MARKOV CHAINS UNDER NONLINEAR EXPECTATION

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Abstract. In this paper, we consider nonlinear continuous-time Markov chains with a finite state space. We define so-called Q-operators as an extension of Q-matrices to a nonlinear setup, where the nonlinearity is due to parameter uncertainty. The main result gives a full characterization of convex Q-operators in terms of a positive maximum principle, a dual representation by means of Q-matrices, continuous-time Markov chains under convex expectations and fully nonlinear ODEs. This extends a well-known characterization of Q-matrices.
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1. Introduction and main result

Let S be a finite state space with cardinality $|S| = d \in \mathbb{N}$. Throughout, we identify $S = \{1, \ldots, d\}$ and thus the space of all bounded measurable functions $S \to \mathbb{R}$ is $\mathbb{R}^d$. A matrix $q = (q_{ij})_{1\leq i,j\leq d} \in \mathbb{R}^{d\times d}$ is called a *Q-matrix* if it satisfies the following conditions:

(i) $q_{ii} \leq 0$ for all $i \in \{1, \ldots, d\}$,
(ii) $q_{ij} \geq 0$ for all $i, j \in \{1, \ldots, d\}$ with $i \neq j$,
(iii) $\sum_{j=1}^d q_{ij} = 0$ for all $i \in \{1, \ldots, d\}$.

It is well known that every continuous-time Markov chain with certain regularity properties at time $t = 0$ can be related to a Q-matrix and vice versa. More precisely, for a matrix $q \in \mathbb{R}^{d\times d}$ the following statements are equivalent:

(i) q is a Q-matrix.
(ii) There exists a Markov chain $\big(\Omega, \mathcal{F}, \mathbb{P}, (X_t)_{t\geq 0}\big)$ such that

$$qu_0 = \lim_{h\searrow 0} \frac{\mathbb{E}\big(u_0(X_h)\big) - u_0}{h}$$

for all $u_0 \in \mathbb{R}^d$, where $u_0(i)$ is the i-th component of u_0 for $i \in \{1, \ldots, d\}$.

In this case, for each $u_0 \in \mathbb{R}^d$, the function $u\colon [0,\infty) \to \mathbb{R}^d$, $t \mapsto \mathbb{E}\big(u_0(X_t)\big)$ is the unique classical solution $u \in C^1\big([0,\infty); \mathbb{R}^d\big)$ of the initial value problem

$$u'(t) = qu(t), \quad t \geq 0,$$
$$u(0) = u_0,$$

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i.e. $u(t) = e^{tq}u_0$ for all $t \geq 0$, where e^{tq} is the matrix exponential of tq. We refer to Norris [13] for a detailed illustration of this relation.

Moreover, it can be shown that a matrix $q \in \mathbb{R}^{d\times d}$ is a Q-matrix if and only if it satisfies the positive maximum principle and $q1 = 0$, where $1 := (1, \ldots, 1)^T \in \mathbb{R}^d$ denotes the constant 1 vector. Here and throughout this paper, we say that a (possibly nonlinear) operator $\mathcal{Q}\colon \mathbb{R}^d \to \mathbb{R}^d$ satisfies the *positive maximum principle* if for $u = (u_1, \ldots, u_d)^T \in \mathbb{R}^d$ and $i \in \{1, \ldots, d\}$ it holds $(\mathcal{Q}u)_i \leq 0$ whenever $u_i \geq u_j$ for all $j \in \{1, \ldots, d\}$.

The notion of a nonlinear expectation was introduced by Peng [14]. Since then, nonlinear expectations have been widely used in order to describe model uncertainty, or so-called Knightian uncertainty, in a probabilistic way. Prominent examples of nonlinear expectations are the g-expectation, see Coquet et al. [2], and the G-expectation introduced by Peng [15],[16]. Given a measurable space, we consider the space $\mathcal{L}^\infty(\Omega, \mathcal{F})$ of all bounded measurable functions $\Omega \to \mathbb{R}$. A *nonlinear expectation* is then a functional $\mathcal{E}\colon \mathcal{L}^\infty(\Omega, \mathcal{F}) \to \mathbb{R}$ which satisfies $\mathcal{E}(X) \leq \mathcal{E}(Y)$ whenever $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$, and $\mathcal{E}(\alpha 1_\Omega) = \alpha$ for all $\alpha \in \mathbb{R}$. It is well known, see e.g. [8] or [10], that every convex expectation $\mathcal{E}$ admits a dual representation in terms of finitely additive probability measures. If $\mathcal{E}$, however, even admits a dual representation in terms of (countably additive) probability measures, we say that $(\Omega, \mathcal{F}, \mathcal{E})$ is a *convex expectation space*. More precisely, we say that $(\Omega, \mathcal{F}, \mathcal{E})$ is a *convex expectation space* if there exists a set $\mathcal{P}$ of probability measures on $(\Omega, \mathcal{F})$ and a family $(\alpha_\mathbb{P})_{\mathbb{P}\in\mathcal{P}} \subset [0, \infty)$ with $\inf_{\mathbb{P}\in\mathcal{P}} \alpha_\mathbb{P} = 0$ such that

$$\mathcal{E}(X) = \sup_{\mathbb{P}\in\mathcal{P}} \big(\mathbb{E}_\mathbb{P}(X) - \alpha_\mathbb{P}\big)$$

for all $X \in \mathcal{L}^\infty(\Omega, \mathcal{F})$, where $\mathbb{E}_\mathbb{P}$ denotes the expectation w.r.t. a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$. If $\alpha_\mathbb{P} = 0$ for all $\mathbb{P} \in \mathcal{P}$, we say that $(\Omega, \mathcal{F}, \mathcal{E})$ is a *sublinear expectation space*.

If a nonlinear expectation $\mathcal{E}$ is sublinear, then $\rho(X) := \mathcal{E}(-X)$ defines a coherent monetary risk measure as introduced by Artzner et al. [1] and Delbaen [5],[6], see also Föllmer and Schied [10] for an overview of convex monetary risk measures. Another related concept are so-called (Choquet) capacites (see e.g. Dellacherie-Meyer [7]). However, in many applications the functional approach, using nonlinear expectations, has certain advantages, in particular regarding extension theorems or the existence of stochastic processes under model uncertainty, see e.g. Denk et al. [8].

In [14], Peng introduces a first notion of nonlinear Markov chains. However, the existence of stochastic processes under nonlinear expectations has only been considered in terms of finite dimensional nonlinear marginal distributions, whereas completely path-dependent functionals could not be regarded. Markov chains under model uncertainty have been considered amongst others by Hartfiel [11], Škulj [17] and De Cooman et al. [4]. In [11], Hartfiel considers so-called Markov set-chains in discrete time, using matrix intervals in order to describe model uncertainty in the transition matrices. Later, Škulj [17] approached Markov chains under model uncertainty using Choquet capacities, which results in higher-dimensional matrices on the power set, while De Cooman et al. [4] considered imprecise Markov chains using an operator theoretic approach with upper and lower epectations. In [8, Example 5.3], model uncertainty in the transition matrix is being described by a transition operator, which allows the construction of discrete-time Markov chains on the canonical path space. In continuous time, in particular computational aspects of sublinear imprecise markov chains, have been studied amongst others by Škulj [18] or De Bock et al. [3].

In this paper, we now consider continuous-time Markov chains under convex expectations and extend the above relation between nonlinear Markov chains, so-called Q-operators and fully nonlinear ordinary differential equations to the convex case. This relation is established using convex duality, so-called Nisio semigroups (cf. Nisio [12]) and a convex version of Kolmogorov's extension theorem, see Denk et. al. [8], which provides an extension to the whole path space. A similar approach has been used by Denk et al. [9] in order to construct Lévy processes under nonlinear expectations via solutions to fully nonlinear PDEs using Nisio semigroups. Restricting the time parameter to the set of natural numbers leads to a discrete-time Markov chain, in the sense of [8, Example 5.3]. Throughout this paper, we will make use of the following two definitions:

Definition 1.1. A (possibly nonlinear) map $\mathcal{Q}\colon \mathbb{R}^d \to \mathbb{R}^d$ is called a *Q-operator* if the following conditions are satisfied:

- (i) $(\mathcal{Q}\lambda e_i)_i \leq 0$ for all $\lambda > 0$ and all $i \in \{1, \ldots, d\}$,
- (ii) $\big(\mathcal{Q}(-\lambda e_j)\big)_i \leq 0$ for all $\lambda > 0$ and all $i, j \in \{1, \ldots, d\}$ with $i \neq j$,
- (iii) $\mathcal{Q}\alpha = 0$ for all $\alpha \in \mathbb{R}$, where we identify α with $(\alpha, \ldots, \alpha)^T \in \mathbb{R}^d$.

Definition 1.2. A *convex Markov chain* is a quadruple $\big(\Omega, \mathcal{F}, \mathcal{E}, (X_t)_{t\geq 0}\big)$, where

- (i) $(\Omega, \mathcal{F})$ is a measurable space.
- (ii) $X_t\colon \Omega \to \{1, \ldots, d\}$ is $\mathcal{F}$-measurable for all $t \geq 0$.
- $(iiii)$ $\mathcal{E} = (\mathcal{E}_1, \ldots, \mathcal{E}_d)^T$, where $(\Omega, \mathcal{F}, \mathcal{E}_i)$ is a convex expectation space for all $i \in \{1, \ldots, d\}$ and $\mathcal{E}\big(u_0(X_0)\big) = u_0$. Here and in the following we make use of the notation
$$\mathcal{E}(Y) := \big(\mathcal{E}_1(Y), \ldots, \mathcal{E}_d(Y)\big)^T \in \mathbb{R}^d$$
for $Y \in \mathcal{L}^\infty(\Omega, \mathcal{F})$.
- (iv) For all $s, t \geq 0$, $n \in \mathbb{N}$, $0 \leq t_1 < \ldots < t_n \leq s$ and $v_0 \in \big(\mathbb{R}^d\big)^{(n+1)}$ we have that
$$\mathcal{E}\big(v_0(X_{t_1}, \ldots, X_{t_n}, X_{s+t})\big) = \mathcal{E}\left[\mathcal{E}_{X_s,t}\big(v_0(X_{t_1}, \ldots, X_{t_n}, \,\cdot\,)\big)\right]$$
with $\mathcal{E}_{i,t}(u_0) := \mathcal{E}_i\big(u_0(X_t)\big)$ for all $u_0 \in \mathbb{R}^d$ and $i \in \{1, \ldots, d\}$.

We say that the Markov chain $\big(\Omega, \mathcal{F}, \mathcal{E}, (X_t)_{t\geq 0}\big)$ is *linear* or *sublinear* if the mapping $\mathcal{E}\colon \mathcal{L}^\infty(\Omega, \mathcal{F}) \to \mathbb{R}^d$ is additionally linear or sublinear, respectively.

The following main theorem gives a full characterization of convex Q-operators.

Theorem 1.3. *Let $\mathcal{Q}\colon \mathbb{R}^d \to \mathbb{R}^d$ be a mapping. Then the following statements are equivalent:*

- *(i) $\mathcal{Q}$ is a convex Q-operator.*
- *(ii) $\mathcal{Q}$ is convex, satisfies the positive maximum principle and $\mathcal{Q}\alpha = 0$ for all $\alpha \in \mathbb{R}$, where $\alpha := (\alpha, \ldots, \alpha)^T \in \mathbb{R}^d$.*
- *(iii) There exists a set $\mathcal{P} \subset \mathbb{R}^{d\times d}$ of Q-matrices and a family $f = (f_q)_{q\in\mathcal{P}} \subset \mathbb{R}^d$ of vectors with $\sup_{q\in\mathcal{P}} f_q = f_{q_0} = 0$ for some $q_0 \in \mathcal{P}$ such that*
$$\mathcal{Q}u_0 = \sup_{q\in\mathcal{P}} \big(qu_0 + f_q\big) \tag{1.1}$$
for all $u_0 \in \mathbb{R}^d$, where the suprema are to be understood componentwise.
- *(iv) There exists a convex Markov chain $\big(\Omega, \mathcal{F}, \mathcal{E}, (X_t)_{t\geq 0}\big)$ such that*
$$\mathcal{Q}u_0 = \lim_{h\searrow 0} \frac{\mathcal{E}\big(u_0(X_h)\big) - u_0}{h}$$
for all $u_0 \in \mathbb{R}^d$.

In this case, for each $u_0 \in \mathbb{R}^d$, the function $u\colon [0,\infty) \to \mathbb{R}^d$, $t \mapsto \mathcal{E}\big(u_0(X_t)\big)$ is the unique classical solution $u \in C^1\big([0,\infty); \mathbb{R}^d\big)$ of the initial value problem
$$\begin{aligned} u'(t) &= \mathcal{Q}u(t) = \sup_{q\in\mathcal{P}} \big(qu(t) + f_q\big), \quad t \geq 0, \\ u(0) &= u_0, \end{aligned} \tag{1.2}$$
and it holds
$$u(t) = \mathscr{S}(t)u_0$$
for all $t \geq 0$, where the family $\big(\mathscr{S}(t)\big)_{t\geq 0}$ is the Nisio semigroup w.r.t. $(\mathcal{P}, f)$ (see Definition 3.2 below).

Remark 1.4. Consider the situation of Theorem 1.3.

- a) Notice that the Nisio semigroup $\big(\mathscr{S}(t)\big)_{t\geq 0}$ can be construted w.r.t. any dual representation $(\mathcal{P}, f)$ as in (iii) and results in the unique classical solution of (1.2) independent of the choice of the representation $(\mathcal{P}, f)$.
- b) The same equivalence as in Theorem 1.3 holds if convexity is replaced by sublinearity in (i), (ii) and (iv) and $f_q = 0$ for all $q \in \mathcal{P}$ in (iii). In this case, the set $\mathcal{P}$ in (iii) can be chosen to be compact as we will see in the proof of Theorem 1.3.

Notation. Throughout, we consider a finite non-empty state space S with cardinality $d := |S| \in \mathbb{N}$. We endow S with the discrete topology 2^S and w.l.o.g. assume that $S = \{1, \ldots, d\}$. The space of all bounded measurable functions $S \to \mathbb{R}$ can therefore be identified by $\mathbb{R}^d$. A bounded random variable u thus will always be denoted as a vector of the form $u = (u_1, \ldots, u_d)^T \in \mathbb{R}^d$. On $\mathbb{R}^d$ we will always consider the norm
$$\|u\|_\infty := \max_{i=1,\ldots,d} |u_i|$$

for a vector $u \in \mathbb{R}^d$. Moreover, for $\alpha \in \mathbb{R}$ we denote by $\alpha \in \mathbb{R}^d$ the constant vector $u \in \mathbb{R}^d$ with $u_i = \alpha$ for all $i \in \{1, \ldots, d\}$. For a matrix $a = (a_{ij})_{1\le i,j\le d} \in \mathbb{R}^{d\times d}$, we denote by $\|a\|$ the operator norm of $a\colon \mathbb{R}^d \to \mathbb{R}^d$ w.r.t. the norm $\|\cdot\|_\infty$, i.e.

$$\|a\| = \sup_{v\in\mathbb{R}^d\setminus\{0\}} \frac{\|av\|_\infty}{\|v\|_\infty} = \max_{i=1,\ldots,d} \left(\sum_{j=1}^d |a_{ij}| \right).$$

Inequalities of vectors are always understood componentwise, i.e. for $u, v \in \mathbb{R}^d$

$$u \le v \iff \forall i \in \{1, \ldots, d\}\colon u_i \le v_i.$$

All concepts in $\mathbb{R}^d$ that inculde inequalities are to be understood w.r.t. the latter partial ordering. For example, a vector field $F\colon \mathbb{R}^d \to \mathbb{R}^d$ is called convex if

$$F_i\big(\lambda u + (1-\lambda)v\big) \le \lambda F_i(u) + (1-\lambda)F_i(v)$$

for all $i \in \{1, \ldots, d\}$, $u, v \in \mathbb{R}^d$ and $\lambda \in [0, 1]$. A vector field F is called sublinear if it is convex and positive homogeneous of degree 1. Moreover, for a set $M \subset \mathbb{R}^d$ of vectors, we write $u = \sup M$ if $u_i = \sup_{v\in M} v_i$ for all $i \in \{1, \ldots, d\}$.

Structure of the paper. In Section 2, we give a proof of the implications $(iv) \Rightarrow (ii) \Rightarrow (i) \Rightarrow (iii)$ of Theorem 1.3. The main tool we use in this part is convex duality. In Section 3, we prove the implication $(iii) \Rightarrow (iv)$. Here, we use a combination of Nisio semigroups as introduced in [12], a Kolmogorov extension theorem for convex expectations derived in [8] and the theory of ordinary differential equations.

2. Proof of $(iv) \Rightarrow (ii) \Rightarrow (i) \Rightarrow (iii)$

We say that a set $\mathcal{P} \subset \mathbb{R}^{d\times d}$ of matrices is *row-convex* if for any diagonal matrix $\lambda \in \mathbb{R}^{d\times d}$ with $\lambda_i := \lambda_{ii} \in [0, 1]$ for all $i \in \{1, \ldots, d\}$ and all $p, q \in \mathcal{P}$ we have that

$$\lambda p + (I - \lambda)q \in \mathcal{P},$$

where $I = I_d$ is the d-dimensional identity matrix. Note that for all $i \in \{1, \ldots, d\}$ the i-th row of the matrix $\lambda p + (I-\lambda)q$ is the convex combination of the i-th row of p and q with λ_i. Therefore, a set $\mathcal{P}$ is row-convex if for all $p, q \in \mathcal{P}$ the convex combination with different $\lambda \in [0, 1]$ in every row is again an element of $\mathcal{P}$. Note that for example the set of all Q-matrices is row-convex.

Remark 2.1. Let $\mathcal{Q}$ be a convex Q-operator. For every matrix $q \in \mathbb{R}^{d\times d}$ let

$$\mathcal{Q}^*(q) := \sup_{u\in\mathbb{R}^d} \big(qu - \mathcal{Q}(u)\big) \in [0, \infty]$$

be the *conjugate function* of $\mathcal{Q}$. Notice that $0 \le \mathcal{Q}^*(q)$ for all $q \in \mathbb{R}^{d\times d}$ since $\mathcal{Q}(0) = 0$. Let

$$\mathcal{P} := \{q \in \mathbb{R}^{d\times d} \,|\, \mathcal{Q}^*(q) < \infty\}$$

and $f_q := -\mathcal{Q}^*(q)$ for all $q \in \mathcal{P}$. Then, the following facts are well-known results from convex duality theory in $\mathbb{R}^d$.

a) The set $\mathcal{P}$ is row-convex and the mapping $\mathcal{P} \to \mathbb{R}$, $q \mapsto f_q$ is continuous.

b) Let $M > 0$ and $\mathcal{P}_M := \{q \in \mathbb{R}^{d\times d} \,|\, \mathcal{Q}^*(q) \le M\}$. Then, $\mathcal{P}_M \subset \mathbb{R}^{d\times d}$ is compact and row-convex. Therefore,

$$\mathcal{Q}_M\colon \mathbb{R}^d \to \mathbb{R}^d, \quad u \mapsto \max_{q\in\mathcal{P}_M} \big(qu + f_q\big) \tag{2.1}$$

defines a convex operator which is Lipschitz continuous. Notice that the maximum in (2.1) is to be understood componentwise. However, for fixed $u \in \mathbb{R}^d$ the maximum can be attained by a single element simultaneously in every component of $\mathcal{P}_M$ since $\mathcal{P}_M$ is row-convex, i.e. for all $u \in \mathbb{R}^d$ there exists some $q_0 \in \mathcal{P}_M$ with

$$\mathcal{Q}_M = q_0 u + f_{q_0}.$$

c) Let $R > 0$. Then, there exists some $M > 0$ such that

$$\mathcal{Q}u = \max_{q\in\mathcal{P}_M} \big(qu + f_q\big) = \mathcal{Q}_M u$$

for all $u \in \mathbb{R}^d$ with $\|u\|_\infty \le R$. In particular, $\mathcal{Q}$ is locally Lipschitz continuous and admits a representation of the form

$$\mathcal{Q}u = \max_{q\in\mathcal{P}} \big(qu + f_q\big)$$

for all $u \in \mathbb{R}^d$, where for fixed $u \in \mathbb{R}^d$ the maximum can be attained by a single element simultaneously in every component of $\mathcal{P}$. In particular, there exists some $q_0 \in \mathcal{P}$ with $f_{q_0} = \sup_{q\in\mathcal{P}} f_q = \mathcal{Q}(0) = 0$.

Proof of Theorem 1.3.
$(iv) \Rightarrow (ii)$: As $\mathcal{E}_i$ is a convex expectation for all $i \in \{1,\ldots,d\}$, it follows that the operator $\mathcal{Q}$ is convex with $\mathcal{Q}\alpha = 0$ for all $\alpha \in \mathbb{R}$. Now, let $u_0 \in \mathbb{R}^d$ and $i \in \{1,\ldots,d\}$ with $u_{0,i} \ge u_{0,j}$ for all $j \in \{1,\ldots,d\}$. Let $\alpha > 0$ be such that

$$\|u_0 + \alpha\|_\infty = \big(u_0 + \alpha\big)_i = u_{0,i} + \alpha$$

and $v_0 := u_0 + \alpha$. Then,

$$\mathcal{Q}v_0 = \lim_{h\searrow 0} \frac{\mathcal{E}\big(u_0(X_h) + \alpha\big) - v_0}{h} = \lim_{h\searrow 0} \frac{\mathcal{E}\big(u_0(X_h)\big) - u_0}{h} = \mathcal{Q}u_0.$$

Assume that $\big(\mathcal{Q}u_0\big)_i > 0$. Then, there exists some $h > 0$ such that

$$\mathcal{E}_i\big(v_0(X_h)\big) - v_{0,i} > 0,$$

i.e.

$$\big\|\mathcal{E}\big(v_0(X_h)\big)\big\|_\infty \ge \mathcal{E}_i\big(v_0(X_h)\big) > v_{0,i} = \|v_0\|_\infty,$$

which is a contradiction to

$$\big\|\mathcal{E}\big(v_0(X_h)\big)\big\|_\infty \le \|v_0\|_\infty.$$

This shows that $\mathcal{Q}$ satisfies the positive maximum principle.
$(ii) \Rightarrow (i)$: This follows directly from the positive maximum principle, considering the vectors λe_i and $-\lambda e_i$ for all $\lambda > 0$ and $i \in \{1,\ldots,d\}$.
$(i) \Rightarrow (iii)$: Let $\mathcal{Q}$ be a convex Q-operator. Moreover, let $\mathcal{P}$ and $f = (f_q)_{q\in\mathcal{P}}$ as in Remark 2.1. Then, by Remark 2.1 c), it only remains to show that every

$q \in \mathcal{P}$ is a Q-matrix. To this end, fix an arbitrary $q \in \mathcal{P}$. Then, for all $\alpha \in \mathbb{R}$ it holds

$$q\alpha = \frac{1}{\lambda} q(\lambda\alpha) \leq \frac{1}{\lambda}\big(\mathcal{Q}(\lambda\alpha) + \mathcal{Q}^*(q)\big) = \frac{1}{\lambda}\mathcal{Q}^*(q) \to 0,$$

as $\lambda \to \infty$. Therefore, $q\alpha \leq 0$ for all $\alpha \in \mathbb{R}$. Since, q is linear, it follows $q1 = 0$. Now, let $i \in \{1, \ldots, d\}$. Then, by definition of a Q-operator, we obtain that

$$q_{ii} \leq \frac{1}{\lambda}\big(\mathcal{Q}(\lambda e_i) + \mathcal{Q}^*(q)\big)_i \leq \frac{1}{\lambda}\big(\mathcal{Q}^*(q)\big)_i \to 0$$

as $\lambda \to \infty$, i.e. $q_{ii} \leq 0$. Now, let $i, j \in \{1, \ldots, d\}$ with $i \neq j$. Then, again by definition of a Q-operator, it follows that

$$-q_{ij} \leq \frac{1}{\lambda}\big(\mathcal{Q}(-\lambda e_i) + \mathcal{Q}^*(q)\big)_j \leq \frac{1}{\lambda}\big(\mathcal{Q}^*(q)\big)_j \to 0$$

as $\lambda \to \infty$, i.e. $q_{ij} \geq 0$. Therefore, q is a Q-matrix.
It remains to show $(iii) \Rightarrow (iv)$, which is done in the entire next section. □

3. Proof of $(iii) \Rightarrow (iv)$

Throughout, let $\mathcal{P} \subset \mathbb{R}^{d\times d}$ be a set of Q-matrices and $f = (f_q)_{q\in\mathcal{P}} \subset \mathbb{R}^d$ with $\sup_{q\in\mathcal{P}} f_q = f_{q_0} = 0$ for some $q_0 \in \mathcal{P}$ such that

$$\mathcal{Q}\colon \mathbb{R}^d \to \mathbb{R}^d, \quad u_0 \mapsto \sup_{q\in\mathcal{P}} \big(qu_0 + f_q\big)$$

is well-defined. For every $q \in \mathcal{P}$, we consider the linear ODE

$$u'(t) = qu(t) + f_q, \quad t \geq 0 \tag{3.1}$$

with $u(0) = u_0 \in \mathbb{R}^d$. Then, by variation of constant, the solution of (3.1) is given by

$$u(t) = e^{qt}u_0 + \int_0^t e^{qs} f_q \,\mathrm{d}s = u_0 + \int_0^t e^{sq}\big(qu_0 + f_q\big)\,\mathrm{d}s =: S_q(t)u_0 \tag{3.2}$$

for $t \geq 0$, where $e^{tq} \in \mathbb{R}^{d\times d}$ is the matrix exponential of tq for all $t \geq 0$. Then, the family $S_q = \big(S_q(t)\big)_{t\geq 0}$ defines a uniformly continuous semigroup of affine linear operators, i.e.

(i) $S_q(0) = I$, where $I = I_d$ is the d-dimensional identity matrix,
(ii) $S_q(s + t) = S_q(s)S_q(t)$ for all $s, t \geq 0$,
(iii) $\|S_q(t) - I\| \to 0$ as $t \searrow 0$.

Remark 3.1.

a) Note that for all $q \in \mathcal{P}$ and $t \geq 0$ the matrix exponential $e^{tq} \in \mathbb{R}^{d\times d}$ is a *stochastic matrix*, i.e.
   (i) $\big(e^{tq}\big)_{ij} \geq 0$ for all $i, j \in \{1, \ldots, d\}$,
   (ii) $e^{tq}1 = 1$.

   In particular, $e^{tq}u \leq e^{tq}v$ for all $u, v \in \mathbb{R}^d$ with $u \leq v$ and therefore, the semigroup S_q is monotone (see part b) below for a definition).

b) In line with [8, Definition 5.1], we say that a (possibly nonlinear) map $\mathcal{E}\colon \mathbb{R}^d \to \mathbb{R}^d$ is a kernel, if $\mathcal{E}$ is *monotone*, i.e. $\mathcal{E}(u) \le \mathcal{E}(v)$ for all $u, v \in \mathbb{R}^d$ with $u \le v$, and $\mathcal{E}$ *preserves constants*, i.e. $\mathcal{E}(\alpha) = \alpha$ for all $\alpha \in \mathbb{R}$. By part a), it is clear that $e^{tq} \in \mathbb{R}^{d\times d}$ is a linear kernel for all $q \in \mathcal{P}$ and $t \ge 0$.

For the family $(S_q)_{q\in\mathcal{P}}$, or more precisely for $(\mathcal{P}, f)$, we will now construct the respective Nisio semigroup and show that this is the unique classical solution to the nonlinear ODE (1.2). To this end, we consider the set of finite partitions

$$P := \big\{\pi \subset [0,\infty)\,\big|\, 0 \in \pi, |\pi| < \infty\big\}.$$

For a partition $\pi \in P$, $\pi = \{t_0, t_1, \ldots, t_m\}$ with $0 = t_0 < t_1 < \ldots < t_m$ we set

$$|\pi|_\infty := \max_{j=1,\ldots,m} (t_j - t_{j-1}).$$

Moreover, we define $|\{0\}|_\infty := 0$. The set of partitions with end-point t will be denoted by P_t, i.e. $P_t := \{\pi \in P \,|\, \max \pi = t\}$. Note that

$$P = \bigcup_{t\ge 0} P_t.$$

For all $h \ge 0$ and $u \in \mathbb{R}^d$ we define

$$\mathcal{E}_h u := \sup_{q\in\mathcal{P}} S_q(h)u,$$

where the supremum is taken componentwise. Note that $\mathcal{E}_h$ is well-defined since

$$S_q(h)u = e^{hq}u + \int_0^h e^{sq} f_q \,\mathrm{d}s \le e^{hq}u \le \|u\|_\infty$$

for all $q \in \mathcal{P}$, where we used the fact that e^{hq} is stochastic. Moreover, $\mathcal{E}_h$ is a convex kernel as it is monotone and

$$\mathcal{E}_h \alpha = \alpha + \sup_{q\in\mathcal{P}} \int_0^h e^{sq} f_q \,\mathrm{d}s = \alpha$$

since there is some $q_0 \in \mathcal{P}$ with $f_{q_0} = 0$. For a partition $\pi = \{t_0, t_1, \ldots, t_m\} \in P$ with $m \in \mathbb{N}$ and $0 = t_0 < t_1 < \ldots < t_m$, we set

$$\mathcal{E}_\pi := \mathcal{E}_{t_1 - t_0} \ldots \mathcal{E}_{t_m - t_{m-1}}.$$

Moreover, we set $\mathcal{E}_{\{0\}} := \mathcal{E}_0$. Then, $\mathcal{E}_\pi$ is a convex kernel for all $\pi \in P$ being a concatenation of convex kernels.

Definition 3.2. The *Nisio semigroup* $\big(\mathscr{S}(t)\big)_{t\ge 0}$ w.r.t. $(\mathcal{P}, f)$ is defined by

$$\mathscr{S}(t)u := \sup_{\pi\in P_t} \mathcal{E}_\pi u$$

for all $u \in \mathbb{R}^d$ and $t \ge 0$.

Note that $\mathscr{S}(t)\colon \mathbb{R}^d \to \mathbb{R}^d$ is well-defined and a convex kernel for all $t \ge 0$ since $\mathcal{E}_\pi$ is a convex kernel for all $\pi \in P$. In many of the subsequent proofs, we will first concentrate on the case, where the family f is bounded and then use an approximation of the Nisio semigroup by means of Nisio semigroups w.r.t. bounded f. This approximation procedure is specified in the following remark.

Remark 3.3. Let $M \geq 0$, $\mathcal{P}_M := \big\{q \in \mathcal{P} \,\big|\, \|f_q\|_\infty \leq M\big\}$ and $f_M := (f_q)_{q\in\mathcal{P}_M}$. Then, for all $q \in \mathcal{P}_M$ and $u \in \mathbb{R}^d$ with $\|u\|_\infty = 1$ we have that

$$qu \leq \mathcal{Q}u - f_q \leq \|\mathcal{Q}u\|_\infty + \|f_q\|_\infty \leq M + \max_{v\in\mathbb{S}^{d-1}} \|\mathcal{Q}v\|_\infty,$$

where, in the last step, we used that $\mathcal{Q}\colon \mathbb{R}^d \to \mathbb{R}^d$ convex and therefore continuous. This implies that the set $\mathcal{P}_M$ is bounded. Therefore,

$$\sup_{q\in\mathcal{P}} \|qu + f_q\|_\infty \leq \sup_{q\in\mathcal{P}} \big(\|q\|\|u\|_\infty + \|f_q\|_\infty\big) < \infty \tag{3.3}$$

for all $u \in \mathbb{R}^d$ and thus the operator

$$\mathcal{Q}_M\colon \mathbb{R}^d \to \mathbb{R}^d, \quad u \mapsto \sup_{q\in\mathcal{P}_M} \big(qu + f_q\big)$$

is well-defined. Notice that, by assumption, there exists some $q_0 \in \mathcal{P}$ with $f_{q_0} = 0$ and therefore $f_{q_0} \in \mathcal{P}_M$. In particular, $\mathcal{P}_M \neq \emptyset$. Let $\big(\mathscr{S}_M(t)\big)_{t\geq 0}$ be the Nisio semigroup w.r.t. $(\mathcal{P}_M, f_M)$ for all $M \geq 0$. Since

$$\bigcup_{M\geq 0} \mathcal{P}_M = \mathcal{P},$$

it then follows that $\mathcal{Q}_M \nearrow \mathcal{Q}$ and $\mathscr{S}_M(t) \nearrow \mathscr{S}(t)$ as $M \to \infty$ for all $t \geq 0$.

Lemma 3.4. *Assume that the family f is bounded, i.e. $(\mathcal{P}, f) = (\mathcal{P}_M, f_M)$ for some $M \geq 0$. Then, for all $u \in \mathbb{R}^d$ the mapping $[0,\infty) \to \mathbb{R}^d$, $h \mapsto \mathcal{E}_h u$ is continuous.*

Proof. Let $u \in \mathbb{R}^d$ and $0 \leq h_1 < h_2$. Then, by (3.2), for all $q \in \mathcal{P}$ we have that

$$\|S_q(h_2)u - S_q(h_1)u\|_\infty \leq \int_{h_1}^{h_2} \big\|e^{qs}(qu + f_q)\big\|_\infty \,\mathrm{d}s \leq (h_2 - h_1)\|qu + f_q\|_\infty,$$

which implies that

$$\|\mathcal{E}_{h_2}u - \mathcal{E}_{h_1}u\|_\infty \leq \sup_{q\in\mathcal{P}} \|S_q h_2 u - S_q(h_1)u\|_\infty \leq (h_2 - h_1)\bigg(\sup_{q\in\mathcal{P}} \|qu + f_q\|_\infty\bigg). \tag{3.4}$$

Note that $\sup_{q\in\mathcal{P}} \|qu + f_q\|_\infty < \infty$ by (3.3). □

Lemma 3.5. *Assume that the family f is bounded. Then,*

$$\|\mathscr{S}(t)u - u\|_\infty \leq t\bigg(\sup_{q\in\mathcal{P}} \|qu + f_q\|_\infty\bigg)$$

for all $t > 0$ and $u \in \mathbb{R}^d$.

Proof. Let $u \in \mathbb{R}^d$. For a partition $\pi \in P$ of the form $\pi = \{t_0, t_1, \ldots, t_m\}$ with $m \in \mathbb{N}$ and $0 = t_0 < t_1 < \ldots < t_m$, (3.4) then implies that

$$\begin{aligned}\|\mathcal{E}_\pi u - u\|_\infty \leq \sum_{k=1}^m \|\mathcal{E}_{h_k} u - u\|_\infty &\leq \sum_{k=1}^m h_k\bigg(\sup_{q\in\mathcal{P}} \|qu + f_q\|_\infty\bigg) \\ &= t_m\bigg(\sup_{q\in\mathcal{P}} \|qu + f_q\|_\infty\bigg),\end{aligned}$$

where $h_k := t_k - t_{k-1}$ for all $k \in \{1, \ldots, m\}$. By definition of $\mathscr{S}(t)$ for $t \geq 0$ it then follows that

$$\|\mathscr{S}(t)u - u\|_\infty \leq \sup_{\pi \in P_t} \|\mathcal{E}_\pi u - u\|_\infty \leq t\bigg(\sup_{q \in \mathcal{P}} \|qu + f_q\|_\infty\bigg).$$

□

Let $u \in \mathbb{R}^d$. In the following, we want to consider the limit of $\mathcal{E}_\pi u$ when the mesh size of the partition $\pi \in P$ tends to zero. For this, we first remark that for $h_1, h_2 \geq 0$

$$\begin{aligned}\mathcal{E}_{h_1+h_2}u &= \sup_{q \in \mathcal{P}} S_\lambda(h_1 + h_2)u = \sup_{q \in \mathcal{P}} S_\lambda(h_1)S_\lambda(h_2)u \\ &\leq \sup_{q \in \mathcal{P}} S_\lambda(h_1)\mathcal{E}_{h_2}u = \mathcal{E}_{h_1}\mathcal{E}_{h_2}u,\end{aligned}$$

which implies the inequality

$$\mathcal{E}_{\pi_1}u \leq \mathcal{E}_{\pi_2}u \tag{3.5}$$

for $\pi_1, \pi_2 \in P$ with $\pi_1 \subset \pi_2$. The following lemma now shows that $\mathscr{S}(t)$ can be obtained by a pointwise monotone approximation with finite partitions letting the mesh size tend to zero.

Lemma 3.6. *Let $t \geq 0$ and $(\pi_n)_{n\in\mathbb{N}} \subset P_t$ with $\pi_n \subset \pi_{n+1}$ for all $n \in \mathbb{N}$ and $|\pi_n|_\infty \searrow 0$ as $n \to \infty$. Then, for all $u \in \mathbb{R}^d$ it holds that*

$$\mathcal{E}_{\pi_n}u \nearrow \mathscr{S}(t)u, \quad n \to \infty.$$

Proof. Let $u \in \mathbb{R}^d$. For $t = 0$ the statement is trivial. Therefore, assume that $t > 0$ and let

$$v := \sup_{n\in\mathbb{N}} \mathcal{E}_{\pi_n}u. \tag{3.6}$$

As $\pi_n \subset \pi_{n+1}$ for all $n \in \mathbb{N}$, (3.5) implies that

$$\mathcal{E}_{\pi_n}u \nearrow v, \quad n \to \infty.$$

Since $(\pi_n)_{n\in\mathbb{N}} \subset P_t$, we obtain that

$$v \leq \mathscr{S}(t)u.$$

Next, we assume that f is bounded. Let $\pi = \{t_0, t_1, \ldots, t_m\} \in P_t$ with $m \in \mathbb{N}$ and $0 = t_0 < t_1 < \ldots < t_m = t$. Since $|\pi_n|_\infty \searrow 0$ as $n \to \infty$, w.l.o.g. we may assume that $|\pi_n| \geq m+1$ for all $n \in \mathbb{N}$. Moreover, let $0 = t_0^n < t_1^n < \ldots < t_m^n = t$ for all $n \in \mathbb{N}$ with $\pi_n' := \{t_0^n, t_1^n, \ldots, t_m^n\} \subset \pi_n$ and $t_i^n \to t_i$ as $n \to \infty$ for all $i \in \{1, \ldots, m\}$. Then, by Lemma 3.4, we have that

$$\|\mathcal{E}_\pi u - \mathcal{E}_{\pi_n'}u\|_\infty \to 0, \quad n \to \infty$$

and therefore,

$$v \geq \mathcal{E}_{\pi_n}u \geq \mathcal{E}_{\pi_n'}u \geq \mathcal{E}_\pi u - \|\mathcal{E}_\pi u - \mathcal{E}_{\pi_n'}u\|_\infty.$$

Letting $n \to \infty$ we obtain that $v \geq \mathcal{E}_\pi u$. Taking the supremum over all $\pi \in P_t$ yields the assertion for bounded f.

Now, let f again be (possibly) unbounded. It remains to show that $v \geq \mathscr{S}(t)u$. By the previous step, we have that $v \geq v_M = \mathscr{S}_M(t)$ for all $M \geq 0$, where v_M

is given by (3.6) but w.r.t. $(\mathcal{P}_M, f_M)$ instead of $(\mathcal{P}, f)$. Since $\mathscr{S}_M(t) \nearrow \mathscr{S}(t)$ as $M \to \infty$, we obtain that $v \geq \mathscr{S}(t)$, which ends the proof. □

Choosing e.g. $\pi_n = \big\{\frac{kt}{2^n} \colon k \in \{0, \ldots, 2^n\}\big\}$ or $\pi_n = \big\{\frac{kt}{n!} \colon k \in \{0, \ldots, n!\}\big\}$ in Lemma 3.6, we obtain the following corollaries.

Corollary 3.7. *For all $t > 0$ there exists a sequence $(\pi_n)_{n\in\mathbb{N}} \subset P_t$ with*

$$\mathcal{E}_{\pi_n} u \nearrow \mathscr{S}(t)u$$

as $n \to \infty$ for all $u \in \mathbb{R}^d$.

Corollary 3.8. *For all $t \geq 0$ and $u \in \mathbb{R}^d$ we have that*

$$\mathscr{S}(t)u = \sup_{n\in\mathbb{N}} \mathcal{E}_{\frac{1}{n}}^n u = \lim_{n\to\infty} \mathcal{E}_{2^{-n}}^{2^n} u.$$

Proposition 3.9. *The family $(\mathscr{S}(t))_{t\geq 0}$ defines a semigroup of convex kernels from $\mathbb{R}^d$ to $\mathbb{R}^d$. In particular, for all $s, t \geq 0$ we have the dynamic programming principle*

$$\mathscr{S}(s+t) = \mathscr{S}(s)\mathscr{S}(t). \tag{3.7}$$

Proof. It remains to show the semigroup property (3.7). Let $u \in \mathbb{R}^d$. If $s = 0$ or $t = 0$ the statement is trivial. Therefore, let $s, t > 0$, $\pi_0 \in P_{s+t}$ and $\pi := \pi_0 \cup \{s\}$. Then, we have that $\pi \in P_{s+t}$ with $\pi_0 \subset \pi$. Hence, by (3.5), we get that

$$\mathcal{E}_{\pi_0} u \leq \mathcal{E}_\pi u.$$

Let $m \in \mathbb{N}$, $0 = t_0 < t_1 < \ldots t_m = s + t$ with $\pi = \{t_0, \ldots, t_m\}$ and $i \in \{1, \ldots, m\}$ with $t_i = s$. Then, we have that

$$\pi_1 := \{t_0, \ldots, t_i\} \in P_s \quad \text{and} \quad \pi_2 := \{t_i - s, \ldots, t_m - s\} \in P_t$$

with

$$\mathcal{E}_{\pi_1} = \mathcal{E}_{t_1-t_0} \cdots \mathcal{E}_{t_i - t_{i-1}} \quad \text{and} \quad \mathcal{E}_{\pi_2} = \mathcal{E}_{t_{i+1}-t_i} \cdots \mathcal{E}_{t_m - t_{m-1}}.$$

We thus get that

$$\begin{aligned}\mathcal{E}_{\pi_0} u \leq \mathcal{E}_\pi u = \mathcal{E}_{t_1-t_0} \cdots \mathcal{E}_{t_m-t_{m-1}} u &= \big(\mathcal{E}_{t_1-t_0} \cdots \mathcal{E}_{t_i-t_{i-1}}\big)\big(\mathcal{E}_{t_{i+1}-t_i} \cdots \mathcal{E}_{t_m-t_{m-1}} u\big) \\ &= \mathcal{E}_{\pi_1}\mathcal{E}_{\pi_2} u \leq \mathcal{E}_{\pi_1}\big(\mathscr{S}(t)u\big) \leq \mathscr{S}(s)\mathscr{S}(t)u.\end{aligned}$$

Taking the supremum over all $\pi_0 \in P_{s+t}$, we get that $\mathscr{S}(s+t)u \leq \mathscr{S}(s)\mathscr{S}(t)u$. Now, let $(\pi_n)_{n\in\mathbb{N}} \subset P_t$ with $\mathcal{E}_{\pi_n} u \nearrow \mathscr{S}(t)u$ as $n \to \infty$ (see Corollary 3.7) and fix $\pi_0 \in P_s$. Then, for all $n \in \mathbb{N}$ we have that

$$\pi_n' := \pi_0 \cup \{s + \tau \colon \tau \in \pi_n\} \in P_{s+t}$$

with $\mathcal{E}_{\pi_n'} = \mathcal{E}_{\pi_0}\mathcal{E}_{\pi_n}$. We then get that

$$\mathcal{E}_{\pi_0}\big(\mathscr{S}(t)u\big) = \lim_{n\to\infty} \mathcal{E}_{\pi_0}\mathcal{E}_{\pi_n} u = \lim_{n\to\infty} \mathcal{E}_{\pi_n'} u \leq \mathscr{S}(s+t)u.$$

Taking the supremum over all $\pi_0 \in P_s$, we get that $\mathscr{S}(s)\mathscr{S}(t)u \leq \mathscr{S}(s+t)u$. □

Now, Proposition 3.9 and [8, Theorem 5.6] imply the following corollary.

Corollary 3.10. *There exists a convex Markov chain $\big(\Omega, \mathcal{F}, \mathcal{E}, (X_t)_{t\geq 0}\big)$ such that*

$$\big(\mathscr{S}(t)u_0\big)_i = \mathcal{E}_i\big(u_0(X_t)\big)$$

for all $u_0 \in \mathbb{R}^d$, $t \geq 0$ and $i \in \{1, \ldots, d\}$.

Restricting the time parameter of this process to $\mathbb{N}_0$, leads to a discrete-time Markov chain with transition operator $\mathscr{S}(1)$ (cf. [8, Example 5.3]). It remains to show that the Nisio semigroup $\big(\mathscr{S}(t)\big)_{t\geq 0}$ defines the unique classical solution to the nonlinear ODE (1.2).

Remark 3.11. Assume that the set $\mathcal{P}$ is bounded. Note that $\mathcal{P}$ is bounded if and only if $\mathcal{Q}$ is Lipschitz continuous.

a) Then, the Picard-Lindelöf Theorem asserts that, for every $u_0 \in \mathbb{R}^d$, the initial value problem

$$\begin{aligned} u'(t) &= \mathcal{Q}u(t), \quad t \geq 0, \\ u(0) &= u_0, \end{aligned} \tag{3.8}$$

has a unique solution $u \in C^1\big([0,\infty);\mathbb{R}^d\big)$. We will show that this solution u is given by $u(t) = \mathscr{S}(t)u_0$ for all $t \geq 0$. That is, the unique solution of the ODE (3.8) is given by the Nisio semigroup.

b) Since $\mathcal{P}$ is bounded, the mapping

$$\mathfrak{q}\colon \mathbb{R}^d \to \mathbb{R}^d, \quad u \mapsto \sup_{q\in\mathcal{P}} qu$$

is well-defined.

The following key estimate and its proof are taken from the proof of [12, Proposition 5]. Recall that, by Remark 3.3, the boundedness of the family f implies the boundedness of the set $\mathcal{P}$.

Lemma 3.12. *Assume that the family f is bounded. Then,*

$$\mathscr{S}(t)u - u \leq \int_0^t \Sigma(s)\mathcal{Q}u\,\mathrm{d}s$$

for all $u \in \mathbb{R}^d$ and $t \geq 0$. Here, $\big(\Sigma(t)\big)_{t\geq 0}$ is the Nisio semigroup w.r.t. the sublinear Q-operator $\mathfrak{q}$ from the previous remark, or more precisely, the Nisio semigroup w.r.t. $(\mathcal{P}, f)$, where $f_q = 0$ for all $q \in \mathcal{P}$.

Proof. Let $u \in \mathbb{R}^d$ and $h > 0$. Then, by (3.2), we have that

$$S_q(h)u - u = \int_0^h e^{sq}\big(qu + f_q\big)\,\mathrm{d}s \leq \int_0^h \Sigma(s)\mathcal{Q}u\,\mathrm{d}s.$$

Notice that, by Lemma 3.5, the mapping $[0,\infty) \to \mathbb{R}^d$, $t \mapsto \Sigma(t)v$ is continuous and therefore locally integrable for all $v \in \mathbb{R}^d$. Hence, for all $\tau \geq 0$ we have that

$$\mathcal{E}_h u - u \leq \int_0^h \Sigma(s)\mathcal{Q}u\,\mathrm{d}s = \int_\tau^{\tau+h} \Sigma(s-\tau)\mathcal{Q}u\,\mathrm{d}s.$$

Next, we show that

$$\mathcal{E}_\pi u - u \leq \int_0^{\max\pi} \Sigma(s)\mathcal{Q}u\,\mathrm{d}s \tag{3.9}$$

for all $\pi \in P$ by an induction on $m = |\pi|$, where $|\pi|$ denotes the cardinality of π. If $m = 1$, i.e. if $\pi = \{0\}$, the statement is trivial. Hence, assume that

$$\mathcal{E}_{\pi'}u - u \leq \int_0^{\max \pi'} \Sigma(s)\mathcal{Q}u \, \mathrm{d}s$$

for all $\pi' \in P$ with $|\pi'| = m$ for some $m \in \mathbb{N}$. Let $\pi = \{t_0, t_1, \ldots, t_m\}$ with $0 = t_0 < t_1 < \ldots < t_m$ and $\pi' := \pi \setminus \{t_m\}$. Then, we obtain that

$$\begin{aligned}\mathcal{E}_\pi u - \mathcal{E}_{\pi'}u &\leq \Sigma(t_{m-1})\big(\mathcal{E}_{t_m - t_{m-1}}u - u\big) \leq \Sigma(t_{m-1})\bigg(\int_{t_{m-1}}^{t_m} \Sigma(s - t_{m-1})\mathcal{Q}u \, \mathrm{d}s\bigg) \\ &\leq \int_{t_{m-1}}^{t_m} \Sigma(s)\mathcal{Q}u \, \mathrm{d}s,\end{aligned}$$

where we used the sublinearity of $\Sigma(t)$ in the last inequality. Using the induction hypothesis, we thus get that

$$\begin{aligned}\mathcal{E}_\pi u - u &= \big(\mathcal{E}_\pi u - \mathcal{E}_{\pi'}u\big) + \big(\mathcal{E}_{\pi'}u - u\big) \leq \int_{t_{m-1}}^{t_m} \Sigma(s)\mathcal{Q}u \, \mathrm{d}s + \int_0^{t_{m-1}} \Sigma(s)\mathcal{Q}u \, \mathrm{d}s \\ &= \int_0^{\max \pi} \Sigma(s)\mathcal{Q}u \, \mathrm{d}s.\end{aligned}$$

By (3.9), it follows that

$$\mathcal{E}_\pi u - u \leq \int_0^t \Sigma(s)\mathcal{Q}u \, \mathrm{d}s$$

for all $\pi \in P_t$. Taking the supremum over all $\pi \in P_t$ we obtain the assertion. □

The following theorem states that the family $(\mathscr{S}(t))_{t\geq 0}$ is differentiable at zero if the family f is bounded.

Theorem 3.13. *Assume that f is bounded. Then, for all $u \in \mathbb{R}^d$ it holds that*

$$\left\| \frac{\mathscr{S}(h)u - u}{h} - \mathcal{Q}u \right\|_\infty \to 0, \quad h \searrow 0.$$

Proof. Since f is bounded, it follows that $\mathcal{P}$ is bounded (see Remark 3.3). Let $\varepsilon > 0$. Using Lemma 3.5, the boundedness of $\mathcal{P}$ and (3.3), there exists some $h_0 > 0$ such that

$$\begin{aligned}\big\| e^{hq}\big(qu + f_q\big) - \big(qu + f_q\big) \big\|_\infty &\leq \|e^{hq} - I_d\| \cdot \|qu + f_q\|_\infty \\ &\leq \big(e^{\|q\|h} - 1\big)\|qu + f_q\|_\infty \leq \varepsilon\end{aligned}$$

for all $q \in \mathcal{P}$ and

$$\Sigma(h)\mathcal{Q}u - \mathcal{Q}u \leq \varepsilon$$

for all $0 < h \leq h_0$. Let $0 < h \leq h_0$. Then, we get that

$$\mathscr{S}(h)u - u \geq S_q(h)u - u = \int_0^h e^{tq}\big(qu + f_q\big) \, \mathrm{d}s \geq \big(qu + f_q - \varepsilon\big)h$$

for all $q \in \mathcal{P}$. Dividing by h and taking the supremum over all $q \in \mathcal{P}$, it follows that

$$\frac{\mathscr{S}(h)u - u}{h} \geq \mathcal{Q}u - \varepsilon. \tag{3.10}$$

By Lemma 3.12, we have that

$$\mathscr{S}(h)f - f - h\mathcal{Q}f \leq \int_0^h \Sigma(s)\mathcal{Q}f \,\mathrm{d}s - h\mathcal{Q}f = \int_0^h \big(\Sigma(s)\mathcal{Q}f - \mathcal{Q}f\big)\,\mathrm{d}s \leq h\varepsilon.$$

Again, dividing by $h > 0$ yields

$$\frac{\mathscr{S}(h)f - f}{h} - \mathcal{Q}f \leq \varepsilon.$$

Together with (3.10) this implies that

$$\left\| \frac{\mathscr{S}(h)f - f}{h} - \mathcal{Q}f \right\|_\infty \leq \varepsilon.$$

□

Corollary 3.14. *Let f be bounded, $u_0 \in \mathbb{R}^d$ and $u(t) := \mathscr{S}(t)u_0$ for $t \geq 0$. Then, $u \in C^1\big([0,\infty);\mathbb{R}^d\big)$ is the unique classical solution of the ODE*

$$u'(t) = \mathcal{Q}u(t), \quad t \geq 0$$

with $u(0) = u_0$.

Corollary 3.15. *Let f be bounded. Then, there exists some constant $L > 0$ such that*

$$\|\mathscr{S}(t)u_0 - u_0\|_\infty \leq Lt\|u_0\|_\infty$$

for all $t \geq 0$ and $u_0 \in \mathbb{R}^d$.

Proof. Since f is bounded, we have that $\mathcal{P}$ is bounded and therefore $\mathcal{Q}$ is Lipschitz continuous with Lipschitz constant $L := \sup_{q\in\mathcal{P}} \|q\|$. For all $u_0 \in \mathbb{R}^d$ we thus obtain that

$$\|\mathscr{S}(t)u_0 - u_0\|_\infty \leq \int_0^t \|\mathcal{Q}\mathscr{S}(s)u_0\|_\infty \,\mathrm{d}s \leq \int_0^t L\|\mathscr{S}(s)u_0\|_\infty \,\mathrm{d}s \leq Lt\|u_0\|_\infty.$$

□

Finally, in order to end the proof of Theorem 1.3, we have to extend Corollary 3.14 to the unbounded case. We start with the following remark, which is the key observation in order to finish the proof of Theorem 1.3.

Remark 3.16. Let $\mathcal{P}^* := \{q \in \mathbb{R}^{d\times d} \,|\, \mathcal{Q}^*(q) < \infty\}$ and $f_q^* := -\mathcal{Q}^*(q)$ for all $q \in \mathcal{P}^*$. For all $M \geq 0$ let $\mathcal{P}_M^*$, f_M^* and $\mathcal{Q}_M^*$ be as in Remark 3.3 with $\mathcal{P}$ being replaced by $\mathcal{P}^*$. Moreover, let $\big(\mathscr{S}_M^*(t)\big)_{t\geq 0}$ be the Nisio semigroup w.r.t. $(\mathcal{P}_M^*, f_M^*)$ for $M \geq 0$. As

$$\bigcup_{M\geq 0} \mathcal{P}_M^* = \mathcal{P}^*,$$

it follows that $\mathscr{S}_M^*(t) \nearrow \mathscr{S}^*(t)$ as $M \to \infty$ for all $t \geq 0$, where $(\mathscr{S}^*(t))_{t\geq 0}$ is the Nisio semigroup w.r.t. $(\mathcal{P}^*, f^*)$. Let $R > 0$ be fixed. Then, there exists some $M_0 \geq 0$ such that $\mathcal{Q}u = \mathcal{Q}_{M_0}^* u$ for all $u \in \mathbb{R}^d$ with $\|u\|_\infty \leq R$, by choice of $\mathcal{P}^*$

and f^*. Let $u_0\in\mathbb{R}^d$ with $\|u_0\|_\infty\leq R$. Then, it follows that $\|\mathscr{S}_M^*(t)u_0\|_\infty\leq R$ for all $t\geq 0$ and $M\geq 0$, which implies that $\mathscr{S}_M^*(t)u_0=\mathscr{S}_{M_0}^*(t)u_0$ for all $t\geq 0$ and $M\geq M_0$ by the uniqueness obtained in the Picard-Lindelöf Theorem. In particular, $\mathscr{S}^*(t)u_0=\mathscr{S}_{M_0}^*(t)u_0$ for all $t\geq 0$, which shows that the nonlinear ODE (1.2) has a unique classical solution $u^*\in C^1\big([0,\infty);\mathbb{R}^d\big)$ with $u^*(0)=u_0$. This solution is given by $u^*(t)=\mathscr{S}^*(t)u_0$ for all $t\geq 0$. By Corollary 3.15, we thus get that $\mathscr{S}^*(t)\to I$ as $t\searrow 0$ uniformly on compact sets.

We are now able to finish the proof of Theorem 1.3.

Theorem 3.17. *Let $u_0\in\mathbb{R}^d$. Then, $u\colon[0,\infty)\to\mathbb{R}^d$, $t\mapsto\mathscr{S}(t)u_0$ is the unique classical solution $u\in C^1\big([0,\infty);\mathbb{R}^d\big)$ of the initial value problem*

$$\begin{aligned}u'(t)&=\mathcal{Q}u(t),\quad t\geq 0,\\ u(0)&=u_0.\end{aligned}$$

Proof. By Remark 3.16 the initial value problem

$$\begin{aligned}u'(t)&=\mathcal{Q}u(t),\quad t\geq 0,\\ u(0)&=u_0.\end{aligned}$$

has a unique classical solution $u^*\in C^1\big([0,\infty);\mathbb{R}^d\big)$. It remains to show that $u^*(t)=\mathscr{S}(t)u_0$, for all $t\geq 0$. Let $R:=\|u_0\|_\infty$. For all $M\geq 0$ let $\mathcal{P}_M$, f_M, $\mathcal{Q}_M$ and $\big(\mathscr{S}_M(t)\big)_{t\geq 0}$ as in Remark 3.3. Let $\varepsilon>0$. Then, by Dini's lemma, there exists some $M_0\geq 0$ such that

$$\|\mathcal{Q}v-\mathcal{Q}_{M_0}v\|_\infty\leq\varepsilon$$

for all $v\in\mathbb{R}^d$ with $\|v\|_\infty\leq R$. Further, there exists some constant $L>0$ such that

$$\|\mathcal{Q}v_1-\mathcal{Q}v_2\|_\infty\leq L\|v_1-v_2\|_\infty$$

for all $v_1,v_2\in\mathbb{R}^d$ with $\|v_1\|_\infty\leq R$ and $\|v_2\|_\infty\leq R$. Since, $\|u^*(t)\|_\infty\leq R$ and $\|\mathscr{S}_M(t)u_0\|_\infty\leq R$ for all $M\geq 0$ and $t\geq 0$, we obtain that

$$\begin{aligned}\|\mathscr{S}_M(t)u_0-u^*(t)\|_\infty+\frac{\varepsilon}{L}&=\frac{\varepsilon}{L}+\left\|\int_0^t\mathcal{Q}_M\mathscr{S}_M(s)u_0-\mathcal{Q}u^*(s)\,\mathrm{d}s\right\|_\infty\\ &\leq\frac{\varepsilon}{L}+\int_0^t\|\mathcal{Q}_M\mathscr{S}_M(s)u_0-\mathcal{Q}u^*(s)\|_\infty\,\mathrm{d}s\\ &\leq\frac{\varepsilon}{L}+\int_0^t\big(\|\mathcal{Q}\mathscr{S}_M(s)u_0-\mathcal{Q}u^*(s)\|_\infty+\varepsilon\big)\,\mathrm{d}s\\ &\leq\frac{\varepsilon}{L}+\int_0^t L\bigg(\|\mathscr{S}_M(s)u_0-u^*(s)\|_\infty+\frac{\varepsilon}{L}\bigg)\,\mathrm{d}s\end{aligned}$$

for all $t\geq 0$ and $M\geq M_0$. By Gronwall's lemma, we thus get that

$$\|\mathscr{S}_M(t)u_0-u^*(t)\|_\infty\leq\frac{\varepsilon}{L}\big(e^{Lt}-1\big)$$

for all $t\geq 0$ and $M\geq M_0$, showing that $\mathscr{S}_M(t)u_0\to u^*(t)$ as $M\to\infty$ for all $t\geq 0$. However, since $\mathscr{S}_M(t)u_0\nearrow\mathscr{S}(t)u_0$ as $M\to\infty$ for all $t\geq 0$, we obtain that $u^*(t) = \mathscr{S}(t)u_0$. This ends the proof of this theorem and also the proof of Theorem 1.3. $\square$

Corollary 3.18. *The Nisio semigroup $(\mathscr{S}(t))_{t\geq 0}$ is uniformly continuous in the sense that $\mathscr{S}(t) \to I$ as $t \searrow 0$ uniformly on compact sets.*

References

1. P. Artzner, F. Delbaen, J.-M. Eber, and D. Heath. Coherent measures of risk. *Math. Finance*, 9(3):203–228, 1999.
2. F. Coquet, Y. Hu, J. Mémin, and S. Peng. Filtration-consistent nonlinear expectations and related g-expectations. *Probability Theory and Related Fields*, pages 123:1–27, 2002.
3. J. De Bock, T. Krak, and A. Siebes. Imprecise continuous-time Markov chains. *International Journal of Approximate Reasoning*, 88:452–528, 2017.
4. G. De Cooman, F. Hermans, and E. Quaeghebeur. Imprecise markov chains and their limit behavior. *Probability in the Engineering and Informational Sciences*, 23(4):597–635, 2009.
5. F. Delbaen. *Coherent risk measures*. Cattedra Galileiana. [Galileo Chair]. Scuola Normale Superiore, Classe di Scienze, Pisa, 2000.
6. F. Delbaen. Coherent risk measures on general probability spaces. In *Advances in finance and stochastics*, pages 1–37. Springer, Berlin, 2002.
7. C. Dellacherie and P.-A. Meyer. *Probabilities and potential*, volume 29 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam-New York; North-Holland Publishing Co., Amsterdam-New York, 1978.
8. R. Denk, M. Kupper, and M. Nendel. Kolmogorov-type and general extension results for nonlinear expectations. *Preprint*, 2015.
9. R. Denk, M. Kupper, and M. Nendel. A semigroup approach to nonlinear Lévy processes. *Preprint*, 2017.
10. H. Föllmer and A. Schied. *Stochastic finance*. Walter de Gruyter & Co., Berlin, extended edition, 2011. An introduction in discrete time.
11. D. J. Hartfiel. *Markov set-chains*, volume 1695 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1998.
12. M. Nisio. *Stochastic control theory*, volume 72 of *Probability Theory and Stochastic Modelling*. Springer, Tokyo, second edition, 2015. Dynamic programming principle.
13. J. R. Norris. *Markov chains*, volume 2 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, 1998. Reprint of 1997 original.
14. S. Peng. Nonlinear expectations and nonlinear Markov chains. *Chinese Ann. Math. Ser. B*, 26(2):159–184, 2005.
15. S. Peng. G-expectation, G-Brownian motion and related stochastic calculus of Itô type. *Stochastic Analysis and Applications*, Volume 2 of Abel Symposium:541–567, 2007.
16. S. Peng. Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation. *Stochastic Process. Appl.*, 118(12):2223–2253, 2008.
17. D. Škulj. Discrete time markov chains with interval probabilities. *International journal of approximate reasoning*, 50(8):1314–1329, 2009.
18. D. Škulj. Efficient computation of the bounds of continuous time imprecise Markov chains. *Applied Mathematics and Computation*, 250:165–180, 2015.

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