Construction of Local Regular Dirichlet Form on the Sierpiński Gasket using Γ-Convergence

Meng Yang

Abstract

We construct a self-similar local regular Dirichlet form on the Sierpiński gasket using Γ-convergence of stable-like non-local closed forms. Such a Dirichlet form was constructed previously by Kigami [14], but our construction has the advantage that it is a realization of a more general method of construction of a local regular Dirichlet form that works also on the Sierpiński carpet [8]. A direct consequence of this construction is the fact that the domain of the local Dirichlet form is some Besov space.

1 Introduction

Recently, Alexander Grigor'yan and the author [8] gave a purely analytic construction of a local regular Dirichlet form on the Sierpiński carpet. A natural question is whether this method can be applied on p.c.f. (post critically finite) self-similar sets. The main purpose of this paper is to prove that this method can be applied on the Sierpiński gasket, a typical example of p.c.f. self-similar sets. This also gives a clear picture how the mechanism of this method works.

The Sierpiński gasket (SG) is the simplest self-similar set in some sense. The SG can be obtained as follows. Given an equilateral triangle with sides of length 1. Divide the triangle into four congruent small triangles, each with sides of length 1/2, remove the central one. Divide each of the three remaining small triangles into four congruent triangles, each with sides of length 1/4, remove the central ones. Repeat above procedure infinitely many times, the SG is the compact connect set K that remains.

In general, local regular Dirichlet forms are in one-to-one correspondence to Brownian motions (BM). The construction of BM on the SG was given by Barlow and Perkins [2]. The construction of local regular Dirichlet form on the SG was given by Kigami [12] using difference quotients method which was generalized to p.c.f. self-similar sets in [13, 14]. Subsequently, Strichartz [22] gave the characterization of the Dirichlet form and the Laplacian using the averaging method.

The local regular Dirichlet form $\mathcal{E}_{\mathrm{loc}}$ on the SG admits a heat kernel $p_t(x,y)$ satisfying

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta^*}} \exp\left(-c\left(\frac{|x-y|}{t^{1/\beta^*}}\right)^{\frac{\beta^*}{\beta^*-1}}\right) \tag{1}$$

for all $x, y \in K, t \in (0,1)$, where $\alpha = \log 3/\log 2$ is the Hausdorff dimension of the SG and

$$\beta^* = \frac{\log 5}{\log 2}$$

is the *walk dimension of BM* which is frequently denoted also by d_w. The estimates (1) were obtained by Barlow and Perkins [2].

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The domain $\mathcal{F}_{\text{loc}}$ of $\mathcal{E}_{\text{loc}}$ is some Besov space. This was given by Jonsson [11]. Later on, this kind of characterization was generalized to simple nested fractals by Pietruska-Pałuba [18] and p.c.f. self-similar sets by Hu and Wang [10]. This kind of characterization was also given by Pietruska-Pałuba [19], Grigor'yan, Hu and Lau [7], Kumagai and Sturm [16] if local regular Dirichlet forms on metric measure spaces admit sub-Gaussian heat kernel estimates. Here, we reprove this characterization as a direct corollary of our construction.

Consider the following stable-like non-local quadratic form

$$\begin{aligned} \mathfrak{E}_\beta(u,u) &= \int_K\int_K \frac{(u(x)-u(y))^2}{|x-y|^{\alpha+\beta}}\nu(\mathrm{d}x)\nu(\mathrm{d}y),\\ \mathcal{F}_\beta &= \left\{u\in L^2(K;\nu):\mathfrak{E}_\beta(u,u)<+\infty\right\}, \end{aligned} \tag{2}$$

where $\alpha=\dim_{\mathcal{H}}K$ as above, ν is the normalized Hausdorff measure on K of dimension α and $\beta>0$ is so far arbitrary.

Using the estimates (1) and subordination technique, it was proved by Pietruska-Pałuba [20] that

$$\varliminf_{\beta\uparrow\beta^*}(\beta^*-\beta)\mathfrak{E}_\beta(u,u)\asymp\mathcal{E}_{\text{loc}}(u,u)\asymp\varlimsup_{\beta\uparrow\beta^*}(\beta^*-\beta)\mathfrak{E}_\beta(u,u) \tag{3}$$

for all $u\in\mathcal{F}_{\text{loc}}$. This is similar to the following classical result

$$\lim_{\beta\uparrow2}(2-\beta)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(u(x)-u(y))^2}{|x-y|^{n+\beta}}\mathrm{d}x\mathrm{d}y=C(n)\int_{\mathbb{R}^n}|\nabla u(x)|^2\mathrm{d}x$$

for all $u\in W^{1,2}(\mathbb{R}^n)$, where $C(n)$ is some positive constant, see [6, Example 1.4.1]. Recently, the author [23] gave an alternative proof of (3) using discretization method. Here, we reprove (3) as a direct corollary of our construction.

The main purpose of this paper is to give a construction of a local regular Dirichlet form $\mathcal{E}_{\text{loc}}$ on the SG using Γ-convergence of stable-like non-local closed forms of type (2) as $\beta\uparrow\beta^*$. This is our main result Theorem 2.1. The local regular Dirichlet form given here coincides with that given by Kigami due to the uniqueness result given by Sabot [21]. Kusuoka and Zhou [17] gave a construction using the averaging method and approximation of Markov chains.

The idea of our construction of $\mathcal{E}_{\text{loc}}$ is as follows. First, we use the averaging method to construct another quadratic form $\mathcal{E}_\beta$, equivalent to $\mathfrak{E}_\beta$, which turns out to be a regular closed form for all $\beta\in(\alpha,\beta^*)$. Second, we construct a regular closed form $\mathcal{E}$ as a Γ-limit of a sequence $\{(\beta^*-\beta_n)\mathcal{E}_{\beta_n}\}$ with $\beta_n\uparrow\beta^*$. However, $\mathcal{E}$ is not necessarily Markovian, local or self-similar. Third, we use a standard method from [17] to construct $\mathcal{E}_{\text{loc}}$ from $\mathcal{E}$.

The main difficulty in our construction is that we do not have monotonicity property as in Kigami's construction. Nevertheless we have weak monotonicity that allows to obtain the characterization of the Γ-limit. To prove the non-triviality and the regularity of the Γ-limit, we construct on the SG functions with controlled energy and with separation property that are called *good* functions.

The ultimate purpose of the current paper and [8] is to provide a new unified method of construction of local regular Dirichlet forms on a wide class of fractals that uses only self-similar property and ideally should be independent of other specific properties, in particular, p.c.f. property.

This paper is organized as follows. In Section 2, we give statement of the main results. In Section 3, we give resistance estimates and introduce good functions. In Section 4, we give weak monotonicity result. In Section 5, we prove Theorem 2.1.

The paper [15] of Kumagai contains partially similar arguments in spirit.

2 Statement of the Main Results

Consider the following points in $\mathbb{R}^2$: $p_0=(0,0)$, $p_1=(1,0)$, $p_2=(1/2,\sqrt{3}/2)$. Let $f_i(x)=(x+p_i)/2$, $x\in\mathbb{R}^2$. Then the Sierpiński gasket (SG) is the unique non-empty compact set K satisfying $K=f_0(K)\cup f_1(K)\cup f_2(K)$. Let ν be the normalized Hausdorff measure on K of dimension $\alpha=\log 3/\log 2$. Let

$$V_0=\{p_0,p_1,p_2\},V_{n+1}=f_0(V_n)\cup f_1(V_n)\cup f_2(V_n)\text{ for all }n\ge0.$$

Then $\{V_n\}$ is an increasing sequence of finite sets and K is the closure of $V^* = \cup_{n=0}^\infty V_n$.

Let

$$\mathfrak{E}_n(u,u) = \left(\frac{5}{3}\right)^n \sum_{\substack{x,y\in V_n\\|x-y|=2^{-n}}} (u(x)-u(y))^2, n \ge 0, u \in l(K),$$

where $l(S)$ is the set of all real-valued functions on the set S. Then $\mathfrak{E}_n(u,u)$ is monotone increasing in n for all $u \in l(K)$. Let

$$\mathfrak{E}_{\mathrm{loc}}(u,u) = \lim_{n\to+\infty} \mathfrak{E}_n(u,u) = \lim_{n\to+\infty} \left(\frac{5}{3}\right)^n \sum_{\substack{x,y\in V_n\\|x-y|=2^{-n}}} (u(x)-u(y))^2,$$

$$\mathfrak{F}_{\mathrm{loc}} = \left\{u \in C(K) : \mathfrak{E}_{\mathrm{loc}}(u,u) < +\infty\right\},$$

then $(\mathfrak{E}_{\mathrm{loc}}, \mathfrak{F}_{\mathrm{loc}})$ is a self-similar local regular Dirichlet form on $L^2(K;\nu)$, see [12, 13, 14].

Let $W_0 = \{\emptyset\}$ and

$$W_n = \{w = w_1 \dots w_n : w_i = 0,1,2, i = 1,\dots,n\} \text{ for all } n \ge 1.$$

For all $w^{(1)} = w^{(1)}_1 \dots w^{(1)}_m \in W_m, w^{(2)} = w^{(2)}_1 \dots w^{(2)}_n \in W_n$, denote $w^{(1)}w^{(2)}$ as $w = w_1 \dots w_{m+n} \in W_{m+n}$ with $w_i = w^{(1)}_i$ for all $i = 1,\dots,m$ and $w_{m+i} = w^{(2)}_i$ for all $i = 1,\dots,n$. For all $i = 0,1,2$, denote i^n as $w = w_1 \dots w_n \in W_n$ with $w_k = i$ for all $k = 1,\dots,n$. For all $w = w_1 \dots w_{n-1}w_n \in W_n$, denote $w^- = w_1 \dots w_{n-1} \in W_{n-1}$.

For all $w = w_1 \dots w_n \in W_n$, let

$$f_w = f_{w_1} \circ \dots \circ f_{w_n},$$

$$V_w = f_w(V_0), K_w = f_w(K), P_w = f_{w^-}(p_{w_n}),$$

where $f_\emptyset = \mathrm{id}$ is the identity map.

For all $n \ge 1$, let X_n be the graph with vertex set W_n and edge set H_n given by

$$H_n = \left\{(w^{(1)}, w^{(2)}) : w^{(1)}, w^{(2)} \in W_n, w^{(1)} \ne w^{(2)}, K_{w^{(1)}} \cap K_{w^{(2)}} \ne \emptyset\right\}.$$

Denote $w^{(1)} \sim_n w^{(2)}$ if $(w^{(1)}, w^{(2)}) \in H_n$.

For all $u \in L^2(K;\nu)$, $n \ge 1$, let $P_n u : W_n \to \mathbb{R}$ be given by

$$P_n u(w) = \frac{1}{\nu(K_w)} \int_{K_w} u(x)\nu(\mathrm{d}x) = \int_K (u \circ f_w)(x)\nu(\mathrm{d}x), w \in W_n.$$

Our main result is as follows.

Theorem 2.1. *There exists a self-similar strongly local regular Dirichlet form* $(\mathcal{E}_{\mathrm{loc}}, \mathcal{F}_{\mathrm{loc}})$ *on* $L^2(K;\nu)$ *satisfying*

$$\mathcal{E}_{\mathrm{loc}}(u,u) \asymp \sup_{n\ge1} \left(\frac{5}{3}\right)^n \sum_{w^{(1)}\sim_n w^{(2)}} \left(P_n u(w^{(1)}) - P_n u(w^{(2)})\right)^2,$$

$$\mathcal{F}_{\mathrm{loc}} = \left\{u \in L^2(K;\nu) : \sup_{n\ge1} \left(\frac{5}{3}\right)^n \sum_{w^{(1)}\sim_n w^{(2)}} \left(P_n u(w^{(1)}) - P_n u(w^{(2)})\right)^2 < +\infty\right\}.$$

Remark 2.2. *Above theorem was also proved by Kusuoka and Zhou [17, Theorem 7.19, Example 8.4] using approximation of Markov chains. Here, we use* Γ*-convergence of stable-like non-local closed forms.*

Let us introduce the notion of Besov spaces. Let (M,d,μ) be a metric measure space and $\alpha,\beta > 0$ two parameters. Define

$$[u]_{B^{2,2}_{\alpha,\beta}(M)} = \sum_{n=1}^\infty 2^{(\alpha+\beta)n} \int\limits_M \int\limits_{d(x,y)<2^{-n}} (u(x)-u(y))^2 \mu(\mathrm{d}y)\mu(\mathrm{d}x),$$

$$[u]_{B^{2,\infty}_{\alpha,\beta}(M)} = \sup_{n\ge1} 2^{(\alpha+\beta)n} \int\limits_M \int\limits_{d(x,y)<2^{-n}} (u(x)-u(y))^2 \mu(\mathrm{d}y)\mu(\mathrm{d}x),$$

and

$$B^{2,2}_{\alpha,\beta}(M) = \left\{u \in L^2(M;\mu) : [u]_{B^{2,2}_{\alpha,\beta}(M)} < +\infty\right\},$$
$$B^{2,\infty}_{\alpha,\beta}(M) = \left\{u \in L^2(M;\mu) : [u]_{B^{2,\infty}_{\alpha,\beta}(M)} < +\infty\right\}.$$

It is easily proved that $B^{2,2}_{\alpha,\beta}(K)$ and $B^{2,\infty}_{\alpha,\beta}(K)$ have the following equivalent semi-norms.

Lemma 2.3. *([9, Lemma 3.1],[23, Lemma 2.1]) For all $\beta \in (0,+\infty), u \in L^2(K;\nu)$*

$$\mathcal{E}_\beta(u,u) \asymp \mathfrak{E}_\beta(u,u) \asymp [u]_{B^{2,2}_{\alpha,\beta}(K)},$$

$$\sup_{n\ge1} 2^{(\beta-\alpha)n} \sum_{w^{(1)}\sim_n w^{(2)}} \left(P_n u(w^{(1)}) - P_n u(w^{(2)})\right)^2 \asymp [u]_{B^{2,\infty}_{\alpha,\beta}(K)},$$

where

$$\mathcal{E}_\beta(u,u) = \sum_{n=1}^{\infty} 2^{(\beta-\alpha)n} \sum_{w^{(1)}\sim_n w^{(2)}} \left(P_n u(w^{(1)}) - P_n u(w^{(2)})\right)^2,$$
$$\mathfrak{E}_\beta(u,u) = \int_K \int_K \frac{(u(x)-u(y))^2}{|x-y|^{\alpha+\beta}} \nu(\mathrm{d}x)\nu(\mathrm{d}y).$$

We have the following two corollaries whose proofs are obvious by Lemma 2.3 and the proof of Theorem 2.1.

First, we have the characterization of the domain of the local Dirichlet form.

Corollary 2.4. *$\mathcal{F}_{\mathrm{loc}} = B^{2,\infty}_{\alpha,\beta^*}(K)$ and $\mathcal{E}_{\mathrm{loc}}(u,u) \asymp [u]_{B^{2,\infty}_{\alpha,\beta^*}(K)}$ for all $u \in \mathcal{F}_{\mathrm{loc}}$, where $\alpha = \log 3/\log 2$ is the Hausdorff dimension and $\beta^* = \log 5/\log 2$ is the walk dimension of BM.*

Second, we have the approximation of non-local forms to the local form.

Corollary 2.5. *There exists some positive constant C such that for all $u \in \mathcal{F}_{\mathrm{loc}}$*

$$\frac{1}{C}\mathcal{E}_{\mathrm{loc}}(u,u) \le \varliminf_{\beta\uparrow\beta^*} (\beta^*-\beta)\mathcal{E}_\beta(u,u) \le \varlimsup_{\beta\uparrow\beta^*} (\beta^*-\beta)\mathcal{E}_\beta(u,u) \le C\mathcal{E}_{\mathrm{loc}}(u,u),$$

$$\frac{1}{C}\mathcal{E}_{\mathrm{loc}}(u,u) \le \varliminf_{\beta\uparrow\beta^*} (\beta^*-\beta)\mathfrak{E}_\beta(u,u) \le \varlimsup_{\beta\uparrow\beta^*} (\beta^*-\beta)\mathfrak{E}_\beta(u,u) \le C\mathcal{E}_{\mathrm{loc}}(u,u),$$

$$\frac{1}{C}\mathcal{E}_{\mathrm{loc}}(u,u) \le \varliminf_{\beta\uparrow\beta^*} (\beta^*-\beta)[u]_{B^{2,2}_{\alpha,\beta}(K)} \le \varlimsup_{\beta\uparrow\beta^*} (\beta^*-\beta)[u]_{B^{2,2}_{\alpha,\beta}(K)} \le C\mathcal{E}_{\mathrm{loc}}(u,u).$$

3 Resistance Estimates and Good Functions

First, we give resistance estimates. We need two techniques from electrical network. The first is the well-known Δ-Y transform, see [14, Lemma 2.1.15]. The second is shorting and cutting technique, see [5].

For all $n \ge 1$, let us introduce an energy on W_n given by

$$E_n(u,u) = \sum_{w^{(1)}\sim_n w^{(2)}} \left(u(w^{(1)}) - u(w^{(2)})\right)^2, u \in l(W_n).$$

For all $w^{(1)}, w^{(2)} \in W_n$, let effective resistance be given by

$$\begin{aligned} R_n(w^{(1)},w^{(2)}) &= \inf\left\{E_n(u,u) : u(w^{(1)}) = 1, u(w^{(2)}) = 0, u \in l(W_n)\right\}^{-1} \\ &= \sup\left\{\frac{\left(u(w^{(1)}) - u(w^{(2)})\right)^2}{E_n(u,u)} : E_n(u,u) \ne 0, u \in l(W_n)\right\}. \end{aligned}$$

It is obvious that R_n is a metric on W_n.

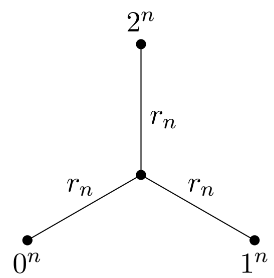

Figure 1: An Equivalent Electrical Network

Theorem 3.1. *Considering effective resistances between any two of $0^n, 1^n, 2^n$, we have the electrical network X_n is equivalent to the electrical network in Figure 1, where*

$$r_n = \frac{1}{2}\left(\frac{5}{3}\right)^n - \frac{1}{2}.$$

Proof. The proof is elementary using Δ-Y transform. □

Remark 3.2. *For all $n \geq 1$, we have*

$$R_n(0^n, 1^n) = R_n(1^n, 2^n) = R_n(0^n, 2^n) = 2r_n = \left(\frac{5}{3}\right)^n - 1.$$

Proposition 3.3. *For all $n \geq 1, w \in W_n$, we have*

$$R_n(w, 0^n), R_n(w, 1^n,), R_n(w, 2^n) \leq \frac{5}{2}\left(\frac{5}{3}\right)^n.$$

Proof. By symmetry, we only need to consider $R_n(w, 0^n)$. Letting $w = w_1 \dots w_{n-2}w_{n-1}w_n$, we construct a finite sequence in W_n as follows.

$$\begin{aligned}
&w^{(1)} = w_1 \dots w_{n-2}w_{n-1}w_n = w, w^{(2)} = w_1 \dots w_{n-2}w_{n-1}w_{n-1},\\
&w^{(3)} = w_1 \dots w_{n-2}w_{n-2}w_{n-2}, \dots,\\
&w^{(n)} = w_1 \dots w_1 w_1 w_1, w^{(n+1)} = 0 \dots 000.
\end{aligned}$$

For all $i = 1, \dots, n-1$, by cutting technique, we have

$$\begin{aligned}
&R_n(w^{(i)}, w^{(i+1)})\\
=&R_n(w_1 \dots w_{n-i-1}w_{n-i}w_{n-i+1} \dots w_{n-i+1}, w_1 \dots w_{n-i-1}w_{n-i}w_{n-i} \dots w_{n-i})\\
\leq&R_i(w_{n-i+1} \dots w_{n-i+1}, w_{n-i} \dots w_{n-i}) \leq R_i(0^i, 1^i) = \left(\frac{5}{3}\right)^i - 1 \leq \left(\frac{5}{3}\right)^i.
\end{aligned}$$

Since

$$R_n(w^{(n)}, w^{(n+1)}) = R_n(w_1^n, 0^n) \leq R_n(0^n, 1^n) = \left(\frac{5}{3}\right)^n - 1 \leq \left(\frac{5}{3}\right)^n,$$

we have

$$R_n(w, 0^n) = R_n(w^{(1)}, w^{(n+1)}) \leq \sum_{i=1}^{n} R_n(w^{(i)}, w^{(i+1)}) \leq \sum_{i=1}^{n}\left(\frac{5}{3}\right)^i \leq \frac{5}{2}\left(\frac{5}{3}\right)^n.$$

□

Second, we introduce good functions with energy property and separation property.

For all $x_0, x_1, x_2 \in \mathbb{R}$, let $U^{(x_0,x_1,x_2)} : K \to \mathbb{R}$ be the standard harmonic function with boundary value x_0, x_1, x_2 on p_0, p_1, p_2, respectively, see [12].

Let

$$\mathcal{U} = \left\{ U^{(x_0,x_1,x_2)} : x_0, x_1, x_2 \in \mathbb{R} \right\}.$$

For all $u \in L^2(K;\nu)$, $n \geq 1$, let

$$A_n(u) = E_n(P_n u, P_n u) = \sum_{w^{(1)} \sim_n w^{(2)}} \left(P_n u(w^{(1)}) - P_n u(w^{(2)}) \right)^2.$$

We have energy property as follows.

Theorem 3.4. *([22, Theorem 3.1]) For all $U = U^{(x_0,x_1,x_2)} \in \mathcal{U}$, $n \geq 1$, we have*

$$A_n(U) = \frac{2}{3} \left[\left(\frac{3}{5}\right)^n - \left(\frac{3}{5}\right)^{2n} \right] \left((x_0 - x_1)^2 + (x_1 - x_2)^2 + (x_0 - x_2)^2 \right).$$

We have separation property as follows.

Proposition 3.5. *$\mathcal{U}$ separates points, that is, for all $x, y \in K$ with $x \neq y$, there exists $U \in \mathcal{U}$ such that $U(x) \neq U(y)$.*

Proof. Without lose of generality, we may assume that $x \in K_0 \backslash K_1$ and $y \in K_1 \backslash K_0$. Take $U = U^{(1,0,0)} \in \mathcal{U}$, then $U(x) \in [\frac{2}{5}, 1]$ and $U(y) \in [0, \frac{2}{5})$, hence $U(x) > U(y)$. □

4 Weak Monotonicity Result

In this section, we give weak monotonicity result using resistance estimates.

For all $u \in L^2(K;\nu)$, $n \geq 1$, let

$$D_n(u) = \left(\frac{5}{3}\right)^n A_n(u) = \left(\frac{5}{3}\right)^n \sum_{w^{(1)} \sim_n w^{(2)}} \left(P_n u(w^{(1)}) - P_n u(w^{(2)}) \right)^2.$$

The weak monotonicity result is as follows.

Theorem 4.1. *There exists some positive constant C such that*

$$D_n(u) \leq C D_{n+m}(u) \text{ for all } u \in L^2(K;\nu), n, m \geq 1.$$

Indeed, we can take $C = 36$.

Remark 4.2. *In Kigami's construction, the energies are monotone, that is, the constant $C = 1$. Hence above result is called weak monotonicity result.*

Theorem 4.1 can be reduced as follows.

For all $n \geq 1$, let

$$G_n(u) = \left(\frac{5}{3}\right)^n E_n(u,u) = \left(\frac{5}{3}\right)^n \sum_{w^{(1)} \sim_n w^{(2)}} \left(u(w^{(1)}) - u(w^{(2)}) \right)^2, u \in l(W_n).$$

For all $n, m \geq 1$, let $M_{n,m} : l(W_{n+m}) \to l(W_n)$ be a mean value operator given by

$$(M_{n,m} u)(w) = \frac{1}{3^m} \sum_{v \in W_m} u(wv), w \in W_n, u \in l(W_{n+m}).$$

Theorem 4.3. *There exists some positive constant C such that*

$$G_n(M_{n,m} u) \leq C G_{n+m}(u) \text{ for all } u \in l(W_{n+m}), n, m \geq 1.$$

Proof of Theorem 4.1 using Theorem 4.3. Note $P_n u = M_{n,m}(P_{n+m} u)$, hence

$$\begin{aligned} D_n(u) &= \left(\frac{5}{3}\right)^n \sum_{w^{(1)} \sim_n w^{(2)}} \left(P_n u(w^{(1)}) - P_n u(w^{(2)}) \right)^2 = G_n(P_n u) \\ &= G_n(M_{n,m}(P_{n+m} u)) \leq C G_{n+m}(P_{n+m} u) \\ &= C \left(\frac{5}{3}\right)^{n+m} \sum_{w^{(1)} \sim_{n+m} w^{(2)}} \left(P_{n+m} u(w^{(1)}) - P_{n+m} u(w^{(2)}) \right)^2 = C D_{n+m}(u). \end{aligned}$$

□

Proof of Theorem 4.3. Fix $n \geq 1$. Assume that $W \subseteq W_n$ is connected, that is, for all $w^{(1)}, w^{(2)} \in W$, there exists a finite sequence $\left\{v^{(1)}, \ldots, v^{(k)}\right\} \subseteq W$ with $v^{(1)} = w^{(1)}, v^{(k)} = w^{(2)}$ and $v^{(i)} \sim_n v^{(i+1)}$ for all $i = 1, \ldots, k-1$. Let us introduce an energy on W given by

$$E_W(u,u) = \sum_{\substack{w^{(1)}, w^{(2)} \in W \\ w^{(1)} \sim_n w^{(2)}}} (u(w^{(1)}) - u(w^{(2)}))^2, u \in l(W).$$

For all $w^{(1)}, w^{(2)} \in W$, let effective resistance be given by

$$\begin{aligned} R_W(w^{(1)}, w^{(2)}) &= \inf\left\{E_W(u,u) : u(w^{(1)}) = 1, u(w^{(2)}) = 0, u \in l(W)\right\}^{-1} \\ &= \sup\left\{\frac{(u(w^{(1)}) - u(w^{(2)}))^2}{E_W(u,u)} : E_W(u,u) \neq 0, u \in l(W)\right\}. \end{aligned}$$

It is obvious that R_W is a metric on W.

By definition, we have

$$\begin{aligned} G_n(M_{n,m}u) &= \left(\frac{5}{3}\right)^n \sum_{w^{(1)} \sim_n w^{(2)}} \left(\frac{1}{3^m} \sum_{v \in W_m} \left(u(w^{(1)}v) - u(w^{(2)}v)\right)\right)^2 \\ &\leq \left(\frac{5}{3}\right)^n \sum_{w^{(1)} \sim_n w^{(2)}} \frac{1}{3^m} \sum_{v \in W_m} \left(u(w^{(1)}v) - u(w^{(2)}v)\right)^2. \end{aligned}$$

For all $w^{(1)} \sim_n w^{(2)}$, there exist $i, j = 0, 1, 2$ such that $w^{(1)} i^m \sim_{n+m} w^{(2)} j^m$. For all $v \in W_m$, we have

$$\left(u(w^{(1)}v) - u(w^{(2)}v)\right)^2 \leq R_{w^{(1)}W_m \cup w^{(2)}W_m}(w^{(1)}v, w^{(2)}v) E_{w^{(1)}W_m \cup w^{(2)}W_m}(u,u).$$

By cutting technique and Proposition 3.3, we have

$$\begin{aligned} & R_{w^{(1)}W_m \cup w^{(2)}W_m}(w^{(1)}v, w^{(2)}v) \\ \leq & R_{w^{(1)}W_m \cup w^{(2)}W_m}(w^{(1)}v, w^{(1)}i^m) + R_{w^{(1)}W_m \cup w^{(2)}W_m}(w^{(1)}i^m, w^{(2)}j^m) \\ + & R_{w^{(1)}W_m \cup w^{(2)}W_m}(w^{(2)}j^m, w^{(2)}v) \\ \leq & R_m(v, i^m) + 1 + R_m(v, j^m) \leq 5\left(\frac{5}{3}\right)^m + 1 \leq 6\left(\frac{5}{3}\right)^m, \end{aligned}$$

hence

$$\begin{aligned} & (u(w^{(1)}v) - u(w^{(2)}v))^2 \leq 6\left(\frac{5}{3}\right)^m E_{w^{(1)}W_m \cup w^{(2)}W_m}(u,u) \\ = & 6\left(\frac{5}{3}\right)^m \left(E_{w^{(1)}W_m}(u,u) + E_{w^{(2)}W_m}(u,u) + \left(u(w^{(1)}i^m) - u(w^{(2)}j^m)\right)^2\right). \end{aligned}$$

Hence

$$\begin{aligned} & \frac{1}{3^m} \sum_{v \in W_m} \left(u(w^{(1)}v) - u(w^{(2)}v)\right)^2 \\ \leq & 6\left(\frac{5}{3}\right)^m \left(E_{w^{(1)}W_m}(u,u) + E_{w^{(2)}W_m}(u,u) + \left(u(w^{(1)}i^m) - u(w^{(2)}j^m)\right)^2\right). \end{aligned}$$

In the summation with respect to $w^{(1)} \sim_n w^{(2)}$, the terms $E_{w^{(1)}W_m}(u,u)$ and $E_{w^{(2)}W_m}(u,u)$ are summed at most 6 times, hence

$$G_n(M_{n,m}u) \leq 6\left(\frac{5}{3}\right)^n 6\left(\frac{5}{3}\right)^m E_{n+m}(u,u) = 36\left(\frac{5}{3}\right)^{n+m} E_{n+m}(u,u) = 36 G_{n+m}(u).$$

□

5 Proof of Theorem 2.1

We need the following result for preparation.

Theorem 5.1. *([7, Theorem 4.11 (iii)]) For all $u \in L^2(K;\nu)$, let*

$$E(u) = \sum_{n=1}^{\infty} 2^{(\beta-\alpha)n} \sum_{w^{(1)}\sim_n w^{(2)}} \left(P_n u(w^{(1)}) - P_n u(w^{(2)})\right)^2,$$

$$F(u) = \sup_{n\geq 1} 2^{(\beta-\alpha)n} \sum_{w^{(1)}\sim_n w^{(2)}} \left(P_n u(w^{(1)}) - P_n u(w^{(2)})\right)^2.$$

Then for all $\beta \in (\alpha, +\infty)$, there exists some positive constant c such that

$$|u(x) - u(y)| \leq c\sqrt{E(u)}|x-y|^{\frac{\beta-\alpha}{2}}, \tag{4}$$

$$|u(x) - u(y)| \leq c\sqrt{F(u)}|x-y|^{\frac{\beta-\alpha}{2}}, \tag{5}$$

for ν-almost every $x, y \in K$, for all $u \in L^2(K;\nu)$.

Remark 5.2. *If $u \in L^2(K;\nu)$ satisfies $E(u) < +\infty$ or $F(u) < +\infty$, then u has a continuous version in $C^{\frac{\beta-\alpha}{2}}(K)$.*

We collect some basic facts about Γ-convergence. In what follows, K is a locally compact separable metric space and ν is a Radon measure on K with full support.

We say that $(\mathcal{E}, \mathcal{F})$ is a *closed form on $L^2(K;\nu)$ in the wide sense* if $\mathcal{F}$ is complete under the inner product $\mathcal{E}_1$ but $\mathcal{F}$ is not necessary to be dense in $L^2(K;\nu)$. If $(\mathcal{E}, \mathcal{F})$ is a closed form on $L^2(K;\nu)$ in the wide sense, we extend $\mathcal{E}$ to be $+\infty$ outside $\mathcal{F}$, hence the information of $\mathcal{F}$ is encoded in $\mathcal{E}$.

Definition 5.3. *Let $\mathcal{E}^n, \mathcal{E}$ be closed forms on $L^2(K;\nu)$ in the wide sense. We say that $\mathcal{E}^n$ is Γ-convergent to $\mathcal{E}$ if the following conditions are satisfied.*

(1) For all $\{u_n\} \subseteq L^2(K;\nu)$ that converges strongly to $u \in L^2(K;\nu)$, we have

$$\varliminf_{n\to+\infty} \mathcal{E}^n(u_n, u_n) \geq \mathcal{E}(u, u).$$

(2) For all $u \in L^2(K;\nu)$, there exists a sequence $\{u_n\} \subseteq L^2(K;\nu)$ converging strongly to u in $L^2(K;\nu)$ such that

$$\varlimsup_{n\to+\infty} \mathcal{E}^n(u_n, u_n) \leq \mathcal{E}(u, u).$$

Proposition 5.4. *([4, Proposition 6.8, Theorem 8.5, Theorem 11.10, Proposition 12.16]) Let $\{(\mathcal{E}^n, \mathcal{F}^n)\}$ be a sequence of closed forms on $L^2(K;\nu)$ in the wide sense, then there exist some subsequence $\{(\mathcal{E}^{n_k}, \mathcal{F}^{n_k})\}$ and some closed form $(\mathcal{E}, \mathcal{F})$ on $L^2(K;\nu)$ in the wide sense such that $\mathcal{E}^{n_k}$ is Γ-convergent to $\mathcal{E}$.*

In what follows, K is the SG in $\mathbb{R}^2$ and ν is the normalized Hausdorff measure on K.

We have non-local regular closed forms and Dirichlet forms as follows.

For all $\beta > 0$, let

$$\mathcal{E}_\beta(u, u) = \sum_{n=1}^{\infty} 2^{(\beta-\alpha)n} \sum_{w^{(1)}\sim_n w^{(2)}} \left(P_n u(w^{(1)}) - P_n u(w^{(2)})\right)^2,$$

$$\mathcal{F}_\beta = \left\{ u \in L^2(K;\nu) : \sum_{n=1}^{\infty} 2^{(\beta-\alpha)n} \sum_{w^{(1)}\sim_n w^{(2)}} \left(P_n u(w^{(1)}) - P_n u(w^{(2)})\right)^2 < +\infty \right\},$$

denote $\mathscr{E}_\beta(u, u) = [u]_{B^{2,2}_{\alpha,\beta}(K)}$ for simplicity.

Theorem 5.5. *For all $\beta \in (\alpha, \beta^*)$, $(\mathcal{E}_\beta, \mathcal{F}_\beta)$ is a non-local regular closed form on $L^2(K;\nu)$, $(\mathscr{E}_\beta, \mathcal{F}_\beta), (\mathfrak{E}_\beta, \mathcal{F}_\beta)$ are non-local regular Dirichlet forms on $L^2(K;\nu)$. For all $\beta \in [\beta^*, +\infty)$, $\mathcal{F}_\beta$ consists only of constant functions.*

Remark 5.6. *$\mathcal{E}_\beta$ does not have Markovian property but $\mathscr{E}_\beta, \mathfrak{E}_\beta$ do have Markovian property. An interesting problem in non-local analysis is for which exponent $\beta > 0$, $(\mathfrak{E}_\beta, \mathcal{F}_\beta)$ is a regular Dirichlet form on $L^2(K;\nu)$. The critical exponent*

$$\beta_* = \sup\left\{\beta > 0 : (\mathfrak{E}_\beta, \mathcal{F}_\beta) \text{ is a regular Dirichlet form on } L^2(K;\nu)\right\}$$

is called the walk dimension of the SG *with Euclidean metric and Hausdorff measure. A classical approach to determine β_* is using the estimates (1) and subordination technique to have*

$$\beta_* = \beta^* = \frac{\log 5}{\log 2},$$

see [19]. The following proof provides an alternative approach without using BM.

Proof of Theorem 5.5. By Fatou's lemma, it is obvious that $(\mathcal{E}_\beta, \mathcal{F}_\beta)$ is a closed form on $L^2(K;\nu)$ in the wide sense.

For all $\beta \in (\alpha, \beta^*)$. By Theorem 5.1, $\mathcal{F}_\beta \subseteq C(K)$. We only need to show that $\mathcal{F}_\beta$ is uniformly dense in $C(K)$, then $\mathcal{F}_\beta$ is dense in $L^2(K;\nu)$, hence $(\mathcal{E}_\beta, \mathcal{F}_\beta)$ is a regular closed form on $L^2(K;\nu)$.

Indeed, by Theorem 3.4, for all $U = U^{(x_0,x_1,x_2)} \in \mathcal{U}$, we have

$$\begin{aligned}
\mathcal{E}_\beta(U,U) &= \sum_{n=1}^{\infty} 2^{(\beta-\alpha)n} \frac{2}{3} \left[\left(\frac{3}{5}\right)^n - \left(\frac{3}{5}\right)^{2n}\right] \left((x_0 - x_1)^2 + (x_1 - x_2)^2 + (x_0 - x_2)^2\right)\\
&\le \frac{2}{3}\left((x_0 - x_1)^2 + (x_1 - x_2)^2 + (x_0 - x_2)^2\right) \sum_{n=1}^{\infty} 2^{(\beta-\alpha)n} \left(\frac{3}{5}\right)^n < +\infty,
\end{aligned}$$

hence $U \in \mathcal{F}_\beta$, $\mathcal{U} \subseteq \mathcal{F}_\beta$. By Proposition 3.5, $\mathcal{F}_\beta$ separates points. It is obvious that $\mathcal{F}_\beta$ is a sub-algebra of $C(K)$, that is, for all $u, v \in \mathcal{F}_\beta, c \in \mathbb{R}$, we have $u + v, cu, uv \in \mathcal{F}_\beta$. By Stone-Weierstrass theorem, $\mathcal{F}_\beta$ is uniformly dense in $C(K)$.

Since $\mathscr{E}_\beta, \mathfrak{E}_\beta$ do have Markovian property, by above, $(\mathscr{E}_\beta, \mathcal{F}_\beta), (\mathfrak{E}_\beta, \mathcal{F}_\beta)$ are non-local regular Dirichlet forms on $L^2(K;\nu)$.

For all $\beta \in [\beta^*, +\infty)$. Assume that $u \in \mathcal{F}_\beta$ is not constant, then there exists some integer $N \ge 1$ such that $D_N(u) > 0$. By Theorem 4.1, we have

$$\mathcal{E}_\beta(u,u) = \sum_{n=1}^{\infty} 2^{(\beta-\alpha)n} \left(\frac{3}{5}\right)^n D_n(u) \ge \frac{1}{C} \sum_{n=N+1}^{\infty} 2^{(\beta-\alpha)n} \left(\frac{3}{5}\right)^n D_N(u) = +\infty,$$

contradiction! Hence $\mathcal{F}_\beta$ consists only of constant functions. □

We need an elementary result as follows which will be frequently used.

Proposition 5.7. *Let $\{x_n\}$ be a sequence of nonnegative real numbers.*

(1)

$$\varliminf_{n\to+\infty} x_n \le \varliminf_{\lambda\uparrow 1} (1-\lambda) \sum_{n=1}^{\infty} \lambda^n x_n \le \varlimsup_{\lambda\uparrow 1} (1-\lambda) \sum_{n=1}^{\infty} \lambda^n x_n \le \varlimsup_{n\to+\infty} x_n \le \sup_{n\ge1} x_n.$$

(2) If there exists some positive constant C such that

$$x_n \le C x_{n+m} \text{ for all } n, m \ge 1,$$

then

$$\sup_{n\ge1} x_n \le C \varliminf_{n\to+\infty} x_n.$$

Proof. The proof is elementary using ε-N argument. □

Take $\{\beta_n\} \subseteq (\alpha, \beta^*)$ with $\beta_n \uparrow \beta^*$. By Proposition 5.4, there exist some subsequence still denoted by $\{\beta_n\}$ and some closed form $(\mathcal{E}, \mathcal{F})$ on $L^2(K;\nu)$ in the wide sense such that $(\beta^* - \beta_n)\mathcal{E}_{\beta_n}$ is Γ-convergent to $\mathcal{E}$. Without lose of generality, we may assume that

$$0 < \beta^* - \beta_n < \frac{1}{n+1} \text{ for all } n \ge 1.$$

We have the characterization of $(\mathcal{E}, \mathcal{F})$ on $L^2(K;\nu)$ as follows.

Theorem 5.8.

$$\mathcal{E}(u,u) \asymp \sup_{n\ge1} D_n(u) = \sup_{n\ge1}\left(\frac{5}{3}\right)^n \sum_{w^{(1)}\sim_n w^{(2)}} \left(P_n u(w^{(1)}) - P_n u(w^{(2)})\right)^2,$$

$$\mathcal{F} = \left\{ u \in L^2(K;\nu) : \sup_{n\ge1}\left(\frac{5}{3}\right)^n \sum_{w^{(1)}\sim_n w^{(2)}} \left(P_n u(w^{(1)}) - P_n u(w^{(2)})\right)^2 < +\infty \right\}.$$

Moreover, $(\mathcal{E},\mathcal{F})$ is a regular closed form on $L^2(K;\nu)$ and

$$\frac{1}{2(\log 2)C^2} \sup_{n\ge1} D_n(u) \le \mathcal{E}(u,u) \le \frac{1}{\log 2} \sup_{n\ge1} D_n(u).$$

Proof. Recall that

$$\mathcal{E}_\beta(u,u) = \sum_{n=1}^\infty 2^{(\beta-\alpha)n} A_n(u) = \sum_{n=1}^\infty 2^{(\beta-\beta^*)n} D_n(u).$$

On the one hand, for all $u \in L^2(K;\nu)$

$$\begin{aligned}
\mathcal{E}(u,u) &\le \varliminf_{n\to+\infty} (\beta^*-\beta_n)\mathcal{E}_{\beta_n}(u,u) = \varliminf_{n\to+\infty} (\beta^*-\beta_n) \sum_{k=1}^\infty 2^{(\beta_n-\beta^*)k} D_k(u) \\
&= \frac{1}{\log 2} \varliminf_{n\to+\infty} (1-2^{\beta_n-\beta^*}) \sum_{k=1}^\infty 2^{(\beta_n-\beta^*)k} D_k(u) \le \frac{1}{\log 2} \sup_{k\ge1} D_k(u).
\end{aligned}$$

On the other hand, for all $u \in L^2(K;\nu)$, there exists $\{u_n\} \subseteq L^2(K;\nu)$ converging strongly to u in $L^2(K;\nu)$ such that

$$\begin{aligned}
&\mathcal{E}(u,u) \ge \varlimsup_{n\to+\infty} (\beta^*-\beta_n)\mathcal{E}_{\beta_n}(u_n,u_n) = \varlimsup_{n\to+\infty} (\beta^*-\beta_n) \sum_{k=1}^\infty 2^{(\beta_n-\beta^*)k} D_k(u_n) \\
\ge& \frac{1}{C} \varlimsup_{n\to+\infty} (\beta^*-\beta_n) \sum_{k=n+1}^\infty 2^{(\beta_n-\beta^*)k} D_n(u_n) = \frac{1}{C} \varlimsup_{n\to+\infty} \left[(\beta^*-\beta_n) \frac{2^{(\beta_n-\beta^*)(n+1)}}{1-2^{\beta_n-\beta^*}} D_n(u_n) \right].
\end{aligned}$$

Since $0 < \beta^*-\beta_n < 1/(n+1)$, we have $2^{(\beta_n-\beta^*)(n+1)} > 1/2$. Since

$$\lim_{n\to+\infty} \frac{\beta^*-\beta_n}{1-2^{\beta_n-\beta^*}} = \frac{1}{\log 2},$$

we have

$$\mathcal{E}(u,u) \ge \frac{1}{2C} \varlimsup_{n\to+\infty} \frac{\beta^*-\beta_n}{1-2^{\beta_n-\beta^*}} D_n(u_n) \ge \frac{1}{2(\log 2)C} \varlimsup_{n\to+\infty} D_n(u_n).$$

Since $u_n \to u$ in $L^2(K;\nu)$, for all $k \ge 1$, we have

$$D_k(u) = \lim_{n\to+\infty} D_k(u_n) = \lim_{k\le n\to+\infty} D_k(u_n) \le C \varliminf_{n\to+\infty} D_n(u_n).$$

Taking supremum with respect to $k \ge 1$, we have

$$\sup_{k\ge1} D_k(u) \le C \varliminf_{n\to+\infty} D_n(u_n) \le C \varlimsup_{n\to+\infty} D_n(u_n) \le 2(\log 2)C^2 \mathcal{E}(u,u).$$

By Theorem 5.1, $\mathcal{F} \subseteq C(K)$. We only need to show that $\mathcal{F}$ is uniformly dense in $C(K)$, then $\mathcal{F}$ is dense in $L^2(K;\nu)$, hence $(\mathcal{E},\mathcal{F})$ is a regular closed form on $L^2(K;\nu)$.

Indeed, by Theorem 3.4, for all $U = U^{(x_0,x_1,x_2)} \in \mathcal{U}$, we have

$$\begin{aligned}
\sup_{n\ge1} D_n(U) &= \sup_{n\ge1} \left(\frac{5}{3}\right)^n \frac{2}{3} \left[\left(\frac{3}{5}\right)^n - \left(\frac{3}{5}\right)^{2n} \right] \left((x_0-x_1)^2 + (x_1-x_2)^2 + (x_0-x_2)^2 \right) \\
&\le \frac{2}{3} \left((x_0-x_1)^2 + (x_1-x_2)^2 + (x_0-x_2)^2 \right) < +\infty,
\end{aligned}$$

hence $U \in \mathcal{F}$, $\mathcal{U} \subseteq \mathcal{F}$. By Proposition 3.5, $\mathcal{F}$ separates points. It is obvious that $\mathcal{F}$ is a sub-algebra of $C(K)$. By Stone-Weierstrass theorem, $\mathcal{F}$ is uniformly dense in $C(K)$. □

Now we prove Theorem 2.1 using a standard method as follows.

Proof of Theorem 2.1. For all $u \in L^2(K;\nu)$, $n, k \geq 1$, $w \in W_n$, $w^{(1)} \sim_k w^{(2)}$, we have

$$P_{n+k}u(ww^{(i)}) = \int_K (u \circ f_{ww^{(i)}})(x)\nu(\mathrm{d}x) = \int_K (u \circ f_w \circ f_{w^{(i)}})(x)\nu(\mathrm{d}x) = P_k(u \circ f_w)(w^{(i)}),$$

hence

$$\begin{aligned}\sum_{w \in W_n} A_k(u \circ f_w) &= \sum_{w \in W_n} \sum_{w^{(1)} \sim_k w^{(2)}} \left(P_k(u \circ f_w)(w^{(1)}) - P_k(u \circ f_w)(w^{(2)})\right)^2 \\ &\leq \sum_{w^{(1)} \sim_{n+k} w^{(2)}} \left(P_{n+k}u(w^{(1)}) - P_{n+k}u(w^{(2)})\right)^2 = A_{n+k}(u),\end{aligned}$$

and

$$\left(\frac{5}{3}\right)^n \sum_{w \in W_n} D_k(u \circ f_w) = \left(\frac{5}{3}\right)^{n+k} \sum_{w \in W_n} A_k(u \circ f_w) \leq \left(\frac{5}{3}\right)^{n+k} A_{n+k}(u) = D_{n+k}(u).$$

For all $u \in \mathcal{F}, n \geq 1, w \in W_n$, we have

$$\sup_{k \geq 1} D_k(u \circ f_w) \leq \sup_{k \geq 1} \sum_{w \in W_n} D_k(u \circ f_w) \leq \left(\frac{3}{5}\right)^n \sup_{k \geq 1} D_{n+k}(u) \leq \left(\frac{3}{5}\right)^n \sup_{k \geq 1} D_k(u) < +\infty,$$

hence $u \circ f_w \in \mathcal{F}$.

For all $u \in L^2(K;\nu), n \geq 1$, let

$$\begin{aligned}\overline{\mathcal{E}}(u,u) &= \sum_{i=0}^{2} \left(u(p_i) - \int_K u(x)\nu(\mathrm{d}x)\right)^2, \\ \overline{\mathcal{E}}^{(n)}(u,u) &= \left(\frac{5}{3}\right)^n \sum_{w \in W_n} \overline{\mathcal{E}}(u \circ f_w, u \circ f_w).\end{aligned}$$

By Theorem 5.1, we have

$$\begin{aligned}\overline{\mathcal{E}}(u,u) &\leq \sum_{i=0}^{2} \int_K \left(u(p_i) - u(x)\right)^2 \nu(\mathrm{d}x) \\ &\leq \sum_{i=0}^{2} \int_K c^2 |p_i - x|^{\beta^* - \alpha} \left(\sup_{k \geq 1} D_k(u)\right) \nu(\mathrm{d}x) \leq 3c^2 \sup_{k \geq 1} D_k(u),\end{aligned}$$

hence

$$\begin{aligned}\overline{\mathcal{E}}^{(n)}(u,u) &\leq \left(\frac{5}{3}\right)^n \sum_{w \in W_n} 3c^2 \sup_{k \geq 1} D_k(u \circ f_w) \leq 3c^2 C \left(\frac{5}{3}\right)^n \sum_{w \in W_n} \varliminf_{k \to +\infty} D_k(u \circ f_w) \\ &\leq 3c^2 C \left(\frac{5}{3}\right)^n \varliminf_{k \to +\infty} \sum_{w \in W_n} D_k(u \circ f_w) \leq 3c^2 C \varliminf_{k \to +\infty} D_{n+k}(u) \leq 3c^2 C \sup_{k \geq 1} D_k(u).\end{aligned} \tag{6}$$

On the other hand, for all $u \in L^2(K;\nu), n \geq 1$, we have

$$D_n(u) = \left(\frac{5}{3}\right)^n \sum_{w^{(1)} \sim_n w^{(2)}} \left(\int_K (u \circ f_{w^{(1)}})(x)\nu(\mathrm{d}x) - \int_K (u \circ f_{w^{(2)}})(x)\nu(\mathrm{d}x)\right)^2.$$

For all $w^{(1)} \sim_n w^{(2)}$, there exist $i, j = 0, 1, 2$ such that

$$K_{w^{(1)}} \cap K_{w^{(2)}} = \{f_{w^{(1)}}(p_i)\} = \{f_{w^{(2)}}(p_j)\}.$$

Hence

$$
\begin{aligned}
D_n(u) \le & 2\left(\frac{5}{3}\right)^n \sum_{w^{(1)}\sim_n w^{(2)}} \left[\left((u\circ f_{w^{(1)}})(p_i) - \int_K (u\circ f_{w^{(1)}})(x)\nu(\mathrm{d}x)\right)^2\right. \\
& \left. + \left((u\circ f_{w^{(2)}})(p_j) - \int_K (u\circ f_{w^{(2)}})(x)\nu(\mathrm{d}x)\right)^2\right] \\
\le & 6\left(\frac{5}{3}\right)^n \sum_{w\in W_n}\sum_{i=0}^{2}\left((u\circ f_w)(p_i) - \int_K (u\circ f_w)(x)\nu(\mathrm{d}x)\right)^2 \\
= & 6\left(\frac{5}{3}\right)^n \sum_{w\in W_n} \overline{\mathcal{E}}(u\circ f_w, u\circ f_w) = 6\overline{\mathcal{E}}^{(n)}(u,u).
\end{aligned}
\tag{7}
$$

For all $u\in L^2(K;\nu), n\ge 1$, we have

$$
\begin{aligned}
\overline{\mathcal{E}}^{(n+1)}(u,u) = & \left(\frac{5}{3}\right)^{n+1} \sum_{w\in W_{n+1}} \overline{\mathcal{E}}(u\circ f_w, u\circ f_w) \\
= \left(\frac{5}{3}\right)^{n+1} & \sum_{i=0}^{2}\sum_{w\in W_n} \overline{\mathcal{E}}(u\circ f_i\circ f_w, u\circ f_i\circ f_w) = \frac{5}{3}\sum_{i=0}^{2}\overline{\mathcal{E}}^{(n)}(u\circ f_i, u\circ f_i).
\end{aligned}
\tag{8}
$$

Let

$$
\tilde{\mathcal{E}}^{(n)}(u,u) = \frac{1}{n}\sum_{l=1}^{n}\overline{\mathcal{E}}^{(l)}(u,u), u\in L^2(K;\nu), n\ge 1.
$$

By Equation (6), we have

$$
\tilde{\mathcal{E}}^{(n)}(u,u) \le 3c^2 C \sup_{k\ge 1} D_k(u) \asymp \mathcal{E}(u,u) \text{ for all } u\in\mathcal{F}, n\ge 1.
$$

Since $(\mathcal{E},\mathcal{F})$ is a regular closed form on $L^2(K;\nu)$, by [3, Definition 1.3.8, Remark 1.3.9, Definition 1.3.10, Remark 1.3.11], we have $(\mathcal{F},\mathcal{E}_1)$ is a separable Hilbert space. Let $\{u_i\}_{i\ge 1}$ be a dense subset of $(\mathcal{F},\mathcal{E}_1)$. For all $i\ge 1$, $\left\{\tilde{\mathcal{E}}^{(n)}(u_i,u_i)\right\}_{n\ge 1}$ is a bounded sequence. By diagonal argument, there exists a subsequence $\{n_k\}_{k\ge 1}$ such that $\left\{\tilde{\mathcal{E}}^{(n_k)}(u_i,u_i)\right\}_{k\ge 1}$ converges for all $i\ge 1$. Hence $\left\{\tilde{\mathcal{E}}^{(n_k)}(u,u)\right\}_{k\ge 1}$ converges for all $u\in\mathcal{F}$. Let

$$
\mathcal{E}_{\mathrm{loc}}(u,u) = \lim_{k\to+\infty}\tilde{\mathcal{E}}^{(n_k)}(u,u) \text{ for all } u\in\mathcal{F}_{\mathrm{loc}} := \mathcal{F}.
$$

Then

$$
\mathcal{E}_{\mathrm{loc}}(u,u) \le 3c^2 C \sup_{k\ge 1} D_k(u) \text{ for all } u\in\mathcal{F}_{\mathrm{loc}} = \mathcal{F}.
$$

By Equation (7), for all $u\in\mathcal{F}_{\mathrm{loc}} = \mathcal{F}$, we have

$$
\mathcal{E}_{\mathrm{loc}}(u,u) = \lim_{k\to+\infty}\tilde{\mathcal{E}}^{(n_k)}(u,u) \ge \varliminf_{n\to+\infty}\overline{\mathcal{E}}^{(n)}(u,u) \ge \frac{1}{6}\varliminf_{k\to+\infty} D_k(u) \ge \frac{1}{6C}\sup_{k\ge 1} D_k(u).
$$

Hence

$$
\mathcal{E}_{\mathrm{loc}}(u,u) \asymp \sup_{k\ge 1} D_k(u) \text{ for all } u\in\mathcal{F}_{\mathrm{loc}} = \mathcal{F}.
$$

Hence $(\mathcal{E}_{\mathrm{loc}},\mathcal{F}_{\mathrm{loc}})$ is a regular closed form on $L^2(K;\nu)$. Since $1\in\mathcal{F}_{\mathrm{loc}}$ and $\mathcal{E}_{\mathrm{loc}}(1,1)=0$, by [6, Lemma 1.6.5, Theorem 1.6.3], $(\mathcal{E}_{\mathrm{loc}},\mathcal{F}_{\mathrm{loc}})$ on $L^2(K;\nu)$ is conservative.

For all $u\in\mathcal{F}_{\mathrm{loc}}=\mathcal{F}$, we have $u\circ f_i\in\mathcal{F}=\mathcal{F}_{\mathrm{loc}}$ for all $i=0,1,2$. Moreover, by Equation (8), we have

$$\begin{aligned}
&\frac{5}{3}\sum_{i=0}^{2}\mathcal{E}_{\mathrm{loc}}(u\circ f_i,u\circ f_i)=\frac{5}{3}\sum_{i=0}^{2}\lim_{k\to+\infty}\tilde{\mathcal{E}}^{(n_k)}(u\circ f_i,u\circ f_i)\\
&=\lim_{k\to+\infty}\frac{1}{n_k}\sum_{l=1}^{n_k}\left[\frac{5}{3}\sum_{i=0}^{2}\overline{\mathcal{E}}^{(l)}(u\circ f_i,u\circ f_i)\right]=\lim_{k\to+\infty}\frac{1}{n_k}\sum_{l=1}^{n_k}\overline{\mathcal{E}}^{(l+1)}(u,u)\\
&=\lim_{k\to+\infty}\left[\frac{1}{n_k}\sum_{l=1}^{n_k}\overline{\mathcal{E}}^{(l)}(u,u)+\frac{1}{n_k}\overline{\mathcal{E}}^{(n_k+1)}(u,u)-\frac{1}{n_k}\overline{\mathcal{E}}^{(1)}(u,u)\right]\\
&=\lim_{k\to+\infty}\tilde{\mathcal{E}}^{(n_k)}(u,u)=\mathcal{E}_{\mathrm{loc}}(u,u).
\end{aligned}$$

Hence $(\mathcal{E}_{\mathrm{loc}},\mathcal{F}_{\mathrm{loc}})$ on $L^2(K;\nu)$ is self-similar.

For all $u,v\in\mathcal{F}_{\mathrm{loc}}$ satisfying $\mathrm{supp}(u),\mathrm{supp}(v)$ are compact and v is constant in an open neighborhood U of $\mathrm{supp}(u)$, we have $K\backslash U$ is compact and $\mathrm{supp}(u)\cap(K\backslash U)=\emptyset$, hence $\delta=\mathrm{dist}(\mathrm{supp}(u),K\backslash U)>0$. Taking sufficiently large $n\ge 1$ such that $2^{1-n}<\delta$, by self-similarity, we have

$$\mathcal{E}_{\mathrm{loc}}(u,v)=\left(\frac{5}{3}\right)^n\sum_{w\in W_n}\mathcal{E}_{\mathrm{loc}}(u\circ f_w,v\circ f_w).$$

For all $w\in W_n$, we have $u\circ f_w=0$ or $v\circ f_w$ is constant, hence $\mathcal{E}_{\mathrm{loc}}(u\circ f_w,v\circ f_w)=0$, hence $\mathcal{E}_{\mathrm{loc}}(u,v)=0$, that is, $(\mathcal{E}_{\mathrm{loc}},\mathcal{F}_{\mathrm{loc}})$ on $L^2(K;\nu)$ is strongly local.

For all $u\in\mathcal{F}_{\mathrm{loc}}$, it is obvious that $u^+,u^-,1-u,\overline{u}=(0\vee u)\wedge 1\in\mathcal{F}_{\mathrm{loc}}$ and

$$\mathcal{E}_{\mathrm{loc}}(u,u)=\mathcal{E}_{\mathrm{loc}}(1-u,1-u).$$

Since $u^+u^-=0$ and $(\mathcal{E}_{\mathrm{loc}},\mathcal{F}_{\mathrm{loc}})$ on $L^2(K;\nu)$ is strongly local, we have $\mathcal{E}_{\mathrm{loc}}(u^+,u^-)=0$. Hence

$$\begin{aligned}
\mathcal{E}_{\mathrm{loc}}(u,u)&=\mathcal{E}_{\mathrm{loc}}(u^+-u^-,u^+-u^-)=\mathcal{E}_{\mathrm{loc}}(u^+,u^+)+\mathcal{E}_{\mathrm{loc}}(u^-,u^-)-2\mathcal{E}_{\mathrm{loc}}(u^+,u^-)\\
&=\mathcal{E}_{\mathrm{loc}}(u^+,u^+)+\mathcal{E}_{\mathrm{loc}}(u^-,u^-)\ge\mathcal{E}_{\mathrm{loc}}(u^+,u^+)=\mathcal{E}_{\mathrm{loc}}(1-u^+,1-u^+)\\
&\ge\mathcal{E}_{\mathrm{loc}}((1-u^+)^+,(1-u^+)^+)=\mathcal{E}_{\mathrm{loc}}(1-(1-u^+)^+,1-(1-u^+)^+)=\mathcal{E}_{\mathrm{loc}}(\overline{u},\overline{u}),
\end{aligned}$$

that is, $(\mathcal{E}_{\mathrm{loc}},\mathcal{F}_{\mathrm{loc}})$ on $L^2(K;\nu)$ is Markovian. Hence $(\mathcal{E}_{\mathrm{loc}},\mathcal{F}_{\mathrm{loc}})$ is a self-similar strongly local regular Dirichlet form on $L^2(K;\nu)$. □

Remark 5.9. *The idea of the standard method is from [17, Section 6]. The proof of Markovian property is from the proof of [1, Theorem 2.1].*

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Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany.

E-mail address: `ymeng@math.uni-bielefeld.de`