies the following condition

- (k) $D \in L^\infty(\mathbb{R}^d), \operatorname{div} D \in L^\infty(\mathbb{R}^d)$.

It turns out (see, e.g., [3]) that, under assumptions (i) and (k), for $u_0 \in L^1 \cap L^\infty$, equation (5.1) has a unique solution $u \in C([0, T]; L^1) \cap L^\infty((0, T) \times \mathbb{R}^n)$ in sense of distributions, that is,

$$\begin{aligned} u_t + \operatorname{div}(Du) - \Delta\beta(u) &= 0 \text{ in } \mathcal{D}'((0, T) \times \mathbb{R}^d), \\ u(0) &= u_0. \end{aligned} \tag{5.2}$$

Moreover, $u \geq 0$ if $u_0 \geq 0$ and $\int_{\mathbb{R}^d} u(t, x)dx = \int_{\mathbb{R}^d} u_0(x)dx$, $\forall t \in [0, T]$.

The solution u is obtained as in the previous case as limit of solutions u_ε to the approximating equation

$$\begin{aligned} (u_\varepsilon)_t + \operatorname{div}(Du_\varepsilon) - \Delta(\beta_\varepsilon(u_\varepsilon) + \varepsilon u_\varepsilon) &= 0 \text{ in } (0, T) \times \mathbb{R}^d, \\ u_\varepsilon(0) &= u_0. \end{aligned}$$

Theorem 5.1. *Let u be the solution to (5.2) for $u_0 \in L^1 \cap L^\infty \cap H^1 \cap W_{\text{loc}}^{1, \infty}$, $u_0 \geq 0$. Then*

$$u = \mathcal{L}(Y), \text{ a.e. in } (0, T) \times \mathbb{R}^d, \tag{5.3}$$

where $\mathcal{L}(Y)$ is the probability density law of a weak solution Y to the stochastic differential equation

$$\begin{aligned} dY &= D(Y(t))dt + \mathcal{X}^u(t, Y(t))dW, \\ Y(0) &= Y_0, \end{aligned} \tag{5.4}$$

where \mathcal{X}^u is as in Section 4.

The proof follows as in the case of Theorem 4.1 by the results of Section 2.

Remark 5.2. We note also that the bosons equations ($\beta(u) = \gamma \ln(u + 1)$) enters within the applicability field of Theorem 5.1. As a matter of fact, it applies to each monotone continuous function β with polynomial growth, $\beta(r) \leq Cr^m$, $\forall r \in \mathbb{R}$. Also, the self-organized criticality model presented in Section 1 is covered by Theorems 4.1 and 5.1.

6 An application to 1– D degenerate parabolic equation

Consider the equation

$$\begin{aligned} y_t(t, x) + a(y_x(t, x)) - (\beta(y_x(t, x)))_x &= 0, \quad t \geq 0, \quad x \in \mathbb{R}, \\ y(0, x) &= y_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{6.1}$$

where $a(r) \equiv b(r)r$ and $b : \mathbb{R} \rightarrow \mathbb{R}$, $\beta : \mathbb{R} \rightarrow \mathbb{R}$ satisfy assumptions (i), (ii), (iii). By the substitution $u = y_x$, we reduce equation (6.1) to a Fokker-Planck equation of the form (1.1), namely

$$\begin{aligned} u_t + (a(u))_x - (\beta(u))_{xx} &= 0, \quad t \geq 0, \quad x \in \mathbb{R}, \\ u(0, x) &= y'_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{6.2}$$

By Theorem 3.2, equation (6.2) has an entropic solution u which satisfies (3.15). Then

$$y(t, x) = \int_{-\infty}^x u(t, \xi) d\xi, \quad t \geq 0, \quad x \in \mathbb{R},$$

can be viewed as a generalized solution to (6.1). Moreover, by Theorem 4.1, it follows that, if

$$y'_0 \geq 0, \quad y_0(+\infty) - y_0(-\infty) = 1, \tag{6.3}$$

then y has the probabilistic representation

$$y(t, x) = y(t, 0) + \mathbb{P}[0 \leq Y(t) \leq x], \quad \forall t \geq 0, \quad x \in \mathbb{R}, \tag{6.4}$$

where Y is the solution to the stochastic differential equation (4.3).

An example. The equation

$$\begin{aligned} y_t + a(y_x) - a_1(y_x)(U'(y_x))_x &= 0, \quad t > 0, \quad x \in \mathbb{R}, \\ y(0, x) &= y_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{6.5}$$

where $U : \mathbb{R} \rightarrow \mathbb{R}$ is a convex, and smooth function, $U(0) = 0$, $a_1 \in C_b(\mathbb{R})$, $a_1 \geq 0$ and $a \in C(\mathbb{R})$, was proposed as a model for the evolution law of interfacial curves arising in material science and crystal growth. (See [13] for its physical significance and an analytic treatment via comparison arguments.)

By the substitution

$$\beta(r) = \int_0^r a_1(s)U''(s)ds, \quad \forall r \in \mathbb{R},$$

one reduces (6.5) to equation (6.1) and so, if $U'' \in L^\infty(\mathbb{R})$, we have for the solution y to (6.5) the probabilistic representation formula (6.4), where Y is the solution to SDE

$$dY = \frac{a(Y)}{Y} dt + \left(\frac{2}{Y} \int_0^Y a_1(s)U''(s) \right) dW.$$

As far as concerns the existence for (6.5), the condition $U' \in W^{1,\infty}(\mathbb{R})$ can be weakened to $U \in W^{1,\infty}(\mathbb{R})$, which leads to a maximal monotone graph $\beta : \mathbb{R} \rightarrow 2^{\mathbb{R}}$ everywhere defined on \mathbb{R} .

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