AN OPTIMAL DIVIDEND PROBLEM WITH CAPITAL INJECTIONS OVER A FINITE HORIZON

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Abstract. In this paper we propose and solve an optimal dividend problem with capital injections over a finite time horizon. The surplus dynamics obeys a linearly controlled drifted Brownian motion that is reflected at zero, dividends give rise to time-dependent instantaneous marginal profits, whereas capital injections are subject to time-dependent instantaneous marginal costs. The aim is to maximize the sum of a liquidation value at terminal time and of the total expected profits from dividends, net of the total expected costs for capital injections. Inspired by the study in [13] on reflected follower problems, we relate the optimal dividend problem with capital injections to an optimal stopping problem for a drifted Brownian motion that is absorbed at zero. We show that whenever the optimal stopping rule is triggered by a time-dependent boundary, the value function of the optimal stopping problem gives the derivative of the value function of the optimal dividend problem. Moreover, the optimal dividends' distribution strategy is also triggered by the moving boundary of the associated stopping problem. The properties of this boundary are then investigated in a case study in which instantaneous marginal profits and costs from dividends and capital injections are constants discounted at a constant rate.

Keywords: optimal dividend problem; capital injections; singular stochastic control; optimal stopping; free boundary.

MSC2010 subject classification: 93E20, 60G40, 62P05, 91G10, 60J65

1. Introduction

The literature on optimal dividend problems started in 1957 with the work of Bruno de Finetti [10], where, for the first time, it was proposed to measure an insurance portfolio by the discounted value of its future dividends' payments. Since then, the literature in Mathematics and Actuarial Mathematics experienced many scientific contributions on the optimal dividend problem, which has been typically modeled as a stochastic control problem subject to different specifications of the control processes and of the surplus dynamics (see, among many others, the early [16], the more recent [1], [9] and [18], the review [2], and the book [30]).

Starting from the observation that ruin occurs almost surely when the fund's manager pays dividends by following the optimal strategy of de Finetti's problem, in [11] the Authors proposed several modifications to the original formulation of the optimal dividend problem. In particular, in [11] it has been suggested a model in which the shareholders are obliged to inject capital in order to avoid bankruptcy. This is the so-called *optimal dividend problem with capital injections*.

The literature on the optimal dividend problem with capital injections is not as rich as that on the classical de Finetti's problem. In [23] the Authors study an optimal dividend problem with capital injections in which the surplus process is reflected at zero, and on $(0, \infty)$ evolves according to a classical Cramér-Lundberg risk model. In [25], in absence of any interventions, the surplus process follows a Brownian motion with drift, whereas in the [15], [32] and [31] it evolves as a general one-dimensional diffusion. In all those papers the optimal dividend problem with capital injections is formulated as a singular stochastic control problem for a

Date: April 13, 2018.

reflected process (i.e. a so-called "reflected follower problem") over an infinite time horizon. Given the stationarity of the setting, in those works it is shown that (a part a possible initial lump sum payment) it is optimal to pay just enough dividends in order to keep the surplus process in the interval $[0, b]$, for some constant $b > 0$ endogenously determined.

In this paper we propose and solve, for the first time in the literature, an optimal dividend problem with capital injections over a finite time horizon $T \in (0, \infty)$. This horizon might be seen as a pre-specified future date at which the fund is liquidated.

As it is common in the literature (see [1], [9] and [25], among many others), also in our problem, in absence of any intervention, the surplus process evolves as Brownian motion with drift μ and volatility σ. This dynamics for the fund's value can be obtained as a suitable (weak) limit of a classical dynamics à la Cramér-Lundberg (see Appendix D.3 in [30] for details). We also assume that, after time-dependent transaction costs/taxes have been paid, shareholders receive a time-dependent instantaneous net proportion of leakages f from the surplus. Moreover, shareholders are asked to inject capital whenever the surplus attempts to become negative, and injecting capital they incur a time-dependent marginal administration cost m. Finally, a surplus-dependent liquidation reward g is obtained at liquidation time T.

Within this setting, the fund's manager takes the point of view of the shareholders and thus aims at solving

$$V(t,x) := \sup_{D} \mathbb{E}\Big[\int_0^{T-t} f(t+s)\ dD_s - \int_0^{T-t} m(t+s)\ dI^D_s + g(T, X^D_{T-t}(x))\Big], \tag{1.1}$$

for any initial time $t \in [0, T]$ and any initial value of the fund $x \in \mathbb{R}_+$. In (1.1) the fund's value evolves as

$$X^D_s(x) = x + \mu s + \sigma W_s - D_s + I^D_s, \quad s \geq 0,$$

and the optimization is performed over a suitable class of nondecreasing processes D. In fact, the quantity D_s represents the cumulative amount of dividends paid to shareholders up to time s, whereas I^D_s is the cumulative amount of capital injected by the shareholders up to time s. Roughly speaking, I^D is the minimal nondecreasing process which ensures that X^D stays nonnegative, and it is flat off $\{t \geq 0 : X^D_t = 0\}$.

If we attempt to tackle problem (1.1) via a dynamic programming approach, we will find that the dynamic programming equation for V takes the form of a parabolic partial differential equation (PDE) with gradient constraint (i.e. a variational inequality), and with a Neumann boundary condition at $x = 0$ (the latter is due to the fact that the state process X is reflected at zero through the capital injections process). Proving that a solution to this PDE problem has enough regularity to characterize an optimal control is far from being trivial.

Starting from the observation that the optimal dividend problem with capital injections (1.1) is actually a reflected follower problem (see, e.g., [3], [12] and [20] as early contributions) with costly reflection at zero, and inspired by the findings of [13], here we solve (1.1) without relying on PDE methods, but relating (1.1) to a (still complex but) more tractable optimization problem; i.e., to an optimal stopping problem with absorption at zero.

In particular, let $S(x) := \inf\{t \geq 0 : x + \mu s + \sigma W_s = 0\}$, $x \geq 0$, take f, m and g suitable nonnegative functions (see Assumption 2.2 below for details), and for any $(t, x) \in [0, T] \times \mathbb{R}_+$ introduce the optimal stopping problem

$$\begin{aligned} u(t,x) := \sup_{\tau \in [0, T-t]} \mathbb{E}\Big[& f(t+\tau)\mathbb{1}_{\{\tau < (T-t)\wedge S(x)\}} + m(t+S(x))\mathbb{1}_{\{\tau \geq S(x)\}} \\ & + g_x(T, x + \mu(T-t) + \sigma W_{T-t})\mathbb{1}_{\{\tau = T-t < S(x)\}}\Big]. \end{aligned} \tag{1.2}$$

Given that the optimization runs up to the (random) horizon $(T-t)\wedge S(x)$, problem (1.2) can be viewed as an optimal stopping problem for the absorbed drifted Brownian motion process

$$A_s(x) := \begin{cases} x + \mu s + \sigma W_s, & s < S(x), \\ \Delta, & s \geq S(x), \end{cases} \tag{1.3}$$

for any $s \geq 0$, and where Δ is a cemetery state isolated from $\mathbb{R}_+$ (see Section 3 below for details). Then, if the optimal stopping time of (1.2) is given as the first hitting time of the time-space process $(t+s, A_s(x))_{s\geq 0}$ to a continuous and strictly positive time-dependent boundary $b(\,\cdot\,)$ (cf. the structural Assumption 3.1 below), then one has that $V_x = u$, and the optimal dividends' payments strategy $D^\star$ is triggered by b (see Theorem 3.2 below). In fact, if the optimization starts at time $t \in [0,T]$, the couple $(D^\star, I^{D^\star})$ keeps at any instant in time $s \in [0, T-t]$ the optimally controlled fund's value $X_s^{D^\star}$nonnegative and below the time-dependent critical level $b(s+t)$.

This result is obtained by performing an almost exclusively probabilistic study in which we suitably integrate in the space variable two different representations of the value u of (1.2). It is worth noticing that although we borrow arguments from the study in [13] on the connection between reflected follower problems and questions of optimal stopping (see also [20]), differently to [13], in our performance criterion (1.1) we also have a cost of reflection and this requires a careful and not immediate adaptation of the ideas and results of [13].

We then show that the structural Assumption 3.1, needed to prove the relation between (1.1) and (1.2), does indeed hold in a canonical formulation of the optimal dividend problem with capital injections in which marginal benefits and costs are constants discounted at a constant rate, and the liquidation value at time T is proportional to the terminal value of the fund. In particular, we show that the optimal dividend strategy is given in terms of an optimal boundary b that is decreasing, continuous, bounded, and null at terminal time. To the best of our knowledge, also this result appears here for the first time.

The rest of the paper is organized as follows. In Section 2 we set up the problem, and in Section 3 we state the connection between (1.1) and (1.2). Its proof is then performed in Section 4. In Section 5 we consider the case study with (discounted) constant marginal benefits and costs, whereas in the Appendices we collect the proofs of some results needed in the paper.

2. Problem Formulation

In this section we introduce the optimal dividend problem that is the object of our study. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\mathbb{F} := (\mathcal{F}_s)_{s\geq 0}$ which satisfies the usual conditions. We assume that the fund's value is described by the one-dimensional process

$$X_s^D(x) = x + \mu s + \sigma W_s - D_s + I_s^D, \quad s \geq 0, \tag{2.1}$$

where $x \geq 0$ is the initial value of the fund, $\mu \in \mathbb{R}$, $\sigma > 0$, and W is a standard Brownian motion. For any $s \geq 0$, D_s represents the cumulative amount of dividends paid to shareholders up to time s, whereas I_s^D is the cumulative amount of capital injected by the shareholders up to time s in order to avoid bankruptcy of the fund.

Remark 2.1. *In absence of any dividends' payment and capital injections the fund's value evolves as a Brownian motion with drift μ and volatility σ. Such a dynamics is typical in the literature on the optimal dividend problem (see [1], [9] and [25], among many others), and it can be obtained as a diffusion approximation of a classical risk model à la Cramér-Lundberg (see Appendix D.3 in [30] for details).*

The fund's manager can pick a dividend distributions' strategy in the (nonempty) set

$$\mathcal{A} = \Big\{ \nu : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+, \mathbb{F} - \text{adapted s.t. } s \mapsto \nu_s(\omega) \text{ is a.s.}$$
$$\text{nondecreasing and left-continuous, and } \nu_0 = 0 \text{ a.s.} \Big\}.$$

For any $D \in \mathcal{A}$ the process $X^D(x)$ is reflected at zero through the capital injections process $I^D \in \mathcal{A}$. In fact, for any $x \geq 0$ and $D \in \mathcal{A}$ the couple (X^D, I^D) is the solution to the discontinuous reflection problem (see, e.g., [7] and [26]):

$$\text{Find } (X^D, I^D) \text{ s.t. } \begin{cases} I^D \in \mathcal{A}, \quad X^D_s = x + \mu s + \sigma W_s - D_s + I^D_s, \quad s \geq 0, \\ X^D_s \geq 0 \quad \text{a.s. for any } s \geq 0, \\ \int_0^\infty X^D_s d(I^D_s)^c = 0 \quad \text{a.s.,} \\ \Delta I^D_s = I^D_{s+} - I^D_s = 2X_{s+} \quad \text{a.s.} \end{cases} \tag{2.2}$$

Here, $(I^D)^c$ denotes the continuous part of I^D. Roughly speaking, I^D is the minimal nondecreasing process which ensures that X^D stays nonnegative, and which is flat off $\{t \geq 0 : X^D_t = 0\}$.

We assume that a.s.

$$D_{s+} - D_s \leq X^D_s \quad \text{for all } s \geq 0; \tag{2.3}$$

that is, bankruptcy cannot be obtained with a single lump sum dividends' payment. Notice that under (2.3), it is shown in Proposition 2 in [8] that the unique $\big(X^D, I^D\big)$ solving problem (2.2) is such that

$$I^D_t = 0 \vee \sup_{0 \leq s \leq t} \left(D_s - (x + \mu s + \sigma W_s) \right).$$

Moreover, $t \mapsto I^D_t$ is continuous.

Given a time horizon $T \in (0, \infty)$ representing, e.g., a finite liquidation time, the fund's manager takes the point of view of the shareholders, and is faced with the problem of choosing a dividend distributions' strategy D maximizing the performance criterion

$$\mathcal{J}(D; t, x) = \mathbb{E}\left[\int_0^{T-t} f(t+s) \; dD_s - \int_0^{T-t} m(t+s) \; dI^D_s + g(T, X^D_{T-t}(x)) \right], \tag{2.4}$$

for $(t, x) \in [0, T] \times \mathbb{R}_+$ given and fixed. That is, the fund's manager aims at solving

$$V(t, x) := \sup_{D \in \mathcal{D}(t,x)} \mathcal{J}(D; t, x), \quad (t, x) \in [0, T] \times \mathbb{R}_+. \tag{2.5}$$

Here, for any $(t, x) \in [0, T] \times \mathbb{R}_+$, $\mathcal{D}(t, x)$ denotes the class of dividend payments belonging to $\mathcal{A}$ and satisfying (2.3), when the surplus process X^D starts from level x and the optimization runs up to time $T - t$. In the following, any $D \in \mathcal{D}(t, x)$ will be called *admissible* for $(t, x) \in [0, T] \times \mathbb{R}_+$.

In the reward functional (2.4) the term $\mathbb{E}[\int_0^{T-t} f(t+s) \; dD_s]$ is the total expected cash-flow from dividends. The function f might be seen as a time-dependent instantaneous net proportion of leakages from the surplus received by the shareholders after time-dependent transaction costs/taxes have been paid. The term $\mathbb{E}[\int_0^{T-t} m(t+s) \; dI^D_s]$ gives the total expected costs of capital injections, and m is a time-dependent marginal administration cost for capital injections. Finally, $\mathbb{E}\left[g(T, X^D_{T-t}(x))\right]$ is a liquidation value.

The functions f, m and g satisfy the following conditions.

Assumption 2.2. *$f : [0,T] \to \mathbb{R}_+, m : [0,T] \to \mathbb{R}_+, g : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ are continuous, f and m are continuously differentiable with respect to t, and g is continuously differentiable with respect to x. Moreover,*

(i) $g_x(T,x) \geq f(T)$ *for any* $x \in (0,\infty)$,
(ii) $m(t) > f(t)$ *for any* $t \in [0,T]$.

Remark 2.3. *Requirement (i) ensures that the marginal liquidation value is at least as high as the marginal profits from dividends. This will ensure that the value function of the optimal stopping problem considered below is not discontinuous at terminal time.*

Condition (ii) means that the marginal costs for capital injections are bigger than the marginal profits from dividends. Notice that in the extreme case in which if $m < f$ the value function might be infinite, as it shown in the next example. Take $f(s) = \eta$, $m(s) = \kappa$ for all $s \in [0,T]$, and $\eta > \kappa$. For arbitrary $\beta > 0$ consider the admissible strategy $\widehat{D}_s := \beta s$, and notice that $\widehat{I}^D_s = \sup_{0\leq u\leq s}(-x - \mu u - \sigma B_u + \beta u) \vee 0$. Then $\widehat{I}^D_s \leq \beta s + Y_s$, with $Y_s := \sup_{0\leq u\leq s}(-x - \mu u - \sigma B_u) \vee 0$, and using that $g \geq 0$ we obtain for the sub-optimal strategy $\widehat{D}$

$$\begin{aligned} V(t,x) &\geq \beta\eta(T-t) - \beta\kappa(T-t) - \kappa\mathbb{E}[Y_{T-t}] \\ &= \beta(T-t)(\eta-\kappa) - \kappa\mathbb{E}[Y_{T-t}]. \end{aligned}$$

However, the latter expression can be made arbitrarily large by increasing β if $\eta > \kappa$.

On the other hand, by taking $m(t) = f(t) = e^{-rt}$, is has been recently shown in [15] for a problem with $T = +\infty$ (see Theorem 3.8 therein) that an optimal control may not exist, but only an ε-optimal control does exist.

In order to avoid pathological situations as the ones described above, here we assume Assumption 2.2-(ii).

Remark 2.4. *Notice that our formulation is general enough to accommodate also a problem in which profits and costs are discounted at a deterministic time-dependent discount rate $(r_s)_{s\geq 0}$. Indeed, if we consider the optimal dividend problem with capital injections*

$$\widehat{V}(t,x) := \sup_{D\in\mathcal{D}(t,x)} \mathbb{E}\bigg[\int_0^{T-t} e^{-\int_t^{t+s} r_\alpha d\alpha}\, \widehat{f}(t+s)\; dD_s - \int_0^{T-t} e^{-\int_t^{t+s} r_\alpha d\alpha}\, \widehat{m}(t+s)\; dI^D_s \\ + e^{-\int_t^T r_\alpha d\alpha} \widehat{g}(T, X^D_{T-t}(x))\bigg],$$

then, for any $(t,x) \in [0,T] \times \mathbb{R}_+$ we can set

$$f(t) := e^{-\int_0^t r_\alpha d\alpha}\, \widehat{f}(t), \quad m(t) := e^{-\int_0^t r_\alpha d\alpha}\, \widehat{m}(t), \quad g(t,x) := e^{-\int_0^t r_\alpha d\alpha} \widehat{g}(t,x),$$

and $V(t,x) := e^{-\int_0^t r_\alpha d\alpha}\, \widehat{V}(t,x)$ is of the form (2.5).

In Section 5 we will consider a problem with constant marginal profits and costs discounted at a constant rate $r > 0$ (see (5.1), (5.2) and (5.3) in Section 5).

Remark 2.5. *Notice that in our model shareholders are forced to inject capital whenever the surplus process attempts to become negative; that is, the capital injection process is not a control variable of their, and shareholders do not choose when and how invest in the company.*

Injecting capital at zero, under the condition that bankruptcy is not allowed, can be shown to be optimal in the canonical formulation of the optimal dividend problem of Section 5 in which marginal costs and profits are constants discounted at a constant interest rate. Indeed, in such a case, due to discounting, shareholders will inject capital as late as possible in order to minimize the total costs of injections. More in general, the policy "inject capital at zero" is optimal when m is decreasing and $\min_{t\in[0,T]} m(t) > g_x(T,x)$ for all $x \in \mathbb{R}_+$. Under these *conditions, shareholders postpone injection of capital, and inject only as much as necessary since any additional injection cannot be compensated by the reward at terminal time.*

The dynamic programming equation for V takes the form of a parabolic partial differential equation (PDE) with gradient constraint, and with a Neumann boundary condition at $x = 0$ (the latter is due to the fact that the state process X is reflected at zero through the capital injections process). Indeed, it reads

$$\max\left\{\partial_t U + \frac{1}{2}\sigma^2\partial_{xx}U + \mu\partial_x U, f - \partial_x U\right\} = 0, \quad \text{on} \quad [0,T) \times (0,\infty),$$

with boundary conditions $\partial_x U(0,t) = m(t)$ for all $t \in [0,T]$, and $U(T,x) = g(T,x)$ for any $x \in (0,\infty)$. Proving that such a PDE problem admits a solution that has enough regularity to characterize an optimal control is far from being trivial.

In order to solve optimal dividend problem (2.5) we then follow a different approach, and we relate (2.5) to an optimal stopping problem with absorbing condition at $x = 0$. This is obtained by borrowing arguments from the study in [13] on the connection between reflected follower problems and questions of optimal stopping (see also [3] and [20]). However, differently to [13], in our performance criterion (2.4) we also have a cost of reflection which requires a careful and not immediate adaptation of the ideas and results of [13].

In particular, introducing a problem of optimal stopping with absorption at zero, we show that a proper integration of the value function of the latter leads to the value function of the optimal control problem (2.5). This result is stated in the next section, and then proved in Section 4.

3. The Main Result

Let $S(x) := \inf\{s \geq 0 : x + \mu s + \sigma W_s = 0\}$, $x \geq 0$, and for any $s \geq 0$, introduce the absorbed drifted Brownian motion

$$A_s(x) := \begin{cases} x + \mu s + \sigma W_s, & s < S(x), \\ \Delta, & s \geq S(x), \end{cases} \tag{3.1}$$

where Δ is a cemetery state isolated from $\mathbb{R}_+$ (i.e. $\Delta < 0$).

Introducing the convention $g_x(T,\Delta) := 0$, for $(t,x) \in [0,T] \times \mathbb{R}_+$, consider the optimal stopping problem

$$\begin{aligned} u(t,x) := & \sup_{\tau \in [0,T-t]} \mathbb{E}\Big[f(t+\tau)\mathbb{1}_{\{\tau < (T-t)\wedge S(x)\}} + m(t+S(x))\mathbb{1}_{\{\tau \geq S(x)\}} \\ & \qquad + g_x(T, x + \mu(T-t) + \sigma W_{T-t})\mathbb{1}_{\{\tau = T-t < S(x)\}}\Big] \\ = & \sup_{\tau \in \Lambda(T-t)} \mathbb{E}\Big[f(t+\tau)\mathbb{1}_{\{A_\tau(x) > 0\}}\mathbb{1}_{\{\tau < T-t\}} + m(t+S(x))\mathbb{1}_{\{A_\tau(x) \leq 0\}} \\ & \qquad + g_x(T, A_{T-t}(x))\mathbb{1}_{\{\tau = T-t\}}\Big], \end{aligned} \tag{3.2}$$

where $\Lambda(T-t)$ denotes the set of all $\mathbb{F}$-stopping times with values in $[0, T-t]$ a.s. Problem (3.2) is an optimal stopping problem for the absorbed process A.

To establish the relation between (2.5) and (3.2) we need the following *structural assumption*, which will be standing in this section and in Section 4. Its validity has to be verified on a case by case basis. In particular, it holds in the optimal dividend problem considered in Section 5.

Assumption 3.1. *Assume that the continuation region of the stopping problem* (3.2) *is given by*

$$\mathcal{C} := \{(t,x) \in [0,T) \times (0,\infty) : u(t,x) > f(t)\} = \{(t,x) \in [0,T) \times (0,\infty) : x < b(t)\}, \tag{3.3}$$

and that its stopping region by

$$
\begin{aligned}
\mathcal{S} &:= \{(t,x) \in [0,T) \times (0,\infty) : u(t,x) \le f(t)\} \cup \big(\{T\} \times (0,\infty)\big) \\
&= \{(t,x) \in [0,T) \times (0,\infty) : x \ge b(t)\} \cup \big(\{T\} \times (0,\infty)\big),
\end{aligned}
\tag{3.4}
$$

for a continuous function $b : [0,T) \to (0,\infty)$. We refer to the function b as to the optimal stopping boundary of problem (3.2). Further, assume that the stopping time

$$
\tau^\star(t,x) := \inf\{s \in [0,T-t) : A_s(x) \ge b(t+s)\} \wedge (T-t)
\tag{3.5}
$$

(with the usual convention $\inf \emptyset = +\infty$) is optimal; that is,

$$
\begin{aligned}
u(t,x) = \mathbb{E}\Big[& f(t+\tau^\star(t,x))\mathbb{1}_{\{\tau^\star(t,x)<(T-t)\wedge S(x)\}} + m(t+S(x))\mathbb{1}_{\{\tau^\star(t,x)\ge S(x)\}} \\
& + g_x(T, x+\mu(T-t)+\sigma W_{T-t})\mathbb{1}_{\{\tau^\star(t,x)=T-t<S(x)\}}\Big].
\end{aligned}
\tag{3.6}
$$

For any $(t,x) \in [0,T] \times \mathbb{R}_+$, and with b the optimal stopping boundary of problem (3.2) (cf. Assumption 3.1), we define the processes $I^\star(t,x)$ and $D^\star(t,x)$ through the system

$$
\begin{cases}
D^\star_s(t,x) := \max\left\{0, \max\limits_{0\le\theta\le s} \big(x+\mu\theta+\sigma W_\theta + I^\star_\theta(t,x) - b(t+\theta)\big)\right\}, \\
I^\star_s(t,x) := \max\left\{0, \max\limits_{0\le\theta\le s} \big(-x-\mu\theta-\sigma W_\theta + D^\star_\theta(t,x)\big)\right\},
\end{cases}
\tag{3.7}
$$

for any $s \in [0,T-t]$, and with initial values $D^\star_0(t,x) = I^\star_0(t,x) = 0$ a.s. The existence and uniqueness of the solution to system (3.7) can be proved by an application of Tarski's fixed point theorem following arguments as those employed in the proof of Proposition 7 in Section 8 of [19]. Moreover, $I^\star$ has continuous paths thanks to (2.3), whereas $t \mapsto D^\star_t$ is continuous a part for a possible initial jump at time zero of amplitude $(x-b(t))^+$. We can now state the following result.

Theorem 3.2. *Let Assumption 3.1 hold true. Then, the process $D^\star$ defined through (3.7) provides the optimal dividends' distribution policy, and the value function V of (2.5) is such that*

$$
V(t,b(t)) - V(t,x) = \int_x^{b(t)} u(t,y)\, dy, \quad (t,x) \in [0,T] \times \mathbb{R}_+.
\tag{3.8}
$$

Consistently with the result of [13] (see also [20]), we find that also in our problem with costly reflection at zero the value of an optimal stopping problem (namely, problem (3.2)) gives the marginal value of the value function (2.5). Moreover, the optimal stopping boundary b triggers the timing at which it is optimal to pay an additional unit of dividends. The proof of Theorem 3.2 is quite lengthy and technical, and it is delegated to Section 4.

4. On the Proof of Theorem 3.2

This section is entirely devoted to the proof of Theorem 3.2. This is done through a series of intermediate results which are proved by employing mostly probabilistic arguments. Assumption 3.1 will be standing throughout this section.

4.1. **On a Representation of the Optimal Stopping Value Function.** Here we derive an alternative representation for the value function of the optimal stopping problem (3.2), by borrowing ideas from [13], Section 3. In the following we set $g_x(T,\Delta) = 0$.

The idea that we adopt here is to rewrite the optimal stopping problem (3.2) in terms of the function b of Assumption 3.1. To accomplish that, for given $(t,x) \in [0,T] \times \mathbb{R}_+$, define the payoff associated to the admissible stopping rule "never stop" as

$$G(t,x) := \mathbb{E}\left[m(t+S(x))\mathbb{1}_{\{S(x)\leq T-t\}} + g_x(T, A_{T-t}(x))\right], \tag{4.1}$$

where we have used that $g_x(T, A_{T-t}(x))\mathbb{1}_{\{T-t<S(x)\}} = g_x(T, A_{T-t}(x))$ because of (3.1) and the fact that $g_x(T,\Delta) = 0$.

Also, introduce the function $\tilde{g} : [0,T] \times [0,\infty] \times \mathbb{R}_+ \to \mathbb{R}$ (depending parametrically on (t,x)) as

$$\tilde{g}(\alpha, q, y; t, x) := \begin{cases} g_x(T,y), & \alpha < q \\ m(t+q), & \alpha \geq q, \end{cases} \tag{4.2}$$

and notice that $v := u - G$ admits the representation

$$v(t,x) = \sup_{\tau \in \Lambda(T-t)} \mathbb{E}\left[\left(f(t+\tau) - \tilde{g}(T-t, S(x), A_{T-t}(x); t,x)\right)\mathbb{1}_{\{\tau < S(x)\wedge T-t\}\}}\right]. \tag{4.3}$$

Clearly, the stopping time $\tau^\star$ defined by (3.5) is also optimal for v since G is independent of $\tau \in \Lambda(T-t)$. Therefore, we can expect that v can be expressed in terms of the optimal stopping boundary b. Following [13], we obtain such a representation for v by means of the theory of dual previsible projections ("balayée prévisible"), as it is shown in the following. From now on, $(t,x) \in [0,T] \times \mathbb{R}_+$ will be given and fixed.

We define the process $(C_\alpha)_{\alpha\in[0,T]}$ such that for any $\alpha \in [0, T-t]$

$$\begin{aligned} C_\alpha(t,x) := &- \int_0^{\alpha\wedge S(x)\wedge T-t} f'(t+\theta)d\theta \\ &+ \left[f(T\wedge(t+S(x))) - \tilde{g}(T-t, S(x), A_{T-t}(x); t,x)\right]\mathbb{1}_{\{0<T-t\wedge S(x)\leq\alpha\}}, \end{aligned} \tag{4.4}$$

as well as the stopping time

$$\sigma_\alpha(t,x) := \inf\{\theta \in [\alpha, T-t) : A_\theta(x) \geq b(t+\theta)\} \wedge (T-t), \tag{4.5}$$

with the convention $\inf \emptyset = +\infty$. The process $C_\cdot(t,x)$ is absolutely continuous on $[0, T-t) \wedge S(x)$ with a possible jump at $(T-t)\wedge S(x)$, and $\alpha \mapsto \sigma_\alpha(t,x)$ is a.s. nondecreasing and right-continuous.

Since the stopping time $\sigma_0(t,x)$ is optimal for $u(t,x)$ by Assumption 3.1, and therefore also for $v(t,x) = (u-G)(t,x)$, by using (4.4) we can write from (4.3)

$$v(t,x) = \mathbb{E}\left[C_{T-t}(t,x) - C_{\sigma_0(t,x)}(t,x)\right] = \mathbb{E}\Big[\widetilde{C}_{T-t}(t,x)\Big], \tag{4.6}$$

where we have introduced

$$\widetilde{C}_\alpha(t,x) := C_{\sigma_\alpha(t,x)}(t,x) - C_{\sigma_0(t,x)}(t,x), \quad \alpha \in [0, T-t]. \tag{4.7}$$

The process $\widetilde{C}_\cdot(t,x)$ is of bounded variation, since it is the composition of the process of bounded variation $C_\cdot(t,x)$ and of the nondecreasing process $\sigma_\cdot(t,x)$, but it is not $\mathbb{F}$-adapted. However, being v an excessive function, it is also the potential of an adapted, nondecreasing process $\Theta_\cdot(t,x)$ (cf. Section IV.4 in [5]) which is the dual predictable (or previsible) projection of $\widetilde{C}_\cdot(t,x)$ (see, e.g., [29], Chapter VI, Theorem 21.1, for further details on the dual predictable projection). In the following we provide the explicit representation of $\Theta_\cdot(t,x)$. This is obtained by employing the methodology of [14], Section 7.

Theorem 4.1. *The dual predictable projection $\Theta(t,x)$ of $\widetilde{C}(t,x)$ exists, is nondecreasing and it is given by*

$$\begin{aligned}\Theta_\alpha(t,x) &= \int_0^\alpha -f'(t+\theta)\mathbb{1}_{\{A_\theta(x)>b(t+\theta)\}}\,d\theta\\ &\quad+\Big[f(T\wedge(t+S(x)))-\tilde{g}(T-t,S(x),A_{T-t}(x);t,x)\Big]\mathbb{1}_{\{A_{T-t}(x)>b(T)\}}\mathbb{1}_{\{0<T-t\wedge S(x)\leq\alpha\}}\\ &=\int_0^{\alpha\wedge S(x)} -f'(t+\theta)\mathbb{1}_{\{x+\mu\theta+\sigma W_\theta>b(t+\theta)\}}\,d\theta\\ &\quad+\Big[f(T\wedge(t+S(x)))-\tilde{g}(T-t,S(x),A_{T-t}(x);t,x)\Big]\mathbb{1}_{\{A_{T-t}(x)>b(T)\}}\mathbb{1}_{\{0<T-t\wedge S(x)\leq\alpha\}}\end{aligned}\tag{4.8}$$

for any $\alpha\in[0,T-t]$.

Theorem 4.1 can be proved by carefully adapting to our case the techniques presented in Section 7 of [14] (see also, Section 3 of [13]). In particular, differently to Section 7 of [14], here we deal with an absorbed drifted Brownian motion as a state variable of the optimal stopping problem (3.2) (instead of a Brownian motion). However, all the arguments and proofs of Section 7 of [14] carry over also to our setting with random time horizon $(T-t)\wedge S(x)$ (up to which the process A is in fact a drifted Brownian motion) upon using representation (4.3) of v (in which the function $\tilde{g}$ takes care of the random time horizon $(T-t)\wedge S(x)$) together with (4.5) and (4.7).

A consequence of Theorem 4.1 is the next result.

Corollary 4.2. *It holds that*

(i) $\big[f(T\wedge(t+S(x)))-\tilde{g}(T-t,S(x),A_{T-t}(x);t,x)\big]\mathbb{1}_{\{A_{T-t}(x)>b(T)\}}=0$ *a.s.*

(ii) $\{t\in[0,T):f'(t)\leq 0\}\supseteq\mathcal{S}$*;*

Proof. (i) On the set $\{A_{T-t}(x)>b(T)\}$ we obtain by the definition of $\tilde{g}$ (see (4.2)) that

$$f(T\wedge(t+S(x)))-\tilde{g}(T-t,S(x),A_{T-t}(x);t,x)=f(T)-g_x(T,A_{T-t}(x)).\tag{4.9}$$

Since $\Theta_\cdot(t,x)$ is nondecreasing, the last term in (4.9) has to be positive, thus implying $f(T)-g_x(T,A_{T-t}(x))\geq 0$ on $\{A_{T-t}(x)>b(T)\}$. However, by Assumption 2.2-(i) one has $f(T)\leq g_x(T,x)$ for all $x\in(0,\infty)$. Hence the claim follows.

(ii) Since $\alpha\mapsto\Theta_\alpha(t,x)$ is a.s. nondecreasing, it follows from (i) above and (4.8) that $f'(t+\theta)\mathbb{1}_{\{A_\theta(x)>b(t+\theta)\}}\leq 0$ a.s. for a.e. $\theta\in[0,T-t]$. But $f'(\cdot)$, $A_\cdot(x)$ and $b(t+\cdot)$ are continuous up to $T-t\wedge S(x)$, and therefore the latter actually holds a.s. for all $\theta\in[0,T-t]$. Hence, $\{t\in[0,T):f'(t)\leq 0\}\supseteq\mathcal{S}$. □

We can now obtain an alternative representation of the value function u of problem (3.2).

Theorem 4.3. *For any $(t,x)\in[0,T]\times\mathbb{R}_+$ one has*

$$\begin{aligned}u(t,x)=\mathbb{E}\bigg[\int_0^{(T-t)\wedge S(x)} &-f'(t+\theta)\mathbb{1}_{\{x+\mu\theta+\sigma W_\theta\geq b(t+\theta)\}}\,d\theta\\ &+m(t+S(x))\mathbb{1}_{\{S(x)\leq T-t\}}+g_x(T,A_{T-t}(x))\bigg].\end{aligned}\tag{4.10}$$

Proof. Since by Theorem 4.1 $\Theta(t,x)$ is the dual predictable projection of $\widetilde{C}(t,x)$, from (4.6) we can write for any $(t,x)\in[0,T]\times\mathbb{R}_+$

$$v(t,x)=\mathbb{E}\left[\widetilde{C}_{T-t}(t,x)\right]=\mathbb{E}\left[\Theta_{T-t}(t,x)\right].\tag{4.11}$$

Due to (4.8) and Corollary 4.2-(i), (4.11) gives

$$v(t,x)=\mathbb{E}\left[\int_0^{(T-t)\wedge S(x)} -f'(t+\theta)\mathbb{1}_{\{x+\mu\theta+\sigma W_\theta\geq b(t+\theta)\}}\,d\theta\right]. \tag{4.12}$$

Here we have also used that the joint law of $S(x)$ and of the drifted Brownian motion is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^2$ (cf. (A.2)) to replace $\mathbb{1}_{\{x+\mu\theta+\sigma W_\theta>b(t+\theta)\}}$ with $\mathbb{1}_{\{x+\mu\theta+\sigma W_\theta\geq b(t+\theta)\}}$ inside the expectation in (4.8).

However, since by definition $v=u-G$, we obtain from (4.12) and (4.1) the alternative representation

$$\begin{aligned}u(t,x)=v(t,x)+G(t,x)=\mathbb{E}\bigg[&\int_0^{(T-t)\wedge S(x)} -f'(t+\theta)\mathbb{1}_{\{x+\mu\theta+\sigma W_\theta\geq b(t+\theta)\}}\,d\theta\\&+m(t+S(x))\mathbb{1}_{\{S(x)\leq T-t\}}+g_x(T,A_{T-t}(x))\bigg].\end{aligned}$$

□

Remark 4.4. *Notice that representation (4.10) coincides with that one might obtain by an application of Itô's formula if u were $C^{1,2}([0,T)\times(0,\infty))\cap C([0,T]\times\mathbb{R}_+)$, and satisfies (as it is customary in optimal stopping problems) the free-boundary problem*

$$\begin{cases}\partial_t u+\frac{1}{2}\sigma^2\partial^2_{xx}u+\mu\partial_x u=0, & 0<x<b(t),\ t\in[0,T)\\ u=f, & x\geq b(t),\ t\in[0,T)\\ u(T,x)=g_x(T,x), & x>0\\ u(t,0)=m(t), & t\in[0,T].\end{cases} \tag{4.13}$$

Indeed, in such a case an application of Dynkin's formula gives

$$\mathbb{E}\left[u(t+(T-t)\wedge S(x),Z_{(T-t)\wedge S(x)}(x))\right]=u(t,x)+\mathbb{E}\left[\int_0^{(T-t)\wedge S(x)} f'(t+\theta)\mathbb{1}_{\{Z_\theta(x)\geq b(t+\theta)\}}\,d\theta\right],$$

where we have set $Z_s(x):=x+\mu s+\sigma W_s$, $s\geq 0$, to simplify exposition. Hence, using (4.13) we have from the latter

$$\begin{aligned}u(t,x)&=\mathbb{E}\bigg[m(t+S(x))\mathbb{1}_{\{S(x)\leq T-t\}}+g_x(T,x+\mu(T-t)+\sigma W_{T-t})\mathbb{1}_{\{S(x)>T-t\}}\\&-\int_0^{(T-t)\wedge S(x)} f'(t+\theta)\mathbb{1}_{\{Z_\theta(x)\geq b(t+\theta)\}}\,d\theta\bigg]=\mathbb{E}\bigg[m(t+S(x))\mathbb{1}_{\{S(x)\leq T-t\}}\\&+g_x(T,A_{T-t}(x))\mathbb{1}_{\{S(x)>T-t\}}-\int_0^{(T-t)\wedge S(x)} f'(t+\theta)\mathbb{1}_{\{Z_\theta(x)\geq b(t+\theta)\}}\,d\theta\bigg]\\&=\mathbb{E}\bigg[m(t+S(x))\mathbb{1}_{\{S(x)\leq T-t\}}+g_x(T,A_{T-t}(x))-\int_0^{(T-t)\wedge S(x)} f'(t+\theta)\mathbb{1}_{\{Z_\theta(x)\geq b(t+\theta)\}}\,d\theta\bigg],\end{aligned}$$

where in the last step we have used that $g_x(T,A_{T-t}(x))\mathbb{1}_{\{S(x)>T-t\}}=g_x(T,A_{T-t}(x))$ because of (3.1) and the fact that $g_x(T,\Delta)=0$.

Remark 4.5. *Notice that representation (4.10) immediately gives an integral equation for the optimal stopping boundary b. Indeed, since (4.10) holds for any $(t,x)\in[0,T]\times\mathbb{R}_+$, by taking $x=b(t)$, $t\leq T$, on both sides of (4.10), and by recalling that $u(t,b(t))=f(t)$, we find that b*

solves

$$f(t) = \mathbb{E}\bigg[\int_0^{(T-t)\wedge S(b(t))} -f'(t+\theta) \mathbb{1}_{\{b(t)+\mu\theta+\sigma W_\theta \geq b(t+\theta)\}} \, d\theta$$
$$+ m(t+S(b(t)))\mathbb{1}_{\{S(b(t))\leq T-t\}} + g_x(T, A_{T-t}(b(t)))\bigg]. \tag{4.14}$$

By following arguments as those in Section 25 of [27] *based on the superharmonic characterization of* u*, one might then prove that* b *is the unique solution to* (4.14) *among a suitable class of continuous and positive functions.*

The next result follows from (4.10) by expressing the expected value as an integral with respect to the probability densities of the involved processes and random variables. Its proof can be found in the Appendix for the sake of completeness.

Corollary 4.6. *One has that* $u(t,\cdot)$ *is continuously differentiable on* $(0,\infty)$ *for all* $t \in [0,T)$.

In the next section we will suitably integrate the two alternative representations of u (3.6) and (4.10) with respect to the space variable, and we will show that such integrations give the value function (2.5) of the optimal dividend problem. As a byproduct, we will also obtain the optimal dividend strategy $D^\star$.

4.2. **Integrating the Optimal Stopping Value Function.** In the next two propositions we integrate with respect to the space variable the two representations of u given by (3.6) and (4.10). The proofs will employ pathwise arguments. However, in order to simplify exposition, we will not stress the ω-dependence of the involved random variables and processes.

Proposition 4.7. *Let* b *the optimal stopping boundary of problem* (3.2)*, set*

$$I^0_s(x) := \max_{0\leq\theta\leq s}\{0, -x-\mu\theta-\sigma W_\theta\}, \quad s\geq 0,$$

and define

$$R_s(x) := x+\mu s+\sigma W_s + I^0_s(x), \quad s \geq 0. \tag{4.15}$$

Then for any $(t,x) \in [0,T]\times\mathbb{R}_+$ *one has*

$$\int_x^{b(t)} u(t,y)\, dy = N(t,b(t)) - N(t,x), \tag{4.16}$$

where

$$N(t,x) := \mathbb{E}\bigg[-\int_0^{T-t} \big(R_s(x) - b(t+s)\big)^+ f'(t+s)\, ds - \int_0^{T-t} m(t+s)\, dI^0_s(x)$$
$$+ g(T, R_{T-t}(x))\bigg]. \tag{4.17}$$

Proof. To prove (4.16) we use representation (4.10) of the value function of the optimal stopping problem (3.2). Using Fubini-Tonelli's Theorem we obtain

$$\int_x^{b(t)} u(t,y)\, dy = \int_x^{b(t)} \mathbb{E}\bigg[\int_0^{(T-t)\wedge S(y)} -f'(t+s)\mathbb{1}_{\{y+\mu s+\sigma W_s\geq b(t+s)\}}\, ds$$
$$+ m(t+S(y))\mathbb{1}_{\{S(y)\leq T-t\}} + g_x(T, A_{T-t}(y))\bigg]\, dy$$
$$= \mathbb{E}\bigg[-\int_0^{(T-t)} f'(t+s)\bigg(\int_x^{b(t)} \mathbb{1}_{\{y+\mu s+\sigma W_s\geq b(t+s)\}}\mathbb{1}_{\{s\leq S(y)\}}\, dy\bigg)\, ds \tag{4.18}$$
$$+ \int_x^{b(t)} m(t+S(y))\mathbb{1}_{\{S(y)\leq T-t\}}\, dy + \int_x^{b(t)} g_x(T, A_{T-t}(y))\, dy\bigg].$$

In the following we investigate separately the three summands of the last term on the right-hand side of (4.18).

Recalling $S(x) = \inf\{u \geq 0 : x + \mu u + \sigma W_u = 0\}$ it is clear that

$$S(y) \geq s \Leftrightarrow M_s \leq y \tag{4.19}$$

for any $(s, y) \in \mathbb{R}_+ \times (0, \infty)$, where we have defined

$$M_s := \max_{0 \leq \theta \leq s} (-\mu\theta - \sigma W_\theta), \quad s \geq 0. \tag{4.20}$$

We can then rewrite (4.15) in terms of (4.20) and obtain

$$R_s(x) = (x \vee M_s) + \mu s + \sigma W_s, \quad s \geq 0. \tag{4.21}$$

By using (4.19) we find

$$\begin{aligned}
&\int_x^{b(t)} \mathbb{1}_{\{y+\mu s+\sigma W_s \geq b(t+s)\}} \mathbb{1}_{\{S(y)\geq s\}}\, dy = \int_{x\vee\left[b(t+s)-\mu s-\sigma W_s\right]}^{b(t)\vee\left[b(t+s)-\mu s-\sigma W_s\right]} \mathbb{1}_{\{S(y)\geq s\}}\, dy \\
&= \int_{x\vee\left[b(t+s)-\mu s-\sigma W_s\right]}^{b(t)\vee\left[b(t+s)-\mu s-\sigma W_s\right]} \mathbb{1}_{\{M_s\leq y\}}\, dy \\
&= \left[(b(t) \vee (b(t+s) - \mu s - \sigma W_s) \vee M_s) - (x \vee (b(t+s) - \mu s - \sigma W_s) \vee M_s)\right] \\
&= \left[(b(t) \vee M_s) \vee (b(t+s) - \mu s - \sigma W_s) - (x \vee M_s) \vee (b(t+s) - \mu s - \sigma W_s)\right] \\
&= \left[\big([(b(t) \vee M_s) + \mu s + \sigma W_s] \vee b(t+s)\big) - \big([(x \vee M_s) + \mu s + \sigma W_s] \vee b(t+s)\big)\right] \\
&= \left[\big(R_s(b(t)) \vee b(t+s)\big) - \big(R_s(x) \vee b(t+s)\big)\right] \\
&= \left[\big(R_s(b(t)) - b(t+s)\big)^+ - \big(R_s(x) - b(t+s)\big)^+\right].
\end{aligned} \tag{4.22}$$

For the third summand of the last term of the right-hand side of (4.18) we have, due to the fact that $g_x(T, \Delta) = 0$,

$$\begin{aligned}
\int_x^{b(t)} g_x(T, A_{T-t}(y))dy &= \int_x^{b(t)} g_x(T, y + \mu(T-t) + \sigma W_{T-t}) \mathbb{1}_{\{S(y)>T-t\}} dy \\
&= \int_x^{b(t)} g_x(T, y + \mu(T-t) + \sigma W_{T-t}) \mathbb{1}_{\{M_{T-t}<y\}} dy \\
&= \int_{x\vee M_{T-t}}^{b(t)\vee M_{T-t}} g_x(T, y + \mu(T-t) + \sigma W_{T-t}) dy \\
&= g(T, R_{T-t}(b(t))) - g(T, R_{T-t}(x)),
\end{aligned} \tag{4.23}$$

where in the last step we use (4.21). To prove that

$$\int_x^{b(t)} m(t + S(y)) \mathbb{1}_{\{S(y)\leq T-t\}} dy = \int_0^{T-t} m(t+s) dI_s^0(x) - \int_0^{T-t} m(t+s) dI_s^0(b(t)) \tag{4.24}$$

we have to distinguish two cases. In the following we let $(t, x) \in [0, T] \times \mathbb{R}_+$ be given and fixed, and we prove (4.24) by taking $x < b(t)$. The arguments are exactly the same if $b(t) < x$ by reversing the roles of x and $b(t)$.

Case 1. Here we take $x \in \{y \in \mathbb{R}_+ : S(y) \geq T - t\}$; that is, the initial point $x > 0$ is such that the drifted Brownian motion is not reaching 0 before the time horizon. This implies that $R_s(x)$ in (4.15) equals $x + \mu s + \sigma W_s$ and so $I_s^0(x) = 0$ for all $s \in [0, T - t]$. Hence, we can write

$$\int_x^{b(t)} m(t+S(y))\mathbb{1}_{\{S(y)\leq T-t\}}dy = 0 = \int_0^{T-t} m(t+s)dI_s^0(x) - \int_0^{T-t} m(t+s)dI_s^0(b(t)), \tag{4.25}$$

where we have used that $S(y) > S(x) \geq T-t$ for any $y > x$ and $\{x\}$ has zero Lebesgue measure to obtain the first equality, and the fact that $0 = I_s^0(x) \geq I_s^0(b(t)) \geq 0$ since $x < b(t)$.

Case 2. Here we take $x \in \{y \in \mathbb{R}_+ : S(y) < T-t\}$; i.e., the drifted Brownian motion reaches 0 before the time horizon. Define

$$z := \inf\{y \in \mathbb{R}_+ : S(y) \geq T-t\}, \tag{4.26}$$

with the usual convention $\inf \emptyset = +\infty$. In the sequel we assume that $z < +\infty$, since otherwise there is no need for the following analysis to be performed. Note that, by continuity in time and in the initial datum of the paths of the drifted Brownian motion, we have $S(z) \leq T-t$. Furthermore, it holds for all $y \in [x,z]$ that (cf. (4.20))

$$y + I_s^0(y) = M_s, \quad \forall s \geq S(y), \tag{4.27}$$

$$I_s^0(y) = 0, \quad \forall s < S(y). \tag{4.28}$$

Using (4.27), (4.28), (4.19), and the change of variable formula of Section 4 in Chapter 0 of [28] (see also equation (4.7) in [4]) we obtain

$$\begin{aligned}
&\int_x^{z\wedge b(t)} m(t+S(y))\mathbb{1}_{\{S(y)\leq T-t\}}dy = \int_x^{z\wedge b(t)} m(t+S(y))dy \\
&= \int_{S(x)}^{S(z\wedge b(t))} m(t+s)dM_s = \int_{S(x)}^{S(z\wedge b(t))} m(t+s)\left(dI_s^0(x) - dI_s^0(z\wedge b(t))\right) \\
&= \int_0^{T-t} m(t+s)\left(dI_s^0(x) - dI_s^0(z\wedge b(t))\right) \\
&= \int_0^{T-t} m(t+s)dI_s^0(x) - \int_0^{T-t} m(t+s)dI_s^0(z\wedge b(t)).
\end{aligned} \tag{4.29}$$

For the integral $\int_{z\wedge b(t)}^{b(t)} m(t+S(y))\mathbb{1}_{\{S(y)\leq T-t\}}\,dy$ we can use the result of Case 1 due to the definition of z (4.26). Then, combining (4.25) and (4.29) leads to (4.24).

By (4.22), (4.23) and (4.24), and recalling (4.17) and (4.18) we obtain (4.16). □

Proposition 4.8. *Let $(D^\star, I^\star)$ be the solution to system (3.7). Then, for any $(t,x) \in [0,T]\times\mathbb{R}_+$ one has*

$$\int_x^{b(t)} u(t,y)\ dy = M(t,b(t)) - M(t,x), \tag{4.30}$$

where b is the optimal stopping boundary of problem (3.2) and

$$M(t,x) := \mathbb{E}\left[\int_0^{T-t} f(t+s)\ dD_s^\star(t,x) - \int_0^{T-t} m(t+s)\ dI_s^\star(t,x) + g(T, X_{T-t}^{D^\star}(x))\right]. \tag{4.31}$$

Proof. For this proof we use instead the representation of u (cf. (3.6))

$$u(t,x) = \mathbb{E}\Big[f(t+\tau^\star(t,x))\mathbb{1}_{\{\tau^\star(t,x)<T-t\wedge S(x)\}} + m(t+S(x))\mathbb{1}_{\{\tau^\star(t,x)\geq S(x)\}} + g_x(T, A_{T-t}(x))\mathbb{1}_{\{\tau^\star(t,x)=T-t<S(x)\}}\Big]. \tag{4.32}$$

The proof is quite long and technical and it is organized in four steps. Moreover, in order to simplify exposition from now we set $t=0$. Indeed, all the following arguments remain valid if $t\in(0,T]$ by obvious modifications.

If $x\geq b(0)$ then (4.30) clearly holds. Indeed, $\int_x^{b(0)} u(0,y)\,dy = -(x-b(0))f(0)$ since $\tau^\star(0,y)=0$ for any $y\geq b(0)$. Also, from (4.31) $M(0,b(0))-M(0,x) = M(0,b(0)) - \big[(x-b(0))f(0)+M(0,b(0))\big]$, since $D^\star(0,x)$ has an initial jump of size $(x-b(0))$ which is such that $X^{D^\star}_{0+}(x)=b(0)$. Hence, in the following we prove (4.30) assuming that $x<b(0)$.

Step 1. Here we take $x\in\{y\in\mathbb{R}_+ : \tau^\star(0,y)<S(y)\}$; that is, the initial point $x>0$ is such that either the drifted Brownian motion reaches the boundary before hitting the origin, or the time horizon arises before hitting the origin. Define the process $(L_s)_{s\geq 0}$ such that

$$L_s := \max_{0\leq\theta\leq s}\{\mu\theta+\sigma W_\theta - b(\theta)\}, \qquad 0\leq s\leq T. \tag{4.33}$$

Then we have that for all $y\in[x,b(0)]$

$$\{\tau^\star(0,y)\leq s\} = \{L_s\geq -y\}, \tag{4.34}$$

$$\{\tau^\star(0,y)=T\} = \{L_T\leq -y\}, \tag{4.35}$$

$$D^\star_s(0,y) = \begin{cases} 0, & 0\leq s\leq\tau^\star(0,y),\\ y+L_s, & \tau^\star(0,y)\leq s\leq S(y),\end{cases} \tag{4.36}$$

and

$$X^{D^\star}_s(y) = \begin{cases} y+\mu s+\sigma W_s, & 0\leq s\leq\tau^\star(0,y),\\ \mu s+\sigma W_s - L_s, & \tau^\star(0,y)\leq s\leq S(y),\end{cases} \tag{4.37}$$

and in particular (cf. (3.7)) $I^\star_s(0,y) = I^\star_s(0,b(0)) = 0$ for any $s\in[0,\tau^\star(0,y)]$.

Moreover it follows by definition of $\tau^\star(0,x)$, $S(x)$ and $X^{D^\star}(x)$ that for all $y\in[x,b(0)]$ we have

$$0=\tau^\star(0,b(0))\leq\tau^\star(0,y)\leq\tau^\star(0,x), \tag{4.38}$$

$$\tau^\star(0,y)<\tau^\star(0,x)<S(x)\leq S(y), \tag{4.39}$$

and

$$\text{on } \{\tau^\star(0,x)<T\}:\quad X^{D^\star}_s(y) = X^{D^\star}_s(x), \quad \forall s>\tau^\star(0,x). \tag{4.40}$$

With these results at hand, we now show that for all $x\in[0,b(0)]$ such that $\tau^\star(0,x)<S(x)$ it holds that

$$\int_x^{b(0)} f(\tau^\star(0,y))1_{\{\tau^\star(0,y)<S(y)\}}dy = \int_0^T f(s)\,dD^\star_s(0,b(0)) - \int_0^T f(s)\,dD^\star_s(0,x), \tag{4.41}$$

$$\int_x^{b(0)} g_x(T,y+\mu T+\sigma W_T)\mathbb{1}_{\{\tau^\star(0,y)=T<S(y)\}}\,dy = g(T,X^{D^\star}_T(b(0))) - g(T,X^{D^\star}_T(x)) \tag{4.42}$$

and

$$\int_x^{b(0)} m(S(y))1_{\{\tau^\star(0,y)\geq S(y)\}}dy = \int_0^T m(s)\ dI^\star_s(0,x) - \int_0^T m(s)\ dI^\star_s(0,b(0)). \tag{4.43}$$

We start with (4.41). By (4.40) we have that $dD^\star_s(0,x) = dD^\star_s(0,b(0))$ for all $\tau^\star(0,x) < s \leq T$. By (4.36), and since $\tau^\star(0,b(0)) = 0$ one also has

$$D^\star_s(0,b(0)) = b(0) + L_s,\ \forall s \in [0, S(b(0))]. \tag{4.44}$$

Hence the right-hand side of (4.41) rewrites as

$$\begin{aligned}&\int_0^T f(s)\ dD^\star_s(0,b(0)) - \int_0^T f(s)\ dD^\star_s(0,x) = \int_0^{\tau^\star(0,x)} f(s)\ dD^\star_s(0,b(0))\\ &- \int_0^{\tau^\star(0,x)} f(s)\ dD^\star_s(0,x) = \int_0^{\tau^\star(0,x)} f(s)\ dD^\star_s(0,b(0)) = \int_0^{\tau^\star(0,x)} f(s)\ dL_s,\end{aligned} \tag{4.45}$$

where we have used that $dD^\star_s(0,x) = 0$ for all $s \in [0, \tau^\star(0,x)]$ by (4.36). However, by using a change of variable formula as in [4], equation (4.7), we obtain

$$\int_x^{b(0)} f(\tau^\star(0,y))1_{\{\tau^\star(0,y)<S(y)\}}dy = \int_x^{b(0)} f(\tau^\star(0,y))dy = \int_0^{\tau^\star(0,x)} f(s)\ dL_s, \tag{4.46}$$

where we have used (4.39) in the first step, and the fact that $L_\cdot$ is the left-continuous inverse of $\tau^\star(0,y)$ (cf. (4.34)) in the last equality. Combining (4.45) and (4.46) equation (4.41) holds.

Next we show (4.42). Using (4.44) and again (4.40) we obtain for the right-hand side of (4.42) that

$$g(T, X^{D^\star}_T(b(0))) - g(T, X^{D^\star}_T(x)) = [g(T, \mu T + \sigma W_T - L_T) - g(T, x + \mu T + \sigma W_T)]\, \mathbb{1}_{\{\tau^\star(0,x)=T\}}.$$

Also, (4.35) and (4.39) yields

$$\begin{aligned}&\int_x^{b(0)} g_x(T, y + \mu T + \sigma W_T)\mathbb{1}_{\{\tau^\star(0,y)=T\}}\ dy = \int_x^{b(0)} g_x(T, y + \mu T + \sigma W_T)\mathbb{1}_{\{y\leq -L_T\}}\ dy\\ &= [g(T, \mu T + \sigma W_T - L_T) - g(T, x + \mu T + \sigma W_T)]\, \mathbb{1}_{\{\tau^\star(0,x)=T\}}.\end{aligned}$$

Hence, we obtain (4.42).

Finally, for (4.43) there is nothing to show. In fact, the left-hand side is equal 0 by (4.39), while the right-hand side is zero since the processes $I^\star(0,x) = I^\star(0,b(0))$ coincide (cf. (4.40)).

Step 2. Here we take $x \in \{y \in \mathbb{R}_+ : \tau^\star(0,y) > S(y),\ \tau^\star(0,q) < S(q)\ \forall q \in (y, b(0))\}$. For a realization like that, such an x is such that the drifted Brownian motion touches the origin before hitting the boundary, but it does not cross the origin. This in particular implies that $I^\star_s(0,x) = 0$ for all $s \leq \tau^\star(0,x)$. Hence the same arguments employed in Step 1 hold true, and (4.41) – (4.43) follow.

Step 3. Here we take $x \in \{y \in \mathbb{R}_+ : \tau^\star(0,y) > S(y)\}$; that is, the drifted Brownian motion hits the origin before reaching the boundary.

Define

$$z := \inf\{y \in [0, b(0)] : \tau^\star(0,y) < S(y)\} \tag{4.47}$$

which exists finite since $y \mapsto \tau^\star(0,y) - S(y)$ is decreasing and $\tau^\star(0,b(0)) = 0$ and $S(0) = 0$ a.s. We want to prove that

$$\int_x^z m(S(y))\mathbb{1}_{\{\tau^\star(0,y)\geq S(y)\}}\ dy = \int_0^T m(s)\ dI^\star_s(0,x) - \int_0^T m(s)\ dI^\star_s(0,z), \tag{4.48}$$

$$\int_x^z f(\tau^\star(0,y))1_{\{\tau^\star(0,y)<S(y)\}}dy = \int_0^T f(s)\ dD^\star_s(0,z) - \int_0^T f(s)\ dD^\star_s(0,x), \tag{4.49}$$

and

$$\begin{aligned}&\int_x^z g_x(T, y+\mu T+\sigma W_T)\mathbb{1}_{\{\tau^\star(0,y)=T<S(y)\}}\ dy \\ &= \left[g(T, X_T^{D^\star}(z)) - g(T, X_T^{D^\star}(x))\right].\end{aligned} \tag{4.50}$$

Recall the process $(M_s)_{s\geq 0}$ of (4.20) such that

$$M_s = \max_{0\leq\theta\leq s}(-\mu\theta - \sigma W_\theta), \quad s \geq 0,$$

and (cf. (4.19))

$$\{M_s \geq x\} = \{S(x) \leq s\} \quad \forall s \geq 0.$$

For all $y \in [x,z)$ and $s \in [0, \tau^\star(0,y)]$ we have

$$I^\star_s(0,y) = \begin{cases} 0, & 0 \leq t \leq S(y) \\ M_s - y, & S(y) \leq s \leq \tau^\star(0,y), \end{cases} = (M_s - y)^+ \tag{4.51}$$

and

$$X_s^{D^\star}(y) = \begin{cases} y+\mu s+\sigma W_s, & 0 \leq s \leq S(y) \\ \mu s + \sigma W_s + M_s, & S(y) \leq s \leq \tau^\star(0,y), \end{cases} = (y \vee M_s) + \mu s + \sigma W_s. \tag{4.52}$$

Also, it follows by (4.52) and (4.51) that for all $y \in [x,z)$

$$X_s^{D^\star}(y) = X_s^{D^\star}(z)\ \forall s \geq S(z). \tag{4.53}$$

Moreover, recall that

$$S(x) \leq S(y) \leq S(z), \tag{4.54}$$

$$\tau^\star(0,y) > S(y), \tag{4.55}$$

With these observation at hand we can now show (4.48)-(4.50).

By (4.53) we have that $dI^\star_s(0,x) = dI^\star_s(0,z)$ for all $s \geq S(z)$. Further, we have that $I^\star_s(0,z) = 0$ for all $s \leq S(z)$. Therefore, by (4.54) $I^\star_s(0,z) = I^\star_s(0,x) = 0$ for $s \leq S(x)$, and the right-hand side of (4.48) rewrites as

$$\begin{aligned}\int_0^T m(s)\ dI^\star_s(0,x) - \int_0^T m(s)\ dI^\star_s(0,z) &= \int_{S(x)}^{S(z)} m(s)\ [dI^\star_s(0,x) - dI^\star_s(0,z)] \\ = \int_{S(x)}^{S(z)} m(s)\ dI^\star_s(0,x) &= \int_{S(x)}^{S(z)} m(s)\ dM_s.\end{aligned} \tag{4.56}$$

Here we have used (4.51) with $y = x$.

On the other hand, for the left-hand side of (4.48), we use the change of variable formula of Section 4 in Chapter 0 of [28]. This leads to

$$\int_x^z m(S(y))\mathbb{1}_{\{\tau^\star(0,y)\geq S(y)\}}\ dy = \int_x^z m(S(y))\ dy = \int_{S(x)}^{S(z)} m(s)\ dM_s, \tag{4.57}$$

where we use (4.55), the fact that $\{z\}$ is a Lebesgue zero set, and that M is the right-continuous inverse of S (see (4.19)). Combining (4.56) and (4.57) proves (4.48).

Equation (4.49) follows by observing that (4.53)–(4.54) imply that the processes $D^\star(0,z)$ and $D^\star(0,x)$ coincide, and the left-hand side equals 0 by definition. Notice that for such an argument particular care has to be put when considering z of (4.47) as a starting point for the drifted Brownian motion. In particular, if the realization of the Brownian motion is such that $\tau^\star(0,z) < S(z)$, then by definition of z, the drifted Brownian motion only touches the boundary at time $\tau^\star(0,z)$ but does not cross it. Hence, we still have $D^\star_s(0,z) = 0$ for all $s \leq S(z)$, which implies (4.53) and therefore still $D^\star_s(0,z) = D^\star_s(0,x)$. In turn, this gives again that (4.49) holds true also for such a particular realization of the Brownian motion.

Finally, to prove equation (4.50) remember that $x \in \{y \in \mathbb{R}_+ : \tau^\star(0,y) > S(y)\}$. By definition of z we obtain $\tau^\star(0,y) \geq S(y)$ for all $y \in [x,z)$ and the left-hand side of (4.50) equals zero. By (4.53) the processes $X^{D^\star}_s(z) = X^{D^\star}_s(x)$ coincides for all $s \geq S(z)$ and $S(z) \leq T$ a.s. by Lemma A.1 in the Appendix. Therefore, the right-hand side of (4.50) equals zero as well.

Step 4. For $x \in \{y \in \mathbb{R}_+ : \tau^\star(0,y) < S(y)\}$ (4.30) follows by the results of Step 1. If, instead, $x \in \{y \in \mathbb{R}_+ : \tau^\star(0,y) > S(y)\}$ then we can integrate u separately in the intervals $[x,z]$ and $[z,b(0)]$. When integrating u in the interval $[x,z]$ we use the results of Step 3. On the other hand, integrating u over $[z,b(0)]$ we have to distinguish two cases. Now, if z belongs to $\{y \in \mathbb{R}_+ : \tau^\star(0,y) < S(y)\}$ then we can still apply the results of Step 1 to conclude. If z belongs to $\{y \in \mathbb{R}_+ : \tau^\star(0,y) > S(y),\ \tau^\star(0,q) < S(q)\ \forall q \in (y,b(0))\}$ we can employ the findings of Step 2 to obtain the claim. Thus, in any case, (4.30) holds true. □

We now prove that the two functions N and M of (4.17) and (4.31), respectively, are such that $N = M$. To accomplish that we preliminary notice that by their definitions and strong Markov property, one has that the processes

$$N(t+s\wedge\tau^\star(t,x), R_{s\wedge\tau^\star(t,x)}(x)) - \int_0^{s\wedge\tau^\star(t,x)} m(t+\theta)\, dI^0_\theta(x),\ 0 \leq s \leq T-t, \tag{4.58}$$

and

$$M(t+s\wedge\tau^\star(t,x), R_{s\wedge\tau^\star(t,x)}(x)) - \int_0^{s\wedge\tau^\star(t,x)} m(t+\theta)\, dI^\star_\theta(t,x),\ 0 \leq s \leq T-t, \tag{4.59}$$

are $\mathbb{F}$-martingales for any $(t,x) \in [0,T] \times \mathbb{R}_+$. Moreover, by (4.16) one has $N(t,x) = N(t,b(t)) - \int_x^{b(t)} u(t,y)\, dy$ and, due to (4.30), $M(t,x) = M(t,b(t)) - \int_x^{b(t)} u(t,y)\, dy$. Hence,

$$\Psi(t) := M(t,x) - N(t,x), \quad t \in [0,T], \tag{4.60}$$

is independent of the x variable. We now prove that one actually has $\Psi = 0$ and therefore $N = M$.

Theorem 4.9. *It holds $\Psi(t) = 0$ for all $t \in [0,T]$. Therefore, $N = M$ on $[0,T] \times \mathbb{R}_+$.*

Proof. Since $(N-M)$ is independent of x, it suffices to show that $(N-M)(t,x) = 0$ at some x for any $t \leq T$. To accomplish that we show $\Psi'(t) = 0$ for any $t < T$, since by (4.16) and (4.30) we already know that

$$\Psi(T) = N(T,x) - M(T,x) = g(T,x) - g(T,x) = 0.$$

Then take $0 < x_1 < x_2, t_0 \in [0,T)$ and $\varepsilon > 0$ such that $t_0 + \varepsilon < T$ given and fixed, consider the rectangular domain $\mathcal{R} := (t_0-\varepsilon, t_0+\varepsilon) \times (x_1,x_2)$ such that $cl(\mathcal{R}) \subset \mathcal{C}$ (where $\mathcal{C}$ has been defined in (3.3)). Also, denote by $\partial_0\mathcal{R} := \partial\mathcal{R} \setminus (\{t_0-\varepsilon\} \times (x_1,x_2))$. Then consider the problem

$$(P)\ \begin{cases} h_t(t,x) = \mathcal{L}h(t,x), & (t,x) \in \mathcal{R}, \\ h(t,x) = (N-M)(t,x), & (t,x) \in \partial_0\mathcal{R}, \end{cases}$$

where $\mathcal{L}$ is the second-order differential operator that acting on $\varphi \in C^{1,2}([0,T] \times \mathbb{R})$ gives

$$(\mathcal{L}\varphi)(t,x) = \mu \frac{\partial \varphi}{\partial x}(t,x) + \frac{1}{2}\sigma^2 \frac{\partial^2 \varphi}{\partial x^2}(t,x), \ (t,x) \in [0,T] \times \mathbb{R}.$$

By reversing time, $t \mapsto T - t$, Problem (P) corresponds to a classical initial value problem with uniformly elliptic operator (notice that $\sigma^2 > 0$) and parabolic boundary $\partial_0 \mathcal{R}$. Since $N - M$ is continuous, and all the coefficients in the first equation of (P) are smooth (actually constant), by classical theory of partial differential equations of parabolic type (see, e.g. [24], Chapter V) problem (P) admits a unique solution h that is continuous, with continuous derivatives h_t, h_x, h_{xx}. Moreover, by the Feynman-Kac's formula, such a solution admits the representation

$$h(t,x) = \mathbb{E}[(N - M)(t + \widehat{\tau}(t,x), Z_{\widehat{\tau}(t,x)}(x))],$$

where

$$\widehat{\tau}(t,x) := \inf\{s \in [0, T-t) : (t+s, Z_s(x)) \in \partial_0 \mathcal{R}\} \wedge (T-t),$$

and $Z_s(x) = x + \mu s + \sigma W_s$, $s \geq 0$. Notice that we have $\widehat{\tau}(t,x) \leq \tau^\star(t,x)$ a.s., since $cl(\mathcal{R}) \subset \mathcal{C}$. Also, the integral terms in (4.58) and (4.59) are equal since $dI^0_\theta(x) = dI^\star_\theta(t,x) = 0$ for any $\theta \leq \widehat{\tau}(t,x) \leq \tau^\star(t,x)$. Hence by the martingale property of (4.58) and (4.59) we have

$$h(t,x) = (N - M)(t,x) \ \text{ in } \mathcal{R}, \tag{4.61}$$

and, by arbitrariness of $\mathcal{R}$,

$$\Psi(t) = (N - M)(t,x) = h(t,x) \ \text{ in } \mathcal{C}.$$

Therefore, since $\Psi = N - M$ is independent of x, continuous in t and solves the first equation of (P) in $\mathcal{C}$, we obtain $\Psi'(t) = 0$ for any $t < T$. Hence $\Psi(t) = 0$ for any $t \leq T$ since $\Psi(T) = 0$, and thus $N(t,x) - M(t,x) = 0$ for any $t \leq T$ and for any $x \in (0, \infty)$.

□

In the following we show that the function N is an upper bound for the value function V of (2.5). We first prove the following result.

Theorem 4.10. *For any $(t,x) \in \mathbb{R}_+ \times [0,T]$ the process*

$$\widetilde{N}_s := N(t+s, R_s(x)) - \int_0^s m(t+u) \ dI^0_u(x), \quad 0 \leq s \leq T - t, \tag{4.62}$$

is an $\mathbb{F}$-supermartingale.

Proof. It is enough to show that $\mathbb{E}\big[\widetilde{N}_\theta\big] \leq \mathbb{E}[\widetilde{N}_\tau]$ for all bounded $\mathbb{F}$-stopping times θ, τ such that $\theta \geq \tau$ (see [21], Chapter 1, Problem 3.26).

By the strong Markov property and the definition of N (4.17), we get that for any bounded $\mathbb{F}$-stopping time ρ one has

$$\begin{aligned}
\mathbb{E}[\widetilde{N}_\rho] &= \mathbb{E}\left[N(t+\rho, R_\rho(x)) - \int_0^\rho m(t+s) \ dI^0_s(x)\right] \\
&= \mathbb{E}\left[-\int_\rho^{T-t} f'(t+s)[R_s(x) - b(t+s)]^+ ds - \int_0^{T-t} m(t+s) \ dI^0_s(x) + g(R_{T-t}(x))\right] \\
&= N(t,x) + \mathbb{E}\left[\int_0^\rho f'(t+s)\big(R_s(x) - b(t+s)\big)^+ ds\right] =: N(t,x) + \Delta_\rho,
\end{aligned}$$

for any $(t,x) \in [0,T] \times \mathbb{R}_+$. Hence, taking θ, τ such that $T - t \geq \theta \geq \tau$ we get from the latter that $\mathbb{E}[\widetilde{N}_\theta] = N(t,x) + \Delta_\theta \leq N(t,x) + \Delta_\tau = \mathbb{E}[\widetilde{N}_\tau]$, where the inequality is due to the fact that $f' \leq 0$ on $\mathcal{S}$ (cf. Corollary 4.2-(ii)). This proves the claimed supermartingale property. □

To proceed further, we need the following properties of the function N of (4.17). Its proof is delegated to the Appendix.

Lemma 4.11. *The function* $N \in C^{1,2}([0,T) \times (0,\infty)) \cap C^0([0,T] \times \mathbb{R}_+)$.

Thanks to Lemma 4.11, an application of Itô's formula allows us to obtain the following (unique) Doob-Meyer decomposition (see, e.g., [21], Chapter 1, Theorem 4.10) of the $\mathbb{F}$-supermartingale $\widetilde{N}$ (cf. (4.62)).

Corollary 4.12. *The* $\mathbb{F}$*-supermartingale* $\widetilde{N}$ *of* (4.62) *is such that for all* $(t,x) \in [0,T] \times \mathbb{R}_+$ *and* $s \in [0, T-t]$

$$N(t+s, R_s(x)) - \int_0^s m(t+\theta)\, dI^0_\theta(x) = N(t,x) + \sigma \int_0^s u(t+\theta, R_\theta(x))\, dW_\theta + A_s(t,x), \tag{4.63}$$

where $A_\cdot(t,x)$ *is a continuous, nonincreasing and* $\mathbb{F}$*-adapted process.*

Proof. By the Doob-Meyer decomposition, the $\mathbb{F}$-supermartingale in (4.62) can be (uniquely) written as the sum of an $\mathbb{F}$-martingale and a continuous, $\mathbb{F}$-adapted nonincreasing process $(A_s)_{s\geq 0}$. Applying the martingale representation theorem to the martingale part of $\widetilde{N}$, yields the decomposition

$$\widetilde{N}_s = N(t,x) + \int_0^s \phi_\theta\, dW_\theta + A_s(t,x), \tag{4.64}$$

for some $\phi \in L^2(\Omega \times [0,T], \mathbb{P} \otimes dt)$. Finally, an application of Itô's lemma shows that $\phi_\theta = \sigma u(t+\theta, R_\theta(x))$ a.s. □

Theorem 4.13. *For any process* $D \in \mathcal{D}(t,x)$ *and any* $(t,x) \in [0,T] \times \mathbb{R}_+$*, the process*

$$Q_s(D;t,x) := \int_{[0,s]} f(t+\theta)\, dD_\theta - \int_0^s m(t+\theta)\, dI^D_\theta + N(t+s, X^D_s(x)), \tag{4.65}$$

$s \in [0, T-t]$*, is such that*

$$\mathbb{E}\left[Q_s(D;t,x)\right] \leq N(t,x) \quad \textit{for any } s \in [0,T-t]. \tag{4.66}$$

Proof. The proof is organized in 3 steps.

Step 1. For $D \equiv 0$, the proof is given by Theorem 4.10.

Step 2. Let $D_s := \int_0^s z_u\, du$, $s \geq 0$, where z is a bounded, nonnegative, $\mathbb{F}$-progressively measurable process. To show (4.66) we use Girsanov's Theorem and we rewrite the state process $X^D_s(x) = x + \mu s + \sigma W_s + D_s - I^D_s$ as a new drifted Brownian motion reflected at the origin. We therefore introduce the exponential martingale

$$Z_s = \exp\left(\int_0^s \frac{z_u}{\sigma}\, dW_u - \frac{1}{2\sigma^2}\int_0^s z_u^2\, du\right), \quad s \geq 0,$$

and we obtain that under the measure $\widehat{\mathbb{P}} = Z_T \mathbb{P}$, the process

$$\widehat{W}_s := W_s - \frac{1}{\sigma}\int_0^s z_u du, \quad s \geq 0,$$

is an $\mathbb{F}$- Brownian motion.

We can now rewrite the process Q of (4.65) under $\widehat{\mathbb{P}}$ as

$$Q_s(D;t,x) = \int_{[0,s]} f(t+\theta)\, dD_\theta - \int_0^s m(t+\theta)\, d\widehat{I}^D_\theta + N(t+s, \widehat{R}_s(x)), \tag{4.67}$$

for any $s \in [0,T-t]$, where under $\widehat{\mathbb{P}}$

$$\widehat{X}^D_s(x) = x + \mu s + \sigma \widehat{W}_s + \widehat{I}^D_s =: \widehat{R}_s(x).$$

Here $\widehat{I}^D_\cdot$ is flat off $\{s \geq 0 : \widehat{R}_s(x) = 0\}$ and reflects the drifted Brownian motion at zero. By employing (4.63), equation (4.67) reads as

$$Q_s(D;t,x) = N(t,x) + \sigma \int_0^s u(t+u, \widehat{R}_u(x))d\widehat{W}_u + \widehat{A}_s(t,x), \quad s \in [0, T-t], \tag{4.68}$$

where we have set

$$\widehat{A}_s(t,x) := A_s(t,x) + \int_0^s \Big(f(t+\theta) - u(t+\theta, R_\theta(x))\Big) z_\theta d\theta, \quad s \in [0, T-t]. \tag{4.69}$$

Since $\widehat{A}$ is nonincreasing due to the fact that $u \geq f$ and $A_\cdot(t,x)$ is nonincreasing, we can take expectations in (4.68) so to obtain

$$\mathbb{E}\left[Q_s(D;t,x)\right] \leq N(t,x), \quad \forall s \in [0, T-t].$$

Step 3. Since any arbitrary $D \in \mathcal{D}(t,x)$ can be approximated by an increasing sequence $(D^n)_{n\in\mathbb{N}}$ of absolutely continuous processes as the ones considered in Step 2 (see [12], Lemmata 5.4, 5.5 and Proposition 5.6), we have for all $n \in \mathbb{N}$

$$\mathbb{E}\left[Q_s(D^n;t,x)\right] \leq N(t,x).$$

Applying monotone and dominated convergence theorem, this property holds true for $Q(D;t,x)$ as well, for any $D \in \mathcal{D}(t,x)$. □

By Theorem 4.13 and the definition of Q as in (4.65) we immediately obtain

$$V(t,x) = \sup_{D \in \mathcal{D}(t,x)} \mathcal{J}(D;t,x) = \sup_{D \in \mathcal{D}(t,x)} \mathbb{E}\left[Q_{T-t}(D;t,x)\right] \leq N(t,x). \tag{4.70}$$

Moreover, by definition (4.31) one has

$$M(t,x) = \mathcal{J}(D^\star(t,x);t,x) \leq V(t,x). \tag{4.71}$$

With all these results at hand, we can now finally prove Theorem 3.2.

Proof of Theorem 3.2. By combining (4.70), (4.71), and Theorem 4.9 we obtain the series of inequalities

$$N(t,x) \geq V(t,x) \geq M(t,x) = N(t,x)$$

which proves the claim that $V = M$, and the optimality of $D^\star$. □

Remark 4.14. *As a byproduct of the fact that $V = N$ and of Lemma 4.11, we have that $V \in C^{1,2}([0,T) \times (0,\infty)) \cap C^0([0,T] \times \mathbb{R}_+)$. Moreover, from (3.8) and (3.2) we have that V satisfies the Neumann boundary condition $V_x(t,0) = m(t)$ for all $t \in [0,T]$.*

Remark 4.15. *The pathwise approach followed in this section seems to suggest that some of the intermediate results needed to prove Theorem 3.2 remain valid also in a more general setting in which profits and costs in (2.5) are discounted at a stochastic rate. We leave the analysis of this interesting problem for future work.*

5. Verifying Assumption 3.1: A Case Study with Discounted Constant Marginal Profits and Costs

In this section we consider the optimal dividend problem with capital injections

$$\begin{aligned} \widehat{V}(t,x) &:= \sup_{D \in \mathcal{D}(t,x)} \mathbb{E}\left[\int_0^{T-t} \eta e^{-rs}\, dD_s - \int_0^{T-t} \kappa e^{-rs}\, dI^D_s + \eta e^{-r(T-t)} X^D_{T-t}(x)\right] \\ &= e^{rt} V(t,x), \end{aligned} \tag{5.1}$$

where we have defined

$$V(t,x) := \sup_{D\in\mathcal{D}(t,x)} \mathbb{E}\left[\int_0^{T-t} \eta e^{-r(t+s)}\, dD_s - \int_0^{T-t} \kappa e^{-r(t+s)}\, dI_s^D + \eta e^{-rT} X_{T-t}^D(x)\right]. \tag{5.2}$$

It is clear from (5.2) and (2.4) that such a problem can be accommodated in our general setting (2.5) by taking (cf. Assumption 2.2)

$$f(t) = \eta e^{-rt}, \quad m(t) = \kappa e^{-rt}, \quad g(t,x) = \eta e^{-rt}x, \tag{5.3}$$

for some $\kappa > \eta$ (see also Remark 2.4).

In $\widehat{V}$ of (5.1) the coefficient κ can be seen as a constant proportional administration cost for capital injections. On the other hand, if we immagine that transaction costs or taxes have to be paid on dividends, the coefficient η measures a constant net proportion of leakages from the surplus received by the shareholders.

Remark 5.1. *Problem (5.1) is perhaps the most common formulation of the optimal dividend problem with capital injections (see, e.g., [23], [25], [32] and references therein). However, to the best of our knowledge, no previous work has considered such a problem in the case of a finite time horizon, whereas problem (5.1) has been extensively studied when $T = +\infty$ (see, e.g., [15] and references therein). In particular, it has been shown, e.g., in [15] that in the case $T = +\infty$ the optimal dividend strategy is triggered by a boundary $b_\infty > 0$ that can be characterized as the solution to a nonlinear algebraic equation (see Proposition 3.2 in [15]). In Proposition 3.6 of [15] such a trigger value is also shown to be the optimal stopping boundary of problem (5.4) below (when the optimization is performed over all the $\mathbb{F}$-stopping times).*

Thanks to Theorem 3.2 we know that, whenever Assumption 3.1 is satisfied, the optimal control $D^\star$ for problem (5.2) is triggered by the optimal stopping boundary b of the optimal stopping problem

$$\begin{aligned} u(t,x) &= \sup_{\tau\in\Lambda(T-t)} \mathbb{E}\Big[e^{-r\tau}\eta \mathbb{1}_{\{\tau<S(x)\}} + e^{-rS(x)}\kappa\mathbb{1}_{\{\tau\geq S(x)\}}\Big] \\ &= \sup_{\tau\in\Lambda(T-t)} \mathbb{E}\Big[e^{-r\tau}\eta \mathbb{1}_{\{A_\tau(x)>0\}} + e^{-rS(x)}\kappa\mathbb{1}_{\{A_\tau(x)\leq 0\}}\Big]. \end{aligned} \tag{5.4}$$

In the following we study optimal stopping problem (5.4) and verify the requirements of Assumption 3.1.

Moreover, by taking the sub-optimal stopping time $\tau = 0$ in (5.4) clearly gives $u(t,x) \geq \eta$ for $(t,x) \in [0,T] \times (0,\infty)$. Therefore, we can define the continuation and the stopping region of problem (5.4) as

$$\mathcal{C} := \{(t,x) \in [0,T) \times (0,\infty) : u(t,x) > \eta\}, \quad \mathcal{S} := \{(t,x) \in [0,T] \times (0,\infty) : u(t,x) = \eta\}.$$

Also, notice that we have $u(t,x) \leq \kappa$ for $(t,x) \in [0,T] \times \mathbb{R}_+$ since $\eta < \kappa$.

Since the reward process $\phi_t := e^{-rt}\eta\mathbb{1}_{\{t<S(x)\}} + e^{-rS(x)}\kappa\mathbb{1}_{\{t\geq S(x)\}}$ is upper semicontinuous in expectation along stopping times (thanks to the fact that $\eta < \kappa$), Theorem 2.9 in [22] ensures that the first time the value process (i.e. the Snell envelope of the reward process) equals the reward process is optimal. In our Markovian setting we thus have that the stopping time

$$\tau^\star(t,x) := \inf\{s \in [0, T-t) :\ (t+s, A_s(x)) \in \mathcal{S}\} \wedge (T-t), \quad (t,x) \in [0,T] \times \mathbb{R}_+, \tag{5.5}$$

is optimal. Further, defining $Z_s(x) := x + \mu s + \sigma W_s$, $s \geq 0$, the process

$$e^{-r(s\wedge\tau^\star(t,x)\wedge S(x))}u(t + (s\wedge\tau^\star(t,x)\wedge S(x)), Z_{(s\wedge\tau^\star(t,x)\wedge S(x))}(x)),\ s\in[0,T-t], \tag{5.6}$$

is an $\mathbb{F}$-martingale (cf. Proposition 1.6 and Remark 1.7 in [22]).

The next proposition proves some preliminary properties of u.

Proposition 5.2. *The value function u of (5.4) satisfies the following:*

(i) $u(T,x)=\eta$ *for any* $x>0$ *and* $u(t,0)=\kappa$ *for any* $t\in[0,T]$*;*
(ii) $t\mapsto u(t,x)$ *is nonincreasing for any* $x>0$*;*
(iii) $x\mapsto u(t,x)$ *is nonincreasing for any* $t\in[0,T]$.

Proof. We prove each item separately.

(i) The first property easily follows from definition (5.4).

(ii) The second property is due to the fact that $\Lambda(T-\cdot)$ shrinks and the expected value on the right-hand side of (5.4) is independent of $t\in[0,T]$.

(iii) Fix $t\in[0,T]$, $x_2>x_1\geq 0$ and notice that $S(x_2)>S(x_1)$. Then, from (5.4) we can write

$$\begin{aligned}
&u(t,x_2)-u(t,x_1)\\
&\leq \sup_{\tau\in\Lambda(T-t)}\mathbb{E}\Big[e^{-r\tau}\eta\mathbb{1}_{\{\tau<S(x_2)\}}-e^{-r\tau}\eta\mathbb{1}_{\{\tau<S(x_1)\}}+e^{-rS(x_2)}\kappa\mathbb{1}_{\{\tau\geq S(x_2)\}}-e^{-rS(x_1)}\kappa\mathbb{1}_{\{\tau\geq S(x_1)\}}\Big]\\
&=\sup_{\tau\in\Lambda(T-t)}\mathbb{E}\Big[\mathbb{1}_{\{S(x_1)\leq\tau<S(x_2)\}}\Big(e^{-r\tau}\eta-e^{-rS(x_1)}\kappa\Big)+\Big(e^{-rS(x_2)}-e^{-rS(x_1)}\Big)\kappa\mathbb{1}_{\{\tau\geq S(x_2)\}}\Big]\\
&\leq \sup_{\tau\in\Lambda(T-t)}\mathbb{E}\Big[e^{-rS(x_1)}(\eta-\kappa)\mathbb{1}_{\{S(x_1)\leq\tau<S(x_2)\}}+\Big(e^{-rS(x_2)}-e^{-rS(x_1)}\Big)\kappa\mathbb{1}_{\{\tau\geq S(x_2)\}}\Big]\leq 0,
\end{aligned}$$

where we have used that $\eta<\kappa$ in the last step. □

Since $x\mapsto u(t,x)$ is nonincreasing for each $t\in[0,T]$, setting

$$b(t):=\inf\{x>0: u(t,x)=\eta\},\quad t\in[0,T], \tag{5.7}$$

it is clear that

$$\mathcal{C}=\{(t,x)\in[0,T)\times[0,\infty):0<x<b(t)\}\,,\quad \mathcal{S}=\{(t,x)\in[0,T]\times[0,\infty):x\geq b(t)\}\,. \tag{5.8}$$

Moreover, the optimal stopping time of (5.5) reads

$$\tau^\star(t,x):=\inf\{s\in[0,T-t):\ A_s(x)\geq b(t+s)\}\wedge(T-t). \tag{5.9}$$

In the following we will refer to b as to the *free boundary.* The next theorem proves preliminary properties of b.

Proposition 5.3. *The free boundary b is such that*

(i) $t\mapsto b(t)$ *is nonincreasing;*
(ii) *One has* $b(t)>0$ *for all* $t\in[0,T)$*. Moreover, there exists* $b_\infty>0$ *such that* $b(t)\leq b_\infty$ *for any* $t\in[0,T]$.

Proof. We prove each item separately.

(i) The claimed monotonicity of b immediately follows from (ii) of Proposition 5.2.

(ii) To show that $b(t)>0$ for any $t\in[0,T)$ it is enough to observe that $u(t,0)=\kappa>\eta$ for all $t\in[0,T)$.

To prove $b(t)<\infty$ notice that $u(t,x)\leq u_\infty(x)$ for all $(t,x)\in[0,T]\times\mathbb{R}_+$, where

$$u_\infty(x):=\sup_{\tau\geq 0}\mathbb{E}\left[\eta e^{-r\tau}\mathbb{1}_{\{\tau<S(x)\}}+\kappa e^{-rS(x)}\mathbb{1}_{\{\tau\geq S(x)\}}\right].$$

Hence, setting $b_\infty:=\inf\{x>0:u_\infty(x)=\eta\}$ (which exists finite, e.g., by Proposition 3.2 in [15]; see also Remark 5.1 above), we have $b(t)\leq b_\infty$ for all $t\in[0,T]$. □

The proof of the next proposition is quite lenghty, and it is therefore postponed in the Appendix in order to simplify the exposition.

Proposition 5.4. *One has that $(t,x) \mapsto u(t,x)$ is lower semicontinuous on $[0,T) \times (0,\infty)$.*

The lower semicontinuity of u implies that the martingale of (5.6) has right-continuous sample paths, and that the stopping region is closed. The latter fact in turn plays an important role when proving continuity of the free boundary, as it is shown in the next proposition.

Proposition 5.5. *The free boundary b is such that $t \mapsto b(t)$ is continuous on $[0,T)$. Moreover, $b(T-) := \lim_{t\uparrow T} b(t) = 0$.*

Proof. We prove the two properties separately.

Here we show that b is continuous, and this proof is divided in two parts. We start with the right-continuity. Note that, by lower semicontinuity of u (cf. Proposition 5.4), the stopping region $\mathcal{S}$ is closed. Then fix an arbitrary point $t \in [0,T)$, take any sequence $(t_n)_{n\geq 1}$ such that $t_n \downarrow t$, and notice that $(t_n, b(t_n)) \in \mathcal{S}$, by definition. Setting $b(t+) := \lim_{t_n \downarrow t} b(t_n)$ (which exists due to Proposition 5.3-(i)), we have $(t_n, b(t_n)) \to (t, b(t+))$, and since $\mathcal{S}$ is closed $(t, b(t+)) \in \mathcal{S}$. Therefore, it holds $b(t+) \geq b(t)$ by definition (5.7) of b. However, $b(\cdot)$ is nonincreasing, and therefore $b(t) = b(t+)$.

Next we show left-continuity for all $t \in (0,T)$. Suppose that b makes a jump at some $t \in (0,T)$. By Proposition 5.3-(i) we have $\lim_{t_n \uparrow t} b(t_n) := b(t-) \geq b(t)$. We employ a contradiction scheme to show $b(t-) = b(t)$, and we assume $b(t-) > b(t)$. Let $x := \frac{b(t-)+b(t)}{2}$, recall $Z_s(x) = x + \mu s + \sigma W_s$, $s \geq 0$, and define

$$\tau_\varepsilon := \inf\{s \geq 0 : Z_s(x) \notin (b(t-), b(t))\} \wedge \varepsilon$$

for $\varepsilon \in (0,t)$. Then noticing that $\tau_\varepsilon < \tau^\star(t-\varepsilon, x) \wedge S(x)$, by the martingale property of (5.6) we can write

$$\begin{aligned}
u(t-\varepsilon, x) &= \mathbb{E}\left[e^{-r\tau_\varepsilon} u(t-\varepsilon+\tau_\varepsilon, Z_{\tau_\varepsilon}(x))\right] \\
&= \mathbb{E}\left[e^{-r\varepsilon} u(t, Z_\varepsilon(x)) \mathbb{1}_{\{\tau_\varepsilon = \varepsilon\}} + e^{-r\tau_\varepsilon} u(t-\varepsilon+\tau_\varepsilon, Z_{\tau_\varepsilon}(x)) \mathbb{1}_{\{\tau_\varepsilon < \varepsilon\}}\right] \\
&\leq \mathbb{E}\left[e^{-r\varepsilon} \eta \mathbb{1}_{\{\tau_\varepsilon = \varepsilon\}} + e^{-r\tau_\varepsilon} \kappa \mathbb{1}_{\{\tau_\varepsilon < \varepsilon\}}\right] \\
&\leq e^{-r\varepsilon}\eta + \kappa \mathbb{P}\left(\tau_\varepsilon < \varepsilon\right),
\end{aligned}$$

where the last step follows from the fact that $u \leq \kappa$, and that $Z_{\tau_\varepsilon}(x) \geq b(t)$ on the set $\{\tau_\varepsilon = \varepsilon\}$. Since $e^{-r\varepsilon}\eta + \kappa\mathbb{P}(\tau_\varepsilon < \varepsilon) = \eta(1-r\varepsilon) + \kappa o(\varepsilon)$ as $\varepsilon \downarrow 0$, we have found a contradiction to $u(t,x) \geq \eta$. Therefore, $b(t-) = b(t)$ and b is continuous on $[0,T)$.

To prove the claimed limit, notice that if $b(T-) := \lim_{t\uparrow T} b(t) > 0$, then any point (T,x) with $x \in (0, b(T-))$ belongs to $\mathcal{C}$. However, we know that $(T,x) \in \mathcal{S}$ for all $x > 0$, and we thus reach a contradiction. □

Thanks to the previous results all the requirements of Assumption 3.1 are satisfied for problem (5.4). Hence Theorem 3.2 holds, and one has that V of (5.2) and u of (5.4) are such that $V_x = u$ on $[0,T] \times \mathbb{R}_+$. In particular, by (5.1) and Theorem 3.2 we can write

$$\widehat{V}(t, b(t)) - \widehat{V}(t,x) = e^{rt} \int_x^{b(t)} u(t,y)\, dy.$$

Moreover, the optimal dividend distributions' policy $D^\star$ given through (3.7) is triggered by the free boundary b whose properties have been derived in Theorem 5.5.

Acknowledgments

Financial support by the German Research Foundation (DFG) through the Collaborative Research Centre 1283 "Taming uncertainty and profiting from randomness and low regularity in analysis, stochastics and their applications" is gratefully acknowledged.

Both the authors thank Miryana Grigorova and Hanspeter Schmidli for fruitful discussions and comments. Part of this work has been finalized while the first author was visiting the Department of Mathematics of the University of Padova thanks to the program "Visiting Scientists 2018". Giorgio Ferrari thanks the Department of Mathematics of the University of Padova for the hospitality.

Appendix A. Appendix

A.1. **Proof of Corollary 4.6.** Notice that from (4.10) we can write for any $x > 0$ and $t \in [0,T]$

$$
\begin{aligned}
u(t,x) &= \mathbb{E}\bigg[\int_0^{T-t} -f'(t+\theta)\mathbb{1}_{\{x+\mu\theta+\sigma W_\theta \geq b(t+\theta)\}}\mathbb{1}_{\{\theta<S(x)\}}\,d\theta \\
&\quad + m(t+S(x))\mathbb{1}_{\{S(x)\leq T-t\}} + g_x(T, A_{T-t}(x))\bigg] \\
&= \int_0^{T-t} -f'(t+\theta)\mathbb{P}\big(x+\mu\theta+\sigma W_\theta \geq b(t+\theta), S(x)>\theta\big)\,d\theta \\
&\quad + \mathbb{E}\big[m(t+S(x))\mathbb{1}_{\{S(x)\leq T-t\}}\big] + \mathbb{E}\big[g_x(T, A_{T-t}(x))\big],
\end{aligned} \tag{A.1}
$$

where Fubini's theorem and the fact that f' is deterministic has been used for the integral term above.

We now investigate the three summands separately. By using Proposition 3.2.1.1 in [17], and recalling that the stopping boundary b is strictly positive by Assumption 3.1, we have

$$
\begin{aligned}
&\mathbb{P}\Big(x+\mu\theta+\sigma W_\theta \geq b(t+\theta), S(x)>\theta\Big) \\
&= \mathbb{P}\Big(x+\mu\theta+\sigma W_\theta \geq b(t+\theta), \inf_{s\leq\theta}(x+\mu s+\sigma W_s)>0\Big) \\
&= \mathbb{P}\Big(\frac{\mu}{\sigma}\theta + W_\theta \geq \frac{b(t+\theta)-x}{\sigma}, \inf_{s\leq\theta}\Big(\frac{\mu}{\sigma}s+W_s\Big) > -\frac{x}{\sigma}\Big) \\
&= \mathcal{N}\Bigg(\frac{\frac{x-b(t+\theta)}{\sigma}+\frac{\mu}{\sigma}\theta}{\sqrt{\theta}}\Bigg) - e^{-2\frac{\mu x}{\sigma^2}}\mathcal{N}\Bigg(\frac{-\frac{b(t+\theta)+x}{\sigma}+\frac{\mu}{\sigma}\theta}{\sqrt{\theta}}\Bigg).
\end{aligned} \tag{A.2}
$$

Here $\mathcal{N}(\,\cdot\,)$ denotes the cumulative distribution function of a standard Gaussian random variable. Note that the last term in (A.2) is continuously differentiable with respect to x for any $\theta > 0$.

For the second summand in the last expression on the right-hand side of (A.1) we first rewrite $S(x)$, for $x \geq 0$, as

$$
\begin{aligned}
S(x) &= \inf\{s\geq 0 : x+\mu s+\sigma W_s = 0\} = \inf\{s\geq 0 : \frac{\mu}{\sigma}s + W_s = -\frac{x}{\sigma}\} \\
&\overset{\mathcal{L}}{=} \inf\{s\geq 0 : -\frac{\mu}{\sigma}s + \widehat{W}_s = \frac{x}{\sigma}\}.
\end{aligned} \tag{A.3}
$$

where $\widehat{W}$ is a standard Brownian motion. Hence equation (3.2.3) in [17] applies and allows us to write the probability density of $S(x)$ as

$$\rho_{S(x)}(u) := \frac{d\mathbb{P}(S(x) \in du)}{du} = \frac{x}{\sigma\sqrt{2\pi u^3}} e^{-\frac{(\frac{x}{\sigma}+\frac{\mu}{\sigma}u)^2}{2u}}, \quad u \geq 0. \tag{A.4}$$

For the third summand we notice that the absorbed process $A_{T-t}(x)$ of (3.1) is the drifted Brownian motion started in x and killed at zero. Denote by $\rho_A(t,x,y)$ its transition density of moving from x to y in t units of time. Then, by employing the result of Section 15 in Appendix 1 of [6] (suitably adjusted to our case with $\sigma \neq 1$) we obtain

$$\begin{aligned}\rho_A(T-t,x,y) := \frac{d\mathbb{P}(A_{T-t}(x) \in dy)}{dy} &= \frac{1}{\sqrt{2\pi(T-t)\sigma^2}} \exp\left(-\left(\frac{\mu(x-y)}{\sigma^2}\right) - \frac{\mu^2}{2\sigma^2}(T-t)\right) \\ &\times \left(\exp\left(-\frac{(x-y)^2}{2\sigma^2(T-t)}\right) - \exp\left(-\frac{(x+y)^2}{2\sigma^2(T-t)}\right)\right).\end{aligned} \tag{A.5}$$

Feeding (A.2), (A.4) and (A.5) back into (A.1) we obtain

$$\begin{aligned}u(t,x) &= \int_0^{T-t} -f'(t+\theta)\left[\mathcal{N}\left(\frac{\frac{x-b(t+\theta)}{\sigma}+\frac{\mu}{\sigma}\theta}{\sqrt{\theta}}\right) - e^{-2\frac{\mu x}{\sigma^2}}\mathcal{N}\left(\frac{-\frac{b(t+\theta)+x}{\sigma}+\frac{\mu}{\sigma}\theta}{\sqrt{\theta}}\right)\right] d\theta \\ &+ \int_0^\infty m(t+u)\rho_{S(x)}(u)\ du + \int_0^\infty g_x(T,y)\rho_A(T-t,x,y)\ dy,\end{aligned} \tag{A.6}$$

and it is easy to see by the dominated convergence theorem that $x \mapsto u(t,x)$ is continuously differentiable on $(0,\infty)$ for any $t < T$.

A.2. **Proof of Lemma 4.11.** By (4.16) and Corollary 4.6 the function N of (4.17) is twice-continuously differentiable with respect to x on $(0,\infty)$. To show that N is also continuously differentiable with respect to t on $[0,T)$ we express the expected value on the right-hand side of (4.17) as an integral with respect to the probability densities of the involved processes. We thus start computing the transition density of the reflected Brownian motion R of (4.21), which we call ρ_R. By Appendix 1, Chapter 14, in [6] (easily adapted to our case with $\sigma \neq 1$) we have

$$\begin{aligned}\rho_R(u,x,y) := \frac{d\mathbb{P}(R_u(x) \in dy)}{dy} &= \frac{1}{\sqrt{2\pi u\sigma^2}} \exp\left(-\frac{\mu}{\sigma}\left(\frac{x-y}{\sigma}\right) - \frac{\mu^2}{2\sigma^2}u\right) \times \\ &\left(\exp\left(-\frac{(x-y)^2}{2\sigma^2 u}\right) - \exp\left(-\frac{(x+y)^2}{2\sigma^2 u}\right)\right) - \frac{\mu}{2\sigma}\mathrm{Erfc}\left(\frac{x+y+\mu u}{\sqrt{2\sigma^2 u}}\right),\end{aligned} \tag{A.7}$$

where $\mathrm{Erfc}(x) := \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\ dy$ for $x \in \mathbb{R}$. Hence, by using Fubini's Theorem, (4.17) reads as

$$
\begin{aligned}
N(t,x) &= \mathbb{E}\Big[-\int_0^{T-t} (R_s(x)-b(t+s))^+ \, f'(t+s)\; ds - \int_0^{T-t} m(t+s)\; dI^0_s(x) \\
&\quad + g(T, R_{T-t}(x)) \Big] = -\int_t^T \mathbb{E}\Big[(R_{u-t}(x)-b(u))^+ \Big] f'(u)\; du \\
&\quad - \mathbb{E}\Big[\int_0^{T-t} m(t+s)\; dI^0_s(x) \Big] + \mathbb{E}\Big[g(T, R_{T-t}(x)) \Big] \\
&= -\int_t^T \Big(\int_0^\infty (y-b(u))^+ \, \rho_R(u-t,x,y)\; dy \Big) f'(u)\; du - \mathbb{E}\Big[\int_t^T m(u)\; dI^0_{u-t}(x) \Big] \qquad \text{(A.8)} \\
&\quad + \int_0^\infty g(T,y) \rho_R(T-t,x,y)\; dy.
\end{aligned}
$$

Recalling that m is continuously differentiable by Assumption 2.2 and using an integration by parts, we can write

$$
\begin{aligned}
\mathbb{E}\Big[\int_t^T m(u)\; dI^0_{u-t}(x) \Big] &= \mathbb{E}\Big[m(T) I^0_{T-t}(x) - \int_t^T I^0_{u-t}(x) m'(u)\; du \Big] \\
&= m(T)\mathbb{E}\big[I^0_{T-t}(x) \big] - \int_t^T \mathbb{E}\big[I^0_{u-t}(x) \big] m'(u)\; du \\
&= m(T)\mathbb{E}\big[0 \vee (\sigma \xi_{T-t} - x) \big] - \int_t^T \mathbb{E}\big[0 \vee (\sigma \xi_{u-t} - x) \big] m'(u)\; du,
\end{aligned}
$$

where we have used that $I^0_s(x) = 0 \vee (\sigma \xi_s - x)$ with $\xi_s := \sup_{\theta \le s}(-\frac{\mu}{\sigma}\theta - W_\theta)$. Since (cf. Chapter 3.2.2 in [17])

$$
\mathbb{P}\left(\xi_s \le z\right) = \mathcal{N}\left(\frac{z - \frac{\mu}{\sigma} s}{\sqrt{s}} \right) - \exp\left(2\frac{\mu}{\sigma} z \right) \mathcal{N}\left(\frac{-z - \frac{\mu}{\sigma} s}{\sqrt{s}} \right), \qquad \text{(A.9)}
$$

we get

$$
\mathbb{E}\Big[0 \vee (\sigma \xi_{u-t} - x) \Big] = \int_{\frac{x}{\sigma}}^\infty (\sigma z - x) \rho_\xi(u-t,z)\; dz, \qquad \text{(A.10)}
$$

where we have defined $\rho_\xi(s,z) := \frac{d\mathbb{P}(\xi_s \le z)}{dz}$. Because $\rho_\xi(\cdot, z)$ and $\rho_R(\cdot, x, y)$ are continuously differentiable on $(0,T]$, it follows that $N(t,x)$ as in (A.8) is continuously differentiable with respect to t, for any $t < T$. The continuity of N on $[0,T] \times \mathbb{R}_+$ also follows from the previous equations.

A.3. **Proof of Proposition 5.4.** Let $(t,x) \in [0,T) \times (0,\infty)$ be given and fixed, and take any sequence $(t_n, x_n) \subset [0,T) \times (0,\infty)$ such that $(t_n, x_n) \to (t,x)$. Then, let $\tau^\star := \tau^\star(t,x)$ be the optimal stopping time for $u(t,x)$ of (5.9). From (5.4) and the fact that $\tau^\star \le T - t$ a.s. we then find

$$
\begin{aligned}
u(t,x) - u(t_n,x_n) &\le \mathbb{E}\Big[\eta e^{-r\tau^\star} \mathbb{1}_{\{\tau^\star < S(x)\}} + \kappa e^{-rS(x)} \mathbb{1}_{\{\tau^\star \ge S(x)\}} \\
&\qquad - \eta e^{-r(\tau^\star \wedge (T-t_n))} \mathbb{1}_{\{\tau^\star \wedge (T-t_n) < S(x_n)\}} - \kappa e^{-rS(x_n)} \mathbb{1}_{\{\tau^\star \wedge (T-t_n) \ge S(x_n)\}} \Big] \\
&= \mathbb{E}\Big[\mathbb{1}_{\{\tau^\star \le T-t_n\}} \Big\{ \eta e^{-r\tau^\star} \left(\mathbb{1}_{\{\tau^\star \ge S(x_n)\}} - \mathbb{1}_{\{\tau^\star \ge S(x)\}} \right) \\
&\qquad + \kappa \left(e^{-rS(x)} \mathbb{1}_{\{\tau^\star \ge S(x)\}} - e^{-rS(x_n)} \mathbb{1}_{\{\tau^\star \ge S(x_n)\}} \right) \Big\} \Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\mathbb{E}\Bigg[\mathbb{1}_{\{\tau^\star > T-t_n\}}\Bigg\{\eta e^{-r\tau^\star}\mathbb{1}_{\{\tau^\star < S(x)\}} - \eta e^{-r(T-t_n)}\mathbb{1}_{\{T-t_n < S(x_n)\}} \\
&+ \kappa\left(e^{-rS(x)}\mathbb{1}_{\{\tau^\star \geq S(x)\}} - e^{-rS(x_n)}\mathbb{1}_{\{T-t_n \geq S(x_n)\}}\right)\Bigg\}\Bigg] \\
&\leq \mathbb{E}\Bigg[\mathbb{1}_{\{\tau^\star \leq T-t_n\}}\Bigg\{\eta e^{-r\tau^\star}\mathbb{1}_{\{S(x_n)\leq \tau^\star < S(x)\}} \\
&+ \kappa\left(\left|e^{-rS(x)} - e^{-rS(x_n)}\right|\mathbb{1}_{\{\tau^\star \geq S(x_n)\vee S(x_n)\}} + e^{-rS(x)}\mathbb{1}_{\{S(x_n) > \tau^\star \geq S(x)\}}\right)\Bigg\}\Bigg] \\
&+\mathbb{E}\Bigg[\mathbb{1}_{\{\tau^\star > T-t_n\}}\Bigg\{\eta e^{-r(T-t_n)}\left(\mathbb{1}_{\{T-t_n < S(x)\}} - \mathbb{1}_{\{T-t_n < S(x_n)\}}\right) \\
&+ \kappa\mathbb{1}_{\{T-t > S(x)\}}\left(e^{-rS(x)}\mathbb{1}_{\{\tau^\star \geq S(x)\}} - e^{-rS(x_n)}\mathbb{1}_{\{T-t_n \geq S(x_n)\}}\right) \\
&+ \kappa\mathbb{1}_{\{T-t = S(x)\}}\left(e^{-rS(x)}\mathbb{1}_{\{\tau^\star \geq S(x)\}} - e^{-rS(x_n)}\mathbb{1}_{\{T-t_n \geq S(x_n)\}}\right) \\
&+ \kappa\mathbb{1}_{\{T-t < S(x)\}}\left(e^{-rS(x)}\mathbb{1}_{\{\tau^\star \geq S(x)\}} - e^{-rS(x_n)}\mathbb{1}_{\{T-t_n \geq S(x_n)\}}\right)\Bigg\}\Bigg] \\
&\leq \mathbb{E}\left[\eta e^{-r\tau^\star}\mathbb{1}_{\{S(x_n)\leq \tau^\star < S(x)\}} + \kappa\left(\left|e^{-rS(x)} - e^{-rS(x_n)}\right| + \mathbb{1}_{\{S(x_n) > \tau^\star \geq S(x)\}}\right)\right] \\
&+\mathbb{E}\Bigg[\mathbb{1}_{\{\tau^\star > T-t_n\}}\Bigg\{\eta e^{-r(T-t_n)}\mathbb{1}_{\{S(x_n)\leq T-t_n < S(x)\}} \\
&+ \kappa\mathbb{1}_{\{T-t > S(x)\}}\left(e^{-rS(x)}\mathbb{1}_{\{T-t \geq S(x)\}} - e^{-rS(x_n)}\mathbb{1}_{\{T-t_n \geq S(x_n)\}}\right) \\
&+ \kappa\mathbb{1}_{\{T-t = S(x)\}} + \kappa\mathbb{1}_{\{T-t < S(x)\}}\mathbb{1}_{\{\tau^\star \geq S(x)\}}\Bigg\}\Bigg].
\end{aligned}
$$

Rearranging terms and taking limit inferior as $n \uparrow \infty$ on both sides one obtains

$$
\begin{aligned}
\underline{\lim}_{n\to\infty} u(t_n, x_n) &\geq u(t,x) - \overline{\lim}_{n\to\infty}\mathbb{E}\Bigg[\eta e^{-r\tau^\star}\mathbb{1}_{\{S(x_n)\leq \tau^\star < S(x)\}} \\
&+ \kappa\left(\left|e^{-rS(x)} - e^{-rS(x_n)}\right| + \mathbb{1}_{\{S(x_n) > \tau^\star \geq S(x)\}}\right)\Bigg] \\
&- \overline{\lim}_{n\to\infty}\mathbb{E}\Bigg[\mathbb{1}_{\{\tau^\star > T-t_n\}}\Bigg\{\eta e^{-r(T-t_n)}\mathbb{1}_{\{S(x_n)\leq T-t_n < S(x)\}} \\
&+ \kappa\mathbb{1}_{\{T-t > S(x)\}}\left(e^{-rS(x)}\mathbb{1}_{\{T-t \geq S(x)\}} - e^{-rS(x_n)}\mathbb{1}_{\{T-t_n \geq S(x_n)\}}\right) \\
&+ \kappa\mathbb{1}_{\{T-t = S(x)\}} + \kappa\mathbb{1}_{\{S(x)\leq \tau^\star \leq T-t < S(x)\}}\Bigg\}\Bigg] \\
&\geq u(t,x) - \mathbb{E}\left[\kappa\mathbb{1}_{\{S(x)=\tau^\star\}}\right] - \mathbb{E}\left[\eta e^{-r(T-t)}\mathbb{1}_{\{T-t=S(x)\}} + \kappa\mathbb{1}_{\{T-t=S(x)\}}\right] \\
&= u(t,x) - \kappa\mathbb{P}\left(\tau^\star = S(x)\right) - \left(\eta e^{-r(T-t)} + \kappa\right)\mathbb{P}\left(T-t = S(x)\right).
\end{aligned}
$$

The last inequality follows by interchanging expectations and limits by the dominated convergence theorem, using that $S(x_n) \to S(x)$, carefully investigating the involved limits superior, and observing that $\{\tau^\star \geq T-t\} = \{\tau^\star = T-t\}$ since $\tau^\star \in \Lambda(T-t)$.

Using now that $\{T-t = S(x)\}$ is a $\mathbb{P}$-null set by (A.4), and the fact that $\mathbb{P}(\tau^\star = S(x)) = 0$ since the free boundary is strictly positive on $[0,T)$, we then obtain

$$
\underline{\lim}_{n\to\infty} u(t_n, x_n) \geq u(t,x), \tag{A.11}
$$

which proves the claimed lower semicontinuity of u on $[0,T) \times (0,\infty)$.

A.4. **Lemma A.1.**

Lemma A.1. *Recall that (cf. (4.47))*

$$z = \inf\{y \in [0, b(0)] : \tau^\star(0,y) < S(y)\}.$$

Then it holds that

$$S(z) \leq T \quad a.s. \tag{A.12}$$

Proof. In order to simplify exposition, in the following we shall stress the dependence on ω only when strictly necessary. Suppose that there exists a set $\Omega_0 \subset \Omega$ s.t. $\mathbb{P}(\Omega_0) > 0$, and that for any $\omega \in \Omega_0$ we have $S(z) > T$. Then take $\omega_0 \in \Omega_0$, recall that $Z_s(x) = x + \mu s + \sigma W_s$ for any $x > 0$ and $s \geq 0$, and notice that $\min_{0\leq s\leq T} Z_s(z;\omega_0) = \ell := \ell(\omega_0) > 0$. Then, defining $\widehat{z}(\omega_o) := \widehat{z} = z - \frac{\ell}{2}$, one has

$$\begin{aligned}\min_{0\leq s\leq T} Z_s(\widehat{z};\omega_0) &= \min_{0\leq s\leq T}\left(z + \mu s + \sigma W_s(\omega_0) - \frac{\ell}{2}\right)\\ &= \ell - \frac{\ell}{2} = \frac{\ell}{2} > 0.\end{aligned}$$

Hence, $S(\widehat{z}) > T \geq \tau^\star(0,\widehat{z})$, but this contradicts the definition of z since $\widehat{z} < z$. Therefore we conclude that $S(z) \leq T$ a.s. □

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