Continuity equation in LlogL for the 2D Euler equations under the enstrophy measure

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Abstract

The 2D Euler equations with random initial condition has been investigates by S. Albeverio and A.-B. Cruzeiro in [1] and other authors. Here we prove existence of solutions for the associated continuity equation in Hilbert spaces, in a quite general class with LlogL densities with respect to the enstrophy measure.

1 Introduction

We consider the 2D Euler equations on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, formulated in terms of the vorticity ω

$$\partial_t \omega + u \cdot \nabla \omega = 0 \tag{1}$$

where u is the velocity, divergence free vector field such that $\omega = \partial_2 u_1 - \partial_1 u_2$. We consider this equation in the following abstract Wiener space structure. We set $H = L^2\left(\mathbb{T}^2\right)$ with scalar product $\langle \cdot, \cdot \rangle_H$ and norm $\|\cdot\|_H$. Given $\delta > 0$, we consider the negative order Sobolev space $B := H^{-1-\delta}\left(\mathbb{T}^2\right)$, its dual $B^* = H^{1+\delta}\left(\mathbb{T}^2\right)$, and we write $\langle \cdot, \cdot \rangle$ for the dual pairing between elements of B and B^*. More generally, we shall use the notation $\langle \cdot, \cdot \rangle$ also for the dual pairing between elements of $C^\infty\left(\mathbb{T}^2\right)'$ and $C^\infty\left(\mathbb{T}^2\right)$; in all cases $\langle \cdot, \cdot \rangle$ reduces to $\langle \cdot, \cdot \rangle_H$ when both elements are in H. Let μ be the so called "enstrophy measure", the centered Gaussian measure on B (in fact it is supported on $H^{-1-}\left(\mathbb{T}^2\right) = \cap_{\delta>0} H^{-1-\delta}\left(\mathbb{T}^2\right)$; but not on $H^{-1}\left(\mathbb{T}^2\right)$) such that

$$\int_B \langle \omega, \phi \rangle \langle \omega, \psi \rangle \mu(d\omega) = \langle \phi, \psi \rangle_H$$

for all $\phi, \psi \in C^\infty\left(\mathbb{T}^2\right)$. Equation (1) has been investigated in this framework and it has been proved that, with a suitable interpretation of the nonlinear term of the equation, it

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has a (possibly non unique) solution for μ-almost every initial condition in B. Moreover, on a suitable probability space $(\Xi, \mathcal{F}, \mathbb{P})$, there exists a stationary process with continuous trajectories in B, with marginal law μ at every time t (in this sense we could say that μ is invariant for equation (1); see also the infinitesimal invariance [2]), whose trajectories are solutions of equation (1) in that suitable specified sense. These results have been proved first by Albeverio and Cruzeiro in [1] and proved with a different concept of solution (used below) in [12].

We want to study the *continuity equation*, associated to equation (1), for a density $\rho_t(\omega)$ with respect to μ. Let us introduce the notation

$$b(\omega) = -u(\omega) \cdot \nabla\omega$$

for the drift in equation (1), where we stress by writing $u(\omega)$ the fact that u depends on ω. The precise meaning of $b(\omega)$ is a nontrivial problem discussed below; for the time being, let us take it as an heuristic notation. Let $\mathcal{FC}^1_{b,T}$ be the set of all functionals $F : [0,T] \times C^\infty(\mathbb{T}^2)' \to \mathbb{R}$ of the form $F(t,\omega) = \sum_{i=1}^m \widetilde{f}_i(\langle \omega, \phi_1 \rangle, ..., \langle \omega, \phi_n \rangle) g_i(t)$, with $\phi_1, ..., \phi_n \in C^\infty(\mathbb{T}^2)$, $\widetilde{f}_i \in C^1_b(\mathbb{R}^n)$, $g_i \in C^1([0,T])$ with $g_i(T) = 0$. The weak form of the continuity equation is

$$\int_0^T \int_B (\partial_t F(t,\omega) + \langle b(\omega), DF(t,\omega) \rangle) \rho_t(\omega) \mu(d\omega) dt = -\int_B F(0,\omega) \rho_0(\omega) \mu(d\omega). \quad (2)$$

The most critical term, which requires a careful definition, is $\langle b(\omega), DF(t,\omega) \rangle$. Let us discuss this issue.

When $F(t,\omega) = \sum_{i=1}^m \widetilde{f}_i(\langle \omega, \phi_1 \rangle, ..., \langle \omega, \phi_n \rangle) g_i(t)$ as above, given any element $\eta \in C^\infty(\mathbb{T}^2)'$ the limit

$$\lim_{\epsilon \to 0} \epsilon^{-1} (F(t, \omega + \epsilon\eta) - F(t,\omega))$$

exists for every $(t,\omega) \in [0,T] \times C^\infty(\mathbb{T}^2)'$ and it is equal to

$$\sum_{i=1}^m \sum_{j=1}^n \partial_j \widetilde{f}_i(\langle \omega_t, \phi_1 \rangle, ..., \langle \omega_t, \phi_n \rangle) g_i(t) \langle \eta, \phi_j \rangle.$$

Assume we have defined $\langle b(\omega), \phi \rangle$ when ω is a typical element under μ and $\phi \in C^\infty(\mathbb{T}^2)$. Then we set

$$\langle b(\omega), DF(t,\omega) \rangle := \sum_{i=1}^m \sum_{j=1}^n \partial_j \widetilde{f}_i(\langle \omega_t, \phi_1 \rangle, ..., \langle \omega_t, \phi_n \rangle) g_i(t) \langle b(\omega), \phi_j \rangle. \quad (3)$$

To complete the meaning of $\langle b(\omega), DF(t,\omega) \rangle$ we thus have to give a meaning to $\langle b(\omega), \phi \rangle$ for every $\phi \in C^\infty(\mathbb{T}^2)$. Formally

$$\langle b(\omega), \phi \rangle = -\langle u(\omega) \cdot \nabla\omega, \phi \rangle.$$

In Theorem 7 of Section 2 we shall define (for each $\phi \in C^{\infty}(\mathbb{T}^2)$) a random variable $\omega \mapsto \langle b(\omega), \phi \rangle$ on the space $(B, \mathcal{B}, \mu)$ ($\mathcal{B}$ being the Borel σ-field on B). With this definition, identity (3) provides a rigorous definition of the measurable map $(\omega, t) \mapsto \langle b(\omega), DF(t, \omega) \rangle$, with certain integrability properties in ω coming from the results of Section 2.

Remark 1 *To help the intuition, let us heuristically write equation (2) in the form*

$$\partial_t \rho_t + \operatorname{div}_\mu (\rho_t b) = 0 \tag{4}$$

with initial condition $\rho_0(\omega)$, where $\operatorname{div}_\mu(v)$, when defined, for a vector field v on B, is (heuristically) defined by the identity

$$\int_B F(\omega) \operatorname{div}_\mu (v(\omega)) \mu(d\omega) = -\int_B \langle v(\omega), DF(\omega) \rangle \mu(d\omega) \tag{5}$$

for all $F \in \mathcal{FC}_b^1$, where $\mathcal{FC}_b^1$ is defined as $\mathcal{FC}_{b,T}^1$ but without the time-dependent components g_i.

In [12] it is proved that the random variable $\omega \mapsto \langle b(\omega), \phi \rangle$ on $(B, \mathcal{B}, \mu)$ has all finite moments; here we improve the result and show that it is exponentially integrable: given $\phi \in C^{\infty}(\mathbb{T}^2)$, it holds

$$\int_B e^{\epsilon |\langle b(\omega), \phi \rangle|} \mu(d\omega) < \infty \tag{6}$$

for some $\epsilon > 0$, which depends only on $\|\phi\|_\infty$; see Theorem 8 in Section 2 below.

This exponential integrability is a key ingredient to extend, to the 2D Euler equations, the result of the authors [7] for abstract equations in Hilbert spaces (in that work the measure μ is not necessarily Gaussian, but the nonlinearity is bounded). Indeed, we aim to prove existence in the class of densities $\rho_t(\omega)$ such that

$$\sup_{t \in [0,T]} \int_B \rho_t(\omega) \log \rho_t(\omega) \mu(d\omega) < \infty. \tag{7}$$

Since $ab \leq e^{\epsilon a} + \epsilon^{-1} b \left(\log \epsilon^{-1} b - 1 \right)$, if $\rho_t(\omega)$ satisfies (7) and property (6) is proved, then

$$\int_B \langle b(\omega), DF(t, \omega) \rangle \rho_t(\omega) \mu(d\omega)$$

is well defined. With these preliminaries we can give the following definition.

Definition 2 *Given a measurable function $\rho_0 : B \to [0, \infty)$ such that $\int_B \rho_0(\omega) \log \rho_0(\omega) \mu(d\omega) < \infty$, we say that a measurable function $\rho : [0,T] \times B \to [0, \infty)$ is a solution of equation (4) of class LlogL if property (7) is satisfied and identity (2) holds for every $F \in \mathcal{FC}_{b,T}^1$.*

Our main result, proved in Section 3, is:

Theorem 3 *If*

$$\int_B \rho_0(\omega) \log \rho_0(\omega) \mu(d\omega) < \infty$$

then there exists a solution of equation (4) of class LlogL.

2 Definition and properties of $\langle b(\omega), \phi \rangle$

We denote by $\{e_n\}$ the complete orthonormal system in $L^2(\mathbb{T}^2; \mathbb{C})$ given by $e_n(x) = e^{2\pi i n \cdot x}$, $n \in \mathbb{Z}^2$. As already said in the Introduction, given a distribution $\omega \in C^\infty(\mathbb{T}^2)'$ and a test function $\phi \in C^\infty(\mathbb{T}^2)$, we denoted by $\langle \omega, \phi \rangle$ the duality between ω and ϕ (namely $\omega(\phi)$), and we use the same symbol for the inner product of $L^2(\mathbb{T}^2)$. We set $\widehat{\omega}(n) = \langle \omega, e_n \rangle$, $n \in \mathbb{Z}^2$ and we define, for each $s \in \mathbb{R}$, the space $H^s(\mathbb{T}^2)$ as the space of all distributions $\omega \in C^\infty(\mathbb{T}^2)'$ such that

$$\|\omega\|_{H^s}^2 := \sum_{n \in \mathbb{Z}^2} \left(1 + |n|^2\right)^s |\widehat{\omega}(n)|^2 < \infty.$$

We use similar definitions and notations for the space $H^s(\mathbb{T}^2, \mathbb{C})$ of complex valued functions.

We want to define, for every $\phi \in C^\infty(\mathbb{T}^2)$, the random variable

$$\begin{aligned}\langle b(\omega), \phi \rangle = -\langle u(\omega) \cdot \nabla \omega, \phi \rangle &= -\int_{\mathbb{T}^2} u(\omega)(x) \cdot \nabla \omega(x) \phi(x) \, dx \\ &= \int_{\mathbb{T}^2} \omega(x) u(\omega)(x) \cdot \nabla \phi(x) \, dx\end{aligned}$$

where we have used integration by parts and the condition $\operatorname{div} u = 0$ (the computation is heuristic, or it holds for smooth periodic functions; we are still looking for a meaningful definition). Recall that u is divergence free and associated to ω by $\omega = \partial_2 u_1 - \partial_1 u_2$. This relation can be inverted using the so called Biot-Savart law:

$$u(x) = \int_{\mathbb{T}^2} K(x - y) \omega(y) \, dy$$

where $K(x, y)$ is the Biot-Savart kernel; in full space it is given by $K(x-y) = \frac{1}{2\pi} \frac{(x-y)^\perp}{|x-y|^2}$; on the torus its form is less simple but we still have K smooth for $x \neq y$, $K(y - x) = -K(x - y)$,

$$|K(x - y)| \leq \frac{C}{|x - y|}$$

for small values of $|x - y|$. See for instance [14] for details.

The difficulty in the definition of $\langle b(\omega), \phi \rangle$ is that ω is of class $H^{-1-\delta}(\mathbb{T}^2)$ and u of class $H^{-\delta}(\mathbb{T}^2)$, so we need to multiply distributions. The following remark recalls a trick used in several works on measure-valued solutions of 2D Euler equations, like [9], [10], [13], [14], [15].

Remark 4 *If ω is sufficiently smooth and periodic, using Biot-Savart law we can write*

$$\langle b(\omega), \phi \rangle = \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \omega(x) \omega(y) K(x-y) \cdot \nabla \phi(x) \, dx dy.$$

Since the double integral, when we rename x by y and y by x, is the same (the renaming doesn't affect the value), and $K(y-x) = -K(x-y)$, we get

$$\langle b(\omega), \phi \rangle = \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \omega(x) \omega(y) H_\phi(x,y) \, dx dy$$

where

$$H_\phi(x,y) := \frac{1}{2} K(x-y) \cdot (\nabla \phi(x) - \nabla \phi(y)).$$

The advantage of this symmetrization is that H_ϕ (opposite to $K(x-y) \cdot \nabla \phi(x)$) is a bounded function. It is smooth outside the diagonal $x = y$, discontinuous on the diagonal; more precisely, we can write

$$H_\phi(x,y) = \frac{1}{2\pi} \left\langle D^2 \phi(x) \frac{x-y}{|x-y|}, \frac{(x-y)^\perp}{|x-y|} \right\rangle + R_\phi(x,y) \tag{8}$$

where $R_\phi(x,y)$ is Lipschitz continuous, with

$$|R_\phi(x,y)| \leq C |x-y|.$$

To summarize, when ω is sufficiently smooth and periodic, we have

$$\langle b(\omega), \phi \rangle = \langle \omega \otimes \omega, H_\phi \rangle_{L^2(\mathbb{T}^2 \times \mathbb{T}^2)}$$

where $\omega \otimes \omega : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{R}$ is defined as $(\omega \otimes \omega)(x,y) = \omega(x) \omega(y)$.

Remark 5 *The previous expression is meaningful when ω is a measure, since H_ϕ is Borel bounded. When ω is only a distribution, of class $H^{-1-\delta}(\mathbb{T}^2)$, one can define $\omega \otimes \omega$ as the unique element of $H^{-2-2\delta}(\mathbb{T}^2 \times \mathbb{T}^2)$ such that*

$$\langle \omega \otimes \omega, f \rangle = \langle \omega, \varphi \rangle \langle \omega, \psi \rangle$$

for every smooth $f : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{R}$ of the form $f(x,y) = \varphi(x) \psi(y)$, where the dual pairing $\langle \omega \otimes \omega, f \rangle$ is on $\mathbb{T}^2 \times \mathbb{T}^2$. But H_ϕ is not of class $H^{2+2\delta}(\mathbb{T}^2 \times \mathbb{T}^2)$, hence there is no simple deterministic meaning for $\langle \omega \otimes \omega, H_\phi \rangle$ when $\omega \in H^{-1-\delta}(\mathbb{T}^2)$. It is here that probability will play the essential role.

In [12] the following result has been proved. As remarked above, when $f \in H^{2+2\delta}\left(\mathbb{T}^2 \times \mathbb{T}^2\right)$, $\langle \omega \otimes \omega, f \rangle$ is well defined for all $\omega \in H^{-1-\delta}\left(\mathbb{T}^2\right)$, hence for a.e. ω with respect to the Entrophy measure μ.

Lemma 6 *Assume $f \in H^{2+\epsilon}\left(\mathbb{T}^2 \times \mathbb{T}^2\right)$ for some $\epsilon > 0$. One has*

$$\int_B |\langle \omega \otimes \omega, f \rangle|^p \, \mu(d\omega) \leq \frac{(2p)!}{2^p p!} \|f\|_\infty^p$$

for every positive integer $p \geq 2$,

$$\int_B \langle \omega \otimes \omega, f \rangle \, \mu(d\omega) = \int_{\mathbb{T}^2} f(x,x)\, dx$$

and, when f is also symmetric,

$$\int_B \left| \langle \omega \otimes \omega, f \rangle - \int_{\mathbb{T}^2} f(x,x)\, dx \right|^2 \mu(d\omega) = 2 \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} f(x,y)^2 \, dx dy.$$

The consequence proved in [12] is:

Theorem 7 *Let $\omega : \Xi \to C^\infty\left(\mathbb{T}^2\right)'$ be a white noise and $\phi \in C^\infty\left(\mathbb{T}^2\right)$ be given. Assume that $H_\phi^n \in H^{2+}\left(\mathbb{T}^2 \times \mathbb{T}^2\right)$ are symmetric and approximate H_ϕ in the following sense:*

$$\lim_{n\to\infty} \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \left(H_\phi^n - H_\phi\right)^2 (x,y)\, dx dy = 0$$

$$\lim_{n\to\infty} \int_{\mathbb{T}^2} H_\phi^n (x,x)\, dx = 0.$$

Then the sequence of r.v.'s $\left\langle \omega \otimes \omega, H_\phi^n \right\rangle$ is a Cauchy sequence in mean square. We denote by

$$\langle b(\omega), \phi \rangle = \langle \omega \otimes \omega, H_\phi \rangle$$

its limit. Moreover, the limit is the same if H_ϕ^n is replaced by $\widetilde{H}_\phi^n$ with the same properties and such that $\lim_{n\to\infty} \int \int \left(H_\phi^n - \widetilde{H}_\phi^n\right)^2 (x,y)\, dx dy = 0$.

A simple example of functions H_ϕ^n with these properties is given in [12]. In addition to these fact, here we prove exponential integrability, see (6).

Theorem 8 *Given a bounded measurable f with $\|f\|_\infty \leq 1$, we have*

$$\int_B e^{\epsilon |\langle \omega \otimes \omega, f \rangle|} \mu(d\omega) < \infty$$

for all $\epsilon < \frac{1}{2}$.

Proof.

$$\mathbb{E}\left[e^{\epsilon|\langle\omega\otimes\omega,f\rangle|}\right]=\sum_{p=0}^{\infty}\frac{\epsilon^p\mathbb{E}\left[|\langle\omega\otimes\omega,f\rangle|^p\right]}{p!}\leq\sum_{p=0}^{\infty}\left(\frac{\epsilon}{2}\right)^p\frac{(2p)!}{p!p!}.$$

This series converges for $\epsilon<\frac{1}{2}$ because (using ratio test)

$$\frac{\left(\frac{\epsilon}{2}\right)^{p+1}\frac{(2(p+1))!}{(p+1)!(p+1)!}}{\left(\frac{\epsilon}{2}\right)^p\frac{(2p)!}{p!p!}}=\frac{\epsilon}{2}\frac{(2p+2)(2p+1)}{(p+1)(p+1)}\to 2\epsilon.$$

■

3 Proof of Theorem 3

3.1 Approximate problem

Recall from the Introduction that $\delta>0$ is fixed and we set $B=H^{-1-\delta}\left(\mathbb{T}^2\right)$, $H=L^2\left(\mathbb{T}^2\right)$; recall also from Section 2 that we write $e_n(x)=e^{2\pi in\cdot x}$, $x\in\mathbb{T}^2$, $n\in\mathbb{Z}^2$, that is a complete orthonormal system in $H^{\mathbb{C}}:=L^2\left(\mathbb{T}^2;\mathbb{C}\right)$. Given $N\in\mathbb{N}$, let $H_N^{\mathbb{C}}$ be the span of e_n for $|n|_\infty\leq N$, $|n|_\infty:=\max\left(|n_1|,|n_2|\right)$ for $n=(n_1,n_2)$; it is a subspace of $H^{\mathbb{C}}$. Let H_N be the subspace of $H_N^{\mathbb{C}}$ made of real-valued elements; it is a subspace of H and is characterized by the following property: $\omega=\sum_{|n|_\infty\leq N}\omega_n e_n$ is in H_N if and only if $\overline{\omega_n}=\omega_{-n}$, for all n such that $|n|_\infty\leq N$.

Let π_N be the orthogonal projection of H onto H_N. It is given by $\pi_N\omega=\sum_{|n|_\infty\leq N}\langle\omega,e_n\rangle_H e_n$, for all $\omega\in H$. We extend π_N to an operator on B by setting

$$\pi_N:B\to H_N$$
$$\pi_N\omega=\sum_{|n|_\infty\leq N}\langle\omega,e_n\rangle e_n$$

where now $\langle\omega,e_n\rangle$ is the dual pairing. We may introduce the Dirichlet kernel

$$\theta_N(x_1,x_2)=\sum_{n_1=-N}^{N}\sum_{n_2=-N}^{N}e^{2\pi i(n_1x_1+n_2x_2)}=\sum_{|n|_\infty\leq N}^{N}e^{2\pi in\cdot x}\tag{9}$$

for $x=(x_1,x_2)\in\mathbb{T}^2$, and check that

$$\pi_N\omega=\theta_N*\omega.$$

We define the operator

$$b_N:B\to H_N$$

as

$$b_N(\omega) = -\pi_N \left(u(\pi_N \omega) \cdot \nabla \pi_N \omega \right), \qquad \omega \in B$$

where $u(\pi_N \omega)$ denotes the result of Biot-Savart law applied to $\pi_N \omega$,

$$u(\pi_N \omega)(x) := \int_{\mathbb{T}^2} K(x-y)(\pi_N \omega)(y)\, dy.$$

The operator b_N has the following properties. We denote by $\operatorname{div} b_N(\omega)$ the function

$$\operatorname{div} b_N(\omega) = \sum_{|n|_\infty \leq N} \partial_n \langle b_N(\omega), e_n \rangle_H$$

where, when defined, $\partial_n F(\omega) = \lim_{\epsilon \to 0} \epsilon^{-1} (F(\omega + \epsilon e_n) - F(\omega))$, for a function F defined on B. We say that $\operatorname{div} b_N(\omega)$ exists if $\partial_n \langle b_N(\omega), e_n \rangle_H$ exists for all $|n|_\infty \leq N$. Moreover, we set

$$\operatorname{div}_\mu b_N(\omega) := \operatorname{div} b_N(\omega) - \langle \omega, b_N(\omega) \rangle$$

where $\langle \omega, b_N(\omega) \rangle$ is the dual pairing. It is easy to check that this definition is coherent with the general one (5) given in the Introduction.

Lemma 9 *The divergence* $\operatorname{div} b_N(\omega)$ *exists for all* $\omega \in B$ *and*

$$\operatorname{div} b_N(\omega) = 0$$
$$\langle \omega, b_N(\omega) \rangle = 0$$

and thus

$$\operatorname{div}_\mu b_N(\omega) = 0.$$

Proof. Step 1: A basic identity is

$$\langle \omega, b_N(\omega) \rangle = 0$$

for all $\omega \in B$, where as usual $\langle .,. \rangle$ denotes dual pairing. This identity holds because

$$\langle \omega, \pi_N (u(\pi_N \omega) \cdot \nabla \pi_N \omega) \rangle = \langle \pi_N \omega, u(\pi_N \omega) \cdot \nabla \pi_N \omega \rangle_H = 0$$

where the first equality can be checked by writing $\omega = \sum \langle \omega, e_n \rangle e_n$ (the series converges in B), and the second equality is true because

$$\langle v \cdot \nabla f, f \rangle = \frac{1}{2} \int_{\mathbb{T}^2} v(x) \cdot \nabla f^2(x)\, dx = -\frac{1}{2} \int_{\mathbb{T}^2} \operatorname{div} v(x)\, f^2(x)\, dx = 0$$

for all sufficiently smooth divergence free vector field v (we take $v = u(\pi_N \omega)$ that is a smooth divergence free vector field) and all sufficiently smooth functions f (we take $f = \pi_N \omega$).

Step 2: Recall that $u(e_n)(x)$ is periodic, divergence free, and such that $\nabla^{\perp}\cdot u(e_n) = e_n$ (it is also gven by the Biot-Savart law $u(e_n)(x) := \int_{\mathbb{T}^2} K(x-y)\, e_n(y)\, dy$). Then we have

$$u(e_n)(x) \cdot \nabla e_n(x) = 0$$

for every $n \in \mathbb{Z}^2$. Indeed,

$$u(e_n)(x) \cdot \nabla e_n(x) = 2\pi i\, (u(e_n)(x) \cdot n)\, e_n(x)$$

and this is zero because $u(e_n)(x) \cdot n = 0$. To prove the latter property, it is necessary to understand the shape of $u(e_n)(x)$. It is

$$u(e_n)(x) = \frac{n^{\perp}}{|n|^2} e_n(x)$$

(which implies $u(e_n)(x) \cdot n = 0$ because $n^{\perp} \cdot n = 0$). Indeed, this function u is periodic, divengence free (one has $\operatorname{div} u(e_n)(x) = \frac{n^{\perp}}{|n|^2} e_n(x) \cdot n = 0$) and $\nabla^{\perp} \cdot u(e_n)(x) = \frac{n^{\perp}}{|n|^2} e_n(x) \cdot n^{\perp} = e_n(x)$.

Step 3: Finally we can prove that $\operatorname{div} b_N(\omega) = 0$. It is

$$\operatorname{div} b_N(\omega) = -\sum_{|n| \leq N} \partial_n \left\langle \pi_N \left(u(\pi_N \omega) \cdot \nabla \pi_N \omega\right), e_n \right\rangle_H .$$

We have

$$\begin{aligned} &\partial_n \left\langle \pi_N \left(u(\pi_N \omega) \cdot \nabla \pi_N \omega\right), e_n \right\rangle_H \\ &= \partial_n \left\langle u(\pi_N \omega) \cdot \nabla \pi_N \omega, e_n \right\rangle_H \\ &= -\partial_n \left\langle \pi_N \omega, u(\pi_N \omega) \cdot \nabla e_n \right\rangle_H \end{aligned}$$

(we have used integration by parts and $\operatorname{div} u(\pi_N \omega) = 0$ in the last identity)

$$\begin{aligned} &= -\partial_n \left\langle \sum_{|n'| \leq N} \langle \omega, e_{n'} \rangle e_{n'}, \sum_{|n''| \leq N} \langle \omega, e_{n''} \rangle u(e_{n''}) \cdot \nabla e_n \right\rangle_H \\ &= -\left\langle e_n, u(\pi_N \omega) \cdot \nabla e_n \right\rangle_H - \left\langle \pi_N \omega, u(e_n) \cdot \nabla e_n \right\rangle_H . \end{aligned}$$

The first term is zero by the same general rule recalled in Step 1. The second term is zero by Step 2. Therefore $\operatorname{div} b_N(\omega) = 0$. ■

Consider the finite dimensional ordinary differential equation in the space H_N defined as

$$\frac{d\omega_t^N}{dt} = b_N\left(\omega_t^N\right), \qquad \omega_0^N \in H_N. \tag{10}$$

The function b_N, in H_N, is differentable, bounded with bounded derivative on bounded sets. Hence, for every $\omega_0^N \in H_N$, there is a unique local solution ω_t^{N,ω_0^N} of equation (10) and the flow map $\omega_0^N \mapsto \omega_t^{N,\omega_0^N}$, where defined, is continuously differentiable, invertible with continuously differentiable inverse. The solution is global because of the energy estimate

$$\frac{d\left\|\omega_t^N\right\|_H^2}{dt} = 2\left\langle b_N\left(\omega_t^N\right), \omega_t^N\right\rangle_H = 0$$

which implies $\sup_{t\in[0,\tau]}\left\|\omega_t^N\right\|_H^2 \leq \left\|\omega_0^N\right\|_H^2$ on any interval $[0,\tau]$ of local existence; the property $\left\langle b_N\left(\omega_t^N\right), \omega_t^N\right\rangle_H = 0$ holds by Lemma 9. We denote by $\Phi_t^N : H_N \to H_N$ the global flow defined as $\Phi_t^N\left(\omega_0^N\right) = \omega_t^{N,\omega_0^N}$.

Denote by $\mu^N(d\omega)$ the image measure, on H_N, of $\mu(d\omega)$ under the projection π_N. This measure is invariant under the flow Φ_t^N, because $\operatorname{div}_\mu b_N(\omega) = 0$: for every smooth $F : H_N \to [0,\infty)$, bounded with bounded derivatives,

$$\begin{aligned}\int_{H_N} \langle b_N(\omega), DF(\omega)\rangle_{H_N} \mu^N(d\omega) &= \int_B \langle b_N(\omega), DF(\pi_N\omega)\rangle_H \mu(d\omega) \\ &= -\int_B F(\pi_N\omega)\operatorname{div}_\mu b_N(\omega)\mu(d\omega) = 0.\end{aligned}$$

3.2 Continuity equation for the approximate problem

Given a measurable function $\rho_0^N : H_N \to [0,\infty)$, with $\int_B \rho_0^N(\pi_N\omega)\mu(d\omega) < \infty$, consider the measure $\rho_0^N(\pi_N\omega)\mu^N(d\omega)$ and its push forward under the flow map Φ_t^N; denote it by ν_t^N. By definition, for bounded measurable $F : H_N \to [0,\infty)$,

$$\int_{H_N} F(\omega)\nu_t^N(d\omega) = \int_{H_N} F\left(\Phi_t^N(\omega)\right)\rho_0^N(\omega)\mu^N(d\omega).$$

From the invariance of μ^N under the flow Φ_t^N, we have

$$\int_{H_N} F(\omega)\nu_t^N(d\omega) = \int_{H_N} F(\omega)\rho_0^N\left(\left(\Phi_t^N\right)^{-1}(\omega)\right)\mu^N(d\omega)$$

hence

$$\nu_t^N(d\omega) = \rho_t^N(\pi_N\omega)\mu^N(d\omega)$$

where

$$\rho_t^N(\omega) = \rho_0^N\left(\left(\Phi_t^N\right)^{-1}(\omega)\right), \qquad \omega \in H_N. \tag{11}$$

We have partially proved the following statement.

Lemma 10 *Consider equation (10) in H_N, with the associated flow Φ_t^N. Given at time zero a measure of the form $\rho_0^N(\pi_N\omega)\mu^N(d\omega)$ with $\int_B \rho_0^N(\pi_N\omega)\mu(d\omega) < \infty$, its push forward at time t, under the flow map Φ_t^N, is a measure of the form $\rho_t^N(\pi_N\omega)\mu^N(d\omega)$, with $\int_B \rho_t^N(\pi_N\omega)\mu(d\omega) < \infty$. If in addition $\int_B \rho_0^N(\pi_N\omega)\log\rho_0^N(\pi_N\omega)\mu(d\omega) < \infty$, the same is true at time t and*

$$\int_B \rho_t^N(\pi_N\omega)\log\rho_t^N(\pi_N\omega)\mu(d\omega) = \int_B \rho_0^N(\pi_N\omega)\log\rho_0^N(\pi_N\omega)\mu(d\omega). \tag{12}$$

If in addition ρ_0^N is bounded, then $\rho_t^N \leq \left\|\rho_0^N\right\|_\infty$. Finally. ρ_t^N satisfies the continuity equation

$$\int_0^T \int_B \left(\partial_t F(t,\omega) + \langle DF(t,\omega), b_N(\omega)\rangle_H\right)\rho_t^N(\pi_N\omega)\mu(d\omega)\,dt = -\int_B F(0,\omega)\rho_0^N(\pi_N\omega)\mu(d\omega) \tag{13}$$

for all $F \in \mathcal{FC}_{b,T}^1$ of the form $F(t,\omega) = \sum_{i=1}^m \widetilde{f}_i(\langle\omega, e_n\rangle, |n|_\infty \leq N)\, g_i(t)$.

Proof. The integrability of ρ_t^N comes from the invariance of μ^N under Φ_t^N, as well as the LlogL property; let us check this latter one. Using (11) we have

$$\begin{aligned}\int_B \rho_t^N(\pi_N\omega)\log\rho_t^N(\pi_N\omega)\mu(d\omega) &= \int_{H_N} \rho_t^N(\omega)\log\rho_t^N(\omega)\mu^N(d\omega)\\ &= \int_{H_N} \rho_0^N\left(\left(\Phi_t^N\right)^{-1}(\omega)\right)\log\rho_0^N\left(\left(\Phi_t^N\right)^{-1}(\omega)\right)\mu^N(d\omega)\\ &= \int_{H_N} \rho_0^N(\omega)\log\rho_0^N(\omega)\mu^N(d\omega)\\ &= \int_B \rho_0^N(\pi_N\omega)\log\rho_0^N(\pi_N\omega)\mu(d\omega).\end{aligned}$$

When ρ_0^N is bounded, we have

$$\rho_t^N(\omega) = \rho_0^N\left(\left(\Phi_t^N\right)^{-1}(\omega)\right) \leq \left\|\rho_0^N\right\|_\infty.$$

Finally, form the chain rule applied to $F\left(t, \Phi_t^N(\omega)\right)$, $\omega \in H_N$, we get the weak form of the continuity equation. ■

Remark 11 *We may construct ρ_t^N and prove (12) also by the following procedure, closer to [7]. We study the transport equation in H_N*

$$\partial_t\rho_t^N + \left\langle b_N, D\rho_t^N\right\rangle_H = 0$$

with initial condition ρ_0^N, which has the solution (11) by the method of characteristics. Its weak form reduces to (13) because (for F like those of the Lemma)

$$\int_{H_N} F(t,\omega) \left\langle b_N(\omega), D\rho_t^N(\omega) \right\rangle_H \mu^N(d\omega)$$
$$= \int_B F(t,\omega) \left\langle b_N(\omega), D\rho_t^N(\pi_N\omega) \right\rangle_H \mu(d\omega)$$
$$= -\int_B \langle DF(t,\omega), b_N(\omega) \rangle_H \rho_t^N(\pi_N\omega) \mu(d\omega)$$

where we have used the property $\operatorname{div}_\mu b_N(\omega) = 0$. Finally, to prove (12) as in [7], we compute

$$\frac{d}{dt}\int_{H_N} \rho_t^N \left(\log \rho_t^N - 1\right) d\mu^N$$
$$= \int_{H_N} \log \rho_t^N \partial_t \rho_t^N d\mu^N = -\int_{H_N} \log \rho_t^N \left\langle b_N, D\rho_t^N \right\rangle d\mu^N$$
$$= -\int_{H_N} \left\langle b_N, D\left[\rho_t^N \left(\log \rho_t^N - 1\right)\right] \right\rangle d\mu^N$$
$$= \int_{H_N} \left[\rho_t^N \left(\log \rho_t^N - 1\right)\right] \operatorname{div}_\mu b_N d\mu^N = 0.$$

3.3 Construction of a solution to the limit problem

3.3.1 First case: bounded ρ_0

Consider first the case when ρ_0 is a bounded measurable function on B. Define the sequence of equibounded functions ρ_0^N on H_N by setting $\rho_0^N(\pi_N\omega) = \rho_0(\pi_N\omega)$. For each one of them, consider the associated function $\rho_t^N(\pi_N\omega)$ given by Lemma 10. There is a subsequence, still denoted for simplicity by $\rho_t^N(\pi_N\omega)$ which converges to some function ρ_t weak* in $L^\infty([0,T] \times B)$; moreover we have (12) which implies (see [7] for similar computations)

$$\int_B \rho_t(\omega) \log \rho_t(\omega) \mu(d\omega) \leq \int_B \rho_0(\omega) \log \rho_0(\omega) \mu(d\omega).$$

Finally we have to prove that ρ_t satisfies the weak formulation. We have to pass to the limit in (13). The only problem is the term

$$\int_0^T \int_B \langle b_N(\omega), DF(t,\omega) \rangle_H \rho_t^N(\pi_N\omega) \mu(d\omega) dt.$$

We add and subtract the term

$$\int_0^T \int_B \langle b(\omega), DF(t,\omega) \rangle \rho_t^N(\pi_N\omega) \mu(d\omega) dt$$

and use integrability of $\langle b(\omega), D_H F(t,\omega)\rangle$ and weak* convergence of $\rho_t^N(\pi_N\omega)$ to $\rho_t(\omega)$ to pass to the limit in one addend. It remains to prove that

$$\lim_{k\to\infty}\int_0^T\int_B \left(\langle b_N(\omega), DF(t,\omega)\rangle_H - \langle b(\omega), DF(t,\omega)\rangle\right)\rho_t^N(\pi_N\omega)\,\mu(d\omega)\,dt = 0.$$

Keeping in mind again the weak* convergence of $\rho_t^N(\pi_N\omega)$, it is sufficient to prove that $\int_B \langle b_N(\omega), D_H F(t,\omega)\rangle_H$ converges strongly to $\langle b(\omega), D_H F(t,\omega)\rangle$ in $L^1\left(0,T;L^1(B,\mu)\right)$. Due to the form of F, it is sufficient to prove the following claim: given $\phi\in C^\infty\left(\mathbb{T}^2\right)$,

$$\lim_{k\to\infty}\int_B \left|\langle b_N(\omega),\phi\rangle_H - \langle b(\omega),\phi\rangle\right|\mu(d\omega) = 0.$$

The remainder of this subsection is devoted to the proof of this claim.

It is not restrictive to assume that $\phi\in H_{N_0}$ for some N_0. Hence, for N large enough so that $\pi_N\phi=\phi$,

$$\begin{aligned}\langle b_N(\omega),\phi\rangle_H &= -\langle \pi_N\left(u(\pi_N\omega)\cdot\nabla\pi_N\omega\right),\phi\rangle_H\\ &= -\langle u(\pi_N\omega)\cdot\nabla\pi_N\omega,\phi\rangle_H\\ &= \langle \pi_N\omega, u(\pi_N\omega)\cdot\nabla\phi\rangle_H\\ &= \langle(\pi_N\omega)\otimes(\pi_N\omega), H_\phi\rangle\end{aligned}$$

where the last identity is proved as in Remark 4. We have

$$\langle(\pi_N\omega)\otimes(\pi_N\omega), H_\phi\rangle = \left\langle \omega\otimes\omega, (H_\phi)_N\right\rangle$$

where

$$(H_\phi)_N(x,y) = \sum_{|n|_\infty\le N}\sum_{|n'|_\infty\le N} e_n(x)\,e_{n'}(y)\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} e_{n'}\left(y'\right)e_n\left(x'\right)H_\phi\left(x',y'\right)dx'dy'.$$

Therefore, our aim is to prove that, given $\phi\in C^\infty\left(\mathbb{T}^2\right)$,

$$\lim_{k\to\infty}\int_B\left|\left\langle\omega\otimes\omega,(H_\phi)_N - H_\phi\right\rangle\right|\mu(d\omega) = 0.$$

Thanks to Lemma 6 and Theorem 7, with a simple argument on Cauchy sequences one can see that it is sufficient to prove that $(H_\phi)_N\to H_\phi$ in $L^2\left(\mathbb{T}^2\times\mathbb{T}^2\right)$ and

$$\int_{\mathbb{T}^2}(H_\phi)_N(x,x)\,dx\to 0. \tag{14}$$

From the theory of Fourier series, $(H_\phi)_N\to H_\phi$ in $L^2\left(T^2\times T^2\right)$. The limit property (14) requires more work. The result is included in the next lemma, which completes the proof that ρ_t is a weak solution, in the case when ρ_0 is bounded.

Lemma 12 *i) The Dirichlet kernel (9) has the two properties*

$$\theta_N(x_1,x_2)=\theta_N(x_2,x_1)$$
$$\theta_N(-x_1,x_2)=\theta_N(x_1,x_2).$$

ii) If a kernel $\theta_N(x)$, $x\in T^2$, has these two properties, the the kernel $W_N=\theta_N*\theta_N$ has the same properties.

iii) It follows that, for any symmetric matrix S,

$$\int_{T^2}W_N(x)\left\langle S\frac{x}{|x|},\frac{x^\perp}{|x|}\right\rangle dx=0.$$

iv) It follows also that

$$\lim_{N\to\infty}\int_{T^2}\int_{T^2}W_N(x-y)\,H_\phi(x,y)\,dxdy=0.$$

In the case when θ_N is the Dirichlet kernel, this property is the limit property (14).

Proof. Property (i) is obvious. The proof of (ii) is elementary, but we give the computations for completeness:

$$\begin{aligned}W_N(x_1,x_2)&=\int_{T^2}\theta_N(x_1-y_1,x_2-y_2)\,\theta_N(y_1,y_2)\,dy_1dy_2\\&=\int_{T^2}\theta_N(x_2-y_2,x_1-y_1)\,\theta_N(y_2,y_1)\,dy_1dy_2\\&=W_N(x_2,x_1)\end{aligned}$$

$$\begin{aligned}W_N(-x_1,x_2)&=\int_{T^2}\theta_N(-x_1-y_1,x_2-y_2)\,\theta_N(y_1,y_2)\,dy_1dy_2\\&=\int_{T^2}\theta_N(x_1+y_1,x_2-y_2)\,\theta_N(y_1,y_2)\,dy_1dy_2\\&=\int_{T^2}\theta_N(x_1-y_1,x_2-y_2)\,\theta_N(-y_1,y_2)\,dy_1dy_2\\&=\int_{T^2}\theta_N(x_1-y_1,x_2-y_2)\,\theta_N(y_1,y_2)\,dy_1dy_2\\&=W_N(x_1,x_2).\end{aligned}$$

Let us prove (iii). We can write

$$\left\langle S\frac{x}{|x|},\frac{x^\perp}{|x|}\right\rangle=(S_{11}+S_{22})\frac{x_1x_2}{|x|^2}+S_{12}\frac{x_2^2-x_1^2}{|x|^2}.$$

Let us show that the integrals corresponding to each one of the two terms vanish. We have

$$\int_{T^2} W_N(x) \frac{x_1 x_2}{|x|^2} dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} W_N(x) \frac{x_1 x_2}{|x|^2} dx_1 dx_2$$

The integration in the second quadrant,

$$\int_0^{\frac{1}{2}} \int_{-\frac{1}{2}}^{0} W_N(x) \frac{x_1 x_2}{|x|^2} dx_1 dx_2$$

cancels with the integration in the first quadrant,

$$\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} W_N(x) \frac{x_1 x_2}{|x|^2} dx_1 dx_2$$

because of property $W_N(-x_1, x_2) = W_N(x_1, x_2)$ (point (ii)); similarly for the integrations in the other quadrants. So $\int_{T^2} W_N(x) \frac{x_1 x_2}{|x|^2} dx = 0$. For the other integral, just by renaming the variables we have

$$\int_{T^2} W_N(x_1, x_2) \frac{x_1^2}{|x|^2} dx_1 dx_2 = \int_{T^2} W_N(x_2, x_1) \frac{x_2^2}{|x|^2} dx_2 dx_1$$

and then, using $W_N(x_1, x_2) = W_N(x_2, x_1)$ (point (ii))

$$= \int_{T^2} W_N(x_1, x_2) \frac{x_2^2}{|x|^2} dx_1 dx_2$$

hence $\int_{T^2} W_N(x) \frac{x_2^2 - x_1^2}{|x|^2} dx = 0$. We have proved (iii).

Finally, the limit in (iv) is a consequence of the decompositon (8). Indeed,

$$\int_{T^2} \int_{T^2} W_N(x-y) \left\langle D^2\phi(x) \frac{x-y}{|x-y|}, \frac{(x-y)^\perp}{|x-y|} \right\rangle dxdy$$

$$= \int_{T^2} \left(\int_{T^2} W_N(z) \left\langle D^2\phi(x) \frac{z}{|z|}, \frac{z^\perp}{|z|} \right\rangle dz \right) dx = 0$$

by (iii), and

$$\lim_{N\to\infty} \int_{T^2} \int_{T^2} W_N(x-y) R_\phi(x,y) \, dxdy = 0$$

because $R_\phi(x,y)$ is Lipschitz continuous with $|R_\phi(x,y)| \le C|x-y|$. To complete the proof of the claims of part (iv), let us check that, when θ_N is the Dirichlet kernel, the property stated in (iv) coincides with the limit property (14). We have

$$\begin{aligned}\int_{T^2} (H_\phi)_N (x,x)\, dx &= \sum_{|n'|_\infty \le N}^{N} \sum_{|n|_\infty \le N}^{N} \int_{T^2}\int_{T^2}\int_{T^2} e^{2\pi i n'\cdot(x-x')} e^{2\pi i n\cdot(x-y')} H_\phi (x',y')\, dy' dx' dx \\ &= \int_{T^2}\int_{T^2} \left(\sum_{|n'|_\infty \le N}^{N} \sum_{|n|_\infty \le N}^{N} \int_{T^2} e^{2\pi i n'\cdot(x'-x)} e^{2\pi i n\cdot(x-y')} dx \right) H_\phi (x',y')\, dy' dx' \\ &= \int_{T^2}\int_{T^2} W_N (x'-y')\, H_\phi (x',y')\, dy' dx'.\end{aligned}$$

The proof is complete. ■

3.3.2 General case: ρ_0 of class LlogL

Assume now that ρ_0 satisfies only the assumptions of the main theorem. Define $\rho_0^n = \rho_0 \wedge n$. For each n, apply the result of the first case and construct a weak solution ρ_t^n, which fulfills in particular

$$\int_B \rho_t^n (\omega) \log \rho_t^n (\omega)\, \mu (d\omega) \le \int_B \rho_0^n (\omega) \log \rho_0^n (\omega)\, \mu (d\omega) \le \int_B \rho_0 (\omega) \log \rho_0 (\omega)\, \mu (d\omega).$$

From this inequality we deduce the existence of a subsequence, still denoted for simplicity by $\rho_t^n (\omega)$ which converges to some function ρ_t weak* in $L^1 \left(0, T; L^1 (B,\mu)\right)$, which satisfies property (7), and moreover, from the duality of Orlicz spaces, such that

$$\int_0^T \int_B G(t,\omega)\, \rho_t^n (\pi_N \omega)\, \mu (d\omega)\, dt \to \int_0^T \int_B G(t,\omega)\, \rho_t (\omega)\, \mu (d\omega)\, dt$$

for all G such that, for some $\epsilon > 0$,

$$\sup_{t\in[0,T]} \int_B e^{\epsilon |G(t,\omega)|} \mu (d\omega) < \infty. \tag{15}$$

Due to these fact, in order to prove that ρ_t satisfies the weak formulation of the continuity equation, we have only to prove that

$$\int_0^T \int_B \langle b(\omega), DF(t,\omega)\rangle\, \rho_t^n (\omega)\, \mu (d\omega)\, dt \to \int_0^T \int_B \langle b(\omega), DF(t,\omega)\rangle\, \rho_t (\omega)\, \mu (d\omega)\, dt.$$

Since $G(t,\omega) := \langle b(\omega), DF(t,\omega)\rangle$ has property (15) by Theorem 8, this is true, and the proof is complete.

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