# Variational solutions to nonlinear stochastic differential equations in Hilbert spaces

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**Abstract**

One introduces a new variational concept of solution for the stochastic differential equation $dX + A(t)X\,dt + \lambda X\,dt = X\,dW$, $t \in (0,T)$; $X(0) = x$ in a real Hilbert space where $A(t) = \partial\varphi(t)$, $t \in (0,T)$, is a maximal monotone subpotential operator in $H$ while $W$ is a Wiener process in $H$ on a probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$. In this new context, the solution $X = X(t,x)$ exists for each $x \in H$, is unique, and depends continuously on $x$. This functional scheme applies to a general class of stochastic PDE not covered by the classical variational existence theory ([15], [16], [17]) and, in particular, to stochastic variational inequalities and parabolic stochastic equations with general monotone nonlinearities with low or superfast growth to $+\infty$.

**Keywords:** Brownian motion, maximal monotone operator, subdifferential, random differential equation, minimizastion problem.

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# 1 Introduction

Here, for $\lambda \in (0,\infty)$, we consider the stochastic differential equation

$$\begin{aligned} & dX(t) + A(t)X(t)dt + \lambda X(t)dt \ni X(t)dW_t,\ t \in (0,T), \\ & X(0) = x \in H, \end{aligned} \tag{1.1}$$

in a real Hilbert spaced $H$ whose elements are generalized functions on a bounded domain $\mathcal{O} \subset \mathbb{R}^d$ with a smooth boundary $\partial\mathcal{O}$. In examples, we have in mind that $H$ is e.g. $L^2(\mathcal{O})$ or $H^1_0(\mathcal{O})$, $H^1(\mathcal{O})$, $H^{-1}(\mathcal{O})$.

The norm of $H$ is denoted by $|\cdot|_H$, its scalar product by $(\cdot,\cdot)$ and its Borel $\sigma$-algebra by $\mathcal{B}(H)$.

$W$ is a Wiener process of the form

$$W(t,\xi) = \sum_{j=1}^{\infty} \mu_j e_j(\xi)\beta_j(t),\ \xi \in \mathcal{O},\ t \geq 0, \tag{1.2}$$

where $\{\beta_j\}_{j=1}^{\infty}$ is an independent system of real $(\mathcal{F}_t)$-Brownian motions on a probability space $\{\Omega,\mathcal{F},\mathbb{P}\}$ with natural filtration $(\mathcal{F}_t)_{t\geq 0}$ and $\{e_j\}$ is an orthonormal basis in $H$ such that both $c_j$ and $e_j^2$, $j \in \mathbb{N}$, are multipliers in $H$, while $\mu_j \in \mathbb{R}$, $j = 1,2,...$, satisfy (1.9) below.

As regards the nonlinear (multivalued) operator $A = A(t,\omega) : H \to H$, the following hypotheses will be assumed below.

(i) *Let $\varphi : [0,T]\times H\times\Omega \to \overline{\mathbb{R}} = ]-\infty,+\infty]$ be convex lower semicontinuous in $y \in H$ and progressively measurable, i.e., for each $t \in [0,T]$ the function $\varphi$ restricted to $[0,t]\times H\times\Omega$ is $\mathcal{B}([0,t])\otimes\mathcal{B}(H)\otimes\mathcal{F}_t$ measurable, and let*

$$A(t,\omega) = \partial\varphi(t,\omega),\ \forall (t,\omega) \in [0,T]\times\Omega. \tag{1.3}$$

*In particular, $y \to A(t,\omega,y)$ is maximal monotone in $H\times H$ for all $(t,\omega) \in [0,T]\times\Omega$. Furthermore, $\varphi$ is such that there exists $\alpha \in L^2([0,T]\times\Omega;H)$ and $\beta \in L^2([0,T]\times\Omega)$ such that*

$$\varphi(t,y,\omega) \geq (\alpha(t,\omega),y) - \beta(t,\omega) \text{ for } dt\otimes\mathbb{P}-a.e.,\ (t,\omega) \in [0,T]\times\Omega.$$

(ii) *$e^{\pm W(t)}$ is a multiplier in $H$ such that there is an $(\mathcal{F}_t)_{t\geq 0}$-adapted $\mathbb{R}_+$-valued process $Z(t)$, $t \in [0,T]$, with*

$$\begin{aligned} \sup_{t\in[0,T]} |Z(t)| &< \infty, \qquad \mathbb{P}\text{-}a.s., \\ |e^{\pm W(t)}y|_H &\leq Z(t)|y|_H,\ \ \forall t \in [0,T],\ y \in H, \end{aligned} \tag{1.4}$$

$$t \to e^{\pm W(t)} \in L(H,H) \text{ is continuous.} \tag{1.5}$$

Recall that a multivalued mapping $A : D(A) \subset H \to H$ is said to be maximal monotone if it is monotone, that is, for $u_1, u_2 \in D(A)$,

$$(z_1 - z_2, u_1 - u_2) \geq 0,\ \forall z_i \in Au_i,\ i = 1, 2,$$

and the range $R(\lambda I + A)$ is all of $H$ for each $\lambda > 0$.

If $\varphi : H \to \overline{\mathbb{R}}$ is a convex, lower semicontinuous function, then its subdifferential $\partial\varphi : H \to H$

$$\partial\varphi(u) = \{v \in H;\ \varphi(u) \leq \varphi(\bar{u}) + (v, u - \bar{u}),\ \forall \bar{u} \in H\} \tag{1.6}$$

is maximal monotone (see, e.g., [1]).

The conjugate $\varphi^*$ of $H$ defined by

$$\varphi^*(v) = \sup\{(u, v) - \varphi(u);\ u \in H\} \tag{1.7}$$

satisfies

$$\begin{array}{rcll} \varphi(u) + \varphi^*(v) & \geq & (u, v), & \forall u, v \in H, \\ \varphi(u) + \varphi^*(v) & = & (u, v), & \text{iff } v \in \partial\varphi(u). \end{array} \tag{1.8}$$

As regards the basis $\{e_j\}$ arising in the definition of the Wiener process $W$, we assume also that, for the multipliers $e_j^2$, we have

(iii) *For* $\gamma_j = \max\{\sup\{|ue_j|_H;\ |u|_H = 1\}, (\sup\{ue_j^2|_H;\ |u|_H = 1\})^{\frac{1}{2}}, 1\}$, *we assume*

$$\nu = \sum_{j=1}^{\infty} \mu_j^2 \gamma_j^2 < \infty, \tag{1.9}$$

*and that* $\lambda > \nu$.

Clearly, then

$$\mu = \frac{1}{2} \sum_{j=1}^{\infty} \mu_j^2 e_j^2 \tag{1.10}$$

is a multiplier in $H$.

It should be noted that the condition $\lambda > \nu$ in (1.9) is made only for convenience. In fact, by the substitution $X \to \exp(-\lambda t)X$ and replacing $A(t)$ by $u \to e^{-\lambda t}A(t)(e^{\lambda t}u)$ we can always change $\lambda$ in (1.1) to a big enough $\lambda$ which satisfies $\lambda > \nu$. It should be emphasized that a general existence and uniqueness result for equation (1.1) is known only for the special case where $A(t)$ are monotone and demicontinuous operators from $V$ to $V'$, where

$(V, V')$ is a pair of reflexive Banach spaces in duality with the Hilbert space $H$ as pivot space, that is, $V \subset H(\equiv H') \subset V'$ densely and continuously. If, in addition, for $\alpha_1 \in (0, \infty)$, $\alpha_2, \alpha_3 \in \mathbb{R}$,

$$
\begin{aligned}
{}_{V'}(A(t)u, u)_V &\geq \alpha_1 \|u\|_V^p + \alpha_2 |u|_H^2,\ \forall u \in V, && (1.11)\\
\|A(t)u\|_{V'} &\leq \alpha_3 \|u\|_V^{p-1},\ \forall u \in V, && (1.12)
\end{aligned}
$$

where $1 < p < \infty$, then equation (1.1) has under assumptions (i)–(iii) a unique strong solution $X \in L^p((0,T) \times \Omega; V)$ (see [15], [16], [17], [18]). We noted before that assumption (i) implies that $A(t, \omega)$ is maximal monotone in $H$ for all $t \in [0,T]$, though not every maximal monotone operator $A(t) : D(A(t)) \subset H \to H$ has a realization in a convenient pair of spaces $(V, V')$ such that (1.8)–(1.9) hold. Though assumptions (1.11)–(1.12) hold for a large class of stochastic parabolic equations in Sobolev spaces $W^{1,p}(\mathcal{O})$, $1 \leq p < \infty$ (see [6]), some other important stochastic PDEs are not covered by this functional scheme. For instance, the variational stochastic differential equations, nonlinear parabolic stochastic equations in $W^{1,1}(\mathcal{O})$, in Orlicz-Sobolev spaces on $\mathcal{O}$ or in $BV(\mathcal{O})$ (bounded variation stochastic flows) cannot be treated in this functional setting. As a matter of fact, contrary to what happens for deterministic infinite differential equations, there is no general existence theory for equation (1.1) under assumption (i)–(iii). The definition of a convenient concept of a weak solution to be unique and continuous with respect to data is a challenging objective of the existence theory of the infinite dimensional SDE. In this paper, we introduce such a solution $X$ for (1.1) which is defined as a minimum point of a certain convex functional defined on a suitable space of $H$-valued processes on $(0,T)$. This idea was developed in [11] for nonlinear operators $A(t) : V \to V'$ satisfying condition (1.11)–(1.12) and is based on the so-called *Brezis–Ekeland variational principle* [11]. Such a solution in the sequel will be called the *variational solution* to (1.1). (Along these lines see also [2], [3], [5], [6].)

# 2 The variational solution to equation (1.1)

First, we transform equation (1.1) into a random differential equation via the substitution

$$X(t) = e^{W(t)}(y(t) + x),\ t \in [0,T], \tag{2.1}$$

which, by Itô's product rule,

$$dX = e^W dy + e^W (y + x)dW + \mu e^W (y + x)dt,$$

leads to

$$\begin{aligned} &\frac{dy(t)}{dt} + e^{-W(t)} A(t)(e^{W(t)}(y(t) + x)) + (\mu + \lambda)(y(t) + x) \ni 0, \\ &\hspace{20em} t \in (0, T), \\ &y(0) = 0. \end{aligned} \tag{2.2}$$

(In the following, we shall omit $\omega$ from the notation $A(t, \omega)$.)

As a matter of fact, the equivalence between (1.1) and (2.2) is true only for a smooth solution $y$ to (2.2), that is, for pathwise absolutely continuous strong solutions to (2.2) (see [9], [10]). In the sequel, we shall define a generalized (variational) solution for the random Cauchy problem (2.2) and will call the corresponding process $X$ defined by (2.1) *the variational solution to* (1.1).

We shall treat equation (2.2) by the operator method developed in [10]. Namely, consider the space $\mathcal{H}$ of all $H$-valued processes $y : [0, T] \to H$ such that

$$|y|_{\mathcal{H}} = \left( \mathbb{E} \int_0^s |e^{W(t)} y(t)|_H^2 dt \right)^{\frac{1}{2}} < \infty,$$

which have an $(\mathcal{F}_t)_{t \geq 0}$-adapted version. Here $\mathbb{E}$ denotes the expectation with respect to $\mathbb{P}$. The space $\mathcal{H}$ is a Hilbert space with the scalar product

$$\langle y, z \rangle = \mathbb{E} \int_0^s (e^{W(t)} y(t), e^{W(t)} z(t)) dt, \ \ y, z \in \mathcal{H}.$$

We set $\delta = \frac{1}{2}(\lambda - \nu)$. Now, consider the operators $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ and $\mathcal{B} : D(\mathcal{B}) \subset \mathcal{H} \to \mathcal{H}$ defined by

$$\begin{aligned} (\mathcal{A}y)(t) \ &= \ e^{-W(t)} A(t)(e^{W(t)}(y(t) + x)) + \delta(y + x), \ \ \forall y \in D(\mathcal{A}), \\ &\hspace{18em} t \in [0, T], \\ D(\mathcal{A}) \ &= \ \{y \in \mathcal{H}; \ e^{W(t)}(y(t) + x) \in D(A(t)), \ \forall t \in [0, T] \text{ and} \\ &\hspace{12em} e^{-W} A(e^W (y + x)) \in \mathcal{H}\}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} (\mathcal{B}y)(t) &= \frac{dy}{dt}(t) + (\mu + \nu + \delta)(y + x), \ \text{ a.e. } t \in (0, T), \ y \in D(\mathcal{B}), \\ D(\mathcal{B}) &= \left\{ y \in \mathcal{H}; \ y \in W_0^{1,2}([0, T]; H), \ \mathbb{P}\text{-a.s.}, \ \frac{dy}{dt} \in \mathcal{H} \right\}. \end{aligned} \tag{2.4}$$

Here, $W_0^{1,2}([0,T];H)$ denotes the space $\{y \in W^{1,2}([0,T];H);\ y(0)=0\}$, where $W^{1,2}([0,T];H)$ is the Sobolev space $\left\{y \in L^2(0,T;H),\ \frac{dy}{dt} \in L^2(0,T;H)\right\}$. We recall that $W^{1,2}([0,T];H) \subset AC([0,T];H)$, the space of all $H$-valued absolutely continuous functions on $[0,T]$.

Then we may rewrite equation (2.2) as

$$\mathcal{B}y + \mathcal{A}y \ni 0. \tag{2.5}$$

(If $A(t)$ is multivalued, we replace $A(t)(e^W(y+x))$ in (2.3) by $\{\eta(t);\ \eta(t) \in A(t)(e^{W(t)}(y(t)+x)),\ \text{a.e. } (t,\omega) \in (0,T)\times\Omega\}$.)

Consider the functions $\Phi : \mathcal{H} \to \overline{\mathbb{R}}$ defined by

$$\Phi(y) = \mathbb{E}\int_0^T (\varphi(t, e^{W(t)}(y(t)+x)) + \frac{\delta}{2}\,|e^{W(t)}(y(t)+x)|_H^2)dt,\ \forall y \in \mathcal{H}. \tag{2.6}$$

It is easily seen that $\Phi$ is convex, lower-semicontinuous and

$$\partial\Phi = \mathcal{A}. \tag{2.7}$$

As regards the operator $\mathcal{B}$, we have

**Lemma 2.1** *For each $y \in D(\mathcal{B})$ we have*

$$\begin{aligned} \langle \mathcal{B}y, y\rangle &= \frac{1}{2}\,\mathbb{E}|e^{W(T)}y(T)|_H^2 + (\nu+\delta)|y|_{\mathcal{H}}^2 \\ &\quad -\frac{1}{2}\,\mathbb{E}\int_0^T \sum_{j=1}^\infty |e^W y e_j|_H^2 \mu_j^2 dt \\ &\geq \frac{1}{2}\,\mathbb{E}|e^{W(T)}y(T)|_H^2 + \frac{\lambda}{2}\,|y|_{\mathcal{H}}^2. \end{aligned} \tag{2.8}$$

**Proof.** We have

$$\begin{aligned} \langle \mathcal{B}y, y\rangle &= \mathbb{E}\int_0^T \left(e^{W(t)}\frac{dy}{dt}\,(t), e^{W(t)}y(t)\right) dt \\ &\quad +\mathbb{E}\int_0^T ((\mu+\nu+\delta)e^W y, e^W y)dt. \end{aligned} \tag{2.9}$$

Taking into account that

$$d(e^W y) = e^W\,dy + e^W y\,dW + \mu e^W y\,dt,\ \forall y \in D(\mathcal{B}),$$

we get via Itô's formula that (see [6])

$$\begin{aligned}\frac{1}{2}\,d|e^W y|_H^2 &= \left(e^W\,\frac{dy}{dt}, e^W y\right)dt + (e^W y, e^W y\,dW) + (\mu e^W y, e^W y)dt\\ &\quad + \frac{1}{2}\sum_{j=1}^{\infty}\mu_j^2|e^W y e_j|_H^2 dt.\end{aligned}$$

Hence

$$\begin{aligned}\mathbb{E}\int_0^T\left(e^W\frac{dy}{dt}, e^W y\right)dt &= \frac{1}{2}\,\mathbb{E}|e^{W(T)}y(T)|_H^2 - \mathbb{E}\int_0^T(\mu e^W y, e^W y)dt\\ &\quad - \frac{1}{2}\,\mathbb{E}\int_0^T\sum_{j=1}^{\infty}|e^W y e_j|_H^2\mu_j^2\,dt,\end{aligned}$$

and so, because $\lambda > \nu$, by (1.9), (2.9), we get (2.8), as claimed. ∎

Consider now the conjugate $\Phi^* : \mathcal{H} \to \overline{\mathbb{R}}$ of functions $\Phi$, that is,

$$\Phi^*(z) = \sup\{\langle z, y\rangle_{\mathcal{H}} - \Phi(y);\ y \in \mathcal{H}\}.$$

By (2.6), we see that (see [19])

$$\Phi^*(z) = \mathbb{E}\int_0^T(\psi^*(t, e^{W(t)}z(t)) - (e^{W(t)}z(t), e^{W(t)}x))dt, \tag{2.10}$$

where $\psi^*$ is the conjugate of the function

$$\psi(t, y) = \varphi(t, y) + \frac{\delta}{2}\,|y|_H^2, \tag{2.11}$$

that is,

$$\psi^*(t, v) = \sup\{(v, y) - \varphi(t, y) - \frac{\delta}{2}\,|y|_H^2;\ y \in H\}. \tag{2.12}$$

We recall (see (1.8)) that

$$\Phi(y) + \Phi^*(u) \geq \langle y, u\rangle\,,\ \forall y, u \in \mathcal{H}, \tag{2.13}$$

with equality if and only if $u \in \partial\Phi(y)$. We infer that $y^*$ is *a solution to equation* (2.5) *if and only if*

$$\begin{aligned}y^* &= \arg\min_{(y,u)\in\mathcal{D}(\mathcal{B})\times\mathcal{H}}\{\Phi(y) + \Phi^*(u) - \langle y, u\rangle\,;\ \mathcal{B}y + u = 0\}\\ &= \arg\min_{(y,u)\in\mathcal{D}(\mathcal{B})\times\mathcal{H}}\{\Phi(y) + \Phi^*(u) + \langle\mathcal{B}y, y\rangle\,;\ \mathcal{B}y + u = 0\}\end{aligned} \tag{2.14}$$

and

$$\Phi(y^*) + \Phi^*(u^*) + \langle \mathcal{B}y^*, y^* \rangle = 0. \tag{2.15}$$

Taking into account (2.10) and recalling (2.6), (2.8), we have

$$\begin{aligned} y^* = \arg \min_{(y,u)\in\mathcal{D}(\mathcal{B})\times\mathcal{H}} \Bigg\{ &\mathbb{E} \int_0^T \Big( \varphi(t, e^{W(t)}(y(t)+x)) + \frac{\delta}{2}\, |e^{W(t)}(y(t)+x)|_H^2 \\ &+ \psi^*(t, e^{W(t)}u(t)) - (e^{W(t)}u(t), e^{W(t)}x) \\ &+ \eta(e^{W(t)}y(t)) \Big) dt + \frac{1}{2}\, \mathbb{E}|e^{W(T)}y(T)|_H^2;\ \mathcal{B}y + u = 0 \Bigg\}, \end{aligned} \tag{2.16}$$

where

$$\eta(z) = (\nu+\delta)|z|_H^2 - \frac{1}{2} \sum_{j=1}^{\infty} |ze_j|_H^2 \mu_j^2. \tag{2.17}$$

We note also that, by Itô's product rule, we have, for $u \in \mathcal{H}$, $y \in \mathcal{D}(\mathcal{B})$,

$$\begin{aligned} &-\mathbb{E} \int_0^T (e^{W(t)}u(t), e^{W(t)}x)dt \\ &\quad= \mathbb{E} \int_0^T \left( e^W x, e^W \left( \frac{dy}{dt} + (\mu+\nu+\delta)(y+x) \right) \right) dt \\ &\quad= \mathbb{E} \int_0^T (e^W x, d(e^W y)) - \int_0^T (e^W x, \mu e^W y - (\mu+\nu+\delta)(y+x)e^W)dt \\ &\quad= \mathbb{E} \int_0^T (e^W x, (\mu+\delta+\nu)(y+x)e^W)dt \\ &\quad+\mathbb{E} \int_0^T d(e^W x, e^W y) - \int_0^T \big( (e^W y, e^W(1+\mu)x) - (\mu e^W y, e^W x) \big)\, dt \\ &\quad= \mathbb{E}(e^{W(T)}x, e^{W(T)}y(T)) - \int_0^T \big( (e^W y, e^W(1+\mu)x) - (\mu e^W y, e^W x) \big)\, dt \\ &\quad+\mathbb{E} \int_0^T (e^W x, (\mu+\nu+\delta)(y+x)e^W)dt \\ &= \mathbb{E}(e^{W(T)}x, e^{W(T)}y(T)) + \mathbb{E} \int_0^T (e^W x, ((\mu+\nu+\delta)(y+x) - \mu y)e^W)dt \\ &\quad - \int_0^T (e^W y, e^W(1+\mu)x)dt \\ &\quad= \mathbb{E} \int_0^T (e^W((\nu+\delta)(y+x)+\mu x), e^W x)dt + \mathbb{E}(e^{W(T)}y(T), e^{W(T)}x) \\ &\quad - \int_0^T (e^W y, e^W(1+\mu)x)dt. \end{aligned}$$

Let $\mathcal{H}_0$ denote the set of all $u \in L^2([0,T] \times \Omega; H)$ which have an $(\mathcal{F}_t)_{t\geq 0}$-adapted version. We set, for $y \in \mathcal{H},\ u \in \mathcal{H}_0$,

$$\begin{aligned} G_1(y) = \ & \mathbb{E}\int_0^T \varphi(t, e^{W(t)}(y(t)+x))dt \\ & + \mathbb{E}\int_0^T ((e^{W(t)}((\nu+\delta)(y(t)+x)+\mu x), e^{W(t)}x) \\ & + \frac{\delta}{2}\,|e^{W(t)}(y(t)+x)|_H^2 + \eta(e^{W(t)}y(t)))dt \\ & + \frac{1}{2}\,\mathbb{E}|e^{W(T)}y(T)|_H^2 + \mathbb{E}(e^{W(T)}y(T), e^{W(T)}x) \\ & - \mathbb{E}\int_0^T (e^{W(t)}y(t), e^{W(t)}(1+\mu)x)dt, \end{aligned} \tag{2.18}$$

$$G_2(u) = \mathbb{E}\int_0^T \psi^*(t, u(t))dt, \tag{2.19}$$

where $\psi^*$ is given by (2.12).

By (2.16) it follows that $y^*$ is a solution to equation (2.5) *if and only if*

$$y^* = \arg \min_{(y,u)\in\mathcal{D}(\mathcal{B})\times\mathcal{H}_0} \{G_1(y) + G_2(u);\ e^W\mathcal{B}y + u = 0\} \tag{2.20}$$

*and*

$$G_1(y^*) + G_2(u^*) = 0. \tag{2.21}$$

It should be said, however, that under our assumptions the convex minimization problem (2.20) might have no solution $(y^*, u^*)$ because, in general, $G_2$ is not coercive on the space $\mathcal{H}$. ($G_2$ is, however, coercive if $\varphi$ is bounded on bounded sets of $H$. But such a condition is too restrictive for applications to PDEs.) So, we are led to replace (2.20) by a relaxed optimization problem to be defined below.

Let

$$\mathcal{X} = L^2(\Omega; (W^{1,2}([0,T]; H))'), \tag{2.22}$$

where $(W^{1,2}([0,T]; H))'$ is the dual space of $W^{1,2}([0,T]; H)$.

Define the operator $\widetilde{\mathcal{B}} : \mathcal{H} \times L^2(\Omega; H) \to \mathcal{X}$ by

$$
\begin{aligned}
(\widetilde{\mathcal{B}}(y,y_1))(\theta) = {} & \mathbb{E}(e^{W(T)}y_1,\theta(T)) + \mathbb{E}\int_0^T ((\nu+\delta)(y(t)+x) \\
& +\mu x)e^{W(t)},\theta(t))dt - \mathbb{E}\int_0^T \left(e^{W(t)}y(t), \frac{d\theta}{dt}(t)\right)dt, \\
& \forall \theta \in L^2(\Omega; W^{1,2}([0,T];H)).
\end{aligned} \tag{2.23}
$$

We note that $y_1(\omega) \in H$ can be viewed as the trace of $y(\omega)$ at $t = T$.

Indeed, if $y \in \mathcal{D}(\mathcal{B})$, we have via Itô's formula

$$
\begin{aligned}
\mathbb{E}\int_0^T (e^W \mathcal{B}y,\theta)dt = {} & \mathbb{E}\left(\int_0^T (d(e^W y),\theta) - \int_0^T (e^W \mu y,\theta)dt\right) \\
& +\mathbb{E}\int_0^T (e^W(\mu+\nu+\delta)(y+x),\theta)dt \\
= {} & \mathbb{E}(e^{W(T)}y(T),\theta(T)) \\
& +\mathbb{E}\int_0^T (e^W((y+x)(\nu+\delta)+\mu x),\theta)dt \\
& -\mathbb{E}\int_0^T \left(e^W y, \frac{d\theta}{dt}\right)dt,\ \forall \theta \in L^2(\Omega; W^{1,2}([0,T];H)).
\end{aligned}
$$

This means that $\widetilde{\mathcal{B}}(y,y(T)) = e^W \mathcal{B}y$, $\forall y \in \mathcal{D}(\mathcal{B})$. We set

$$
\begin{aligned}
\widetilde{G}_1(y,y_1) = {} & \mathbb{E}\int_0^T \varphi(t, e^{W(t)}(y(t)+x))dt \\
& +\mathbb{E}\int_0^T \Big((e^{W(t)}((\nu+\delta)(y(t)+x)+\mu x), e^{W(t)}x) \\
& +\frac{\delta}{2}\,|e^{W(t)}(y(t)+x)|_H^2 + \eta(e^{W(t)}y(t))\Big)dt \\
& -\mathbb{E}\int_0^T (e^{W(t)}y(t), e^{W(t)}(1+\mu)x)dt + \frac{1}{2}\,\mathbb{E}|e^{W(T)}y_1|_H^2 \\
& +\mathbb{E}(e^{W(T)}y_1, e^{W(T)}x), \quad \forall (y,y_1) \in \mathcal{H} \times L^2(\Omega;H)
\end{aligned} \tag{2.24}
$$

and note that $\widetilde{G}_1(y;y(T)) = G_1(y)$, $\forall y \in \mathcal{D}(\mathcal{B})$.

We note also that, if $y_n \in \mathcal{D}(\mathcal{B})$ such that $y_n \to y$ weakly in $\mathcal{H}$ and $y_n(T) \to y_1$ weakly in $L^2(\Omega;H)$, then

$$
e^W \mathcal{B}y_n \to \widetilde{\mathcal{B}}(y,y_1) \text{ weakly in } \mathcal{X}. \tag{2.25}
$$

Let $\overline{G} : \mathcal{H} \times L^2(\Omega; H) \times \mathcal{X} \to \overline{\mathbb{R}}$ be the lower semicontinuous closure of the function $G(y, y_1, u) = \widetilde{G}_1(y, y_1) + G_2(u)$ in $\mathcal{H} \times L^2(\Omega; H) \times \mathcal{X}$, on the set $\{(y, y_1, u) \in \mathcal{H} \times L^2(\Omega; H) \times \mathcal{X};\ e^W \mathcal{B}y + u = 0\}$, that is,

$$\begin{aligned}\overline{G}(y, y_1, u) = \liminf\{&G(z, z(T), u);\ z(T) \to y_1 \text{ in } L^2(\Omega; H),\\ &z \in \mathcal{D}(\mathcal{B}),\ (z, v) \to (y, u) \text{ in } \mathcal{H} \times \mathcal{X};\ e^W \mathcal{B}z + v = 0\}.\end{aligned} \tag{2.26}$$

(Here and everywhere in the following, by $\to$ we mean weak convergence.)

Taking into account that the function $\widetilde{G}_1$ is convex and lower semicontinuous in $\mathcal{H} \times L^2(\Omega; H)$, we have by (2.26)

$$\begin{aligned}\overline{G}(y, y_1, u) = \widetilde{G}_1(y, y_1) + \liminf\{&G_2(v); (z, v) \to (y, u) \text{ in } \mathcal{H} \times \mathcal{X},\\ &e^W \mathcal{B}z + v = 0\}.\end{aligned} \tag{2.27}$$

Now, we relax (2.20) to the convex minimization problem

(P) $\quad \mathrm{Min}\{\overline{G}(y, y_1, u);\ \widetilde{\mathcal{B}}(y, y_1) + u = 0;\ (y, y_1, u) \in \mathcal{H} \times L^2(\Omega; H) \times \mathcal{X}\}.$

We have

**Theorem 2.2** *Let $x \in H$. Then problem* (P) *has a unique solution* $(y^*, y_1^*, u^*) \in \mathcal{H} \times L^2(\Omega; H) \times \mathcal{X}$, *with* $u^* = -\widetilde{\mathcal{B}}(y^*, y_1^*)$. *Moreover,* $\varphi(\cdot, e^W(y^* + x)) \in L^1((0, T) \times \Omega)$.

**Proof.** Let $m$ be the infimum in (P) and let $(y_n, u_n) \in \mathcal{D}(\mathcal{B}) \times \mathcal{H}$ be such that

$$m \le G(y_n, y_n(T), u_n) \le m + \frac{1}{n}, \quad \forall n \in \mathbb{N}, \tag{2.28}$$

$$e^W \mathcal{B}y_n + u_n = 0. \tag{2.29}$$

Since, by assumption (iii), for some $C_1, C_2 \in ]0, \infty[$,

$$\widetilde{G}_1(y_n, y_n(T)) \ge C_1(|y_n|^2_{\mathcal{H}} + \mathbb{E}|e^{W(T)} y_n(T)|^2_H) - C_2,$$

we have along a subsequence

$$y_n \longrightarrow y^* \text{ weakly in } \mathcal{H}, \quad y_n(T) \to y_1^* \text{ weakly in } L^2(\Omega; H),$$

and so, by (2.25), we have

$$u_n \longrightarrow u^* = -\widetilde{\mathcal{B}}(y^*, y_1^*) \text{ weakly in } \mathcal{X}.$$

As $\overline{G}$ is weakly lower semicontinuous on $\mathcal{H} \times L^2(\Omega; H) \times \mathcal{X}$, we see by (2.28) that

$$\overline{G}(y^*, y_1^*, u^*) = m,$$

as claimed. The uniqueness of $(y^*, y_1^*, u^*)$ is immediate because the function $\overline{G}(\cdot, \cdot, u)$ is strictly convex on $\mathcal{H} \times L^2(\Omega; H)$ for all $u \in \mathcal{X}$. ∎

**Definition 2.3** A pair $(y^*, y_1^*)$ such that $(y^*, y_1^*, u^*) \in \mathcal{H} \times L^2(\Omega; H) \times \mathcal{X}$, $u^* = -\widetilde{B}(y^*, y_1^*)$, is a solution to problem (P), is called *the variational solution to equation* (2.2), and $X^* = e^W(y^* + x)$ is called the *variational solution to equation* (1.1).

The variational solution $X^* : (0, T) \to H$ is an $(\mathcal{F}_t)_{t \geq 0}$-adapted process.

Theorem 2.2 can be rephrased as:

**Theorem 2.4** *Under hypotheses* (i)–(iii)*, equation* (1.1) *has a unique variational solution* $X^* \in L^2((0, T) \times \Omega; H)$ *with* $\varphi(t, X^*) \in L^1((0, T) \times \Omega)$.

It should be noted that $y^*$ and $X^*$, as well, are not pathwise continuous on $[0, T]$. As seen later on, this happens, however, in some specific cases with respect to a weaker topology.

In the next section, we shall see how problem (P) looks like in a few important examples of stochastic PDEs.

**Remark 2.5** The above formulation of the variational solution $X^*$ is strongly dependent on the subdifferential form (1.3) of the operator $A(t)$. The extension of the above technique to a general maximal monotone function $A(t) : H \to H$ remains to be done using the Fitzpatrick formalism (see [20]).

# 3 Nonlinear parabolic stochastic differential equations

We consider here the stochastic differential equation

$$\begin{aligned} & dX - \mathrm{div}_\xi(a(t, \nabla X))dt + \lambda X\, dt = X\, dW \text{ in } (0, T) \times \mathcal{O}, \\ & X = 0 \text{ on } (0, T) \times \partial\mathcal{O}, \\ & X(0, \xi) = x(\xi),\ \xi \in \mathcal{O} \subset \mathbb{R}^d, \end{aligned} \tag{3.1}$$

where $x \in H$, $W$ is the Wiener process (1.2) in $H = L^2(\mathcal{O})$, $\mathcal{O}$ is a bounded and open subset of $\mathbb{R}^d$ with smooth boundary $\partial\mathcal{O}$, and $a : (0,T) \times \mathbb{R}^d \to \mathbb{R}^d$ is a nonlinear mapping of the form

$$a(t,z) = \partial_z j(t,z),\ \forall z \in \mathbb{R}^d,\ t \in [0,T], \tag{3.2}$$

where $j : (0,T) \times \mathbb{R}^d \to \mathbb{R}$ is measurable, convex, lower semicontinuous in $z$ and

$$\lim_{|z|\to\infty} \frac{j(t,z)}{|z|} = +\infty, \quad t \in [0,T], \tag{3.3}$$

$$\lim_{|v|\to\infty} \frac{j^*(t,v)}{|v|} = +\infty, \quad t \in [0,T], \tag{3.4}$$

uniformly with respect to $t \in [0,T]$.

We note that, if the function $(t,y) \to j(t,y)$ is bounded on bounded subsets of $[0,T]\times\mathbb{R}^d$, then (3.4) automatically holds by the conjugacy formula (1.8), that is,

$$j^*(t,v) \geq v \cdot z - j(t,z),\ \forall v,z \in \mathbb{R}^d,\ t \in [0,T].$$

It should be noted that equation (3.1) cannot be treated in the functional setting (1.11)-(1.12) which require polynomial growth and boundedness for $j(t,\cdot)$, while assumptions (3.3)–(3.4) allow nonlinear diffusions $a$ with slow growth to $+\infty$ as well as superlinear growth of the form

$$a(t,z) = a_0 \exp(a_1|z|^p \mathrm{sgn}\ z).$$

We note also that assumptions (3.2)–(3.4) do not preclude multivalued mappings $a$. Such an example is

$$\begin{aligned} j(t,z) &\equiv |z|(\log(|z|+1)), \\ a(t,z) &= \left(\log(|z|+1) + \frac{1}{|z|+1}\right) \mathrm{sign}\ z, \quad \forall z \in \mathbb{R}^d. \end{aligned}$$

By (2.1), one reduces equation (3.1) to the random parabolic differential equation

$$\begin{aligned} &\frac{\partial y}{\partial t} - e^{-W} \mathrm{div}_\xi a(t, \nabla(e^W(y+x)) + (\lambda+\mu)(y+x)) = 0 \\ &\hspace{18em} \text{in } (0,T)\times\mathcal{O}, \\ &y = 0 \text{ on } (0,T)\times\partial\mathcal{O}, \\ &y(0,\xi) = 0, \quad \xi \in \mathcal{O}. \end{aligned} \tag{3.5}$$

We are under the conditions of Section 2, where

$$\begin{array}{rcl} H &=& L^2(\mathcal{O}),\\ A(t)y &=& -\mathrm{div}_\xi a(t,\nabla y),\\ D(A(t)) &=& \{y\in W_0^{1,1}(\mathcal{O});\ \mathrm{div}_\xi a(t,\nabla y)\in L^2(\mathcal{O})\}\\ \varphi(t,y) &=& \displaystyle\int_{\mathcal{O}} j(t,\nabla y(\xi))d\xi. \end{array}$$

By (2.12), we have

$$\psi^*(t,v)=\int_{\mathcal{O}}(a(t,\nabla z)\cdot\nabla z-j(t,\nabla z)+\frac{\delta}{2}\,z^2)d\xi,\ \forall v\in L^2(\mathcal{O}), \tag{3.6}$$

where $z$ is the solution to the equation

$$\begin{array}{rcll} -\mathrm{div}\ a(t,\nabla z)+\delta z &=& v & \text{in } \mathcal{O},\\ z &=& 0 & \text{on } \partial\mathcal{O}, \end{array} \tag{3.7}$$

or, equivalently,

$$z=\arg\min_{\tilde z\in W_0^{1,1}(\mathcal{O})}\left\{\int_{\mathcal{O}} j(t,\nabla\tilde z)d\xi-\int_{\mathcal{O}} v\tilde z\,d\xi+\frac{\delta}{2}\int_{\mathcal{O}}\tilde z^2 d\xi\right\}. \tag{3.8}$$

By (3.3), it follows that (3.8) has, for each $v\in L^2(\mathcal{O})$ and $t\in[0,T]$, a unique solution $z\in W_0^{1,1}(\mathcal{O})$. In fact, as easily seen, by condition (3.3) it follows that the functional arising in the right side part of (3.8) is convex, lower semicontinuous and coercive on $W_0^{1,1}(\mathcal{O})$. By (2.24)), we have

$$\begin{aligned} \widetilde G_1(y,y_1)=\;&\mathbb{E}\int_0^T\int_{\mathcal{O}}(j(t,\nabla(e^{W(t)}(y(t)+x)))+\frac{\delta}{2}\,|e^{W(t)}(y(t)+x)|_H^2\\ &+e^{W(t)}((\nu+\delta)(y(t)+x)+\mu x)e^{W(t)}x)\,d\xi\,dt\\ &-\mathbb{E}\int_0^T(e^{W(t)}y(t),e^{W(t)}(1+\mu)x)dt\\ &+\mathbb{E}\int_0^T\eta(e^{W(t)}y(t))dt+\frac{1}{2}\,\mathbb{E}\int_{\mathcal{O}}|e^{W(T)}y_1(\xi)|^2d\xi\\ &+\mathbb{E}(e^{W(T)}y_1,e^{W(T)}x),\ (y,y_1)\in\mathcal{H}\times L^2(\Omega;H), \end{aligned} \tag{3.9}$$

where (see (2.17))

$$\eta(z)=(\nu+\delta)\int_{\mathcal{O}}|z|^2d\xi-\frac{1}{2}\sum_{j=1}^\infty\mu_j^2\int_{\mathcal{O}}|ze_j|^2d\xi. \tag{3.10}$$

By (2.19) and (3.6)–(3.7), we also have

$$\begin{aligned} G_2(u)=\mathbb{E}\int_0^T\int_{\mathcal{O}}&(a(t,\nabla z(t,\xi))\cdot\nabla z(t,\xi)) \\ &-j(t,\nabla z(t,\xi))+\frac{\delta}{2}\,z^2(t,\xi)d\xi\,dt,\ u\in\mathcal{H}, \end{aligned}\tag{3.11}$$

where $z(t,\omega)\in W_0^{1,1}(\mathcal{O})$ for $dt\otimes\mathbb{P}$-a.e., $(t,\omega)\in(0,T)\times\Omega$, is given by (see (3.7))

$$\begin{aligned} -\mathrm{div}\ a(t,\nabla z)+\delta z &= u \quad \text{in } \mathcal{O}, \\ z &= 0 \quad \text{on } \partial\mathcal{O}. \end{aligned}\tag{3.12}$$

Taking into account that $a(t,\nabla z)\cdot\nabla z\ge j(t,\nabla z)-j(t,0)$, we see by (3.3) and (3.12) that

$$z\in L^1((0,T)\times\Omega;W^{1,1}(\mathcal{O}))\cap L^2((0,T)\times\mathcal{O}\times\Omega).$$

Recalling (1.7)–(1.8), we have

$$a(t,\nabla z)\cdot\nabla z-j(t,\nabla z)=j^*(t,a(t,\nabla z))\text{ a.e. in }(0,T)\times\mathcal{O},$$

and this yields

$$G_2(u)=\mathbb{E}\int_0^T\int_{\mathcal{O}}(j^*(t,a(t,\nabla z(t,\xi)))+\frac{\delta}{2}\,z^2(t,\xi))d\xi\,dt. \tag{3.13}$$

By (3.4), it follows via the Dunford–Pettis weak compactness theorem in $L^1$ that every level set

$$\left\{v;\mathbb{E}\int_0^T\int_{\mathcal{O}}j^*(t,v(t,\xi))dx\,d\xi\le M\right\},\ M>0,$$

is weakly compact in the space $L^1((0,T)\times\mathcal{O}\times\Omega)$. By (3.12) and (3.13), we see that, if $G_2(u_n)\le M$, where $\{u_n\}\subset L^2((0,T)\times\mathcal{O}\times\Omega\}$ and $z_n$ is the solution to (3.12) with $u_n$ replacing $u$, then, by the Dunford–Pettis theorem, the sequence $\{a(t,\nabla z_n)\}$ is weakly compact in $L^1((0,T)\times\mathcal{O}\times\Omega\}$. Hence $\{u_n\}$ is weakly compact in $L^1((0,T)\times\Omega;W^{-1,\infty}(\mathcal{O}))$.

By (3.13), it follows also that $\{z_n\}$ is weakly compact in $L^2((0,T)\times\mathcal{O}\times\Omega)$. By (2.26), this means that, if $x\in L^2(\mathcal{O})$, then, for $(y,y_1,u)\in\mathcal{H}\times L^2(\Omega;H)\times\mathcal{X}$,

$$\begin{aligned} &\overline{G}(y,y_1,u) \\ &=\widetilde{G}_1(y,y_1)+\mathbb{E}\int_0^T\int_{\mathcal{O}}\left(j^*(t,a(t,\nabla z(t,\xi))+\frac{\delta}{2}\,z^2(t,\xi))\right)d\xi\,dt, \end{aligned}\tag{3.14}$$

where $z \in L^1((0,T)\times\Omega; W_0^{1,-1}(\mathcal{O})) \cap L^2((0,T)\times\mathcal{O}\times\Omega)$ is the solution to (3.12).

Let $(y_n,u_n)\in\mathcal{H}\times\mathcal{H}$ be such that $e^W\mathcal{B}y_n+u_n=0$ and $(y_n,u_n)\to(y,u)$ in $\mathcal{H}\times\mathcal{X}$, $y_n(T)\to y_1)$ in $L^2(\Omega;H)$. Since $\sup\limits_n\{G_1(y_n)\}<\infty$, by (3.3) and (3.9), it follows also that $\{\nabla(e^W(y_n+x)\}$ is weakly compact in $L^1((0,T)\times\mathcal{O}\times\Omega)$, and so $e^W(y+x)\in L^1((0,T)\times\mathcal{O};W_0^{1,1}(\mathcal{O}))$. Moreover, it follows that $\{\frac{dy_n}{dt}\}$ is weakly compact in $L^1((0,T)\times\Omega;W^{-1,\infty}(\mathcal{O}))$, and so $\frac{dy}{dt}\in L^1((0,T)\times\Omega;W^{-1,\infty}(\mathcal{O}))$. This implies that the equation $\widetilde{\mathcal{B}}(y^*,y_1^*)+u^*=0$ reduces to

$$\begin{aligned}&e^W\,\frac{dy^*}{dt}+e^W(\mu+\nu+\delta)(y^*+x)+u^*=0 \text{ in } \mathcal{D}'(0,T),\ \mathbb{P}\text{-a.s.},\\&y^*(0)=0,\quad y^*(T)=y_1^*.\end{aligned}$$

Hence, if $D(\overline{G}_1)=\{(y,y_1,u);\ \overline{G}_1(y,y_1,u)<\infty\}$, then we have

$$\begin{aligned}&D(\overline{G}_1)\subset\{(y,y_1,u)\in\mathcal{H}\times L^2(\Omega;H)\times\mathcal{H};\ e^Wy\in L^1((0,T)\times\Omega;W_0^{1,1}(\mathcal{O}));\\&\frac{dy}{dt}\in L^1((0,T)\times\Omega;W^{-1,\infty}(\mathcal{O}));\ u\in L^1((0,T)\times\Omega;W^{-1,\infty}(\mathcal{O})),\ y_1=y(T)\}.\end{aligned}$$

This means that, in this case, problem (P) can be rewritten as

$$\text{Min}\left\{\begin{array}{l}\overline{G}(y,y(T),u);\ y\in L^2((0,T)\times\mathcal{O}\times\Omega)\cap\mathcal{H},\\ e^W(y+x)\in L^1((0,T)\times\Omega;W_0^{1,1}(\mathcal{O})),\\ \dfrac{dy}{dt}\in L^1((0,T)\times\Omega;W^{-1,\infty}(\mathcal{O})),\\ u\in L^1((0,T)\times\Omega;W^{-1\infty}(\mathcal{O}))\cap\mathcal{X};\\ \text{subject to}\\ \dfrac{dy}{dt}+(\mu+\nu+\delta)(y+x)+e^{-W}u=0 \text{ on } (0,T);\ y(0)=0\end{array}\right\}, \tag{3.15}$$

where $\overline{G}_1$ is defined by (3.14). By Theorem 2.2, there is a unique solution $(y^*,u^*)$ to (3.15). Taking into account that $u^*\in L^1((0,T)\times\Omega;W^{-1,\infty}(\mathcal{O}))$ and that

$$y^*(t)=-\int_0^t e^{-W}u^*(s)ds-\int_0^t(\mu+\nu+\delta)(y^*(s)+x)ds,\ \forall t\in(0,T),$$

we infer that the process $t\to y^*(t)$ in pathwise $W^{-1,\infty}(\mathcal{O})$ continuous on$(0,T)$. By Theorem 2.4, we have, therefore,

**Theorem 3.1** *Assume that $x \in L^2(\mathcal{O})$ and that conditions* (3.2)–(3.4) *hold. Then, equation* (3.1) *has a unique variational solution*

$$X^* \in L^2((0,T) \times \mathcal{O} \times \Omega),\ \ e^W X^* \in L^1((0,T) \times \Omega; W_0^{1,1}(\mathcal{O})). \tag{3.16}$$

*Moreover, the process $t \to X^*(t)$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted and pathwise $W^{-1,\infty}(\mathcal{O})$-valued continuous on $(0,T)$.*

**The total variation flow**

The stochastic differential equation

$$\begin{aligned} & dX - \operatorname{div}\left(\frac{\nabla X}{|\nabla X|_d}\right) dt + \lambda X\, dt = X\, dW \text{ in } (0,T) \times \mathcal{O}, \\ & X(0) = x \text{ in } \mathcal{O}, \\ & X = 0 \text{ on } (0,T) \times \partial\mathcal{O} \end{aligned} \tag{3.17}$$

with $x \in L^2(\mathcal{O})$ is the equation of stochastic variational flow in $\mathcal{O} \subset \mathbb{R}^d$, $1 \leq d \leq 3$. The existence and uniqueness of a generalized solution to (3.17) $X : [0,T] \to BV(\mathcal{O})$ was established in [9] by using some specific approximation techniques. We shall treat now equation (3.17) in the framework of variational solution developed above in the space $H = L^2(\mathcal{O})$ with the norm $|\cdot|_H = |\cdot|_2$ and the scalar product $(\cdot,\cdot)$, and $\varphi : L^2(\mathcal{O}) \to \overline{\mathbb{R}}$ defined by

$$\varphi(y) = \begin{cases} \|Dy\| + \displaystyle\int_{\partial\mathcal{O}} |\gamma_0(y)| d\mathcal{H}^{d-1}, & y \in BV(\mathcal{O}) \setminus L^2(\mathcal{O}), \\ +\infty & \text{otherwise.} \end{cases}$$

Here, $BV(\mathcal{O})$ is the space of functions with bounded variation and $\|Dy\|$ is the total variation of $y \in BV(\mathcal{O})$. (See, e.g., [9].) Then, with the notations of Section 2, we have $Ay = \partial\varphi(y)$, where $\partial\varphi : L^2(\mathcal{O}) \to L^2(\mathcal{O})$ is the subdifferential of $\varphi$ and (see (2.2), (2.18))

$$\begin{aligned} & \frac{\partial y}{\partial t} + e^{-W} A(e^W(y+x)) + \mu(y+x) = 0 \text{ in } (0,T) \times \mathcal{O}, \\ & y(0,\xi) = 0, \quad \xi \in \mathcal{O}, \\ & y = 0 \text{ on } (0,T) \times \partial\mathcal{O}. \end{aligned} \tag{3.18}$$

The function $\widetilde{G}_1$ is given, in this case, by

$$
\begin{aligned}
\widetilde{G}_1(y,y_1) = \mathbb{E}\int_0^T & \Big(\varphi(e^{W(t)}(y(t)+x)) + \frac{\delta}{2}\,|e^{W(t)}(y(t)+x)|_2^2 \\
& + (e^{W(t)}((\nu+\delta)(y(t)+x)+\mu x), e^{W(t)}x)\Big)dt \\
& - \mathbb{E}\int_0^T (e^{W(t)}y(t), e^{W(t)}(1+\mu)x)dt \\
& + \mathbb{E}\int_0^T \eta(e^{W(t)}y(t))dt + \frac{1}{2}\,\mathbb{E}|e^{W(T)}y_1|_2^2 \\
& + \mathbb{E}(e^{W(T)}y_1, e^{W(T)}x),\ (y,y_1)\in\mathcal{H}\times L^2(\Omega;H),
\end{aligned}
\tag{3.19}
$$

where $\eta$ is given by (3.10). We have also (see (2.11), (2.12), (3.6))

$$
\begin{aligned}
\psi(y) &= \varphi(y) + \frac{\delta}{2}\,|y|_2^2, && \forall y\in D(\varphi),\\
\psi^*(v) &= (v,\theta) - \varphi(\theta) - \frac{\delta}{2}\,|\theta|_2^2, && v\in\partial\varphi(\theta)+\delta\theta.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\psi^*(v) &= \frac{\delta}{2}\,|\theta|^2 + (\partial\varphi(z),\theta) - \varphi(\theta)\\
&= \frac{\delta}{2}\,|(\delta I+\partial\varphi)^{-1}v|_2^2 + \varphi^*(v-(\delta I+\partial\varphi)^{-1}v)
\end{aligned}
$$

and, therefore, by (2.19),

$$
G_2(u) = \mathbb{E}\int_0^T \left(\frac{\delta}{2}\,|(\delta I+\partial\varphi)^{-1}(u)|_2^2 + \varphi^*(u-(\delta I+\partial\varphi)^{-1}(u))\right)dt,
$$

where $\varphi^* : L^2(\mathcal{O})\to\overline{\mathbb{R}}$ is the conjugate of the function $\varphi$. This yields

$$
\begin{aligned}
\overline{G}(y,y_1,u) = G_1(y,y_1) + \liminf_{\substack{(z,v)\to(y,u)\\ \text{in }\ \mathcal{H}\times\mathcal{X}}} & \Big\{\mathbb{E}\int_0^T \Big(\frac{\delta}{2}\,|(I+\partial\varphi)^{-1}(v(t))|_2^2 \\
& + \varphi^*(v-(\delta I+\partial\varphi)^{-1}(v(t)))\Big)dt,\ e^W\mathcal{B}z+v=0\Big\},
\end{aligned}
\tag{3.20}
$$

where the space $\mathcal{X}$ is defined by (2.22).

By definition, the solution $(y^*,y_1^*)$ to the minimization problem

$$
\mathrm{Min}\{\overline{G}(y,y_1,u);\ \mathcal{B}(y,y_1)+u=0,\ (y,y_1,u)\in\mathcal{H}\times L^2(\Omega;H)\times\mathcal{X}\} \tag{3.21}
$$

is the *variational solution* to the random differential equation (3.18).

Denote by $V^*$ the dual of the space $V = BV(\mathcal{O}) \cap L^2(\mathcal{O})$. We note that $\varphi^*$ can be extended as a convex lower semicontinuous convex function on $F^*$, and we also have

$$\frac{\varphi^*(u)}{\|u\|_{V^*}} \longrightarrow +\infty \quad \text{as} \quad \|u\|_{V^*} \longrightarrow +\infty.$$

Then, if $(z_n, y_n) \in \mathcal{H} \times \mathcal{H}$ is convergent to $(y, u) \in \mathcal{H} \times \mathcal{X}$, it follows by the Dunford-Pettis compactness criterium (see [12]) that $\{v_n\}$ is weakly compact in $L^1((0,T) \times \Omega; V^*)$. This implies that

$$D(\overline{G}) \subset L^1((0,T) \times \Omega; BV(\mathcal{O})) \times L^2(\Omega; H) \times L^1((0,T) \times \Omega; V^*),$$

and so, in particular, it follows that

$$y \in W^{1,1}([0,T]; V^*),\ \mathbb{P}\text{-a.s.}$$

We have, therefore,

**Theorem 3.2** *Let $x \in BV(\mathcal{O}) \cap L^2(\mathcal{O})$. Then equation* (3.17) *has a unique variational solution $X = e^W(y + x)$ which is $V^*$-valued pathwise continuous and satisfies*

$$\begin{aligned} \varphi(X) &\in L^1((0,T) \times \Omega), && (3.22)\\ X &\in L^2((0,T) \times \mathcal{O} \times \Omega),\ AX \in L^1((0,T) \times \Omega; V^*), && (3.23)\\ e^{-W}X &\in W^{1,1}([0,T]; V^*),\ \mathbb{P}\text{-}a.s. && (3.24)\end{aligned}$$

In [9], it was proved the existence and uniqueness of a generalized solution $X$, also called *the variational solution*, which was obtained as limit $X^* = \lim\limits_{\varepsilon \to 0} X_\varepsilon$ in $L^2(\Omega; C((0,T); L^2(\mathcal{O})))$, where $X_\varepsilon$ is the solution to the approximating equation

$$\begin{aligned} &dX_\varepsilon - \text{div } a_\varepsilon(\nabla X_\varepsilon)dt + \lambda X_\varepsilon = X_\varepsilon dW \text{ in } (0,T) \times \mathcal{O},\\ &X_\varepsilon(0) = x, \quad X_\varepsilon = 0 \text{ on } (0,T) \times \mathcal{O}, \end{aligned} \tag{3.25}$$

where $a_\varepsilon = \nabla j_\varepsilon$ and $j_\varepsilon$ is the Moreau–Yosida approximation of the function $r \to |r|_d$. Since, as strong solution to (3.25), $X_\varepsilon$ is also a variational solution to this equation in sense of Definition 2.3, it is clear by the structural stability of convex minimization problems that, for $\varepsilon \to 0$, we have also $X_\varepsilon \to X$, where $X$ is the variational solution given by Theorem 3.2.

We may infer, therefore, that *the function* $X$ *given by Theorem* 3.2 *is just the generalized solution of* (3.17) *given by Theorem* 3.1 in [9]. In particular, this implies that $X$ is $L^2(\mathcal{O})$-valued pathwise continuous.

In [4], it is developed a direct variational approach to (3.17), which leads via first order conditions of optimality to sharper results. (On these lines, see also [14].)

**Stochastic porous media equations**

Consider the equation

$$\begin{aligned}
&dX - \Delta\beta(X)dt + \lambda X\,dt = X\,dW \text{ in } (0,T)\times\mathcal{O},\\
&X = 0 \text{ on } (0,T)\times\partial\mathcal{O},\\
&X(0,\xi) = x(\xi),\quad \xi\in\mathcal{O},
\end{aligned}\tag{3.26}$$

where $\mathcal{O}$ is a bounded and open domain of $\mathbb{R}^d$, $d \geq 1$, $\lambda > 0$, $W$ is a Wiener process in $H = H^{-1}(\mathcal{O})$ of the form (1.2) and $\beta$ is a continuous and monotonically nondecreasing function such that $\beta(0) = 0$ and

$$\lim_{|r|\to\infty} \frac{j(r)}{|r|} = +\infty. \tag{3.27}$$

In this case,

$$\begin{aligned}
H &= H^{-1}(\mathcal{O}),\\
Ay &= -\Delta\beta(y),\\
D(A) &= \{y \in H^{-1}(\mathcal{O}) \cap L^1(\mathcal{O}),\ \beta(y) \in H_0^1(\mathcal{O})\} \text{ and}\\
A &= \partial\varphi,\ \text{where } \varphi(y) = \int_{\mathcal{O}} j(y(\xi))d\xi.
\end{aligned}$$

By (2.11), we have also

$$\psi^*(v) = \int_{\mathcal{O}} j^*(\beta(\theta))d\xi + \frac{\delta}{2}\,|\theta|_{-1}^2,\ v \in L^2(\mathcal{O}),$$

where $\theta \in H^{-1}(\mathcal{O}) \cap L^1(\mathcal{O})$,

$$\begin{aligned}
2\delta\theta - \Delta\beta(\theta) &= v \quad \text{in } \mathcal{O},\\
\theta &= 0 \quad \text{on } \partial\mathcal{O},
\end{aligned}$$

and $|\cdot|_{-1}$ is the norm of $H^{-1}(\mathcal{O})$. Then we have

$$\begin{aligned}
\widetilde{G}_1(y,y_1) = \mathbb{E}\int_0^T &\left(\int_{\mathcal{O}} j(e^{W(t)}(y(t)+x))d\xi + \frac{\delta}{2}\,|e^{W(t)}(y(t)+x)|_{-1}^2\right) d\xi\, dt\\
&+\mathbb{E}\int_0^T\int_{\mathcal{O}} e^{W(t)}((\nu+\delta)(y(t)+x)+\mu x)e^{W(t)}x\, d\xi\, dt\\
&-\mathbb{E}\int_0^T (e^{W(t)}y(t), e^{W(t)}(1+\mu)x)dt\\
&+\mathbb{E}\int_0^T \eta(e^W y)dt + \frac{1}{2}\,\mathbb{E}|e^{W(T)}y(T)|_{-1}^2\\
&+(e^{W(T)}y_1, e^{W(T)}x)_{-1},
\end{aligned}$$

while

$$G_2(u) = \mathbb{E}\int_0^T \left(\int_{\mathcal{O}} j^*(\beta(\widetilde{z}))d\xi + \frac{\delta}{2}\,|\widetilde{z}(t)|_{-1}^2\right)dt,$$

where

$$\begin{aligned}
\delta\widetilde{z} - \Delta\beta(\widetilde{z}) &= u \quad \text{in } \mathcal{O},\\
\widetilde{z} &= 0 \quad \text{on } \partial\mathcal{O}.
\end{aligned} \tag{3.28}$$

(Here, $(\cdot,\cdot)_{-1}$ is the scalar product of $H^{-1}(\mathcal{O})$.)

Taking into account that $\frac{j^*(r)}{|r|} \to +\infty$ as $|r|\to\infty$, it follows, as in the previous case, for each $M>0$, the set

$$\left\{\beta(\widetilde{z});\ \mathbb{E}\int_0^T\int_{\mathcal{O}} j^*(\beta(\widetilde{z}))dt\, d\xi \le M\right\}$$

is weakly compact in $L^1((0,T)\times\mathcal{O}\times\Omega)$, we infer that

$$\overline{G}(y,y_1,u) = \widetilde{G}_1(y,y_1) + \mathbb{E}\int_0^T \left(\int_{\mathcal{O}} j^*(\beta(\widetilde{z}))d\xi + \frac{\delta}{2}\,|\widetilde{z}(t)|_{-1}^2\right)dt, \tag{3.29}$$

where $\widetilde{z}$ is the solution to (3.28). This implies that

$$D(\overline{G}) \subset \{(y,y_1,u)\in\mathcal{H}\times L^2(\Omega;H)\times\mathcal{X};\ u\in L^1((0,T)\times\Omega;\mathcal{Z})\}.$$

Here $\mathcal{Z} = (-\Delta)^{-1}(L^1(\mathcal{O}))\subset W_0^{1,p}(\mathcal{O}),\ 1\le p<\frac{d}{d-1}$, where $\Delta$ is the Laplace operator with homogeneous Dirichlet conditions and

$$D(\overline{G}) = \{(y,y_1,u);\ \overline{G}(y,y_1,u)<\infty\}.$$

We define, as above, the solution to (3.26) as $X^* = e^W y^*$, where $(y^*, y_1^*, u^*)$ is the solution to the minimization problem

$$\begin{aligned}\mathrm{Min}\Big\{&\overline{G}(y,y_1,u); \frac{dy}{dt}+(\mu+\nu+\delta)(y+x)+e^{-W}u=0,\ y(0)=0,\\ &\qquad y(T)=y_1,\ (y,y_1,u)\in\mathcal{H}\times L^2(\Omega;H)\times\mathcal{X}\Big\}.\end{aligned} \tag{3.30}$$

(Here, $\frac{dy}{dt}$ is taken in sense of distributions, i.e., in $\mathcal{D}'(0,T;H)$.) We have, therefore,

**Theorem 3.3** *Assume that $x \in L^2(\mathcal{O})$. Then equation (3.26) has a unique variational solution $X^*$,*

$$\begin{gathered}X^*\in L^2((0,T)\times\mathcal{O}\times\Omega);\ \varphi(X^*)\in L^1((0,T)\times\mathcal{O}\times\Omega),\\ e^{-W}X\in W^{1,1}([0,T];W_0^{1,1}(\mathcal{O})),\ \mathbb{P}\text{-}a.s.\end{gathered}$$

*Moreover, the process $t \to X^*(t)$ is pathwise $W_0^{1,1}(\mathcal{O})$-valued continuous on $(0,T)$.*

**Remark 3.4** A different treatment of equation (3.26) under the general assumptions (3.27) was developed in [7] (see also [8], Ch. 5).

# 4 Stochastic variational inequalities

Consider the stochastic differential equation

$$\begin{aligned}&dX + A_0X\,dt + N_K(X)dt + \lambda X\,dt \ni X\,dW,\ t\in(0,T),\\ &X(0)=x,\end{aligned} \tag{4.1}$$

in a real Hilbert space $H$ with the scalar product $(\cdot,\cdot)$ and the norm $|\cdot|$. Assume that $x \in H$ and

- (j) *$A_0 : D(A_0) \subset H \to H$ is a linear self-adjoint, positive definite operator in $H$.*
- (jj) *$W$ is the Wiener process (1.2) and $\lambda > \nu$.*
- (jjj) *$K$ is a closed, convex subset of $H$ such that $0 \in K$, $(I+\lambda A_0)^{-1}K \subset K$, $\forall\lambda > 0$.*

Here, $N_K : H \to 2^H$ is the normal cone to $K$, that is,

$$N_K(u) = \{\eta \in H;\ (\eta, u - v) \geq 0,\ \forall v \in K\}. \tag{4.2}$$

By the transformation (2.1), equation (4.1) reduces to the nonlinear random differential equation

$$\begin{aligned} &\frac{dy}{dt} + e^{-W} A_0(e^W(y+x)) + e^{-W} N_K(e^W(y+x)) + \mu(y+x) = 0, \\ &\hspace{20em} t \in (0,T), \\ &y(0) = 0. \end{aligned} \tag{4.3}$$

(We note that, if $W(t) = \sum\limits_{j=1}^{N} \mu_j \beta_j(t)$, then (4.3) reduces to a deterministic variational inequality.)

To represent this problem as an optimization problem of the form (P), we set

$$\varphi(u) = \frac{1}{2}\,(A_0 u, u) + I_K(u),\ \forall u \in H,$$

where $I_K$ is the indicator function

$$I_K(u) = \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{otherwise.} \end{cases}$$

The function $\varphi : H \to ]-\infty, +\infty]$ is convex and lower semicontinuous. Then, by (2.6), (2.18), (2.19), we have

$$\begin{aligned} \widetilde{G}_1(y, y_1) \ &= \ \mathbb{E} \int_0^T \bigg( \frac{1}{2}\,(A_0(e^{W(t)}(y(t)+x)), e^{W(t)}(y(t)+x)) \qquad (4.4) \\ &\quad + e^{2W}((\nu+\delta)(y+x) + \mu y)x + \frac{\delta}{2}\,|e^{W(t)}(y(t)+x)|_H^2 \\ &\quad + I_K(e^{W(t)}(y(t)+x)) + \eta(e^{W(t)}y(t)) \bigg) dt \\ &\quad - \mathbb{E} \int_0^T (e^W y, e^W(1+\mu)y)dt \qquad (4.5) \\ &\quad + \frac{1}{2}\,\mathbb{E}|e^{W(T)}y_1|_H^2 + \mathbb{E}(e^{W(T)}y_1, e^{W(T)}x), \\ G_2(u) \ &= \ \mathbb{E} \int_0^T \psi^*(u(t))dt, \qquad (4.6) \end{aligned}$$

where, by (2.11)-(2.12), we have

$$\begin{aligned}\psi^*(e^W u) &= \sup\left\{(e^W u, v) - \frac{1}{2}\,(A_0 v, v) - \frac{\delta}{2}\,|v|^2;\ v \in K\right\}, \\ &= (e^W u, z) - \frac{1}{2}\,(A_0 z, z) - \frac{\delta}{2}|z|^2,\end{aligned} \tag{4.7}$$

where $A_0 z+\delta z+N_K(z) \ni e^W u$. (We note that, by (iii), $z$ is uniquely defined.)

By (4.4)-(4.6), we see that

$$\overline{G}(y, y_1, u) = \widetilde{G}_1(y, y_1) + \frac{1}{2}\ \liminf_{n\to\infty} \mathbb{E} \int_0^T \left((A_0 z_n, z_n) + \frac{\delta}{2}\,|z_n|^2\right) dt,$$

where

$$\begin{aligned}&A_0 z_n + \delta z_n + N_K(z_n) \ni u_n,\ e^W B y_n + u_n = 0, \\ &y_n \to y \text{ in } \mathcal{H},\ y_n(T) \to y_1 \text{ in } L^2(\Omega; H),\ u_n \to u \in \mathcal{X}.\end{aligned} \tag{4.8}$$

This yields

$$\mathbb{E}\int_0^T |A_0^{\frac{1}{2}} z_n|^2 dt \le C < \infty,\ \forall n \in \mathbb{N}, \tag{4.9}$$

and, therefore, we have

$$\begin{aligned}\overline{G}(y, y_1, u) &= \widetilde{G}_1(y, y_1) + \frac{1}{2}\,\mathbb{E}\int_0^T (|A_0^{\frac{1}{2}} z|^2 + \delta|z|^2) dt, \\ z &= w - \lim_{n\to\infty} z_n \text{ in } L^2((0,T)\times\Omega; V),\end{aligned} \tag{4.10}$$

where $V = D(A_0^{\frac{1}{2}})$. We note that $D(\widetilde{G}_1) \subset L^2((0,T)\times\Omega; V)$.

We may conclude, therefore, by Theorem 2.4 that

**Theorem 4.1** *Under hypotheses* (j)–(jjj)*, there is a unique variational solution $X^*(t) \in K$, a.e. $t \in (0,T)$, $X^* \in L^2((0,T)\times V; \Omega)$ to equation* (4.1).

More insight into the problem can be gained in the following two special cases.

**Stochastic parabolic variational inequalities**

The stochastic differential equation

$$\begin{aligned}&dX - \Delta X\, dt + \lambda X\, dt + N_K(X) dt \ni X\, dW \text{ in } (0,T)\times\mathcal{O}, \\ &X(0) = x \text{ in } \mathcal{O}, \\ &X = 0 \text{ on } (0,T)\times\partial\mathcal{O},\end{aligned} \tag{4.11}$$

where $N_K(X) \subset L^2(\mathcal{O})$ is the normal cone to the closed convex set $K$ of $L^2(\mathcal{O})$,

$$K=\{z\in L^2(\mathcal{O});\ z\geq 0,\ \text{a.e. in } \mathcal{O}\},\ \alpha\in\mathbb{R},$$

can be treated following the above infinite-dimensional scheme in the space $H=L^2(\mathcal{O})$, where $A_0u=-\Delta u$, $u\in D(A_0)=H_0^1(\mathcal{O})\cap H^2(\mathcal{O})$.

Then the variational solution to (4.11) is defined by $X=e^W y$, where $y$ is given by (**??**) and $\overline{G}$ is given by

$$\overline{G}(y,y_1,u)=\widetilde{G}_1(y,y_1)+\frac{1}{2}\,\mathbb{E}\int_0^T\int_{\mathcal{O}}(|\nabla z^2|^2+\delta|z|^2)d\xi\,dt,$$

where $G_1$ is defined y (4.4) and $z=w-\lim\limits_{n\to\infty} z_n$ in $L^2((0,T)\times\Omega;H_0^1(\mathcal{O}))$,

$$\begin{gathered}-\Delta z_n+\delta z_n+\eta_n=u_n,\ e^W\mathcal{B}y_n+y_n=0,\\ \eta_n\in N_K(z_n),\ \mathbb{E}\int_0^T\int_{\mathcal{O}}|\nabla z_n|^2d\xi\,dt\leq C,\ \forall n.\end{gathered}\tag{4.12}$$

Since $u_n\to u$ in $\mathcal{D}'(0,T;L^2(\mathcal{O}))$ and $\eta_n(t,\xi)\leq 0$ a.e. $(t,\xi)\in(0,T)\times\mathcal{O}$, by (4.12), we infer that

$$-\Delta z+\delta z+\eta=u \text{ in } \mathcal{D}'((0,T)\times\mathcal{O}),$$

where $\eta,u$ are in $\mathcal{M}((0,T)\times\mathcal{O})$ the space of bounded measures on $(0,T)\times\mathcal{O}$. If we denote by $\eta_a,u_a\in L^1((0,T)\times\mathcal{O})$ the absolutely continuous parts of $\eta$ and $u$, we get

$$\begin{gathered}-\Delta z+\delta z+\eta_a=u_a \text{ in } L^1(\mathcal{O}),\\ z\in H_0^1(\mathcal{O}) \text{ and } \eta_a(t,\xi)=0,\ \text{a.e. on } [z(t,\xi)>0]\\ \eta_a(t,\xi)\geq 0, \text{ a.e. on } [z(t,\xi)=0].\end{gathered}$$

Then the process $X=e^W(y+x)$ is the variational solution to (4.11) and so, by Theorem 4.1, we have

**Corollary 4.2** *There is a unique variational solution* $X\in L^2((0,T)\times\Omega;H_0^1(\mathcal{O}))$, $X\geq 0$, *a.e. on* $(0,T)\times\Omega$.

### Finite dimensional stochastic variational inequalities

Consider equation (4.1) in the special case $K\subset\mathbb{R}^d$, int $K\neq\emptyset$, $0\in K$, $W=\sum\limits_{i=1}^N\mu_i\beta_i$ and $A_0\in L(\mathbb{R}^d,\mathbb{R}^d)$, $A_0=A_0^*$. Then, as easily seen by (2.13), we have

$$\psi^*(u) \geq \alpha_1|u| - \alpha_2,\ \forall u \in \mathbb{R}^d. \tag{4.13}$$

Let $z_n$ be the solution to (see (4.8))

$$A_0 z_n + \delta z_n + N_K(z_n) \ni u_n. \tag{4.14}$$

Since, by (4.13)-(4.14), the sequence $\{u_n\}$ is bounded in $L^1((0,T)\times\Omega,\mathbb{R}^d)$, it follows that it is weak-star compact in $\mathcal{M}(0,T;\mathbb{R}^d)$, $\forall\varepsilon > 0$, and so $u \in \mathcal{M}(0,T;\mathbb{R}^d)$. (Here, $\mathcal{M}(0,T;\mathbb{R}^d)$ is the space of $\mathbb{R}^d$-valued bounded measures on $(0,T)$. Letting $n \to \infty$ in (4.14), we get

$$A_0 z + \delta z + \zeta = u, \tag{4.15}$$

where $u \in \mathcal{M}(0,T;\mathbb{R}^d)$, $\forall\varepsilon > 0$, and $\zeta \in \mathcal{M}((0,T);\mathbb{R}^d)$, $\mathbb{P}$-a.s. By the Lebesgue decomposition theorem, we have

$$\begin{aligned} \zeta &= \zeta_a + \zeta_s, \quad \zeta_a \in L^1(0,T;\mathbb{R}^d),\\ u &= u_a + u_s, \quad u_a \in L^1(0,T;\mathbb{R}^d), \end{aligned}$$

where $u_s$ and $\zeta_s$ are singular measures and $\zeta_a \in N_K(z)$. Hence, by (4.15), we have

$$z = (A_0 + \delta I + N_K)^{-1}(u_a) = F(u_a),\ \zeta_s = u_s. \tag{4.16}$$

As a matter of fact, the singular measure $\zeta_s$ belongs to the normal cone $N_{\mathcal{K}}(z) \subset \mathcal{M}(0,T;\mathbb{R}^d)$ to the set $\mathcal{K} = \{\widetilde{z} \in C([0,T];\mathbb{R}^d); \widetilde{z}(t) \in K, \forall t \in [0,T]\}$ and it is concentrated on the set of $t$-values for which $z(t)$ defined by (4.16) lies on the boundary $\partial K$ of $K$.

By (2.22)-(2.23), we have

$$\overline{G}(y,y_1,u) = \widetilde{G}_1(y,y_1) + \mathbb{E}\int_0^T \left(\frac{1}{2}(A_0F(u_a),F(u_a)) + \frac{\delta}{2}\,|F(u_a)|^2\right)dt, \tag{4.17}$$

where $\widetilde{G}_1$ is given by (4.4) and $y \in \mathcal{H}$ is solution to the equation

$$\begin{aligned} &y = y_a + y_s,\ y_a \in AC([0,T];\mathbb{R}^d),\ y_s \in BV([0,T];\mathbb{R}^d),\ \mathbb{P}\text{-a.s.},\\ &\frac{dy_a}{dt} + (\mu+\nu+\delta)(y_a + x) + e^{-W}u_a = 0,\ \text{a.e. on } (0,T),\\ &y_a(0) = 0,\\ &\frac{dy_s}{dt} + e^{-W}u_s = 0 \text{ in } \mathcal{D}'(0,T;\mathbb{R}^d), \end{aligned} \tag{4.18}$$

where $BV([0,T];\mathbb{R}^d)$ is the space of functions with founded variations on $[0,T]$. We note that, by (4.17), it follows also that

$$D(\overline{G}) \subset \{(y,y_1,u) \in \mathcal{H} \times L^2(\Omega;H) \times \mathcal{X};\ y \in BV([0,T];\mathbb{R}^d),\ \mathbb{P}\text{-a.s.}, \\ F(u_a) \in L^2((0,T) \times \Omega \times \mathbb{R}^d)\},$$

where $D(\overline{G}) = \{(y,y_1,u);\ G(y,y_1,u) < \infty\}$. We have, therefore,

**Theorem 4.3** *The minimization problem*

$$\mathrm{Min}\{\overline{G}(y,y_1,u);\ (y,y_1,u) \in \mathcal{H} \times L^2(\Omega;\mathbb{R}^d) \times \mathcal{X},\ \textit{subject to } (4.18)\} \tag{4.19}$$

*has a unique solution* $(y^*,y_1^*) \in \mathcal{H} \times L^2(\Omega;\mathbb{R}^d)$ *satisfying* (4.18). *The process* $X^* = e^W y^*$ *is the solution to the variational solution to* (4.17).

**Remark 4.4** Since $y^* \in BV([0,T];\mathbb{R}^d)$ and, as seen by (4.18), the singular measure $\zeta_s = u_s \neq 0$, it follows that the process $X^*$ is not pathwise continuous on $[0,T]$. However, by the Lebesgue decomposition, we have, $\mathbb{P}$-a.s., $X^*(t) = X_a^*(t) + X_1^*(t) + X_2^*(t)$, $\forall t \in [0,T]$, where $t \to X_a^*(t)e^{-W(t)}$ is absolutely continuous, $X_1^*$ is a jump function and $X_2^*$ is a singular function, that is, $X_2^* = e^W y_2$, where $\frac{dy_2}{dt} = 0$. a.e.

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# References

[1] Barbu, V., *Nonlinear Differential Equations of Monotone Type in Banach Spaces*, Springer Monographs in Mathematics, Springer, New York, 2010.

[2] Barbu, V., A variational approach to stochastic nonlinear problems, *J. Math. Anal. Appl.*, 384 (2011), 2-15.

[3] Barbu, V., Optimal control approach to nonlinear diffusion equations driven by Wiener noise, *J. Optim. Theory Appl.*, 153 (2012), 1-26.

[4] Barbu, V., A variational approach to nonlinear stochastic diffrential equations with linear multiplicative noise (submitted).

[5] Barbu, Existence for nonlinear finite dimensional stochastic differential equations of subgradient type, *Mathematical Control and Related Fields* (to appear).

[6] Barbu, V., Brzezniak, Z., Hausenblas, E., Tubaro, L., Existence and convergence results for infinite dimensional nonlinear stochastic equations with multiplicative noise, *Stoch. Processes and Their Appl.*, 123 (2013), 984-951.

[7] Barbu, V., Da Prato, G., Röckner, M., Existence of strong solutions for stochastic porous media equations under general monotonicity conditions, *Ann. Probab.*, 37 (2) (2009), 428-452.

[8] Barbu, V., Da Prato, G., Röckner, M., *Stochastic Porous Media Equations*, Lecture Notes in Mathematics, 2163, Springer, 2016.

[9] Barbu, V., Röckner, M., Stochastic variational inequalities and applications to the total variation flow perturbed by linear multiplicative noise, *Archive Rational Mech. Anal.*, 209 (2013), 797-834.

[10] Barbu, V., Röckner, M., An operatorial approach to stochastic partial differential equations driven by linear multiplicative noise, *J. European Math. Soc.*, 17 (2015), 1789-1815.

[11] Brezis, H., Ekeland, I., Un principe variationnel associé à certains équations paraboliques, le cas indépendent du temps, *C.R. Acad. Sci. Paris*, 282 (1976), 971-974.

[12] Brooks, J.K., Dinculeanu, N., Weak compactness in spaces of Bochner integrable functions and applications, *Advances in math.*, 24 (1977), 172-188.

[13] Da Prato, G., Zabczyk, J., *Stochastic Equations in Infinite Dimensions*, 1992. Second Edition, Cambridge University Press, Cambridge, 2008.

[14] Gess, B., Röckner, M., Stochastic variational inequalities and regularity for degenerate stochastic partial differential equations, *Trans. Amer. Math. Soc.*, 369 (2017), 3017-3045.

[15] Krylov, N.V., Rozovskii, B.L., Stochastic evolution equations, *J. Soviet Math.*, 16 (1981), 1233-1277.

[16] Liu, W., Röckner, M., *Stochastic Partial Differential Equations: An Introduction*, Springer, 2015.

[17] Pardoux, E., Equations aux dérivées partielles stochastiques non-linéaires monotones, Thèse, Orsay, 1972.

[18] Prevot, C., Röckner, M., *A Concise Course on Stochastic Partial Differential Equations*, Lecture Notes in Mathematics, 1905, Springer, Berlin 2007.

[19] Rockafellar, R.T., Integrals which are convex functionals, *Pacific J. Math.*, 24 (1968), 525-539.

[20] Visintin, A., Extension of the Brezis-Ekeland-Nayroles principle to monotone operators, *Adv. Math. Sci. Appl.*, 18 (2008), 633-680.