ON THE STATIONARY DISTRIBUTION OF THE BLOCK COUNTING PROCESS FOR POPULATION MODELS WITH MUTATION AND SELECTION

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Abstract. We consider two population models subject to the evolutionary forces of selection and mutation, the Moran model and the Λ-Wright-Fisher model. In such models the block counting process traces back the number of potential ancestors of a sample of the population at present. Under some conditions the block counting process is positive recurrent and its stationary distribution is described via a linear system of equations. In this work, we first characterise the measures Λ leading to a geometric stationary distribution, the Bolthausen-Sznitman model being the most prominent example having this feature. Next, we solve the linear system of equations corresponding to the Moran model. For the Λ-Wright-Fisher model we show that the probability generating function associated to the stationary distribution of the block counting process satisfies an integro differential equation. We solve the latter for the Kingman model and the star-shaped model.

Keywords: Kingman coalescent; star-shaped coalescent; Bolthausen-Sznitman coalescent; Wright–Fisher model; Moran model.

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1. Introduction

There is a large variety of population models describing the interplay between mutation and selection forwards in time. Understanding the underlying ancestral processes is a major challenge in population genetics. In neutral population models, ancestries are typically described by coalescent processes. The most important example is Kingman’s coalescent [14], which only allows for mergers of pairs of ancestral lineages. Kingman shows in [14] that this process arises as the limit of the genealogies of the neutral Moran and Wright-Fisher models when the population size tends to infinity. Convergence to the Kingman coalescent holds for a wide class of neutral population models (see [19]). However, in some situations Kingman’s coalescent is not a suitable approximation, which leads to consider coalescent processes that allow for multiple mergers. Exchangeable coalescents with multiple mergers (but without simultaneous multiple mergers) are characterised by a finite measure Λ on [0, 1], and therefore called Λ-coalescents (see [21, 23] for further details). An important role is played by the block counting process, which counts the number of ancestors of a given sample of the population. The case Λ = δ₀ corresponds to Kingman’s coalescent, the case Λ = δ₁ to the star-shaped coalescent, and the case where Λ is the uniform distribution on (0, 1) to the Bolthausen-Sznitman coalescent [4]. Formulas for the infinitesimal rates of the block counting process are provided in [12] for the Λ-coalescents and in [10] for the full class of exchangeable coalescents.

The ancestral selection graph (ASG) describes the ancestries of a sample of individuals in the Wright-Fisher diffusion model with selection (see [16, 20]). The coalescence mechanism is given by Kingman’s coalescent. Additionally, selection introduces binary branching at constant rate per ancestral line. The lines in the ASG represent potential ancestors of a given sample of the population. The joint effects of selection and mutation in the ancestries have been described in [18] with the help of a new ancestral process called the pruned lookdown ancestral selection graph (pLD-ASG). This construction was extended to the Λ-Wright-Fisher model with selection and mutation in [3] and to the Moran model and its deterministic limit in [5] (see also [2]). In this context, the block counting process with selection and mutation refers to the process that counts the number of lines in the corresponding pLD-ASG.

In the Wright-Fisher diffusion and the Moran model the block counting process is positive recurrent for any strictly positive selection parameter. For the Λ-Wright-Fisher model, there is a critical value σₐ such that the block counting process is positive recurrent for any selection parameter σ ∈ (0, σₐ)
Moreover, for the Wright-Fisher diffusion, the stationary tail probabilities of the block counting process are characterised via a two-step recurrence relation (see [13]), which is referred to in the literature as Fearnhead’s recursion (see also [8, 24]). Linear systems of equations that characterise the stationary tail probabilities of the block counting process are provided in [3] for the Λ-Wright-Fisher model and in [5] for the Moran model. We refer to these linear systems as Fearnhead-type recursions.

On the basis of the Fearnhead-type recursions, we aim to identify the measures Λ such that the stationary distribution of the block counting process is geometrically distributed, the measure Λ = 0 being known to have this property [2]. Next, we aim to provide explicit expressions for the stationary distribution of the block counting process for the Moran model and some particular cases of the Λ-model, namely, the Kingman model, the star-shaped model and the Bolthausen-Sznitman model. For the general Λ-model we will characterise the probability generating function of the stationary distribution of the block counting process via an integro differential equation.

The paper is organised as follows. In Section 2 we briefly describe the Moran model and the Λ-Wright-Fisher model with selection and mutation together with their corresponding block counting process. For both models we recall the characterisation of the stationary tail probabilities of the block counting process via the Fearnhead-type recursions. In Section 3 we characterise the measures Λ leading to a geometric distribution and we provide a class of measures having this feature. In Section 4 we treat the Moran model with mutation and selection. We obtain formulas for the probability mass function, the probability generating function, the mean and the factorial moments of the stationary distribution of the block counting process. In Section 5 we characterise the probability generating function of the stationary distribution of the block counting process for the Λ-Wright-Fisher model by an integro differential equation. In Section 6 we obtain formulas for the probability mass function, the probability generating function, the mean and the factorial moments of the stationary distribution of the block counting process for the Moran model and some particular cases of the Λ-model, namely, the Kingman model, the star-shaped model and the Bolthausen-Sznitman model. For the general Λ-model we will characterise the probability generating function of the stationary distribution of the block counting process via an integro differential equation.

2. Preliminaries: Fearnhead-type recursions

In the two-types Moran model of size \( N > 1 \) each individual is characterised by a type \( i \in \{0, 1\} \). If an individual reproduces, its single offspring inherits the parent’s type and replaces a uniformly chosen individual, possibly its own parent. The replaced individual dies, keeping the size of the population constant. Individuals of type 1 reproduce at rate 1, whereas individuals of type 0 reproduce at rate \( 1 + s, s > 0 \). Mutation occurs independently of reproduction. Moreover, each individual mutates to type \( j \in \{0, 1\} \) at rate \( u_j \geq 0 \). Hence, the total rate of mutation per individual is \( u := u_0 + u_1 \).

Backward in time potential ancestors of a sample of the population are traced back with the help of the pruned lookdown ASG (see [5]). The number of potential ancestors of a given sample of individuals is described by the block counting process \( L^N := (L^N_t)_{t \geq 0} \). The latter is the continuous time Markov chain with state space \( \mathbb{N} := \{1, \ldots, N\} \) and infinitesimal rates

\[
q_N(i, j) := \frac{i(N - i)}{N}1_{(i = j + 1)} + \left(\frac{(i - 1)}{N} + (i - 1)u_1 + u_0\right)1_{(j = i - 1)} + u_01_{(j \leq i - 2)}, \quad i, j \in \mathbb{N}.
\]

The process \( L^N \) is irreducible, and has hence a unique stationary distribution \( (p^N_0)_{n \in \{\mathbb{N}\}} \). Moreover, the stationary tail probabilities \( a^N_n := P(L^N_\infty > n), n \in [N - 1]_0 := \{0, \ldots, N - 1\} \), are characterised by the recurrence relation (see [5, Prop. 4.7])

\[
\left(\frac{n}{N} + u_1\right)a^N_n = \left(\frac{n}{N} + \frac{N - n + 1}{N} s + u\right)a^N_{n-1} - \frac{N - n + 1}{N} s a^N_{n-2}, \quad n \in \{2, \ldots, N - 1\},
\]

together with the boundary conditions

\[
a^N_0 = 1 \quad \text{and} \quad \left(1 + u + \frac{s}{N}\right)a^N_{N-1} = \frac{s}{N} a^N_{N-2}.
\]

Depending on the strengths of selection and mutation two standard limits of large populations arise in the Moran model. The first one assumes that the parameters of selection and mutation remain constant with respect to the size of the population (strong selection - strong mutation). In this case, the proportion of fit individuals converges to the solution of the haploid mutation-selection equation (see [2, 6] for more details)

\[
z'(t) = sz(t)(1 - z(t)) + u_0(1 - z(t)) - u_1z(t), \quad t \geq 0.
\]
The other asymptotic regime arises when \( s \sim \sigma/N \), \( u_0 \sim \theta_0/N \) and \( u_1 \sim \theta_1/N \), for some \( \sigma, \theta_1, \theta_0 \geq 0 \) (weak selection - weak mutation). In this case, rescaling time by \( N \), the proportion of fit individuals converges to the Wright-Fisher diffusion process with infinitesimal generator

\[
\mathcal{A}_y f(x) := x(1-x)f''(x) + [\sigma x(1-x) + \theta_0(1-x) - \theta_1 x] f'(x), \quad f \in C^2([0,1]), \quad x \in [0,1].
\]

These two infinite population models are particular cases of the \( \Lambda \)-Wright-Fisher model. The \( \Lambda \)-Wright-Fisher model describes a two-types infinite population evolving according to random reproduction, two-way mutation and fertility selection. The parameters of the model are (1) a finite measure \( \Lambda \) on \([0,1]\) modelling the neutral reproduction, (2) the selective advantage \( \sigma \in \mathbb{R}_+ := [0,\infty) \) and (3) the mutation rates \( \theta_0, \theta_1 \in \mathbb{R}_+ \). The process \( X \) describing the frequency of type 0 in the population has the generator

\[
A_X f(x) := \int_{[0,1]} \frac{\Lambda(dx)}{x^2} [xf(x+z(1-x)) - f(x)] + \frac{\Lambda(\{0\})}{2} (1-x)f''(x) + [\sigma x(1-x) + \theta_0(1-x) - \theta_1 x] f'(x), \quad f \in C^2([0,1]), \quad x \in [0,1].
\]

Note that the case \( \Lambda = 2\delta_0 \), where \( \delta_0 \) is the Dirac mass at 0, corresponds to the Wright-Fisher diffusion model. The degenerate case \( \Lambda \equiv 0 \), meaning that there is no neutral reproduction, corresponds to the deterministic limit of the Moran model, i.e. the solution of the ODE (2.14) with \( s = \sigma, \ u_0 = \theta_0 \) and \( u_1 = \theta_1 \). One may think of a population of seeds which do not reproduce, but forces like mutation and selection may still act on the seeds since they are exposed to heat, chemicals or radiation. We refer to this model as the seed bank model.

In [3] the \( \Lambda \)-pruned lookdown ASG was defined in order to trace back ancestries in the \( \Lambda \)-Wright-Fisher model. The corresponding block counting process \( L^\Lambda := (L^\Lambda_t)_{t \geq 0} \) is the continuous time Markov chain with state space \( \mathbb{N} := \{1,2,\ldots\} \) and infinitesimal generator

\[
G_{L^\Lambda} g(k) := \sum_{\ell=1}^{k-1} \frac{\Lambda(dx)}{x^2} \left[ k \lambda_{k-\ell+1} [g(\ell) - g(k)] + k \sigma [g(k+1) - g(k)] + (k-1) \theta_0 [g(k-1) - g(k)] + \sum_{\ell=1}^{k-1} \theta_0 [g(k-\ell) - g(k)] \right], \quad g : \mathbb{N} \to \mathbb{R}, \quad k \in \mathbb{N},
\]

where \( \lambda_{k,j} := \int_{[0,1]} x^j (1-x)^{k-j} x^{-2} \Lambda(dx), \) \( 2 \leq j \leq k \).

Let \( \sigma_\Lambda := -\int_{[0,1]} \log(1-x) \frac{\Lambda(dx)}{x^2} \). In [3] it is shown that if \( \sigma \in (0,\sigma_\Lambda) \), then the process \( L^\Lambda \) is positive recurrent. The next result improves this condition for \( \theta := \theta_0 + \theta_1 > 0 \).

**Lemma 2.1.** Assume that \( \sigma > 0 \). If \( \theta_0 > 0 \) or \( \sigma < \sigma_\Lambda + \theta_1 \), then the process \( L^\Lambda \) is positive recurrent.

**Proof.** If \( \sigma_\Lambda = \infty \) the result is already covered in [3]. In the case \( \sigma_\Lambda < \infty \), we follow the proof of [3] Lemma 2.4. Let \( T_k := \inf\{s \geq 0 : L^\Lambda_s < k\}, \ k > 1 \). We will show that there exists \( n_0 \in \mathbb{N} \), such that for all \( n \geq n_0, \ E_n[T_{n_0}] < \infty \). If \( \theta_0 > 0 \) and \( L^\Lambda_0 = n \geq 2 \), \( T_2 \) is dominated by an exponential time with parameter \( \theta_0 \), and the result follows in this case. Now we assume that \( \theta_0 = 0 \) and that \( \sigma - \theta_1 < \sigma_\Lambda \).

If \( \theta_1 > \sigma \), \( L^\Lambda \) is dominated by a birth and death process with birth rate \( \sigma \) and death rate \( \theta_1 \), which is positive recurrent. Hence \( L^\Lambda \) is positive recurrent. At last we consider the case where \( \sigma \geq \theta_1 \). We define for \( n \geq 2 \) and \( \ell \in \mathbb{N} \)

\[
\delta(n) := -n \int_{[0,1]} \log \left( 1 - \frac{1}{n} (ax - 1 + (1-x)^n) \right) \frac{\Lambda(dx)}{x^2}, \quad f(\ell) := \sum_{k=\ell}^{n} \frac{k}{\delta(k)} \log \left( \frac{k}{k-1} \right).
\]

A slight modification of the proof of [3] Lemma 2.3] permits to show that

\[
G_{L^\Lambda} f(\ell) \leq -1 + (\sigma - \theta_1) \frac{\ell}{\delta(\ell)}, \quad \ell \geq 2.
\]

Set \( f_N(\ell) := f(\ell) 1_{\ell \leq N+1}, \ N \in \mathbb{N} \). By Dynkin’s formula the process \((f_N(L^\Lambda_t) - f'_0 \int_0^t G_{L^\Lambda} f_N(L^\Lambda_s) ds)_{t \geq 0}\) is a martingale. Since \( \lim_{n \to \infty} \delta(n)/n = \sigma_\Lambda^{-1} \) (see [3] Remark 4.3]), we infer that for any \( \epsilon > 0 \) there is \( n_0 \in \mathbb{N} \) such that for all \( \ell \geq n_0, \ell/\delta(\ell) \leq \sigma_\Lambda^{-1} + \epsilon \). Consider the stopping time \( S_N := \inf\{s \geq 0 : L^\Lambda_s > N\} \). Applying the optional stopping theorem and using that \( G_{L^\Lambda} f_N(\ell) = G_{L^\Lambda} f(\ell) \) for \( \ell \leq N \) yields for
In this section, we aim to characterise the measures $\Lambda$ such that $L^\Lambda_{n_0}$ is geometrically distributed. Therefore,

$$E_n[f_N(L^\Lambda_{n_0} \land S_{n_0} \land k)] = f_N(n) + \int_0^{T_{n_0} \land S_{n_0} \land k} G_{L^\Lambda} f_N(L^\Lambda_s) ds$$

$$\leq f_N(n) + \int_0^{T_{n_0} \land S_{n_0} \land k} \left[-1 + \left(\sigma - \theta_1 \right) \frac{L^\Lambda_s}{\sigma} \right] ds$$

$$\leq f_N(n) + \left(-1 + \frac{(\sigma - \theta_1)}{\sigma} + \epsilon(\sigma - \theta_1) \right) E_n[T_{n_0} \land S_{n_0} \land k].$$

Therefore,

$$\left(1 - \sigma - \theta_1 \right) \frac{-\epsilon(\sigma - \theta_1)}{\sigma} \leq f_N(n) - E_n[f_N(L^\Lambda_{n_0} \land S_{n_0} \land k)] \leq f_N(n).$$

We choose $\epsilon > 0$ such that $1 - (\sigma - \theta_1)(\sigma^{-1} + \epsilon) > 0$. Since the process $L^\Lambda$ is non-explosive (it is dominated by a Yule process with parameter $\sigma$), $S_{n_0} \to \infty$ as $N \to \infty$. Letting $N \to \infty$ in the previous inequality yields, for all $n \geq n_0$,

$$E_n[T_{n_0} \land S_{n_0} \land k] \leq f(n).$$

Letting $k \to \infty$ yields $E_n[T_{n_0}] \leq f(n)$. The proof is achieved. \qed

Under the assumptions of Lemma 2.1, the process $L^\Lambda$ admits a unique stationary measure $(p^\Lambda_n)_{n \in \mathbb{N}}$. We denote by $L^\Lambda_{n_0}$ a random variable distributed according to $(p^\Lambda_n)_{n \in \mathbb{N}}$. From [3, Theorem 2.4] we know that the sequence of stationary tail probabilities $a^\Lambda_n := P(L^\Lambda > n)$, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, is the unique solution of the system of equations

$$\sum_{k=2}^{\infty} \frac{1}{n} \binom{k+n-1}{k} \lambda_{k+n,k}(a^\Lambda_n - a^\Lambda_{n+1}) + \left(\frac{m_1}{n} + \sigma + \theta \right) a^\Lambda_n = \sigma a^\Lambda_{n-1} + \theta_1 a^\Lambda_{n+1}, \quad n \in \mathbb{N}, \quad (2.5)$$

with $m_1 := \Lambda(\{1\})$, together with the boundary conditions $a^\Lambda_0 = 1$ and $\lim_{n \to \infty} a^\Lambda_n = 0$. For $\Lambda = 2\delta_0$, (2.5) reduces to Fearnhead’s recursion ([8, 24, 18]). For this reason, we refer to (2.2) and to (2.5) as Fearnhead-type recursions.

It will be convenient to decompose $\Lambda := m_0 \delta_0 + \Lambda_0$, where $\Lambda_0$ has no mass at 0.

### 3. Geometric law in the $\Lambda$-model

For the seed bank model ($\Lambda \equiv 0$), it has been shown in [2] (see also [5]) that the block counting process is positive recurrent if and only if $\theta_0 > 0$ or $\theta_0 = 0$ and $\theta_1 > \sigma$, in which case its stationary distribution is geometric with parameter $1 - p$, i.e. $p^\Lambda_n = (1 - p) p^{n-1}$, $n \in \mathbb{N}$, where

$$p := \begin{cases} \frac{\sigma + \theta_0}{\sigma + \theta + \sqrt{(\sigma - \theta)^2 + 16\theta_0}} & \text{if } \theta_1 = 0, \\ \frac{\sigma + \theta + \sqrt{(\sigma - \theta)^2 + 16\theta_0}}{2\theta} & \text{if } \theta_1 > 0. \end{cases} \quad (3.1)$$

In this section, we aim to characterise the measures $\Lambda$ such that $L^\Lambda_{n_0}$ is geometrically distributed.

**Proposition 3.1.** Let $\rho \in (0, 1)$. The following assertions are equivalent

1. The random variable $L^\Lambda_{n_0}$ is geometrically distributed with parameter $1 - \rho$.
2. $m_0 = 0$ and for all $n \in \mathbb{N},$
   $$\frac{1}{n} \int_{[0,1]} \left[ (1 - \rho)(1 - x)^n + \rho - \left(1 - \frac{x}{1 - \rho x} \right)^n \right] \frac{\Lambda_0(dx)}{x^2} = \theta_1 \rho^2 - (\sigma + \theta) \rho + \sigma. \quad (3.2)$$
3. $m_0 = m_1 = 0$ and for all $n \in \mathbb{N}_0,$
   $$\left(1 - \rho\right) \int_{[0,1]} \left[ \frac{1 - x}{1 - \rho x} \right]^n \frac{1}{1 - \rho x} - (1 - x)^n \right] \frac{\Lambda_0(dx)}{x} = \theta_1 \rho^2 - (\sigma + \theta) \rho + \sigma. \quad (3.3)$$
4. $m_0 = m_1 = 0$ and for all $n \in \mathbb{N}_0,$
   $$\int_{[0,1]} (1 - x)^n \Lambda_0(dx) = (1 - \rho) \int_{[0,1]} \frac{1 - x}{1 - \rho x} \right]^n \frac{\Lambda_0(dx)}{(1 - \rho x)^2}, \quad (3.4)$$
   and
   $$\int_{[0,1]} \frac{\Lambda_0(dx)}{1 - \rho x} = \frac{\theta_1 \rho^2 - (\sigma + \theta) \rho + \sigma}{\rho(1 - \rho)}. \quad (3.5)$$
If in addition $\Lambda \neq 0$, then $\int x^{-1}\Lambda_0(dx) = +\infty$, corresponding to a dust-free component.

Proof. From Eq. (2.5), we see that $L_\infty^\Lambda \sim \text{Geom}(1-\rho)$ if and only if
\[
\frac{1}{n} \sum_{k=2}^{\infty} \frac{(n)^\updownarrow}{k!} \lambda_{k+n,k}(\rho - \rho^k) + \frac{m_1}{n} \rho = \theta_1 \rho^2 - (\sigma + \theta)\rho + \sigma, \quad n \in \mathbb{N},
\]
where $(\cdot)^\updownarrow$ is the rising factorial (see Appendix A). Now, let us define
\[
I_n := \frac{1}{n} \int_{(0,1)} \left(1 - \rho\right)(1-x)^n + \rho - \frac{1 - x}{1 - \rho x} \frac{n}{\rho} \Lambda_0(dx), \quad n \in \mathbb{N},
\]
\[
J_n := (1-\rho) \int_{(0,1)} \left(1 - \rho\right) \frac{1 - x}{1 - \rho x} \frac{(1-x)^n}{x} (1-x)^n \Lambda_0(dx), \quad n \in \mathbb{N}_0,
\]
\[
K_n := (1-\rho) \int_{(0,1)} \left(1 - \rho\right) \frac{1 - x}{1 - \rho x} \frac{1 - x}{x} (1-x)^n \Lambda_0(dx), \quad n \in \mathbb{N}_0.
\]
Using Fubini’s theorem, we see that
\[
\frac{1}{n} \sum_{k=2}^{\infty} \frac{(n)^\updownarrow}{k!} \lambda_{k+n,k}(\rho - \rho^k) = \frac{m_0(n+1)}{2} \rho - \rho^2 + I_n.
\]
Therefore, $L_\infty^\Lambda \sim \text{Geom}(1-\rho)$ if and only if
\[
\frac{m_0(n+1)}{2} \rho - \rho^2 + I_n + \frac{m_1}{n} \rho = \theta_1 \rho^2 - (\sigma + \theta)\rho + \sigma, \quad n \in \mathbb{N}.
\]
Since the left-hand side of (3.8) equals $I_n + m_1 \rho/n$, clearly (2) implies (1). Now assume that $L_\infty^\Lambda \sim \text{Geom}(1-\rho)$. A straightforward application of Fubini’s theorem shows that
\[
I_n = \frac{1}{n} \sum_{k=2}^{\infty} \frac{(n)^\updownarrow}{k!} \lambda_{k+n,k}(\rho - \rho^k) \int_{(0,1)} x^k(1-x)^n \frac{\Lambda_0(dx)}{x} \geq 0.
\]
Therefore, if $m_0 > 0$, the left-hand side of (3.8) tends to infinity as $n \to \infty$, in contrast to the right-hand side which is constant. We conclude that $m_0 = 0$. Hence Eq. (3.8) yields (3.2). Thus (1) implies (2).

Now we prove that (2) implies (3). Indeed, if (2) holds true, then
\[
k I_k + m_1 \rho = k(\theta_1 \rho^2 - (\sigma + \theta)\rho + \sigma), \quad k \in \mathbb{N}.
\]
Note that $J_n = (n+1)I_{n+1} - nI_n$. Therefore, (3.3), for $n \geq 1$, is obtained by writing down Eq. (3.9) for $k = n$ and $k = n + 1$ and taking the difference of these two equations. In addition, since
\[
nI_n = I_1 + \sum_{k=1}^{n-1} J_n, \quad n \in \mathbb{N},
\]
we deduce that $nI_n = n(\theta_1 \rho^2 - (\sigma + \theta)\rho + \sigma)$. Comparing this equation with (3.9), we get $m_1 = 0$. Since for $m_1 = 0$ we have $I_1 = J_0$, we conclude that (3.2) holds true also for $n = 0$, which ends the proof that (2) implies (3). Moreover, since $I_1 = J_0$ for $m_1 = 0$, (2) follows directly from (3) using (3.10).

It remains to prove the equivalence of (3) and (4), but this follows using that
\[
J_n - J_{n+1} = (1-\rho)K_n, \quad J_n = J_1 - (1-\rho) \sum_{k=0}^{n-1} K_k, \quad \text{and} \quad J_0 = (1-\rho) \int_{(0,1)} \frac{\Lambda_0(dx)}{1-\rho x}.
\]
Finally, let us assume that $L_\infty^\Lambda \sim \text{Geom}(1-\rho)$ and that $\int x^{-1}\Lambda_0(dx) < +\infty$. Applying the dominated convergence theorem, we get that the left-hand side of (3.3) converges to zero as $n \to \infty$. Hence the right-hand side of (3.3) has to be zero. Since the function integrated in (3.3) is non-negative, we conclude that it has to be zero, which is impossible. This finishes the proof. \qed

A first consequence of the previous result is that for the Wright-Fisher diffusion model and for the star-shaped model the distribution of $L_\infty^\Lambda$ is not geometric. Next, we show the existence of a non-trivial $\Lambda$ measure such that $L_\infty^\Lambda$ has the geometric distribution. More precisely, we show that the uniform measure on $[0,1]$ provides such an example.

**Lemma 3.2.** Let $\Lambda$ be the uniform measure on $[0,1]$. Then, there is a unique $\rho := \rho(\sigma, \theta_0, \theta_1) \in (0,1)$ satisfying Eq. (3.3). Moreover,

(1) if $\theta_0 = \theta_1 = 0$, then $\rho = 1 - e^{-\sigma}$.
(2) if \( \theta_0 = 0 \) and \( \theta_1 > 0 \), then \( \varphi = 1 - \theta_1^{-1} W (\theta_1 e^{\theta_1 - \sigma}) \).
(3) if \( \theta_0 > 0 \) and \( \theta_1 = 0 \), then \( \varphi = 1 - \theta_0 \left[ W (\theta_0 e^{\theta_0 + \sigma}) \right]^{-1} \),
where \( W \) denotes the (single-valued) restriction to \( \mathbb{R}_+ \) of the (multi-valued) Lambert-W function.

**Proof.** A straightforward calculation shows that (3.5) is equivalent to \((\sigma + \log(1 - \rho) - \theta_1 \rho)(\rho - 1) + \theta_0 \rho = 0\). Therefore, we only need to show that the function \( r : (0, 1) \to \mathbb{R} \) defined via
\[
r(x) := (\sigma + \log(1 - x) - \theta_1 x)(x - 1) + \theta_0 x, \quad x \in (0, 1),
\]
has a unique root. For this note that for all \( x \in (0, 1) \), we have \( r'(x) = -2\theta_1 - \frac{1}{x^2} < 0 \), and hence \( r' \) is strictly decreasing in \((0, 1)\). Moreover, \( r'(0+) = 1 + \sigma + \theta > 0 \) and \( r'(1-) = -\infty \). We infer that \( r' \) has a unique root \( x_0 \in (0, 1) \) and that \( r \) is strictly decreasing in \((x_0, 1)\). In addition, \( r(1-) = \theta_0 \geq 0 \), and thus, \( r(x) > \theta_0 \) for \( x \in (x_0, 1) \). Since, \( r(0+) = -\sigma < 0 \), this implies that \( r \) has a root in \((0, 1)\). The uniqueness of this root is a consequence of the strict monotonicity of \( r' \).

It remains to show the explicit formulas for \( \rho \) in the cases (1), (2) and (3). Case (1) is trivial. Cases (2) and (3) follow using that the function \( W \) is the inverse of the function \( x \mapsto xe^x \).

**Corollary 3.3.** For the Bolthausen-Sznitman model with selection parameter \( \sigma > 0 \) and mutation parameters \( \theta_0, \theta_1 \geq 0 \), the stationary distribution of the block counting process is geometric with parameter \( 1 - \varphi \), where \( \varphi \) is the unique solution of (3.3).

**Proof.** Using the change of variables \( y = (1 - \rho)x/(1 - \rho x) \), we get
\[
(1 - \rho) \int_0^1 \frac{1 - x}{1 - \rho x}^n \frac{dx}{(1 - \rho x)^2} = \int_0^1 (1 - y)^n dy,
\]
and therefore the uniform measure on \([0, 1]\) satisfies (3.4). Since (3.5) is satisfied for \( \rho = \varphi \), the result follows using Proposition 3.1.

It seems natural to ask if the uniform measure on \([0, 1]\) is the unique (up to multiplicative constant) measure \( \Lambda \) leading to the geometric distribution. This question will be the matter in the rest of this section.

**Lemma 3.4.** Let \( \varphi : [0, 1] \to [0, 1] \) be defined via \( \varphi(x) := (1 - x)/(1 - \rho x) \), \( x \in [0, 1] \). If \( L_\infty^{\Lambda} \sim \text{Geom}(1 - \rho) \), then the moments \( y_k := \int y^k \mu(dy), \ k \in \mathbb{N}_0 \), where \( \mu := \Lambda \circ \varphi^{-1} \) denotes the pushforward of the measure \( \Lambda \) by \( \varphi \), satisfy the linear system of equations
\[
y_n - 2\rho y_{n+1} + \rho^2 y_{n+2} - (1 - \rho)^{n+1} \sum_{k \geq 0} \binom{n + k - 1}{k} \rho^k y_{n+k} = 0, \quad n \in \mathbb{N},
\]
and \( \rho y_0 - 2\rho y_1 + \rho^2 y_2 = 0 \).

**Proof.** Noting that \( \varphi^{-1} = \varphi \), a straightforward calculation shows that (3.4) translates into
\[
(1 - \rho)^n \int_{[0,1]} \frac{y^n}{(1 - \rho y)^n} \mu(dy) = \frac{1}{1 - \rho} \int_{[0,1]} y^n (1 - \rho y)^2 \mu(dy), \quad n \in \mathbb{N}_0.
\]
Moreover,
\[
\int_{[0,1]} y^n (1 - \rho y)^2 \mu(dy) = y_n - 2\rho y_{n+1} + \rho^2 y_{n+2},
\]
and
\[
\int_{[0,1]} \frac{y^n}{(1 - \rho y)^n} \mu(dy) = \int_{[0,1]} y^n \sum_{k \geq 0} \binom{n + k - 1}{k} (\rho y)^k \mu(dy) = \sum_{k \geq 0} \binom{n + k - 1}{k} \rho^k y_{n+k}.
\]
The result follows.

Let us consider the linear operator \( S : \ell^\infty \to \ell^\infty \) on the Banach space \( \ell^\infty := \{ x = (x_i)_{i \in \mathbb{N}_0} \in \mathbb{R}^\infty : \| x \| := \text{sup}_{i \in \mathbb{N}_0} |x_i| < \infty \} \) defined via
\[
(Sx)_n := 2\rho x_{n+1} - \rho^2 x_{n+2} + (1 - \rho) \sum_{k \geq 0} \binom{n + k - 1}{k} \rho^k x_{n+k} \quad \text{and} \quad (Sx)_0 := 2\rho x_1 - \rho^2 x_2 + (1 - \rho) x_0,
\]
for all \( n \in \mathbb{N} \) and \( x \in \ell^\infty \). Note that from Lemma 3.4, if \( L_\infty^{\Lambda} \sim \text{Geom}(1 - \rho) \), then the vector \( y := (y_k)_{k \in \mathbb{N}_0} \) of the moments of \( \mu = \Lambda \circ \varphi^{-1} \) is a fixed point of \( S \). We are then interested on the fixed points of \( S \) arising as the moments of a finite measure. It is a well known result of Hausdorff ([11]) that a non-negative sequence \( x := (x_k)_{k \in \mathbb{N}_0} \) corresponds to a finite measure if and only if \( x \) is a completely
Proposition 3.5. Let \(\mu\) be a finite measure on \([0,1]\) and let \(x := (x_k)_{k \in \mathbb{N}_0} \in K\) be the sequence of moments of \(\mu\), then

\[
(Sx)_k = \int_{[0,1]} y^k \mu_S(dy), \quad k \in \mathbb{N}_0,
\]

where \(\mu_S(dy) := \rho y(2-\rho y)\mu(dy) + (1-\rho)\mu \circ \phi^{-1}(dy)\) and \(\phi : [0,1] \to [0,1]\) is defined via

\[
\phi(y) := \frac{(1-\rho)y}{1-\rho y}, \quad y \in [0,1].
\]

In particular, \(S(K) \subset K\).

Proof. From definition, we have

\[
(Sx)_k = \int_{[0,1]} y^k \left(2\rho y - \rho^2 y^2 + (1-\rho) \left(\frac{1-\rho}{1-\rho y}\right)^k\right) \mu(dy), \quad k \in \mathbb{N}_0.
\]

The first result follows. The second one is a direct consequence of (3.11). \(\square\)

We can now identify the restriction of \(S\) to \(K\) with the operator \(S : \mathcal{M}_f([0,1]) \to \mathcal{M}_f([0,1])\) on the space of finite positive measures defined via

\[
S(\mu)(dy) = \rho y(2-\rho y)\mu(dy) + (1-\rho)\mu \circ \phi^{-1}(dy), \quad \mu \in \mathcal{M}_f([0,1]).
\]

Moreover, fixed points of \(S\) in \(K\) are in a one-to-one relation with fixed points of \(S\). Note that if \(\mu\) is a fixed point of \(S\) then its support is invariant under \(\phi\).

We denote by \(\mathcal{M}_f^c\), \(\mathcal{M}_f^\nu\) and \(\mathcal{M}_f^\sigma\) the subsets of finite positive measures that are absolutely continuous, singular continuous and discrete, respectively. We know that if \(\Lambda\) is the uniform distribution on \([0,1]\), then \(L_1^\infty \sim \text{Geom}(1-\rho)\). Therefore, using Lemma 3.4 and Proposition 3.5, we conclude that the measure \(\mu \in \mathcal{M}_f^c\) with density \(k(y) := (1-\rho)/(1-\rho y)^2, y \in [0,1]\), is a fixed point of \(S\).

For each \(k \in \mathbb{N}\), \(\phi^{(k)}\) denotes the \(k\)-th iteration of the function \(\phi\), and \(\phi^{(-k)}\) denotes its inverse.

Lemma 3.6. For all \(n \in \mathbb{N}\) and \(x \in [0,1]\), we have

\[
\phi^{(n)}(x) = \frac{(1-\rho)^nx}{1-x(1-(1-\rho)^n)} \quad \text{and} \quad \phi^{(-n)}(x) = \frac{x}{(1-\rho)^n + x(1-(1-\rho)^n)}.
\]

Proof. The first identity can be shown by induction. The second one follows from the first one. \(\square\)

Proposition 3.7. Let \(\mu \in \mathcal{M}_f^\sigma\) be a fixed point of \(S\), then \(\mu(\{0\}) = \mu(\{1\}) = 0\). Moreover, if \(x_0 \in (0,1)\) has positive mass, then for all \(k \in \mathbb{Z}\), \(m_k := \mu(\{\phi^{(k)}(x_0)\}) > 0\) and

\[
m_{-k} = \frac{(1-\rho)^k - (1-\rho x_0)^2 m_0}{((1-\rho)^k + x_0(1-(1-\rho)^k))^2} \quad \text{and} \quad m_k = \frac{(1-\rho)^k (1-\rho x_0)^2 m_0}{(1-x_0(1-(1-\rho)^k))^2}, \quad k \in \mathbb{N}.
\]

Proof. Let \(\mu \in \mathcal{M}_f^\sigma\) be a fixed point of \(S\) and assume that there is \(x_0 \in [0,1]\) with \(m_0 := \mu(\{x_0\}) > 0\). From (3.12) we deduce that \(m_0 = \rho x_0(2-\rho x_0)m_0 + (1-\rho)\mu(\{\phi^{-1}(x_0)\})\). Therefore, \(x_0 \in (0,1)\) and \(m_{-1} := \mu(\{\phi^{-1}(x_0)\}) > 0\). Iterating the argument yields, for all \(k \in \mathbb{N}\), \(m_{-k} := \mu(\{\phi^{(-k)}(x_0)\}) > 0\) and

\[
m_{-k} \frac{(1-\rho \phi^{(-k)}(x_0))^2}{1-\rho} = m_{-k-1}, \quad k \in \mathbb{N}.
\]

Hence, \(m_{-k} = m_0 \prod_{i=0}^{k-1} (1-\rho \phi^{(-i)}(x_0))^2/(1-\rho)\). The first identity follows using that

\[
1 - \rho \phi^{(-i)}(x_0) = (1-\rho)\frac{(1-\rho)^{i-1} + x_0(1-(1-\rho)^i)}{(1-\rho)^i + x_0(1-(1-\rho)^i)} \quad \text{for} \quad i \in \mathbb{N}_0.
\]

Similarly, for \(k \in \mathbb{N}\), \(m_k := \mu(\{\phi^{(k)}(x_0)\}) > 0\) and

\[
m_{k+1} = \frac{1}{(1-\rho \phi^{(k+1)}(x_0))^2} m_k, \quad k \in \mathbb{N}.
\]
Thus, \( m_k = m_0 \prod_{i=1}^{k} (1 - \rho)/(1 - \rho \phi(i)(x_0))^2 \). The second identity follows using that 
\[
1 - \rho \phi(i)(x_0) = \frac{1 - x_0(1 - (1 - \rho)^i)}{1 - x_0(1 - (1 - \rho)^i)}, \quad i \in \mathbb{N}_0.
\]

The next result provides a class of fixed points of \( S \) in \( M_f^\mu \) and of discrete measures \( \Lambda \) such that \( L_{\infty}^\Lambda \) is geometrically distributed.

**Proposition 3.8.** For any \( \rho, x_0 \in (0, 1) \) and \( m_0 > 0 \), the measure \( \mu := \mu(\rho, x_0, m_0) \) given by 
\[
\mu := \sum_{k \in \mathbb{Z}} m_k \delta_{\phi(k)(x_0)},
\]
where the coefficients \( (m_k)_{k \in \mathbb{Z}} \) are defined via Eqs. (3.13), is a fixed point of \( S \) in \( M_f^\mu \). In addition, for \( m_0 < \sigma x_0(1 - x_0) \), the equation \( m_0(1 - \rho x_0)^2 - x_0(1 - x_0)(\theta_1 \rho^2 - (\sigma + \theta) \rho + \sigma) = 0 \) has a unique solution \( \rho_* \) in \( (0, 1) \). Setting \( \Lambda := \mu(\rho_*, x_0, m_0) \), we have \( L_{\infty}^\Lambda \sim \text{Geom}(1 - \rho_*) \).

**Proof.** Since \( \lim_{k \to \infty} \phi(k)(x_0) = 0 \) and \( \lim_{k \to \infty} \phi^{(-k)}(x_0) = 1 \), then there exists \( k_0 \in \mathbb{N} \) such that 
\[
m_k \leq c_{x_0} \left( 1 - \frac{\rho}{2} \right)^k m_0 \quad \text{and} \quad m_{-k} \leq C_{x_0} \left( 1 - \frac{\rho}{2} \right)^k m_0, \quad k > k_0,
\]
for some appropriate constants \( c_{x_0}, C_{x_0} > 0 \). Therefore, \( \mu \in M_f^\mu \). Moreover, since the coefficients \( (m_k)_{k \in \mathbb{Z}} \) satisfy (3.13) and (3.15), it follows that for all \( k \in \mathbb{Z} \), \( S\mu(\{\phi^{(-k)}(x_0)\}) = \mu(\{\phi^{(-k)}(x_0)\}) \). Hence, \( \mu \) is a fixed point of \( S \). Now, we assume that \( m_0 < \sigma x_0(1 - x_0) \). Note that the function \( r : [0, 1] \to \mathbb{R} \) defined via \( r(z) := m_0(1 - \rho z)_x^2 - x_0(1 - x_0)((\theta_1 z^2 - (\sigma + \theta) z + \sigma), z \in [0, 1], \) is a quadratic polynomial with \( r(0) = m_0 - \sigma x_0(1 - x_0) < 0 \) and \( r(1) = m_0(1 - \rho x_0)^2 + x_0(1 - x_0)\theta_0 > 0 \), and therefore, it has a unique root \( \rho_* \) in \( (0, 1) \). Let \( \mu := \mu(\rho_*, x_0, m_0) \). Using that \( \mu \) is a fixed point of \( S \), one can easily show that the measure \( \Lambda := \mu \cdot \varphi^{-1} \) satisfies (3.4) for \( \rho = \rho_* \). It remains to show that \( \Lambda \) satisfies (3.5) for \( \rho = \rho_* \). Using that \( \varphi = \varphi^{-1} \) and the definition of \( \mu \), we obtain
\[
\int_{(0,1)} \frac{L_{0}(dx)}{1 - \rho_*x} = \sum_{i \in \mathbb{Z}} a_i \quad \text{where} \quad a_i = m_i \frac{1 - \rho \phi(i)(x_0)}{1 - \rho_*}, \quad i \in \mathbb{Z}.
\]
Moreover, setting \( b_n := 1 - x_0(1 - (1 - \rho_*)^n) \) for \( n \in \mathbb{N}_0 \), we get 
\[
a_n = \frac{m_0(1 - \rho_*)^2(1 - \rho_*)^{-n}}{b_n^2 b_n} = \frac{m_0(1 - \rho_*)^2}{\rho_* (1 - x_0)} \left( \frac{(1 - \rho_*)^{-n-1}}{b_n} - \frac{(1 - \rho_*)^{-n}}{b_n+1} \right).
\]
Hence,
\[
\sum_{i \in \mathbb{N}_0} a_i = \frac{m_0(1 - \rho_*)^2}{\rho_* (1 - \rho_*) (1 - x_0)}.
\]
Similarly, setting \( c_n := (1 - \rho_*)^n + x_0(1 - (1 - \rho_*)^n) \), \( n \in \mathbb{N}_0 \), we obtain 
\[
a_{-n} = \frac{m_0(1 - \rho_*)^{-2}(1 - \rho_*)^{-n-2}}{c_{n-1} c_n} = \frac{m_0(1 - \rho_*)^2}{\rho_* x_0} \left( \frac{(1 - \rho_*)^{-n-2}}{c_{n-1}} - \frac{(1 - \rho_*)^{-n-1}}{c_n} \right).
\]
Therefore,
\[
\sum_{i \in \mathbb{N}} a_{-i} = \frac{m_0(1 - \rho_*)^2}{\rho_* (1 - \rho_*) x_0}.
\]
Summarising, we have
\[
\int_{(0,1)} \frac{L_{0}(dx)}{1 - \rho_*x} = \sum_{i \in \mathbb{Z}} a_i = \frac{m_0(1 - \rho_*)^2}{\rho_* (1 - \rho_*) x_0} = \frac{\theta_1 \rho_*^2 - (\sigma + \theta) \rho_* + \sigma}{\rho_* (1 - \rho_*)},
\]
ending the proof. \( \square \)

As a consequence the dimension of the set of fixed points of \( S \) in \( X \) is infinite. In the next proposition, we show that the measure \( \mu(dy) = h(y)dy \), with \( h(y) = 1/(1 - \rho y)^2 \), \( y \in [0, 1] \), is the unique fixed point of \( S \) (up to multiplicative constant) in \( M_f^\mu \) having a density which is continuous in \([0, 1]\).
Proposition 3.9. Let $h : [0, 1] \to \mathbb{R}_+$ be a continuous function on $[0, 1]$. The measure $\mu(dy) = h(y)dy$ on $[0, 1]$ is a fixed point of $\mathcal{S}$ if and only if

$$h(y) = \left(\frac{1 - \rho}{1 - \rho y}\right)^2 h(1), \quad y \in [0, 1].$$

Proof. Proving that the function $h$ defined in the statement leads to a fixed point of $\mathcal{S}$ is straightforward. Now, assume that $\mu(dy) = h(y)dy$ is a fixed point of $\mathcal{S}$. It follows that

$$h(y) = \frac{1 - \rho}{(1 - \rho y)^2} h\left(\phi^{-1}(y)\right)\phi^{-1}(y) = \left(\frac{1 - \rho}{1 - \rho y} \right)^2 h\left(\phi^{-1}(y)\right), \quad y \in (0, 1).$$

Iterating this equation, we obtain

$$h(y) = \left(\prod_{k=0}^{n} \frac{1 - \rho}{1 - \rho \phi^{-k}(y)} \right)^2 h\left(\phi^{-n+1}(y)\right), \quad y \in (0, 1). \quad (3.16)$$

Using Lemma 3.6, we deduce that

$$\frac{1 - \rho}{(1 - \rho \phi^{-k}(y)) (1 - \rho + \rho \phi^{-k}(y))} = \frac{a_k(y)}{a_{k+1}(y)a_{k-1}(y)}, \quad k \in \mathbb{N}_0,$$

where $a_k(y) = (1 - \rho)^k + y (1 - \rho y)^k$, $k \geq -1$. Hence,

$$\prod_{k=0}^{n} \frac{1 - \rho}{1 - \rho \phi^{-k}(y)} \frac{a_0(y)}{a_{-1}(y) a_{n+1}(y)} = \frac{a_0(y)}{1 - \rho y a_{n+1}(y)}.$$

Letting $n \to \infty$ in (3.16) yields the result. \quad \square

Remark 3.1. Consider a measure of the form $\mu(dy) = h(y)dy$ with $h : [0, 1] \to \mathbb{R}_+$ being measurable in $[0, 1]$ and continuous in an open interval $I \subset (0, 1)$. On can easily check that if $\mu$ is a fixed point of $\mathcal{S}$, then $h$ is continuous in $(0, 1)$. However, for the uniqueness (up to multiplicative constant) of such a fixed point, the continuity of the density function at 1 (or at 0) is crucial. Indeed, let $x_0 \in (0, 1)$ and let $p : [0, 1] \to \mathbb{R}_+$ be a continuous function on $[0, 1]$ such that $p(0) = p(1)$. We define the function $C : (0, 1) \to \mathbb{R}_+$ via

$$C(x) := p\left(\phi^{(k)}(x) - x_0\right), \quad \text{for} \quad x \in [\phi^{-k}(x_0), \phi^{(-k+1)}(x_0)], \quad k \in \mathbb{Z}.$$

The function $C$ is bounded, continuous in $(0, 1)$ and is constant on the sets of the form $\{\phi^{(i)}(y)\}_{i \in \mathbb{Z}}$ for some $y \in (0, 1)$. Hence, the function $h : (0, 1) \to \mathbb{R}_+$ defined via $h(y) := C(y)/(1 - \rho y)^2$, $y \in (0, 1)$, satisfies

$$h(y) = \left(\frac{1 - \rho}{1 - \rho y}\right)^2 h\left(\phi^{-1}(y)\right)\phi^{-1}(y), \quad y \in (0, 1).$$

Therefore, the measure $\mu(dy) := h(y)dy$ is a fixed point of $\mathcal{S}$. Moreover, one can prove that all the fixed points of $\mathcal{S}$ in $\mathcal{M}_1^+$ having a density which is continuous in $(0, 1)$ are of this form. As a consequence, the unique $\beta(a, b)$-model leading to a geometric distribution is the $\beta(1, 1)$-model, i.e. the Bolthausen-Sznitman model.

4. Solving the Fearnhead-type recursion for the Moran model

In the Moran model with mutation and selection, the stationary distribution of the block counting process is characterised by Equations (2.2) and (2.3), which, using that $a_n^N = \sum_{k=n+1}^N p_k^N$, turn out to be equivalent to

$$\left(\frac{n}{N} + u_1\right) p_n^N = \frac{(N - n + 1) s}{N} p_{n-1}^N - u_0 \sum_{\ell=n}^{N} p_{\ell}^N, \quad n \in [N+1], \quad (4.1)$$

together with the boundary conditions

$$\sum_{i=1}^{N} p_i^N = 1 \quad \text{and} \quad (1 + u) p_N^N = \frac{s}{N} p_{N-1}^N. \quad (4.2)$$

Let $D := \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk and set $D_* = D \setminus \{z : \text{Im}(z) = 0, \text{Re}(z) \leq 0\}$. For $z_1, z_2 \in D_*$ and any holomorphic function $f : D_* \to \mathbb{C}$ we denote by $\int_{z_1}^{z_2} f(\xi)d\xi$ the integral of $f$. 
along any smooth path in $D_*$ connecting $z_1$ and $z_2$. The following is the main result of this section. It provides explicit expressions for the stationary distribution $(p_N^n)_{n \in [N]}$ and its probability generating function $p_N : \mathbb{C} \to \mathbb{C}$, defined via $p_N(z) := \sum_{n=1}^N p_N^n z^n$.

**Theorem 4.1.** For the Moran model with selection parameter $s > 0$ and mutation parameters $u_0, u_1 \geq 0$ the following holds

(i) If $u_0 = 0$, then

$$p_N^n = \frac{1}{2F_1\left[\frac{1}{1-N} \mid \frac{N-1}{N u_2+2}, -s\right]} \binom{N-1}{n-1} s^{n-1}, \quad n \in [N],$$

where $2F_1$ is the Gauss hypergeometric function (see Appendix A). In particular, we have

$$p_N(z) = \frac{1}{2F_1\left[\frac{1}{1-N} \mid \frac{N-1}{N u_2+2}, -sz\right]} \binom{N-1}{n-1} s^{n-1}, \quad z \in \mathbb{C}.$$  \hspace{1cm} (4.4)

(ii) If $u_0 > 0$, then for all $z \in D_*$

$$p_N(z) = \frac{N u_0 I_N^N}{s(I_0^N - I_1^N)} \left(z + \frac{1}{2}\right)^{(1+u_1+\rho_0)N} \cdot \frac{\rho_0}{z(1-z)^{N-1}} \int_0^z \left(\frac{I_0^N}{\xi} - \frac{1}{2}\right) \left(\frac{I_1^N}{\xi} + 1\right)^{(1+u_1+\rho_0)N+1} \xi \frac{d\xi}{\xi},$$

with $\rho_0 := u_0/(1+s)$ and $I_N^N := \int_0^1 \frac{q^{N+1}(1+y)^{N+1}}{(y+z)^{N+1}} \rho(1+y)^{N+1} dy$, $i \in \{0,1\}$. Moreover,

$$p_N^n = \frac{N u_0}{I_0^N - I_1^N} \left[\frac{I_1^N}{N u_1 + 1} q_{n,1}^N - \frac{I_0^N}{N u_1 + 2} q_{n,2}^N\right], \quad n \in [N],$$

where $q_{n,1}^N := 1$, $q_{n,2}^N := 0$, and for $n \in \{2, \ldots, N\}$

$$q_{n,1}^N := \sum_{m=0}^{N-1} \frac{(-N + i - 1)^m}{(N u_1 + i + 1)m} \cdot (-s)^m 3F_2 \left[m + 1; 1, 1 - N \rho_0; m - n + i \mid 1 + s\right], \quad i \in \{0,1\},$$

where $3F_2$ is the generalised hypergeometric function (see Appendix A).

**Remark 4.1.** When $u = 0$, Eq. (4.3) implies that $L_N^N$ is a binomial random variable with parameters $N$ and $s/(1+s)$ conditioned to be strictly positive (see also [3, Section 3.1]).

The proof of Theorem 4.1 is based on the following result.

**Lemma 4.2.** The generating function $p_N$ satisfies the ordinary differential equation

$$z(1-z)(1+sz)p_N'(z) = -N(sz^2 - (s + u)z + u_1)p_N(z) + (1 + N u_1) p_N^N z(1-z) - N u_0 z^2,$$

on $D_*$ with boundary conditions $p_N(0) = 0$ and $p_N(1) = 1$.

**Proof.** The boundary conditions follow from the definition of $p_N$. Multiplying (4.1) with $z^n$ and summing over all $n \in \{2, \ldots, N-1\}$ yields

$$\sum_{n=2}^{N-1} \left(\frac{n}{N} + u_1\right) p_N^n z^n = \sum_{n=2}^{N-1} (N-n+1)p_{n-1}^N z^n - u_0 \sum_{n=2}^{N-1} z^n \sum_{\ell=0}^{N} p_{\ell}^N.$$

The left hand side is equal to

$$\sum_{n=2}^{N-1} \left(\frac{n}{N} + u_1\right) p_N^n z^n = \frac{z}{N} p_N'(z) + u_1 p_N(z) - \left(\frac{1}{N} + u_1\right) p_1^N z - (1 + u_1)p_N^N z^N.$$

Moreover,

$$\sum_{n=2}^{N-1} (N-n+1)p_{n-1}^N z^n = -sz^2 p_N(z) + sz p_N(z) - \frac{s}{N} p_{N-1}^N z^N,$$

$$\sum_{n=2}^{N-1} z^n \sum_{\ell=0}^{N} p_{\ell}^N = -\frac{z}{1-z} p_N(z) - p_N^N z^N + \frac{z^2}{1-z}.$$
Proof of Theorem 4.4: (i): Formula 4.5 is obtained by iteration of 4.1 and using 4.2. Formula 4.4 is a direct consequence of Eq. 4.3.

(ii): Since both sides of (4.5) are analytic in $D_*$, it suffices to show that they coincide on the real interval $(0,1)$. Thus, we have to solve (4.7) in $(0,1)$. Separation of variables in the homogeneous equation $x(1-x)(1+sz)f'(x) + N(sx^2 - (s + u)x + u)g(x) = 0$ on $(0,1)$ implies that its basic solution is given by $g(x) = (x + s^{-1}(1+u+\rho_0))N_u^{-1}N_u(1-x)^{-N\rho_0}$, $x \in (0,1)$. Hence, for any $x_0 \in (0,1)$, the variation of constants method yields

$$p_N(x) = \frac{1}{s} \left( x + \frac{1}{2} \right)^{(1+u+\rho_0)N} x^{N\rho_0} \left[ c_{x_0} - \int_x^{x_0} \frac{\beta_N - \alpha_N \xi}{(1+u+\rho_0)N_{\rho_0}} \left( \frac{1}{(1+u+\rho_0)N_{\rho_0} + 1} \right) \xi \right], \ x \in (0, x_0),$$

where $c_{x_0}$ is a constant, $\alpha_N := N_{u_0} + (1 + N_{u_1})p_1^N$ and $\beta_N := (1 + N_{u_1})p_1^N$. Moreover, the boundary condition $p_N(0) = 0$ implies that

$$p_N(x) = \frac{1}{s} \left( x + \frac{1}{2} \right)^{(1+u+\rho_0)N} x^{N\rho_0} \left[ 1 + \int_0^x \frac{\beta_N - \alpha_N \xi}{(1+u+\rho_0)N_{\rho_0}} \left( \frac{1}{(1+u+\rho_0)N_{\rho_0} + 1} \right) \xi \right], \ x \in (0, x_0). \quad (4.8)$$

Since $x_0 \in (0,1)$ is arbitrary, the previous identity holds for all $x \in (0,1)$. Letting $x \to 1$ and using that $p_N(1) = 1$, we infer that $\beta_N = \alpha_N$. Hence, $p_1^N = N_{u_0}F_1^1 + ((N_{u_1} + 1)(I_{\rho_0}^1 - I_{\rho_0}^1))$. The resulting expressions for $\alpha_N$ and $\beta_N$ in 4.8 shows that 4.5 holds in $(0,1)$, and thus in $D_*$. Note that from 4.5 and Corollary 4.3 we have for all $z \in \{w \in D : |w| < 1/\sqrt{1+2s}\}$

$$p_N(z) = \frac{N_{u_0}}{I_{\rho_0}^1 - I_{\rho_0}^1} \left[ \frac{I_{\rho_0}^1}{N_{u_1} + 1} q_{1,1}(z) - \frac{I_{\rho_0}^1}{N_{u_1} + 1} q_{1,2}(z) \right], \quad (4.9)$$

where for $i \in \{1,2\}$

$$q_{i,1}(z) = \frac{(1+s)(1+u+\rho_0)N}{(1-z)^{N_{\rho_0}}} \left( \begin{array}{c} N_{u_1} + i + 1; \ -N + i + 1; \ -z \end{array} \right)_{N_{u_1} + i + 1} F_1 \left( \begin{array}{c} 1; \ -N + i + 1; \ -1 - N_{\rho_0} \end{array} \right)_{N_{u_1} + i + 1}.$$
(2) if \( u_1 = 0 \), then
\[
E \left[ (L^N_{\infty})^n_k \right] = \frac{n!(N-1)^k \sum_{s=1}^{n-1} \left( \frac{s}{1+s} \right)^{n-1} E[L^N_{\infty}] }{(2 + \frac{Nz}{1+s})^{n-1}}.
\]

Proof. Differentiating Eq. (4.10) \( n \) times and using the general Leibniz rule we obtain
\[
(1 + sz) p^{(n+1)}_N(z) + n s p^{(n)}_N(z) = \sum_{k=0}^{n} \frac{(n-k)(z-1)^k}{k!}.
\]

Since \( \lim_{z \to 1} p^{(n)}_N(z) = E[(L^N_{\infty})^n] \), we have
\[
\lim_{z \to 1} \sum_{k=0}^{n} \frac{(n-k)(z-1)^k}{k!} = E \left[ \sum_{k=0}^{n} \frac{(n-k)(N_{\infty})^k}{k!} \right] = E \left[ (L^N_{\infty} - 1)^n \right].
\]

In addition, using \( n \)’s Hôpital’s rule, we get for all \( n \in \mathbb{N} \)
\[
\lim_{z \to 1} \frac{1}{(z-1)^{n+1}} \left( \sum_{k=0}^{n} \frac{p^{(n)}_N(z)(z-1)^k}{k!} - 1 \right)
\]
\[
= \lim_{z \to 1} \frac{1}{(n+1)(z-1)^n} \sum_{k=0}^{n} \frac{p^{(k+1)}_N(z)(z-1)^k}{k!} - \frac{p^{(k)}_N(z)(z-1)^{k-1}}{k!}
\]
\[
= \lim_{z \to 1} \frac{p^{(n+1)}_N(z)(z-1)^n}{n!(n+1)!} = \frac{E[(L^N_{\infty})^n]}{(n+1)!}.
\]

The first statement follows letting \( z \to 1 \) in (4.11) and using (4.12) and (4.13).

Now, we proceed to prove (1). First note that \( E[(L^N_{\infty})^n] = \sum_{j=n}^{\infty} p^j N \).

Using (1.4) we get
\[
E[(L^N_{\infty})^n] = p^1 N N u - 21 \int \left[ \frac{1}{N_{u+2}} \right] x \right] x.
\]

The result follows from [17] p. 241, Eq. 9.2.3.

Assertion (2) follows directly iterating the first statement with \( u_1 = 0 \).

5. The master equation for the Wright-Fisher model

As in the previous section, we denote by \( D := \{ z \in \mathbb{C} : |z| < 1 \} \) the open unit disk. In this section we aim to characterise the probability generating function \( p^N \) via \( p^N(z) := \sum_{k=1}^{\infty} \frac{N}{k} z^k \). Since \( p^N = a^{n+1}_N - a^n_N \), Eq. (2.15) turns into
\[
\left( \frac{m_0(n+1)}{2} + \theta_1 \right) a^{n+1}_N + \sum_{k=n+1}^{\infty} \frac{\sigma}{n} \left( c_{n,k} + \frac{m_1}{n} + \theta_0 \right) a^{n}_N = \sigma p^n_N, \quad n \in \mathbb{N},
\]
\[
\text{where } c_{n,k} := \frac{1}{n} \sum_{l=k+1}^{\infty} \binom{l-1}{l-n} \int_{0.1}^{1} \xi^{l-n-2}(1-\xi)^{n} d\xi, \quad k > n.
\]

The recursion is completed with the condition \( \sum_k p^k_N = 1 \).

For \( z_1, z_2 \in D \) and any analytic function \( f : D \to \mathbb{C} \) we denote by \( \int_{z_1}^{z_2} f(\xi) \, d\xi \) the integral of \( f \) along any smooth path in \( D \) connecting \( z_1 \) and \( z_2 \).

Proposition 5.1 (Master equation 1). For all \( z \in D \setminus \{0\} \),
\[
\frac{m_0}{2} p^N(z) + m_1 \int_0^z \frac{u - p^N(u)}{u(1-u)} du + \frac{\sigma z^2 - (\sigma + \theta) z + \theta_1}{z(1-z)} p^N(z) = \left( \frac{m_0}{2} + \theta_1 \right) p^1_N + \frac{\theta_0 - 1}{1-z} + \sum_{k=1}^{\infty} c_k(z),
\]
\[
\text{where } c_k(z) := \sum_{n=1}^{k-2} c_{n,k} z^n.
\]
Proof. Let \(z \in D \setminus \{0\}\). Multiplying (5.1) with \(z^n\) and summing over all \(n \in \mathbb{N}\) leads to

\[
\sum_{n=1}^{\infty} \left( \frac{m_0(n + 1)}{2} + \theta_1 \right) p_{n+1}^\Lambda z^n + \sum_{n=1}^{\infty} z^n \sum_{k=n+1}^{\infty} p_k^\Lambda \left( c_{n,k} + \frac{m_1}{n} + \theta_0 \right) = \sigma p_\Lambda(z). 
\] (5.2)

Note that \(\sum_{n=1}^{\infty} n p_n^\Lambda z^{n-1} = p_\Lambda'(z)\). Therefore,

\[
\sum_{n=1}^{\infty} \left( \frac{m_0(n + 1)}{2} + \theta_1 \right) p_{n+1}^\Lambda z^n = \frac{m_0}{2} (p_\Lambda'(z) - p_1^\Lambda) + \theta_1 (p_\Lambda(z) - z p_1^\Lambda). 
\] (5.3)

In addition, using Fubini’s theorem, we get

\[
\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{k=n+1}^{\infty} p_k^\Lambda (c_{n,k} + \theta_0) = \sum_{k=2}^{\infty} p_k^\Lambda \left( c_k(z) + \frac{z - z_k}{1 - z} \right) = \sum_{k=2}^{\infty} p_k^\Lambda c_k(z) + \frac{z - p_\Lambda(z)}{1 - z}. 
\] (5.4)

Similarly, we have

\[
\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{k=n+1}^{\infty} p_k^\Lambda = \int_0^z \left( \sum_{n=1}^{\infty} \sum_{k=n+1}^{\infty} p_k^\Lambda \right) du = \int_0^z u - \frac{p_\Lambda(u)}{u(1 - u)} du. 
\] (5.5)

Plugging (5.3), (5.4), (5.5) in (5.2) yields the result. \(\square\)

**Proposition 5.2** (Master equation II). For all \(z \in D \setminus \{0\}\),

\[
\frac{m_0}{2} p_\Lambda'(z) + m_1 \int_0^z \frac{u - p_\Lambda(u)}{u(1 - u)} du + \frac{\sigma z - (\sigma + \theta) z + \theta_1 z}{z(1 - z)} p_\Lambda(z) = \frac{m_0}{2} (p_\Lambda'(z) - p_1^\Lambda) - \frac{\theta_0 z}{1 - z} 
\]

\[
- \int_{(0,1)} \frac{\Lambda_0(\xi)}{\xi^2} \left[ \int_{z(1 - \xi)}^z \frac{u - p_\Lambda(u)}{u(1 - u)} du - \frac{1 - p_\Lambda(u)}{1 - u} du + \int_0^{\xi z(1 - \xi)} \frac{1 - p_\Lambda(u)}{1 - u} du \right].
\]

**Proof.** From Proposition 5.1 it suffices to show that

\[
\sum_{k=2}^{\infty} p_k^\Lambda c_k(z) = \int_{(0,1)} \frac{\Lambda_0(\xi)}{\xi^2} \left( \int_0^z \sum_{k=2}^{\infty} p_k^\Lambda C_k(u, \xi) du \right). 
\] (5.6)

Using Fubini’s theorem, we deduce that

\[
\sum_{k=2}^{\infty} p_k^\Lambda c_k(z) = \int_{(0,1)} \frac{\Lambda_0(\xi)}{\xi^2} \left( \int_0^z \sum_{k=2}^{\infty} p_k^\Lambda C_k(u, \xi) du \right). 
\]

where for \(u \in D \setminus \{0\}\) and \(\xi \in (0,1)\)

\[
C_k(u, \xi) := \sum_{n=0}^{k-2} u^n \sum_{\ell=k}^{\infty} \binom{\ell}{n} \xi^{\ell-n}(1 - \xi)^{n+1} = (1 - \xi) \sum_{n=0}^{k-2} \frac{u(1 - \xi)}{\xi} \sum_{\ell=k}^{\infty} \frac{\ell}{n} \xi^{\ell}.
\]

Since

\[
\sum_{\ell=k}^{\infty} \binom{\ell}{n} \xi^{\ell} = \sum_{\ell=n}^{\infty} \binom{\ell}{n} \xi^{\ell} - \sum_{\ell=0}^{k-1} \binom{\ell}{n} \xi^{\ell} = \frac{\xi^n}{(1 - \xi)^{n+1}} - (1 - \frac{1}{n}) \xi^{k-1} - \sum_{\ell=0}^{k-2} \frac{\ell}{n} \xi^{\ell},
\]

we deduce that

\[
C_k(u, \xi) = \sum_{n=0}^{k-2} u^n - (1 - \xi) \left[ \xi^{k-1} \sum_{n=0}^{k-2} \frac{\ell}{n} \xi^{\ell} + \sum_{n=0}^{k-2} \frac{u(1 - \xi)}{\xi} \sum_{\ell=n}^{k-2} \frac{\ell}{n} \xi^{\ell} \right] 
\]

\[
= \frac{1 - u^{k-1}}{1 - u} - (1 - \xi) \left[ \xi^{k-1} \sum_{n=0}^{k-2} \frac{\ell}{n} \xi^{\ell} + \sum_{n=0}^{k-2} \frac{u(1 - \xi)}{\xi} \sum_{\ell=n}^{k-2} \frac{\ell}{n} \xi^{\ell} \right] 
\]

\[
= \frac{1 - u^{k-1}}{1 - u} - (1 - \xi) \left[ \xi^{k-1} \sum_{n=0}^{k-2} \frac{\ell}{n} \xi^{\ell} + \sum_{n=0}^{k-2} \frac{u(1 - \xi)}{\xi} \sum_{\ell=n}^{k-2} \frac{\ell}{n} \xi^{\ell} \right] 
\]

\[
= \frac{1 - u^{k-1}}{1 - u} - (1 - \xi) \left[ (\xi + u(1 - \xi))^{k-1} - (u(1 - \xi))^{k-1} + \sum_{\ell=0}^{k-2} \frac{\ell}{n} \xi^{\ell} \right] 
\]

\[
= \frac{1 - u^{k-1}}{1 - u} - (1 - \xi) \left[ (\xi + u(1 - \xi))^{k-1} - (u(1 - \xi))^{k-1} + \sum_{\ell=0}^{k-2} (\xi + u(1 - \xi))^{\ell} \right]. 
\]
As a consequence, we obtain
\[
\sum_{k=2}^{\infty} p_k^\Lambda C_k(u, \xi) = \frac{u - p_\Lambda(u)}{u(1-u)} - (1 - \xi) \left[ \frac{1 - p_\Lambda(\xi + u(1-\xi))}{1 - \xi - u(1-\xi)} - \frac{p_\Lambda(u(1-\xi))}{u(1-\xi)} \right].
\]
Integrating over \(u \in (0, z)\) and making appropriate change of variables, we get
\[
\int_0^z \sum_{k=2}^{\infty} p_k^\Lambda C_k(u, \xi)du = \int_0^z \frac{u - p_\Lambda(u)}{u(1-u)}du - \int_\xi^{\xi + z(1-\xi)} \frac{1 - p_\Lambda(v)}{1-v}dv + \int_0^{z(1-\xi)} \frac{p_\Lambda(v)}{v}dv.
\]

The result follows.

As a first application of the results obtained in this section we rediscover the geometric law arising in the seed bank model.

**Corollary 5.3** (The seed bank model). If \(\Lambda \equiv 0\), and \(\theta_0 > 0\) or \(\theta_1 > \sigma\), then for all \(z \in D\)
\[
p_\Lambda(z) = \frac{(1-p)z}{1-pz},
\]
where \(p\) is given in (6.1). In particular, \(L_\infty^\Lambda \sim \text{Geom}(1-p)\).

**Proof.** In this case \(\sigma_\Lambda = 0\) and Lemma 2.14 implies that, if \(\theta_0 > 0\) or \(\theta_1 > \sigma\), the process \(L_\Lambda\) is positive recurrent. Moreover, Proposition 6.2 yields
\[
p_\Lambda(z) = \frac{z[\theta_1 p_\Lambda^\Lambda (1-z) - \theta_0 z]}{\sigma z^2 - (\sigma + \theta)z + \theta_1}, \quad z \in D \setminus \{0\}.
\]
The result for \(\theta_1 = 0\) follows directly. For \(\theta_1 > 0\), the map \(z \mapsto \sigma z^2 - (\sigma + \theta)z + \theta_1\) has exactly one root in \(D\), which is given by \(z_0 : = (\sigma + \theta - \sqrt{(\sigma + \theta)^2 - 4\sigma\theta_1})/(2\sigma)\). Since \(p_\Lambda\) is analytic in \(D\), we conclude that \(p_\Lambda^\Lambda = \theta_0 z_0/(\theta_1(1 - z_0))\). Plugging this expression in the formula for \(p_\Lambda\) yields the result.

**6. Solving the Fearnhead recursion for the Wright-Fisher diffusion model**

In this section we assume that the measure \(\Lambda\) is concentrated in 0 with total mass \(m_0 := \Lambda(\{0\})\), i.e. blocks merge according to the Kingman coalescent. In particular, \(\sigma_\Lambda = \infty\), and therefore, the block counting process is positive recurrent for any \(\sigma > 0\). Note that (6.1) reads
\[
\left(\frac{m_0(n+1)}{2} + \theta_1\right) p_{n+1}^\Lambda = \sigma p_n - \theta_0 \sum_{k=n+1}^{\infty} p_k^\Lambda, \quad n \in \mathbb{N}.
\]
The boundary condition \(a_0^\Lambda = 1\) yields \(\sum_{n=1}^{\infty} p_n^\Lambda = 1\). The following is the main result of this section.

**Theorem 6.1.** For the Wright-Fisher diffusion model with selection parameter \(\sigma > 0\) and mutation parameters \(\theta_0, \theta_1 \geq 0\) the following holds

(i) If \(\theta_0 = 0\), then
\[
p_n^\Lambda = \frac{1}{1 F_1 \left[ \begin{array}{c} 1 \\ 2 + \frac{2\sigma}{m_0} \end{array} \right] \left( \frac{2\sigma}{m_0} \right)^{n-1}, \quad n \in \mathbb{N},
\]
where \(1 F_1\) is the confluent hypergeometric function (see Appendix A). In particular, we have
\[
p_\Lambda(z) = \frac{1}{1 F_1 \left[ \begin{array}{c} 1 \\ 2 + \frac{2\sigma}{m_0} \end{array} \right] \left( \frac{2\sigma}{m_0} \right)^z}, \quad z \in \mathbb{C}.
\]
(ii) If \( \theta_0 > 0 \), then for all \( z \in D \)
\[
p_\Lambda(z) = \frac{2 \theta_0 I_0}{\mu_0 (I_0 - I_1)} e^{\frac{\Theta}{\mu_0}} z e^{\frac{\Theta}{\mu_0}} (1 - z) - \int_0^z \left( \frac{I_1}{I_0} - \xi \right) \left( 1 - \xi \right) e^{\frac{\Theta}{\mu_0}} (1 - \xi) e^{\frac{\Theta}{\mu_0}} \xi d\xi,
\]
where \( I_i = \int_0^1 y^{\frac{\Theta}{\mu_0}} (1 - y) e^{-\frac{\Theta}{\mu_0}} y dy, \ i \in \{0, 1\} \). Moreover,
\[
p_n = \frac{2 \theta_0}{(I_0 - I_1)} \left[ \frac{I_0}{2 \theta_1 + m_0} q_{n,1} - \frac{I_0}{2 \theta_1 + m_0} q_{n,2} \right], \ n \in \mathbb{N},
\]
where \( q_{1,1} = 1, q_{1,2} := 0, \) and for \( n \geq 2 \)
\[
q_{n,i} := \sum_{m=0}^{n-i} \left( \frac{2 \mu_0}{m_0 + i + 1} \right)^m \left[ m + 1; 1 - \frac{\mu_0}{m_0}; m - n + i \ ; 1 \right] F_2 \left( m + 1; 1 - \frac{\mu_0}{m_0}; m + i + 1; 1 \right).
\]

**Remark 6.1.** In the case \( \theta = 0 \), Eq. (6.2) implies that \( L^\Lambda_N \) is a Poisson random variable with parameter \( 2\sigma/m_0 \) to be strictly positive (see [22]).

**Remark 6.2.** Note that Proposition 5.2 yields
\[
\frac{m_0}{2}(1 - z) p_\Lambda(z) + \left( \sigma z^2 - (\sigma + \theta)(z + \theta) \right) p_\Lambda(z) = \left( \frac{m_0}{2} + \theta_1 \right) p_\Lambda(z)(1 - z) - \theta_0 z^2, \ z \in D_+.
\]

We can solve this ODE and show Theorem 6.2 following the proof of Theorem 5.2. We provide here an alternative approach based on the results of Section 4 and the following lemma.

**Lemma 6.2.** If \( s = \sigma/N, u_1 = \theta_1/N, u_0 = \theta_0/N \) and \( m_0 = 2 \) then
\[
L^\Lambda_N \xrightarrow{d} L^\Lambda_\infty.
\]

**Proof.** It suffices to show that \( a_n^\Lambda \to a_0^\Lambda \) as \( N \to \infty \) for all \( n \in \mathbb{N}_0 \). We do this by induction on \( n \in \mathbb{N} \). Since \( a_0^\Lambda = 1 = a_0^\Lambda \), the assertion is true for \( n = 0 \). The case \( n = 1 \) follows from [15] Lemma 3. Assume that the assertion holds for all \( k < n \). Then (2.2) implies that the limit of \( a_n^\Lambda \) exists and is related to \( a_{n-1}^\Lambda \) and \( a_{n-2}^\Lambda \) via (2.5). Therefore, \( \lim_{N \to \infty} a_n^\Lambda = a_n^\Lambda \).

**Proof of Theorem 6.1** (i): Identity (6.2) is obtained by iteration of (6.1) and imposing \( \sum_{n \in \mathbb{N}} p_n = 1 \). Formula (6.3) is a direct consequence of Eq. (6.2).

(ii): Since \( L^\Lambda_N(\sigma, \theta_0, \theta_1, m_0) \) is distributed as \( L^\Lambda_\infty(2\sigma/m_0, 2\theta_0/m_0, 2\theta_1/m_0, 2) \), we assume without loss of generality that \( m_0 = 2 \). For the Moran model with parameters \( s = \sigma/N, u_1 = \theta_1/N \) and \( u_0 = \theta_0/N \) Theorem 1.1 yields
\[
p_N(z) = \frac{\theta_0 I_0^{\frac{N}{2}}}{(I_0 - I_1)^{\frac{N}{2}}} \left( 1 + \frac{\sigma^2}{N} \right)^{N + \theta_1} \frac{I_1^{\frac{N}{2}}}{(I_0 - I_1)^{\frac{N}{2}}} \int_0^z \left( \frac{I_1^{\frac{N}{2}} - I_0^{\frac{N}{2}} - \xi}{(1 + \frac{\sigma^2}{N})^{N + \theta_1 + \theta_1 + \theta_1}} \right) d\xi.
\]

Lemma 6.2 implies that \( p_N(z) \to p_\Lambda(z) \) as \( N \to \infty \). In addition, by dominated convergence we get
\[
(N/\sigma)^{N + \theta_1 + \theta_1} \frac{I_0^{\frac{N}{2}}}{(I_0 - I_1)^{\frac{N}{2}}} \xrightarrow{N \to \infty} I_1, \ i \in \{0, 1\}.
\]

Hence, letting \( N \to \infty \) in (6.7) and using dominated convergence yields (6.3). Moreover, a straightforward calculation shows that \( \lim_{N \to \infty} q_{n,i}^N = q_{n,i}, \ i \in \{0, 1\} \). Thus, (6.5) follows by letting \( N \to \infty \) in (4.9).

**Proposition 6.3.** The random variable \( L^\Lambda_\infty \) has mean
\[
E[L^\Lambda_\infty] = \frac{2(\sigma + \theta_0 - \theta_1) + (m_0 + 2\theta_0)p_1}{m_0 + 2\theta_0}.
\]

Moreover, \( L^\Lambda_\infty \) has factorial moments of all orders and they satisfy
\[
((n + 1)m_0 + 2\theta_0) E[L^\Lambda_\infty]^{(n+1)} = 2(n + 1)\sigma E[L^\Lambda_\infty]^{(n)} - 2(n + 1)\theta_1 E[L^\Lambda_\infty - 1]^{(n)}], \ n \in \mathbb{N}.
\]

In addition,
\[
(1) \text{ if } \theta_0 = 0, \text{ then}
\]
\[
E[L^\Lambda_\infty]^{(k)} = k! \left( 1 F_1 \left[ \frac{k + 1 + 2(\sigma + \theta_0 + \theta_1)}{m_0 + 2\theta_0} \right] p_{k+1}^\Lambda + 1 F_1 \left[ \frac{k + 1 + 2(\sigma + \theta_0 + \theta_1)}{m_0 + 2\theta_0} \right] p_1^\Lambda \right), \ k \in \mathbb{N}.
\]
(2) if \( \theta_1 = 0 \), then

\[
E[(L^\Lambda_{\infty})_n^\theta] = \frac{n!}{(2 + \frac{m_0}{m_0})^{n-1}} \left( \frac{2\sigma}{m_0} \right)^{n-1} E[L^\Lambda_{\infty}], \quad n \in \mathbb{N}.
\]

Proof. Without loss of generality we assume that \( m_0 = 2 \). First note that (6.13) implies that \( p^\Lambda_n \leq \sigma^{n-1}/(2 + \theta_1)^n \), \( n \in \mathbb{N} \). Thus, \( L^\Lambda_{\infty} \) admits moments of all orders. Similarly, using (4.11) with \( s = \sigma/N \), \( u_1 = \theta_1/N \), \( u_0 = \theta_0/N \), we get \( p^\Lambda_n \leq \sigma^{n-1}/(2 + \theta_1)^n \). Thus, by dominated convergence and Lemma 6.2 we conclude that

\[
E[(L^\Lambda_{\infty})_n^\theta] = \sum_{n=1}^{\infty} p^\Lambda_n(n)_k \lim_{N \to \infty} \sum_{n=1}^{\infty} p^\Lambda_n(n)_k = E[(L^\Lambda_{\infty})_n^\theta].
\]

The formula for the mean of \( L^\Lambda_{\infty} \) and the recursion (6.8) follow by letting \( N \to \infty \) in Corollary 4.3 and Proposition 4.4 respectively. Now, we prove assertion (1). Note that (6.2) yields

\[
E[(L^\Lambda_{\infty})_n^\theta] = p_n \sum_{n=1}^{\infty} \frac{(2\sigma/m_0)^{n-1}}{2 + \frac{m_0}{m_0}} (n)_k = p_n \left( \frac{2\sigma}{m_0} \right)^{k-1} f^{(k)} \left( \frac{2\sigma}{m_0} \right),
\]

where \( f(x) = \sum_{n=1}^{\infty} x^n / \left( 2 + \frac{m_0}{m_0} \right)_n = \left[ 1 + \frac{1}{x} \right] x \). The result follows from [17] p. 261, Eq. 9.9.5.

Assertion (2) follows iterating (6.8) with \( \theta_1 = 0 \). \( \square \)

7. Solving the Fearnhead-type recursion for the star-shaped model

In this section we assume that the measure \( \Lambda \) is concentrated in 1 with total mass \( m_1 := \Lambda(\{1\}) \), i.e. blocks merge according to the star-shaped coalescent. In particular, \( \sigma_\Lambda = \infty \), and therefore, the block counting process is positive recurrent for any \( \sigma > 0 \). Note that (2.5) reads

\[
\left( \frac{m_1}{n} + \theta + \sigma \right) a_n^\Lambda = \sigma a_{n-1}^\Lambda + \theta_1 a_{n+1}^\Lambda, \quad n \in \mathbb{N}.
\]  

(7.1)

In addition, \( a_0^\Lambda = \sum_{n=1}^{\infty} p^\Lambda_n = 1 \). The following is the main result of this section.

Theorem 7.1. For the star-shaped model with selection parameter \( \sigma > 0 \) and mutation parameters \( \theta_0, \theta_1 \geq 0 \) the following holds

(i) If \( \theta_1 = 0 \), then

\[
p^\Lambda_n = \left( \frac{n\theta_0 + m_1}{n(\sigma + \theta_0) + m_1} \right) \frac{(n-1)!}{(1 + \frac{m_1}{\sigma + \theta_0})^n} \left( \frac{\sigma}{\sigma + \theta_0} \right)^{n-1}, \quad n \in \mathbb{N},
\]  

(7.2)

and

\[
p_\Lambda(z) = 1 - (1 - z) 2F_1 \left[ \begin{array}{c} 1; 1; \frac{\sigma z}{\sigma + \theta_0} \\ 1 + \frac{m_1}{\sigma + \theta_0} \end{array} \right], \quad z \in D.
\]  

(7.3)

(ii) If \( \theta_1 > 0 \), then for all \( z \in D \setminus \{ x_- \} \)

\[
p_\Lambda(z) = z \left( 1 - \frac{\sigma(1-z)}{\sigma z^2 - (\sigma + \theta)z + \theta_1} \right) \frac{m_1}{m_1} \frac{x_-}{x^+} \int_{x_-}^{x^+} \left( \frac{1 - \frac{z}{x}}{1 - \frac{\sigma}{x}} \right)^{m_1} du,
\]  

(7.4)

where \( d := (\sigma + \theta)^2 - 4\sigma \theta_1 \), \( x_- := (\sigma + \theta - d)/(2\sigma) \in (0, 1) \) and \( x_+ := (\sigma + \theta + d)/(2\sigma) > 1 \). In particular,

\[
p_\Lambda^\Lambda = 1 - \frac{\sigma}{\theta_1} \int_0^{x_-} \left( \frac{1 - \frac{z}{x}}{1 - \frac{\sigma}{x}} \right)^{m_1} du.
\]

Proof. (i): In this case, Eq. (7.1) takes the form \((m_1/n + \theta + \sigma) a_n^\Lambda = \sigma a_{n-1}^\Lambda, n \in \mathbb{N}, \) with obvious solution

\[
a_n^\Lambda = \left( \frac{n!}{(1 + \frac{m_1}{\sigma + \theta_0})^n} \right) \left( \frac{\sigma}{\sigma + \theta_0} \right)^n, \quad n \in \mathbb{N}.
\]  

(7.5)
Plugging this expression in $p^n_\lambda = a^n_\lambda - a^{n-1}_\lambda$ yields (7.2). Moreover, (7.3) follows from (7.3) and the identity $\sum_{n=0}^{\infty} a^n_\lambda z^n = (1 - p\lambda(z))/(1 - z)$.

(ii): First note that Proposition (5.2) yields
\[ m_1 z(1 - z) \int_0^z \frac{u - p\lambda(u)}{u(1 - u)} \, du + (\sigma z^2 - (\sigma + \theta)z + \theta_1) p\lambda(z) = \theta_1 p^n_\lambda z(1 - z) - \theta_0 z^2, \quad z \in D \backslash \{0\}. \] (7.6)

Hence, the function $f : D \backslash \{0\} \to \mathbb{C}$ defined via $f(z) := (z - p\lambda(z))/(z(1 - z))$ satisfies
\[ m_1 \int_0^z f(u) \, du - (\sigma z^2 - (\sigma + \theta)z + \theta_1) f(z) = \theta_1 (p^n_\lambda - 1) + \sigma z. \]

Differentiating this equation leads to the following first order differential equation
\[ (m_1 + \sigma + \theta - 2\sigma z) f(z) - (\sigma z^2 - (\sigma + \theta)z + \theta_1) f'(z) = \sigma. \] (7.7)

The solution of the homogeneous differential equation $(m_1 + \sigma + \theta - 2\sigma z) f_0(z) = (\sigma z^2 - (\sigma + \theta)z + \theta_1) f_0'(z)$, is, up to a multiplicative constant, given by
\[ f_0(z) = \frac{1}{\sigma z^2 - (\sigma + \theta)z + \theta_1} \left( \frac{1}{1 - \frac{z}{x_+}} \right)^{m_1} \], \quad z \in D \backslash \{x_-\},

where $d := \sqrt{(\sigma + \theta)^2 - 4\sigma\theta_1}$, $x_- := (\sigma + \theta - d)/(2\sigma)$ and $x_+ := (\sigma + \theta + d)/(2\sigma)$ ($x_-$ and $x_+$ are the roots of the polynomial $z \mapsto \sigma z^2 - (\sigma + \theta)z + \theta_1$). Therefore, the solution of the inhomogeneous differential equation (7.7) is of the form
\[ f(z) = f_0(z) \left( C - \sigma \int_{x_-}^z \left( \frac{1 - \frac{u}{x_+}}{1 - \frac{u}{x_-}} \right)^{m_1} \, du \right), \quad z \in D \backslash \{x_-\}. \]

Since $f_0$ has a singularity at $z = x_-$, but $f$ is analytic in $D \backslash \{0\}$, we get $C = \sigma f_0 \left( \frac{1 - u/x_+}{1 - u/x_-} \right)^{m_1} \, du$. Plugging this value of $C$ into the previous formula for $f$ yields
\[ f(z) = \sigma f_0(z) \int_{x_-}^z \left( \frac{1 - \frac{u}{x_+}}{1 - \frac{u}{x_-}} \right)^{m_1} \, du, \quad z \in D \backslash \{x_-\}. \] (7.8)

Since $p\lambda(z) = z(1 - (1 - z) f(z))$, (7.4) follows. Letting $z \to 0$ in (7.4) yields the expression for $p^n_\lambda$. □

**Remark 7.1.** Making the substitution $y = (x_- - u)/(x_- - z)$ in (7.8) and applying (A.2) we obtain
\[ f(z) = \frac{d}{(m_1 + d)(x_+ - x_-)} \left( \frac{x_+ - z}{x_+ - x_-} \right)^{\gamma - 1} 2F_1 \left[ \begin{array}{c} \frac{m_1 - 1}{m_1 + 2} \quad | \quad x_- - x_+ \end{array} \right] \]
for $z \in B := \{w \in D : |z - x_-| < x_+ - x_- \}$. Moreover, from [17] p.247, Eqs. 9.5.1 and 9.5.2], the previous identity translates into
\[ f(z) = \frac{d}{(m_1 + d)(x_+ - x_-)} \left[ \begin{array}{c} 2 \quad | \quad x_- - x_+ \end{array} \right] \]

From this expression one can easily obtain the coefficients of the series expansion of $f$ around $x_-$. However, a series expansion for $f$ around 0 using this formula is only possible if $2x_- < x_+$ (i.e. $(\sigma + \theta)^2 > 9\theta_1\sigma/2$). In this case, using [17] p. 241, Eq. 9.2.3] we deduce that $f(z) = \sum_{k=0}^{\infty} f_k z^k$, where
\[ f_k = \frac{d}{(m_1 + d)} \left( \frac{2}{2} \right)_k \left( \frac{2}{2} \right)_{k+1} F_1 \left[ \begin{array}{c} 2 + k \quad 1 + k \quad | \quad x_- - x_+ \end{array} \right]. \]

The coefficients $(p_k^n)_k \in \mathbb{N}$ are obtained by setting $p^n_0 = 1 - f_0$ and $p^n_{k+1} = f_{k+1} - f_k$, $k \in \mathbb{N}$.

In the case, where $2x_- \geq x_+$, we can proceed as follows. We set $a^n_1 = 1 - p^n_1$. Then using $a^n_1 = 0$, we obtain the values $a^n_2, a^n_3, \ldots$ by successive substitution in (7.1). Finally we set $p^n_0 = a^n_{n-1} - a^n_n$. 


8. Solving the Fearnhead-type recursion for the Bolthausen-Sznitman model

Let us assume that $\Lambda$ is the uniform measure on $[0,1]$, i.e. blocks merge according to the Bolthausen-Sznitman coalescent. Since in this case $\sigma_\Lambda = \infty$, then $L^\Lambda$ is positive recurrent for any $\sigma > 0$. Moreover, we have shown in Section 2 that $L^\Lambda_n \sim \text{Geom}(1 - \rho)$, where $\rho$ is the unique solution to Eq. (3.5) (see Corollary 3.3). In this section we would like to relate this result with the results obtained in Section 5.

Lemma 8.1 (Carleman integral equation). The function $\rho_\Lambda$ defined via $\rho_\Lambda(x) := p_\Lambda(x)/x$, $x \in (0,1)$, is a solution of the Carleman singular integral equation

$$\alpha(x)\rho_\Lambda(x) - \int_0^1 \frac{\rho_\Lambda(t)}{t-x} dt = f(x), \quad x \in (0,1), \quad (8.1)$$

where $\alpha(x) := \sigma + \log(1-x) - \log(x) - \frac{\theta_1}{x} + \frac{\theta_0}{x}$, $f(x) := \frac{\theta_0 x}{x} - \frac{\theta_1 x}{x}$ and $\int_0^b h(t)dt$ denotes the Cauchy principal value of a function $h$ (provided this value exists).

Proof. Since $c_{n,k} = 1/(k-n)$, we have

$$c_k(x) = \sum_{n=1}^{k-1} \frac{x^n}{k-n} = \int_0^1 u^{k-1} \left( \sum_{n=1}^{k-1} \frac{(x/u)^n}{n} \right) du = x \int_0^1 \frac{u^{k-1} - x^{k-1}}{u-x} du,$$

and hence

$$\sum_{k=2}^{\infty} p_\Lambda^k c_k(x) = x \int_0^1 \frac{\rho_\Lambda(u) - \rho_\Lambda(x)}{u-x} du.$$

Combining this with Proposition 3.1 we obtain

$$\left( \frac{\theta_1}{x} - \frac{\theta_0}{1-x} - \sigma \right) \rho_\Lambda(x) + \int_0^1 \frac{\rho_\Lambda(x) - \rho_\Lambda(t)}{x-t} dt = \frac{\theta_1 p_\Lambda^1}{x} - \frac{\theta_0}{1-x}, \quad x \in (0,1). \quad (8.2)$$

The result follows using that $\int_0^1 \frac{p_\Lambda(x) - p_\Lambda(t)}{x-t} dt = \frac{\theta_1}{x} p_\Lambda^1 - \rho_\Lambda(x) (\log(1-x) - \log(x)).$ \hfill \Box

The solution of (8.1) with boundary conditions $\lim_{x \to 0} \rho_\Lambda(x) = p_\Lambda^1$ and $\lim_{x \to 0} \rho_\Lambda(x) = 1$ can be derived via the method described in [7, Eq. (2.1)] (see also [25, Section 4.4]). This approach involves quite technical calculations and leads to rather complicated formulas for $\rho_\Lambda$ and $p_\Lambda^1$, from which it seems not straightforward to infer that the underlying distribution is geometric. However, knowing that $L^\Lambda_n \sim \text{Geom}(1-\rho)$ for some $\rho \in (0,1)$, we can deduce the value of $\rho$ from Lemma 8.1. Indeed, in this case

$$p_\Lambda^1 = 1-\rho \quad \text{and} \quad \rho_\Lambda(x) = \frac{1-\rho}{1-\rho x}, \quad x \in (0,1).$$

Moreover, one can check that

$$\int_0^1 \frac{\rho_\Lambda(t)}{t-x} dt = \rho_\Lambda(x) (\log(1-x) - \log(x) - \log(1-\rho)),$$

and therefore

$$\alpha(x)\rho_\Lambda(x) - \int_0^1 \frac{\rho_\Lambda(t)}{t-x} dt = \rho_\Lambda(x) \left( \sigma - \frac{\theta_1}{x} + \frac{\theta_0}{1-x} + \log(1-\rho) \right).$$

In addition, we have

$$f(x) = \rho_\Lambda(x) \left( \frac{\theta_0}{1-x} + \frac{\theta_0 \rho}{1-\rho} - \frac{\theta_1}{x} + \theta_1 \rho \right).$$

Since $\rho_\Lambda$ satisfies (8.1), we infer that $(\sigma + \log(1-\rho) - \theta_1 \rho)(1-\rho)$ is not zero, i.e. $\rho = \varrho$ (see Lemma 3.2).

9. A remark on the $\beta(3,1)$-model

There is another instance where the general method in Section 5 leads to a simple ordinary differential equation. This is given by the $\beta(3,1)$-model, i.e. the $\Lambda$-Wright-Fisher model with $\Lambda(dx) = 3x^2 dx$. Indeed, in this case $c_{n,k} = 3/(k+1)$, and hence $c_k(z) = \frac{3}{k+1} \frac{z^{k+1}}{1-z}$. Therefore,

$$\sum_{k=2}^{\infty} p_\Lambda^k c_k(z) = \frac{3}{1-z} \left[ \sum_{k=2}^{\infty} \frac{p_\Lambda^k}{k+1} - \sum_{k=2}^{\infty} \frac{p_\Lambda^k}{k+1} \right] = \frac{3}{1-z} \left[ \sum_{k=1}^{\infty} \frac{p_\Lambda^k}{k+1} - \frac{1}{z} \int_0^z p_\Lambda(u) du \right].$$
In addition, using Eq. (5.11) for \( n = 1 \) we get
\[
3 \sum_{k=1}^{\infty} \frac{p_k^\Lambda}{k+1} = \left( \frac{3}{2} + \sigma + \theta_0 \right) p_1^\Lambda - \theta_1 p_2^\Lambda - \theta_0.
\]
Thus, Proposition 5.1 yields
\[
3 \int_0^z p_\Lambda(u)du - (\sigma z^2 - (\sigma + \theta)z + \theta_1)p_\Lambda(z) = \left[ \left( \frac{3}{2} + \sigma + \theta \right) p_1^\Lambda - \theta_1 p_2^\Lambda \right] z - \theta_1 p_1^\Lambda z.
\]
Differentiating this equation, we deduce that \( p_\Lambda \) solves the ordinary differential equation
\[
(\sigma z^2 - (\sigma + \theta)z + \theta_1)p_\Lambda'(z) + (2\sigma z - \sigma - \theta - 3)p_\Lambda(z) = \theta_1 p_1^\Lambda - ((3 + 2(\sigma + \theta))p_1^\Lambda - 2\theta_1 p_2^\Lambda)z, \quad z \in D_*.
\]
Explicit formulas for \( p_\Lambda \) and \( p_1^\Lambda \) can be obtained solving this equation with the boundary conditions \( p_\Lambda(0) = 0 \) and \( p_\Lambda(1) = 1 \). We leave the details to the reader.

**Appendix A. Some special functions**

The rising and falling factorials \((\cdot)^\uparrow\) and \((\cdot)^\downarrow\) are defined as
\[
(a)_n^\uparrow := \alpha(a+1) \cdots (a+n-1) \quad \text{and} \quad (a)_n^\downarrow := \alpha(a-1) \cdots (a-n+1), \quad n \in \mathbb{N},
\]
and \((a)_0^\downarrow := 1 =: (a)_0^\uparrow\). The Gauss hypergeometric function \( 2F_1 \) is the absolutely convergent power series
\[
\begin{aligned}
2F_1 \left[ \begin{array}{c} \alpha; \beta \\ \gamma \end{array} \right] | z & := \sum_{k=0}^{\infty} \frac{(\alpha)_k^\downarrow (\beta)_k^\uparrow z^k}{(\gamma)_k^\downarrow k!}, \quad z \in D,
\end{aligned}
\]
where \( \alpha, \beta, \gamma \) are parameters which can take real or complex values (provided that \( \gamma \notin -\mathbb{N}_0 \)). The function \( 2F_1 \) admits the integral representation (see [17] p. 239, Eq. 9.1.4)
\[
2F_1 \left[ \begin{array}{c} \alpha; \beta \\ \gamma \end{array} \right] | z = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha} dt, \quad \Re(\gamma) > \Re(\beta) > 0.
\]

The confluent hypergeometric function \( 1F_1 \) is the absolutely convergent power series
\[
\begin{aligned}
1F_1 \left[ \begin{array}{c} \alpha \\ \gamma \end{array} \right] | z & := \frac{(\alpha)_k^\downarrow}{(\gamma)_k^\downarrow k!} z^k, \quad z \in \mathbb{C},
\end{aligned}
\]
where \( \alpha, \gamma \) are parameters which can take real or complex values (provided that \( \gamma \notin -\mathbb{N}_0 \)). The function \( 1F_1 \) admits the integral representation (see [17] p. 266, Eq. 9.11.1))
\[
1F_1 \left[ \begin{array}{c} \alpha \\ \gamma \end{array} \right] | z = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 e^{zt}t^{\alpha-1}(1-t)^{\gamma-\alpha-1} dt, \quad \Re(\gamma) > \Re(\alpha) > 0.
\]

Similarly, the generalised hypergeometric function \( 3F_2 \) is the power series
\[
\begin{aligned}
3F_2 \left[ \begin{array}{c} \alpha; \beta; \gamma \\ \delta; \rho \end{array} \right] | z & := \frac{(\alpha)_k^\downarrow (\beta)_k^\uparrow (\gamma)_k^\downarrow}{(\delta)_k^\downarrow (\rho)_k^\uparrow k!} z^k, \quad z \in D,
\end{aligned}
\]
where \( \delta, \rho \notin -\mathbb{N}_0 \). The functions \( 2F_1 \) and \( 3F_2 \) can be defined outside the disk \( D \) by using analytic continuation. Moreover, when \( \alpha \) or \( \beta \) are nonpositive integers, \( 2F_1 \) reduces to a polynomial, and therefore, is well defined in the whole complex plane. The same holds for \( 3F_2 \) when \( \alpha, \beta \) or \( \gamma \) are nonpositive integers. A natural two variables generalisation of the Gauss hypergeometric function is given by the Appell function \( F_1 \) (see [1]), which is given by
\[
F_1 \left[ \begin{array}{c} a; \ b; \ c \\ d \end{array} \right] | w; z = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_m^\downarrow (b)_n^\uparrow (c)_n^\downarrow z^m w^n}{(d)_{m+n}^\uparrow m! m!}, \quad z, w \in D,
\]
where \( d \) is a non-negative integer. There are four types of Appell functions, but we focus here only on \( F_1 \). The function \( F_1 \) can be expressed in terms of \( 2F_1 \) functions as follows
\[
F_1 \left[ \begin{array}{c} a; \ b; \ c \\ d \end{array} \right] | w; z = \sum_{m=0}^{\infty} \frac{(a)_m^\downarrow (b)_n^\uparrow z^m}{(d)_{m+n}^\uparrow m! m!} F_1 \left[ \begin{array}{c} a+m; \ b; \ c \\ d+m \end{array} \right] | w.
\]

The function \( F_1 \) admits the integral representation (see [1] Eq. 24)
\[
F_1 \left[ \begin{array}{c} a; \ b; \ c \\ d \end{array} \right] | w = \frac{\Gamma(d)}{\Gamma(a)\Gamma(d-a)} \int_0^1 t^{a-1}(1-t)^{d-a-1}(1-zt)^{-b}(1-wt)^{-c} dt, \quad \Re(d) > \Re(a) > 0.
\]
Appendix B. Some integral identities

For $\alpha, \beta, \gamma, \nu > 0$, we define $I(\alpha, \beta, \gamma, \nu; z) := \int_0^z y^\alpha (1 - y)^\beta (y + \nu)^{-\gamma} dy$, $z \in \mathbb{C} \setminus \mathbb{R}_-$.

Lemma B.1. For every $z \in \mathbb{C} \setminus \mathbb{R}_-$, we have

$$I(\alpha, \beta, \gamma, \nu; z) = \nu^{\alpha-\gamma+1} \left( \frac{z}{z + \nu} \right)^{1 + \alpha} \int_0^1 t^{\alpha} \left( 1 - \frac{z}{z + \nu} t \right)^{\gamma - \alpha - \beta - 2} \left( 1 - \frac{(1 + \nu)z}{z + \nu} t \right)^{\beta} dt.$$  

Proof. This follows directly by making the change of variable $t = (z + \nu)y/(z(y + \nu))$. \hfill $\Box$

Corollary B.2. We have

$$I(\alpha, \beta, \gamma, \nu; 1) = \frac{\nu^{1+\alpha-\gamma} \Gamma(1 + \alpha) \Gamma(1 + \beta)}{(1 + \nu)^{1+\alpha} \Gamma(2 + \alpha + \beta)} 2F_1 \left[ \begin{array}{c} 2 + \alpha + \beta - \gamma; 1 + \alpha \end{array} \middle| 1 + \nu \right].$$

Proof. This follows directly from Lemma B.1 and Eq. \ref{eq:lemma_B.1}.

Let $D_\nu := \{z \in D_\nu : |z| < \nu/\sqrt{\nu^2 + 2\nu}\}$. One can easily check that for all $z \in D_\nu$, $(1 + \nu)z/(z + \nu) \in D$.

Corollary B.3. For all $z \in D_\nu$, we have

$$I(\alpha, \beta, \gamma, \nu; z) = \nu^{\alpha-\gamma+1} \left( \frac{z}{z + \nu} \right)^{1 + \alpha} \frac{1}{1 + \alpha} \left( \frac{\nu}{z + \nu} \right)^{2 + \alpha} F_1 \left[ \begin{array}{c} 1 + \alpha; z/(z + \nu) \end{array} \middle| \frac{(1 + \nu)z}{z + \nu} \right].$$

Proof. This follows directly from Lemma B.1 and Eq. \ref{eq:lemma_B.1}.

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