INEQUALITIES BETWEEN RANDOM VARIABLES AND TWO-STEP NILPOTENT LIE SEMIGROUPS

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Abstract We determine all possible vectors $(\mathbf{P}(x<y), \mathbf{P}(y<z), \mathbf{P}(z<x))$ for independent random variables x, y, z satisfying the condition $\mathbf{P}(x=y=z)=0$. Surprisingly, this result is obtained as a probabilistic interpretation of our algebraic result on the free 2-step nilpotent Lie semigroup of rank 3.[1]

1. Introduction

Let x, y, z be random variables. We will assume that

$$\mathbf{P}(x=y=z)=0 \tag{1}$$

and set

$$p=\mathbf{P}(x<y), \quad q=\mathbf{P}(y<z), \quad r=\mathbf{P}(z<x).$$

Then we have

$$0=\mathbf{P}(x \geq y \geq z \geq x) \geq \mathbf{P}(x \geq y)+\mathbf{P}(y \geq z)+\mathbf{P}(z \geq x)-2=1-p-q-r,$$

$$0=\mathbf{P}(x<y<z<x) \geq \mathbf{P}(x<y)+\mathbf{P}(y<z)+\mathbf{P}(z<x)-2=p+q+r-2,$$

Thus, $1 \leq p+q+r \leq 2$. The inequalities

$$0 \leq p, q, r \leq 1, \quad 1 \leq p+q+r \leq 2 \tag{2}$$

define a triangular antiprism in $\mathbb{R}^3$, which we will denote by A. Thus, $(p, q, r) \in A$.

In general there are no other restrictions on the vector (p, q, r). (To show this, it suffices to consider the cases when one of the random variables is constant.) The situation changes if the random variables are supposed to be independent. The following theorem is our main result.

Theorem 1. *For the (pairwise) independent random variables x, y, z satisfying* (1)*, the set of possible vectors (p, q, r) is the curved (non-convex) polyhedron A^{cut} cut from the antiprism A by the inequalities*

$$\min\{p+qr, q+rp, r+pq\} \leq 1, \quad \min\{qr, rp, pq\} \leq p+q+r-1.$$

Note that A^{cut} is a centrally symmetric star body with the center at $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. It is invariant under all permutations of the coordinates p, q, r.

The assertion of the theorem can be formulated in a different way as follows: given the probabilities $\mathbf{P}(x<y)=p$ and $\mathbf{P}(y<z)=q$, then all possible values of the probability $\mathbf{P}(x<z)=1-r$ constitute the interval

$$[\min\{pq, \frac{p+q-1}{p}, \frac{p+q-1}{q}\}, 1],$$

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if $p+q \geq 1$, and the interval

$$[0, \max\{p+q-pq, \frac{q}{1-p}, \frac{p}{1-q}\}],$$

if $p+q \leq 1$. (These two cases reduce to one another by changing the signs of the inequalities.)

It is quite remarkable that this theorem appears as a probabilistic interpretation of an algebraic theorem related to 2-step nilpotent Lie semigroups (see Theorem 3).

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2. Basic objects and formulas

Let $\mathfrak{g} = \mathfrak{g}(n)$ denote the free two-step nilpotent real Lie algebra of rank n with generators $\xi_1, \ldots, \xi_n$, so that the elements

$$\xi_i \, (1 \leq i \leq n), \quad [\xi_i, \xi_j] \, (1 \leq i < j \leq n)$$

constitute a basis of it. Let further $G = G(n)$ be the simply connected Lie group with $\mathrm{Lie}(G) = \mathfrak{g}(n)$. Every element $g \in G$ is uniquely represented in the form

$$g = \exp\bigl(\sum_i d_i(g)\xi_i + \frac{1}{2}\sum_{i<j} d_{ij}(g)[\xi_i, \xi_j]\bigr).$$

The numbers $d_i(g)$ and $d_{ij}(g)$ will be called the *canonical coordinates* of g. For $i > j$, we set $d_{ij}(g) = -d_{ji}(g)$.

It is easy to see that

$$\exp\xi \cdot \exp\eta = \exp(\xi + \eta + \frac{1}{2}[\xi, \eta]) \tag{3}$$

for $\xi, \eta \in \mathfrak{g}$ (a very special case of the Campbell–Hausdorff formula). It follows that, for two elements $g, h \in G$,

$$d_i(gh) = d_i(g) + d_i(h), \quad d_{ij}(gh) = d_{ij}(g) + d_{ij}(h) + \begin{vmatrix} d_i(g) & d_j(g) \\ d_i(h) & d_j(h) \end{vmatrix}. \tag{4}$$

Set $x_i = \exp\xi_i$ and $x_i^t = \exp t\xi_i$ for $t \in \mathbb{R}$. It is easy to see that every element $g \in G$ can be represented in the form

$$g = x_{i_1}^{t_1} \ldots x_{i_m}^{t_m}. \tag{5}$$

It follows from (3) by induction on m that the canonical coordinates of such an element are

$$d_i(g) = \sum_{k:\, i_k = i} t_k, \quad d_{ij}(g) = \sum_{k.l:\, i_k = i,\, i_l = j} \mathrm{sgn}(l-k) t_k t_l. \tag{6}$$

For this reason, the number $d_i(g)$ will be called the *degree* of g in x_i, and the vector $(d_i(g), \ldots, d_n(g))$ will be called the *multidegree* of g.

It follows from (4) that, if the multidegrees of elements $g, h \in G$ are proportional, then the canonical coordinates of the product gh are just the sums of the corresponding canonical coordinates of g and h.

Our main object will be the semigroup $B = B(n) \subset G(n)$ generated by the elements x_i^t with $t \geq 0$, which can be called the *free two-step nilpotent semigroup* of rank n. For $n \geq 3$, finding an explicit description of $B(n)$ in the canonical coordinates turns out to be an interesting and difficult problem. We shall do this for $n = 3$. Unexpectedly, this result can be interpreted in terms of probability theory.

Proposition 1. *The semigroup B lies in the body defined by the inequalities*

$$d_i \geq 0 \quad (1 \leq i \leq n), \quad |d_{ij}| \leq d_i d_j \quad (1 \leq i < j \leq n).$$

Proof. For any $g \in B$ represented in the form (5) with $t_1, \dots, t_m \geq 0$ we have according to (6)

$$d_i(g) = \sum_{k:\, i_k = i} t_k \geq 0, \quad |d_{ij}(g)| \leq \sum_{k.l:\, i_k = i,\, i_l = j} t_k t_l = \Big(\sum_{k:\, i_k = i} t_k\Big)\Big(\sum_{l:\, i_l = j} t_l\Big) = d_i(g) d_j(g).$$

□

3. The case $n = 2$

The following description of the semigroup $B(2)$ is known: see, for example, [1], Section 2.1.

Proposition 2. *In the canonical coordinates, the semigroup $B(2)$ is defined by the inequalities*

$$d_1 \geq 0, \quad d_2 \geq 0, \quad |d_{12}| \leq d_1 d_2. \tag{7}$$

Moreover, every element $g \in B(2)$ can be represented in the form

$$g = x_1^{s_1} x_2^{t} x_1^{s_2} \quad \text{with} \quad s_1, s_2, t \geq 0. \tag{8}$$

Proof. According to Proposition 1, all the elements of $B(2)$ satisfy the inequalities (7). On the other hand, calculating the canonical coordinates of the element (8), we obtain

$$d_1(g) = s_1 + s_2, \quad d_2(g) = t, \quad d_{12}(g) = (s_1 - s_2)t.$$

It is easy to see that the equations

$$s_1 + s_2 = d_1, \quad t = d_2, \quad (s_1 - s_2)t = d_{12}$$

have a solution in non-negative numbers for any d_1, d_2, d_{12} satisfying the inequalities (7).

□

Some authors call $B(2)$ the *Heisenberg beak*.

4. Automorphisms

The following transformations of the generators $x_1, \dots, x_n$ extend to automorphisms of the group G leaving invariant the semigroup B:

1) any permutation of $x_1, \dots, x_n$;
2) any transformation $\mu(c_1, \dots, c_n) : x_i \mapsto x_i^{c_i}$ with $c_1, \dots, c_n > 0$.

It is clear that

$$d_i(\mu(c_1, \dots, c_n)g) = c_i d_i(g), \quad d_{ij}(\mu(c_1, \dots, c_n)g) = c_i c_j d_{ij}(g). \tag{9}$$

We set for brevity

$$g^{(c)} = \mu(c. \dots, c)g.$$

for $c > 0$. Obviously,

$$d_i(g^{(c)}) = c d_i(g), \quad d_{ij}(g^{(c)}) = c^2 d_{ij}(g). \tag{10}$$

Similarily, one can define the transformation $\mu(c_1, \dots, c_n)$ for any $c_1, \dots, c_n \geq 0$. This will be an endomorphism of the group G taking the semigroup B into itself.

For the element $g \in G$ given by (5), the element

$$g^\top = x_{i_m}^{t_m} \dots x_{i_1}^{t_1} \in G \tag{11}$$

will be called the *transpose* of g. It is easy to see that

$$d_i(g^\top) = d_i(g), \quad d_{ij}(g^\top) = -d_{ij}(g) \tag{12}$$

In particular, this shows that the element $g^\top$ does not depend on the expression of g in terms of the generators. The map $g \mapsto g^\top$ is an antiautomorphism of the group G leaving the semigroup B invariant.

5. The section

It follows from (9) that if all the degrees $d_i(g)$ are strictly positive, then by means of an automorphism of the form $\mu(c_1, \dots, c_n)$ the element g is equivalent to a unique element of multidegree $(1, \dots, 1)$.

Let us call the *section* of B and denote by $B_1 = B_1(n)$ the set of all elements $g \in B$ of multidegree $(1, \dots, 1)$. The following proposition shows that the semigroup B is uniquely reconstructed from B_1.

Proposition 3.

$$B = \bigcup_{c_1, \dots, c_n \geq 0} \mu(c_1, \dots, c_n) B_1.$$

Proof. Obviously, $B \supset \bigcup_{c_1, \dots, c_n \geq 0} \mu(c_1, \dots, c_n) B_1$. Conversely, take any $g \in B$. Assume for deficiency that

$$d_i(g) = c_i > 0 \quad \text{for} \quad i \leq k, \quad d_i(g) = 0 \quad \text{for} \quad i > k.$$

Then $g = \mu(c_1, \dots, c_k, 0, \dots, 0) g_1$, where

$$g_1 = \mu(c_1^{-1}, \dots, c_k^{-1}, 1, \dots, 1) g x_{k+1} \dots x_n \in B_1.$$

□

Theorem 2. *In the canonical coordinates d_{ij}, the section B_1 is a centrally symmetric star body in $\mathbb{R}^{\frac{n(n-1)}{2}}$ contained in the cube*

$$|d_{ij}| \leq 1 \quad (1 \leq i < j \leq n). \tag{13}$$

Proof. Note that, for any element g represented as a palindrome in $x_1, \dots, x_n$, we have $g^\top = g$ and, hence, all the coordinates $d_{ij}(g)$ are equal to zero. Taking any palindrome of multidegree $(1, \dots, 1)$ (for example, $x_1^{1/2} \dots x_{n-1}^{1/2} x_n x_{n-1}^{1/2} \dots x_1^{1/2}$), we obtain an element $g_0 \in B_1$ with all $d_{ij}(g_0) = 0$, i.e. the origin of the coordinate space $\mathbb{R}^{\frac{n(n-1)}{2}}$.

For any $g \in B_1$ we also have $g^\top \in B_1$. In view of (12), this means that B_1 is centrally symmetric.

Making use of (10) and (4), for any $g \in B_1$ and $0 < c < 1$ we obtain

$$h = g^{(c)} g_0^{(1-c)} \in B_1$$

and

$$d_{ij}(h) = c^2 d_{ij}(g).$$

This shows that B_1 is a star body.

The last assertion of the theorem follows from Proposition 1. □

6. The probabilistic interpretation

Definition 1. *A word in $x_1, \dots, x_n$ of pattern $[x_{i_1}, \dots, x_{i_m}]$ is a sequence $w = (x_{i_1}^{t_1}, \dots, x_{i_m}^{t_m})$ with $t_1, \dots, t_m \in \mathbb{R}$. It is called non-negative, if $t_1, \dots, t_m \geq 0$, and normalized, if $\sum_{k;\, i_k = i} t_k = 1$ for each i.*

The product

$$g(w) = x_{i_1}^{t_1} \dots x_{i_m}^{t_m}$$

is an element of the group G, which belongs to B, if the word w is non-negative, and to B_1, if, moreover, it is normalized.

To each normalized non-negative word, one can associate n random variables, which we will denote by the same letters $x_1, \dots, x_n$ as the generators of G, assuming that x_i takes the value k with probability t_k, if $i_k = i$, and 0 otherwise. We will suppose that these random variables are (pairwise) independent. Then the formula (6) means that

$$d_{ij}(g(w)) = \mathbf{P}(x_i < x_j) - \mathbf{P}(x_i > x_j) = 2\mathbf{P}(x_i < x_j) - 1.$$

Making use of this interpretation, one can prove

Proposition 4. *The elements of the section B_1 satisfy the inequalities*

$$|d_{ij} + d_{jk} + d_{ki}| \leq 1 \quad (1 \leq i, j, k \leq n). \tag{14}$$

Proof. Since $x_i < x_j$ and $x_j < x_k$ implies $x_i < x_k$,

$$\mathbf{P}(x_i < x_k) \geq \mathbf{P}(x_i < x_j) + \mathbf{P}(x_j < x_k) - 1$$

and, therefore,

$$d_{ik} \geq d_{ij} + d_{jk} - 1$$

or, equivalently,

$$d_{ij} + d_{jk} + d_{ki} \leq 1.$$

Similarly, one can prove that

$$-(d_{ij} + d_{jk} + d_{ki}) = d_{ik} + d_{kj} + d_{ji} \leq 1.$$

□

Remark 1. *Nothing will change, if we take any increasing sequence $a_1, \dots, a_m$ instead of $1, \dots, m$ for values of our random variables.*

7. The case $n = 3$

From now on, we assume that $n = 3$. For more convenience, set

$$x_1 = x, \quad x_2 = y, \quad x_3 = z.$$

Besides, assuming that $d_1 = d_2 = d_3 = 1$, we will use the *probabilistic coordinates* p, q, r defined by

$$d_{12} = 2p - 1, \quad d_{23} = 2q - 1, \quad d_{31} = 2r - 1, \tag{15}$$

instead of the canonical coordinates d_{12}, d_{23}, d_{31}. For an element $g \in B_1$ represented by a normalized non-negative word, they have the following sense:

$$p = \mathbf{P}(x < y), \quad q = \mathbf{P}(y < z), \quad r = \mathbf{P}(z < x). \tag{16}$$

In the probablistic coordinates, the inequalities (13) and (14) take the form

$$0 \le p, q, r \le 1, \quad 1 \le p + q + r \le 2. \tag{17}$$

They define a triangular antiprism A with the center $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. The central symmetry is given by

$$(p, q, r) \mapsto (1 - p, 1 - q, 1 - r).$$

Proposition 5. *All the side faces of the antiprism A (coming from the faces of the cube $0 \le p, q, r \le 1$) lie in B_1.*

Proof. By symmetry, it suffices to consider just one side face. For the face F lying in the plane $p = 1$, the proof is achieved by a direct calculation of the probabilistic coordinates of the elements $g \in B_1$ represented by normalized non-negative words of pattern $[zxyzy]$, i.e. of the products of the form

$$g = z^s xy^t z^{1-s} y^{1-t} \quad \text{with} \quad 0 \le s, t \le 1. \tag{18}$$

Namely, we have

$$p = 1, \quad q = t(1 - s), \quad r = s,$$

and it is easy to see that every point of F is obtained for some admissible values of s, t. □

Proposition 6. *The elements of B_1 represented by normalized non-negative words of patterns $[xyzy]$, $[yzxz]$, $[zxyx]$ lie on the side edges of A (which are edges of the cube $0 \le p, q, r \le 1$).*

Proof. By symmetry, it suffices to consider the products of the form $xy^t zy^{1-t} \quad (0 \le t \le 1)$. They are just the products (18) with $s = 0$, so their probabilistic coordinates are $1, t, 0$. □

Proposition 7. *The intersection of A with each of the quadric surfaces*

$$p + qr = 1, \quad q + rp = 1, \quad r + pq = 1 \tag{19}$$

lies in B_1.

Proof. For the surface $p+qr = 1$, the proof is achieved by a calculation of the probabilistic coordinates of the normalized non-negative words of the pattern $[xyzxy]$. Namely, for

$$g = x^s y^t zx^{1-s} y^{1-t} \quad \text{with} \quad 0 \le s, t \le 1 \tag{20}$$

we have

$$p = 1 - t + st, \quad q = t, \quad r = 1 - s,$$

and it easy to see that every point of the intersection of A with the surface $r + pq = 1$ is obtained for some admissible values of s, t. □

The plane $p+q+r = \frac{3}{2}$ parallel to the bases of the antiprism A divide it into two parts, say A_+ and A_-, the former being defined by the inequality $p + q + r \ge \frac{3}{2}$ and the latter by the opposite inequality. They are symmetric to one another with respect to the center of A.

Proposition 8. *The intersection of A with the surface*

$$\min\{p + qr, q + rp, r + pq\} = 1 \tag{21}$$

strictly lies in A_+.

Proof. Let the point $(p,q,r) \in A$ satisfy (21). Assume that $p \le q,r$. Then $p+qr \le q+rp, r+pq$, so $p+qr=1$ and

$$p+q+r \ge p+2\sqrt{qr} = p+2\sqrt{1-p}.$$

It is easy to see that the function $f(t)=t+2\sqrt{1-t}$ is decreasing on $[0,1]$ and $f(\frac{1}{2}) > \frac{3}{2}$. Thus, if $p \le \frac{1}{2}$, we are done. But if $p > \frac{1}{2}$, then also $q,r > \frac{1}{2}$ and, hence, $p+q+r > \frac{3}{2}$. $\square$

Cut the "upper" base of A lying in the plane $p+q+r=2$, along the surface (21). Then we shall obtain three curved quadrilateral faces instead of the former plane triangular face. Each of them contains one vertex and halves of two sides of the former face, and all three new faces intersect in one point. Making use of the central symmetry, let us do the same with the "lower" base of A. As a result, we shall obtain some curved polyhedron with 6 plane triangular faces and 6 curved quadrilateral faces. Denote it by A^{cut}. According to the above, the boundary of A^{cut} is contained in B_1. Since B_1 is a star polyhedron, it follows that $A^{\mathrm{cut}} \subset B_1$.

Let A_p, A_q, A_r be the bodies defined in A by the inequalities

$$p+qr \le 1, \quad q+rp \le 1, \quad r+pq \le 1, \tag{22}$$

respectively, and A'_p, A'_q, A'_r be symmetric to A_p, A_q, A_r with respect to the center of A. It follows from Proposition 8 that

$$A_p \cup A_q \cup A_r \supset A_-, \quad A'_p \cup A'_q \cup A'_r \supset A_+$$

and

$$A^{\mathrm{cut}} = (A_p \cup A_q \cup A_r) \cap (A'_p \cup A'_q \cup A'_r). \tag{23}$$

Theorem 3. *$B_1 = A^{\mathrm{cut}}$.*

Proof. For each $t \ge 0$, define a map

$$L_x(t) : B_1 \to B_1$$

as the left multiplication by x^t with the subsequent dividing all the exponents of x by $1+t$, which makes the total degree in x equal to 1. In the coordinates p,q,r it looks as follows:

$$(p,q,r) \mapsto (\frac{p+t}{1+t}, q, \frac{r}{1+t},). \tag{24}$$

In a similar way, define $L_y(t)$ and $L_z(t)$.

Clearly, any element of B_1 can by obtained by a consecutive application of maps $L_x(t)$, $L_y(t)$, $L_z(t)$ to the elements considered in Proposition 6. Thus, it suffices to prove that A^{cut} is invariant under all these maps. By symmetry, it suffices to prove this for $L_x(t)$ only.

Note that the permutation of y and z, which obviously commutes with $L_x(t)$, acts by the formula

$$(p,q,r) \mapsto (1-r, 1-q, 1-p)$$

and, hence, takes A_p, A_q, A_r to A'_r, A'_q, A'_p, respectively. In particular, it takes $A_p \cup A_q \cup A_r$ to $A'_p \cup A'_q \cup A'_r$. In view of (23), it follows that it suffices to prove that the body $A_p \cup A_q \cup A_r$ is invariant under $L_x(t)$.

If $(p,q,r) \in A_p$, i.e., $p+qr \leq 1$, then

$$\frac{p+t}{1+t} + \frac{qr}{1+t} = \frac{p+qr+t}{1+t} \leq 1,$$

so the body A_p is invariant under $L_t(x)$.

If $(p,q,r) \in A_r$, i.e., $r+pq \leq 1$, then

$$\frac{r}{1+t} + \frac{(p+t)q}{1+t} \leq \frac{r+pq+t}{1+t} \leq 1,$$

so the body A_r is also invariant under $L_t(x)$.

Let now $(p,q,r) \in A_q$, i.e., $q+rp \leq 1$. Then

$$q + \frac{r(p+t)}{(1+t)^2} \leq 1 \quad \text{for} \quad p \geq \frac{1}{2},$$

since under the latter condition $\frac{p+t}{(1+t)^2} \leq p$.

Finally, let $(p,q,r) \in A_q$, but $p < \frac{1}{2}$. Let us prove that in this case $L_x(t)(p,q,r) \in A_p$. Assume that $p+qr > 1$. Then

$$p^2 - q^2 = p(p+qr) - q(q+rp) \geq p - q,$$

whence $p \geq q$ (since $p+q \geq p+qr > 1$). Thus, $q \leq p < \frac{1}{2}$. But then $p+qr \leq 1$, a contradiction. □

8. Proof of the main theorem

Making use of the probabilistic interpretation (16) of the coordinates p,q,r, one can deduce Theorem 1 from Theorem 3.

If x,y,z take only finitely many values and the sets of their values are disjoint, the assertion directly follows from Theorem 3 (see also Remark 1). In particular, we obtain that any vector of A^{cut} serves as (p,q,r) for suitable random variables x,y,z (taking only finitely many values).

Thus, it remains to prove that $(p,q,r) \in A^{\text{cut}}$ for any independent random variables x,y,z satisfying (1). This will be achieved by approximation.

Let us first get rid of the restriction that the sets of the values of x,y,z are disjoint (though assuming that the intersection of all of them is empty). Suppose that, say, x and y take some value a with positive probabilities. Consider a new random variable x' taking the value $a' > a$ each time when x takes the value a, and being equal to x in all the other cases. If a' is close enough to a, then

$$\mathbf{P}(x' < y) = \mathbf{P}(x < y) \quad \text{and} \quad \mathbf{P}(z < x') = \mathbf{P}(z < x).$$

Proceeding in this way, in several steps one can obtain new random variables, whose sets of values are disjoint, without changing the probabilities p,q,r. Thus, the assertion of Theorem 1 is true for any random variables taking only finitely many values, provided that the condition (1) holds.

Let now x,y,z be arbitrary independent random values satisfying the condition (1). Take any $n \in \mathbb{N}$ and consecutively determine numbers $a_1 < a_2 < \dots$ such that for any $u \in \{x,y,z\}$ and any i

$$\mathbf{P}(a_{i-1} < u < a_i) \leq \frac{1}{n},$$

assuming that $a_0 = -\infty$, and for any i there exists $u \in \{x, y, z\}$ such that

$$\mathbf{P}(a_{i-1} < u \leq a_i) \geq \frac{1}{n}.$$

Clearly, the process will stop at some a_m with $m \leq 3n$, and for any $u \in \{x, y, z\}$ we shall have

$$\mathbf{P}(a_m < u < +\infty) \leq \frac{1}{n}.$$

Choose three different points x_i, y_i, z_i in each interval (a_{i-1}, a_i) for $i = 1, \ldots, m, m+1$ (assuming that $a_{m+1} = +\infty$) and consider new random variables x', y', z' taking the values a_i with the same probabilities as x, y, z and the values x_i, y_i, z_i with probabilities

$$\mathbf{P}(a_{i-1} < x < a_i), \quad \mathbf{P}(a_{i-1} < y < a_i), \quad \mathbf{P}(a_{i-1} < z < a_i),$$

respectively. Representing the line as a disjoint union of the intervals (a_{i-1}, a_i) and points a_i, we see that

$$|\mathbf{P}(x < y) - \mathbf{P}(x' < y')| = |\sum_{i=1}^{m+1} \mathbf{P}(a_{i-1} < x < y < a_i) - \epsilon_i \mathbf{P}(a_{i-1} < x < a_i)\mathbf{P}(a_{i-1} < y < a_i)| \leq \frac{3n+1}{n^2},$$

where $\epsilon_i = 1$, if $x_i < y_i$ and 0 otherwise. Similar inequalities are obtained for $|\mathbf{P}(y < z) - \mathbf{P}(y' < z')|$ and $|\mathbf{P}(z < x) - \mathbf{P}(z' < x')|$.

Thus, the point $(\mathbf{P}(x < y), \mathbf{P}(y < z), \mathbf{P}(z < x))$ can be approximated by points of A^{cut} and, hence, lies in A^{cut}.

References

[1] Hilgert J., Neeb K.-H. Lie semigroups and their applications. Lecture Notes in Mathemetics, v.1552. Springer-Verlag, 1993.