Preemptive Investment under Uncertainty☆

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April 8, 2018

Abstract

This paper provides a general characterization of subgame-perfect equilibria for strategic timing problems, where two firms have the (real) option to make an irreversible investment. Profit streams are uncertain and depend on the market structure. The analysis is based directly on the inherent economic structure of the model. In particular, determining equilibria with preemptive investment is reduced to solving a single class of constrained optimal stopping problems. Further tools are derived for analyzing Markovian state-space models. Applications to typical models from the literature complete commonly insufficient equilibrium arguments, show when uncertainty leads to qualitatively different behavior, and establish additional equilibria that are Pareto improvements.

Keywords: preemption, real options, irreversible investment, subgame perfect equilibrium, optimal stopping

JEL subject classification: C61, C73, D21, D43, L12, L13

MSC2010 subject classification: 60G40, 91A25, 91A55, 91A60

1 Introduction

Preemption is a well-known phenomenon in the context of irreversible investment. In their seminal paper, Fudenberg and Tirole (1985) argue that the commitment power of irreversibility and subgame-perfectness together imply that the first firm that adopts a new technology in some industry can deter other firms from adopting soon; the second adopter's benefits will be reduced by competition and thus not worth the immediate adoption cost. In consequence, the firms try to preempt each other in order to win the (temporary) monopoly profit.[1]

Such preemption is of particular interest when it counteracts an incentive to wait. In the deterministic model of Fudenberg and Tirole (1985), the cost of adoption is decreasing in time. Alternatively, if uncertainty is considered, then preemption eliminates the (real) option value of waiting for sufficiently good states. A sizable literature thus argues for the drastic impact of competition on the valuation of real options, typically using ideas from Fudenberg and Tirole (1985) and applying them to certain (value) functions of a stochastic state instead of time.[2]

Several issues result from exploiting such analogies. First, additional arguments are needed for, e.g., the optimality of waiting when obvious monotonicities hold in the deterministic case. Second, observing similar geometries of value functions for different familiar classes of dynamics brings up the question of common,

☆I would like to thank three anonymous referees and the advisory editor at Games and Economic Behavior for their valuable comments, and also participants of GAMES 2016. Financial support by the German Research Foundation [CRC 1283] is gratefully acknowledged.

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[1] This effect does not appear in simple Nash equilibria as studied by Reinganum (1981), where firms precommit to adoption times.

[2] See Azevedo and Paxson (2014) for an extensive survey or specifically the papers mentioned in the following.

deeper economic principles. Third, because equilibrium behavior in many stochastic models as well as the deterministic one can be described by thresholds for a state that drives profitability, it is also important to elaborate on *qualitative* differences.

Figures 1 and 2 illustrate some important principles and limitations of analogies. Figure 1 shows the values of the firms in Fudenberg and Tirole (1985), discounted to time $t_0 = 0$, if the first adoption happens at $t \geq 0$. If a single firm is the first to adopt, its value is $L(t)$ and that of the other firm $F(t)$; the value obtained from simultaneous adoption is $M(t)$. The strategic structure is quite clear: Initially, it is optimal to wait, to benefit from the increase in L if the opponent does not adopt and from $F > L > M$ else; then, there is a phase with first-mover advantage $L > F$ that may induce preemption; eventually, all payoffs are identical and decreasing, and adoption becomes dominant if is has not happened before.

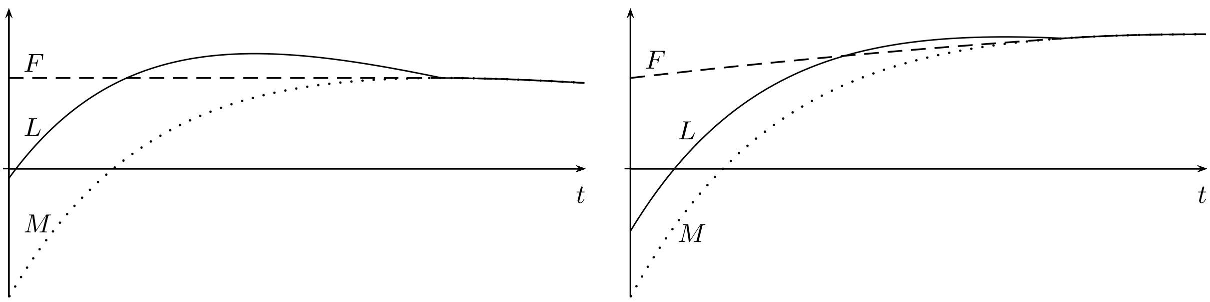

Figure 1: Value functions from the model of Fudenberg and Tirole (1985).[3]

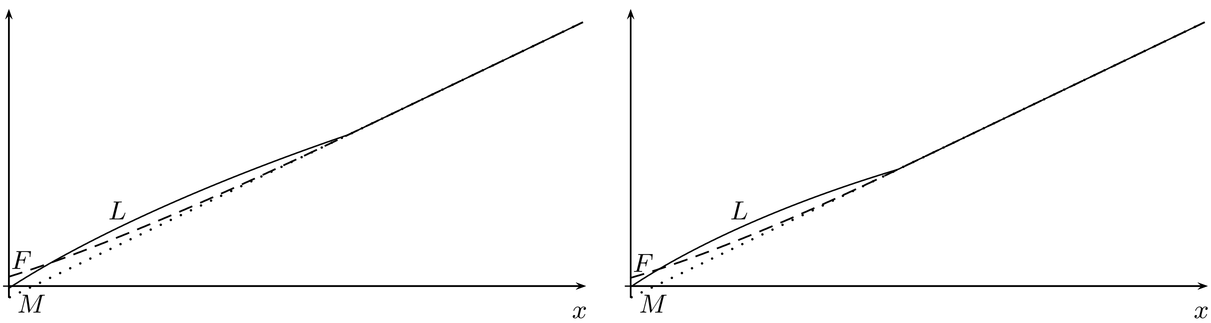

Figure 2: Value functions from a typical stochastic model and its deterministic limit.[4]

Similar local orders can be seen in Figure 2, showing *current* values as functions of a stochastic exogenous state x for a typical Markovian model. With a stochastically evolving state, however, the dynamics of expected discounted values cannot be read from the figure, and intertemporal comparisons require different arguments. To hint at the role of discounting, the volatility is reduced to zero in the right panel, where x grows deterministically over the shown range. Then the values implicitly become functions of time, and discounting leads back to the left panel of Figure 1. F, e.g., which is convex in x, becomes concave in t and eventually decreases.

In this paper, we formulate a strategic investment model based on revenue streams that keeps the stochastic structure completely general and analyze it by intertemporal tradeoffs with immediate economic

[3] The model of Fudenberg and Tirole (1985) that underlies these value functions is presented in Section 4.1, fn. 19. For completeness, the specification on the left is $\pi_0(0) = \pi_0(1) = 0$, $\pi_1(1) = 0.03$, $\pi_1(2) = 0.012$, $r = 0.02$, $c(t) = e^{-(r+a)t}$, $a = 0.08$ and on the right the same, except $\pi_0(0) = 0.006$, $\pi_1(1) = 0.022$.

[4] The model underlying these value functions is the main model in Section 4.1. For completeness, the specification on the left is $D_{00} = D_{01} = 0$, $D_{10} = 2.5$, $D_{11} = I^1 = I^2 = 1$, $r = 0.1$, $\mu = 0.08$, $\sigma = 0.2$ and on the right the same, except $\sigma = 0$.

meaning. Mainly comparing revenue streams and implied opportunity costs, the verification of subgame-perfect equilibria with preemption is reduced to solving a single class of non-strategic optimal stopping problems for one firm. Thereby, on the one hand, we provide a method to generalize findings from specific models in the literature to large classes of underlying stochastic shocks, and, on the other hand, a unified view that yields more detailed economic insights into equilibrium behavior; many economically quite diverse models can be nested. Because mutual preemption destroys value, we furthermore establish some principles for when it can be avoided, and we identify times when it is impossible to delay investment in equilibrium.

Alongside, important general questions for equilibria of real-option games are addressed, such as:

- At what times is there a first-mover advantage for both firms that they may fight for by trying to preempt each other?
- When and how is the first investment affected by a threat of preemption?
- Will a firm ever want to invest when it has a second-mover advantage?

Answers to these questions will be found by studying suitable optimal stopping problems.

More specific characterizations can be obtained for Markovian state-space models. We develop tools that would apply to many other models than those from the literature, but we use them for two typical ones, those of Grenadier (1996) and Pawlina and Kort (2006). In fact, these and other papers neglect to verify equilibria in parts of the state space that are relevant for their results, in particular the optimality of waiting; see Section 4 for details. Our results ensure a complete coverage. We also address neglected equilibrium behavior that actually distinguishes stochastic models qualitatively from deterministic ones by a particular risk that uncertainty can induce. We finally identify further equilibria for each model that may be Pareto improvements and thus more plausible.

More generally, some models that can be nested here are those of Reinganum (1981) and Fudenberg and Tirole (1985), which are deterministic; Mason and Weeds (2010), where revenue is linear in a geometric Brownian motion; Pawlina and Kort (2006), adding asymmetry in investment costs; Boyarchenko and Levendorskiĭ (2014), a further extension to exponential Lévy processes; Weeds (2002), which includes Poisson arrivals of R&D success but is formally equivalent to a symmetric setting with geometric Brownian motion again, and similarly Grenadier (1996), including a construction delay.[5]

The paper is organized as follows. The general model is presented in Section 2. Section 3 characterizes equilibria with and without preemption. The implications for typical state-space models are illustrated in Section 4. Section 5 concludes. Some technical results are collected in Appendix A and proofs in Appendix B. Appendix C elaborates on *necessary* conditions for equilibria, in particular on when investment cannot be delayed any further.

2 The model

Consider two firms $i \in \{1, 2\}$ that each can choose when to make one irreversible investment. For instance, firm i may wish to enter some new market or to improve present operations by updating technology or expanding production capacity. Each firm's investment has a potential effect on both firms' revenues. Assume therefore that as long as no firm has invested, firm i's per-period revenues are given by a process (π_t^{0i}) that may depend on an exogenous state of the world. When firm i invests before its opponent, its revenues switch to the process (π_t^{Li}), whereas when the opponent invests first, firm i's revenues switch to the process (π_t^{Fi}). Once both firms have invested (possibly simultaneously), firm i's revenues follow the process (π_t^{Bi}). In order to analyze opportunity costs of waiting by comparing revenue streams, the revenues $\pi_\cdot^{Li}$ and $\pi_\cdot^{Bi}$ applying after firm i's investment are understood net of any capitalized investment cost.[6] All revenues are given in time $t = 0$ units.

[5] Thijssen et al. (2012) have a similar structure of the state space inspired by Fudenberg and Tirole (1985), but by direct assumptions on value functions and not from modeling revenue streams.

[6] Any discounted investment cost that is strictly decreasing in time, like $c(t)$ in Fudenberg and Tirole (1985), can be capitalized by a change of variable, $c(t) = e^{-ry} = \int_y^\infty e^{-rz} r\, dz$. If the discounted investment cost is stochastic and strictly

Time is continuous, $t \in \mathbb{R}_+$, so only accrued revenues in intervals of time matter. Allowing for an exogenous state of the world, assume the revenues thus to be product-measurable w.r.t. a given probability space $(\Omega, \mathscr{F}, P)$ and time. Assume them in fact to be $P \otimes dt$-integrable, i.e., $E[\int_0^\infty |\pi_t^{0i}|\, dt] < \infty$ etc., to ensure finite expectations throughout. Correspondingly, any (in-)equalities between revenue processes are understood to hold $P \otimes dt$-a.e. and those between random variables P-a.s.

There is dynamic information about the state of the world, modeled by a filtration $(\mathscr{F}_t)$ satisfying the *usual conditions* of right-continuity and completeness. Assume that the revenues (potentially) accrued up to any time $t \in \mathbb{R}_+$ are $\mathscr{F}_t$-measurable, i.e., the processes $(\int_0^t \pi_s^{0i}\, ds)$ etc. are *adapted* to $(\mathscr{F}_t)$.[7]

As a final economic assumption, the following orders are imposed. First, a single firm's investment cannot enhance the revenue of the opponent, i.e., for both $i = 1, 2$, $\pi_\cdot^{Li} \geq \pi_\cdot^{Bi}$ (e.g., as the first investor loses a monopoly premium when the laggard invests) and $\pi_\cdot^{0i} \geq \pi_\cdot^{Fi}$ (e.g., as the first investor steals some business from the laggard). The special case $\pi_\cdot^{0i} = \pi_\cdot^{Fi}$ is typical for market entry models and has additional implications that will be pointed out. Second, firm 2 has a disadvantage in the sense of smaller investment gains relative to being laggard, formally $\pi_\cdot^{B2} - \pi_\cdot^{F2} \leq \pi_\cdot^{B1} - \pi_\cdot^{F1}$ and $\pi_\cdot^{L2} - \pi_\cdot^{F2} \leq \pi_\cdot^{L1} - \pi_\cdot^{F1}$. This disadvantage arises, e.g., from a higher capitalized investment cost. Given the first part of the disadvantage, that firm 2's investment gain as laggard is at most that of firm 1, the second part also obtains if $\pi_\cdot^{L2} - \pi_\cdot^{B2}$, firm 2's potential revenue loss as first investor due to the laggard's investment, is not greater than firm 1's, $\pi_\cdot^{L1} - \pi_\cdot^{B1}$.

The firms' payoffs are expected revenues. Therefore, the investment timing decisions are strategic if some firm's investment indeed affects the other's revenue, i.e., if $\{\pi_\cdot^{Li} > \pi_\cdot^{Bi}\}$ or $\{\pi_\cdot^{0i} > \pi_\cdot^{Fi}\}$ have positive measure for some $i \in \{1, 2\}$. We will formulate the problem as a dynamic game in continuous time.

2.1 The investment timing game

It is well known that continuous time games do not admit extensive forms based on histories of actions – "invest" and "wait" in our case – unless reactions are restricted (see Simon and Stinchcombe, 1989, or Alós-Ferrer and Ritzberger, 2008). A typical approach for analyzing timing games dynamically is thus that players first make plans when to perform their single move that are *conditional on no-one moving before* (cf. Fudenberg and Tirole, 1985, who call them "simple strategies", or Laraki et al., 2005). From these plans, only the time of the first move and the identity of the first movers are determined – by examining whose planned time is minimal. The actual move times of any remaining players are then determined via conditional continuation problems. To rule out non-credible threats, the game is reconsidered whenever a move could potentially occur, under the hypothesis that no move has happened, yet, and plans for different starting times are required to be consistent.

In this spirit, we use the following framework for stochastic models developed in Riedel and Steg (2017), based on Fudenberg and Tirole's (1985) approach.[8] The central concept for dealing with uncertainty is a *stopping time* with respect to the filtration $(\mathscr{F}_t)$, i.e., a random variable $\tau \colon \Omega \to [0, \infty] := \mathbb{R}_+ \cup \{\infty\}$ satisfying $\{\tau \leq t\} \in \mathscr{F}_t$ for all $t \in \mathbb{R}_+$. Let $\mathscr{T}$ denote the set of all stopping times. They are plans that depend on the dynamic exogenous information (e.g., on perceived demand) by requiring the information at any time t to reveal if the planned time τ has been reached or not. To determine behavior also off path, each stopping time is furthermore nominated as the start of a *subgame* with the connotation that no-one has moved, yet;[9] in this second role, they are commonly denoted by ϑ. A feasible plan for the subgame starting at some $\vartheta \in \mathscr{T}$ is then another $\tau \in \mathscr{T}$ satisfying $\tau \geq \vartheta$. A complete *strategy* (for determining the first move) is then a family of plans $(\tau^\vartheta; \vartheta \in \mathscr{T})$ satisfying $\tau^\vartheta \geq \vartheta$ and *time-consistency*, meaning that planned move

decreasing in expectation (a strict supermartingale), then one can use the monotone part of its Doob-Meyer decomposition in place of $c(t)$.

[7] This property holds, e.g., if the processes (π_t^{0i}) etc. are *progressively measurable*, i.e., if the restricted mappings $\pi_\cdot^{0i} \colon \Omega \times [0, T] \to \mathbb{R}$ etc. are $\mathscr{F}_T \otimes \mathcal{B}([0, T])$-measurable for all $T \in \mathbb{R}_+$.

[8] A different approach inspired by Fudenberg and Tirole's is formulated in Thijssen et al. (2012) and adopted by Boyarchenko and Levendorskiĭ (2014). Their strategies are unconditional on other players' moves, but profiles must be jointly consistent.

[9] In contrast to discrete time, it is not enough to consider deterministic times $t \in \mathbb{R}_+$ and their information sets $\mathscr{F}_t$ (like in a tree); see Riedel and Steg (2017).

times are not changed before they are reached, i.e., for all $\vartheta, \vartheta' \in \mathscr{T}$ one has

$$\vartheta \le \vartheta' \text{ (a.s.)} \quad \Rightarrow \quad \tau^{\vartheta} \le \tau^{\vartheta'} \text{ (a.s.), with equality on } \{\tau^{\vartheta} \ge \vartheta'\}. \tag{2.1}$$

2.2 Continuation problems at first investment

Having defined plans that will determine the first move (resp. investment), we next stipulate continuation payoffs for every possible *outcome*, i.e., state-dependent time and identity of the chosen firm(s). To specify optimal conditional reactions, suppose that the opponent of firm i is the first to invest at arbitrary $\tau \in \mathscr{T}$ (e.g., due to a pair of initial plans where firm i's is not minimal). Given that the opponent has invested, firm i is now free to invest at any stopping time $\tau' \ge \tau$, aiming to attain the conditional *follower payoff*

$$F^i(\tau) := \int_0^{\tau} \pi_s^{0i}\, ds + \operatorname*{ess\,sup}_{\tau' \ge \tau} E\biggl[\int_{\tau}^{\tau'} \pi_s^{Fi}\, ds + \int_{\tau'}^{\infty} \pi_s^{Bi}\, ds \biggm| \mathscr{F}_{\tau}\biggr].[10] \tag{2.2}$$

The problem in (2.2) is equivalent to that for the reward process $(\int_{\tau}^{t}(\pi_s^{Fi} - \pi_s^{Bi})\, ds)$, which has an optimal stopping time due to continuity and integrability. We can even devise optimal $\tau_*, \tau^* \in \mathscr{T}$ such that any optimal τ' satisfies $\tau \le \tau_* \le \tau' \le \tau^*$.[11] Therefore, requiring optimal stopping times to be minimal (maximal), resp. earliest (latest), uniquely defines a follower reaction attaining $F^i(\tau)$. We fix the *latest* for technical convenience (see Remark 2.1) and denote it by $\tau_F^i(\tau) \in \mathscr{T}$. This choice is of course innocuous if $\tau_* = \tau^*$, like for all applications mentioned in the Introduction and similar models based on diffusions.

Now suppose on the contrary that firm i is the first to invest at $\tau \in \mathscr{T}$. Then the other firm $j \in \{1, 2\}\backslash\{i\}$ is assumed to follow suit at $\tau_F^j(\tau)$, in order to attain $F^j(\tau)$, which yields firm i the conditional *leader payoff*

$$L^i(\tau) := \int_0^{\tau} \pi_s^{0i}\, ds + E\biggl[\int_{\tau}^{\tau_F^j(\tau)} \pi_s^{Li}\, ds + \int_{\tau_F^j(\tau)}^{\infty} \pi_s^{Bi}\, ds \biggm| \mathscr{F}_{\tau}\biggr]. \tag{2.3}$$

Finally, if the firms invest simultaneously at $\tau \in \mathscr{T}$, as their plans coincide, then firm i's conditional payoff is

$$M^i(\tau) := \int_0^{\tau} \pi_s^{0i}\, ds + E\biggl[\int_{\tau}^{\infty} \pi_s^{Bi}\, ds \biggm| \mathscr{F}_{\tau}\biggr] \le \min\bigl\{F^i(\tau), L^i(\tau)\bigr\}. \tag{2.4}$$

The inequality holds because $\tau' = \tau$ is feasible for any follower and $\pi_{\cdot}^{Li} \ge \pi_{\cdot}^{Bi}$. In particular, if no firm invests in finite time, then firm i obtains

$$M^i(\infty) = \int_0^{\infty} \pi_s^{0i}\, ds = F^i(\infty) = L^i(\infty).$$

Remark 2.1. Before using the continuation payoffs at arbitrary times for mapping initial plans to expected payoffs, note the following regularity properties. Instead of families like $(F^i(\tau); \tau \in \mathscr{T})$, it is much more convenient to work with well-behaved, adapted *processes* (L_t^i), (F_t^i) and (M_t^i) for $t \in [0, \infty]$ that, if evaluated at any stopping time $\tau \in \mathscr{T}$, yield the right-hand sides of (2.2), (2.3) and (2.4) and correspond to the hypothesized follower behavior. In Lemma A.5 in Appendix A we establish such processes with right-continuous paths (employing for (L_t^i) that every $\tau_F^i(\tau)$ is the latest time attaining $F^i(\tau)$). The payoffs are moreover sufficiently integrable to be bounded in expectation and such that pathwise limits at any stopping time induce the corresponding limit in expectation.

[10] A random variable is measurable w.r.t. $\mathscr{F}_{\tau} := \{A \in \mathscr{F} \,|\, \forall t \in \mathbb{R}_+ \colon A \cap \{\tau \le t\} \in \mathscr{F}_t\}$ if its value is known whenever τ has occurred. The value of a stochastic process at $\tau \in \mathscr{T}$ is an $\mathscr{F}_{\tau}$-measurable random variable if the process is progressively measurable (cf. fn. 7), which holds for $(\int_0^t \pi_s^{0i}\, ds)$ by adaptedness and path continuity.

[11] See, e.g., El Karoui (1981): If $U = (U_t)$ denotes the Snell envelope of the reward process, i.e., the supermartingale satisfying $U_{\tau} = \operatorname{ess\,sup}_{\tau' \ge \tau} E[\int_0^{\tau'} (\pi_s^{Fi} - \pi_s^{Bi})\, ds \,|\, \mathscr{F}_{\tau}]$ for every $\tau \in \mathscr{T}$, and $U = M - A$ its Doob-Meyer decomposition into martingale $M = (M_t)$ and nondecreasing compensator $A = (A_t)$, then $\tau_* = \inf\{t \ge \tau \,|\, U_t = \int_0^t (\pi_s^{Fi} - \pi_s^{Bi})\, ds\}$ and $\tau^* = \inf\{t \ge \tau \,|\, A_t > A_{\tau}\}$.

2.3 Payoffs, randomization and equilibrium

In the subgame starting at $\vartheta \in \mathscr{T}$, if firms $i, j \in \{1,2\}$, $i \neq j$, pick feasible plans $\tau_i, \tau_j \geq \vartheta$, then firm i's expected payoff will be

$$E\Big[L^i_{\tau_i}\mathbf{1}_{\{\tau_i<\tau_j\}} + F^i_{\tau_j}\mathbf{1}_{\{\tau_i>\tau_j\}} + M^i_{\tau_j}\mathbf{1}_{\{\tau_i=\tau_j\}}\,\Big|\,\mathscr{F}_\vartheta\Big]. \tag{2.5}$$

This space of payoffs is, however, too small to obtain equilibria when there are mutual preemption incentives due to first-mover advantages $L^i_\cdot > F^i_\cdot$ for both firms (like in Figures 1 and 2). We need to allow for randomization and even some degree of coordination. We thus apply the *extended mixed strategies* defined in Riedel and Steg (2017). However, the details are only provided in Appendix A for completeness, in order to focus on standard optimal stopping problems here. Indeed, preemption continuation payoffs are available from a general result in Riedel and Steg (2017), and in Proposition 2.3 we will show how to assemble the corresponding continuation strategies with the solutions of our stopping problems, such that the compound strategies remain equilibria when randomization is allowed throughout.

To wit, there are in principle two – linked – randomization tools in every subgame. First, for every $\vartheta \in \mathscr{T}$, any firm i can specify a cumulative distribution function $G^\vartheta_i(t)$ over time $t \in [\vartheta, \infty]$ that may react to the dynamic information $(\mathscr{F}_t)$.[12] These will here only take the degenerate form $G^\vartheta_i(t) = \mathbf{1}_{\{t\geq\tau\}}$ for *pure* plans $\tau \in \mathscr{T}$.[13] Second, partial coordination is facilitated by "atoms" $\alpha^\vartheta_i(t) \in [0,1]$ on *every* point in continuous time $t \in [\vartheta, \infty]$. These are interpreted as probabilities of moving *at* each t if no-one has moved before and, given enough regularity, allow the assignment of crucial limit outcomes from discrete time that lack with standard distributions over continuous time, cf. Subsection 3.1.2.

Formally, an extended mixed strategy for a subgame, denoted by σ^ϑ_i, thus consists of a pair of adapted processes $(G^\vartheta_i(t), \alpha^\vartheta_i(t))$ satisfying regularity conditions given in Definition A.1. The set of all such strategies is denoted by $\mathscr{S}^\vartheta$. A strategy for the full game is then again a family $\sigma_i = (\sigma^\vartheta_i; \vartheta \in \mathscr{T})$ satisfying $\sigma^\vartheta_i \in \mathscr{S}^\vartheta$ and time-consistency, meaning that the conditional probability of moving at any fixed time is not changed before the latter is reached (Definition A.2). With only degenerate G^ϑ_i, their time-consistency is (2.1) for the corresponding pure plans; all processes α^ϑ_i must essentially be identical except for dropping the respective past. The *payoff* of firm i from a pair of extended mixed strategies $(\sigma^\vartheta_i, \sigma^\vartheta_j)$ in the subgame starting at $\vartheta \in \mathscr{T}$ is a linear extension of (2.5) (Definition A.3) and denoted by $V^\vartheta_i(\sigma^\vartheta_i, \sigma^\vartheta_j)$; it equals (2.5) if $\sigma^\vartheta_i, \sigma^\vartheta_j$ correspond to pure plans $\tau_i, \tau_j \in \mathscr{T}$ (with $\alpha^\vartheta_i(t) = \alpha^\vartheta_j(t) = 0$ for all $t \in \mathbb{R}_+$).

Definition 2.2 (Riedel and Steg, 2017, Definition 2.14)**.** A *subgame-perfect equilibrium* is a profile $(\sigma_1, \sigma_2) = ((\sigma^\vartheta_1, \sigma^\vartheta_2); \vartheta \in \mathscr{T})$ of time-consistent extended mixed strategies such that for all $\vartheta \in \mathscr{T}$, $i, j \in \{1,2\}$, $i \neq j$, and extended mixed strategies $\sigma^\vartheta_a \in \mathscr{S}^\vartheta$ it holds that

$$V^\vartheta_i(\sigma^\vartheta_i, \sigma^\vartheta_j) \geq V^\vartheta_i(\sigma^\vartheta_a, \sigma^\vartheta_j),$$

i.e., such that each pair $(\sigma^\vartheta_1, \sigma^\vartheta_2)$ is an *equilibrium* in the subgame starting at $\vartheta \in \mathscr{T}$.

The following result allows us to employ continuation equilibrium payoffs at some $\vartheta' \geq \vartheta$ that involve extended mixed strategies, but to ignore randomization otherwise. In that context, it is for any $\tau_i, \tau_j \in \mathscr{T}$ with values in $[\vartheta, \vartheta']$ understood that the pure plans τ_i, τ_j are carried out on $\{\tau_i \wedge \tau_j < \vartheta'\}$ and the continuation strategies for ϑ' on $\{\tau_i = \tau_j = \vartheta'\}$. When treating all subgames this way, then time-consistency must hold, but we will be given a single, aggregating process that specifies randomization for preemption.

Proposition 2.3.

[12] Touzi and Vieille (2002) show that such distribution functions are payoff-equivalent to randomizing over stopping times before the start of the game.

[13] See Steg and Thijssen (2015) for an application where the players additionally randomize with a hazard rate when they have second-mover advantages.

(i) *Suppose that* $\vartheta, \vartheta' \in \mathscr{T}$, $\vartheta \le \vartheta'$, $\sigma^{\vartheta'} = (\sigma_1^{\vartheta'}, \sigma_2^{\vartheta'}) \in \mathscr{S}^{\vartheta'} \times \mathscr{S}^{\vartheta'}$ *and* $\tau_1, \tau_2 \in \mathscr{T}$ *with* $\tau_1, \tau_2 \in [\vartheta, \vartheta']$. *Let*

$$V_i^{\vartheta}\big(\tau_i, \tau_j; \sigma^{\vartheta'}\big) := E\Big[\mathbf{1}_{\{\tau_i \wedge \tau_j < \vartheta'\}}\Big(L^i_{\tau_i}\mathbf{1}_{\{\tau_i < \tau_j\}} + F^i_{\tau_j}\mathbf{1}_{\{\tau_i > \tau_j\}} + M^i_{\tau_j}\mathbf{1}_{\{\tau_i = \tau_j\}}\Big) + \mathbf{1}_{\{\tau_i = \tau_j = \vartheta'\}} V_i^{\vartheta'}\big(\sigma_i^{\vartheta'}, \sigma_j^{\vartheta'}\big) \,\Big|\, \mathscr{F}_\vartheta\Big]$$

for any $i, j \in \{1, 2\}$, $i \ne j$.[14] *Then there is an extended mixed strategy* $\sigma_k^{\vartheta} \in \mathscr{S}^{\vartheta}$ *for each* $k = 1, 2$ *(given by* $G_k^{\vartheta}(t) = \mathbf{1}_{\{\tau_k < \vartheta'\}}\mathbf{1}_{\{t \ge \tau_k\}} + \mathbf{1}_{\{\tau_k = \vartheta'\}} G_k^{\vartheta'}(t)$ *for every* $t \in \mathbb{R}_+$ *and* $\alpha_i^{\vartheta} = \alpha_i^{\vartheta'}$*), such that time-consistency with* $\sigma_k^{\vartheta'}$ *holds and*

$$V_i^{\vartheta}\big(\sigma_i^{\vartheta}, \sigma_j^{\vartheta}\big) = V_i^{\vartheta}\big(\tau_i, \tau_j; \sigma^{\vartheta'}\big). \tag{2.6}$$

(ii) *If* $\sigma_i^{\vartheta'}$ *in* (i) *is a best reply for firm* i *to* $\sigma_j^{\vartheta'}$ *at* ϑ' *and if* τ_i *attains*

$$\operatorname*{ess\,sup}_{\tau \in \mathscr{T}, \tau \in [\vartheta, \vartheta']} V_i^{\vartheta}\big(\tau, \tau_j; \sigma^{\vartheta'}\big), \tag{2.7}$$

then σ_i^{ϑ} *is a best reply for firm* i *to* σ_j^{ϑ} *at* ϑ.

(iii) *Suppose* $(\sigma_k^{\vartheta}; \vartheta \in \mathscr{T})$ *is constructed as in* (i) *for* $k \in \{1, 2\}$*, where, for each* ϑ*,* $\vartheta' = \tau_c(\vartheta) \ge \vartheta$ *from a family of stopping times* $(\tau_c(\vartheta); \vartheta \in \mathscr{T})$ *with associated* $\sigma_k^{\tau_c(\vartheta)} \in \mathscr{S}^{\tau_c(\vartheta)}$*, and* $\tau_k = \tau_k^{\vartheta} \in [\vartheta, \tau_c(\vartheta)]$ *for stopping times* $(\tau_k^{\vartheta}; \vartheta \in \mathscr{T})$ *satisfying time-consistency condition* (2.1)*. Then* $(\sigma_k^{\vartheta}; \vartheta \in \mathscr{T})$ *is a time-consistent extended mixed strategy if all* $\sigma_k^{\tau_c(\vartheta)}$ *are such that* $G_k^{\tau_c(\vartheta)}(t) = \mathbf{1}_{\{t \ge \tau_c(\vartheta)\}}$ *and* $\alpha_k^{\tau_c(\vartheta)}(t) = \mathbf{1}_{\{t \ge \tau_c(\vartheta)\}}\alpha_k^o(t)$ *for a fixed process* α_k^o *satisfying* $\alpha_k^o(t) = 0$ *for all* $t \in [\vartheta, \tau_c(\vartheta))$ *for any* $\vartheta \in \mathscr{T}$.

If $\vartheta' \equiv \infty$, then $V_i^{\vartheta'}(\sigma_i^{\vartheta'}, \sigma_j^{\vartheta'}) = M^i_\infty$ and problem (2.7) simplifies to that of maximizing (2.5) over $\tau_i \ge \vartheta$, which means that equilibria in pure strategies persist if extended mixed strategies are admitted. To verify time-consistency of randomization occurring (only) in the continuation equilibria, the process α_k^o aggregating them must not charge $[\vartheta, \tau_c(\vartheta))$ in accordance with pure plans applying there.

Remark 2.4. The proof of Proposition 2.3 only assumes that the processes (L^i_t), (F^i_t) and (M^i_t) are, as ours, measurable and satisfying the mild integrability condition "class (D)" that is standard for optimal stopping and verified in Lemma A.5; cf. Remark 2.1.

3 Equilibrium characterization

The assumed orders between different revenues have important consequences for equilibria of the timing game, independently of any more specific model for the uncertainty. This section illuminates the structure of possible equilibria just by comparing revenue streams, to provide more detailed economic insights than analyses based on reduced functional forms of payoffs for specific state-space models, and to provide complete equilibrium verification arguments. We show that it suffices to solve a particular class of constrained optimal stopping problems in order to construct subgame-perfect equilibria with preemption. As mutual preemption may destroy option values unnecessarily, we also consider alternative equilibria that avoid preemption and provide further arguments to simplify their verification. See also Appendix C, elaborating on when it is indeed impossible to delay investment in any equilibrium.

3.1 Sufficient equilibrium conditions

In order to construct subgame-perfect equilibria, it is first determined when immediate investment is an equilibrium, possibly due to a mutual preemption scheme.

[14] We deliberately abuse notation by reusing $V_i^{\vartheta}(\cdot)$ with different argument, intending to emphasize identity (2.6).

3.1.1 Simultaneous investment

Immediate investment by both firms is an equilibrium at $\vartheta \in \mathscr{T}$ if both follower options are worthless, i.e., if $F^i_\vartheta = M^i_\vartheta$ for both $i = 1, 2$. First consider only pure plans, so that the payoffs are given by (2.5) with $\tau_j = \vartheta$. If one firm i deviated to any plan $\tau_i > \vartheta$, it would become follower and actually invest at $\tau^i_F(\vartheta)$, which still attains $F^i_\vartheta = M^i_\vartheta$. In particular, if $\vartheta = \tau^i_F(\vartheta)$ for both $i = 1, 2$, then a unilateral deviation would not even change the physical outcome and firm i's payoff would stay $F^i_\vartheta = M^i_\vartheta = L^i_\vartheta$. Note, however, that even in this case, when a follower would incur a loss by any hesitation, each firm i may only be willing to invest *proactively* by the plan $\tau_i = \vartheta$ because the other firm does so. If firm i's investment was only *triggered* by $\tau^i_F(\tau_j) = \tau_j < \tau_i$, then firm j might want to delay investment (cf. Appendix C).

With Proposition 2.3, simultaneous investment can also be sustained on the event $\{F^1_\vartheta = M^1_\vartheta\} \cap \{F^2_\vartheta = M^2_\vartheta\}$ if its probability is not 1, and if extended mixed strategies appear elsewhere. For both purposes we consider a continuation equilibrium at some $\vartheta' \geq \vartheta$, such that simultaneous investment occurs on $\{\vartheta < \vartheta'\}$, whereas the "continuation" equilibrium is carried out on $\{\vartheta = \vartheta'\}$.

Lemma 3.1. *Suppose that $\vartheta, \vartheta' \in \mathscr{T}$, $\vartheta \leq \vartheta'$, and that $\sigma^{\vartheta'} = (\sigma^{\vartheta'}_1, \sigma^{\vartheta'}_2)$ are an equilibrium at ϑ'. If $\vartheta = \vartheta'$ off $\{F^1_\vartheta = M^1_\vartheta\} \cap \{F^2_\vartheta = M^2_\vartheta\}$, then the strategies from Proposition 2.3 for $\tau_1 = \tau_2 = \vartheta$ are an equilibrium at ϑ. Firm i's payoff at ϑ then is $\mathbf{1}_{\{\vartheta<\vartheta'\}} M^i_\vartheta + \mathbf{1}_{\{\vartheta=\vartheta'\}} V^{\vartheta'}_i(\sigma^{\vartheta'}_i, \sigma^{\vartheta'}_j)$.*

To see when simultaneous investment can be sustained, we can show that firm 1's follower option is not worth more than firm 2's by $\pi^{B2}_\cdot - \pi^{F2}_\cdot \leq \pi^{B1}_\cdot - \pi^{F1}_\cdot$, so both are worthless if and only if $\tau' = \vartheta$ attains F^2_ϑ. Similarly, firm 1's follower reaction time never exceeds firm 2's.

Lemma 3.2. *$\tau^1_F(\tau) \leq \tau^2_F(\tau)$ and $F^1_\tau - M^1_\tau \leq F^2_\tau - M^2_\tau$ for any $\tau \in \mathscr{T}$.*

Lemma 3.2 results from the followers' opportunity cost of waiting being given by $\pi^{Bi}_\cdot - \pi^{Fi}_\cdot$, which for firm 1 is not less than for firm 2. Thus, if firm 1 is follower, it cannot wait longer than firm 2 could. More generally, firm 1 cannot gain more from waiting until any time than firm 2 could, so firm 1's option value $F^1_\tau - M^1_\tau$ as follower is at most what firm 2's would be.

3.1.2 Preemption

Critical phases of a timing game occur when both players have a first-mover advantage, i.e., in the set $\mathcal{P} := \{L^1_\cdot > F^1_\cdot\} \cap \{L^2_\cdot > F^2_\cdot\} \subseteq \Omega \times \mathbb{R}_+$. If any player plans to become leader in such a phase, then a preemption scheme is triggered with both players trying to move before each other, to become leader. Therefore, $\mathcal{P}$ is called *preemption region*.

Preemption incentives may cause equilibrium failure if only standard mixed strategies are considered. Such problems arise when moving simultaneously is not an equilibrium, and if each player would prefer to wait without preemptive pressure, like in Figure 1 (see Hendricks and Wilson, 1992). With an interval of positive atoms $\alpha^\vartheta_i(t)$, however, each firm i can build up a crucial "threat" to move very quickly if the other firm hesitates for a positive amount of time, but controlling to some extent the risk of moving simultaneously (resp. *unconditionally*) if the other uses such atoms, too. The outcomes are then reminiscent of discrete-time limits, such that joint investment occurs only with some probability, whereas each firm also becomes leader with some probability.

In the following equilibria established in Riedel and Steg (2017), both firms are in fact indifferent to move or not. An exception occurs if one firm is indifferent to become leader or follower but the other not; then the latter wins for sure. Let $\tau_{\mathcal{P}}(\vartheta) := \inf\{t \geq \vartheta \,|\, L^1_t > F^1_t \text{ and } L^2_t > F^2_t\} \in \mathscr{T}$ denote the first hitting time of $\mathcal{P}$ from any $\vartheta \in \mathscr{T}$; so if we set $\vartheta' = \tau_{\mathcal{P}}(\vartheta)$, then it satisfies $\vartheta' = \tau_{\mathcal{P}}(\vartheta')$.

Lemma 3.3.

(i) *Whenever $\vartheta' = \tau_{\mathcal{P}}(\vartheta)$ for $\vartheta \in \mathscr{T}$, then there is $(\sigma^{\vartheta'}_1, \sigma^{\vartheta'}_2) \in \mathscr{S}^{\vartheta'} \times \mathscr{S}^{\vartheta'}$ forming an equilibrium at ϑ', such that firm i's payoff is $V^{\vartheta'}_i(\sigma^{\vartheta'}_i, \sigma^{\vartheta'}_j) = F^i_{\vartheta'} \mathbf{1}_{\{L^j_{\vartheta'} > F^j_{\vartheta'}\}} + L^i_{\vartheta'} \mathbf{1}_{\{L^j_{\vartheta'} = F^j_{\vartheta'}\}}$ (for any $i, j \in \{1, 2\}$, $i \neq j$).*

(ii) *The* $\sigma_k^{\vartheta'}$ *from* (i) *for* $k=1,2$ *are specifically such that* $G_k^{\vartheta'}(t)=\mathbf{1}_{\{t\geq\vartheta'\}}$ *and* $\alpha_k^{\vartheta'}(t)=\mathbf{1}_{\{t\geq\vartheta'\}}\alpha_k^o(t)$ *for* $\alpha_k^o=\alpha_i^{\tau_{\mathcal{P}}(\vartheta_0)}$ *with* $\vartheta_0\equiv 0$*, which then also satisfies* $\alpha_k^o(t)=0$ *for all* $t\in[\vartheta,\tau_{\mathcal{P}}(\vartheta))$ *for any* $\vartheta\in\mathscr{T}$*.*

Lemma 3.3 follows directly from Proposition 3.1 in Riedel and Steg (2017), because their Assumption 2.1 is satisfied by our payoff processes (cf. Remark 2.1), and because $F_t^i\geq M_t^i$ for all $t\in\mathbb{R}_+$ (a.s.) due to right-continuity for both $i=1,2$.

Assertion (i) gives us continuation payoffs to apply in Proposition 2.3, and if we do so for all $\vartheta\in\mathscr{T}$, then assertion (ii) ensures that the corresponding $(\sigma_i^{\vartheta};\vartheta\in\mathscr{T})$ satisfy time-consistency if the related, degenerate G_i^{ϑ} do so via condition (2.1) for pure plans. From a practical point of view, it may thus be assumed that both firms plan to move no later than at $\tau_{\mathcal{P}}(\vartheta)$ in any subgame, and then for preemptive purposes, and the corresponding payoffs result.

We can characterize the preemption region further by showing that firm 1's first-mover advantage is never less than firm 2's, given that $\tau_F^1(\cdot)\leq\tau_F^2(\cdot)$ and the assumption $\pi_\cdot^{L1}-\pi_\cdot^{F1}\geq\pi_\cdot^{L2}-\pi_\cdot^{F2}$.

Lemma 3.4. $L_\tau^1-F_\tau^1\geq L_\tau^2-F_\tau^2$ *for any* $\tau\in\mathscr{T}$*, and thus* $\mathcal{P}=\{L_\cdot^2>F_\cdot^2\}$ *and* $\tau_{\mathcal{P}}(\vartheta)=\inf\{t\geq\vartheta\,|\,L_t^2>F_t^2\}$ *for every* $\vartheta\in\mathscr{T}$*.*

Lemma 3.4 uses the fact that the revenue difference between being leader or follower is $\pi_\cdot^{Li}-\pi_\cdot^{Fi}$ before any follower would invest, which for firm 1 is not less than for firm 2. Firm 1 also prefers to be leader between the own follower reaction time and that of firm 2, because it earns $\pi_\cdot^{L1}$ instead of $\pi_\cdot^{B1}$. Firm 2, on the contrary, cannot gain from being leader between those two times, as it can only obtain $\pi_\cdot^{B2}$ instead of $\pi_\cdot^{F2}$, which is never a gain before its own follower reaction time.

For firm 2 to have a first-mover advantage, $\pi_\cdot^{B1}-\pi_\cdot^{F1}$ must not be too profitable for firm 1: if $\vartheta=\tau_F^1(\vartheta)$, then $L_\vartheta^2=M_\vartheta^2\leq F_\vartheta^2$.[15] Moreover, investment must be sufficiently profitable in terms of the revenue difference $\pi_\cdot^{L2}-\pi_\cdot^{F2}$ – firm 2's only potential gain from being leader instead of follower. Firm 2 can in fact only have a first-mover advantage if it still does when it is optimal to start the stream $\pi_\cdot^{L2}-\pi_\cdot^{F2}$, because starting it earlier cannot be an additional gain. This argument provides a criterion for whether $\mathcal{P}=\emptyset$ that will be formalized in Section 3.2.1. In particular, $\mathcal{P}=\emptyset$ if $\pi_\cdot^{L2}-\pi_\cdot^{F2}\leq\pi_\cdot^{B1}-\pi_\cdot^{F1}$, because then firm 1 would follow immediately at the latest optimal time to start $\pi_\cdot^{L2}-\pi_\cdot^{F2}$. $\pi_\cdot^{L2}$ must thus exceed $\pi_\cdot^{B2}$ enough for firm 2 to have any first-mover advantage, as by assumption $\pi_\cdot^{B2}-\pi_\cdot^{F2}\leq\pi_\cdot^{B1}-\pi_\cdot^{F1}$.

3.1.3 Subgame-perfect equilibria with preemption

The subsequent equilibrium construction is facilitated by the fact that independently of what happens in the preemption region, no firm ever wants to invest when it has a second-mover advantage, so we can focus on subsequent continuation problems. This finding results from the assumption that investment does not benefit the other firm; in contrast to some suggestions in the literature, a second-mover advantage alone does not suffice to delay investment in general. For the following formal statement also for extended mixed strategies, note that $\mathscr{S}^{\vartheta'}\subseteq\mathscr{S}^{\vartheta}$ for $\vartheta\leq\vartheta'$, and that using some strategy from $\mathscr{S}^{\vartheta'}$ (e.g., corresponding to a pure plan $\tau\geq\vartheta'$) in the subgame starting at ϑ means remaining idle on $[\vartheta,\vartheta')$.

Proposition 3.5. *In any subgame and for any firm* $i\in\{1,2\}$*, it is never optimal to become first investor (sole or joint, and with any positive probability) while* $F_\cdot^i>L_\cdot^i$*.*

Furthermore, letting $\tau_{L>F}^i(\vartheta)=\inf\{t\geq\vartheta\,|\,L_t^i>F_t^i\}$ *for arbitrary* $\vartheta\in\mathscr{T}$ *and* $\vartheta'=\min\{\tau_{L>F}^i(\vartheta),\tau_F^i(\vartheta)\}$*, then, in Proposition 2.3,* $\tau_i=\vartheta'$ *attains (2.7) for every* $\tau_j\in\mathscr{T}$ *with* $\tau_j\in[\vartheta,\vartheta']$ *whenever* $\sigma_i^{\vartheta'}$ *is a best reply to* $\sigma_j^{\vartheta'}$ *at* ϑ'*.*

More generally, given ϑ' *as before, it is no loss for firm* i *to consider only strategies from* $\mathscr{S}^{\vartheta'}$ *at* ϑ*, i.e., for any* $(\sigma_i^{\vartheta},\sigma_j^{\vartheta})\in\mathscr{S}^{\vartheta}\times\mathscr{S}^{\vartheta}$ *there is* $\sigma_a^{\vartheta'}\in\mathscr{S}^{\vartheta'}$ *with*

$$V_i^{\vartheta}(\sigma_a^{\vartheta'},\sigma_j^{\vartheta})\geq V_i^{\vartheta}(\sigma_i^{\vartheta},\sigma_j^{\vartheta}),$$

[15] If $\vartheta=\tau_F^1(\vartheta)$, then it is indeed not even on the boundary of $\mathcal{P}$ if $\tau'=\vartheta$ does not attain F_ϑ^2, as then $L_\vartheta^2=M_\vartheta^2<F_\vartheta^2$ and hence $\vartheta<\tau_{\mathcal{P}}(\vartheta)$ by right-continuity of the processes.

and $\sigma_a^{\vartheta'}$ is a best reply for firm i at ϑ to σ_j^{ϑ} if $\sigma_a^{\vartheta'} \in \mathscr{S}^{\vartheta'}$ and if it is a best reply at ϑ' to some $\sigma_j^{\vartheta'} \in \mathscr{S}^{\vartheta'}$ that is time-consistent with σ_j^{ϑ}.

Idleness on $[\vartheta, \vartheta')$, where $F^i_\cdot \geq L^i_\cdot \geq M^i_\cdot$, is no loss because firm i can always secure at least the follower payoff in expectation by planning to invest at the follower reaction time. Indeed, the follower payoff is nondecreasing in expectation (a *submartingale*) until that time – if the opponent invests in the meantime, that does not affect firm i's reaction and can only defer the laggard revenue $\pi^{Fi}_\cdot \leq \pi^{0i}_\cdot$ – and at the own reaction time, investing regardlessly is at least as good as becoming follower by $\pi^{Li}_\cdot \geq \pi^{Bi}_\cdot$.

By Lemma 3.4, $\tau^2_{L>F}(\vartheta) = \tau_{\mathcal{P}}(\vartheta)$, so we may let firm 2's plan for any $\vartheta \in \mathscr{T}$ be to wait until $\vartheta' = \min\{\tau_{\mathcal{P}}(\vartheta), \tau^2_F(\vartheta)\}$ by Proposition 3.5. Indeed, we have a continuation equilibrium at ϑ' with $G_i^{\vartheta'}(t) = \mathbf{1}_{\{t\geq\vartheta'\}}$ for each $i = 1, 2$ by Lemmas 3.1 and 3.2, with simultaneous investment on $\{\tau^2_F(\vartheta) < \tau_{\mathcal{P}}(\vartheta)\}$, preemption on $\{\tau_{\mathcal{P}}(\vartheta) \leq \tau^2_F(\vartheta)\}$, and payoffs at $\tau_{\mathcal{P}}(\vartheta)$ provided by Lemma 3.3 (i). In case of symmetric revenues, we obtain an equilibrium at ϑ by switching roles. Otherwise, firm 1 may have a strict first-mover advantage before $\tau_{\mathcal{P}}(\vartheta)$ and may want to exploit it. Specifically, given the preemption payoffs at $\tau_{\mathcal{P}}(\vartheta)$ and $L^1_\cdot = F^1_\cdot = M^1_\cdot$ at $\tau^2_F(\vartheta)$, firm 1 can obtain $L^1_\cdot$ anywhere before or at $\min\{\tau_{\mathcal{P}}(\vartheta), \tau^2_F(\vartheta)\}$, except when $L^2_\cdot > F^2_\cdot$ at $\tau_{\mathcal{P}}(\vartheta)$: then firm 1 will obtain $F^1_\cdot$. As $L^2_t > F^2_t$ not before $\tau_{\mathcal{P}}(\vartheta)$, the best reply problem (2.7) for firm 1 can thus be written as

$$\operatorname*{ess\,sup}_{\vartheta\leq\tau\leq\tau_{\mathcal{P}}(\vartheta)\wedge\tau^2_F(\vartheta)} E\Big[L^1_\tau \mathbf{1}_{\{L^2_\tau\leq F^2_\tau\}} + F^1_\tau \mathbf{1}_{\{L^2_\tau > F^2_\tau\}} \,\Big|\, \mathscr{F}_\vartheta\Big]. \tag{3.1}$$

If problem (3.1) has a solution $\tau^*_1(\vartheta)$, then its value is firm 1's equilibrium payoff at ϑ by Proposition 2.3, and that of firm 2 is $E[F^2_{\tau^*_1(\vartheta)} \,|\, \mathscr{F}_\vartheta]$, who obtains the follower payoff (in expectation) also when $\tau^*_1(\vartheta) = \min\{\tau_{\mathcal{P}}(\vartheta), \tau^2_F(\vartheta)\}$. We can summarize as follows.

Theorem 3.6. *If there is a family of solutions $(\tau^*_1(\vartheta); \vartheta \in \mathscr{T})$ to (3.1) that satisfies time-consistency condition (2.1), and if we let $\tau^*_2(\vartheta) = \min\{\tau_{\mathcal{P}}(\vartheta), \tau^2_F(\vartheta)\}$ for every $\vartheta \in \mathscr{T}$, then there is a subgame-perfect equilibrium in which the strategy for each firm $i = 1, 2$ in the subgame starting at any $\vartheta \in \mathscr{T}$, σ_i^ϑ, is such that $G_i^\vartheta(t) = \mathbf{1}_{\{t\geq\tau^*_i(\vartheta)\}}$ and $\alpha_i^\vartheta(t) = \mathbf{1}_{\{t\geq\vartheta\}}\alpha_i^o(t) = \mathbf{1}_{\{t\geq\tau_{\mathcal{P}}(\vartheta)\}}\alpha_i^o(t)$ from Lemma 3.3.*

If all revenues are symmetric, then there is a symmetric subgame-perfect equilibrium in which both firms use the given strategy for firm 2.

Time-consistency of the $\tau^*_1(\vartheta)$ can be ensured whenever there are solutions to (3.1), because then there are respectively earliest ones due to right-continuity; it holds automatically for the $\tau^*_2(\vartheta)$.[16] For firm 1 it holds alternatively if each $\tau^*_1(\vartheta)$ is a latest solution to (3.1), or if each $\tau^*_1(\vartheta)$ is of threshold-type in a state-space model.

The existence of a solution to (3.1) is generally not clear, however, because the process to be stopped has a discontinuity at $\tau_{\mathcal{P}}(\vartheta)$ when $\vartheta < \tau_{\mathcal{P}}(\vartheta) < \tau^2_F(\vartheta)$ and $L^2_{\tau_{\mathcal{P}}(\vartheta)} > F^2_{\tau_{\mathcal{P}}(\vartheta)}$: then also $L^1_{\tau_{\mathcal{P}}(\vartheta)} > F^1_{\tau_{\mathcal{P}}(\vartheta)}$ by Lemma 3.4 and preemption causes a drop. A solution does exist if the process $L^2_\cdot - F^2_\cdot$ is lower semi-continuous, because then $L^2_{\tau_{\mathcal{P}}(\vartheta)} = F^2_{\tau_{\mathcal{P}}(\vartheta)}$ on $\{\vartheta < \tau_{\mathcal{P}}(\vartheta)\}$, such that (3.1) reduces to

$$\operatorname*{ess\,sup}_{\vartheta\leq\tau\leq\tau_{\mathcal{P}}(\vartheta)\wedge\tau^2_F(\vartheta)} E\Big[L^1_\tau \,\Big|\, \mathscr{F}_\vartheta\Big]. \tag{3.2}$$

Indeed, the solutions of (3.2) coincide with the solutions of the conceptually simpler constrained stopping problem

$$\operatorname*{ess\,sup}_{\vartheta\leq\tau\leq\tau_{\mathcal{P}}(\vartheta)\wedge\tau^2_F(\vartheta)} E\bigg[\int_0^\tau \pi^{01}_s\,ds + \int_\tau^\infty \pi^{L1}_s\,ds \,\bigg|\, \mathscr{F}_\vartheta\bigg], \tag{3.3}$$

because the follower reaction time $\tau^2_F(\tau)$ in L^1_τ remains constant for $\tau \in [\vartheta, \tau^2_F(\vartheta)]$. (3.3) has a solution by continuity.

[16] The families $(\tau_{\mathcal{P}}(\vartheta); \vartheta \in \mathscr{T})$ and $(\tau^2_F(\vartheta); \vartheta \in \mathscr{T})$ satisfy time-consistency by construction and thus also $(\tau^*_2(\vartheta); \vartheta \in \mathscr{T})$. As the latter are the constraints in (3.1), any family of earliest solutions $(\tau^*_1(\vartheta); \vartheta \in \mathscr{T})$ will then be time-consistent, too.

Proposition 3.7. *Assume that $L^2_\cdot - F^2_\cdot$ is lower semi-continuous from the left. Then there exists a subgame-perfect equilibrium as in Theorem 3.6, with each $\tau_1^*(\vartheta)$ the respectively earliest solution of* (3.3).

In this equilibrium, each firm either plans to invest because that is the opponent's plan (for preemption or as the follower options become worthless), or firm 1 exploits that waiting is dominant for firm 2 and thus acts like a constrained monopolist. Indeed, problem (3.3) is a constrained version of the *monopoly problem*

$$\operatorname*{ess\,sup}_{\tau\geq\vartheta} E\biggl[\int_0^\tau \pi_s^{0i}\,ds + \int_\tau^\infty \pi_s^{Li}\,ds \biggm| \mathscr{F}_\vartheta\biggr] \tag{3.4}$$

for $i = 1$. The two problems' solutions are of course linked; cf. Section 4 and Appendix C. For instance, whenever it is optimal to invest in (3.4), so it must be in the constrained problem (3.3). Therefore, only the constraint $\tau \leq \tau_{\mathcal{P}}(\vartheta)$ matters in (3.3) if $\pi_\cdot^{L1} - \pi_\cdot^{01} \geq \pi_\cdot^{B1} - \pi_\cdot^{F1}$, like for market entry with $\pi_\cdot^{01} = \pi_\cdot^{F1}$, because then the solution of (3.4) is to invest no later than at $\tau_F^1(\vartheta) \leq \tau_F^2(\vartheta)$.

3.2 Avoiding preemption

There can be other equilibria without preemption, even if the region $\mathcal{P}$ of *potential* preemption is non-empty. Preemption can be avoided by profitable continuation equilibria, and this will be a Pareto improvement. For instance, joint investment at a future time $\tau_J \in \mathscr{T}$ can be an equilibrium if it yields at least the same expected payoff as becoming leader earlier on, like in the right panel of Figure 1. Therefore, τ_J needs to be an (at least constrained) optimal time for the problems

$$\operatorname*{ess\,sup}_{\tau\geq\vartheta} E\Bigl[M_\tau^i \Bigm| \mathscr{F}_\vartheta\Bigr] = \operatorname*{ess\,sup}_{\tau\geq\vartheta} E\biggl[\int_0^\tau \pi_s^{0i}\,ds + \int_\tau^\infty \pi_s^{Bi}\,ds \biggm| \mathscr{F}_\vartheta\biggr]. \tag{3.5}$$

The firms can also plan to invest sequentially if one accepts to become follower when the other invests. Either equilibria depend on the relative magnitudes of the revenue processes, however, so existence cannot be ensured by simple regularity properties like continuity in Proposition 3.7. On the contrary, if $\pi_\cdot^{Fi} = \pi_\cdot^{0i}$, then $F_\cdot^i$ is nonincreasing in expectation (a supermartingale), as becoming follower later only leaves less possibilities to invest optimally. Thus, if $L_\vartheta^i > F_\vartheta^i$, then firm i would not accept to obtain only a follower payoff later on. For firm i to wait, it is therefore necessary that $\pi_\cdot^{Fi} < \pi_\cdot^{0i}$ occurs (e.g., due to the first investment stealing business from the other firm).

In the remainder of this section, we present some tools that help to verify whether preemption is avoidable. These tools greatly reduce the number of stopping problems to consider. In particular for state-space models, it may suffice to evaluate payoffs at a single threshold, like in Section 4. We do not assume any particular stochastic structure here, yet, so the following tools read abstractly. However, they can then be applied to many more complex stochastic shock processes than those illustrated in Section 4, and they nevertheless still follow a clear intuition based on opportunity costs.

3.2.1 Characterizing the preemption region

First, to see if the preemption region is empty, it suffices to consider stopping times that are optimal for some simple stopping problems. They are the solutions of the monopoly problem (3.4) if $\pi_\cdot^{0i} = \pi_\cdot^{Fi}$ (like in a market entry model).

Lemma 3.8. *For any $\vartheta \in \mathscr{T}$, $L_\vartheta^2 > F_\vartheta^2$ only if $E[L^2_{\tau^i_\Delta} - F^2_{\tau^i_\Delta} \mid \mathscr{F}_\vartheta] > 0$ for every time $\tau_\Delta^i \in \mathscr{T}$ that attains*

$$\operatorname*{ess\,sup}_{\tau\geq\vartheta} E\biggl[\int_0^\tau \pi_s^{Fi}\,ds + \int_\tau^\infty \pi_s^{Li}\,ds \biggm| \mathscr{F}_\vartheta\biggr] \tag{3.6}$$

for any $i \in \{1, 2\}$. When $\tau_\Delta^2 = \vartheta$ attains (3.6) for $i = 2$, then $L_\vartheta^2 - F_\vartheta^2 \geq E[L_\tau^2 - F_\tau^2 \mid \mathscr{F}_\vartheta]$ for all $\tau \in [\vartheta, \tau_F^1(\vartheta)]$.

Lemma 3.8 rests on the fact that for any $\tau \in [\vartheta, \tau_F^2(\vartheta)]$, the difference between L_ϑ^2 and F_ϑ^2 on $[\vartheta, \tau]$ is that between the monopoly or duopoly revenue and the laggard's revenue, so at most $\pi_\cdot^{L2} - \pi_\cdot^{F2}$. That difference is nonpositive in expectation up to any solution of (3.6), where indeed $\tau_\Delta^2 \leq \tau_F^2(\vartheta)$ by $\pi_\cdot^{L2} \geq \pi_\cdot^{B2}$. Moreover, the revenue difference between L_ϑ^2 and F_ϑ^2 on $[\tau_\Delta^2, \infty)$ is at most that between $L_{\tau_\Delta^2}^2$ and $F_{\tau_\Delta^2}^2$, because firm 2's follower reaction remains the same and, by becoming leader later, firm 2 receives the monopoly revenue at least until the same time.

For state-space models like in Section 4, we get the following characterization. First, as noted in Subsection 3.1.2, a follower threshold for either firm i, say $x_F^i \in \mathbb{R}$, is never contained in the preemption region,[17] not even in its closure if investment at x_F^1 is not optimal for firm 2. As $L_\cdot^2 \leq F_\cdot^2$ for all states above such x_F^i, the latter must lie above any non-empty preemption region. Second, by Lemma 3.8, any non-empty preemption region must intersect the stopping regions from (3.6) for both $i = 1, 2$; a threshold solving that problem, say $x_\Delta^i \in \mathbb{R}$, cannot lie above the preemption region. In particular, if $x_\Delta^2 \geq x_F^1$, then $\mathcal{P} = \emptyset$. Third, if firm 2 has no first-mover advantage at x_Δ^2, then it has none at any value that the state will attain before crossing x_F^1. Thus, if the state, starting from some $x_\Delta^2 < x_F^1$, will attain any intermediate value before reaching x_F^1, then it suffices to check whether there is a first-mover advantage for firm 2 at x_Δ^2; otherwise the preemption region is empty, because x_Δ^2 cannot lie above it.

3.2.2 Verification of equilibria without preemption

For equilibria without preemption, but with delayed joint investment or sequential investment, it needs to be verified that waiting at least as long as the opponent is optimal. Proposition 3.9 reduces such a verification to stopping problems less complex than maximizing the leader payoff directly; cf. also Appendix C. For state-space models, it may again suffice to consider deviations at a single threshold. Recall that, when verifying an equilibrium corresponding to pure plans by Proposition 2.3, then problem (2.7) becomes maximizing (2.5) over τ_i.

Proposition 3.9. *Let $\vartheta, \tau_*^i, \tau_*^j \in \mathscr{T}$ with $\vartheta \leq \tau_*^j \leq \tau_*^i$ for some $i, j \in \{1, 2\}$, $i \neq j$. Suppose the strategies of firms i and j for the subgame at $\vartheta \in \mathscr{T}$ correspond to the pure plans τ_*^i, τ_*^j, respectively. Then firm i's strategy is a best reply to firm j's if $F_{\tau_*^j}^i = M_{\tau_*^j}^i$ on $\{\tau_*^i = \tau_*^j\}$ and*

(i) $E[F_{\tau_*^j}^i \mid \mathscr{F}_\vartheta] \geq \operatorname{ess\,sup}_{\tau \in [\vartheta, \tau_*^j]} E[M_\tau^i \mid \mathscr{F}_\vartheta]$ *and*

(ii) *for each stopping time $\vartheta' \geq \vartheta$, on $\{\vartheta' < \tau_*^j\}$, one of the solutions $\tau_D^i(\vartheta') \in \mathscr{T}$ of the problem*

$$\operatorname*{ess\,sup}_{\tau \in [\vartheta', \tau_*^j \vee \vartheta']} E\left[\int_0^\tau \pi_s^{0i}\, ds + \int_\tau^\infty \pi_s^{Li}\, ds \,\middle|\, \mathscr{F}_{\vartheta'}\right] \tag{3.7}$$

satisfies $\tau_D^i(\vartheta') < \tau_F^j(\vartheta') \Rightarrow L_{\tau_D^i(\vartheta')}^i - E[F_{\tau_*^j}^i \mid \mathscr{F}_{\tau_D^i(\vartheta')}] \leq 0$ (a.s.).

When ϑ' attains (3.7)*, then $L_{\vartheta'}^i - E[F_{\tau_*^j}^i \mid \mathscr{F}_{\vartheta'}] \geq E[L_\tau^i - F_{\tau_*^j}^i \mid \mathscr{F}_{\vartheta'}]$ for all stopping times $\tau \in [\vartheta', \tau_F^j(\vartheta')]$.*

Furthermore, if $\pi_\cdot^{L1} - \pi_\cdot^{01} \geq \pi_\cdot^{L2} - \pi_\cdot^{02}$, $\pi_\cdot^{B1} - \pi_\cdot^{01} \geq \pi_\cdot^{B2} - \pi_\cdot^{02}$, $F_{\tau_*^2}^2 = M_{\tau_*^2}^2$ and (i), (ii) *hold for $i = 1$, then the strategies corresponding to the pure plans $\tau_*^1 = \tau_*^2$ are an equilibrium at ϑ.*

Condition (i) is also necessary, as the terminal payoff is at most $F_{\tau_*^j}^i$ (without preemption as modeled in Section 3.1.2) and $L_\cdot^i \geq M_\cdot^i$. Condition (ii) addresses the leader payoff via the constrained monopoly problems (3.7) (cf. (3.2) and (3.3)), saying that it suffices to check for deviations at solutions $\tau_D^i(\vartheta') < \tau_F^j(\vartheta')$; so there is nothing to check when $\vartheta' = \tau_F^j(\vartheta')$. The next claim implies that for threshold-type models it is typically enough to consider $\vartheta' = \tau_D^i(\vartheta)$: If firm i does not want to become leader then, it does not at any

[17] Here "the preemption region" refers to an area in the same state space in which the thresholds are defined, which is of course an abuse of terminology regarding the previous definition of $\mathcal{P}$.

value that the state process will attain before crossing firm j's follower threshold that determines $\tau_F^j(\vartheta)$. For states above that threshold, no deviations need to be considered.

Proposition 3.9 can be applied for equilibria of joint investment at some time $\tau_J = \tau_*^1 = \tau_*^2 \geq \vartheta$. Then, on the one hand, $F^2_{\tau_J} = M^2_{\tau_J}$ is necessary, which automatically implies $F^1_{\tau_J} = M^1_{\tau_J}$ by Lemma 3.2. On the other hand, (i) is then the clearly necessary condition that τ_J must be an (at least constrained) optimal time for maximizing the expected joint investment payoff $E[M^i_{\tau_J} \,|\, \mathscr{F}_\vartheta]$ (cf. (3.5) and also Lemma C.3). Given such τ_J, an equilibrium can be verified by condition (ii), where it suffices to consider firm 1 if the additional revenue order holds.

Proposition 3.9 simplifies as follows for sequential investment.

Corollary 3.10. *Let $\vartheta \in \mathscr{T}$, $\tau_*^2 := \tau_F^2(\vartheta)$, and $\tau_*^1 := \tau_S$, where $\tau_S \in \mathscr{T}$ is one of the solutions of* (3.7) *for $\vartheta' = \vartheta$, $i = 1$, and $j = 2$. Then the strategies corresponding to the pure plans τ_*^1, τ_*^2 are an equilibrium at ϑ if condition* (ii) *of Proposition 3.9 is satisfied for $i = 2$ and $j = 1$.*

Moreover, if $\pi_\cdot^{L1} - \pi_\cdot^{01} \geq \pi_\cdot^{L2} - \pi_\cdot^{02}$, then $\tau_D^2(\vartheta') = \tau_S$ attains (3.7) *whenever $\vartheta' \leq \tau_*^1 = \tau_S$.*

Note that, in the setting of Corollary 3.10, condition (ii) of Proposition 3.9 holds if firm 2 does not have a local first-mover advantage when $\tau_D^2(\vartheta') < \tau_F^1(\vartheta')$ attains (3.7), as (F_t^2) is a submartingale on $[\vartheta', \tau_F^2(\vartheta')]$. Under the additional revenue order in Corollary 3.10, this simply amounts to $[\tau_S]$ not being in the preemption region $\mathcal{P}$.

4 Applications

As an illustration, the previous general results will now be applied to two typical models from the strategic real options literature, in order to provide complete proofs for basic equilibrium outcomes that are discussed extensively in the literature, to derive neglected equilibria that may constitute Pareto improvements or actually display behavior that qualitatively differs from deterministic models, and to argue that some equilibria analyzed in the literature only exist under additional restrictions, if at all. The model of Pawlina and Kort (2006) first serves as the main vehicle, because, as we allow for weak orders among its parameters, it then also nests the models of Weeds (2002) and Fudenberg and Tirole (1985) (cf. fn. 19). Afterwards, the results of Grenadier (1996) will be revisited using the same arguments, although his economic setting is quite different.

4.1 Irreversible investment with asymmetric costs

The model of Pawlina and Kort (2006) is quite prototypic for the real options literature, but its equilibrium analysis is not complete.[18] Theorem 3.6 yields proper subgame-perfect equilibria. We will analyze them in detail, in order to show some remarkable neglected behavior and to make the arguments applicable to other models as well. The revenue streams for firm $i \in \{1, 2\}$ in Pawlina and Kort (2006) are

$$\left.\begin{aligned} \pi_t^{0i} &= e^{-rt} x_t D_{00}, & \pi_t^{Li} &= e^{-rt}(x_t D_{10} - rI^i), \\ \pi_t^{Fi} &= e^{-rt} x_t D_{01}, & \pi_t^{Bi} &= e^{-rt}(x_t D_{11} - rI^i), \end{aligned}\right\} \tag{4.1}$$

with discount factor $r > 0$ and demand uncertainty reflected by a geometric Brownian motion (x_t) satisfying

$$dx_t = \mu x_t \, dt + \sigma x_t \, dB_t, \tag{4.2}$$

where (B_t) is Brownian noise, $\mu < r$ the expected growth rate and $\sigma > 0$ the volatility. The constants $D_{10} \geq D_{11}$ and $D_{00} \geq D_{01}$ capture a negative impact of investment on the opponent's revenue. The firms'

[18] There are two different, unrelated issues. First, their proposed preemption equilibrium investment, with the high cost firm 2 investing at the follower threshold x_F^2, can only be seen as an outcome, but not as an equilibrium strategy; firm 1 would only be willing to invest at the preemption point if there was a preemption threat. Second, and also in other papers like Grenadier (1996) or Weeds (2002), noting a current second-mover advantage does not justify optimality of waiting, and only subgames with low initial states are considered, despite the aim for subgame-perfectness.

investment costs $I^2 \geq I^1 > 0$ are also constant and capitalized here, because our general results were based on comparisons of revenue streams and their implied opportunity costs of waiting (cf. fn. 6). This will also help to solve constrained stopping problems in the following. By assuming only the given weak inequalities, we can here nest the models of Weeds (2002) and Fudenberg and Tirole (1985).[19] The present instances of the follower problems (2.2) and the monopoly problems (3.4) are solved by investing when x_t exceeds some thresholds x_F^i and x_L^i, respectively, and the payoff processes (L_t^i), (F_t^i) and (M_t^i) are continuous (as functions of the state x_t).[20] In particular, simultaneous investment is an equilibrium for all states $x_\vartheta \geq x_F^2$.

If the preemption region in this model is non-empty, it is characterized by an open interval $(\underline{x}, \bar{x})$ of the state space $\mathbb{R}_+$ with $\bar{x} \leq x_F^1 \leq x_F^2$ (where both inequalities are strict if $I^2 > I^1$ and $D_{10} > D_{11} > D_{01}$), such that we can simply call $(\underline{x}, \bar{x})$ preemption region. The proof of the following proposition generalizes to other models driven by a continuous Markov process that affects revenues monotonically.

Proposition 4.1. *Consider the specification* (4.1). *There are two numbers* $\underline{x} \leq \bar{x} \in (0, x_F^1]$ *such that* $L_t^2 > F_t^2 \Leftrightarrow x_t \in (\underline{x}, \bar{x})$ *for all* $t \in \mathbb{R}_+$, *with* $\bar{x} = x_F^2$ *if* $I^1 = I^2$.

By Lemma 3.8 in Section 3.2.1 and the discussion thereafter it is enough to check if $L_0^2 - F_0^2 > 0$ for $x_0 = x_\Delta^2$, the threshold solving (3.6), which is the case if the cost-disadvantage I^2/I^1 is not too large; otherwise firm 2 prefers to invest much later than firm 1 and the preemption region is empty (in particular if $x_\Delta^2 \geq x_F^1$, when firm 1 would follow immediately).[21]

We can now characterize the equilibria of Theorem 3.6 for this model, which also have remarkable outcomes not captured in Pawlina and Kort (2006). Existence is guaranteed by Proposition 3.7 thanks to continuity, and it suffices to solve the simpler constrained monopoly problems (3.3). By the strong Markov property, this amounts to finding the region in the state space $\mathbb{R}_+$ where immediate investment is optimal in the problem for $t = 0$,

$$\sup_{\tau \leq \tau_{\mathcal{P}}(0) \wedge \tau_F^2(0)} E\left[\int_\tau^\infty e^{-rs}\big(x_s(D_{10} - D_{00}) - rI^1\big)\, ds\right]. \tag{4.3}$$

The constraint here takes the form $\min\{\tau_{\mathcal{P}}(0), \tau_F^2(0)\} = \inf\{t \geq 0 \,|\, x_t \in (\underline{x}, \bar{x}) \cup [x_F^2, \infty)\} = \inf\{t \geq 0 \,|\, x_t \in [\underline{x}, \bar{x}] \cup [x_F^2, \infty)\}$ (P-a.s.). Problem (3.3) is then solved by investing once the state x_t hits the investment region $\{x \in \mathbb{R}_+ \,|\, \tau = 0 \text{ attains (4.3) for } x_0 = x\}$ from time ϑ.

First, consider a non-empty preemption region $(\underline{x}, \bar{x})$ that is connected to the unconstrained monopoly investment region $[x_L^1, \infty)$, as it holds for the market entry variant of the model with $D_{01} = D_{00}$, cf. Lemma 3.8. Then immediate investment is optimal in (4.3) for any state $x_0 \geq \bar{x} \geq x_L^1$, as it is in the unconstrained

[19] In Weeds (2002), investment starts an R&D project with success arrival rate $h > 0$. The expected payoffs are equivalent to those from (4.1) with augmented discount rate $r + h$ instead of r, $D_{00} = D_{01} = 0$, $D_{10} = h$, $D_{11} = h(r + h - \mu)/(r + 2h - \mu)$ and $I^1 = I^2 = K$. The model of Fudenberg and Tirole (1985) with their concrete discounted cost function $c(t) = e^{-(r+a)t}$ is equivalent to (4.1) with $D_{00} = \pi_0(0)$, $D_{01} = \pi_0(1)$, $D_{10} = \pi_1(1)$, $D_{11} = \pi_1(2)$, $\mu = a$, augmented discount rate $r + a$ instead of r and $\sigma = 0$. The solutions derived for $\sigma > 0$ in Section 4.1 converge to the solutions for the deterministic case as $\sigma \to 0$; see fn. 20.

[20] If $D_{11} > D_{01}$, then $x_F^i = \frac{\beta_1}{\beta_1 - 1} \cdot \frac{I^i(r-\mu)}{D_{11} - D_{01}}$, where $\beta_1 > 1$ is the positive root of $\frac{1}{2}\sigma^2\beta(\beta-1) + \mu\beta - r = 0$. If $D_{11} \leq D_{01}$, then $x_F^i = \infty$. Analogously, $x_L^i = \frac{\beta_1}{\beta_1 - 1} \cdot \frac{I^i(r-\mu)}{(D_{10} - D_{00})^+}$. These are standard from option pricing, as are the following representations. $L_t^i = \int_0^t e^{-rs} x_s D_{00}\, ds + e^{-rt}\big(x_t D_{10}/(r - \mu) - I^i + (x_t/(x_F^j \vee x_t))^{\beta_1}(x_F^j \vee x_t)(D_{11} - D_{10})/(r - \mu)\big)$, $F_t^i = \int_0^t e^{-rs} x_s D_{00}\, ds + e^{-rt}\big(x_t D_{01}/(r-\mu) + (x_t/(x_F^i \vee x_t))^{\beta_1}((x_F^i \vee x_t)(D_{11} - D_{01})/(r-\mu) - I^i)\big)$ and $M_t^i = \int_0^t e^{-rs} x_s D_{00}\, ds + e^{-rt}(x_t D_{11}/(r-\mu) - I^i)$. If $\sigma \to 0$, then the terms for $\sigma > 0$ converge to their deterministic counterparts. In particular, β_1 increases to $r/(\mu^+)$, so the investment thresholds converge to those for the deterministic case by $\beta_1/(\beta_1 - 1) \to r/(r - \mu^+)$, as does the expected discount factor for the first time that the state x_t exceeds a threshold $x > x_0$, $(x_0/x)^{\beta_1}$.

[21] The precise condition $(I^2/I^1)^{\beta_1 - 1} < ((1 + c)^{\beta_1} - 1)/(\beta_1 c)$ if $c := (D_{10} - D_{11})/(D_{11} - D_{01}) \in (0, \infty)$ is obtained by plugging $x_0 = x_\Delta^2 = \frac{\beta_1}{\beta_1 - 1} \cdot \frac{I^2(r-\mu)}{(D_{10} - D_{01})^+}$ into the expressions for L_0^2 and F_0^2 in fn. 20; Pawlina and Kort (2006) obtain the same condition by a graphical argument. The condition implies $x_\Delta^2 < x_F^1$. The constraint on the cost ratio strictly exceeds 1 and is strictly increasing in c to infinity by $\beta_1 > 1$. If $D_{10} > D_{01} \geq D_{11}$, then $x_F^1 = \infty$ and the preemption region is non-empty for all $I^2 \geq I^1$. Finally, if $D_{10} \leq \max\{D_{11}, D_{01}\}$, then $x_\Delta^2 \geq x_F^1$ and the preemption region is empty.

problem. For states $x_0 < \underline{x}$, the preemption constraint in (4.3) is a constant upper threshold, so it is optimal to wait there until x_t exceeds either the constraint $\underline{x}$ or the unconstrained threshold x_L^1; see Lemma A.7 in Appendix A. The subgame-perfect equilibrium is complete in this case: no investment for states strictly below $\min\{\underline{x}, x_L^1\}$, preemptive investment in $[\underline{x}, \bar{x}]$ as described in Section 3.1.2, firm 1 investing as the leader in $[x_L^1, x_F^2) \setminus [\underline{x}, \bar{x}]$, and simultaneous investment for all states in $[x_F^2, \infty)$.

Next, if the preemption region is empty, then firm 1 only faces the upper constraint x_F^2 in (4.3). Again by Lemma A.7, it is then optimal for firm 1 to invest as soon as x_t exceeds either the constraint x_F^2 or the unconstrained monopoly threshold x_L^1. Note that for the market entry variant with $D_{00} = D_{01} < D_{11}$, $x_L^1 \leq x_F^1 < x_F^2 < \infty$. However, even if firm 1 uses the unconstrained monopoly threshold, it is still constrained by firm 2's plan. Firm 1 can only maximize the leader payoff subject to firm 2 investing also *proactively* in $[x_F^2, \infty)$.

The necessary conditions derived in Appendix C imply that preemption cannot be avoided if $D_{00} = D_{01}$ (like for market entry) and neither simultaneous investment in $[x_F^2, \infty)$ then by Lemma C.3, and the equilibrium in each of the previous cases is unique. Indeed, if the preemption region is non-empty, then it must contain the optimal stopping region for the continuous process $L_t^2 - F_t^2$, which takes positive values only there. Then also L_t^2 must be stopped in that stopping region (the problem considered in Lemma C.1), because $L_t^2 = (L_t^2 - F_t^2) + F_t^2$ and F_t^2 is nonincreasing in expectation (a supermartingale) now.

So far, with $x_L^1 \leq \bar{x}$ or $\mathcal{P} = \emptyset$, investment occurs if and only if demand is high enough, i.e., if the state is at least $\min\{\underline{x}, x_L^1\}$ or $\min\{x_F^2, x_L^1\}$. This behavior is the same for the stochastic model and its deterministic version, e.g., that in Fudenberg and Tirole (1985) (cf. fn. 19).

4.1.1 Preemption when demand falls

Qualitatively different behavior can be observed in the remaining case, a monopoly threshold lying above a non-empty preemption region, $x_L^1 > \bar{x} > \underline{x}$, which requires a sufficiently high pre-investment revenue level $D_{00} > D_{01}$. Firm 1 may then remain inactive even when it would invest immediately as follower (in states above x_F^1), because it has higher opportunity costs as prospective leader. This phenomenon is not addressed by Pawlina and Kort (2006), who only consider states below $\underline{x}$, where the same behavior as before holds: firm 1 waits until x_t hits the constraint $\underline{x} < x_L^1$. Problem (4.3) becomes more interesting for states in $(\bar{x}, x_F^2)$, where both constraints may be binding if that interval intersects the continuation region $[0, x_L^1)$ of the unconstrained problem, and behavior may be more complex.

A lower constraint, like presently $\bar{x}$, has a much stronger effect than any upper constraint as considered before. Two cases can be distinguished for the problem of delaying the revenue change $\pi_t^{L1} - \pi_t^{01} = e^{-rt}(x_t(D_{10} - D_{00}) - rI^1)$ in $[\bar{x}, x_F^2]$. The easier one is that $x_t(D_{10} - D_{00}) > rI^1$ on all of $(\bar{x}, x_F^2)$, because then any delay is a loss of revenue and it is optimal to invest immediately everywhere on $(\bar{x}, x_F^2)$. Nevertheless, this is already an effect of the lower constraint $\bar{x}$. Without the constraint, it would be optimal to forego some positive revenue, i.e., to wait when $x_t(D_{10} - D_{00})$ exceeds rI^1 only little, to avoid the risk of negative revenues from a decreasing state.[22] With the constraint, however, the revenues from lower states cannot be escaped. The more difficult case is that $x(D_{10} - D_{00}) < rI^1$ near the preemption region. Firm 1 must wait when this inequality holds, in order not to start with running losses, so one has to determine the investment region towards the upper constraint, x_F^2. It may then in fact be optimal to invest far before the constraint is reached.

Proposition 4.2. *Consider the specification* (4.1) *and suppose the corresponding preemption region* $(\underline{x}, \bar{x}) \subseteq (0, x_F^1]$ *from Proposition 4.1 is non-empty. If* $\bar{x}(D_{10} - D_{00}) \geq rI^1$*, then the solution of problem* (4.3) *for all states* x_0 *in* $(\bar{x}, x_F^2)$ *is to invest immediately, whereas if* $D_{10} - D_{00} \leq 0$*, the solution is to wait until the state exits* $(\bar{x}, x_F^2)$.

If $0 < \bar{x}(D_{10} - D_{00}) < rI^1$*, then there is a unique threshold* $\hat{x} \in [rI^1/(D_{10} - D_{00}), x_L^1)$ *solving*

$$(\beta_1 - 1)A(x)x^{\beta_1} + (\beta_2 - 1)B(x)x^{\beta_2} = I^1 \tag{4.4}$$

[22] The unconstrained optimal threshold x_L^i from fn. 20 exceeds $rI^i/(D_{10} - D_{00})$ for $D_{10} > D_{00}$ and $\sigma^2 > 0$.

with

$$\begin{pmatrix} A(x) \\ B(x) \end{pmatrix} = \left[\bar{x}^{\beta_1} x^{\beta_2} - x^{\beta_1} \bar{x}^{\beta_2}\right]^{-1} \begin{pmatrix} x^{\beta_2} & -\bar{x}^{\beta_2} \\ -x^{\beta_1} & \bar{x}^{\beta_1} \end{pmatrix} \begin{pmatrix} \bar{x}\frac{D_{10}-D_{00}}{r-\mu} - I^1 \\ x\frac{D_{10}-D_{00}}{r-\mu} - I^1 \end{pmatrix} \tag{4.5}$$

and $\beta_1 > 1$ *and* $\beta_2 < 0$ *the roots of* $\frac{1}{2}\sigma^2\beta(\beta-1) + \mu\beta - r = 0$, *and the solution of problem* (4.3) *for all states* x_0 *in* $(\bar{x}, x_F^2)$ *is to invest when* (x_t) *exits* $(\bar{x}, \hat{x} \wedge x_F^2)$.

The "smooth-pasting" condition, which is frequently used to guess value functions, only holds in the last case and only if $\hat{x} \leq x_F^2$. If $x_F^2(D_{10} - D_{00}) \leq rI^1$, then $\hat{x} \geq x_F^2$ and the solution is to wait until the state exits $(\bar{x}, x_F^2)$. It is easy to compute the solutions $\hat{x}$ of (4.4), which are typically much lower than the upper constraint x_F^2 or the unconstrained threshold x_L^1. Thus, the risk of getting trapped at $\bar{x}$ by preemption induces much earlier investment; this is illustrated in Section 4.1.4. The effect cannot be observed in the deterministic version of the model with a growing market (or falling cost).

4.1.2 Joint investment equilibria

If $D_{00} > D_{01}$, then there are potentially many more equilibria than those from Theorem 3.6, as one can now drop the premise that preemption occurs in the preemption region, or that simultaneous investment occurs everywhere above x_F^2.

First, Proposition 3.9 is now applied to verify equilibria of delayed joint investment, which cannot happen below x_F^2 for firm 2 to invest simultaneously. The highest expected value of joint investment can be achieved by solving (3.5), which yields a maximal threshold, say x_M^1 for firm 1. But one can also consider constrained versions of that problem, with some investment threshold $x_J \in [x_F^2, x_M^1]$. Joint investment triggered by x_J is an equilibrium if firm 1 does not want to become leader at the threshold solving problem (3.7), which is $\min\{x_J, x_L^1\}$ by Lemma A.7 again.

Proposition 4.3. *Consider the specification* (4.1) *and let* $x_M^1 \geq x_L^1 \in [0, \infty]$ *denote the threshold solving problem* (3.5) *for firm* 1.[23] *Suppose* $x_M^1 \geq x_F^2$. *Then there exists a subgame-perfect equilibrium of simultaneous investment triggered by the threshold* $x_J \in [x_F^2, x_M^1]$ *iff that yields firm* 1 *at least the expected payoff* L_0^1 *for* $x_0 = x_L^1 < x_F^2$, *which is iff*

$$x_L^1 \geq x_F^2 \quad \Leftrightarrow \quad D_{10} \leq D_{00} \quad \textit{or} \quad \frac{I^2}{I^1} \leq \frac{(D_{11} - D_{01})^+}{D_{10} - D_{00}}$$

or if

$$\left(\frac{I^2}{I^1}\right)^{\beta_1 - 1} \left[1 + \left(\frac{x_L^1}{x_J}\right)^{\beta_1} \left(\beta_1 - 1 - \frac{x_J}{x_L^1}\beta_1 \frac{D_{11} - D_{00}}{D_{10} - D_{00}}\right)\right] \leq \beta_1 \frac{D_{10} - D_{11}}{D_{10} - D_{00}} \left(\frac{(D_{11} - D_{01})^+}{D_{10} - D_{00}}\right)^{\beta_1 - 1} \tag{4.6}$$

with $\beta_1 > 1$ *from Proposition 4.2. The left-hand side of* (4.6) *is strictly positive and strictly decreasing in* $x_J \in [x_L^1, x_M^1]$ *if* $x_L^1 < x_F^2$.

If $D_{10} \leq D_{00}$, then it is never better to become leader than to invest jointly with the follower, which is not better than maximizing the simultaneous investment payoff. Note that $x_L^1 < x_F^2$ implies $D_{10} > D_{00}$, and then the simultaneous investment equilibrium exists if and only if the cost disadvantage is not too large. Otherwise, firm 2's follower threshold becomes too large for firm 1 to give up the leader markup that it can then obtain. If $D_{10} > D_{00}$, then the second restriction on I^2/I^1 in Proposition 4.3 is weaker than the first for $x_J = x_L^1$, and it is further relaxed if x_J increases. If $x_J = x_M^1 < \infty$, then (4.6) coincides with the maximal bound on I^2/I^1 identified by a graphical argument in Pawlina and Kort (2006), who impose $D_{11} > D_{00}$.[24] Proposition 4.3 also applies for $D_{11} \leq D_{00}$, when the firms, after both have invested, end up

[23] $x_M^1 = \frac{\beta_1}{\beta_1 - 1} \cdot \frac{I^1(r-\mu)}{(D_{11} - D_{00})^+}$, cf. fn. 20.

[24] $x_M^1 < \infty \Leftrightarrow D_{11} > D_{00}$, and then $x_J = x_M^1$ implies $x_J / x_L^1 = (D_{10} - D_{00})/(D_{11} - D_{00})$.

no better than before. It can then still be optimal to invest at some threshold x_J only because the other firm does, although both would prefer that neither invests.

Indeed, there may be many equilibria with "inefficient" joint investment in states above x_F^2 and where the expected joint investment payoff could be improved. If $(D_{11} - D_{00})x_F^2 < rI^1$, then M_t^i increases in expectation for states in the interval $[x_F^2, rI^1/(D_{11} - D_{00})^+)$, because investment decreases revenue, and it is hence optimal to wait in any constrained version of problem (3.5). Therefore, one can partition the latter interval into arbitrary subintervals of alternating joint investment and idleness.

4.1.3 Sequential investment equilibria

Sequential investment without preemption may also be an equilibrium if the preemption region is non-empty, which is a Pareto improvement compared to the equilibria of Pawlina and Kort (2006) if delayed joint investment as in Section 4.1.2 is not feasible. Such an equilibrium can be verified by Corollary 3.10, and it exists for the current specification if and only if firm 2 does not have a strict first-mover advantage at x_L^1, when firm 1 first invests.

Proposition 4.4. *Consider the specification* (4.1) *and suppose $x_L^1 < x_F^2$ (whence $D_{10} > D_{00}$). Then there exists a subgame-perfect equilibrium with firm 1 planning to invest as soon as x_t exceeds x_L^1, and firm 2 planning to invest when x_t exceeds x_F^2, iff $x_L^1 \notin (\underline{x}, \bar{x})$ from Proposition 4.1, which is iff*

$$x_L^1 \geq x_F^1 \quad \Leftrightarrow \quad (D_{10} - D_{00})^+ \leq (D_{11} - D_{01})^+$$

or

$$(\beta_1 - 1)\frac{I^2}{I^1} + \left(\frac{I^2}{I^1}\right)^{1-\beta_1}\left(\frac{(D_{11} - D_{01})^+}{D_{10} - D_{00}}\right)^{\beta_1} \geq \beta_1\left[\frac{D_{10} - D_{01}}{D_{10} - D_{00}} - \frac{D_{10} - D_{11}}{D_{10} - D_{00}}\left(\frac{(D_{11} - D_{01})^+}{D_{10} - D_{00}}\right)^{\beta_1 - 1}\right] \quad (4.7)$$

with $\beta_1 > 1$ from Proposition 4.2. The left-hand side of (4.7) *is strictly increasing in I^2/I^1 and the right-hand side is strictly positive if $x_L^1 < x_F^1$.*

If $x_L^1 < x_F^1$, i.e., if firm 1 would invest earlier as monopolist than as follower, then it is not profitable for firm 2 to become leader instead of follower at x_L^1 if and only if the cost disadvantage is large enough. As the comparison is made at x_L^1, the investment time as leader is fixed, and then a high investment cost favors the follower value due to the option to choose the optimal investment time.

Finally, there may be equilibria with sequential investment as in Proposition 4.4 or preemption as in Proposition 4.2 and where joint investment is delayed to some threshold $x_J > x_F^2$, such that firm 1 can optimize the leader payoff over larger intervals. This may separate the investment region from the sequential equilibrium into one where firm 1 invests as leader and one where simultaneous investment occurs, with a gap in between. Such equilibria are more difficult to characterize explicitly. If x_F^2 is between two investment regions, the non-constant follower reaction prevents the simplifications used in the previous propositions.

4.1.4 Comparison of leader investment regions

In order to illustrate the potentially strong impact of preemption on states in $(\bar{x}, x_F^2)$ for varying parameter values in Figure 3, the model is re-parameterized as follows. First, r, μ and σ determine $\beta_{1,2}$ and together with the ratio $I^1/(D_{11} - D_{01})$ also firm 1's follower threshold x_F^1, which we fix and which is an upper bound for $\bar{x}$.

The distance between $\bar{x}$ and x_F^2, which is the region where firm 1 can invest as leader, grows in I^2. Indeed, x_F^2 obviously grows in I^2, and if the preemption region $(\underline{x}, \bar{x})$ is non-empty, it strictly shrinks if I^2 grows;[25]

[25] Suppose $x_0 < x_F^2$, such that firm 2's first-mover advantage $L_0^2 - F_0^2$ is non-trivial. If I^2 is increased, that has two negative effects on $L_0^2 - F_0^2$. First, it increases the investment cost stream $e^{-rt}rI^2$ up to firm 2's former follower investment time $\tau_F^2(0)$, which reduces L_0^2. Second, it delays $\tau_F^2(0)$. The new revenue stream difference $e^{-rt}(x_t(D_{11} - D_{01}) - rI^2)$ (with increased I^2) between the former and the new $\tau_F^2(0)$ has non-positive expectation by optimality of the new $\tau_F^2(0)$, and thus reduces $L_0^2 - F_0^2$.

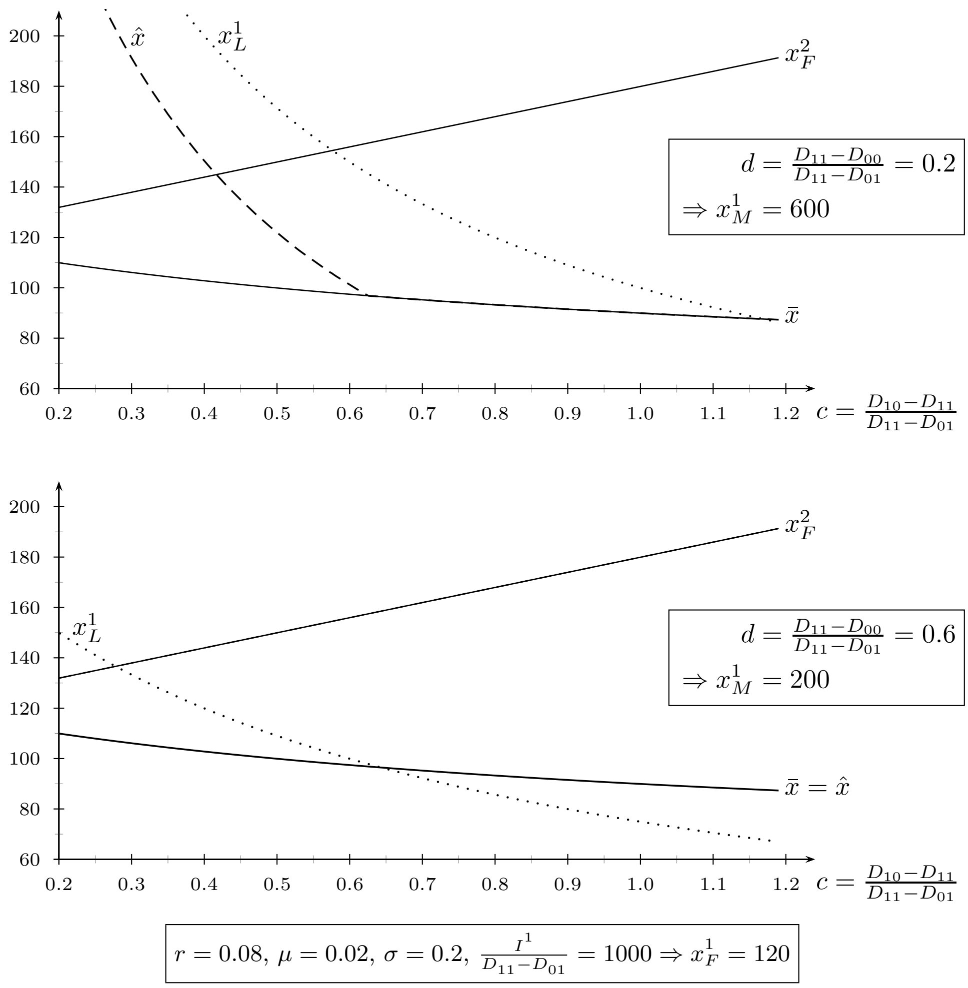

Figure 3: Constrained leader stopping regions.

$(\underline{x}, \bar{x})$ collapses when $I^2/I^1 = x_F^2/x_F^1$ reaches a bound given in fn. 21 in terms of $c = (D_{10}-D_{11})/(D_{11}-D_{01})$, the loss of a monopolist relative to the gain of the follower when the latter invests. We pick those limit values for I^2 and x_F^2 for simplicity, thus making both functions of c, although then just $\underline{x} = \bar{x} = x_\Delta^2$, the threshold solving (3.6). Now c also determines $\bar{x}$ by $x_\Delta^2 = x_F^2/(1+c)$.

Equation (4.4) for $\hat{x}$ can be reduced to the parameters $\beta_{1,2}$ and x_L^1, the unconstrained monopoly threshold, which is an upper bound on $\hat{x}$ and itself satisfies $x_L^1 = x_F^1/(c+d)$ with $d := (D_{11}-D_{00})/(D_{11}-D_{01})$. The latter ratio comes close to 1 if the leader's investment has not much influence on the follower's revenue, like in a market entry situation; it becomes small when the leader steals considerable business from the follower, like by a drastic innovation. d also controls the best simultaneous investment threshold by $x_M^1 = x_F^1/d$.

In the equilibria from Theorem 3.6, firm 1 can freely decide when to invest in the interval $(\bar{x}, x_F^2)$. Without the threat of preemption, it would not invest below $\min\{x_L^1, x_F^2\}$. However, given the threat of preemption, firm 1 already invests when the state exceeds $\hat{x}$, which may be much earlier as Figure 3 shows. In the upper panel with a low value of d, the threat of preemption strongly matters for $c \geq 0.45$. Firm 1 never chooses to wait at all in the lower panel with a moderate value of d. Joint investment at x_M^1 is an equilibrium avoiding preemption if $x_L^1 \geq x_F^2$; it is not an equilibrium for $d = 0.6$ and $c \geq 0.45$.

4.2 Strategic real estate development with construction time

Similar reasoning as before shows on the one hand that equilibria discussed in Grenadier (1996) only exist under certain parameter restrictions and on the other hand that there exist additional equilibria that are Pareto improvements.

Grenadier (1996) models a real option game between two symmetric real estate owners, who may each invest in redeveloping their property in order to earn higher rents. His model needs a slight translation for our framework, as it includes a delay of construction: if an owner invests, it takes $\delta \geq 0$ time units until the new building yields any revenues. The observations that we are going to point out do not depend on δ being large or small. Before investment by any owner, both earn the deterministic rent $R \geq 0$. Investment at cost $I > 0$ terminates that rent, reduces the rent of the opponent to $(1-\gamma)R$ with $\gamma \in [0,1]$ and initiates new own rent $D_1 x_t$ after the delay δ. (x_t) is a geometric Brownian motion as in (4.2). Once both new buildings are completed, each owner earns the rent $D_2 x_t$, with $0 < D_2 \leq D_1$.

Grenadier's model is strategically equivalent to specifying

$$\begin{aligned} \pi_t^{0i} &= e^{-rt}R, & \pi_t^{Li} &= e^{-rt}(D_1 e^{-(r-\mu)\delta} x_t - rI), \\ \pi_t^{Fi} &= e^{-rt}(1-\gamma)R, & \pi_t^{Bi} &= e^{-rt}(D_2 e^{-(r-\mu)\delta} x_t - rI) \end{aligned}$$

in our framework. The equilibria proposed in Grenadier (1996) are justified by the insufficient argument that waiting is optimal if the current follower payoff exceeds the current leader payoff. Nevertheless, there exists a subgame-perfect equilibrium as in Theorem 3.6 by symmetry; it can be characterized as follows. The follower problems (2.2) are again solved by investing once x_t exceeds a threshold $x_F > 0$, whence simultaneous investment is an equilibrium for all states $x_\vartheta \geq x_F$.[26] Problem (3.6) is solved by a threshold $x_\Delta = x_F D_2/D_1$, and the preemption region $\mathcal{P}$ is in fact non-empty if and only if $D_2 < D_1$. $\mathcal{P}$ can be represented by an interval $(\underline{x}, \bar{x})$ of the state space by the same arguments as in the proof of Proposition 4.1, where now $\bar{x} = x_F$.

4.2.1 Qualification of further equilibria

Depending on the parameter values, there may be other equilibria with delayed simultaneous investment, with or without preemption in $\mathcal{P}$. Let x_L denote the threshold solving the present instance of the unconstrained monopoly problem (3.4).[27] For states above $\bar{x} = x_F$, any investment will be simultaneous. Contrarily to the claim made in Grenadier (1996), simultaneous investment cannot be delayed past the threshold $x_M = x_L D_1/D_2 \geq x_F$ solving problem (3.5). Indeed, in any equilibrium with preemption in $\mathcal{P}$, both firms get by symmetry at most the follower payoff at the time of investment. The same holds for any equilibrium with only joint investment. In either case, investment must occur as soon as the state exceeds x_M, because then investing regardlessly gives the unique, maximal remaining payoff and any delay would be a loss (cf. Lemma C.3 for details).

With preemption occurring in $\mathcal{P}$, one can only consider delaying simultaneous investment in the interval $[\bar{x}, x_M]$, i.e., delaying the revenue change $\pi_t^{Bi} - \pi_t^{0i} = e^{-rt}(D_2 e^{-(r-\mu)\delta} x_t - rI - R)$. This problem has the same form as the one with two-sided constraint considered in Proposition 4.2 (recall also the illustration in Section 4.1.4), with $D_2 e^{-(r-\mu)\delta}$ replacing $D_{10} - D_{00}$, $I + R/r$ replacing I^1 and x_M replacing x_F^2. Thus, given now $\bar{x} = x_F$, if $D_2 e^{-(r-\mu)\delta} x_F \geq rI + R$, which means if

$$\gamma \leq \left(\frac{rI}{R} + 1\right)\left(1 - \frac{\beta_1 - 1}{\beta_1(r-\mu)}\right), \tag{4.8}$$

then investment cannot be delayed at all for states above x_F, which is not recognized in Grenadier (1996). In this case, the preemption region extends to such high states that any foregone revenue above it is a loss. Note that the right-hand side of (4.8) is strictly positive.

[26] $x_F = \frac{\beta_1}{\beta_1 - 1} \cdot e^{(r-\mu)\delta}(I + (1-\gamma)R/r)(r-\mu)/D_2$ with $\beta_1 > 1$ from fn. 20.

[27] $x_L = \frac{\beta_1}{\beta_1 - 1} e^{(r-\mu)\delta}(I + R/r)(r-\mu)/D_1$ with $\beta_1 > 1$ from fn. 20. This should not be confused with X_L in Grenadier (1996), which corresponds to the present $\underline{x}$.

Only if (4.8) fails will there exist a solution $\hat{x} \in [(rI+R)e^{(r-\mu)\delta}/D_2, x_M)$ to the current version of (4.4), such that investment can be held back in $(x_F, \hat{x})$. Only then the phenomenon discussed extensively in Section V of Grenadier (1996) can arise, that preemption occurs when demand *falls* to x_F.

However, if γ is sufficiently large to violate (4.8), then delayed joint investment may be attractive enough to avoid preemption altogether, which will be a Pareto improvement w.r.t. Grenadier (1996). By the same arguments as for Proposition 4.3, preemption can be avoided in an equilibrium of joint investment with the threshold $x_M \geq x_F$ if and only if that yields firm 1 at least the expected payoff L^1_0 for $x_0 = x_L < x_F$, which is if and only if

$$x_L \geq x_F \quad \Leftrightarrow \quad \gamma \geq \Big(\frac{rI}{R}+1\Big)\Big(1-\frac{D_2}{D_1}\Big)$$

or if

$$\gamma \geq \Big(\frac{rI}{R}+1\Big)\bigg(1-D_2\Big(\beta_1\frac{D_1-D_2}{D_1^{\beta_1}-D_2^{\beta_1}}\Big)^{\frac{1}{\beta_1-1}}\bigg)$$

with $\beta_1 > 1$ from fn. 20. The last restriction on γ is indeed weaker than the previous one.

5 Conclusion

The equilibrium analysis of the general model in Section 3 was based directly on its primitives and not on derived analytic properties of value functions, as it frequently happens in the growing literature on real option games. By this more general perspective, there is on the one hand less risk to neglect verification problems for equilibria and on the other hand a more detailed view of their economic structure. For models that satisfy the general assumptions made here, the number of equilibrium verification problems has been reduced considerably by economically meaningful arguments and it remains to solve a single class of optimal stopping problems for one firm. Theorem 3.6 applies to many more examples from the literature than the ones revisited in Section 4 (e.g., to those listed in the Introduction). The presented applications, which have quite distinctive economic properties, show how the general results act in typical state-space models. By the more complete approach, some neglected equilibrium behavior that qualitatively distinguishes stochastic from deterministic models has been identified. In particular, two-sided constraints induce feedback effects when the state evolves randomly. The arguments developed for the identification of additional equilibria that may be Pareto improvements also generalize to other models, e.g., for the source of uncertainty.

Therefore, the general perspective taken here provides a foundation for a more complete analysis of models of preemptive investment that fit into the framework and, moreover, a guideline for the analysis of other models that do not satisfy the revenue orders assumed here.

A Some technical details

Definition A.1 (Riedel and Steg, 2017, Definition 2.7)**.** An *extended mixed strategy* for firm $i \in \{1,2\}$ for the subgame starting at $\vartheta \in \mathscr{T}$ (in which no-one has moved, yet) is denoted by σ_i^ϑ and consists of a pair of processes $(G_i^\vartheta, \alpha_i^\vartheta)$ that each take values in $[0,1]$ and satisfy the following.

(i) G_i^ϑ is adapted. It is a.s. non-decreasing, right-continuous, and satisfying $G_i^\vartheta(s) = 0$ for all $s < \vartheta$.

(ii) α_i^ϑ is progressively measurable.[28] It is a.s. right-continuous in all $t \in \mathbb{R}_+$ for which $\alpha_i^\vartheta(t) \in (0,1)$ and satisfying $\alpha_i^\vartheta(s) = 0$ for all $s < \vartheta$.

(iii)

$$\alpha_i^\vartheta(t) > 0 \Rightarrow G_i^\vartheta(t) = 1 \quad \text{for all } t \geq 0 \quad \text{a.s.}$$

[28] Progressive measurability (cf. fn. 7) is generally stronger than adaptedness, but G_i^ϑ is also progressively measurable by being adapted and right-continuous.

For every extended mixed strategy, define also $G_i^\vartheta(0-) \equiv 0$, $G_i^\vartheta(\infty) \equiv 1$ and $\alpha_i^\vartheta(\infty) \equiv 1$. Let $\mathscr{S}^\vartheta$ denote the set of all extended mixed strategies for the subgame starting at ϑ.

Condition (iii) reflects that if a positive $\alpha_i^\vartheta(t)$ is reached, and the other firm does not move at t, then firm i is sure to move; cf. Subsection 3.1.2.[29]

Time-consistency for randomized plans means that conditional probabilities of moving at any given time (as determined by Bayes' rule) agree across subgames whenever possible.

Definition A.2 (Riedel and Steg, 2017, Definition 2.13)**.** A *time-consistent extended mixed strategy* for firm $i \in \{1,2\}$ for the timing game is a family of extended mixed strategies for all subgames, $\sigma_i := (\sigma_i^\vartheta; \vartheta \in \mathscr{T}) = ((G_i^\vartheta, \alpha_i^\vartheta); \vartheta \in \mathscr{T})$, such that for all $\vartheta, \vartheta', \tau \in \mathscr{T}$ with $\vartheta \leq \vartheta' \leq \tau$ it holds that (a.s.)

$$G_i^\vartheta(t) = G_i^\vartheta(\vartheta'-) + \big(1 - G_i^\vartheta(\vartheta'-)\big) G_i^{\vartheta'}(t) \text{ for all } t \geq \vartheta' \quad \text{and} \quad \alpha_i^\vartheta(\tau) = \alpha_i^{\vartheta'}(\tau).$$

To define expected payoffs from randomized plans for the first move, denote any mass points of the cumulative distribution functions by $\Delta G_i^\vartheta(\hat\tau^\vartheta) = G_i^\vartheta(\hat\tau^\vartheta) - G_i^\vartheta(\hat\tau^\vartheta-)$.

Definition A.3 (Riedel and Steg, 2017, Definition 2.11)**.** Given a profile of extended mixed strategies $(\sigma_1^\vartheta, \sigma_2^\vartheta) \in \mathscr{S}^\vartheta \times \mathscr{S}^\vartheta$ and $i, j \in \{1,2\}$, $i \neq j$, the *payoff* of firm i in the subgame starting at $\vartheta \in \mathscr{T}$ is

$$\begin{aligned} V_i^\vartheta(\sigma_i^\vartheta, \sigma_j^\vartheta) := E\Big[& \int_{[0,\hat\tau^\vartheta)} \big(1 - G_j^\vartheta(s)\big) L_s^i \, dG_i^\vartheta(s) + \int_{[0,\hat\tau^\vartheta)} \big(1 - G_i^\vartheta(s)\big) F_s^i \, dG_j^\vartheta(s) \\ & + \sum_{s \in [0,\hat\tau^\vartheta)} \Delta G_i^\vartheta(s) \Delta G_j^\vartheta(s) M_s^i + \lambda_{L,i}^\vartheta L_{\hat\tau^\vartheta}^i + \lambda_{L,j}^\vartheta F_{\hat\tau^\vartheta}^i + \lambda_M^\vartheta M_{\hat\tau^\vartheta}^i \,\Big|\, \mathscr{F}_\vartheta \Big], \end{aligned}$$

where $\hat\tau^\vartheta = \inf\{t \geq \vartheta \,|\, \alpha_1^\vartheta(t) + \alpha_2^\vartheta(t) > 0\}$ and $\lambda_{L,1}^\vartheta$, $\lambda_{L,2}^\vartheta$ and λ_M^ϑ are the outcome probabilities at $\hat\tau^\vartheta$ from Definition A.4, which satisfy $\lambda_{L,i}^\vartheta + \lambda_{L,j}^\vartheta + \lambda_M^\vartheta = (1 - G_i^\vartheta(\hat\tau^\vartheta-))(1 - G_j^\vartheta(\hat\tau^\vartheta-))$.

$\lambda_{L,1}^\vartheta$, $\lambda_{L,2}^\vartheta$ and λ_M^ϑ denote the additional outcome probabilities from the extensions α_i^ϑ, of only firm 1, only firm 2 or both moving first. Their definition is based on the interpretation that $\alpha_i^\vartheta(t) > 0$ means an atom at t, and then, by right-continuity, infinitely many follow immediately. Therefore, let the functions μ_L and μ_M from $[0,1]^2 \setminus \{(0,0)\}$ to $[0,1]$ be defined by

$$\mu_L(x,y) := x(1-y) \sum_{n=0}^{\infty} [(1-x)(1-y)]^n = \frac{x(1-y)}{x+y-xy} \qquad \text{and} \qquad \mu_M(x,y) := \frac{xy}{x+y-xy}.$$

$\mu_L(a_i, a_j)$ is the probability that player i stops first if players i and j stop with probabilities a_i and a_j, respectively, in every stage of a repeated game. $\mu_M(a_i, a_j)$ is the probability of simultaneous stopping and $1 - \mu_L(a_i, a_j) - \mu_M(a_i, a_j) = \mu_L(a_j, a_i)$ is that of player j stopping first. Extra first-round behavior occurs if an interval of atoms meets an isolated atom $\Delta G_i^\vartheta(\hat\tau^\vartheta) = G_i^\vartheta(\hat\tau^\vartheta) - G_i^\vartheta(\hat\tau^\vartheta-)$ of a cumulative distribution function. Special treatment is moreover required when atoms become arbitrarily small, because μ_L lacks a continuous extension at the origin; see Riedel and Steg (2017) for more details. Note that the probability that $\hat\tau^\vartheta$ is reached from ϑ with no firm having moved before is $(1 - G_1^\vartheta(\hat\tau^\vartheta-))(1 - G_2^\vartheta(\hat\tau^\vartheta-))$ and that Definition A.1 (iii) implies $(1 - G_i^\vartheta(\hat\tau^\vartheta-)) = \Delta G_i^\vartheta(\hat\tau^\vartheta)$ if $\hat\tau^\vartheta = \inf\{t \geq \vartheta \,|\, \alpha_i^\vartheta(t) > 0\}$. If $(1 - G_i^\vartheta(\hat\tau^\vartheta-)) = 0$, then it is for notational convenience understood that $(1 - G_i^\vartheta(\hat\tau^\vartheta-))/(1 - G_i^\vartheta(\hat\tau^\vartheta-)) := 0$.

Definition A.4 (Riedel and Steg, 2017, Definition 2.9)**.** Given $\vartheta \in \mathscr{T}$ and a pair of extended mixed strategies $(G_1^\vartheta, \alpha_1^\vartheta)$, $(G_2^\vartheta, \alpha_2^\vartheta)$, the *outcome probabilities* $\lambda_{L,1}^\vartheta$, $\lambda_{L,2}^\vartheta$ and λ_M^ϑ for firm 1 becoming leader, firm 2 becoming leader and simultaneous moving, respectively, at $\hat\tau^\vartheta$ are defined as follows. Let $i, j \in \{1,2\}$, $i \neq j$.

[29]The intuition is that by condition (ii) then either $\alpha_i^\vartheta(t) = 1$, so firm i moves for sure when t is reached, or right-continuity implies that infinitely many positive atoms occur before any time past t, so that firm i will have moved if the other has not.

If $\hat\tau^\vartheta < \hat\tau_j^\vartheta := \inf\{t \ge \vartheta \,|\, \alpha_j^\vartheta(t) > 0\}$, then

$$\lambda_{L,i}^\vartheta := \Delta G_i^\vartheta(\hat\tau^\vartheta)\big(1 - G_j^\vartheta(\hat\tau^\vartheta-)\big)\left[1 - \frac{\Delta G_j^\vartheta(\hat\tau^\vartheta)}{1 - G_j^\vartheta(\hat\tau^\vartheta-)}\right] = \Delta G_i^\vartheta(\hat\tau^\vartheta)\big(1 - G_j^\vartheta(\hat\tau^\vartheta)\big),$$

$$\lambda_M^\vartheta := \Delta G_i^\vartheta(\hat\tau^\vartheta)\big(1 - G_j^\vartheta(\hat\tau^\vartheta-)\big)\alpha_i^\vartheta(\hat\tau^\vartheta)\frac{\Delta G_j^\vartheta(\hat\tau^\vartheta)}{1 - G_j^\vartheta(\hat\tau^\vartheta-)} = \Delta G_i^\vartheta(\hat\tau^\vartheta)\Delta G_j^\vartheta(\hat\tau^\vartheta)\alpha_i^\vartheta(\hat\tau^\vartheta).$$

If $\hat\tau^\vartheta < \hat\tau_i^\vartheta := \inf\{t \ge \vartheta \,|\, \alpha_i^\vartheta(t) > 0\}$, then

$$\lambda_{L,i}^\vartheta := \big(1 - G_i^\vartheta(\hat\tau^\vartheta-)\big)\Delta G_j^\vartheta(\hat\tau^\vartheta)\frac{\Delta G_i^\vartheta(\hat\tau^\vartheta)}{1 - G_i^\vartheta(\hat\tau^\vartheta-)}\big(1 - \alpha_j^\vartheta(\hat\tau^\vartheta)\big) = \Delta G_i^\vartheta(\hat\tau^\vartheta)\Delta G_j^\vartheta(\hat\tau^\vartheta)\big(1 - \alpha_j^\vartheta(\hat\tau^\vartheta)\big),$$

$$\lambda_M^\vartheta := \big(1 - G_i^\vartheta(\hat\tau^\vartheta-)\big)\Delta G_j^\vartheta(\hat\tau^\vartheta)\frac{\Delta G_i^\vartheta(\hat\tau^\vartheta)}{1 - G_i^\vartheta(\hat\tau^\vartheta-)}\alpha_j^\vartheta(\hat\tau^\vartheta) = \Delta G_i^\vartheta(\hat\tau^\vartheta)\Delta G_j^\vartheta(\hat\tau^\vartheta)\alpha_j^\vartheta(\hat\tau^\vartheta).$$

If $\hat\tau^\vartheta = \hat\tau_1^\vartheta = \hat\tau_2^\vartheta$ and either $\max\{\alpha_1^\vartheta(\hat\tau^\vartheta), \alpha_2^\vartheta(\hat\tau^\vartheta)\} = 1$ or $\min\{\alpha_1^\vartheta(\hat\tau^\vartheta), \alpha_2^\vartheta(\hat\tau^\vartheta)\} > 0$, then

$$\lambda_{L,i}^\vartheta := \Delta G_i^\vartheta(\hat\tau^\vartheta)\Delta G_j^\vartheta(\hat\tau^\vartheta)\mu_L(\alpha_i^\vartheta(\hat\tau^\vartheta), \alpha_j^\vartheta(\hat\tau^\vartheta)), \quad \lambda_M^\vartheta := \Delta G_i^\vartheta(\hat\tau^\vartheta)\Delta G_j^\vartheta(\hat\tau^\vartheta)\mu_M(\alpha_1^\vartheta(\hat\tau^\vartheta), \alpha_2^\vartheta(\hat\tau^\vartheta)).$$

If $\hat\tau^\vartheta = \hat\tau_1^\vartheta = \hat\tau_2^\vartheta$, $\max\{\alpha_1^\vartheta(\hat\tau^\vartheta), \alpha_2^\vartheta(\hat\tau^\vartheta)\} < 1$ and $\min\{\alpha_1^\vartheta(\hat\tau^\vartheta), \alpha_2^\vartheta(\hat\tau^\vartheta)\} = 0$, then

$$\begin{aligned}\lambda_{L,i}^\vartheta := {} & \Delta G_i^\vartheta(\hat\tau^\vartheta)\Delta G_j^\vartheta(\hat\tau^\vartheta)\big(1 - \alpha_j^\vartheta(\hat\tau^\vartheta)\big) \\ & \cdot \Bigg(\alpha_i^\vartheta(\hat\tau^\vartheta) + \big(1 - \alpha_i^\vartheta(\hat\tau^\vartheta)\big)\frac{1}{2}\Bigg\{\liminf_{\substack{t \searrow \hat\tau^\vartheta \\ \alpha_i^\vartheta(t) + \alpha_j^\vartheta(t) > 0}} \mu_L(\alpha_i^\vartheta(t), \alpha_j^\vartheta(t)) + \limsup_{\substack{t \searrow \hat\tau^\vartheta \\ \alpha_i^\vartheta(t) + \alpha_j^\vartheta(t) > 0}} \mu_L(\alpha_i^\vartheta(t), \alpha_j^\vartheta(t))\Bigg\}\Bigg),\end{aligned}$$

$$\begin{aligned}\lambda_M^\vartheta &:= \Delta G_i^\vartheta(\hat\tau^\vartheta)\Delta G_j^\vartheta(\hat\tau^\vartheta) - \lambda_{L,i}^\vartheta - \lambda_{L,j}^\vartheta \\ &= \Delta G_i^\vartheta(\hat\tau^\vartheta)\Delta G_j^\vartheta(\hat\tau^\vartheta)\big(1 - \alpha_i^\vartheta(\hat\tau^\vartheta)\big)\big(1 - \alpha_j^\vartheta(\hat\tau^\vartheta)\big)\mu_M(\alpha_1^\vartheta(\hat\tau^\vartheta+), \alpha_2^\vartheta(\hat\tau^\vartheta+)) \quad \text{if } \alpha_1^\vartheta(\hat\tau^\vartheta+), \alpha_2^\vartheta(\hat\tau^\vartheta+) \text{ exist.}\end{aligned}$$

The last case uses $\mu_M(0,0) := 0$, the function's continuous extension to the origin. $\lambda_{L,i}^\vartheta$ is of course also the probability of firm j becoming follower at $\hat\tau^\vartheta$.

Lemma A.5. *In the setting of Section 2, consider four processes $(\pi_t^m) \in L^1(dt \otimes P)$, $m = 0, L, F, B$, such that each process $(\int_0^t \pi_s^m\, ds)$ is adapted, and let $\{\tau_O(\tau), \tau \in \mathscr{T}\}$ be a family of stopping times satisfying $\tau \le \tau_O(\tau) \le \tau_O(\tau')$ a.s. for all $\tau, \tau' \in \mathscr{T}$ with $\tau \le \tau'$ a.s. Then there exist optional processes*[30] *(L_t) and (F_t) that are of class* (D)[31] *and which satisfy*

$$L_\tau = L(\tau) := \int_0^\tau \pi_s^0\, ds + E\left[\int_\tau^{\tau_O(\tau)} \pi_s^L\, ds + \int_{\tau_O(\tau)}^\infty \pi_s^B\, ds \,\middle|\, \mathscr{F}_\tau\right]$$

and

$$F_\tau = F(\tau) := \int_0^\tau \pi_s^0\, ds + \operatorname*{ess\,sup}_{\tau' \ge \tau} E\left[\int_\tau^{\tau'} \pi_s^F\, ds + \int_{\tau'}^\infty \pi_s^B\, ds \,\middle|\, \mathscr{F}_\tau\right]$$

[30]This means that the processes are not only measurable on $\Omega \times \mathbb{R}_+$ w.r.t. $\mathscr{F} \otimes \mathcal{B}(\mathbb{R}_+)$, but also w.r.t. the optional σ-field, generated by all right-continuous processes that are adapted to $(\mathscr{F}_t)$ (which are thus optional themselves).

[31]This is a mild integrability condition: A measurable process X is of class (D) if the family $\{X_\tau \,|\, \tau \in \mathscr{T}, \tau < \infty\}$ is uniformly integrable, so that the family is bounded in $L^1(P)$ and pointwise convergence of X at a stopping time implies convergence in $L^1(P)$ as well.

a.s. for every $\tau \in \mathscr{T}$. In particular, the process (F_t) can be chosen right-continuous. If $\lim \tau_O(\tau^n) = \tau_O(\tau)$ a.s. for any $\tau \in \mathscr{T}$ and sequence $(\tau^n)_{n\in\mathbb{N}} \subseteq \mathscr{T}$ with $\tau^n \searrow \tau$ a.s., then also (L_t) can be chosen right-continuous.

All conditions are met when letting each $\tau_O(\tau)$ be the latest stopping time attaining the value of $F(\tau)$ or when letting each $\tau_O(\tau) = \tau$.

Proof. First, rewrite $F(\tau)$ as

$$F(\tau) = \int_0^\tau \left(\pi_s^0 - \pi_s^F\right) ds + E\left[\int_0^\infty \pi_s^B \, ds \,\middle|\, \mathscr{F}_\tau\right] + \operatorname*{ess\,sup}_{\tau' \geq \tau} E\left[\int_0^{\tau'} \left(\pi_s^F - \pi_s^B\right) ds \,\middle|\, \mathscr{F}_\tau\right]. \tag{A.1}$$

The first term on the right-hand side is a continuous process evaluated at τ, which is by assumption adapted and bounded by $\int_0^\infty (|\pi_s^0| + |\pi_s^F|)\, ds \in L^1(P)$, hence optional and of class (D). The second and third terms are (super-)martingale-systems (cf. El Karoui, 1981, Proposition 2.26) of class (D) – particularly the latter bounded by the family $(E[\int_0^\infty (|\pi_s^F| + |\pi_s^B|)\, ds \,|\, \mathscr{F}_\tau]; \tau \in \mathscr{T})$ of class (D). Thus, there exist optional processes of class (D) that aggregate the two (super-)martingale-systems, respectively. The former, being a martingale, may be chosen right-continuous. The latter is in fact the Snell envelope U_Y of the continuous process $(Y_t) := (\int_0^t (\pi_s^F - \pi_s^B)\, ds)$, whence U_Y is (right-)continuous in expectation and may thus be taken to have right-continuous paths, a.s.

$L(\tau)$ can be written like (A.1), with a third term $X(\tau) := E[\int_0^{\tau_O(\tau)} (\pi_s^L - \pi_s^B)\, ds \,|\, \mathscr{F}_\tau]$. Suppose first that $\pi_s^L - \pi_s^B \geq 0$ for all $s \in \mathbb{R}_+$, a.s. In this case,

$$E\big[X(\tau') \,\big|\, \mathscr{F}_\tau\big] = X(\tau) + E\left[\int_{\tau_O(\tau)}^{\tau_O(\tau')} \left(\pi_s^L - \pi_s^B\right) ds \,\middle|\, \mathscr{F}_\tau\right] \geq X(\tau)$$

for all stopping times $\tau' \geq \tau$ (as $\tau_O(\tau') \geq \tau_O(\tau)$), so $X := (X(\tau); \tau \in \mathscr{T})$ is a submartingale-system. X is bounded by $(E[\int_0^\infty (|\pi_s^L| + |\pi_s^B|)\, ds \,|\, \mathscr{F}_\tau]; \tau \in \mathscr{T})$, hence of class (D). In general, the last argument applies separately to $(\pi_s^L - \pi_s^B)^+$ and $(\pi_s^L - \pi_s^B)^-$, showing that X is the difference of two submartingale-systems, which can be aggregated by two optional processes of class (D).

If $\lim \tau_O(\tau^n) = \tau_O(\tau)$ a.s. for any sequence $(\tau^n)_{n\in\mathbb{N}} \subseteq \mathscr{T}$ with $\tau^n \searrow \tau$ a.s., then X, being of class (D), is right-continuous in expectation, and the aggregating submartingales can be chosen with right-continuous paths.

Finally, as the process (Y_t) defined above is continuous, the latest stopping time after τ that attains $F(\tau)$, $\tau_F(\tau)$, is the first time the monotone part of the Snell envelope U_Y increases. That monotone part inherits continuity from (Y_t). Thus chosen, $\tau \leq \tau_F(\tau) \leq \tau_F(\tau')$ on $\{\tau \leq \tau'\}$ for all $\tau, \tau' \in \mathscr{T}$. Now consider a sequence of stopping times $\tau^n \searrow \tau$ a.s., whence also $\tau_F(\tau^n)$ decreases in n. By construction, we can only have $\lim \tau_F(\tau^n) > \tau_F(\tau) \geq \tau$ when the monotone part of U_Y is constant on $(\tau_F(\tau), \lim \tau_F(\tau^n)]$. By continuity, it must then be constant on $[\tau_F(\tau), \lim \tau_F(\tau^n)]$. However, the monotone part of U_Y increases at $\tau_F(\tau)$ by definition, so we must have $\tau_F(\tau) = \lim \tau_F(\tau^n)$ a.s. ☐

Remark A.6. As the proof of Lemma A.5 relies on the aggregation of supermartingales of class (D), we may furthermore assume that the processes (L_t) and (F_t) have left limits at any time t (see El Karoui, 1981, Proposition 2.27).

Lemma A.7. *Let (x_t) be a geometric Brownian motion on $(\Omega, \mathscr{F}, P)$, satisfying*

$$dx_t = \mu x_t \, dt + \sigma x_t \, dB_t$$

for a Brownian motion (B_t) adapted to $(\mathscr{F}_t)$. Moreover, let $\tau_{\tilde{x}} := \inf\{t \geq 0 \,|\, x_t \geq \tilde{x}\}$ for any given constant $\tilde{x} \in \mathbb{R}_+$. Then the problem

$$\sup_{\tau \in \mathscr{T},\, \tau \leq \tau_{\tilde{x}}} E\left[\int_\tau^\infty e^{-rt} (Dx_t - rI)\, dt\right] \tag{A.2}$$

with $r > \max\{\mu, 0\}$, $D \in \mathbb{R}$ and $I > 0$ is solved by $\tau^* := \inf\{t \geq 0 \,|\, x_t \geq \tilde{x} \wedge x^*\}$, where

$$x^* = \frac{\beta_1}{\beta_1 - 1} \cdot \frac{I(r-\mu)}{D^+}$$

and $\beta_1 > 1$ is the positive root of $\frac{1}{2}\sigma^2\beta(\beta-1) + \mu\beta - r = 0$.

Proof. If $D \leq 0$, then the integrand in (A.2) is always negative and the latest feasible stopping time is optimal, which indeed satisfies $\tau_{\tilde{x}} = \tau^*$ as now $x^* = \infty$. For $D > 0$, Lemma A.7 is a special case of Proposition 4.6 in Steg and Thijssen (2015), setting their $Y_0 = Dx_0$, $\mu_Y = \mu$, $\sigma_Y = \sigma$, $X_0 = c_0 = c_B = 0$ and $y_{\mathcal{P}} = (r - \mu_Y)(I - c_A/r) = \tilde{x}$. □

B Proofs

Proof of Proposition 2.3. For (i), let $\vartheta, \vartheta' \in \mathscr{T}$ with $\vartheta \leq \vartheta'$. Consider first two arbitrary extended mixed strategies for each $i = 1, 2$, $\sigma_i^\vartheta \in \mathscr{S}^\vartheta$ and $\sigma_i^{\vartheta'} \in \mathscr{S}^{\vartheta'}$, that are time-consistent (as in Definition A.2). Then $dG_i^\vartheta(t) = (1 - G_i^\vartheta(\vartheta'-))dG_i^{\vartheta'}(t)$ and $(1 - G_i^\vartheta(t)) = (1 - G_i^\vartheta(\vartheta'-))(1 - G_i^{\vartheta'}(t))$ for every $t \geq \vartheta'$. Suppose that in fact $\alpha_i^{\vartheta'}(t) = \mathbf{1}_{\{t \geq \vartheta'\}}\alpha_i^\vartheta(t)$ for all $t \in \mathbb{R}_+$. Then $\hat{\tau}^{\vartheta'} = \hat{\tau}^\vartheta$ on $\{\hat{\tau}^\vartheta \geq \vartheta'\}$, and Definition A.4 implies $\lambda_{L,i}^\vartheta = (1 - G_i^\vartheta(\vartheta'-))(1 - G_j^\vartheta(\vartheta'-))\lambda_{L,i}^{\vartheta'}$; analogously for λ_M^ϑ. Together, the payoffs from Definition A.3 then satisfy

$$\begin{aligned} V_i^\vartheta(\sigma_i^\vartheta, \sigma_j^\vartheta) = E\Big[&\int_{[\vartheta, \hat{\tau}^\vartheta \wedge \vartheta')} \big(1 - G_j^\vartheta(s)\big) L_s^i \, dG_i^\vartheta(s) + \int_{[\vartheta, \hat{\tau}^\vartheta \wedge \vartheta')} \big(1 - G_i^\vartheta(s)\big) F_s^i \, dG_j^\vartheta(s) \\ &+ \sum_{s \in [\vartheta, \hat{\tau}^\vartheta \wedge \vartheta')} \Delta G_i^\vartheta(s) \Delta G_j^\vartheta(s) M_s^i + \mathbf{1}_{\{\hat{\tau}^\vartheta < \vartheta'\}} \Big(\lambda_{L,i}^\vartheta L_{\hat{\tau}^\vartheta}^i + \lambda_{L,j}^\vartheta F_{\hat{\tau}^\vartheta}^i + \lambda_M^\vartheta M_{\hat{\tau}^\vartheta}^i \Big) \\ &+ \big(1 - G_i^\vartheta(\vartheta'-)\big)\big(1 - G_j^\vartheta(\vartheta'-)\big) V_i^{\vartheta'}\big(\sigma_i^{\vartheta'}, \sigma_j^{\vartheta'}\big) \,\Big|\, \mathscr{F}_\vartheta \Big] \end{aligned} \quad \text{(B.1)}$$

(using iterated expectations). The last term needs no indicator for $\{\hat{\tau}^\vartheta \geq \vartheta'\}$, because by Definition A.1 (iii) this case must hold whenever $(1 - G_i^\vartheta(\vartheta'-))(1 - G_j^\vartheta(\vartheta'-)) > 0$.

Now let $\sigma_1^{\vartheta'}, \sigma_2^{\vartheta'} \in \mathscr{S}^{\vartheta'}$ and $\tau_1, \tau_2 \in \mathscr{T}$ with $\tau_1, \tau_2 \in [\vartheta, \vartheta']$, and construct σ_k^ϑ as indicated in Proposition 2.3 for $k = 1, 2$. The latter satisfy Definition A.1: Condition (ii) holds by $\vartheta' \geq \vartheta$ and condition (iii) by $G_k^\vartheta(t) \geq G_k^{\vartheta'}(t)$ for all $t \in \mathbb{R}_+$. G_k^ϑ satisfies condition (i) for the starting time τ_k, because it is the composition of two processes clearly satisfying it, based on the events $\{\tau_k < \vartheta'\}, \{\tau_k = \vartheta'\} \in \mathscr{F}_{\tau_k}$; G_k^ϑ thus satisfies condition (i) also for the starting time $\vartheta \leq \tau_k$. Time-consistency with $G_k^{\vartheta'}$ holds by construction. Now $\hat{\tau}^\vartheta \geq \vartheta'$ and $G_1^\vartheta(t), G_2^\vartheta(t) \in \{0, 1\}$ for $t < \vartheta'$, so (B.1) yields (2.6).

For (ii), suppose that $\sigma_i^{\vartheta'}$ is a best reply for firm i to $\sigma_j^{\vartheta'}$ at ϑ'. To show optimality of σ_i^ϑ against σ_j^ϑ at ϑ, let $\sigma_a^\vartheta \in \mathscr{S}^\vartheta$ be arbitrary. σ_a^ϑ is time-consistent with $\sigma_a^{\vartheta'}$ defined by $G_a^{\vartheta'}(t) = \mathbf{1}_{\{t \geq \vartheta'\}}(\mathbf{1}_{\{G_a^\vartheta(\vartheta'-) < 1\}}(G_a^\vartheta(t) - G_a^\vartheta(\vartheta'-))/(1 - G_a^\vartheta(\vartheta'-)) + \mathbf{1}_{\{G_a^\vartheta(\vartheta'-) = 1\}})$ and $\alpha_a^{\vartheta'}(t) = \mathbf{1}_{\{t \geq \vartheta'\}}\alpha_a^\vartheta(t)$ for every $t \in \mathbb{R}_+$. The latter satisfies Definition A.1 for the starting time ϑ' because $G_a^\vartheta(\vartheta'-)$ is $\mathscr{F}_{\vartheta'}$-measurable, so $V_i^{\vartheta'}(\sigma_a^{\vartheta'}, \sigma_j^{\vartheta'}) \leq V_i^{\vartheta'}(\sigma_i^{\vartheta'}, \sigma_j^{\vartheta'})$. Moreover, as σ_j^ϑ is time-consistent with $\sigma_j^{\vartheta'}$, (B.1) applied with $\hat{\tau}_a^\vartheta := \inf\{t \geq \vartheta \,|\, \alpha_a^\vartheta(t) + \alpha_j^\vartheta(t) > 0\}$ and

$G_j^\vartheta(t) \in \{0,1\}$ for $t < \vartheta'$ yields

$$V_i^\vartheta(\sigma_a^\vartheta, \sigma_j^\vartheta) = E\Big[\int_{[\vartheta, \hat\tau_a^\vartheta \wedge \vartheta' \wedge \tau_j)} L_s^i \, dG_a^\vartheta(s) + \mathbf{1}_{\{\tau_j < \hat\tau_a^\vartheta \wedge \vartheta'\}} \Big(\big(1 - G_a^\vartheta(\tau_j)\big) F_{\tau_j}^i + \Delta G_a^\vartheta(\tau_j) M_{\tau_j}^i \Big)$$
$$+ \mathbf{1}_{\{\hat\tau_a^\vartheta < \vartheta'\}} \Big(\lambda_{L,a}^\vartheta L_{\hat\tau_a^\vartheta}^i + \lambda_{L,j}^\vartheta F_{\hat\tau_a^\vartheta}^i + \lambda_M^\vartheta M_{\hat\tau_a^\vartheta}^i \Big)$$
$$+ \big(1 - G_a^\vartheta(\vartheta'-)\big)\big(1 - G_j^\vartheta(\vartheta'-)\big) V_i^{\vartheta'}(\sigma_a^{\vartheta'}, \sigma_j^{\vartheta'}) \,\Big|\, \mathscr{F}_\vartheta \Big].$$

We aim to eliminate α_a^ϑ for $t < \vartheta'$ without changing the payoff. On $\{\hat\tau_a^\vartheta < \tau_j\}$, $G_j^\vartheta(\hat\tau_a^\vartheta) = 0$ and thus $\lambda_{L,a}^\vartheta = \Delta G_a^\vartheta(\hat\tau_a^\vartheta) = 1 - G_a^\vartheta(\hat\tau_a^\vartheta-)$ by Definition A.4. Now also $dG_a^\vartheta(s) = 0$ for all $s > \hat\tau_a^\vartheta$, so we can move $\mathbf{1}_{\{\hat\tau_a^\vartheta < \tau_j\}} \mathbf{1}_{\{\hat\tau_a^\vartheta < \vartheta'\}} L_{\hat\tau_a^\vartheta}^i \Delta G_a^\vartheta(\hat\tau_a^\vartheta)$ from the second line into the first integral by integrating up to $\vartheta' \wedge \tau_j$. Moreover, then $\lambda_{L,j}^\vartheta = \lambda_M^\vartheta = 0$, so we can multiply the second line by $\mathbf{1}_{\{\hat\tau_a^\vartheta \geq \tau_j\}}$. On $\{\hat\tau_a^\vartheta > \tau_j\}$, $\Delta G_j^\vartheta(\hat\tau_a^\vartheta) = 1 - G_j^\vartheta(\hat\tau_a^\vartheta) = 0$ and thus $\lambda_{L,a}^\vartheta = \lambda_{L,j}^\vartheta = \lambda_M^\vartheta = 0$, so we can multiply the second line also by $\mathbf{1}_{\{\hat\tau_a^\vartheta \leq \tau_j\}}$ or in total by $\mathbf{1}_{\{\hat\tau_a^\vartheta = \tau_j\}}$. Then, on $\{\hat\tau_a^\vartheta = \tau_j < \vartheta'\}$, as $\vartheta' \leq \inf\{t \geq \vartheta \,|\, \alpha_j^{\vartheta'}(t) > 0\} = \inf\{t \geq \vartheta \,|\, \alpha_j^\vartheta(t) > 0\}$, Definition A.4 implies $\lambda_{L,a}^\vartheta = 0$, $\lambda_M^\vartheta = \Delta G_a^\vartheta(\tau_j)\alpha_a^\vartheta(\tau_j) = (1 - G_a^\vartheta(\tau_j-))\alpha_a^\vartheta(\tau_j)$ and $\lambda_{L,j}^\vartheta = (1 - G_a^\vartheta(\tau_j-))(1 - \alpha_a^\vartheta(\tau_j))$. In the third line, $(1 - G_j^\vartheta(\vartheta'-)) = \mathbf{1}_{\{\tau_j \geq \vartheta'\}}$. These steps yield

$$V_i^\vartheta(\sigma_a^\vartheta, \sigma_j^\vartheta) = E\Big[\int_{[\vartheta, \vartheta' \wedge \tau_j)} L_s^i \, dG_a^\vartheta(s) + \mathbf{1}_{\{\tau_j < \hat\tau_a^\vartheta \wedge \vartheta'\}} \Big(\big(1 - G_a^\vartheta(\tau_j)\big) F_{\tau_j}^i + \Delta G_a^\vartheta(\tau_j) M_{\tau_j}^i \Big)$$
$$+ \mathbf{1}_{\{\tau_j = \hat\tau_a^\vartheta < \vartheta'\}} \big(1 - G_a^\vartheta(\tau_j-)\big) \Big(\big(1 - \alpha_a^\vartheta(\tau_j)\big) F_{\tau_j}^i + \alpha_a^\vartheta(\tau_j) M_{\tau_j}^i \Big)$$
$$+ \mathbf{1}_{\{\tau_j \geq \vartheta'\}} \big(1 - G_a^\vartheta(\vartheta'-)\big) V_i^{\vartheta'}(\sigma_a^{\vartheta'}, \sigma_j^{\vartheta'}) \,\Big|\, \mathscr{F}_\vartheta \Big].$$

If we now modify α_a^ϑ by multiplying it with $\mathbf{1}_{\{t \geq \vartheta'\}}$, which does not inflict Definition A.1 and induces $\hat\tau_a^\vartheta \geq \vartheta'$, two things happen. The first indicator switches from 0 to 1 on $\{\hat\tau_a^\vartheta \leq \tau_j < \vartheta'\}$. This has no effect on the subset $\{\hat\tau_a^\vartheta < \tau_j < \vartheta'\}$, because then $\hat\tau_a^\vartheta = \inf\{t \geq \vartheta \,|\, \alpha_a^\vartheta(t) > 0\}$ by $\tau_j \leq \inf\{t \geq \vartheta \,|\, \alpha_j^\vartheta(t) > 0\}$, implying $G_a^\vartheta(\hat\tau_a^\vartheta) = 1$ and $(1 - G_a^\vartheta(\tau_j)) = \Delta G_a^\vartheta(\tau_j) = 0$. On the subset $\{\hat\tau_a^\vartheta = \tau_j < \vartheta'\}$, however, we may gain some payoff. The other effect is that the second line vanishes, where we lose some payoff. Gain and loss can be equated by also modifying G_a^ϑ on $\{\hat\tau_a^\vartheta = \tau_j < \vartheta'\}$, precisely by setting $G_a^\vartheta(t) = G_a^\vartheta(\tau_j-) + \alpha_a^\vartheta(\tau_j)(1 - G_a^\vartheta(\tau_j-))$ for every $t \in [\tau_j, \vartheta')$, where $\alpha_a^\vartheta(\tau_j)$ is the original one.[32] Hence, we may assume $\hat\tau_a^\vartheta \geq \vartheta'$ without changing the payoff.

The cumulative distribution is now dealt with by a change of variable. Defining $\tau_a^G(x) := \inf\{s \geq 0 \,|\, G_a^\vartheta(s) > x\}$ for $x \in [0,1)$, it holds that $\int_{[0,\tau)} L_s^i \, dG_a^\vartheta(s) = \int_0^1 L_{\tau_a^G(x)}^i \mathbf{1}_{\{\tau_a^G(x) \in [0,\tau)\}} \, dx$ (a.s.) for any $\tau \in \mathscr{T}$;

[32] To see that the modified G_a^ϑ is still adapted and consistent with $G_a^{\vartheta'}$, notice that $\{\hat\tau_a^\vartheta = \tau_j < \vartheta'\} \in \mathscr{F}_{\tau_j}$ and that $G_a^\vartheta(\tau_j-)$ and $\alpha_a^\vartheta(\tau_j)$ are $\mathscr{F}_{\tau_j}$-measurable, so we can define an adapted process $G_a^{\tau_j}$ by

$$G_a^{\tau_j}(t) = \mathbf{1}_{\{t \geq \tau_j\}} \mathbf{1}_{\{\hat\tau_a^\vartheta = \tau_j < \vartheta'\}} \Big(G_a^\vartheta(\tau_j-) + \alpha_a^\vartheta(\tau_j)\big(1 - G_a^\vartheta(\tau_j-)\big) \Big) + \mathbf{1}_{\{t \geq \tau_j\}} \Big(1 - \mathbf{1}_{\{\hat\tau_a^\vartheta = \tau_j < \vartheta'\}} \Big) G_a^\vartheta(t).$$

Then the modified G_a^ϑ is the process given by $\mathbf{1}_{\{t \notin [\tau_j, \vartheta')\}} G_a^\vartheta(t) + \mathbf{1}_{\{t \in [\tau_j, \vartheta')\}} G_a^{\tau_j}(t)$ for every $t \in \mathbb{R}_+$ and, thus, also adapted. The latter process is time-consistent with $G_a^{\vartheta'}$, because we have $G_a^\vartheta(\vartheta') = G_a^{\vartheta'}(\vartheta') = 1$ on $\{\hat\tau_a^\vartheta = \tau_j < \vartheta'\}$ due to $G_a^\vartheta(\hat\tau_a^\vartheta) = 1$.

see Lemma B.2 in Riedel and Steg (2017) for details. Treating the other terms analogously yields

$$V_i^\vartheta(\sigma_a^\vartheta,\sigma_j^\vartheta) = E\Big[\int_0^1 L^i_{\tau_a^G(x)}\mathbf{1}_{\{\tau_a^G(x)\in[\vartheta,\vartheta'\wedge\tau_j)\}}\,dx + \mathbf{1}_{\{\tau_j<\vartheta'\}}\int_0^1\Big(F^i_{\tau_j}\mathbf{1}_{\{\tau_a^G(x)\in(\tau_j,\infty]\}} + M^i_{\tau_j}\mathbf{1}_{\{\tau_a^G(x)\in[\tau_j,\tau_j]\}}\Big)\,dx + \mathbf{1}_{\{\tau_j\geq\vartheta'\}}\int_0^1 V_i^{\vartheta'}(\sigma_a^{\vartheta'},\sigma_j^{\vartheta'})\mathbf{1}_{\{\tau_a^G(x)\in[\vartheta',\infty]\}}\,dx\,\Big|\,\mathscr{F}_\vartheta\Big].$$

Using the optimal payoff at ϑ', we obtain an upper bound. Collecting the integrals then yields

$$V_i^\vartheta(\sigma_a^\vartheta,\sigma_j^\vartheta) \leq E\Big[\int_0^1\Big(L^i_{\tau_a^G(x)}\mathbf{1}_{\{\tau_a^G(x)<\vartheta'\wedge\tau_j\}} + \mathbf{1}_{\{\tau_j<\vartheta'\}}\Big(F^i_{\tau_j}\mathbf{1}_{\{\tau_a^G(x)>\tau_j\}} + M^i_{\tau_j}\mathbf{1}_{\{\tau_a^G(x)=\tau_j\}}\Big) + \mathbf{1}_{\{\tau_j\geq\vartheta'\}}V_i^{\vartheta'}(\sigma_i^{\vartheta'},\sigma_j^{\vartheta'})\mathbf{1}_{\{\tau_a^G(x)\geq\vartheta'\}}\Big)\,dx\,\Big|\,\mathscr{F}_\vartheta\Big]. \tag{B.2}$$

As necessarily $\tau_a^G(x)\geq\vartheta$ for every $x\in(0,1]$, we can bound the conditional expectation of the integrand, and thus that of the integral, and therefore also the left-hand side of (B.2) by[33]

$$V_i^\vartheta(\sigma_a^\vartheta,\sigma_j^\vartheta) \leq \operatorname*{ess\,sup}_{\tau\geq\vartheta} E\Big[L^i_\tau\mathbf{1}_{\{\tau<\vartheta'\wedge\tau_j\}} + \mathbf{1}_{\{\tau_j<\vartheta'\}}\Big(F^i_{\tau_j}\mathbf{1}_{\{\tau>\tau_j\}} + M^i_{\tau_j}\mathbf{1}_{\{\tau=\tau_j\}}\Big) + \mathbf{1}_{\{\tau\wedge\tau_j\geq\vartheta'\}}V_i^{\vartheta'}(\sigma_i^{\vartheta'},\sigma_j^{\vartheta'})\,\Big|\,\mathscr{F}_\vartheta\Big].$$

In the essential supremum, the payoff is the same for $\tau\wedge\vartheta'$ in place of τ, and then, as $\tau_j\leq\vartheta'$ and $\{\tau<\vartheta'\wedge\tau_j\}=\{\tau<\tau_j\}\cap\{\tau\wedge\tau_j<\vartheta'\}$, the same as in (2.7). If the latter is attained by τ_i, then (2.6) implies $V_i^\vartheta(\sigma_i^\vartheta,\sigma_j^\vartheta)\geq V_i^\vartheta(\sigma_a^\vartheta,\sigma_j^\vartheta)$ for any other $\sigma_a^\vartheta\in\mathscr{S}^\vartheta$.

As to (iii), note that any σ_k^ϑ as hypothesized is such that $G_k^\vartheta(t)=\mathbf{1}_{\{t\geq\tau_k^\vartheta\}}$, clearly satisfying Definition A.1 (i), and such that $\alpha_k^\vartheta=\alpha_k^{\tau_c(\vartheta)}$, inheriting (ii) from the latter; condition (iii) holds by $\tau_k^\vartheta\leq\tau_c(\vartheta)$. The family $(\alpha_k^\vartheta;\vartheta\in\mathscr{T})$ satisfies the time-consistency property from Definition A.2, because $\alpha_k^\vartheta=\alpha_k^{\tau_c(\vartheta)}$ implies $\alpha_k^\vartheta(t)=\mathbf{1}_{\{t\geq\vartheta\}}\alpha_k^o(t)$ under the hypothesis. The family $(G_k^\vartheta;\vartheta\in\mathscr{T})$ satisfies the time-consistency property from Definition A.2, because $G_k^\vartheta(t)=\mathbf{1}_{\{t\geq\tau_k^\vartheta\}}$ and $(\tau_k^\vartheta;\vartheta\in\mathscr{T})$ satisfies (2.1). □

Proof of Lemma 3.1. Given $\tau_j=\vartheta$ and the hypothesis, (2.7) becomes

$$\operatorname*{ess\,sup}_{\tau\in\mathscr{T},\tau\in[\vartheta,\vartheta']} E\Big[\mathbf{1}_{\{\vartheta<\vartheta'\}}\Big(F^i_\vartheta\mathbf{1}_{\{\tau>\vartheta\}} + M^i_\vartheta\mathbf{1}_{\{\tau=\vartheta\}}\Big) + \mathbf{1}_{\{\vartheta=\vartheta'\}}V_i^{\vartheta'}(\sigma_i^{\vartheta'},\sigma_j^{\vartheta'})\,\Big|\,\mathscr{F}_\vartheta\Big]$$
$$= \operatorname*{ess\,sup}_{\tau\in\mathscr{T},\tau\in[\vartheta,\vartheta']} E\Big[\mathbf{1}_{\{\vartheta<\vartheta'\}}M^i_\vartheta + \mathbf{1}_{\{\vartheta=\vartheta'\}}V_i^{\vartheta'}(\sigma_i^{\vartheta'},\sigma_j^{\vartheta'})\,\Big|\,\mathscr{F}_\vartheta\Big] = \mathbf{1}_{\{\vartheta<\vartheta'\}}M^i_\vartheta + \mathbf{1}_{\{\vartheta=\vartheta'\}}V_i^{\vartheta'}(\sigma_i^{\vartheta'},\sigma_j^{\vartheta'}),$$

which is attained by $\tau_i=\vartheta$, so the claim follows from Proposition 2.3. □

Proof of Lemma 3.2. The stopping problem in (2.2) is equivalent to $\operatorname{ess\,inf}_{\tau'\geq\tau} E[\int_\tau^{\tau'}(\pi_s^{Bi}-\pi_s^{Fi})\,ds\,|\,\mathscr{F}_\tau]$ (up to a constant). Optimality of $\tau_F^i(\tau)$ and iterated expectations thus imply $E[\int_{\tau'}^{\tau_F^i(\tau)}(\pi_s^{Bi}-\pi_s^{Fi})\,ds\,|\,\mathscr{F}_{\tau'}]\leq$

[33]The last step actually follows by way of contradiction, because we are integrating over uncountably many random variables in (B.2) and bound their conditional expectations, each of which is defined only up to nullsets. Specifically, for a process (X_t) of class (D), let $Y_\vartheta := \operatorname{ess\,sup}_{\tau\geq\vartheta}E[X_\tau\,|\,\mathscr{F}_\vartheta]$ and $A=\{Y_\vartheta<E[\int_0^1 X_{\tau_a^G(x)}\,dx\,|\,\mathscr{F}_\vartheta]\}$. If we had $P[A]>0$, then $E[\mathbf{1}_AY_\vartheta]<E[\mathbf{1}_A\int_0^1 X_{\tau_a^G(x)}\,dx]$ by $A\in\mathscr{F}_\vartheta$ and iterated expectations, and hence $E[\mathbf{1}_AY_\vartheta]<\int_0^1 E[\mathbf{1}_AX_{\tau_a^G(x)}]\,dx$ by Fubini. There would thus exist an $x\in(0,1)$ with $E[\mathbf{1}_AY_\vartheta]<E[\mathbf{1}_AX_{\tau_a^G(x)}]=E[\mathbf{1}_AE[X_{\tau_a^G(x)}\,|\,\mathscr{F}_\vartheta]]$, contradicting $Y_\vartheta\geq E[X_{\tau_a^G(x)}\,|\,\mathscr{F}_\vartheta]$.

0 for all $\tau' \in [\tau, \tau_F^i(\tau)]$ and $E[\int_{\tau_F^i(\tau)}^{\tau'} (\pi_s^{Bi} - \pi_s^{Fi})\, ds \,|\, \mathscr{F}_{\tau_F^i(\tau)}] \geq 0$ for all $\tau' \geq \tau_F^i(\tau)$, strictly on $\{\tau' > \tau_F^i(\tau)\}$, as $\tau_F^i(\tau)$ is the latest time attaining (2.2). With $\tau' = \min\{\tau_F^1(\tau), \tau_F^2(\tau)\}$ and $\pi_\cdot^{B2} - \pi_\cdot^{F2} \leq \pi_\cdot^{B1} - \pi_\cdot^{F1}$, we thus have

$$0 \leq E\biggl[\int_{\tau'}^{\tau_F^1(\tau)} (\pi_s^{B2} - \pi_s^{F2})\, ds \,\bigg|\, \mathscr{F}_{\tau'}\biggr] \leq E\biggl[\int_{\tau'}^{\tau_F^1(\tau)} (\pi_s^{B1} - \pi_s^{F1})\, ds \,\bigg|\, \mathscr{F}_{\tau'}\biggr] \leq 0.$$

The first inequality is strict on $\{\tau_F^2(\tau) < \tau_F^1(\tau)\}$ (up to a P-nullset), so $\tau_F^1(\tau) \leq \tau_F^2(\tau)$ (P-a.s.).

Finally, $F_\tau^i - M_\tau^i = \operatorname{ess\,sup}_{\tau' \geq \tau} E[\int_\tau^{\tau'} (\pi_s^{Fi} - \pi_s^{Bi})\, ds \,|\, \mathscr{F}_\tau]$ is not greater for $i = 1$ than for $i = 2$. □

Proof of Lemma 3.4. For any $\tau \in \mathscr{T}$ we have

$$L_\tau^2 - F_\tau^2 = E\biggl[\int_\tau^{\tau_F^1(\tau)} (\pi_s^{L2} - \pi_s^{F2})\, ds + \int_{\tau_F^1(\tau)}^{\tau_F^2(\tau)} (\pi_s^{B2} - \pi_s^{F2})\, ds \,\bigg|\, \mathscr{F}_\tau\biggr] \tag{B.3}$$

and

$$L_\tau^1 - F_\tau^1 = E\biggl[\int_\tau^{\tau_F^1(\tau)} (\pi_s^{L1} - \pi_s^{F1})\, ds + \int_{\tau_F^1(\tau)}^{\tau_F^2(\tau)} (\pi_s^{L1} - \pi_s^{B1})\, ds \,\bigg|\, \mathscr{F}_\tau\biggr],$$

where $\tau_F^1(\tau) \leq \tau_F^2(\tau)$ by Lemma 3.2. By the optimality of $\tau_F^2(\tau)$ for stopping the stream $(\pi_s^{B2} - \pi_s^{F2})$, the second integral on the right-hand side of (B.3) has non-positive conditional expectation, cf. the proof of Lemma 3.2. The first claim now follows from the assumptions $\pi_\cdot^{L1} - \pi_\cdot^{F1} \geq \pi_\cdot^{L2} - \pi_\cdot^{F2}$ and $\pi_\cdot^{L1} \geq \pi_\cdot^{B1}$. The remaining claims then follow from $(L_t^1 - F_t^1) - (L_t^2 - F_t^2) \geq 0$ for all $t \in \mathbb{R}_+$ a.s. by the first claim and right-continuity. □

Proof of Proposition 3.5. The following facts are the main arguments for the proof. First, letting $\tau_{L>F}^i(\tau) = \inf\{t \geq \tau \,|\, L_t^i > F_t^i\}$ for every $\tau \in \mathscr{T}$ and $i = 1, 2$, we have $M_\cdot^i \leq L_\cdot^i \leq F_\cdot^i$ on $[\tau, \tau_{L>F}^i(\tau))$ by $\pi_\cdot^{Li} \geq \pi_\cdot^{Bi}$ and definition of $\tau_{L>F}^i(\tau)$, respectively. Second, for all $\tau, \tau' \in \mathscr{T}$ with $\tau \leq \tau' \leq \tau_F^i(\tau)$ we have $F_\tau^i \leq E[F_{\tau'}^i \,|\, \mathscr{F}_\tau]$, because $\tau_F^i(\tau) = \tau_F^i(\tau')$ attains both F_τ^i, $F_{\tau'}^i$, and $\pi_\cdot^{0i} \geq \pi_\cdot^{Fi}$. Third, if $\tau = \tau_F^i(\tau)$, then firm i's payoff from the strategy corresponding to the pure plan $\tau_i = \tau$, i.e., σ_i^τ given by $G_i^\tau(t) = \mathbf{1}_{\{t \geq \tau\}}$ and $\alpha_i^\tau(t) = 0$ for every $t \in \mathbb{R}_+$, yields at least the payoff F_τ^i. Indeed, then $L_\tau^i \geq M_\tau^i = F_\tau^i$ by $\pi_\cdot^{Li} \geq \pi_\cdot^{Bi}$ and τ attaining F_τ^i, so for any strategy $\sigma_j^\tau \in \mathscr{S}^\tau$, $V_i^\tau(\sigma_i^\tau, \sigma_j^\tau)$ is $(1 - G_j^\tau(\tau))L_\tau^i + G_j^\tau(\tau)M_\tau^i \geq F_\tau^i$ on $\{\tau < \hat\tau^\tau\}$, and on $\{\tau = \hat\tau^\tau\}$ it is at least $F_\tau^i(\lambda_{L,i}^\tau + \lambda_{L,j}^\tau + \lambda_M^\tau) = F_\tau^i(1 - G_i^\tau(\hat\tau^\tau-))(1 - G_j^\tau(\hat\tau^\tau-)) = F_\tau^i$. Thanks to the minimal payoff $F_{\tau_F^i(\tau)}^i$ at $\tau_F^i(\tau)$, firm i can also ensure the minimal payoff F_τ^i for $\tau \leq \tau_F^i(\tau)$, in particular if $\tau = \min\{\tau_{L>F}^i(\vartheta), \tau_F^i(\vartheta)\}$ for some $\vartheta \in \mathscr{T}$. Showing this and completing the proof for pure strategies would be mainly a matter of suitable conditional expectations. For extended mixed strategies, however, we need tools like in the proof of Proposition 2.3 in order to apply the same arguments.

We start with the third claim. Let any $\vartheta \in \mathscr{T}$, $i, j \in \{1, 2\}$, $i \neq j$, and $\sigma_i^\vartheta, \sigma_j^\vartheta \in \mathscr{S}^\vartheta$ be given and set $\vartheta' = \min\{\tau_{L>F}^i(\vartheta), \tau_F^i(\vartheta)\}$. We want to show that it is no loss for firm i to remain idle on $[\vartheta, \vartheta')$. First, to ensure a sufficient continuation payoff, construct $\sigma_i^{\vartheta'}$ that is time-consistent with σ_i^ϑ by $G_i^{\vartheta'}(t) = \mathbf{1}_{\{t \geq \vartheta'\}}(\mathbf{1}_{\{G_i^\vartheta(\vartheta'-) < 1\}}(G_i^\vartheta(t) - G_i^\vartheta(\vartheta'-))/(1 - G_i^\vartheta(\vartheta'-)) + \mathbf{1}_{\{G_i^\vartheta(\vartheta'-) = 1\}})$ and $\alpha_i^{\vartheta'}(t) = \mathbf{1}_{\{t \geq \vartheta'\}}\alpha_i^\vartheta(t)$ for every $t \in \mathbb{R}_+$. Construct $\sigma_j^{\vartheta'}$ likewise, and let $A_{V<F}^{\vartheta'}$ denote the event $\{V_i^{\vartheta'}(\sigma_i^{\vartheta'}, \sigma_j^{\vartheta'}) < F_{\vartheta'}^i\} \in \mathscr{F}_{\vartheta'}$. Now define the alternative continuation strategy $\sigma_a^{\vartheta'} \in \mathscr{S}^{\vartheta'}$ by

$$G_a^{\vartheta'}(t) = \mathbf{1}_{\{\vartheta' > \vartheta\}}\mathbf{1}_{A_{V<F}^{\vartheta'}}\mathbf{1}_{\{t \geq \tau_F^i(\vartheta')\}} + \Bigl(1 - \mathbf{1}_{\{\vartheta' > \vartheta\}}\mathbf{1}_{A_{V<F}^{\vartheta'}}\Bigr)G_i^{\vartheta'}(t)$$

and $\alpha_a^{\vartheta'}(t) = (1 - \mathbf{1}_{\{\vartheta' > \vartheta\}}\mathbf{1}_{A_{V<F}^{\vartheta'}})\mathbf{1}_{\{t \geq \vartheta'\}}\alpha_i^\vartheta(t)$ for every $t \in \mathbb{R}_+$. $\sigma_a^{\vartheta'}$ agrees with $\sigma_i^{\vartheta'}$ on $\{\vartheta' = \vartheta\}$ and off $A_{V<F}^{\vartheta'}$, and then the continuation payoff is $V_i^{\vartheta'}(\sigma_i^{\vartheta'}, \sigma_j^{\vartheta'})$. Otherwise, on $\{\vartheta' > \vartheta\} \cap A_{V<F}^{\vartheta'}$, it is

$V_i^{\vartheta'}(\sigma_a^{\vartheta'},\sigma_j^{\vartheta'}) \geq F^i_{\vartheta'}$. The latter bound can be obtained along the lines followed below, but needs fewer details, so we save spelling out the argument; it uses that the continuation payoff at $\tau^i_F(\vartheta')$ is at least $F^i_{\tau^i_F(\vartheta')}$ as shown at the beginning of the proof.

Next, set $\sigma_a^\vartheta = \sigma_a^{\vartheta'} \in \mathscr{S}^{\vartheta'} \subseteq \mathscr{S}^\vartheta$, which does not charge $[\vartheta,\vartheta')$ and agrees with σ_i^ϑ on $\{\vartheta'=\vartheta\}$. In perspective of (B.1), note that, although the continuation payoffs from $\sigma_a^{\vartheta'}$ and $\sigma_i^{\vartheta'}$ are also the same off $A^{\vartheta'}_{V<F}$, they can then accrue with different probabilities, because $G_a^\vartheta(\vartheta'-) = 0 \leq G_i^\vartheta(\vartheta'-)$ can be strict on $\{\vartheta'>\vartheta\}$. Now consider $\hat\tau_j^\vartheta = \inf\{t\geq\vartheta \,|\, \alpha_j^\vartheta(t)>0\}$. We have $\hat\tau_j^\vartheta < \inf\{t\geq\vartheta\,|\,\alpha_a^\vartheta(t)>0\}$ on $\{\hat\tau_j^\vartheta<\vartheta'\}$ and, as then $\vartheta<\vartheta'$ and thus $\Delta G_a^\vartheta(\hat\tau_j^\vartheta)=0$, the probability that firm i obtains $L^i_\cdot$ or $M^i_\cdot$ at $\hat\tau_j^\vartheta$ is null by Definition A.4, whereas $F^i_\cdot$ is obtained with probability $(1-G_a^\vartheta(\hat\tau_j^\vartheta-))(1-G_j^\vartheta(\hat\tau_j^\vartheta-)) = \Delta G_j^\vartheta(\hat\tau_j^\vartheta)$. On $\{\hat\tau_j^\vartheta\geq\vartheta'\}$, we have $\vartheta'\leq\inf\{t\geq\vartheta\,|\,\alpha_a^\vartheta(t)+\alpha_j^\vartheta(t)>0\}$ and the continuation payoff applies. (B.1) thus yields

$$\begin{aligned}
&V_i^\vartheta(\sigma_i^\vartheta,\sigma_j^\vartheta) - V_i^\vartheta(\sigma_a^\vartheta,\sigma_j^\vartheta)\\
&= E\bigg[\mathbf{1}_{\{\vartheta'>\vartheta\}}\bigg(\int_{[\vartheta,\hat\tau^\vartheta\wedge\vartheta')}\big(1-G_j^\vartheta(s)\big)L^i_s\,dG_i^\vartheta(s) + \int_{[\vartheta,\hat\tau^\vartheta\wedge\vartheta')}\big(1-G_i^\vartheta(s)\big)F^i_s\,dG_j^\vartheta(s)\\
&\quad+\sum_{s\in[\vartheta,\hat\tau^\vartheta\wedge\vartheta')}\Delta G_i^\vartheta(s)\Delta G_j^\vartheta(s)M^i_s + \mathbf{1}_{\{\hat\tau^\vartheta<\vartheta'\}}\Big(\lambda^\vartheta_{L,i}L^i_{\hat\tau^\vartheta}+\lambda^\vartheta_{L,j}F^i_{\hat\tau^\vartheta}+\lambda^\vartheta_M M^i_{\hat\tau^\vartheta}\Big)\\
&\quad-\int_{[\vartheta,\hat\tau_j^\vartheta\wedge\vartheta')}F^i_s\,dG_j^\vartheta(s) - \mathbf{1}_{\{\hat\tau_j^\vartheta<\vartheta'\}}\Delta G_j^\vartheta(\hat\tau_j^\vartheta)F^i_{\hat\tau_j^\vartheta} + \mathbf{1}_{A^{\vartheta'}_{V<F}}\big(1-G_i^\vartheta(\vartheta'-)\big)\big(1-G_j^\vartheta(\vartheta'-)\big)V_i^{\vartheta'}(\sigma_i^{\vartheta'},\sigma_j^{\vartheta'})\\
&\quad-\Big(1-\mathbf{1}_{A^{\vartheta'}_{V<F}}\Big)G_i^\vartheta(\vartheta'-)\big(1-G_j^\vartheta(\vartheta'-)\big)V_i^{\vartheta'}(\sigma_i^{\vartheta'},\sigma_j^{\vartheta'}) - \mathbf{1}_{A^{\vartheta'}_{V<F}}\big(1-G_j^\vartheta(\vartheta'-)\big)V_i^{\vartheta'}(\sigma_a^{\vartheta'},\sigma_j^{\vartheta'})\bigg)\,\bigg|\,\mathscr{F}_\vartheta\bigg].
\end{aligned}$$

Now we can apply the estimate $F^i_\cdot \geq L^i_\cdot \geq M^i_\cdot$ on $[\vartheta,\vartheta')$ and collect the first six terms in the first two integrals. The sum, when written as an integral, becomes $\int_{[0,\hat\tau^\vartheta\wedge\vartheta')}\Delta G_j^\vartheta(s)F^i_s\,dG_i^\vartheta(s)$ and can be included in the first integral, becoming $\int_{[0,\hat\tau^\vartheta\wedge\vartheta')}(1-G_j^\vartheta(s-))F^i_s\,dG_i^\vartheta(s)$. Next, it always holds that

$$\begin{aligned}
\lambda^\vartheta_{L,i}+\lambda^\vartheta_{L,j}+\lambda^\vartheta_M &= \big(1-G_i^\vartheta(\hat\tau^\vartheta-)\big)\big(1-G_j^\vartheta(\hat\tau^\vartheta-)\big)\\
&= \big(1-G_i^\vartheta(\hat\tau^\vartheta)\big)\big(1-G_j^\vartheta(\hat\tau^\vartheta-)\big) + \Delta G_i^\vartheta(\hat\tau^\vartheta)\big(1-G_j^\vartheta(\hat\tau^\vartheta-)\big)\\
&= \big(1-G_i^\vartheta(\hat\tau^\vartheta)\big)\Delta G_j^\vartheta(\hat\tau^\vartheta) + \Delta G_i^\vartheta(\hat\tau^\vartheta)\big(1-G_j^\vartheta(\hat\tau^\vartheta-)\big),
\end{aligned}$$

where the last step is due to $G_i^\vartheta(\hat\tau^\vartheta)<1 \Rightarrow G_j^\vartheta(\hat\tau^\vartheta)=1$. The estimate of the fourth term can thus be included in the first two integrals by integrating over $[\vartheta,\vartheta')$ in each. The fifth and sixth term can also be combined as one integral over $[\vartheta,\vartheta')$ and then included in the second. Using also the definition of $A^{\vartheta'}_{V<F}$ in the second and third to last terms and $V_i^{\vartheta'}(\sigma_a^{\vartheta'},\sigma_j^{\vartheta'})\geq F^i_{\vartheta'}$ on $\{\vartheta'>\vartheta\}\cap A^{\vartheta'}_{V<F}$, we obtain

$$\begin{aligned}
&V_i^\vartheta(\sigma_i^\vartheta,\sigma_j^\vartheta) - V_i^\vartheta(\sigma_a^\vartheta,\sigma_j^\vartheta) \qquad\qquad \text{(B.4)}\\
&\leq E\bigg[\mathbf{1}_{\{\vartheta'>\vartheta\}}\bigg(\int_{[\vartheta,\vartheta')}\big(1-G_j^\vartheta(s-)\big)F^i_s\,dG_i^\vartheta(s) - \int_{[\vartheta,\vartheta')}G_i^\vartheta(s)F^i_s\,dG_j^\vartheta(s) - G_i^\vartheta(\vartheta'-)\big(1-G_j^\vartheta(\vartheta'-)\big)F^i_{\vartheta'}\bigg)\,\bigg|\,\mathscr{F}_\vartheta\bigg].
\end{aligned}$$

In order to estimate the expectation on the right-hand side, we can perform a change of variable on G_i^ϑ and G_j^ϑ as in the proof of Proposition 2.3 with $\tau_i^G(x) = \inf\{s\geq 0\,|\,G_i^\vartheta(s)>x\}$ and $\tau_j^G(y)$ analogously for $x,y\in[0,1)$. The expectation then becomes

$$\begin{aligned}
&E\bigg[\mathbf{1}_{\{\vartheta'>\vartheta\}}\bigg(\int_0^1\mathbf{1}_{\{\tau_i^G(x)\in[\vartheta,\vartheta')\}}\big(1-G_j^\vartheta(\tau_i^G(x)-)\big)F^i_{\tau_i^G(x)}\,dx - \int_0^1\mathbf{1}_{\{\tau_j^G(y)\in[\vartheta,\vartheta')\}}G_i^\vartheta(\tau_j^G(y))F^i_{\tau_j^G(y)}\,dy\\
&\quad-\int_0^1\mathbf{1}_{\{\tau_i^G(x)\in[\vartheta,\vartheta')\}}\big(1-G_j^\vartheta(\vartheta'-)\big)F^i_{\vartheta'}\,dx\bigg)\,\bigg|\,\mathscr{F}_\vartheta\bigg]
\end{aligned}$$

$$
\begin{aligned}
&= E\Big[\mathbf{1}_{\{\vartheta'>\vartheta\}}\Big(\int_0^1\int_0^1 \mathbf{1}_{\{\tau_i^G(x)\in[\vartheta,\vartheta')\}}\mathbf{1}_{\{\tau_j^G(y)\in[\tau_i^G(x),\infty]\}}F^i_{\tau_i^G(x)}\,dy\,dx \\
&\quad -\int_0^1\int_0^1 \mathbf{1}_{\{\tau_j^G(y)\in[\vartheta,\vartheta')\}}\mathbf{1}_{\{\tau_i^G(x)\in[\vartheta,\tau_j^G(y)]\}}F^i_{\tau_j^G(y)}\,dx\,dy \\
&\quad -\int_0^1\int_0^1 \mathbf{1}_{\{\tau_i^G(x)\in[\vartheta,\vartheta')\}}\mathbf{1}_{\{\tau_j^G(y)\in[\vartheta',\infty]\}}F^i_{\vartheta'}\,dy\,dx\Big)\Big|\mathscr{F}_\vartheta\Big] \\
&= E\Big[\mathbf{1}_{\{\vartheta'>\vartheta\}}\int_0^1\int_0^1\Big(\mathbf{1}_{\{\tau_i^G(x)<\vartheta'\}}\mathbf{1}_{\{\tau_j^G(y)\geq\tau_i^G(x)\}}F^i_{\tau_i^G(x)}-\mathbf{1}_{\{\tau_j^G(y)<\vartheta'\}}\mathbf{1}_{\{\tau_i^G(x)\leq\tau_j^G(y)\}}F^i_{\tau_j^G(y)} \\
&\quad -\mathbf{1}_{\{\tau_i^G(x)<\vartheta'\}}\mathbf{1}_{\{\tau_j^G(y)\geq\vartheta'\}}F^i_{\vartheta'}\Big)\,dx\,dy\Big|\mathscr{F}_\vartheta\Big] \\
&= E\Big[\int_0^1\int_0^1 \mathbf{1}_{\{\vartheta'>\vartheta\}}\Big(\mathbf{1}_{\{\tau_i^G(x)<\vartheta'\}}\mathbf{1}_{\{\tau_i^G(x)\leq\tau_j^G(y)\}}\Big(F^i_{\tau_i^G(x)}-F^i_{\tau_j^G(y)\wedge\vartheta'}\Big)\Big)\,dx\,dy\Big|\mathscr{F}_\vartheta\Big].
\end{aligned}
$$

For any $x,y\in[0,1)$, the integrand has nonpositive $\mathscr{F}_\vartheta$-conditional expectation, because we can use the estimate $F^i_{\tau_i^G(x)}\leq E[F^i_{\tau_j^G(y)\wedge\vartheta'}\,|\,\mathscr{F}_{\tau_i^G(x)}]$ on $\{\tau_i^G(x)\leq\tau_j^G(y)\wedge\vartheta'\}$ (as noted at the beginning of the proof) with iterated expectations thanks to $\{\vartheta'>\vartheta\}\cap\{\tau_i^G(x)<\vartheta'\}\cap\{\tau_i^G(x)\leq\tau_j^G(y)\}\in\mathscr{F}_{\tau_i^G(x)}$; the conditional expectation of the (double) integral is thus nonpositive as well (cf. fn. 33). As $\sigma_a^\vartheta=\sigma_a^{\vartheta'}\in\mathscr{S}^{\vartheta'}$, the proof of the first part of the third claim is complete. The second part is an immediate consequence of the first one and (B.1).

The second claim now follows from $\sigma_i^\vartheta=\sigma_i^{\vartheta'}$ for $\tau_i=\vartheta'$ in Proposition 2.3 and time-consistency of σ_j^ϑ with $\sigma_j^{\vartheta'}$ for any $\tau_j\in\mathscr{T}$ with $\tau_j\in[\vartheta,\vartheta')$.

As to the first claim, suppose σ_i^ϑ is a best reply to $\sigma_j^\vartheta\in\mathscr{S}^\vartheta$ at $\vartheta\in\mathscr{T}$ for some $i,j\in\{1,2\}$, $i\neq j$. We want to show that firm i never obtains the payoffs $L^i_\cdot$ or $M^i_\cdot$ while $F^i_\cdot>L^i_\cdot$, i.e.,

$$
E\Big[\int_{[\vartheta,\hat\tau^\vartheta)}\big(1-G_j^\vartheta(s)\big)\mathbf{1}_{\{F_s^i>L_s^i\}}\,dG_i^\vartheta(s)+\sum_{s\in[\vartheta,\hat\tau^\vartheta)}\Delta G_i^\vartheta(s)\Delta G_j^\vartheta(s)\mathbf{1}_{\{F_s^i>L_s^i\}}+\Big(\lambda_{L,i}^\vartheta+\lambda_M^\vartheta\Big)\mathbf{1}_{\{F^i_{\hat\tau^\vartheta}>L^i_{\hat\tau^\vartheta}\}}\Big]=0.
$$

Suppose by way of contradiction that the left-hand side is positive. We are going to use the arguments from the second claim, but we need to find a suitable starting point as we cannot guarantee that a violation occurs (already) on $[\vartheta,\vartheta')$ as defined before. Therefore, rewrite the sum on the left-hand side as $\int_{[\vartheta,\hat\tau^\vartheta)}\Delta G_j^\vartheta(s)\mathbf{1}_{\{F_s^i>L_s^i\}}\,dG_i^\vartheta(s)$ and combine it with the integral. A change of variable like before on G_i^ϑ in the combined integral, using again $\tau_i^G(x)$, yields

$$
E\Big[\int_{[\vartheta,\hat\tau^\vartheta)}\big(1-G_j^\vartheta(s-)\big)\mathbf{1}_{\{F_s^i>L_s^i\}}\,dG_i^\vartheta(s)\Big]=\int_0^1 E\Big[\mathbf{1}_{\{\tau_i^G(x)<\hat\tau^\vartheta\}}\big(1-G_j^\vartheta(\tau_i^G(x)-)\big)\mathbf{1}_{\{F^i_{\tau_i^G(x)}>L^i_{\tau_i^G(x)}\}}\Big]\,dx. \tag{B.5}
$$

If this is positive, then the last expectation must be positive for all x from a subset of $[0,1)$ with positive measure. In this case, fix an x_0 from that subset such that the subset still has positive measure on $(x_0,1)$, and set $\tau_i=\tau_i^G(x_0)$. Otherwise, by hypothesis $\lambda_{L,i}^\vartheta+\lambda_M^\vartheta>0$ with positive probability on $\{F^i_{\hat\tau^\vartheta}>L^i_{\hat\tau^\vartheta}\}$; in this case, set $\tau_i=\hat\tau^\vartheta$. In order to use the estimate (B.4) – to show that it is strict – suppose first $\tau_i=\vartheta$. We then have in either case $F^i_\vartheta>L^i_\vartheta$ with positive probability, which implies $\vartheta<\vartheta'=\min\{\tau^i_{L>F}(\vartheta),\tau^i_F(\vartheta)\}$ as noted at the beginning of the proof. In the second case, the estimate (B.4) is then strict with positive probability, as also $M^i_\vartheta\leq L^i_\vartheta<F^i_\vartheta$. To see that it is also strict in the first case, note that we may then assume $G_j^\vartheta(\vartheta)<1$ or $\Delta G_i^\vartheta(\vartheta)>0$ with positive probability on $\{F^i_\vartheta>L^i_\vartheta\}$. Indeed, if $\Delta G_i^\vartheta(\vartheta)=0$ and $G_j^\vartheta(\vartheta)=1$ on that set, then $\tau_i^G(x)>\vartheta$ for all $x\in(x_0,1)$ and thus $G_j^\vartheta(\tau_i^G(x)-)=1$, which contradicts the definition of x_0. By definition of x_0, ϑ additionally satisfies $\vartheta<\hat\tau^\vartheta$ (still with positive probability in the first case). Therefore, the estimate of the first integral in (B.4) becomes strict. Indeed, by right-continuity we still have $F^i_\cdot > L^i_\cdot \geq M^i_\cdot$ on an interval to the right of ϑ. If $G^\vartheta_j(\vartheta) < 1$, then $G^\vartheta_j(t) < 1$ on an interval by right-continuity, which is charged by dG^ϑ_i because $G^\vartheta_i(t) > G^\vartheta_i(\vartheta-)$ for all $t > \vartheta$ by definition of $\tau^G_i(x_0)$; otherwise, at least $G^\vartheta_j(\vartheta-) < 1$, and we must have $\Delta G^\vartheta_i(\vartheta) > 0$ (all with positive probability). We would thus obtain $E[V^\vartheta_i(\sigma^\vartheta_i, \sigma^\vartheta_j) - \operatorname{ess\,sup}_{\sigma^{\vartheta'}_b \in \mathscr{S}^{\vartheta'}} V^\vartheta_i(\sigma^{\vartheta'}_b, \sigma^\vartheta_j)] < 0$, contradicting the hypothesis. Finally, to remove the assumption $\tau_i = \vartheta$, note that in both cases $(1-G^\vartheta_i(\tau_i-))(1-G^\vartheta_j(\tau_i-)) > 0$ with positive probability – in the second case, because always $\lambda^\vartheta_M + \lambda^\vartheta_{L,i} + \lambda^\vartheta_{L,j} = (1-G^\vartheta_i(\hat\tau^\vartheta-))(1-G^\vartheta_j(\hat\tau^\vartheta-))$ and $\lambda^\vartheta_{L,j} \geq 0$. Therefore, if we define time-consistent continuation strategies $\sigma^{\tau_i}_i$ and $\sigma^{\tau_i}_j$ as before by $G^{\tau_i}_i(t) = \mathbf{1}_{\{G^\vartheta_i(\tau_i-)<1\}}(G^\vartheta_i(t) - G^\vartheta_i(\tau_i-))/(1 - G^\vartheta_i(\tau_i-)) + \mathbf{1}_{\{G^\vartheta_i(\tau_i-)=1\}}\mathbf{1}_{\{t\geq\tau_i\}}$ and $\alpha^{\tau_i}_i(t) = \mathbf{1}_{\{t\geq\tau_i\}}\alpha^\vartheta_i(t)$, and analogously for j, then (B.1) implies that $\sigma^{\tau_i}_i$ must be a best reply to $\sigma^{\tau_i}_j$. But now, $\lambda^{\tau_i}_{L,i} + \lambda^{\tau_i}_M = (\lambda^\vartheta_{L,i} + \lambda^\vartheta_M)/[(1 - G^\vartheta_i(\tau_i-))(1 - G^\vartheta_j(\tau_i-))]$ whenever the denominator is positive, which implies also $\tau_i \leq \hat\tau^\vartheta = \hat\tau^{\tau_i}$ as well as $(1 - G^{\tau_i}_j(s-))dG^{\tau_i}_i(s) = (1 - G^\vartheta_j(s-))dG^\vartheta_i(s)/[(1 - G^\vartheta_j(\tau_i-))(1 - G^\vartheta_i(\tau_i-))]$ for all $s \geq \tau_i$. By definition of x_0, if B.5 is positive, then the integral on the right-hand side is still positive over $x \in [x_0, 1)$, and it is equivalent to the integral on the left-hand side over $[\tau_i, \hat\tau^\vartheta)$. Now all the previous arguments apply at τ_i. ☐

Proof of Lemma 3.8. First, note that there are solutions $\tau^i_\Delta \leq \tau^i_F(\vartheta) \leq \tau^2_F(\vartheta)$ to (3.6) for $i = 1, 2$, as the respective process to be stopped is continuous and integrable. The estimate follows from the assumption $\pi^{Li}_\cdot - \pi^{Fi}_\cdot \geq \pi^{Bi}_\cdot - \pi^{Fi}_\cdot$, cf. the proof of Lemma 3.2.

By the optimality of τ^i_Δ in (3.6), $E[\int_\vartheta^{\tau^i_\Delta}(\pi^{Li}_s - \pi^{Fi}_s)\,ds \,|\, \mathscr{F}_\vartheta] \leq 0$. Therefore, as $\pi^{L2}_\cdot - \pi^{F2}_\cdot \leq \pi^{Li}_\cdot - \pi^{Fi}_\cdot$, (B.3) can only be strictly positive if

$$E\Big[\int_{\tau^i_\Delta}^{\tau^1_F(\vartheta)} (\pi^{L2}_s - \pi^{F2}_s)\,ds + \int_{\tau^1_F(\vartheta)}^{\tau^2_F(\vartheta)} (\pi^{B2}_s - \pi^{F2}_s)\,ds \,\Big|\, \mathscr{F}_\vartheta\Big] > 0$$

(which can in fact only be the case if $P[\tau^i_\Delta < \tau^1_F(\vartheta)] > 0$), and which implies

$$E\Big[L^2_{\tau^i_\Delta} - F^2_{\tau^i_\Delta} \,\Big|\, \mathscr{F}_\vartheta\Big] = E\Big[\int_{\tau^i_\Delta}^{\tau^1_F(\tau^i_\Delta)} (\pi^{L2}_s - \pi^{F2}_s)\,ds + \int_{\tau^1_F(\tau^i_\Delta)}^{\tau^2_F(\vartheta)} (\pi^{B2}_s - \pi^{F2}_s)\,ds \,\Big|\, \mathscr{F}_\vartheta\Big] > 0,$$

because $\tau^1_F(\tau^i_\Delta) \geq \tau^1_F(\vartheta)$, $\tau^2_F(\tau^i_\Delta) = \tau^2_F(\vartheta)$ and $\pi^{L2}_\cdot \geq \pi^{B2}_\cdot$.

For all stopping times $\tau \in [\vartheta, \tau^1_F(\vartheta)]$, indeed $\tau^i_F(\tau) = \tau^i_F(\vartheta)$, $i = 1, 2$, and thus $L^2_\vartheta - F^2_\vartheta - E[L^2_\tau - F^2_\tau \,|\, \mathscr{F}_\vartheta] = E[\int_\vartheta^\tau (\pi^{L2}_s - \pi^{F2}_s)\,ds \,|\, \mathscr{F}_\vartheta] \geq 0$ if $\tau^2_\Delta = \vartheta$ attains (3.6). ☐

Proof of Proposition 3.9. For the first claim, it suffices by Proposition 2.3 to verify that τ^i_* maximizes $E[L^i_{\tau^i}\mathbf{1}_{\{\tau^i<\tau^j_*\}} + M^i_{\tau^i}\mathbf{1}_{\{\tau^i=\tau^j_*\}} + F^i_{\tau^j_*}\mathbf{1}_{\{\tau^i>\tau^j_*\}} \,|\, \mathscr{F}_\vartheta] \leq E[L^i_{\tau^i}\mathbf{1}_{\{\tau^i<\tau^j_*\}} + F^i_{\tau^j_*}\mathbf{1}_{\{\tau^i_*\geq\tau^j_*\}} \,|\, \mathscr{F}_\vartheta]$ over all stopping times $\tau^i \geq \vartheta$. The right-hand side is attainable by the stopping time $\tau^i\mathbf{1}_{\{\tau^i<\tau^j_*\}} + \infty\mathbf{1}_{\{\tau^i_*\geq\tau^j_*\}}$, so τ^i_* is a best reply to τ^j_* if and only if $F^i_{\tau^j_*} = M^i_{\tau^j_*}$ on $\{\tau^i_* = \tau^j_*\}$ and $\tau = \tau^j_*$ attains

$$\operatorname*{ess\,sup}_{\vartheta\leq\tau\leq\tau^j_*} E\Big[L^i_\tau\mathbf{1}_{\{\tau<\tau^j_*\}} + F^i_{\tau^j_*}\mathbf{1}_{\{\tau\geq\tau^j_*\}} \,\Big|\, \mathscr{F}_\vartheta\Big].$$

By iterated expectations, this is equivalent to $L^i_{\vartheta'} - E[F^i_{\tau^j_*} \,|\, \mathscr{F}_{\vartheta'}] \leq 0$ on $\{\vartheta' < \tau^j_*\}$ for all stopping times $\vartheta' \geq \vartheta$. To establish the latter under conditions (i) and (ii), fix arbitrary $\vartheta' \geq \vartheta$ and let $\tau^i_D(\vartheta') \in \mathscr{T}$ attain (3.7) (such $\tau^i_D(\vartheta')$ exists by continuity and integrability of the process to be stopped), whence $E[\int_{\vartheta'}^{\tau^i_D(\vartheta')}(\pi^{Li}_s - \pi^{0i}_s)\,ds \,|\, \mathscr{F}_{\vartheta'}] \leq 0$. On $\{\vartheta' < \tau^j_*\}$, we then have

$$L^i_{\vartheta'} - E\Big[M^i_{\tau^j_*} \,\Big|\, \mathscr{F}_{\vartheta'}\Big] = E\Big[\int_{\vartheta'}^{\tau^j_F(\vartheta')} (\pi^{Li}_s - \pi^{0i}_s)\,ds + \int_{\tau^j_F(\vartheta')}^{\tau^j_*} (\pi^{Bi}_s - \pi^{0i}_s)\,ds \,\Big|\, \mathscr{F}_{\vartheta'}\Big] \tag{B.6}$$

$$
\begin{aligned}
&\le E\Big[\int_{\vartheta'}^{\tau_F^j(\vartheta')\vee\tau_D^i(\vartheta')}(\pi_s^{Li}-\pi_s^{0i})\,ds+\int_{\tau_F^j(\vartheta')\vee\tau_D^i(\vartheta')}^{\tau_*^j}(\pi_s^{Bi}-\pi_s^{0i})\,ds\,\Big|\,\mathscr{F}_{\vartheta'}\Big]\\
&\le E\Big[\int_{\tau_D^i(\vartheta')}^{\tau_F^j(\vartheta')\vee\tau_D^i(\vartheta')}(\pi_s^{Li}-\pi_s^{0i})\,ds+\int_{\tau_F^j(\vartheta')\vee\tau_D^i(\vartheta')}^{\tau_*^j}(\pi_s^{Bi}-\pi_s^{0i})\,ds\,\Big|\,\mathscr{F}_{\vartheta'}\Big]\\
&=E\Big[\mathbf{1}_{\{\tau_D^i(\vartheta')<\tau_F^j(\vartheta')\}}\Big(L^i_{\tau_D^i(\vartheta')}-M^i_{\tau_*^j}\Big)+\mathbf{1}_{\{\tau_D^i(\vartheta')\ge\tau_F^j(\vartheta')\}}\Big(M^i_{\tau_D^i(\vartheta')}-M^i_{\tau_*^j}\Big)\,\Big|\,\mathscr{F}_{\vartheta'}\Big].
\end{aligned}
$$

The first equality uses the convention $\int_b^a\cdot\,ds=-\int_a^b\cdot\,ds$ for $a<b$. The first inequality is due to $\pi_\cdot^{Li}\ge\pi_\cdot^{Bi}$ and the second due to the optimality of $\tau_D^i(\vartheta')$. The last equality is analogous to the first, using iterated expectations and $\tau_D^i(\vartheta')<\tau_F^j(\vartheta')\Rightarrow\tau_F^j(\tau_D^i(\vartheta'))=\tau_F^j(\vartheta')$. After replacing $M^i_{\tau_*^j}$ by $F^i_{\tau_*^j}$ in the first and last terms of (B.6), conditions (i) and (ii) make the last nonpositive (taking iterated expectations at $\tau_D^i(\vartheta')$), and thus also $L^i_{\vartheta'}-E[F^i_{\tau_*^j}\,|\,\mathscr{F}_{\vartheta'}]\le 0$.

To prove the next claim, note that, for any stopping time $\tau\in[\vartheta',\tau_F^j(\vartheta')]$, we have $\tau_F^j(\tau)=\tau_F^j(\vartheta')$ and thus $L^i_{\vartheta'}-E[L^i_\tau\,|\,\mathscr{F}_{\vartheta'}]=E[\int_{\vartheta'}^{\tau}(\pi_s^{Li}-\pi_s^{0i})\,ds\,|\,\mathscr{F}_{\vartheta'}]\ge 0$ when ϑ' attains (3.7).

For the final claim, consider any stopping time $\tau_*^2\ge\vartheta$ such that $F^2_{\tau_*^2}=M^2_{\tau_*^2}$; then also $F^1_{\tau_*^2}=M^1_{\tau_*^2}$ by Lemma 3.2. Suppose furthermore that (i) and (ii) hold for $i=1$, so $\tau_*^1=\tau_*^2$ is a best reply for firm 1. To prove that τ_*^2 is also a best reply for firm 2 to $\tau_*^1=\tau_*^2$ if $\pi_\cdot^{L1}-\pi_\cdot^{01}\ge\pi_\cdot^{L2}-\pi_\cdot^{02}$ and $\pi_\cdot^{B1}-\pi_\cdot^{01}\ge\pi_\cdot^{B2}-\pi_\cdot^{02}$, we show that (B.6) is then not greater for $i=2$ than for $i=1$. Therefore, note that for each $i=1,2$, $F^i_{\tau_*^2}=M^i_{\tau_*^2}$ implies $E[\mathbf{1}_A\int_{\tau_*^2}^{\tau_F^i(\vartheta')}(\pi_s^{Bi}-\pi_s^{Fi})\,ds\,|\,\mathscr{F}_{\vartheta'}]=0$ for any set $A\subseteq\{\tau_F^i(\vartheta')\ge\tau_*^2\}$ (taking iterated expectations at τ_*^2), in particular for $A=\{\tau_F^1(\vartheta')>\tau_*^2\}$, as $\tau_F^2(\vartheta')\ge\tau_F^1(\vartheta')$. Moreover, $E[\mathbf{1}_{\{\tau_F^1(\vartheta')>\tau_*^2\}}\int_{\tau_F^1(\vartheta')}^{\tau_F^2(\vartheta')}(\pi_s^{B2}-\pi_s^{F2})\,ds\,|\,\mathscr{F}_{\vartheta'}]\le 0$ by optimality of $\tau_F^2(\vartheta')$ (and iterated expectations at $\tau_F^1(\vartheta')$), so $E[\mathbf{1}_{\{\tau_F^1(\vartheta')>\tau_*^2\}}\int_{\tau_*^2}^{\tau_F^1(\vartheta')}(\pi_s^{B2}-\pi_s^{F2})\,ds\,|\,\mathscr{F}_{\vartheta'}]\ge 0$.

Now, rewriting (B.6) for $i=2$, we obtain

$$
\begin{aligned}
&E\Big[\int_{\vartheta'}^{\tau_F^1(\vartheta')\wedge\tau_*^2}(\pi_s^{L2}-\pi_s^{02})\,ds+\mathbf{1}_{\{\tau_F^1(\vartheta')\le\tau_*^2\}}\int_{\tau_F^1(\vartheta')}^{\tau_*^2}(\pi_s^{B2}-\pi_s^{02})\,ds\\
&\qquad+\mathbf{1}_{\{\tau_F^1(\vartheta')>\tau_*^2\}}\int_{\tau_*^2}^{\tau_F^1(\vartheta')}(\pi_s^{L2}-\pi_s^{B2})\,ds\,\Big|\,\mathscr{F}_{\vartheta'}\Big]\\
&\le E\Big[\int_{\vartheta'}^{\tau_F^1(\vartheta')\wedge\tau_*^2}(\pi_s^{L1}-\pi_s^{01})\,ds+\mathbf{1}_{\{\tau_F^1(\vartheta')\le\tau_*^2\}}\int_{\tau_F^1(\vartheta')}^{\tau_*^2}(\pi_s^{B1}-\pi_s^{01})\,ds\\
&\qquad+\mathbf{1}_{\{\tau_F^1(\vartheta')>\tau_*^2\}}\int_{\tau_*^2}^{\tau_F^1(\vartheta')}(\pi_s^{L2}-\pi_s^{F2})\,ds\,\Big|\,\mathscr{F}_{\vartheta'}\Big]\\
&\le E\Big[\int_{\vartheta'}^{\tau_F^1(\vartheta')\wedge\tau_*^2}(\pi_s^{L1}-\pi_s^{01})\,ds+\mathbf{1}_{\{\tau_F^1(\vartheta')\le\tau_*^2\}}\int_{\tau_F^1(\vartheta')}^{\tau_*^2}(\pi_s^{B1}-\pi_s^{01})\,ds\\
&\qquad+\mathbf{1}_{\{\tau_F^1(\vartheta')>\tau_*^2\}}\int_{\tau_*^2}^{\tau_F^1(\vartheta')}(\pi_s^{L1}-\pi_s^{F1})\,ds+\int_{\tau_F^1(\vartheta')}^{\tau_F^2(\vartheta')}(\pi_s^{L1}-\pi_s^{B1})\,ds\,\Big|\,\mathscr{F}_{\vartheta'}\Big].
\end{aligned}
\tag{B.7}
$$

The last inequality uses the assumption $\pi_\cdot^{L1}-\pi_\cdot^{F1}\ge\pi_\cdot^{L2}-\pi_\cdot^{F2}$, as well as $\tau_F^1(\vartheta')\le\tau_F^2(\vartheta')$ and $\pi_\cdot^{L1}\ge\pi_\cdot^{B1}$. Rearranging (B.7) using $E[\mathbf{1}_{\{\tau_F^1(\vartheta')>\tau_*^2\}}\int_{\tau_*^2}^{\tau_F^i(\vartheta')}(\pi_s^{Bi}-\pi_s^{Fi})\,ds\,|\,\mathscr{F}_{\vartheta}]=0$ yields (B.6) for $i=1$. □

Proof of Corollary 3.10. We only need to verify optimality for firm $i=2$ by applying Proposition 3.9 with $\tau_*^1=\tau_S\le\tau_F^2(\vartheta)=\tau_*^2$. Then indeed $F^2_{\tau_*^2}=M^2_{\tau_*^2}$. Moreover, condition (i) is satisfied as $M_\cdot^2\le F_\cdot^2$ and

(F^2_t) is a submartingale on $[\vartheta, \tau^2_F(\vartheta)]$ by $\pi^{F2}_\cdot \leq \pi^{02}_\cdot$. Hence, τ^2_* is optimal if the remaining condition (ii) is satisfied.

For the second claim, note that if $\pi^{L1}_\cdot - \pi^{01}_\cdot \geq \pi^{L2}_\cdot - \pi^{02}_\cdot$, then $E[\int_\tau^{\tau_S} (\pi^{L2}_s - \pi^{02}_s)\,ds \,|\, \mathscr{F}_\tau] \leq E[\int_\tau^{\tau_S} (\pi^{L1}_s - \pi^{01}_s)\,ds \,|\, \mathscr{F}_\tau] \leq 0$ for any stopping time $\tau \in [\vartheta, \tau_S]$ by the optimality of τ_S, and $\tau^2_D(\vartheta') = \tau_S \vee \vartheta'$ thus attains the current instance of (3.7). □

Proof of Proposition 4.1. By the strong Markov property, it suffices to consider $t = 0$. If the preemption region is empty, then we can set $\underline{x} = \bar{x}$ and pick any number in $(0, x^1_F]$. The upper and lower bounds for a non-empty preemption region are obtained as follows. First, note that $L^2_0 = M^2_0 \leq F^2_0$ for all $x_0 \geq x^1_F$. Second, for all $x_0 > 0$, $L^2_0 \leq E[\int_0^\infty e^{-rs}(x_s D_{10} - rI^2)\,ds] = x_0 D_{10}/(r-\mu) - I^2$ by $D_{10} \geq D_{11}$, and $F^2_0 \geq E[\int_0^\infty e^{-rs} x_s D_{01}\,ds] = x_0 D_{01}/(r-\mu)$ – the value of never investing as follower. Thus, $L^2_0 - F^2_0 \leq x_0(D_{10} - D_{01})/(r-\mu) - I^2 \leq 0$ on the non-empty interval $(0, (r-\mu)I^2/(D_{10} - D_{01})^+)$.

Now suppose $L^2_0 > F^2_0$ for some $x_0 = \hat{x} \in (0, x^1_F)$ and also for some $x_0 = \check{x} < \hat{x}$, and assume by way of contradiction that $L^2_0 \leq F^2_0$ for $x_0 = x' \in (\check{x}, \hat{x})$. We must then have $x' > rI^2/(D_{10} - D_{01})^+$, because otherwise $L^2_0 - F^2_0 = E[\int_0^{\tau'} e^{-rs}(x_s(D_{10} - D_{01}) - rI^2)\,ds] + E[L^2_{\tau'} - F^2_{\tau'}] \leq 0$ if $x_0 = \check{x}$ and $x' \in (\check{x}, rI^2/(D_{10} - D_{01})^+ \wedge x^1_F]$, where $\tau' := \inf\{s \geq 0 \,|\, x_s \geq x'\} \leq \tau^1_F(0)$. By the same argument, we must also have $L^2_0 > F^2_0$ for $x_0 = \check{x} \vee rI^2/(D_{10} - D_{01}) < x'$. But then, if we set $x_0 = x'$ and $\hat{\tau} := \inf\{s \geq 0 \,|\, x_s \notin (\check{x} \vee rI^2/(D_{10} - D_{01}), \hat{x})\} \leq \tau^1_F(0)$, we obtain $L^2_0 - F^2_0 = E[\int_0^{\hat{\tau}} e^{-rs}(x_s(D_{10} - D_{01}) - rI^2)\,ds] + E[L^2_{\hat{\tau}} - F^2_{\hat{\tau}}] > 0$, whence the set $\{x > 0 \,|\, L^2_0 > F^2_0 \text{ given } x_0 = x\}$ is convex. Moreover, that set is open, as $L^2_0 - F^2_0$ is continuous in x_0.

Suppose finally that $I^2 = I^1$ and that the preemption region is non-empty, i.e., by Lemma 3.8 and the discussion thereafter, that the threshold solving (3.6) satisfies $x^2_\Delta < x^1_F = x^2_F$. For any $x_0 \in [x^2_\Delta, x^2_F)$ then $L^2_0 - F^2_0 = E[\int_0^{\tau^2_F(0)} (x_s(D_{10} - D_{01}) - rI^2)\,ds] > 0$, as x^2_Δ solves (3.6) uniquely. □

Proof of Proposition 4.2. $\bar{x} < x^2_F$ can be any two numbers from $(0, \infty]$ in this proof, i.e., we only assume $\bar{x}$ finite. For initial states $x_0 \in (\bar{x}, x^2_F)$, the constraint $\tau_{\mathcal{P}}(0) \wedge \tau^2_F(0)$ in problem (4.3) is the exit time from the given interval, and (4.3) is equivalent to

$$\sup_{\tau \leq \inf\{s \geq 0 \,|\, x_s \notin (\bar{x}, x^2_F)\}} E\left[\int_\tau^\infty e^{-rs}\big(x_s(D_{10} - D_{00}) - rI^1\big)\,ds\right]. \tag{B.8}$$

If $\bar{x}(D_{10} - D_{00}) \geq rI^1$, then the expected payoff difference between stopping at time 0 and any feasible $\tau \geq 0$ is $E[\int_0^\tau e^{-rs}(x_s(D_{10} - D_{00}) - rI^1)\,ds] \geq 0$, such that immediate stopping is optimal. If $D_{10} - D_{00} \leq 0$, also $E[\int_\tau^{\tau_{\mathcal{P}}(0) \wedge \tau^2_F(0)} e^{-rs}(x_s(D_{10} - D_{00}) - rI^1)\,ds] \leq 0$ for any $\tau \leq \tau_{\mathcal{P}}(0) \wedge \tau^2_F(0)$, such that waiting until the constraint is optimal.

Now suppose $0 < \bar{x}(D_{10} - D_{00}) < rI^1$, whence $D_{10} > D_{00}$ and $x^1_L < \infty$. Note that

$$E\left[\int_0^\infty e^{-rs}\big(x_s(D_{10} - D_{00}) - rI^1\big)\,ds\right] = x_0 \frac{D_{10} - D_{00}}{r-\mu} - I^1$$

is the value of stopping immediately in (B.8). Letting $x_0 = x$, we will first verify that the value function of problem (B.8) is

$$V(x) := \begin{cases} A(\hat{x})x^{\beta_1} + B(\hat{x})x^{\beta_2} & \text{if } x \in (\bar{x}, \hat{x}), \\ x\frac{D_{10} - D_{00}}{r-\mu} - I^1 & \text{else,} \end{cases} \tag{B.9}$$

and thus $(\bar{x}, \hat{x})^c$ the sought stopping region, under the hypothesis that either $\hat{x} \in [rI^1/(D_{10} - D_{00}), x^2_F)$ solves (4.4) or "≤" holds for $\hat{x} = x^2_F$. Afterwards, we will establish existence of a unique such $\hat{x}$.

$V(x)$ as defined in (B.9) is continuous, because $A(\hat{x})$ and $B(\hat{x})$ given by (4.5) are the solution to the continuity conditions

$$A\bar{x}^{\beta_1} + B\bar{x}^{\beta_2} = \bar{x}\frac{D_{10} - D_{00}}{r-\mu} - I^1, \qquad A\hat{x}^{\beta_1} + B\hat{x}^{\beta_2} = \hat{x}\frac{D_{10} - D_{00}}{r-\mu} - I^1. \tag{B.10}$$

$V(x)$ is also twice continuously differentiable on $(\bar{x}, x_F^2)$, except possibly at $\hat{x}$. At $\hat{x} < x_F^2$, the first derivative of V is continuous, however, because (4.4) is the differentiability condition $\beta_1 A\hat{x}^{\beta_1-1} + \beta_2 B\hat{x}^{\beta_2-1} = (D_{10} - D_{00})/(r-\mu)$ multiplied by $\hat{x}$, minus the second continuity condition in (B.10). We can thus apply Itō's lemma to see that $(e^{-rt}V(x_t))$ is a continuous, bounded supermartingale until $\tau = \inf\{t \geq 0 \,|\, x_t \notin (\bar{x}, x_F^2)\}$, with zero drift for $x_t \in (\bar{x}, \hat{x})$ and drift $e^{-rt}(rI^1 - x_t(D_{10} - D_{00}))\, dt < 0$ for $x_t \in (\hat{x}, x_F^2)$. As that supermartingale coincides with the payoff process at $\tau = \inf\{t \geq 0 \,|\, x_t \notin (\bar{x}, x_F^2)\}$, it remains to show that $V(x)$ dominates the payoff process for $x \in (\bar{x}, x_F^2)$, which it does by construction for $x \in [\hat{x}, x_F^2]$.

For $x \in (\bar{x}, \hat{x})$, $V''(x) = x^{\beta_2-2}[\beta_1(\beta_1 - 1)A(\hat{x})x^{\beta_1-\beta_2} + \beta_2(\beta_2 - 1)B(\hat{x})]$. As $\beta_k(\beta_k - 1) > 0$, $k = 1, 2$, the difference $V(x) - x(D_{10} - D_{00})/(r - \mu) + I^1$ would be convex if $A(\hat{x}), B(\hat{x}) \geq 0$, and it vanishes at both endpoints $\bar{x}, \hat{x}$. By (4.4), that difference's derivative is non-positive at $\hat{x}$, where the difference would thus take its minimum. It would hence vanish on all of $[\bar{x}, \hat{x}]$, but $V(x)$ cannot be affine on non-empty $(\bar{x}, \hat{x})$. So, we must have $A(\hat{x}) \wedge B(\hat{x}) < 0$. If we had $B(\hat{x}) \geq 0$, then $A(\hat{x}) < 0$ and $V(x)$ would be strictly decreasing on $(\bar{x}, \hat{x})$, contradicting $V(\hat{x}) \geq V(\bar{x})$; thus, $B(\hat{x}) < 0$. Going back to $V''(x)$, which can switch sign at most once, it must start strictly negative at $\bar{x}$. If it stays non-positive, the difference $V(x) - x(D_{10} - D_{00})/(r - \mu) + I^1$ is concave and thus non-negative on $(\bar{x}, \hat{x})$. If $V''(x)$ eventually becomes positive, then the convex part of $V(x) - x(D_{10} - D_{00})/(r - \mu) + I^1$ takes its minimum 0 at $\hat{x}$ as argued before, such that the difference is non-negative at the transition, and thus non-negative for the first, concave part. In summary, $(e^{-rt}V(x_t))$ is a supermartingale until x_t leaves $(\bar{x}, x_F^2)$, dominating the payoff $e^{-rt}(x_t(D_{10} - D_{00})/(r - \mu) - I^1)$, which it coincides with for $x_t \in \{\bar{x}\} \cup [\hat{x}, x_F^2]$, so the latter is the stopping set in $[\bar{x}, x_F^2]$.

Next, we show that there is a unique threshold $\hat{x} \in [rI^1/(D_{10} - D_{00}), x_L^1)$ solving (4.4), and then finally consider the constraint x_F^2.

As the first step, note that $B(x) < 0$ in (4.5) for all $x \in (\bar{x}, x_L^1]$. Indeed, as the first term $[\bar{x}^{\beta_1}x^{\beta_2} - x^{\beta_1}\bar{x}^{\beta_2}]^{-1}$ is negative for $x > \bar{x}$ by $\beta_1 > 1$ and $\beta_2 < 0$, we have $B(x) < 0 \Leftrightarrow x^{-\beta_1}[x(D_{10} - D_{00})/(r - \mu) - I^1] > \bar{x}^{-\beta_1}[\bar{x}(D_{10} - D_{00})/(r - \mu) - I^1]$. The derivative of the latter function of x can be written as $x^{-\beta_1-1}[\beta_1 I^1 - (\beta_1 - 1)x(D_{10} - D_{00})/(r - \mu)] > 0$ for all $x < x_L^1 = \beta_1(r - \mu)I^1/((\beta_1 - 1)(D_{10} - D_{00}))$.

As the second step, note that, with $A = A(x_L^1)$ and $B = B(x_L^1)$, we have $A\cdot(x_L^1)^{\beta_1} + B\cdot(x_L^1)^{\beta_2} = I^1/(\beta_1-1)$ by using the definition of x_L^1 in (B.10), and thus $(\beta_1-1)A\cdot(x_L^1)^{\beta_1} + (\beta_2-1)B\cdot(x_L^1)^{\beta_2} = I^1 + (\beta_2-\beta_1)B\cdot(x_L^1)^{\beta_2} > I^1$ in contrast to "=" in (4.4).

The third step is to show that "$\leq$" holds in (4.4) for the candidate $\hat{x} = rI^1/(D_{10} - D_{00}) \in (\bar{x}, x_F^2)$, where the inclusion is exactly the current considered case. By similar arguments as above, using the continuity condition (B.10), $V(x)$ then satisfies

$$V(x) = E\Big[\int_{\hat{\tau}}^{\infty} e^{-rs}\big(x_s(D_{10} - D_{00}) - rI^1\big)\, ds\Big], \quad x_0 = x \in [\bar{x}, \hat{x}],$$

where we let $\hat{\tau} := \inf\{s \geq 0 \,|\, x_s \notin (\bar{x}, \hat{x})\}$. For $\hat{x} = rI^1/(D_{10} - D_{00})$, the integrand would be strictly negative until $\hat{\tau}$, so $V(x) > x(D_{10} - D_{00})/(r - \mu) - I^1$ for all $x \in (\bar{x}, \hat{x})$. At $x = \hat{x}$, however, equality holds by (B.10) and thus $V'(\hat{x}-) = \beta_1 A(\hat{x})\hat{x}^{\beta_1-1} + \beta_2 B(\hat{x})\hat{x}^{\beta_2-1} \leq (D_{10} - D_{00})/(r - \mu)$. Together with (B.10), the latter inequality implies also "$\leq$" in (4.4).

As the last step, as the function $(\beta_1 - 1)A(x)x^{\beta_1} + (\beta_2 - 1)B(x)x^{\beta_2}$ is continuous, it must attain I^1 at some $\hat{x} \in [rI^1/(D_{10} - D_{00}), x_L^1)$ by the second and third steps. The latter interval is non-empty by the estimate for x_L^1 at the beginning of the proof.

Concerning uniqueness, suppose $\hat{x}_1, \hat{x}_2 \in [rI^1/(D_{10} - D_{00}), x_L^1)$ solve (4.4). With either solution, as we have proved above, $V(x)$ is the value function of problem (B.8) for any $x_F^2 \geq x_L^1$, and (B.8) is solved by both $\hat{\tau}_k := \inf\{s \geq 0 \,|\, x_s \notin (\bar{x}, \hat{x}_k)\}$, $k = 1, 2$. In particular, for any $x_0 \in [x_1, x_2]$,

$$V(x_0) = x_0\frac{D_{10} - D_{00}}{r - \mu} - I^1 = E\Big[\int_{\hat{\tau}_2}^{\infty} e^{-rs}\big(x_s(D_{10} - D_{00}) - rI^1\big)\, ds\Big]$$

$$\Rightarrow \qquad 0 = E\Big[\int_0^{\hat{\tau}_2} e^{-rs}\big(x_s(D_{10} - D_{00}) - rI^1\big)\, ds\Big].$$

Thus, letting $\check{\tau}_1 := \inf\{s \geq 0 \,|\, x_s \leq \hat{x}_1\} \leq \hat{\tau}_2$ and still $x_0 \in [x_1, x_2]$,

$$\begin{aligned} 0 &= E\biggl[\int_0^{\hat{\tau}_2} e^{-rs}\bigl(x_s(D_{10} - D_{00}) - rI^1\bigr)\,ds\biggr] \\ &= E\biggl[\int_0^{\check{\tau}_1 \wedge \hat{\tau}_2} e^{-rs}\bigl(x_s(D_{10} - D_{00}) - rI^1\bigr)\,ds + \int_{\check{\tau}_1 \wedge \hat{\tau}_2}^{\hat{\tau}_2} e^{-rs}\bigl(x_s(D_{10} - D_{00}) - rI^1\bigr)\,ds\biggr]. \end{aligned}$$

The second integral vanishes itself in expectation, whereas the first integrand is strictly positive for $x_s \in (\hat{x}_1, \hat{x}_2)$. Therefore, the latter interval must be empty.

The proof is complete for $\hat{x} \leq x_F^2$. Finally, if $rI^1/(D_{10} - D_{00}) < x_F^2 < \hat{x}$, then the "$\leq$" in (4.4) that we derived above for the candidate $x = rI^1/(D_{10} - D_{00})$ must be strict, and thus also "$<$" must hold in (4.4) for x_F^2, because otherwise $\hat{x} \leq x_F^2$ by continuity of $(\beta_1 - 1)A(x)x^{\beta_1} + (\beta_2 - 1)B(x)x^{\beta_2}$. Now the verification argument above applies if we consider instead $\hat{x} := x_F^2$ with "$\leq$" in (4.4). □

Proof of Proposition 4.3. The stopping times $\tau_J(\vartheta) := \inf\{t \geq \vartheta \,|\, x_t \geq x_J\}$, $\vartheta \in \mathscr{T}$, satisfy time consistency $\vartheta' \leq \tau_J(\vartheta) \Rightarrow \tau_J(\vartheta') = \tau_J(\vartheta)$ for any two $\vartheta \leq \vartheta' \in \mathscr{T}$ by construction. $\tau_J(\vartheta)$ is a mutual best reply at ϑ if the conditions from Proposition 3.9 hold. By $x_J \geq x_F^2$, $F^2_{\tau_J(\vartheta)} = M^2_{\tau_J(\vartheta)}$. Under the current specification, it suffices to verify conditions (i) and (ii) for firm 1.

Condition (i) holds, as by Lemma A.7, waiting until the threshold $x_J \leq x_M^1$ is optimal for the constrained problem of stopping M_t^1 up to it; cf. the unconstrained problem (3.5). Analogously, the threshold $\min\{x_J, x_L^1\}$ solves problem (3.7). Condition (ii) thus holds if $x_L^1 \geq x_F^2$ or, using the strong Markov property, if $0 \geq D_J(x) := L_0^1 - E[M^1_{\tau_J(0)}]$ given $x_0 = x \in [x_L^1, x_F^2)$.

By Proposition 3.9, if $x_L^1 < x_F^2$ solves (3.7) and we let $\tau(x) = \inf\{t \geq 0 \,|\, x_t \geq x\} \leq \tau_F^2(0)$ for any $x \in [x_L^1, x_F^2)$, then $D_J(x_L^1) \geq E[L^1_{\tau(x)} - M^1_{\tau_J(0)}] = E[D_J(x)]$, where the last identity is due to $x_{\tau(x)} = x$. It thus remains to verify $D_J(x_L^1) \leq 0$ for $x_L^1 < x_F^2$.

If $x_L^1 < x_F^2$, then the former is finite and we can write $\lambda := x_J/x_L^1 \in [1, \infty]$. Then also $x_L^1 < x_J$ and thus (cf. fn. 20, accounting for possibly $x_F^2 = \infty$)

$$\begin{aligned} 0 \overset{!}{\geq} D_J(x_L^1) &= \frac{x_L^1 D_{10}}{r - \mu} - I^1 - \frac{x_F^2(D_{10} - D_{11})}{r - \mu}\Bigl(\frac{x_L^1}{x_F^2}\Bigr)^{\beta_1} - \frac{x_L^1 D_{00}}{r - \mu} - \Bigl(\frac{x_J(D_{11} - D_{00})}{r - \mu} - I^1\Bigr)\Bigl(\frac{x_L^1}{x_J}\Bigr)^{\beta_1} \\ &= \frac{\beta_1}{\beta_1 - 1} I^1 - I^1 - \frac{\beta_1}{\beta_1 - 1} I^1 \frac{D_{10} - D_{11}}{D_{10} - D_{00}}\Bigl(\frac{I^1}{I^2}\frac{(D_{11} - D_{01})^+}{D_{10} - D_{00}}\Bigr)^{\beta_1 - 1} \\ &\quad - \Bigl(\lambda \frac{\beta_1}{\beta_1 - 1} I^1 \frac{D_{11} - D_{00}}{D_{10} - D_{00}} - I^1\Bigr)\lambda^{-\beta_1}. \end{aligned}$$

Rearranging yields condition (4.6). The derivative of the square bracket in (4.6) w.r.t. λ is strictly negative for $\lambda \in (0, x_M^1/x_L^1)$, given $\beta_1 > 1$, where it is important to note that $\lambda(D_{11} - D_{00}) < D_{10} - D_{00}$, because $D_{10} > D_{00}$ for $x_L^1 < x_F^2$ and $(D_{10} - D_{00})/(D_{11} - D_{00}) = x_M^1/x_L^1 > \lambda$ if $D_{11} > D_{00}$. Using the latter fact also shows that, for $\lambda = x_M^1/x_L^1$, the square bracket is either $1 - (x_L^1/x_M^1)^{\beta_1} \geq 0$ or 1, if x_M^1 is finite or not, respectively.

Finally, necessity of $D_J(x_L^1) \leq 0$ for $x_L^1 < x_F^2 \leq x_J$ is obvious. □

Proof of Proposition 4.4. By the hypothesis $x_L^1 < x_F^2$ and Lemma A.7, the solution of problem (3.7) for $\vartheta' = \vartheta$, $i = 1$, and $j = 2$ with $\tau_*^2 = \tau_F^2(\vartheta) = \inf\{t \geq \vartheta \,|\, x_t \geq x_F^2\}$ is $\tau_S(\vartheta) := \tau_L^1(\vartheta) = \inf\{t \geq \vartheta \,|\, x_t \geq x_L^1\} \in \mathscr{T}$. These stopping times for firm 1 satisfy time consistency $\vartheta' \leq \tau_S(\vartheta) \Rightarrow \tau_S(\vartheta') = \tau_S(\vartheta)$ for any two $\vartheta \leq \vartheta' \in \mathscr{T}$ by construction, as do the stopping times $\tau_F^2(\vartheta)$.

To verify the equilibrium at $\vartheta \in \mathscr{T}$ by Corollary 3.10, note that now $\pi_\cdot^{L1} - \pi_\cdot^{01} \geq \pi_\cdot^{L2} - \pi_\cdot^{02}$, whence problem (3.7) is solved by $\tau_D^2(\vartheta') = \tau_S(\vartheta) \vee \vartheta'$. We thus have an equilibrium if $x_L^1 \geq x_F^1$ $(\geq \bar{x})$ or, using the strong Markov property, if $0 \geq D_S(x) := L_0^2 - E[F^2_{\tau_S(0)}]$ given $x_0 = x \in [x_L^1, x_F^1)$.

By Proposition 3.9, if $x_L^1 < x_F^1$ and letting $\tau(x) = \inf\{t \geq 0 \,|\, x_t \geq x\} \leq \tau_F^1(0)$ for any $x \in [x_L^1, x_F^1)$, then $D_S(x_L^1) \geq E[L^2_{\tau(x)} - F^2_{\tau_S(0)}] = E[D_S(x)]$, where the last identity is due to $x_{\tau(x)} = x$. It therefore remains to verify $D_S(x_L^1) \leq 0$ for $x_L^1 < x_F^1$, i.e., $x_L^1 \notin (\underline{x}, \bar{x})$. The latter condition is (cf. fn. 20, accounting for possibly $x_F^1 = x_F^2 = \infty$)

$$0 \overset{!}{\geq} D_S(x_L^1) = \frac{x_L^1 D_{10}}{r-\mu} - I^2 - \frac{x_F^1(D_{10}-D_{11})}{r-\mu}\left(\frac{x_L^1}{x_F^1}\right)^{\beta_1} - \frac{x_L^1 D_{01}}{r-\mu} - \left(\frac{x_F^2(D_{11}-D_{01})}{r-\mu} - I^2\right)\left(\frac{x_L^1}{x_F^2}\right)^{\beta_1}$$

$$= \frac{\beta_1}{\beta_1 - 1} I^1 \frac{D_{10}-D_{01}}{D_{10}-D_{00}} - I^2 - \frac{\beta_1}{\beta_1-1} I^1 \frac{D_{10}-D_{11}}{D_{10}-D_{00}}\left(\frac{(D_{11}-D_{01})^+}{D_{10}-D_{00}}\right)^{\beta_1 - 1}$$

$$- \frac{1}{\beta_1 - 1} I^2 \left(\frac{I^1}{I^2}\frac{(D_{11}-D_{01})^+}{D_{10}-D_{00}}\right)^{\beta_1}.$$

Rearranging yields condition (4.7). The derivative of its left-hand side w.r.t. I^2/I^1 is strictly positive for $x_L^1 < x_F^1$, given $\beta_1 > 1$, because then $(D_{11}-D_{00})^+/(D_{10}-D_{00}) < 1$. By the same fact, the right-hand side of (4.7) is strictly positive.

To show necessity of $x_L^1 \notin (\underline{x}, \bar{x})$, suppose the contrary, whence $x_L^1 < x_F^1$ and $D_S(x_L^1) > 0$ by definition. For any $x \leq x_L^1$,

$$D_S(x) = E\Big[D_S(x_L^1)\Big] + L_0^2 - E\Big[L_{\tau_S(0)}^2\Big] = D_S(x_L^1) + E\left[\int_0^{\tau_S(0)} (\pi_s^{L2} - \pi_s^{02})\, ds\right]$$

$$= D_S(x_L^1) + \frac{x(D_{10}-D_{00})}{r-\mu} - I^2 - \frac{x_L^1(D_{10}-D_{00})}{r-\mu}\left(\frac{x}{x_L^1}\right)^{\beta_1},$$

which converges continuously to $D_S(x_L^1) > 0$ as $x \to x_L^1$, so $D_S(x) > 0$ for some $x < x_L^1$. □

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Supplement to

Preemptive investment under uncertainty

Jan-Henrik Steg

C Necessary equilibrium conditions

In the equilibria derived in Section 3, it may often be the case that investment is only optimal because the other firm plans to invest at the same date. Possibly other equilibria exist, with both firms investing later and that then both prefer, but on which they have to coordinate. Here, we derive times when investment is indeed unavoidable in equilibrium. The proofs of all following results are collected in Subsection C.2.

Equilibria are obviously related to optimally stopping the leader payoff processes, typically subject to certain constraints. The next lemma shows that given the assumptions $\pi^{Li}_\cdot \geq \pi^{Bi}_\cdot$ and $\pi^{0i}_\cdot \geq \pi^{Fi}_\cdot$, equilibrium investment must not happen later than when firm i would invest if it had the exclusive right to invest first, i.e., if it considered the *unconstrained* problem of when to become leader.

Due to the dynamic follower reaction in L^i_τ, this is a complex problem. It may for instance not be optimal to invest when the general circumstances are so favorable that any monopolist or follower would invest immediately: When only $\pi^{Bi}_\cdot$ can be realized, it may be better to invest when the follower will react with a lag.[34] In order to become leader optimally, it is however necessary that a monopolist would invest.

Lemma C.1. *Whenever $\tau = \vartheta$ is the only stopping time attaining*

$$\operatorname*{ess\,sup}_{\tau \geq \vartheta} E\Big[L^i_\tau \,\Big|\, \mathscr{F}_\vartheta\Big] \tag{C.1}$$

for some $i \in \{1, 2\}$, then investment must happen immediately in any equilibrium at $\vartheta \in \mathscr{T}$, i.e., $(\sigma^\vartheta_1, \sigma^\vartheta_2)$ must be such that $\max\{G^\vartheta_1(\vartheta), G^\vartheta_2(\vartheta)\} = 1$. *Whenever $\tau = \vartheta$ attains* (C.1)*, it also attains* (3.4)*, i.e.,*

$$\operatorname*{ess\,sup}_{\tau \geq \vartheta} E\bigg[\int_0^\tau \pi^{0i}_s \, ds + \int_\tau^\infty \pi^{Li}_s \, ds \,\bigg|\, \mathscr{F}_\vartheta\bigg]. \tag{3.4}$$

Lemma C.1 rests on the observation that if it is optimal to become leader immediately in (C.1), then there is no superior future follower payoff, either: If firm i had the choice when to become follower, it would generally prefer times $\tau^i_F(\tau)$ in order to avoid the low revenue $\pi^{Fi}_\cdot \leq \pi^{0i}_\cdot$. At any $\tau^i_F(\tau)$, however, becoming follower is not better than becoming leader due to $\pi^{Bi}_\cdot \leq \pi^{Li}_\cdot$.

Problem (C.1) becomes much easier by fixing continuation equilibria, like simultaneous investment at $\tau^2_F(\vartheta)$, which make it impossible to become leader later. By such a constraint, firm 2's follower reaction will always be the same, and firm 1 will not cannibalize any revenue $\pi^{L1}_\cdot$ past $\tau^2_F(\vartheta)$ if it invests before. Firm 1's leader problem thus becomes equivalent to a constrained monopolist's problem. The following constrained version of Lemma C.1 follows the same logic, but it is important that firm 1 will not regret to receive $\pi^{B1}_\cdot$ from $\tau^2_F(\vartheta)$ on by investing before.[35]

Lemma C.2. *Suppose that firm 2's strategy in an equilibrium for the subgame at $\vartheta \in \mathscr{T}$ induces investment no later than at $\tau^2_F(\vartheta)$. Whenever $\tau = \vartheta$ is the latest stopping time attaining*

$$\operatorname*{ess\,sup}_{\tau \in [\vartheta, \tau^2_F(\vartheta)]} E\Big[L^1_\tau \,\Big|\, \mathscr{F}_\vartheta\Big], \tag{C.2}$$

[34] See furthermore Subsection C.1 on the monopolists' and leaders' problems for standard diffusion models.

[35] Firm 2, on the contrary, may prefer to become follower at $\tau^1_F(\vartheta)$ and effectively invest later. If firm 2 can become leader up to $\tau^2_F(\vartheta)$, it may expect a delayed follower reaction and high revenue $\pi^{L2}_\cdot$ in $(\tau^1_F(\vartheta), \tau^2_F(\vartheta)]$, and the problem cannot be simplified.

then investment must happen immediately in any equilibrium at $\vartheta \in \mathscr{T}$, *i.e.,* $(\sigma_1^\vartheta, \sigma_2^\vartheta)$ *must be such that* $\max\{G_1^\vartheta(\vartheta), G_2^\vartheta(\vartheta)\} = 1$. (C.2) *has the same solutions as*

$$\operatorname*{ess\,sup}_{\tau \in [\vartheta, \tau_F^2(\vartheta)]} E\Big[\int_0^\tau \pi_s^{01}\, ds + \int_\tau^\infty \pi_s^{L1}\, ds \Big| \mathscr{F}_\vartheta\Big]. \tag{C.3}$$

If a monopolist's investment gain $\pi_\cdot^{L1} - \pi_\cdot^{01}$ is not less than a follower's, $\pi_\cdot^{B1} - \pi_\cdot^{F1}$ (like in typical market entry with $\pi_\cdot^{01} = \pi_\cdot^{F1}$), then the latest solution of (C.3) does not exceed $\tau_F^1(\vartheta)$, because then any delay only means foregone revenue for a follower in (2.2), and firm 1 would now lose no less as prospective leader. Then (C.3) has the same solutions as firm 1's unconstrained monopoly problem (3.4) (cf. Lemma C.1).

Another continuation equilibrium that potentially induces earlier investment is preemption at $\tau_{\mathcal{P}}(\vartheta)$ as in Section 3.1.2. In this case (or $\mathcal{P} = \emptyset$), firm 2 can never realize payoffs exceeding $F_\cdot^2$, and investment has to occur immediately at all respectively latest optimal times to *become* follower. Indeed, such times have to satisfy $\tau = \tau_F^2(\tau)$ (as it is otherwise no loss to become follower at $\tau_F^2(\tau)$ and receive $\pi_\cdot^{02} \geq \pi_\cdot^{F2}$ longer), and then firm 2 can enforce the payoff $F_\tau^2 = L_\tau^2 = M_\tau^2$ by investing regardlessly. Moreover, a stopping time satisfying $\vartheta = \tau_F^i(\vartheta)$ can only maximize firm i's follower payoff if it also maximizes the simultaneous investment payoff. Conversely, an optimal time for simultaneous investment must also be optimal for becoming follower, as the opportunity cost of waiting for the former, $\pi_\cdot^{Bi} - \pi_\cdot^{0i}$, is at most that for the latter by $\pi_\cdot^{0i} \geq \pi_\cdot^{Fi}$.

Lemma C.3. *Suppose that both firms' strategies in an equilibrium for the subgame at* $\vartheta \in \mathscr{T}$ *are respectively time-consistent with strategies for* $\vartheta' = \tau_{\mathcal{P}}(\vartheta)$ *that are an equilibrium as in Lemma 3.3. Then investment must happen immediately whenever* $\tau = \vartheta$ *is the only stopping time attaining*

$$\operatorname*{ess\,sup}_{\tau \geq \vartheta} E\Big[F_\tau^i \Big| \mathscr{F}_\vartheta\Big] \tag{C.4}$$

for $i = 2$.

For any $\vartheta \in \mathscr{T}$ *and* $i \in \{1, 2\}$, (C.4) *is attained by every stopping time* $\tau_M^i \geq \vartheta$ *that attains* (3.5), *i.e.,*

$$\operatorname*{ess\,sup}_{\tau \geq \vartheta} E\Big[M_\tau^i \Big| \mathscr{F}_\vartheta\Big] = \operatorname*{ess\,sup}_{\tau \geq \vartheta} E\Big[\int_0^\tau \pi_s^{0i}\, ds + \int_\tau^\infty \pi_s^{Bi}\, ds \Big| \mathscr{F}_\vartheta\Big]. \tag{3.5}$$

If $\tau_M^i \geq \vartheta$ *attains* (C.4), *then* $\tau_F^i(\tau_M^i)$ *also attains* (3.5). *In particular, the respectively latest solutions of* (C.4) *and* (3.5) *agree.*

(C.4) and (3.5) thus have a latest solution $\tau_M^i \geq \tau_F^i(\vartheta)$. That inequality may be strict in general. If $\pi_\cdot^{0i} = \pi_\cdot^{Fi}$, however, like in typical market entry models, then (3.5) equals F_ϑ^i and $\tau_F^i(\vartheta)$ is the latest time attaining (C.4).

C.1 Leader problem for diffusion models

The solutions – and in particular the stopping regions – for the monopoly problem (3.4) and problem (C.1) of when to optimally become leader typically differ. Consider a model in which the profit streams are driven by a diffusion (Y_t) such that each firm i has a follower threshold, say y_F^i solving (2.2) with $\tau_F^i(\tau) = \inf\{t \geq \tau \,|\, Y_t \geq y_F^i\}$, and firm 1 also has a monopoly threshold, say $y_L^1 \leq y_F^1$ solving (3.4), and where L_t^1 can be represented as a continuous function of the state Y_t. Now one can apply arguments of Jacka (1993) relying on the semi-martingale property of (L_t^1), which the proof of Lemma A.5 actually establishes. Denote the finite-variation part of (L_t^1) by (A_t). The Snell envelope (S_t) of (L_t^1), i.e., the value process of optimally stopping (L_t^1), is now continuous (as a function of the state) as well, and its monotone decreasing part (B_t) is given by $dB_t = \mathbf{1}_{\{S_t = L_t^1\}} dA_t + \frac{1}{2} dL_t^0(S_t - L_t^1)$. The last term is the local time of $(S_t - L_t^1)$ spent at 0 (i.e., in the stopping region), which is absolutely continuous w.r.t. $\mathbf{1}_{\{S_t = L_t^1\}} dA_t \leq 0$.

Now suppose the stopping region $\{S_\cdot = L_\cdot^1\}$ is that of the monopoly problem, $\{Y_\cdot \geq y_L^1\}$, whence $dL_t^0(S_t - L_t^1)$ lives on the boundary $\{Y_\cdot = y_L^1\}$. For $Y_t \in [y_L^1, y_F^2)$, (L_t^1) has a drift given by the foregone monopoly profit stream, $dA_t = -\pi_t^{L1}\, dt$, whence $dL_t^0(S_t - L_t^1) \equiv 0$ if (Y_t) has a transition density, cf. Theorem 6 of Jacka (1993).

As (L_t^1) is of class (D), so is (S_t), which thus converges to $S_\infty = L_\infty^1 = 0$ in $L^1(P)$ as $t \to \infty$. Therefore, the martingale part of (S_t) is simply $E[-B_\infty \,|\, \mathscr{F}_t]$, and $S_t = E[-\int_t^\infty \mathbf{1}_{\{S_s = L_s^1\}}\, dA_s \,|\, \mathscr{F}_t]$. Noting moreover that (L_t^1) has a drift given by the foregone duopoly stream for $Y_t > y_F^2$, i.e., $dA_t = -\pi_t^{B1}\, dt$, we then obtain

$$S_t = E\biggl[\int_t^\infty \Bigl(\mathbf{1}_{\{Y_s \in [y_L^1, y_F^2)\}} \pi_s^{L1} + \mathbf{1}_{\{Y_s > y_F^2\}} \pi_s^{B1}\Bigr)\, ds - \int_t^\infty \mathbf{1}_{\{Y_s = y_F^2\}}\, dA_s \,\bigg|\, \mathscr{F}_t\biggr]. \tag{C.5}$$

By applying similar reasoning to firm 1's monopoly problem (3.4), which is solved by $\tau_L^1(t) = \inf\{s \geq t \,|\, Y_s \geq y_L^1\}$, its value is $E\bigl[\int_{\tau_L^1(t)}^\infty \pi_s^{L1}\, ds \,\big|\, \mathscr{F}_t\bigr] = E\bigl[\int_t^\infty \mathbf{1}_{\{Y_s \geq y_L^1\}} \pi_s^{L1}\, ds \,\big|\, \mathscr{F}_t\bigr]$, i.e., $E\bigl[\int_{\tau_L^1(t)}^\infty \mathbf{1}_{\{Y_s < y_L^1\}} \pi_s^{L1}\, ds \,\big|\, \mathscr{F}_t\bigr] = 0$. Therefore, if $Y_t \geq y_L^1$, then (C.5) can be rewritten as

$$S_t = E\biggl[\int_t^\infty \Bigl(\mathbf{1}_{\{Y_s < y_F^2\}} \pi_s^{L1} + \mathbf{1}_{\{Y_s > y_F^2\}} \pi_s^{B1}\Bigr)\, ds - \int_t^\infty \mathbf{1}_{\{Y_s = y_F^2\}}\, dA_s \,\bigg|\, \mathscr{F}_t\biggr].$$

In this hypothesized stopping region for (L_t^1), also $S_t = L_t^1$, in particular for $Y_t \geq y_F^2 \geq y_L^1$, i.e.,

$$S_t = E\biggl[\int_t^\infty \pi_s^{B1}\, ds \,\bigg|\, \mathscr{F}_t\biggr].$$

With y_F^2 in the stopping region, $-\mathbf{1}_{\{Y_s = y_F^2\}}\, dA_s \geq 0$, and by assumption $\pi_\cdot^{L1} \geq \pi_\cdot^{B1}$. Moreover, $\mathbf{1}_{\{Y_s = y_F^2\}}$ is a $P \otimes dt$ nullset if Y has a transition density, such that equating the two last expressions for S_t implies indeed

$$E\biggl[\int_t^\infty \mathbf{1}_{\{Y_s < y_F^2\}} \Bigl(\pi_s^{L1} - \pi_s^{B1}\Bigr)\, ds \,\bigg|\, \mathscr{F}_t\biggr] = 0$$

(and $E\bigl[-\int_t^\infty \mathbf{1}_{\{Y_s = y_F^2\}}\, dA_s \,\big|\, \mathscr{F}_t\bigr] = 0$). This contradicts the typical strict ordering $\pi_\cdot^{L1} > \pi_\cdot^{B1}$.

C.2 Proofs

Proof of Lemma C.1. The key argument is that when $L_\vartheta^i > E[L_\tau^i \,|\, \mathscr{F}_\vartheta]$ for all stopping times $\tau > \vartheta$, then we must also have $L_\vartheta^i \geq E[F_\tau^i \,|\, \mathscr{F}_\vartheta]$ for any $\tau \geq \vartheta$, strictly on $\{\tau > \vartheta\}$, as follows. First note that $F_\tau^i - E[F_{\tau_F^i(\tau)}^i \,|\, \mathscr{F}_\tau] = E[\int_\tau^{\tau_F^i(\tau)} (\pi_s^{Fi} - \pi_s^{0i}) \,|\, \mathscr{F}_\tau] \leq 0$, because $\tau_F^i(\tau_F^i(\tau)) = \tau_F^i(\tau)$. Moreover, note that $L_{\tau_F^i(\tau)}^i \geq F_{\tau_F^i(\tau)}^i$ by $\pi_\cdot^{Li} \geq \pi_\cdot^{Bi}$. Together with the hypothesis, it must thus hold that $L_\vartheta^i > E[F_\tau^i \,|\, \mathscr{F}_\vartheta]$ on $\{\tau > \vartheta\}$ for any $\tau \in \mathscr{T}$ and $L_\vartheta^i \geq F_\vartheta^i$ using $\tau = \vartheta$. Recall that also $F_\tau^i \geq M_\tau^i$ for any $\tau \in \mathscr{T}$, so L_ϑ^i is the highest possible payoff.

Now let $\sigma_1^\vartheta, \sigma_2^\vartheta \in \mathscr{S}^\vartheta$. By definition,

$$\begin{aligned} V_i^\vartheta(\sigma_i^\vartheta, \sigma_j^\vartheta) = E\biggl[&\int_{[\vartheta, \hat{\tau}^\vartheta)} \bigl(1 - G_j^\vartheta(s)\bigr) L_s^i\, dG_i^\vartheta(s) + \int_{[\vartheta, \hat{\tau}^\vartheta)} \bigl(1 - G_i^\vartheta(s)\bigr) F_s^i\, dG_j^\vartheta(s) \\ &+ \sum_{s \in [\vartheta, \hat{\tau}^\vartheta)} \Delta G_i^\vartheta(s) \Delta G_j^\vartheta(s) M_s^i + \lambda_{L,i}^\vartheta L_{\hat{\tau}^\vartheta}^i + \lambda_{L,j}^\vartheta F_{\hat{\tau}^\vartheta}^i + \lambda_M^\vartheta M_{\hat{\tau}^\vartheta}^i \,\bigg|\, \mathscr{F}_\vartheta\biggr]. \end{aligned}$$

For an upper estimate, apply $M_\tau^i \leq F_\tau^i \leq E[F_{\tau_F^i(\tau)}^i \,|\, \mathscr{F}_\tau] \leq E[L_{\tau_F^i(\tau)}^i \,|\, \mathscr{F}_\tau]$ for $\tau = \hat{\tau}^\vartheta$ and define $\tau' \in \mathscr{T}$ by $\tau' = \hat{\tau}^\vartheta$ on $\{L_{\hat{\tau}^\vartheta}^i \geq E[L_{\tau_F^i(\hat{\tau}^\vartheta)}^i \,|\, \mathscr{F}_{\hat{\tau}^\vartheta}]\} \in \mathscr{F}_{\hat{\tau}^\vartheta}$ and $\tau' = \tau_F^i(\hat{\tau}^\vartheta) \geq \hat{\tau}^\vartheta$ otherwise. Then $E[L_{\tau'}^i \,|\, \mathscr{F}_{\hat{\tau}^\vartheta}] =$

$\max\{L^i_{\hat\tau^\vartheta}, E[L^i_{\tau^i_F(\hat\tau^\vartheta)} \,|\, \mathscr{F}_{\hat\tau^\vartheta}]\}$. Moreover, apply $M^i_s \le F^i_s$ (for all $s \in \mathbb{R}_+$ a.s. due to right-continuity), write the sum as $\int_{[\vartheta,\hat\tau^\vartheta)} \Delta G^\vartheta_i(s) F^i_s \, dG^\vartheta_j(s)$, and combine it with the second integral to obtain

$$V^\vartheta_i(\sigma^\vartheta_i, \sigma^\vartheta_j) \le E\Big[\int_{[\vartheta,\hat\tau^\vartheta)} \big(1 - G^\vartheta_j(s)\big) L^i_s \, dG^\vartheta_i(s) + \int_{[\vartheta,\hat\tau^\vartheta)} \big(1 - G^\vartheta_i(s-)\big) F^i_s \, dG^\vartheta_j(s) + (\lambda^\vartheta_{L,i} + \lambda^\vartheta_{L,j} + \lambda^\vartheta_M) L^i_{\tau'} \,\Big|\, \mathscr{F}_\vartheta\Big].$$

In fact, we want to establish the upper bound $L^i_\vartheta(1 - G^\vartheta_j(\vartheta)) + F^i_\vartheta G^\vartheta_j(\vartheta)$. Using $\Delta G^\vartheta_i(\vartheta) = G^\vartheta_i(\vartheta)$, $\Delta G^\vartheta_j(\vartheta) = G^\vartheta_j(\vartheta)$, $G^\vartheta_j(\vartheta-) = 0$ and $\lambda^\vartheta_{L,i} + \lambda^\vartheta_{L,j} + \lambda^\vartheta_M = (1 - G^\vartheta_i(\hat\tau^\vartheta-))(1 - G^\vartheta_j(\hat\tau^\vartheta-))$, and then performing a change of variable on G^ϑ_i and G^ϑ_j as in the proof of Proposition 2.3 with $\tau^G_i(x) = \inf\{s \ge 0 \,|\, G^\vartheta_i(s) > x\}$ and analogously $\tau^G_j(y)$ for $x, y \in [0,1)$, we obtain

$$\begin{aligned}
&V^\vartheta_i(\sigma^\vartheta_i, \sigma^\vartheta_j) - \Big(L^i_\vartheta(1 - G^\vartheta_j(\vartheta)) + F^i_\vartheta G^\vartheta_j(\vartheta)\Big) \\
&\le E\Big[-L^i_\vartheta(1 - G^\vartheta_j(\vartheta))(1 - G^\vartheta_i(\vartheta)) + \int_{(\vartheta,\hat\tau^\vartheta)} \big(1 - G^\vartheta_j(s)\big) L^i_s \, dG^\vartheta_i(s) \\
&\qquad + \int_{(\vartheta,\hat\tau^\vartheta)} \big(1 - G^\vartheta_i(s-)\big) F^i_s \, dG^\vartheta_j(s) + (1 - G^\vartheta_i(\hat\tau^\vartheta-))(1 - G^\vartheta_j(\hat\tau^\vartheta-)) L^i_{\tau'} \,\Big|\, \mathscr{F}_\vartheta\Big] \\
&= E\Big[-\int_0^1\int_0^1 \mathbf{1}_{\{\tau^G_j(y) \in (\vartheta,\infty]\}} \mathbf{1}_{\{\tau^G_i(x) \in (\vartheta,\infty]\}} L^i_\vartheta \, dy \, dx \\
&\qquad + \int_0^1\int_0^1 \mathbf{1}_{\{\tau^G_i(x) \in (\vartheta,\hat\tau^\vartheta)\}} \mathbf{1}_{\{\tau^G_j(y) \in (\tau^G_i(x),\infty]\}} L^i_{\tau^G_i(x)} \, dy \, dx \\
&\qquad + \int_0^1\int_0^1 \mathbf{1}_{\{\tau^G_j(y) \in (\vartheta,\hat\tau^\vartheta)\}} \mathbf{1}_{\{\tau^G_i(x) \in [\tau^G_j(y),\infty]\}} F^i_{\tau^G_j(y)} \, dx \, dy \\
&\qquad + \int_0^1\int_0^1 \mathbf{1}_{\{\tau^G_i(x) \in [\hat\tau^\vartheta,\infty]\}} \mathbf{1}_{\{\tau^G_j(y) \in [\hat\tau^\vartheta,\infty]\}} L^i_{\tau'} \, dy \, dx \,\Big|\, \mathscr{F}_\vartheta\Big].
\end{aligned}$$

For any $x, y \in [0,1)$, the $\mathscr{F}_\vartheta$-conditional expectation of the integrand in the third double integral does not decrease if we replace $F^i_{\tau^G_j(y)}$ by $L^i_{\tau^i_F(\tau^G_j(y))}$, because we can use the estimate $F^i_\tau \le E[F^i_{\tau^i_F(\tau)} \,|\, \mathscr{F}_\tau] \le E[L^i_{\tau^i_F(\tau)} \,|\, \mathscr{F}_\tau]$ for $\tau = \tau^G_j(y)$ with iterated expectations thanks to $\{\tau^G_j(y) \in (\vartheta, \hat\tau^\vartheta)\} \cap \{\tau^G_i(x) \ge \tau^G_j(y)\} \in \mathscr{F}_{\tau^G_j(y)}$; the conditional expectation of the (double) integral thus does not decrease, either (cf. fn. 33). Then, in order to collect the last three double integrals, define $\tau'_{x,y} \in \mathscr{T}$ for each $(x,y) \in [0,1)^2$ by $\tau'_{x,y} = \tau^G_i(x)$ on $\{\tau^G_i(x) \in (\vartheta, \hat\tau^\vartheta)\} \cap \{\tau^G_j(y) > \tau^G_i(x)\}$, $\tau'_{x,y} = \tau^i_F(\tau^G_j(y)) \ge \tau^G_j(y)$ on $\{\tau^G_j(y) \in (\vartheta, \hat\tau^\vartheta)\} \cap \{\tau^G_i(x) \ge \tau^G_j(y)\}$ and $\tau'_{x,y} = \tau' \ge \hat\tau^\vartheta$ on $\{\tau^G_i(x) \ge \hat\tau^\vartheta\} \cap \{\tau^G_j(y) \ge \hat\tau^\vartheta\}$. Note that all these events are contained in $\mathscr{F}_{\tau^G_i(x) \wedge \tau^G_j(y) \wedge \hat\tau^\vartheta}$ and that their union is $\{\tau^G_i(x) \wedge \tau^G_j(y) > \vartheta\} \in \mathscr{F}_\vartheta$. Therefore,

$$V^\vartheta_i(\sigma^\vartheta_i, \sigma^\vartheta_j) - \Big(L^i_\vartheta(1 - G^\vartheta_j(\vartheta)) + F^i_\vartheta G^\vartheta_j(\vartheta)\Big) \le E\Big[\int_0^1\int_0^1 \mathbf{1}_{\{\tau^G_i(x) \wedge \tau^G_j(y) > \vartheta\}} \Big(L^i_{\tau'_{x,y}} - L^i_\vartheta\Big) \, dy \, dx \,\Big|\, \mathscr{F}_\vartheta\Big], \tag{C.6}$$

which is nonpositive when $L^i_\vartheta > E[L^i_\tau \,|\, \mathscr{F}_\vartheta]$ for all stopping times $\tau > \vartheta$ (cf. fn. 33 again). The inequality is strict when investment does not occur immediately for sure, i.e., when $\max\{G^\vartheta_i(\vartheta), G^\vartheta_j(\vartheta)\} < 1$, because then $\tau^G_i(x) \wedge \tau^G_j(y) > \vartheta$ for all $(x,y) \in (G^\vartheta_i(\vartheta), 1) \times (G^\vartheta_j(\vartheta), 1)$ and $\hat\tau^\vartheta > \vartheta$, so also $\tau'_{x,y} > \vartheta$ for all these (x,y). In this case, σ^ϑ_i cannot be optimal. Indeed, consider $\sigma^\vartheta_n \in \mathscr{S}^\vartheta$ that are on $\{\max\{G^\vartheta_i(\vartheta), G^\vartheta_j(\vartheta)\} < 1\}$ given by $G^\vartheta_n(t) = \mathbf{1}_{\{t \ge \vartheta + 1/n\}}$ and $\alpha^\vartheta_n(t) = 0$ for $n \in \mathbb{N}$, $n > 2$. Then, by right-continuity of $L^i_\cdot$ and $\vartheta < \hat\tau^\vartheta$, $\lim_{n\to\infty} V^\vartheta_i(\sigma^\vartheta_n, \sigma^\vartheta_j) = L^i_\vartheta(1 - G^\vartheta_j(\vartheta)) + F^i_\vartheta G^\vartheta_j(\vartheta)$. This proves the first claim.

As to the second claim, suppose by way of contradiction that $\tau = \vartheta$ attains (C.1), but that there exists a stopping time $\tau' \geq \vartheta$ such that $E[\int_\vartheta^{\tau'} (\pi_s^{Li} - \pi_s^{0i})\, ds \,|\, \mathscr{F}_\vartheta] < 0$ with positive probability. On that event,

$$\begin{aligned} L_\vartheta^i &= \int_0^\vartheta \pi_s^{0i}\, ds + E\Big[\int_\vartheta^{\tau_F^j(\vartheta)} \pi_s^{Li}\, ds + \int_{\tau_F^j(\vartheta)}^\infty \pi_s^{Bi}\, ds \,\Big|\, \mathscr{F}_\vartheta\Big] \\ &< \int_0^\vartheta \pi_s^{0i}\, ds + E\Big[\int_\vartheta^{\tau'} \pi_s^{0i}\, ds + \int_{\tau'}^{\tau_F^j(\vartheta)} \pi_s^{Li}\, ds + \int_{\tau_F^j(\vartheta)}^\infty \pi_s^{Bi}\, ds \,\Big|\, \mathscr{F}_\vartheta\Big] \leq E\Big[L_{\tau'}^i \,\Big|\, \mathscr{F}_\vartheta\Big], \end{aligned}$$

because $\tau_F^j(\tau') \geq \tau_F^j(\vartheta)$ and $\pi_\cdot^{Li} \geq \pi_\cdot^{Bi}$, which contradicts the optimality of $\tau = \vartheta$ in (C.1). □

Remark C.4. The $\mathscr{F}$-events on which $\tau > \vartheta \Rightarrow L_\vartheta^i > E[L_\tau^i \,|\, \mathscr{F}_\vartheta]$ a.s. for the stopping times $\tau \geq \vartheta$ can be aggregated into an $\mathscr{F}_\vartheta$-event as follows: With $A(\tau) := \{\tau > \vartheta\} \in \mathscr{F}_\vartheta$ and $B(\tau) := \{L_\vartheta^i > E[L_\tau^i \,|\, \mathscr{F}_\vartheta]\} \in \mathscr{F}_\vartheta$ for any stopping time $\tau \geq \vartheta$, the given property can be written as $\mathbf{1}_{\{B(\tau)\}} - \mathbf{1}_{\{A(\tau)\}} = 0$ a.s. for all $\tau \geq \vartheta$ (as $B(\tau) \subseteq A(\tau)$). The latter holds for any $\mathscr{F}$-event if and only if it is a subset of $C_0 := \{\operatorname{ess\,inf}_{\tau\geq\vartheta}(\mathbf{1}_{\{B(\tau)\}} - \mathbf{1}_{\{A(\tau)\}}) = 0\}$ (up to a nullset). As all $\mathbf{1}_{\{B(\tau)\}} - \mathbf{1}_{\{A(\tau)\}}$ are $\mathscr{F}_\vartheta$-measurable random variables, so is $\operatorname{ess\,inf}_{\tau\geq\vartheta}(\mathbf{1}_{\{B(\tau)\}} - \mathbf{1}_{\{A(\tau)\}})$. Indeed, as $\mathbf{1}_{\{B(\tau)\}} - \mathbf{1}_{\{A(\tau)\}} \geq \operatorname{ess\,inf}_{\tau\geq\vartheta}(\cdot)$, also $\mathbf{1}_{\{B(\tau)\}} - \mathbf{1}_{\{A(\tau)\}} \geq E[\operatorname{ess\,inf}_{\tau\geq\vartheta}(\cdot) \,|\, \mathscr{F}_\vartheta]$ a.s. for all $\tau \geq \vartheta$ and thus $\operatorname{ess\,inf}_{\tau\geq\vartheta}(\cdot) \geq E[\operatorname{ess\,inf}_{\tau\geq\vartheta}(\cdot) \,|\, \mathscr{F}_\vartheta]$ a.s. by the definition of $\operatorname{ess\,inf}(\cdot)$. However, as the left- and right-hand sides have the same expectation, equality holds a.s.

Moreover, there exists a sequence of mutually disjoint sets (C_n) and a sequence of stopping times (τ_n) such that $\bigcup C_n = \Omega \setminus C_0$ (up to a nullset), $\inf \tau_n \geq \vartheta$ and, on each C_n, $\tau_n > \vartheta$ and $L_\vartheta^i = E[L_{\tau_n}^i \,|\, \mathscr{F}_\vartheta]$ a.s. This follows from the fact that the family $\{\mathbf{1}_{\{B(\tau)\}} - \mathbf{1}_{\{A(\tau)\}} \,|\, \tau \geq \vartheta\}$ is directed downwards, as by all $\mathbf{1}_{\{B(\tau)\}} - \mathbf{1}_{\{A(\tau)\}}$ being $\{-1, 0\}$-valued, for any $\tau_1, \tau_2 \geq \vartheta$ also $\tau_3 := \tau_1 + (\mathbf{1}_{\{A(\tau_2)\}} - \mathbf{1}_{\{B(\tau_2)\}})(\tau_2 - \tau_1) \geq \vartheta$ is a stopping time that satisfies $\mathbf{1}_{\{A(\tau_3)\}} - \mathbf{1}_{\{B(\tau_3)\}} = \min(\mathbf{1}_{\{A(\tau_1)\}} - \mathbf{1}_{\{B(\tau_1)\}}, \mathbf{1}_{\{A(\tau_2)\}} - \mathbf{1}_{\{B(\tau_2)\}})$. There thus exists a sequence $(\tau_n) \subseteq \mathscr{T}$ with $\inf \tau_n \geq \vartheta$ and $\mathbf{1}_{\{B(\tau_n)\}} - \mathbf{1}_{\{A(\tau_n)\}} \searrow \operatorname{ess\,inf}_{\tau\geq\vartheta}(\mathbf{1}_{\{B(\tau)\}} - \mathbf{1}_{\{A(\tau)\}})$ a.s., so $P[\{\mathbf{1}_{\{B(\tau_n)\}} = \mathbf{1}_{\{A(\tau_n)\}}\} \setminus C_0] \searrow 0$. Now one can recursively set $C_n = A(\tau_n) \setminus (B(\tau_n) \cup C_{n-1})$.

Proof of Lemma C.2. First, note that there exists an optimal stopping time for (C.3) (and also a latest one), because the process to be stopped is continuous and integrable. For any stopping time $\tau \in [\vartheta, \tau_F^2(\vartheta)]$, $\tau_F^2(\tau) = \tau_F^2(\vartheta)$ and thus $L_\vartheta^1 - E[L_\tau^1 \,|\, \mathscr{F}_\vartheta] = E[\int_\vartheta^\tau (\pi_s^{L1} - \pi_s^{01})\, ds \,|\, \mathscr{F}_\vartheta]$ is the same payoff difference as that between ϑ and τ in (C.3). Therefore, when ϑ is uniquely optimal in (C.3), then also $L_\vartheta^1 > E[L_\tau^1 \,|\, \mathscr{F}_\vartheta]$ on $\{\tau > \vartheta\}$. Regarding the other possible payoffs, as argued in the proof of Lemma C.1, $M_\tau^1 \leq F_\tau^1 \leq E[F_{\tau_F^1(\tau)}^1 \,|\, \mathscr{F}_\tau] \leq E[L_{\tau_F^1(\tau)}^1 \,|\, \mathscr{F}_\tau]$, where now $\tau_F^1(\tau) \leq \tau_F^2(\tau) = \tau_F^2(\vartheta)$ for $\tau \in [\vartheta, \tau_F^2(\vartheta)]$. Hence, L_ϑ^1 exceeds the expectations of L_τ^1, F_τ^1 and M_τ^1 conditional on $\mathscr{F}_\vartheta$ for any stopping time $\tau \in (\vartheta, \tau_F^2(\vartheta)]$. The same property for any $\tau > \vartheta$ was used in the proof of Lemma C.1 only for estimating the right-hand side of (C.6). We obtain the same conclusion now (with $i = 1$, $j = 2$), because if $G_2^\vartheta(\tau_F^2(\vartheta)) = 1$, then $\tau'_{x,y} \leq \tau_F^2(\vartheta)$ for all $(x, y) \in [0, 1)$. Indeed, then $\tau_2^G(y) \leq \tau_F^2(\vartheta)$ for all $y \in [0, 1)$ and thus, as $\tau'_{x,y} \leq \tau_F^i(\tau_j^G(y))$, now $\tau'_{x,y} \leq \tau_F^1(\tau_j^G(y)) \leq \tau_F^2(\tau_j^G(y)) = \tau_F^2(\vartheta)$ by Lemma 3.2. □

Proof of Lemma C.3. When $\tau = \vartheta$ is the only stopping time attaining (C.4), then, as observed before Lemma C.3, $\vartheta = \tau_F^2(\vartheta)$ and thus $F_\vartheta^2 = L_\vartheta^2 = M_\vartheta^2$ by Lemma 3.2. Then firm 2's payoff at ϑ is F_ϑ^2 for $\sigma_2^\vartheta \in \mathscr{S}^\vartheta$ given by $G_2^\vartheta(t) = \mathbf{1}_{\{t\geq\vartheta\}}$ and $\alpha_2^\vartheta(t) = 0$ for all $t \in \mathbb{R}_+$ and any $\sigma_1^\vartheta \in \mathscr{S}^\vartheta$. Any pair of strategies with $\max\{G_1^\vartheta(\vartheta), G_2^\vartheta(\vartheta)\} < 1$ that are respectively time-consistent with strategies for $\vartheta' = \tau_{\mathcal{P}}(\vartheta)$ given by Lemma 3.3 can be shown to yield a lower payoff along the lines of the proof of Lemma C.1. Then $\hat{\tau}^\vartheta \leq \tau_{\mathcal{P}}(\vartheta)$. Moreover, $M_s^2 \leq L_s^2 \leq F_s^2$ for all $s \in [\vartheta, \tau_{\mathcal{P}}(\vartheta))$ a.s. by right-continuity and Lemma 3.4, respectively. Thus, using (B.1) for $\vartheta' = \tau_{\mathcal{P}}(\vartheta)$ and that $\lambda_{L,i}^{\vartheta} + \lambda_{L,j}^{\vartheta} + \lambda_M^{\vartheta} = (1 - G_i^{\vartheta}(\hat{\tau}^{\vartheta}-))(1 - G_j^{\vartheta}(\hat{\tau}^{\vartheta}-))$, we obtain

$$V_2^{\vartheta}(\sigma_2^{\vartheta}, \sigma_1^{\vartheta}) \leq E\Big[\int_{[\vartheta,\hat{\tau}^{\vartheta})} \big(1 - G_1^{\vartheta}(s)\big) F_s^2 \, dG_2^{\vartheta}(s) + \int_{[\vartheta,\hat{\tau}^{\vartheta})} \big(1 - G_2^{\vartheta}(s-)\big) F_s^2 \, dG_1^{\vartheta}(s)$$
$$+ (1 - G_2^{\vartheta}(\hat{\tau}^{\vartheta}-))(1 - G_1^{\vartheta}(\hat{\tau}^{\vartheta}-)) F^2_{\hat{\tau}^{\vartheta} \wedge \tau_{\mathcal{P}}(\vartheta)} \Big| \mathscr{F}_{\vartheta}\Big].$$

Performing the change of variable and setting $\tau'_{x,y} = \min\{\tau_2^G(x), \tau_1^G(y), \hat{\tau}^{\vartheta}\}$, the analogue of (C.6) which we obtain is

$$V_2^{\vartheta}(\sigma_2^{\vartheta}, \sigma_1^{\vartheta}) - F_{\vartheta}^2 \leq \int_0^1 \int_0^1 \mathbf{1}_{\{\tau_2^G(x) \wedge \tau_1^G(y) > \vartheta\}} E\Big[F^2_{\tau'_{x,y}} - F^2_{\vartheta} \Big| \mathscr{F}_{\vartheta}\Big] \, dy \, dx,$$

with analogous conclusions to those after (C.6).

As to the further claims, first note that there exists an optimal stopping time $\tau_M^i \geq \vartheta$ for (3.5) and also a latest one, because the process to be stopped is continuous and integrable. An optimal τ_M^i satisfies the necessary and sufficient conditions $E[\int_{\tau}^{\tau_M^i} (\pi_s^{0i} - \pi_s^{Bi}) \, ds \,|\, \mathscr{F}_{\tau}] \geq 0$ on $\{\tau \leq \tau_M^i\}$ and $E[\int_{\tau_M^i}^{\tau} (\pi_s^{0i} - \pi_s^{Bi}) \, ds \,|\, \mathscr{F}_{\tau_M^i}] \leq 0$ on $\{\tau \geq \tau_M^i\}$ for all stopping times $\tau \geq \vartheta$, the last inequality being strict on $\{\tau > \tau_M^i\}$ if τ_M^i is the latest solution. We will derive the analogous properties for the process (F_t^i); therefore, consider an arbitrary stopping time $\tau \geq \vartheta$.

For the first property, note that on $\{\tau \leq \tau_M^i\}$ we have

$$E\big[F^i_{\tau_M^i \wedge \tau_F^i(\tau)} \,\big|\, \mathscr{F}_{\tau}\big] - F_{\tau}^i = E\bigg[\int_{\tau}^{\tau_M^i \wedge \tau_F^i(\tau)} (\pi_s^{0i} - \pi_s^{Fi}) \, ds \bigg| \mathscr{F}_{\tau}\bigg] \geq 0$$

by $\pi_\cdot^{0i} \geq \pi_\cdot^{Fi}$ and $\tau_F^i(\tau_M^i \wedge \tau_F^i(\tau)) = \tau_F^i(\tau)$. Moreover, on the subset $\{\tau_M^i > \tau_F^i(\tau)\}$ we have

$$E\big[F^i_{\tau_M^i} \,\big|\, \mathscr{F}_{\tau_F^i(\tau)}\big] - F^i_{\tau_F^i(\tau)} = E\bigg[\int_{\tau_F^i(\tau)}^{\tau_M^i} (\pi_s^{0i} - \pi_s^{Bi}) \, ds + \int_{\tau_M^i}^{\tau_F^i(\tau_M^i)} (\pi_s^{Fi} - \pi_s^{Bi}) \, ds \bigg| \mathscr{F}_{\tau_F^i(\tau)}\bigg] \geq 0$$

by the optimality of τ_M^i and the definition of $\tau_F^i(\tau_M^i)$; cf. the proof of Lemma 3.2. Together, $E[F^i_{\tau_M^i} \,|\, \mathscr{F}_{\tau}] - F_{\tau}^i = E[F^i_{\tau_M^i} - F^i_{\tau_M^i \wedge \tau_F^i(\tau)} \,|\, \mathscr{F}_{\tau}] + E[F^i_{\tau_M^i \wedge \tau_F^i(\tau)} \,|\, \mathscr{F}_{\tau}] - F_{\tau}^i \geq 0$.

For the second property, note that $E[F^i_{\tau_F^i(\tau)} \,|\, \mathscr{F}_{\tau}] - F_{\tau}^i = E[\int_{\tau}^{\tau_F^i(\tau)} (\pi_s^{0i} - \pi_s^{Fi}) \, ds \,|\, \mathscr{F}_{\tau}] \geq 0$, again by $\pi_\cdot^{0i} \geq \pi_\cdot^{Fi}$ and $\tau_F^i(\tau_F^i(\tau)) = \tau_F^i(\tau)$, so it suffices to show $E[F^i_{\tau_F^i(\tau)} \,|\, \mathscr{F}_{\tau_M^i}] \leq F^i_{\tau_M^i}$ on $\{\tau \geq \tau_M^i\}$. Then $\tau_F^i(\tau) \geq \tau_F^i(\tau_M^i)$ and hence

$$\begin{aligned} E\big[F^i_{\tau_F^i(\tau)} \,\big|\, \mathscr{F}_{\tau_M^i}\big] - F^i_{\tau_M^i} &= E\bigg[\int_{\tau_M^i}^{\tau_F^i(\tau_M^i)} (\pi_s^{0i} - \pi_s^{Fi}) \, ds + \int_{\tau_F^i(\tau_M^i)}^{\tau_F^i(\tau)} (\pi_s^{0i} - \pi_s^{Bi}) \, ds \bigg| \mathscr{F}_{\tau_M^i}\bigg] \\ &\leq E\bigg[\int_{\tau_M^i}^{\tau_F^i(\tau_M^i)} (\pi_s^{0i} - \pi_s^{Bi}) \, ds + \int_{\tau_F^i(\tau_M^i)}^{\tau_F^i(\tau)} (\pi_s^{0i} - \pi_s^{Bi}) \, ds \bigg| \mathscr{F}_{\tau_M^i}\bigg] \leq 0, \end{aligned}$$

where we have used the definition of $\tau_F^i(\tau_M^i)$ in the first estimate, and the optimality of τ_M^i in the last. The last inequality is strict on $\{\tau > \tau_M^i\}$ if τ_M^i is the latest solution of (3.5).

Now suppose that the stopping time $\tau_M^i \geq \vartheta$ optimally stops (F_t^i) from $\vartheta \in \mathscr{T}$, i.e., it satisfies $E[F^i_{\tau_M^i} \,|\, \mathscr{F}_{\tau}] \geq F_{\tau}^i$ on $\{\tau \leq \tau_M^i\}$ and $E[F_{\tau}^i \,|\, \mathscr{F}_{\tau_M^i}] \leq F^i_{\tau_M^i}$ on $\{\tau \geq \tau_M^i\}$ for all stopping times $\tau \geq \vartheta$. As $E[F^i_{\tau_F^i(\tau_M^i)} \,|\, \mathscr{F}_{\tau_M^i}] \geq F^i_{\tau_M^i}$ as noted above, we must then have equality, i.e., $\tau_F^i(\tau_M^i)$ is optimal, too, and we may set $\tau_M^i = \tau_F^i(\tau_M^i)$ for simplicity to show optimality of $\tau_F^i(\tau_M^i)$ in (3.5). Therefore, consider again an arbitrary stopping time $\tau \geq \vartheta$.

On $\{\tau \le \tau_M^i\}$, then $\tau_F^i(\tau) \le \tau_F^i(\tau_M^i) = \tau_M^i$ and hence

$$0 \le E\big[F^i_{\tau^i_M} \,\big|\, \mathscr{F}_\tau\big] - F^i_\tau = E\bigg[\int_\tau^{\tau_F^i(\tau)} (\pi_s^{0i} - \pi_s^{Fi})\, ds + \int_{\tau_F^i(\tau)}^{\tau_M^i} (\pi_s^{0i} - \pi_s^{Bi})\, ds \,\bigg|\, \mathscr{F}_\tau\bigg]$$
$$\le E\bigg[\int_\tau^{\tau_F^i(\tau)} (\pi_s^{0i} - \pi_s^{Bi})\, ds + \int_{\tau_F^i(\tau)}^{\tau_M^i} (\pi_s^{0i} - \pi_s^{Bi})\, ds \,\bigg|\, \mathscr{F}_\tau\bigg]$$

by the definition of $\tau_F^i(\tau)$, which yields the first optimality property for τ_M^i in (3.5).

On $\{\tau \ge \tau_M^i\}$, we have $\tau_F^i(\tau) \ge \tau_M^i$ and hence

$$0 \ge E\big[F^i_\tau \,\big|\, \mathscr{F}_{\tau^i_M}\big] - F^i_{\tau^i_M} = E\bigg[\int_{\tau_M^i}^{\tau} (\pi_s^{0i} - \pi_s^{Bi})\, ds + \int_\tau^{\tau_F^i(\tau)} (\pi_s^{Fi} - \pi_s^{Bi})\, ds \,\bigg|\, \mathscr{F}_{\tau_M^i}\bigg]$$
$$\ge E\bigg[\int_{\tau_M^i}^{\tau} (\pi_s^{0i} - \pi_s^{Bi})\, ds \,\bigg|\, \mathscr{F}_{\tau_M^i}\bigg]$$

by the definition of $\tau_F^i(\tau)$ again, which yields the second optimality property for τ_M^i in (3.5). □

References

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