

DETERMINISTIC AND STOCHASTIC 2D NAVIER-STOKES EQUATIONS WITH ANISOTROPIC VISCOSITY

Siyu Liang ^{1,2} Ping Zhang ¹, Rongchan Zhu ^{3,4}

ABSTRACT. In this paper, we investigate both deterministic and stochastic 2D Navier Stokes equations with anisotropic viscosity. For the deterministic case, we prove the global well-posedness of the system with initial data in the anisotropic Sobolev space $\tilde{H}^{0,1}$. For the stochastic case, we obtain existence of the martingale solutions and pathwise uniqueness of the solutions, which imply existence of the probabilistically strong solution to this system by the Yamada-Watanabe Theorem.

1. INTRODUCTION

We recall the incompressible classical Navier-Stokes system for incompressible fluids

$$(NS_\nu) \quad \begin{cases} \partial_t u + u \cdot \nabla u - \nu \Delta u = -\nabla p, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \end{cases}$$

where $\nu > 0$ is the viscosity of the fluid, u and p denote the velocity and the pressure of the fluid respectively. In 1934, J. Leray proved global existence of finite energy weak solutions to (NS_ν) in the whole space \mathbb{R}^d , for $d = 2, 3$, in the seminar paper [14]. When $d = 2$, global weak solutions to (NS_ν) are unique. However, when $d = 3$, the regularities and uniqueness of Leray solutions to (NS_ν) are still widely open in the field of mathematical fluid mechanics except the case when the norm of the initial data is small compared to the viscosity ν .

Here we consider the incompressible Navier-Stokes equations with partial dissipation. Systems of this type appear in geophysical fluids (see for instance [6, 17]). In fact, instead of putting the classical viscosity $-\nu \Delta$ in (NS_ν) , meteorologist often modelize turbulent diffusion by putting a viscosity of the form: $-\nu_h \Delta_h - \nu_3 \partial_{x_3}^2$, where ν_h and ν_3 are empiric constants, and ν_3 is usually much smaller than ν_h . We refer to the book of J. Pedlovsky [17], Chapter 4 for a more complete discussion. And in the particular case of the so-called Ekman layers [9, 12] for rotating fluids, $\nu_3 = \epsilon \nu_h$ and ϵ is a very small parameter. In [5, 7, 16], the authors consider the global well-posedness of such system with small initial data in some anisotropic Besov type spaces. However, for this 3D anisotropic Navier-Stokes equation, there is no result concerning global existence of weak solutions. In this paper, we concentrate on the 2D case first.

The aim of this paper is to investigate both the following deterministic system on \mathbb{R}^2 or on the two dimensional torus \mathbb{T}^2

$$(1.1) \quad \begin{cases} \partial_t u + u \cdot \nabla u - \partial_1^2 u = -\nabla p, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \end{cases}$$

¹Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China

²School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China

³Department of Mathematics, Beijing Institute of Technology, Beijing 100081, China

⁴Department of Mathematics, University of Bielefeld, D-33615 Bielefeld, Germany

and the following stochastic system on \mathbb{T}^2

$$(1.2) \quad \begin{cases} du + (u \cdot \nabla u - \partial_1^2 u)dt = \sigma(t, u)dW - \nabla p dt, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \end{cases}$$

where σ is the random external force and W is an ℓ^2 -cylindrical Wiener process, the definition of which will be introduced in Section 4. It is well-known that for the 2D incompressible Euler system with initial data in s -order Sobolev space $H^s(\mathbb{R}^2)$ for $s > 2$, there exists a unique global solution in $L_{\text{loc}}^\infty(\mathbb{R}^+, H^s(\mathbb{R}^2))$ (see [1] for instance). As (1.1) is an intermediate equation between 2D Euler equations and 2D Navier Stokes equations, we also have similar global well-posedness for (1.1) with initial data in $H^s(\mathbb{R}^2)$ for $s > 2$. Since (1.1) is more dissipative than Euler system, we are going to prove its global well-posedness with initial data in $\tilde{H}^{0,1}$ (see Section 2 for the definition of $\tilde{H}^{0,1}$). For the stochastic two dimensional Euler equation, we can only deal with it driven by additive or linear multiplicative noise (See [10]). But for the anisotropic system (1.1), we can solve the martingale problem with general multiplicative noise. The main novelty is an $H^{0,1}$ -uniform estimate, the proof of which depends crucially on the divergence free condition (see (3.2)).

The plan of this paper is as follows. In Section 2, we introduce some notations and recall some preliminaries. In the following two sections, we first study the deterministic equation (1.1) and then we consider (1.2) with stochastic external force which may depend on the velocity u .

Main results for deterministic part: We prove the existence and uniqueness of weak solutions in the space $L^\infty(\mathbb{R}^+; H^{0,1}) \cap L^2(\mathbb{R}^+; \dot{H}^{1,1})$ for the deterministic equation (1.1) (see Theorem 3.1 below). In order to prove existence results, we need both the L^2 as well as $H^{0,1}$ uniform estimate for appropriate approximate solutions to (1.1). (The definition of $H^{0,1}$ and $\dot{H}^{1,1}$ are given in Section 2) Note that the only L^2 uniform estimate does not provide the compactness of the approximate solutions due to the lack of the estimate for $\|\partial_2 u\|_{L^2}$. To obtain the uniform $H^{0,1}$ estimate, we have to use the divergence free condition of the velocity field in the crucial way. The uniqueness of solutions is proved by estimating the difference between any two solutions, $w = u - v$, in the space L^2 .

Main results for martingale solution of stochastic equation: We prove the existence of martingale solutions to (1.2) (see Theorem 4.1 below). Similar 3D equations with a Brickman-Forchheimer term, $|u|^{2\alpha}u$, have been studied in [2]. In this paper we prove the existence and uniqueness of probabilistically strong solutions to (1.2) in dimension 2 without Brickman-Forchheimer term. In order to do so, we first use Galerkin approximations to project (1.2) in finite dimensional space. Then we use Itô's formula to obtain the uniform estimates of u_n in both L^2 and $H^{0,1}$. Similar to the deterministic case, the proof depends heavily on the divergence free condition of the velocity field. However, since we have to take the expectation, we can not use Itô's formula to estimate $\|u_n\|_{H^{0,1}}^2$ directly, instead we shall multiply it by an exponential term $e^{-2c \int_0^t \|\partial_1 u\|_{L^2}^2 dt}$ for some proper c . Then by tightness methods (Skorokhod Theorem), we can obtain the existence of martingale solutions. Here we emphasize that we rely more heavily on the divergence free condition and we could not use similar methods as in [2] since we do not have Brickman-Forchheimer term, (which helps to obtain a better estimate for the solution.) And we have to use the martingale approach. Moreover, we can prove the pathwise uniqueness of solutions in L^2 space. Finally by the Yamada-Watanabe theorem, we obtain the existence and uniqueness of the (probabilistically) strong solution to (1.2).

2. PRELIMINARIES

We first recall some function spaces on \mathbb{R}^2 and on the two dimensional torus \mathbb{T}^2 .

2.1. Function spaces on \mathbb{R}^2 . On \mathbb{R}^2 , we recall the classical Sobolev spaces:

$$H^s(\mathbb{R}^2) := \left\{ u \in \mathcal{S}'(\mathbb{R}^2); \|u\|_{H^s(\mathbb{R}^2)}^2 := \int_{\mathbb{R}^2} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi < \infty \right\},$$

where \hat{u} denotes the Fourier transform of u . Due to the anisotropic properties of (1.1), we also need anisotropic Sobolev spaces. Let us recall the anisotropic Sobolev norms and spaces:

$$H^{s,s'}(\mathbb{R}^2) := \left\{ u \in \mathcal{S}'(\mathbb{R}^2); \|u\|_{H^{s,s'}(\mathbb{R}^2)}^2 := \int_{\mathbb{R}^2} (1 + |\xi_1|^2)^s (1 + |\xi_2|^2)^{s'} |\hat{u}(\xi)|^2 d\xi < \infty \right\},$$

where $\xi = (\xi_1, \xi_2)$.

We remark that the space $H^{s,s'}(\mathbb{R}^2)$ endowed with the norm $\|\cdot\|_{H^{s,s'}(\mathbb{R}^2)}$ is a Hilbert space.

We also recall the horizontally homogeneous anisotropic Sobolev norm and the space:

$$\dot{H}^{s,s'}(\mathbb{R}^2) := \left\{ u \in \mathcal{S}'(\mathbb{R}^2); \|u\|_{\dot{H}^{s,s'}(\mathbb{R}^2)}^2 := \int_{\mathbb{R}^2} |\xi_1|^{2s} (1 + |\xi_2|^2)^{s'} |\hat{u}(\xi)|^2 d\xi < \infty \right\}.$$

In what follows, we shall use ‘h’ to denote the horizontal variable x_1 , and ‘v’ to denote the vertical direction x_2 . Let $\mathbb{R}^2 = (\mathbb{R}_h, \mathbb{R}_v)$. For exponents $p, q \in [1, \infty)$, we denote the space $L^p(\mathbb{R}_h, L^q(\mathbb{R}_v))$ by $L_h^p(L_v^q)(\mathbb{R}^2)$, which is endowed with the norm

$$\|u\|_{L_h^p(L_v^q)(\mathbb{R}^2)} := \left\{ \int_{\mathbb{R}_h} \left(\int_{\mathbb{R}_v} |u(x_1, x_2)|^q dx_2 \right)^{\frac{p}{q}} dx_1 \right\}^{\frac{1}{p}}.$$

Similar notation for $L_v^p(L_h^q)(\mathbb{R}^2)$. Then it follows from Minkowski inequality that

$$\begin{aligned} \|u\|_{L_h^p(L_v^q)(\mathbb{R}^2)} &\leq \|u\|_{L_v^q(L_h^p)(\mathbb{R}^2)} \quad \text{when } 1 \leq q \leq p \leq \infty, \\ \|u\|_{L_v^q(L_h^p)(\mathbb{R}^2)} &\leq \|u\|_{L_h^p(L_v^q)(\mathbb{R}^2)} \quad \text{when } 1 \leq p \leq q \leq \infty. \end{aligned}$$

2.2. Function spaces on \mathbb{T}^2 . Now we recall some function spaces for the two dimensional torus \mathbb{T}^2 . Let $\mathbb{T}^2 = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z} = (\mathbb{T}_h, \mathbb{T}_v)$. Similar to the whole space \mathbb{R}^2 , we recall the anisotropic L^p spaces:

$$\|u\|_{L_h^p(L_v^q)(\mathbb{T}^2)} := \left\{ \int_{\mathbb{T}_h} \left(\int_{\mathbb{T}_v} |u(x_1, x_2)|^q dx_2 \right)^{\frac{p}{q}} dx_1 \right\}^{\frac{1}{p}}.$$

Similar to the whole space, we also have:

$$\begin{aligned} \|u\|_{L_h^p(L_v^q)(\mathbb{T}^2)} &\leq \|u\|_{L_v^q(L_h^p)(\mathbb{T}^2)} \quad \text{when } 1 \leq q \leq p \leq \infty, \\ \|u\|_{L_v^q(L_h^p)(\mathbb{T}^2)} &\leq \|u\|_{L_h^p(L_v^q)(\mathbb{T}^2)} \quad \text{when } 1 \leq p \leq q \leq \infty. \end{aligned}$$

For $u \in L^2(\mathbb{T}^2)$, we consider the Fourier expansion of u :

$$u(x) = \sum_{k \in \mathbb{Z}^2} \hat{u}_k e^{ik \cdot x} \quad \text{with} \quad \hat{u}_k = \overline{\hat{u}_{-k}},$$

where $\hat{u}_k := \frac{1}{(2\pi)^2} \int_{[0, 2\pi] \times [0, 2\pi]} u(x) e^{-ik \cdot x} dx$ denotes the Fourier coefficient of u on \mathbb{T}^2 . It follows from Fourier-Plancherel equality that the series is convergent in $L^2(\mathbb{T}^2)$.

Define the Sobolev norm :

$$\|u\|_{H^s(\mathbb{T}^2)}^2 := \sum_{k \in \mathbb{Z}^2} (1 + |k|^2)^s |\hat{u}_k|^2,$$

and the anisotropic Sobolev norms:

$$\begin{aligned} \|u\|_{H^{s,s'}(\mathbb{T}^2)}^2 &= \sum_{k \in \mathbb{Z}^2} (1 + |k_1|^2)^s (1 + |k_2|^2)^{s'} |\hat{u}_k|^2, \\ \|u\|_{\dot{H}^{s,s'}(\mathbb{T}^2)}^2 &= \sum_{k \in \mathbb{Z}^2} |k_1|^{2s} (1 + |k_2|^2)^{s'} |\hat{u}_k|^2, \end{aligned}$$

where $k = (k_1, k_2)$.

And we also define the Sobolev spaces $H^s(\mathbb{T}^2)$, $H^{s,s'}(\mathbb{T}^2)$ and $\dot{H}^{s,s'}(\mathbb{T}^2)$ as the completion of $C^\infty(\mathbb{T}^2)$ with the norms $\|\cdot\|_{H^s(\mathbb{T}^2)}$, $\|\cdot\|_{H^{s,s'}(\mathbb{T}^2)}$ and $\|\cdot\|_{\dot{H}^{s,s'}(\mathbb{T}^2)}$ respectively.

2.3. Some other notations and definitions. We use D to denote the domain \mathbb{R}^2 or \mathbb{T}^2 . Let us denote

$$\begin{aligned} H &:= \{u \in L^2(D); \operatorname{div} u = 0\}, \\ V &:= \{u \in H^1(D), \operatorname{div} u = 0\}, \\ \tilde{H}^{s,s'} &:= \{u \in H^{s,s'}(D), \operatorname{div} u = 0\}. \end{aligned}$$

Moreover, we use (\cdot, \cdot) or $(\cdot | \cdot)$ to denote the scalar product

$$(u, v) = (u | v) = (u, v)_{L^2(D)} = \sum_{j=1}^2 \int_D u_j(x) v_j(x) dx.$$

We use $(\cdot, \cdot)_{H^{0,1}}$ or $(\cdot, \cdot)_{0,1}$ to denote the inner product

$$(u, v)_{H^{0,1}(D)} = \sum_{j=1}^2 \int_D (u_j(x) v_j(x) + \partial_2 u_j(x) \partial_2 v_j(x)) dx.$$

Let

$\mathbf{P} : L^2(D) \rightarrow H$ is the Leray projection operator to divergence free space.

By applying \mathbf{P} to (1.1), we write

$$\partial_t u = \mathbf{P}(\partial_1^2 u - u \cdot \nabla u).$$

As usual, when $u, v, w \in H^1(D)$, we denote

$$\begin{aligned} B(u, v) &:= u \cdot \nabla v, \\ B(u) &:= u \cdot \nabla u, \\ b(u, v, w) &:= (u \cdot \nabla v, w). \end{aligned}$$

Then we have $b(u, v, w) = -b(u, w, v)$ for $u, v, w \in V$. In particular, $b(u, v, v) = 0$.

Let us end this section by the definition of weak solution to (1.1)

Definition 2.1 (weak solution). *We call u a global weak solution of (1.1) with the initial data u_0 if u satisfies:*

- (i) $u \in L^\infty(\mathbb{R}^+; \tilde{H}^{0,1}(D)) \cap L^2(\mathbb{R}^+; \dot{H}^{1,1}(D))$;
- (ii) for any $\varphi \in C_0^\infty(D)$ with $\operatorname{div} \varphi = 0$, and $t > 0$,

$$(2.1) \quad \int_0^t \{-(u, \partial_t \varphi) + (\partial_1 u, \partial_1 \varphi) + (u \cdot \nabla u, \varphi)\} ds = (u_0, \varphi(0)) - (u(t), \varphi(t)),$$

3. THE DETERMINISTIC CASE

For simplicity, we always omit the domain D in this section.

Theorem 3.1. *Given solenoidal vector field u_0 in $\tilde{H}^{0,1}$, (1.1) has a unique global weak solution $u \in L^\infty(\mathbb{R}^+; \tilde{H}^{0,1}) \cap L^2(\mathbb{R}^+; \dot{H}^{1,1})$ in the sense of Definition 2.1.*

Lemma 3.1. *Let u be a global smooth enough solution to (1.1). Then one has*

$$(3.1) \quad \|u(t)\|_{L^2}^2 + 2 \int_0^t \|\partial_1 u(s)\|_{L^2}^2 ds \leq \|u_0\|_{L^2}^2.$$

Proof. Indeed by taking the L^2 inner product of the momentum equation of (1.1) with u and using $\operatorname{div} u = 0$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2}^2 + \|\partial_1 u\|_{L^2}^2 = 0.$$

Integrating the above inequality over $[0, t]$ leads to (3.1). \square

Lemma 3.2. *Under the same assumption of Lemma 3.1, we have*

$$(3.2) \quad \|\partial_2 u(t)\|_{L^2}^2 + \int_0^t \|\partial_1 \partial_2 u(s)\|_{L^2}^2 ds \leq \|\partial_2 u_0\|_{L^2}^2 e^{C\|u_0\|_{L^2}^2}$$

for some constant $C > 0$.

Proof. By Taking ∂_2 to the momentum equation of (1.1) and then taking L^2 inner product of the resulting equation with $\partial_2 u$, we obtain

$$(3.3) \quad \frac{1}{2} \frac{d}{dt} \|\partial_2 u(t)\|_{L^2}^2 + \|\partial_1 \partial_2 u\|_{L^2}^2 \leq -(\partial_2(u \cdot \nabla u) \mid \partial_2 u).$$

It is easy to observe that

$$(3.4) \quad (\partial_2(u \cdot \nabla u) \mid \partial_2 u) = (\partial_2(u \cdot \nabla u^1) \mid \partial_2 u^1) + (\partial_2(u \cdot \nabla u^2) \mid \partial_2 u^2),$$

where $u = (u^1, u^2)$.

For the first term on the right-hand side of (3.4), we have

$$\begin{aligned} (\partial_2(u \cdot \nabla u^1) \mid \partial_2 u^1) &= (\partial_2(u^1 \partial_1 u^1 + u^2 \partial_2 u^1) \mid \partial_2 u^1) \\ &= (\partial_2 u^1 \partial_1 u^1 \mid \partial_2 u^1) + (u^1 \partial_2 \partial_1 u^1 \mid \partial_2 u^1) \\ &\quad + (\partial_2 u^2 \partial_2 u^1 \mid \partial_2 u^1) + (u^2 \partial_2^2 u^1 \mid \partial_2 u^1), \end{aligned}$$

Yet due to $\operatorname{div} u = 0$, we achieve

$$(\partial_2 u^1 \partial_1 u^1 \mid \partial_2 u^1) + (\partial_2 u^2 \partial_2 u^1 \mid \partial_2 u^1) = 0,$$

and

$$\begin{aligned} (u^1 \partial_2 \partial_1 u^1 \mid \partial_2 u^1) + (u^2 \partial_2^2 u^1 \mid \partial_2 u^1) &= (u \cdot \nabla \partial_2 u^1 \mid \partial_2 u^1) \\ &= \frac{1}{2} \int_D u \cdot \nabla |\partial_2 u^1|^2 dx = -\frac{1}{2} \int_D \operatorname{div} u |\partial_2 u^1|^2 dx = 0. \end{aligned}$$

This leads to

$$(3.5) \quad (\partial_2(u \cdot \nabla u^1) \mid \partial_2 u^1) = 0.$$

For the second term on the right-hand side of (3.4), again due to $\operatorname{div} u = 0$, we have:

$$\begin{aligned} (\partial_2(u \cdot \nabla u^2) \mid \partial_2 u^2) &= (\partial_2(u^1 \partial_1 u^2) \mid \partial_2 u^2) + (\partial_2(u^2 \partial_2 u^2) \mid \partial_2 u^2) \\ &= (\partial_2 u \cdot \nabla u^2 \mid \partial_2 u^2). \end{aligned}$$

The second equality is due to

$$(u^1 \partial_1 \partial_2 u^2 \mid \partial_2 u^2) + (u^2 \partial_2^2 u^2 \mid \partial_2 u^2) = -\frac{1}{2} (\partial_1 u^1 \partial_2 u^2 \mid \partial_2 u^2) - \frac{1}{2} (\partial_2 u^2 \partial_2 u^2 \mid \partial_2 u^2) = 0.$$

Whereas notice that

$$\begin{aligned} &|(\partial_2 u \cdot \nabla u^2 \mid \partial_2 u^2)| \\ &= |(\partial_2 u^1 \partial_1 u^2 \mid \partial_2 u^2) + (\partial_2 u^2 \partial_2 u^2 \mid \partial_2 u^2)| \\ &\leq (\|\partial_2 u^1\|_{L_h^\infty(L_v^2)} \|\partial_1 u^2\|_{L_h^2(L_v^\infty)} + \|\partial_1 u^1\|_{L_h^2(L_v^\infty)} \|\partial_2 u^2\|_{L_h^\infty(L_v^2)}) \|\partial_2 u^2\|_{L^2}, \end{aligned}$$

from which and

$$(3.6) \quad \|u\|_{L_v^2(L_h^\infty)} \lesssim \|u\|_{L^2}^{\frac{1}{2}} \|\partial_{x_1} u\|_{L^2}^{\frac{1}{2}} \quad \text{and} \quad \|u\|_{L_h^2(L_v^\infty)} \lesssim \|u\|_{L^2}^{\frac{1}{2}} \|\partial_{x_2} u\|_{L^2}^{\frac{1}{2}},$$

we infer

$$\begin{aligned} |(\partial_2 u \cdot \nabla u^2 | \partial_2 u^2)| &\lesssim (\|\partial_1 \partial_2 u^1\|_{L^2}^{\frac{1}{2}} \|\partial_2 u^1\|_{L^2}^{\frac{1}{2}} \|\partial_1 u^2\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 u^2\|_{L^2}^{\frac{1}{2}} \\ &\quad + \|\partial_1 \partial_2 u^1\|_{L^2}^{\frac{1}{2}} \|\partial_1 u^1\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 u^2\|_{L^2}^{\frac{1}{2}} \|\partial_2 u^2\|_{L^2}^{\frac{1}{2}}) \|\partial_2 u^2\|_{L^2}. \end{aligned}$$

This together with $\operatorname{div} u = 0$ ensures that

$$|(\partial_2 u \cdot \nabla u^2 | \partial_2 u^2)| \lesssim \|\partial_1 \partial_2 u\|_{L^2} \|\partial_1 u\|_{L^2} \|\partial_2 u\|_{L^2}.$$

Along with (3.5), we achieve

$$|(\partial_2(u \cdot \nabla u) | \partial_2 u)| \lesssim \|\partial_1 \partial_2 u\|_{L^2} \|\partial_1 u\|_{L^2} \|\partial_2 u\|_{L^2}$$

Applying Young's inequality yields

$$|(\partial_2(u \cdot \nabla u) | \partial_2 u)| \leq \frac{1}{2} \|\partial_1 \partial_2 u\|_{L^2}^2 + C \|\partial_1 u\|_{L^2}^2 \|\partial_2 u\|_{L^2}^2.$$

Inserting the above inequality into (3.3) gives rise to

$$\frac{d}{dt} \|\partial_2 u(t)\|_{L^2}^2 + \|\partial_1 \partial_2 u\|_{L^2}^2 \leq 2C \|\partial_1 u\|_{L^2}^2 \|\partial_2 u\|_{L^2}^2.$$

Applying Gronwall's inequality and using (3.1), we obtain

$$\begin{aligned} \|\partial_2 u(t)\|_{L^2}^2 + \int_0^t \|\partial_1 \partial_2 u(s)\|_{L^2}^2 ds &\leq e^{2C \int_0^t \|\partial_1 u\|_{L^2}^2 ds} \|\partial_2 u_0\|_{L^2}^2 \\ &\leq e^{2C \|u_0\|_{L^2}^2} \|\partial_2 u_0\|_{L^2}^2, \end{aligned}$$

which yields (3.2).

It remains to prove (3.6). We only present the proof to the first one, the second one follows along the same line. Indeed observing that

$$\begin{aligned} f^2(x_1, x_2) &= \int_{-\infty}^{x_1} \partial_y f^2(y, x_2) dy = 2 \int_{-\infty}^{x_1} f(y, x_2) \partial_y f(y, x_2) dy \\ &\leq 2 \|f(\cdot, x_2)\|_{L_h^2} \|\partial_{x_1} f(\cdot, x_2)\|_{L_h^2}, \end{aligned}$$

which implies

$$\|f(\cdot, x_2)\|_{L_h^\infty}^2 \leq 2 \|f(\cdot, x_2)\|_{L_h^2} \|\partial_{x_1} f(\cdot, x_2)\|_{L_h^2}.$$

Applying Hölder's inequality in the x_2 variable gives

$$\|f\|_{L_v^2(L_h^\infty)}^2 \leq 2 \|f\|_{L^2} \|\partial_{x_1} f\|_{L^2}.$$

This completes the proof of the lemma. \square

Let us now present the proof of Theorem 3.1.

Proof of Theorem 3.1. We divide the proof of this theorem to the following two parts:

(1) Existence part. It is standard that the first step to prove the existence of weak solutions to some nonlinear partial differential equations is to construct appropriate approximate solutions. Here we consider

$$(3.7) \quad \begin{cases} \partial_t u^\epsilon + u^\epsilon \cdot \nabla u^\epsilon - \partial_1^2 u^\epsilon - \epsilon^2 \partial_2^2 u^\epsilon = -\nabla p^\epsilon \\ \operatorname{div} u^\epsilon = 0 \\ u^\epsilon(0) = u_0 * j_\epsilon \end{cases},$$

where j is a smooth function on \mathbb{R}^2 with

$$j(x) = 1, \quad |x| \leq 1; \quad j(x) = 0, \quad |x| \geq 2,$$

and

$$j_\epsilon(x) = \frac{1}{\epsilon^2} j\left(\frac{x}{\epsilon}\right).$$

It follows from classical theory on Navier-Stokes system that (3.7) has a unique global smooth solution (u^ϵ, p^ϵ) for any fixed ϵ . Furthermore, along the same line to the proof of Lemmas 3.1 and 3.2, we have

$$(3.8) \quad \begin{aligned} \|u^\epsilon(t)\|_{L^2}^2 + 2 \int_0^t \|\partial_1 u^\epsilon\|_{L^2}^2 ds + \epsilon^2 \int_0^t \|\partial_2 u^\epsilon\|_{L^2}^2 ds &\leq \|u_0\|_{L^2}^2, \\ \|\partial_2 u^\epsilon(t)\|_{L^2}^2 + \int_0^t \|\partial_1 \partial_2 u^\epsilon\|_{L^2}^2 ds + \epsilon^2 \int_0^t \|\partial_2^2 u^\epsilon\|_{L^2}^2 ds &\leq \|\partial_2 u_0\|_{L^2}^2 e^{C\|u_0\|_{L^2}^2}. \end{aligned}$$

It is obvious that for $\varphi \in C_0^\infty$ with $\operatorname{div} \varphi = 0$, u^ϵ satisfies the following equation:

$$(3.9) \quad \int_0^t \left(-(u^\epsilon, \partial_t \varphi) + (u^\epsilon \cdot \nabla u^\epsilon, \varphi) + (\partial_1 u^\epsilon, \partial_1 \varphi) + \epsilon^2 (\partial_2 u^\epsilon, \partial_2 \varphi) \right) ds = 0$$

Then for any fixed $T > 0$, $\{u^\epsilon\}_{\epsilon>0}$ is uniformly bounded in $L^\infty([0, T]; H^{0,1}) \cap L^2([0, T]; H^{1,1})$. By interpolation, $\{u^\epsilon\}_{\epsilon>0}$ is uniformly bounded in $L^4([0, T]; H^{\frac{1}{2}})$. Sobolev imbedding implies that $\{u^\epsilon\}_{\epsilon>0}$ is bounded in $L^4([0, T]; L^4)$. Hence the nonlinear term in (3.7) is bounded in $L^2([0, T]; H^{-1})$. Moreover, $\nabla p^\epsilon = \nabla \Delta^{-1} \partial_i \partial_j ((u^\epsilon)^i (u^\epsilon)^j)$ is uniformly bounded in $L^2([0, T]; H^{-1})$. As a result, it comes out that

$$(3.10) \quad \{\partial_t u^\epsilon\}_{\epsilon>0} \quad \text{is uniformly bounded in } L^2([0, T]; H^{-1}).$$

At this stage, we need to use the following Aubin-Lions lemma:

Lemma 3.3 (Aubin-Lions). *Let K be the torus or a smooth bounded domain. If the sequence $(u_n)_{n \in \mathbb{N}}$ is uniformly bounded sequence in $L^q([0, T]; H^1(K))$ for $q \in (1, \infty)$, and $(\partial_t u_n)_{n \in \mathbb{N}}$ is a uniformly bounded sequence in $L^p([0, T]; H^1(K))$ for some $p \in (1, \infty)$, then there exist $u \in L^q([0, T]; H^1(K))$ and a subsequence of $(u_{n_j})_{j \in \mathbb{N}}$ so that $(u_{n_j})_{j \in \mathbb{N}}$ converges strongly to u in $L^q([0, T]; L^2(K))$.*

Let us now take $\epsilon = \frac{1}{n}$ in (3.7). Set $u_n = u^{\frac{1}{n}}$.

(i) For torus \mathbb{T}^2 case, given any $T > 0$, it follows from (3.9), (3.10) and Aubin-Lions Lemma that there is a subsequence, which we still denote by $\{u_n\}_{n \in \mathbb{N}}$ and some $u \in L^\infty([0, T]; H^{0,1}) \cap L^2([0, T]; H^1)$ so that

$$(3.11) \quad u_n \rightarrow u \text{ strongly in } L^2([0, T]; L^2(\mathbb{T}^2)) \text{ as } n \rightarrow \infty.$$

Through a diagonal process with respect to T , we can choose a subsequence, $\{u_n\}_{n \in \mathbb{N}}$, so that (3.11) holds for any $T > 0$. Then we can pass the limit in (3.9) to obtain (2.1).

(ii) For the case that $D = \mathbb{R}^2$, we choose a sequence of compact sets (K_i) , such that $K_i \subset K_{i+1}$, and $\bigcup_{i=1}^\infty K_i = \mathbb{R}^2$. By a classical diagonal methods, we can choose a subsequence of $(u_n)_{n \in \mathbb{N}}$ (which we still denote by $(u_n)_{n \in \mathbb{N}}$ for simplicity) so that

$$u_n \rightarrow u \text{ strongly in } L^2([0, T]; L^2(K_i)) \text{ for any } i.$$

Since the test function φ in (2.1) satisfies $\varphi \in C_0^\infty(D)$, it must be supported in some K_i . Then as in case (i), we can take $n \rightarrow \infty$ in (3.9) to obtain (2.1).

Finally notice that since u^ϵ is uniformly bounded in $L^\infty(\mathbb{R}^+; H^{0,1}) \cap L^2(\mathbb{R}^+; \dot{H}^{1,1})$, we can choose a subsequence of u_n (which we denote by u_n again) and some \tilde{u} , such that

$u_n \rightarrow \tilde{u}$ weakly in $L^2([0, T]; H^{1,1})$ for each $T > 0$, and
 $u_n \rightarrow \tilde{u}$ weakly star in $L^\infty([0, T]; H^{0,1})$ for each $T > 0$.

By the uniqueness of the limits of weak convergence, u and \tilde{u} coincide.

Since u_n is uniformly bounded in $L^\infty(\mathbb{R}^+; H^{0,1}) \cap L^2(\mathbb{R}^+; \dot{H}^{1,1})$, we have

$$\|u\|_{L^\infty(\mathbb{R}^+; H^{0,1})} = \lim_{T \rightarrow \infty} \|u\|_{L^\infty([0, T]; H^{0,1})} \leq \sup_n \|u_n\|_{L^\infty(\mathbb{R}^+; H^{0,1})},$$

and

$$\|u\|_{L^2(\mathbb{R}^+; \dot{H}^{1,1})} = \lim_{T \rightarrow \infty} \|u\|_{L^2([0,T]; \dot{H}^{1,1})} \leq \sup_n \|u_n\|_{L^2(\mathbb{R}^+; \dot{H}^{1,1})}.$$

Thus we actually have $u \in L^\infty(\mathbb{R}^+; H^{0,1}) \cap L^2(\mathbb{R}^+; \dot{H}^{1,1})$.

(2) Uniqueness part. Let $u, v \in L^\infty(\mathbb{R}^+; H^{0,1}) \cap L^2(\mathbb{R}^+; \dot{H}^{1,1})$ be two weak solutions of (1.1). We denote $w := u - v$. Then we have

$$\partial_t w + w \cdot \nabla v + u \cdot \nabla w - \partial_1^2 w = -\nabla p.$$

Taking L^2 inner product of the above equation with w gives

$$(3.12) \quad \frac{1}{2} \frac{d}{dt} \|w(t)\|_{L^2}^2 + \|\partial_1 w\|_{L^2}^2 \leq |(w \cdot \nabla v | w)|.$$

Observing that

$$(3.13) \quad \begin{aligned} |(w \cdot \nabla v | w)| &= |(w^1 \partial_1 v + w^2 \partial_2 v | w)| \\ &\leq (\|w^1\|_{L_h^\infty(L_v^2)} \|\partial_1 v\|_{L_h^2(L_v^\infty)} + \|w^2\|_{L_h^2(L_v^\infty)} \|\partial_2 v\|_{L_h^\infty(L_v^2)}) \|w\|_{L^2}, \end{aligned}$$

where $w = (w^1, w^2)$, from which and (3.6), we deduce that

$$(3.14) \quad \begin{aligned} |(w \cdot \nabla v | w)| &\lesssim (\|w\|_{L^2}^{\frac{1}{2}} \|\partial_1 w\|_{L^2}^{\frac{1}{2}} \|\partial_1 v\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 v\|_{L^2}^{\frac{1}{2}} \\ &\quad + \|w\|_{L^2}^{\frac{1}{2}} \|\partial_1 w\|_{L^2}^{\frac{1}{2}} \|\partial_2 v\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 v\|_{L^2}^{\frac{1}{2}}) \|w\|_{L^2}. \end{aligned}$$

Applying Young's inequality and using the divergence free condition $\partial_2 w^2 = -\partial_1 w^1$ we have

$$|(w \cdot \nabla v | w)| \leq \frac{1}{2} \|\partial_1 w\|_{L^2}^2 + C_0 (\|\partial_1 v\|_{L^2}^{\frac{2}{3}} \|\partial_1 \partial_2 v\|_{L^2}^{\frac{2}{3}} + \|\partial_2 v\|_{L^2}^{\frac{2}{3}} \|\partial_1 \partial_2 v\|_{L^2}^{\frac{2}{3}}) \|w\|_{L^2}^2.$$

Inserting the above inequality into (3.12) and applying Gronwall's inequality we obtain

$$\|w(t)\|_{L^2}^2 \leq \|w_0\|_{L^2}^2 e^{2C_0 \int_0^t (\|\partial_1 v\|_{L^2}^{\frac{2}{3}} \|\partial_1 \partial_2 v\|_{L^2}^{\frac{2}{3}} + \|\partial_2 v\|_{L^2}^{\frac{2}{3}} \|\partial_1 \partial_2 v\|_{L^2}^{\frac{2}{3}}) ds}.$$

This along with the fact that $\|\partial_1 v\|_{L^2}^{\frac{2}{3}} \|\partial_1 \partial_2 v\|_{L^2}^{\frac{2}{3}} + \|\partial_2 v\|_{L^2}^{\frac{2}{3}} \|\partial_1 \partial_2 v\|_{L^2}^{\frac{2}{3}}$ belongs to $L^1([0, T])$ ensures $w(t) = 0$, that is $u = v$. This completes the uniqueness part of the theorem. \square

4. THE STOCHASTIC CASE

For the stochastic case, we consider the equation (1.2) on \mathbb{T}^2 , and again for simplicity of the notation, we always omit the domain \mathbb{T}^2 in what follows.

4.1. Preliminaries and notations. Let $(e_k, k \geq 1)$ be an orthonormal basis of H whose elements belong to H^2 and orthogonal in $\tilde{H}^{0,1}$. For integers $k, l \geq 1$ with $k \neq l$, we deduce that

$$(\partial_2^2 e_k, e_l) = -(\partial_2 e_k, \partial_2 e_l) = 0.$$

Therefore, $\partial_2^2 e_k$ is a constant multiple of e_k .

Let $\mathcal{H}_n = \text{span}(e_1, \dots, e_n)$ and let P_n (resp. \tilde{P}_n) denote the orthogonal projection from H (resp. $\tilde{H}^{0,1}$) to \mathcal{H}_n . We deduce that

$$P_n u = \tilde{P}_n u, \text{ for } u \in \tilde{H}^{0,1}.$$

Indeed, for $v \in \mathcal{H}_n$, we have $\partial_2^2 v \in \mathcal{H}_n$ and for any $u \in \tilde{H}^{0,1}$:

$$(P_n u, v) = (u, v) \text{ and } (\partial_2 P_n u, \partial_2 v) = -(P_n u, \partial_2^2 v) = -(u, \partial_2^2 v) = (\partial_2 u, \partial_2 v).$$

Hence given $u \in \tilde{H}^{0,1}$, we have

$$(P_n u, v)_{H^{0,1}} = (u, v)_{H^{0,1}}, \text{ for any } v \in \mathcal{H}_n.$$

This proves that P_n and \tilde{P}_n coincide on $\tilde{H}^{0,1}$.

Let $(W(t), t \geq 0)$ be an ℓ^2 -cylindrical Wiener process on a stochastic basis (Ω, \mathcal{F}, P) . Let $W_n(t) = \sum_{j=1}^n \psi_j \beta_j(t) := \Pi_n W(t)$, where $\{\beta_j(t)\}$ is a sequence of independent Brownian Motions on (Ω, \mathcal{F}, P) and ψ_j is an orthonormal basis of ℓ^2 .

Let $L^2(\ell^2, \mathbb{U})$ denotes the Hilbert-Schmidt norms from ℓ^2 to \mathbb{U} for Hilbert space \mathbb{U} . For a Polish space \mathbb{V} , let $\mathcal{B}(\mathbb{V})$ denote its Borel σ -algebra and $\mathcal{P}(\mathbb{V})$ denote all the probability measures on $(\mathbb{V}, \mathcal{B}(\mathbb{V}))$. Let σ be a measurable mapping from $([0, T] \times \tilde{H}^{1,1}, \mathcal{B}([0, T] \times \tilde{H}^{1,1}))$ to $(L^2(\ell^2, \tilde{H}^{1,1}), \mathcal{B}(L^2(\ell^2, \tilde{H}^{1,1})))$. Then we introduce probabilistically weak, strong solutions and martingale solutions. Set

$$F(u) := -B(u) + \partial_1^2 u.$$

Definition 4.1 ((Probabilistically) weak solution). *We say that a pair (u, W) is a (probabilistically) weak solution to (1.2) if there exists a stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ such that $u = (u(t))_{t \geq 0}$ is an (\mathcal{F}_t) adapted process and W is an ℓ^2 -cylindrical Wiener process on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ and the following holds:*

- (i) $u \in L^\infty([0, T], \tilde{H}^{0,1}) \cap L^2([0, T], \tilde{H}^{1,1})$ for a.s. P and any $T > 0$;
- (ii) $\int_0^T \|F(u(s))\|_{H^{-1}} ds + \int_0^T \|\sigma(s, u(s))\|_{L^2(\ell^2, H)}^2 ds < +\infty$ a.s. P , for any $T > 0$;
- (iii) For every $l \in C^1(\mathbb{T}^2)$ with $\operatorname{div} l = 0$, a.s. P

$$u(0) = u_0,$$

$$\langle u(t), l \rangle = \langle u_0, l \rangle + \int_0^t \langle -u \cdot \nabla u + \partial_1^2 u, l \rangle ds + \int_0^t \langle \sigma(s, u(s)) dW(s), l \rangle.$$

Here $\langle \cdot, \cdot \rangle$ denotes the duality bracket. $\langle u, v \rangle$ and (u, v) coincide when $u, v \in L^2$.

Now we define the (probabilistically) strong solution of (1.2) and we fix a stochastic basis (Ω, \mathcal{F}, P) and an ℓ^2 -cylindrical Wiener process W on it.

Definition 4.2 ((Probabilistically) strong solution). *We say that u is a (probabilistically) strong solution to the equation (1.2) on the given probability space (Ω, \mathcal{F}, P) with respect to the fixed cylindrical Wiener process W , if it satisfies:*

- (i) u is adapted to the filtration $\hat{\mathcal{F}}_t := \sigma\{u_0 \vee W(s), s \leq t\}$;
- (ii) u satisfies (i), (ii) and (iii) of Definition 4.1.

Finally we define the martingale solutions. For any fixed $T > 0$, let $\Omega^T := C([0, T]; H^{-1})$ be the space of all continuous functions from $[0, T]$ to H^{-1} .

For $0 \leq t \leq T$, define the filtration:

$$\mathcal{F}_t = \sigma\{x(r) : 0 \leq r \leq t, x \in \Omega^T\}.$$

Definition 4.3 (Martingale solution). *We say that a probability measure $P \in \mathcal{P}(C([0, T]; H^{-1}))$ is called a martingale solution of (1.2) with initial value u_0 if*

$$(M1) \ P(u(0) = u_0, u \in L^\infty(0, T; \tilde{H}^{0,1}) \cap L^2(0, T; \tilde{H}^{1,1})) = 1, \text{ and}$$

$$P\{u \in C([0, T], H^{-1}) : \int_0^T \|F(u(s))\|_{H^{-1}} ds + \int_0^T \|\sigma(s, u(s))\|_{L^2(\ell^2, H)}^2 ds < +\infty\} = 1;$$

$$(M2) \text{ For every } l \in C^1(\mathbb{T}^2), \text{ the process}$$

$$M_l(t, u) = \langle u(t), l \rangle - \int_0^t \langle F(u(s)), l \rangle ds$$

is a continuous square integrable \mathcal{F}_t - martingale with respect to P , whose quadratic variation process is $\int_0^t \|\sigma^*(s, u(s))(l)\|_{\ell^2}^2 ds$, where the asterisk denotes the adjoint operator of $\sigma(s, u(s))$;

(M3) We have

$$E^P \left(\sup_{t \in [0, T]} \|u(t)\|_{L^2}^2 + \int_0^T \|u(t)\|_{H^{1,0}}^2 dt \right) \leq C_T (1 + \|u_0\|_{L^2}^2).$$

Remark 4.1. By the above definitions, we know immediately that if u is a (probabilistically) strong solution with respect to the fixed cylindrical Wiener process W , (u, W) is a (probabilistically) weak solution. Moreover, let P denote the law of u in $C([0, T], H^{-1})$, then P is a martingale solution.

Notice that by martingale representation theorem, (see for example [8]) the existence of martingale solution can lead to the existence of (probabilistically) weak solution. And the law of the weak solution gives a martingale solution P .

Definition 4.4 (Condition C). The diffusion coefficient σ is a measurable mapping from $([0, T] \times \tilde{H}^{1,1}, \mathcal{B}([0, T] \times \tilde{H}^{1,1}))$ to $(L^2(\ell^2, \tilde{H}^{1,1}), \mathcal{B}(L^2(\ell^2, \tilde{H}^{1,1})))$ such that :

(i) **Growth condition**

There exist nonnegative constants K'_i , K_i and \tilde{K}_i ($i = 0, 1, 2$) such that for every $t \in [0, T]$ and $u \in \tilde{H}^{1,1}$:

$$\begin{aligned} \|\sigma(t, u)\|_{L^2(\ell^2, H^{-1})}^2 &\leq K'_0 + K'_1 \|u\|_{L^2}^2; \\ \|\sigma(t, u)\|_{L^2(\ell^2, H)}^2 &\leq K_0 + K_1 \|u\|_{L^2}^2 + K_2 \|\partial_1 u\|_{L^2}^2; \\ \|\sigma(t, u)\|_{L^2(\ell^2, H^{0,1})}^2 &\leq \tilde{K}_0 + \tilde{K}_1 \|u\|_{0,1}^2 + \tilde{K}_2 (\|\partial_1 u\|_{L^2}^2 + \|\partial_2 \partial_1 u\|_{L^2}^2). \end{aligned}$$

(ii) **Lipschitz condition**

There exist constants L_1 and L_2 such that for $t \in [0, T]$ and $u, v \in \tilde{H}^{1,1}$:

$$\|\sigma(t, u) - \sigma(t, v)\|_{L^2(\ell^2, H)}^2 \leq L_1 \|u - v\|_{L^2}^2 + L_2 \|\partial_1(u - v)\|_{L^2}^2.$$

Remark 4.2. A typical example of σ satisfying Condition (C) is the following:

First we recall the Hölder space $C^{k+\tau}$ (k is a nonnegative integer and $0 \leq \tau < 1$) as: u has k th derivatives and

$$\|u\|_{C^{k+\tau}} := \sum_{|\alpha| \leq k} \|D^\alpha u\| + \sum_{|\alpha|=k} \sup_{x \neq y} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x - y|^\tau} < \infty.$$

For $u \in H^{1,1}$ and $y \in \ell^2$, let

$$\sigma(t, u)y = \sum_{k=1}^{\infty} (c_k \partial_1 u + b_k g(u)) \langle y, \psi_k \rangle_{\ell^2},$$

where ψ_k , as defined in Section 4.1, is the orthonormal basis of ℓ^2 and $c_k \in C^\rho$, $\sum_{k=1}^{\infty} \|c_k(\xi)\|_{C^\rho}^2 \leq$

M_1 for some $\rho > 2$, $b_k \in L^\infty$, $\partial_2 b_k \in L^\infty$, $\sum_{k=1}^{\infty} |b_k(\xi)|^2 \leq M_2$, and $\sum_{k=1}^{\infty} |\partial_2 b_k(\xi)|^2 \leq M_2$, $\xi \in \mathbb{T}^2$.

We also assume that $\|g\|_{C^1} \leq C(g)$. Here C^ρ and C^1 are the Hölder spaces. And suppose that

$\operatorname{div}(c_k \partial_1 u + b_k g(u)) = 0$. Then we have

$$\begin{aligned}
\|\sigma(t, u)\|_{L^2(\ell^2, H^{-1})} &\leq (\sqrt{M_1} + \sqrt{M_2} C(g)) \|u\|_{L^2} + \sqrt{M_2} C(g); \\
\|\sigma(t, u)\|_{L^2(\ell^2, H)} &\leq \sqrt{M_1} \|\partial_1 u\|_{L^2} + \sqrt{M_2} (C(g) \|u\|_{L^2} + C(g)); \\
\|\sigma(t, u)\|_{L^2(\ell^2, H^{0,1})} &\leq \sqrt{M_1} \|\partial_1 u\|_{L^2} + \sqrt{M_2} (C(g) \|u\|_{L^2} + C(g)) \\
&\quad + \left(\sum_{k=1}^{\infty} \|\partial_2(c_k \partial_1 u)\|_{L^2}^2 \right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty} \|\partial_2(b_k g(u))\|_{L^2}^2 \right)^{\frac{1}{2}} \\
&\leq \sqrt{M_1} \|\partial_1 u\|_{L^2} + \sqrt{M_2} (C(g) \|u\|_{L^2} + C(g)) \\
&\quad + \sqrt{M_1} (\|\partial_1 u\|_{L^2} + \|\partial_1 \partial_2 u\|_{L^2}) + \sqrt{M_2} C(g) (\|u\|_{L^2} + \|\partial_2 u\|_{L^2}); \\
\|\sigma(t, u) - \sigma(t, v)\|_{L^2(\ell^2, H)} &\leq \sqrt{M_1} \|\partial_1(u - v)\|_{L^2} + \sqrt{M_2} C(g) \|u - v\|_{L^2},
\end{aligned}$$

where the first inequality above is due to (2) in page 140, section 2.8.2 [19].

4.2. Main theorems of stochastic cases. In this section we state two theorems about the well-posedness of equation (1.2), which will be proved in the following sections.

Theorem 4.1. Assume that u_0 is a random variable in $L^4(\Omega, \tilde{H}^{0,1})$ and suppose that σ satisfies condition (C) with $K_2 < \frac{2}{11}$ and $\tilde{K}_2 < \frac{2}{5}$. Then (1.2) has a global martingale solution.

Theorem 4.2 (Pathwise uniqueness). Assume that u_0 is a random variable in $L^4(\Omega, \tilde{H}^{0,1})$. Suppose that σ satisfies condition (C) with $K_2 < \frac{2}{11}$, $\tilde{K}_2 < \frac{2}{5}$ and $L_2 < \frac{2}{5}$. If u, v are two weak solutions on the same stochastic basis (Ω, \mathcal{F}, P) . Then we have $u = v$ P -a.s.

Remark 4.3. By the Yamada-Watanabe theorem, (cf. [15]) the existence of (probabilistically) weak solution and pathwise uniqueness lead to the existence of the (probabilistically) strong solution.

4.3. Galerkin Approximation and A Priori Estimates. From now on we use C to denote the constant which can be different from line to line.

Fix $n \geq 1$ and consider the following stochastic ordinary differential equations on \mathcal{H}_n :

$$u_n(0) = P_n u_0,$$

and for $t \in [0, T]$, $v \in \mathcal{H}_n$:

$$(4.1) \quad d(u_n(t), v) = \langle P_n F(u_n(t)), v \rangle dt + (P_n \sigma(t, u_n(t)) \Pi_n dW(t), v).$$

Then for $k = 1, \dots, n$ we have for $t \in [0, T]$:

$$d(u_n(t), e_k) = \langle P_n F(u_n(t)), e_k \rangle dt + \sum_{j=1}^n (P_n \sigma(t, u_n(t)) \psi_j, e_k) d\beta_j(t).$$

Now we use [15] Thm 3.1.1 about existence and uniqueness of solutions to stochastic differential equations. Note that since it is in finite dimensions, there exists some constant $C(n)$ such that $\|v\|_{H^2} \leq C(n) \|v\|_{L^2}$ for $v \in \mathcal{H}_n$.

Let $\varphi, \psi, v \in \mathcal{H}_n$; integration by parts implies that

$$|\langle \partial_1^2 \varphi - \partial_1^2 \psi, v \rangle| \leq \|\varphi - \psi\|_{H^{1,0}} \|v\|_{H^{1,0}} \leq C(n)^2 \|\varphi - \psi\|_{L^2} \|v\|_{L^2}.$$

Moreover, we have

$$\begin{aligned}
|\langle B(\varphi) - B(\psi), v \rangle| &= |-\langle B(\varphi - \psi), v \rangle - \langle B(\psi), \varphi - \psi \rangle| \\
&\leq C \|\varphi - \psi\|_{H^{1,0}} (\|\varphi\|_{H^{1,0}} + \|\psi\|_{H^{1,0}}) \|v\|_{H^{1,1}} \\
&\leq C C(n)^3 \|\varphi - \psi\|_{L^2} (\|\varphi\|_{L^2} + \|\psi\|_{L^2}) \|v\|_{L^2}.
\end{aligned}$$

Hence we know that for $u, v \in \mathcal{H}_n$, and $\|u\|_{L^2}, \|v\|_{L^2} \leq R$,

$$|\langle F(u) - F(v), u - v \rangle| \leq 2RC(n)^3 \|u - v\|_{L^2}^2,$$

The condition (C) implies that for $u, v \in \mathcal{H}_n$, and $\|u\|_{L^2}, \|v\|_{L^2} \leq R$,

$$\begin{aligned} \|P_n(\sigma(t, u) - \sigma(t, v))\|_{L^2(\ell^2, H)}^2 &\leq \|\sigma(t, u) - \sigma(t, v)\|_{L^2(\ell^2, H)}^2 \\ &\leq L_1 \|u - v\|_{L^2}^2 + L_2 \|\partial_1(u - v)\|_{L^2}^2 \\ &\leq C(n)^2 (L_1 + L_2) \|u - v\|_{L^2}^2. \end{aligned}$$

So it satisfies local weak monotonicity. Moreover,

$$\begin{aligned} 2\langle u, P_n F(u) \rangle + \|P_n \sigma(t, u)\|_{L^2(\ell^2, H)}^2 &\leq \|u\|_{H^{1,0}}^2 + \|\sigma\|_{L^2(\ell^2, H)}^2 \\ &\leq C(n)^2 \|u\|_{L^2}^2 + K_0 + K_1 \|u\|_{L^2}^2 + K_2 \|\partial_1 u\|_{L^2}^2 \\ &\leq K_0 + (C(n)^2 + K_1 + K_2 C(n)^2) \|u\|_{L^2}^2. \end{aligned}$$

Thus it satisfies weak coercivity.

Hence by [15] Thm 3.1.1, there exists a unique global strong solution $u_n(t)$ to (4.1). Moreover, $u \in C([0, T], \mathcal{H}_n), P - a.s.$

4.4. The L^2 Energy Estimates. In this section, we give the following a priori estimates.

Lemma 4.1. *We have the following energy estimates under the hypothesis of Thm 4.1:*

$$E\left(\sup_{t \in [0, T]} \|u_n(t)\|_{L^2}^2\right) + E \int_0^T \|u_n(t)\|_{H^{1,0}}^2 dt \leq C(1 + E\|u_0\|_{L^2}^2).$$

Proof. Let $u_n(t)$ be the solution to (4.1) described above. By Itô's formula, we have:

$$\begin{aligned} (4.2) \quad \|u_n(t)\|_{L^2}^2 &= \|P_n u_0\|_{L^2}^2 + 2 \int_0^t (\sigma(s, u_n(s)) dW_n(s), u_n(s)) \\ &\quad - 2 \int_0^t \|\partial_1 u_n(s)\|_{L^2}^2 ds + \int_0^t \|P_n \sigma(s, u_n(s)) \Pi_n\|_{L^2(\ell^2, H)}^2 ds. \end{aligned}$$

The growth condition implies that

$$(4.3) \quad \int_0^t \|P_n \sigma(s, u_n(s)) \Pi_n\|_{L^2(\ell^2, H)}^2 ds \leq \int_0^t [K_0 + K_1 \|u_n(t)\|_{L^2}^2 + K_2 \|\partial_1 u_n(t)\|_{L^2}^2] ds.$$

The Burkholder-Davis-Gundy inequality(see Thm 6.1.2, chapter 6 in [15]) and the Young inequality as well as the growth condition imply that:

$$\begin{aligned} (4.4) \quad &E\left(\sup_{s \leq t} \left| 2 \int_0^s (P_n \sigma(r, u_n(r)) dW_n(r), u_n(r)) \right| \right) \\ &\leq 4E\left\{ \int_0^t \|P_n \sigma(r, u_n(r)) \Pi_n\|_{L^2(\ell^2, H)}^2 \|u_n(r)\|_{L^2}^2 dr \right\}^{\frac{1}{2}} \\ &\leq \beta E(\sup_{s \leq t} \|u_n(s)\|_{L^2}^2) + \frac{4}{\beta} E \int_0^t [K_0 + K_1 \|u_n(s)\|_{L^2}^2 + K_2 \|\partial_1 u_n(t)\|_{L^2}^2] ds. \end{aligned}$$

Since $K_2 < \frac{2}{11}$, we can choose $0 < \beta < 1$ such that $(\frac{4}{\beta} + 1)K_2 - 2 < 0$.

By (4.2)-(4.4) and dropping some negative terms, we deduce:

$$(1 - \beta) E \sup_{s \in [0, t]} \|u_n(s)\|_{L^2}^2 \leq E\|u(0)\|_{L^2}^2 + CK_0 T + CE \int_0^t K_1 \|u_n(s)\|_{L^2}^2 ds.$$

Gronwall's lemma implies that

$$(4.5) \quad E\left(\sup_{t \in [0, T]} \|u_n(t)\|_{L^2}^2\right) \leq C,$$

where C is a constant depending on K_0, K_1, K_2, T but not n .
Inserting (4.5) back to (4.2)-(4.4) yields

$$E\left(\sup_{t \in [0, T]} \|u_n(t)\|_{L^2}^2\right) + E \int_0^t \|u_n(t)\|_{H^{1,0}}^2 ds \leq C(1 + E\|u_0\|_{L^2}^2),$$

where C is a constant depending on K_0, K_1, K_2, T but not n .
This completes the proof. \square

However, it is not enough that we only have $L^2(\Omega)$ estimates. We also need an $L^4(\Omega)$ uniform estimates of u_n .

Lemma 4.2. *We have the following uniform estimates under the hypothesis of Thm 4.1:*

$$E\left(\sup_{t \in [0, T]} \|u_n(t)\|_{L^2}^4\right) + E \int_0^T \|u_n(t)\|_{L^2}^2 \|u_n(t)\|_{H^{1,0}}^2 dt \leq C(1 + E\|u_0\|_{L^2}^4).$$

Proof. Applying once more the Itô's formula to the square of $\|\cdot\|_{L^2}^2$, we obtain:

$$(4.6) \quad \|u_n(t)\|_{L^2}^4 = \|P_n u_0\|_{L^2}^4 - 4 \int_0^t \|\partial_1 u_n(s)\|_{L^2}^2 \|u_n(s)\|_{L^2}^2 ds + I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= 4 \int_0^t (\sigma(s, u_n(s)) dW_n(s), u_n(s)) \|u_n(s)\|_{L^2}^2, \\ I_2 &= 2 \int_0^t \|P_n \sigma(s, u_n(s)) \Pi_n\|_{L^2(\ell^2, H)}^2 \|u_n(s)\|_{L^2}^2 ds, \\ I_3 &= 4 \int_0^t \|(P_n \sigma(s, u_n(s)) \Pi_n)^*(u_n)\|_{\ell^2}^2 ds. \end{aligned}$$

The growth condition implies that

$$(4.7) \quad I_2(t) + I_3(t) \leq 6 \int_0^t (K_0 + K_1 \|u_n(s)\|_{L^2}^2 + K_2 \|\partial_1 u_n(t)\|_{L^2}^2) \|u_n(s)\|_{L^2}^2 ds.$$

The Burkholder-Davis-Gundy inequality, the growth condition and the Young inequality imply that:

$$\begin{aligned} (4.8) \quad E(\sup_{s \leq t} I_1(s)) &\leq 8E \left\{ \int_0^t \|\sigma(r, u_n(r))\|_{L^2(\ell^2, H)}^2 \|u_n(r)\|_{L^2}^6 dr \right\}^{\frac{1}{2}} \\ &\leq \gamma E(\sup_{s \leq t} \|u_n(s)\|_{L^2}^4) \\ &\quad + \frac{16}{\gamma} E \int_0^t (K_0 + K_1 \|u_n(s)\|_{L^2}^2 + K_2 \|\partial_1 u_n(t)\|_{L^2}^2) \|u_n(s)\|_{L^2}^2 ds. \end{aligned}$$

Since $K_2 < \frac{2}{11}$, we can choose $0 < \gamma < 1$, such that $6K_2 + \frac{16}{\gamma} K_2 - 4 < 0$.

Thus combining (4.6)-(4.8) and dropping some negative terms on the right of the inequality, we have:

$$(1 - \gamma) E\left(\sup_{t \in [0, T]} \|u_n(t)\|_{L^2}^4\right) \leq E\|u(0)\|_{L^2}^4 + E \int_0^t C_1 \|u_n(s)\|_{L^2}^4 + C_2 \|u_n(s)\|_{L^2}^2 ds.$$

Since we have obtained $E(\sup_{t \in [0, T]} \|u_n(t)\|_{L^2}^2) \leq C$, the Gronwall inequality yields

$$E\left(\sup_{t \in [0, T]} \|u_n(t)\|_{L^2}^4\right) < \infty.$$

Similar as in the proof of Lemma 4.1, we complete the proof. \square

4.5. Tightness and the Skorokhod Theorem. In this section we use the classical tightness methods. Similar to the deterministic cases, L^2 -estimates are not enough to obtain strong convergence. As a result, we use tightness in the following space \mathcal{X} .

Let \hat{P}_n be the law of u_n on $C([0, T]; H^{-1})$.

Lemma 4.3. *Under the hypothesis of Thm. 4.1, \hat{P}_n is tight in the space*

$$\mathcal{X} = C([0, T]; H^{-1}) \cap L^2([0, T]; H) \cap L_w^2([0, T]; H^{1,1}) \cap L_{w^*}^\infty([0, T]; H^{0,1}),$$

where $L_w^2([0, T]; H^{1,1})$ denotes $L^2([0, T]; H^{1,1})$ with the weak topology and $L_{w^*}^\infty([0, T]; H^{0,1})$ denotes $L^\infty([0, T]; H^{0,1})$ with the weak star topology.

Proof. Firstly, since $\tilde{K}_2 < \frac{2}{5}$, we can choose $\tilde{\alpha}, \tilde{\beta} \in (0, 1)$, such that:

$$\tilde{K}_2 + 2\tilde{\alpha} + \frac{4}{\tilde{\beta}}\tilde{K}_2 < 2.$$

From the calculation in Lemma 3.2, by the Young inequality, we deduce that:

$$(4.9) \quad |(\partial_2(u \cdot \nabla u) \mid \partial_2 u)| \leq \tilde{\alpha} \|\partial_1 \partial_2 u\|_{L^2}^2 + C(\tilde{\alpha}) \|\partial_1 u\|_{L^2}^2 \|\partial_2 u\|_{L^2}^2.$$

Let

$$\begin{aligned} K_R := & \left\{ u \in C([0, T], H^{-1}); \sup_{0 \leq t \leq T} \|u(t)\|_{L^2}^2 + \int_0^T \|u(t)\|_{H^{1,0}}^2 dt + \|u\|_{C^{\frac{1}{8}}([0, T]; H^{-1})} \right. \\ & \left. + \sup_{0 \leq t \leq T} e^{-2C(\tilde{\alpha}) \int_0^t \|\partial_1 u\|_{L^2}^2 ds} \|u(t)\|_{H^{0,1}}^2 + \int_0^T e^{-2C(\tilde{\alpha}) \int_0^t \|\partial_1 u\|_{L^2}^2 ds} \|u\|_{H^{1,1}}^2 dt \leq R \right\}. \end{aligned}$$

Now we want to show that

- (i) For any $R > 0$, K_R is relatively compact in \mathcal{X} ;
- (ii) For any $\epsilon > 0$, there exists $R > 0$, such that $\hat{P}_n(K_R) > 1 - \epsilon$ for any n .

Proof of (i): By the definition of K_R , it is obvious that $u \in K_R$ is bounded in $L^2([0, T]; H^{1,1})$, thus K_R is relatively compact in $L_w^2([0, T]; H^{1,1})$ and $L_{w^*}^\infty([0, T]; H^{0,1})$.

Moreover, by definition, K_R is equicontinuous in $C([0, T]; H^{-1})$. The compactness in $C([0, T]; H^{-1})$ can be obtained by Arzela-Ascoli Lemma.

Finally we prove the compactness in $L^2([0, T]; L^2)$. Let u_n be a sequence in K_R . We can assume that u_n converges to u in $C([0, T]; H^{-1}) \cap L_w^2([0, T]; H^{1,1})$. Then we have:

$$\begin{aligned} \int_0^T \|u_n - u\|_{L^2}^2 dt & \leq \int_0^T \|u_n - u\|_{H^1} \|u_n - u\|_{H^{-1}} dt \\ & \leq \left(\int_0^T \|u_n - u\|_{H^1}^2 dt \right)^{\frac{1}{2}} \left(\int_0^T \|u_n - u\|_{H^{-1}}^2 dt \right)^{\frac{1}{2}} \\ & \leq C_{R,T} \sup_{t \in [0, T]} \|u_n - u\|_{H^{-1}}^2 \\ & \rightarrow 0, \end{aligned}$$

which finishes the proof of (i).

Proof of (ii): By Lemma 4.1 as well as Chebyshev's inequality, we can choose R_0 large enough such that:

$$(4.10) \quad P\left(\sup_{t \in [0, T]} \|u_n(t)\|_{L^2}^2 + \int_0^T \|u_n(t)\|_{H^{1,0}}^2 dt > \frac{R_0}{4}\right) < \frac{\epsilon}{4}.$$

Set $h(t) = 2C(\tilde{\alpha}) \int_0^t \|\partial_1 u_n\|_{L^2}^2 ds$. Now we need another estimate as following:

$$(4.11) \quad \begin{aligned} & E\left(\sup_{t \in [0, T]} (e^{-h(t)} \|u_n(t)\|_{H^{0,1}}^2)\right) + E \int_0^T e^{-h(t)} \|u_n(t)\|_{H^{1,1}}^2 dt \\ & \leq C(\tilde{K}_0, \tilde{K}_1, \tilde{K}_2, T)(1 + E\|u_0\|_{H^{0,1}}^2), \end{aligned}$$

the proof of which is postponed later to Lemma 4.4.

By (4.11) and Chebyshev's Inequality, we can choose R_0 large enough such that:

$$(4.12) \quad \hat{P}_n \left(\sup_{0 \leq t \leq T} e^{-2C(\tilde{\alpha}) \int_0^t \|\partial_1 u\|_{L^2}^2 ds} \|u(t)\|_{H^{0,1}}^2 + \int_0^T e^{-2C(\tilde{\alpha}) \int_0^t \|\partial_1 u\|_{L^2}^2 ds} \|u\|_{H^{1,1}}^2 dt > \frac{R_0}{4} \right) < \frac{\epsilon}{4}.$$

Now we fix R_0 and set

$$\begin{aligned} \hat{K}_{R_0} = \{u \in C([0, T], H^{-1}); \sup_{t \in [0, T]} \|u(t)\|_{L^2}^2 + \int_0^T \|u(t)\|_{H^{1,0}}^2 dt \leq \frac{R_0}{4} \text{ and} \\ \sup_{0 \leq t \leq T} e^{-2C(\tilde{\alpha}) \int_0^t \|\partial_1 u\|_{L^2}^2 ds} \|u(t)\|_{H^{0,1}}^2 + \int_0^T e^{-2C(\tilde{\alpha}) \int_0^t \|\partial_1 u\|_{L^2}^2 ds} \|u\|_{H^{1,1}}^2 dt \leq \frac{R_0}{4}\}. \end{aligned}$$

Then we know $\hat{P}_n(C([0, T], H^{-1}) \setminus \hat{K}_{R_0}) < \frac{\epsilon}{2}$. Now we only consider u in \hat{K}_{R_0} . By Hölder's inequality, we have:

$$(4.13) \quad \begin{aligned} & E^{\hat{P}_n} \left[\sup_{s \neq t \in [0, T]} \left(\frac{\|\int_s^t -\partial_1^2 u(r) + \operatorname{div}(u \otimes u) dr\|_{H^{-1}}^2}{|t - s|} \right) 1_{u \in \hat{K}_{R_0}} \right] \\ & \leq E^{\hat{P}_n} \left[\int_0^T \|-\partial_1^2 u(r) + \operatorname{div}(u \otimes u)\|_{H^{-1}}^2 dr 1_{u \in \hat{K}_{R_0}} \right]. \end{aligned}$$

The boundedness of u in $L^2([0, T]; H^{1,1})$ leads to the boundedness of $\partial_1^2 u$ in $L^2([0, T]; H^{-1})$. By the definition of K_R , u is also bounded in $L^\infty([0, T]; H^{0,1})$. By interpolation, u is bounded in $L^4([0, T]; H^{\frac{1}{2}})$. By Sobolev imbedding, u is bounded in $L^4([0, T]; L^4)$. Thus we obtain:

$$(4.14) \quad E^{\hat{P}_n} \left[\int_0^T \|-\partial_1^2 u(r) + \operatorname{div}(u \otimes u)\|_{H^{-1}}^2 dr 1_{u \in \hat{K}_{R_0}} \right] \leq C(R_0),$$

where $C(R_0)$ is independent of n .

Thus by (4.13) and (4.14), we have

$$(4.15) \quad E^{\hat{P}_n} \left[\sup_{s \neq t \in [0, T]} \left(\frac{\|\int_s^t -\partial_1^2 u(r) + \operatorname{div}(u \otimes u) dr\|_{H^{-1}}^2}{|t - s|} \right) 1_{u \in \hat{K}_{R_0}} \right] \leq C(R_0).$$

Moreover, for any $T \geq t > s \geq 0$ and any $p \in \mathbb{N}$ we have

$$\begin{aligned} E^{\hat{P}_n} \left\| \int_s^t P_n \sigma(r, u(r)) dW_n(r) \right\|_{H^{-1}}^{2p} & \leq C_p E^{\hat{P}_n} \left(\int_s^t \|\sigma(r, u(r))\|_{L^2(\ell^2, H^{-1})}^2 dr \right)^p \\ & \leq C_p |t - s|^{p-1} \int_s^t E^{\hat{P}_n} \|\sigma(r, u(r))\|_{L^2(\ell^2, H^{-1})}^{2p} dr \\ & \leq C_p |t - s|^{p-1} \int_s^t E^{\hat{P}_n} (\|u(r)\|_{L^2}^{2p} + 1) dr \\ & \leq C_{p,T} |t - s|^p (1 + E(\sup_{t \in [0, T]} \|u_n(t)\|_{L^2}^{2p})). \end{aligned}$$

By Kolmogorov's criterion, for any $\alpha \in (0, \frac{p-1}{2p})$, we have:

$$(4.16) \quad E^{\hat{P}_n} \left(\sup_{s \neq t \in [0, T]} \frac{\|\int_s^t P_n \sigma(r, u(r)) dW_n(r)\|_{H^{-1}}^{2p}}{|t - s|^{p\alpha}} \right) \leq C_{p,T} (1 + E(\sup_{t \in [0, T]} \|u_n(t)\|_{L^2}^{2p})).$$

Choose $p = 2$. By (4.15) and (4.16), we get for $\alpha = \frac{1}{8}$:

$$E^{\hat{P}_n} \left(\sup_{s \neq t \in [0, T]} \frac{\|u(t) - u(s)\|_{H^{-1}}}{|t - s|^\alpha} 1_{u \in \hat{K}_{R_0}} \right) < \infty.$$

Similarly, we choose $R > R_0$ large enough and obtain:

$$(4.17) \quad \hat{P}_n(\|u\|_{C^{\frac{1}{8}}([0, T]; H^{-1})} > \frac{R}{4} \text{ and } u \in \hat{K}_{R_0}) < \frac{\epsilon}{4}.$$

Combining (4.10), (4.12) with (4.17) we complete the proof. \square

Lemma 4.4. *Under the hypothesis of Thm. 4.1, the uniform estimates (4.11) holds.*

Proof. Using again the Itô's Formula to $e^{-h(t)}\|u_n(t)\|_{H^{0,1}}^2$, we obtain:

$$(4.18) \quad \begin{aligned} e^{-h(t)}\|u_n(t)\|_{H^{0,1}}^2 &= \|P_n u(0)\|_{H^{0,1}}^2 + \sum_{j=1}^3 T_j(t) - \int_0^t e^{-h(s)} h'(s) \|u_n(s)\|_{H^{0,1}}^2 ds \\ &\quad + \int_0^t e^{-h(s)} [-2\|\partial_1 u_n(s)\|_{L^2}^2 - 2\|\partial_1 \partial_2 u_n(s)\|_{L^2}^2] ds, \end{aligned}$$

where

$$\begin{aligned} T_1(t) &= -2 \int_0^t e^{-h(s)} \langle \partial_2(u_n \cdot \nabla u_n), \partial_2 u_n(s) \rangle, \\ T_2(t) &= 2 \int_0^t e^{-h(s)} (\sigma(s, u_n(s)) dW_n(s), u_n(s))_{H^{0,1}}, \\ T_3(t) &= \int_0^t e^{-h(s)} \|P_n \sigma(s, u_n(s)) \Pi_n\|_{L^2(\ell^2, H^{0,1})}^2 ds. \end{aligned}$$

The growth condition implies that

$$(4.19) \quad T_3(t) \leq \int_0^t e^{-h(s)} [\tilde{K}_0 + \tilde{K}_1 \|u_n(s)\|_{H^{0,1}}^2 + \tilde{K}_2 (\|\partial_1 u_n(s)\|_{L^2}^2 + \|\partial_1 \partial_2 u_n(s)\|_{L^2}^2)] ds.$$

For $T_1(t)$, we use (4.9):

$$(4.20) \quad |T_1(t)| \leq \int_0^t e^{-h(s)} [2\tilde{\alpha} \|\partial_1 \partial_2 u_n\|_{L^2}^2 + 2C(\tilde{\alpha}) \|\partial_1 u_n\|_{L^2}^2 \|\partial_2 u_n\|_{L^2}^2] ds.$$

Similar to (4.8), we have

$$(4.21) \quad \begin{aligned} &E \left(\sup_{s \leq t} \left| 2 \int_0^s e^{-h(r)} (\sigma(r, u_n(r)) dW_n(r), u_n(r))_{H^{0,1}} \right| \right) \\ &\leq 4E \left\{ \int_0^t e^{-h(r)} \|P_n \sigma(r, u_n)(r) \Pi_n\|_{L^2(\ell^2, H^{0,1})}^2 e^{-h(r)} \|u_n(r)\|_{H^{0,1}}^2 dr \right\}^{\frac{1}{2}} \\ &\leq \tilde{\beta} E(\sup_{s \leq t} (e^{-h(s)} \|u_n(s)\|_{H^{0,1}}^2)) \\ &\quad + \frac{4}{\tilde{\beta}} E \int_0^t e^{-h(s)} [\tilde{K}_0 + \tilde{K}_1 \|u_n(s)\|_{H^{0,1}}^2 + \tilde{K}_2 (\|\partial_1 u_n(s)\|_{L^2}^2 + \|\partial_1 \partial_2 u_n(s)\|_{L^2}^2)] ds. \end{aligned}$$

Combining (4.18)-(4.21) and dropping some negative terms, we have:

$$\begin{aligned} &(1 - \tilde{\beta}) E \left(\sup_{t \in [0, T]} e^{-h(t)} \|u_n(t)\|_{H^{0,1}}^2 \right) \\ &\leq E \|P_n u_0\|_{H^{0,1}}^2 + E \int_0^T e^{-h(s)} \left(1 + \frac{4}{\tilde{\beta}} \right) (\tilde{K}_0 + \tilde{K}_1 \|u_n(s)\|_{H^{0,1}}^2) ds \end{aligned}$$

By Gronwall's inequality,

$$(4.22) \quad E\left(\sup_{t \in [0, T]} (e^{-h(t)} \|u_n(t)\|_{H^{0,1}}^2)\right) \leq C(T, \tilde{K}_0, \tilde{K}_1, \tilde{K}_2) \|u_0\|_{H^{0,1}}^2$$

Combining (4.18)-(4.21) again with the estimate (4.22) we obtain:

$$\begin{aligned} E\left(\sup_{t \in [0, T]} (e^{-h(t)} \|u_n(t)\|_{H^{0,1}}^2)\right) + E \int_0^T e^{-h(t)} \|u_n(t)\|_{H^{1,1}}^2 dt \\ \leq C(\tilde{K}_0, \tilde{K}_1, \tilde{K}_2, T)(1 + E\|u_0\|_{H^{0,1}}^2). \end{aligned}$$

□

The classical Skorokhod Theorem can only be used in metric space. We will use the following Jakubowski's version of the Skorokhod Theorem in the form given by Brzeźniak and Ondreját [4] Thm A.1. and it was proved by A. Jakubowski in [13].

Theorem 4.3. *Let \mathcal{Y} be a topological space such that there exists a sequence f_m of continuous functions $f_m : \mathcal{Y} \rightarrow \mathbb{R}$ that separates points of \mathcal{Y} . Let us denote by \mathcal{S} the σ -algebra generated by the maps f_m . Then*

- (j1) *every compact subset of \mathcal{Y} is metrizable;*
- (j2) *if (μ_m) is tight sequence of probability measures on $(\mathcal{Y}, \mathcal{S})$, then there exists a subsequence (m_k) , a probability space (Ω, \mathcal{F}, P) with \mathcal{Y} -valued Borel measurable variables ξ_k, ξ such that μ_{m_k} is the law of ξ_k and ξ_k converges to ξ almost surely on Ω . Moreover, the law of ξ is a Radon measure.*

Now we check the \mathcal{X} defined in Lemma 4.3 satisfies the above condition. It is sufficient to prove that on each space appearing in the definition of \mathcal{X} there exists a countable set of continuous real-valued functions separating points:

Since $C([0, T]; H^{-1})$ and $L^2([0, T]; H)$ are separable Banach spaces, it is easy to see the condition in Thm. 4.3 is satisfied.

For the space $L_w^2([0, T]; H^{1,1})$ it is sufficient to put

$$f_m(u) := \int_0^T (u(t), v_m(t))_{H^{1,1}} dt \in \mathbb{R}, u \in L_w^2([0, T]; H^{1,1}), m \in \mathbb{N},$$

where v_m is a dense subset of $L^2([0, T]; H^{1,1})$.

Similarly for the space $L_{w^*}^\infty([0, T]; H^{0,1})$ it is sufficient to put

$$f_m(u) := \int_0^T (u(t), v_m(t))_{H^{0,1}} dt \in \mathbb{R}, u \in L_{w^*}^\infty([0, T]; H^{0,1}), m \in \mathbb{N},$$

where v_m is a dense subset of $L^1([0, T]; H^{0,1})$.

Now all the conditions of the above Skorokhod theorem are satisfied. By Thm 4.3, there exists another probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and a subsequence \hat{P}_{n_k} as well as random variables \tilde{u}_{n_k} in the space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, such that

- (i) \tilde{u}_{n_k} has the law \hat{P}_{n_k} ;
- (ii) \hat{P}_{n_k} converges weakly to some \hat{P} ;
- (iii) $\tilde{u}_{n_k} \rightarrow \tilde{u}$ in \mathcal{X} $\tilde{\mathbb{P}} - a.s.$ and \tilde{u} has the law $\hat{P} \in \mathcal{P}(C([0, T]; H^{-1}))$.

Remark 4.4. *Since \tilde{u}_{n_k} has the same law as u_{n_k} , we immediately have:*

$$(4.23) \quad \begin{aligned} E^{\tilde{\mathbb{P}}} \left(\sup_{t \in [0, T]} \|\tilde{u}(t)\|_{L^2}^2 \right) + E^{\tilde{\mathbb{P}}} \int_0^T \|\tilde{u}(t)\|_{H^{1,0}}^2 dt &\leq C(1 + E^{\tilde{\mathbb{P}}} \|u_0\|_{L^2}^2). \\ E^{\hat{P}} \left(\sup_{t \in [0, T]} \|u(t)\|_{L^2}^2 \right) + E^{\hat{P}} \int_0^T \|u(t)\|_{H^{1,0}}^2 dt &\leq C(1 + E^{\hat{P}} \|u_0\|_{L^2}^2). \end{aligned}$$

Similarly for $L^4(\Omega)$ estimates, we have:

$$(4.24) \quad E^{\tilde{\mathbb{P}}} \left(\sup_{t \in [0, T]} \|\tilde{u}(t)\|_{L^2}^4 \right) + E^{\tilde{\mathbb{P}}} \int_0^T \|\tilde{u}(t)\|_{L^2}^2 \|\tilde{u}(t)\|_{H^{1,0}}^2 dt \leq C(1 + E^{\tilde{\mathbb{P}}} \|u_0\|_{L^2}^4).$$

$$E^{\hat{P}} \left(\sup_{t \in [0, T]} \|u(t)\|_{L^2}^4 \right) + E^{\hat{P}} \int_0^T \|u(t)\|_{L^2}^2 \|u(t)\|_{H^{1,0}}^2 dt \leq C(1 + E^{\hat{P}} \|u_0\|_{L^2}^4).$$

4.6. Pass to the Limit and the proof of main theorems. In this section we pass the limit as $n \rightarrow \infty$.

Proof of Theorem 4.1. Let us prove \hat{P} satisfies (M1), (M2) and (M3).

(M3) is satisfied by (4.23).

For (M1), noting that $u_n(0) \rightarrow u_0$ in H , we have:

$$\hat{P}(u(0) = u_0) = \tilde{\mathbb{P}}(\tilde{u}(0) = u_0) = \lim_{n \rightarrow \infty} \tilde{\mathbb{P}}(\tilde{u}_n(0) = P_n u_0) = 1,$$

$$\begin{aligned} & \hat{P} \left\{ u \in C([0, T], H^{-1}) : \int_0^T \|F(u(s))\|_{H^{-1}} ds + \int_0^T \|\sigma(s, u(s))\|_{L^2(\ell^2, H)}^2 ds < +\infty \right\} \\ &= \tilde{\mathbb{P}} \left\{ \tilde{u} \in C([0, T], H^{-1}) : \int_0^T \|F(\tilde{u}(s))\|_{H^{-1}} ds + \int_0^T \|\sigma(s, \tilde{u}(s))\|_{L^2(\ell^2, H)}^2 ds < +\infty \right\}. \end{aligned}$$

Since

$$\tilde{u}_n \rightarrow \tilde{u} \text{ in } \mathcal{X} \text{ } \tilde{\mathbb{P}} - a.s.,$$

we have

$$\tilde{u} \in L^2([0, T], H^{1,1}) \cap L^\infty([0, T], H^{0,1}) \text{ } \tilde{\mathbb{P}} - a.s..$$

Thus by the growth condition of σ , we have

$$\int_0^T \|\sigma(s, \tilde{u}(s))\|_{L^2(\ell^2, H)}^2 ds \leq \int_0^T (K_0 + K_1 \|\tilde{u}\|_{L^2}^2 + K_2 \|\partial_1 \tilde{u}\|_{L^2}^2) ds < \infty, \text{ } \tilde{\mathbb{P}} - a.s.$$

Again, we know by interpolation, $\tilde{u} \in L^4([0, T], H^{\frac{1}{2}})$ $\tilde{\mathbb{P}} - a.s.$ By Sobolev imbedding, $\tilde{u} \in L^4([0, T], L^4)$ $\tilde{\mathbb{P}} - a.s.$ Hence $\text{div}(\tilde{u} \otimes \tilde{u}) \in L^2([0, T], H^{-1})$ $\tilde{\mathbb{P}} - a.s.$ And $\partial_1^2 \tilde{u} \in L^1([0, T], H^{-1})$, $F(\tilde{u}(s)) \in L^1([0, T], H^{-1})$ $\tilde{\mathbb{P}} - a.s.$ Thus (M1) is satisfied.

Finally we prove (M2):

Set

$$\begin{aligned} G^{(1)}(t, u) &:= \langle u(t), l \rangle; \\ G^{(2)}(t, u) &:= \int_0^t \langle F(u(s)), l \rangle ds. \end{aligned}$$

Since $\tilde{u}_n \rightarrow \tilde{u}$ in $C([0, T], H^{-1})$ and $l \in C^1(\mathbb{T}^2)$, we have for $\tilde{\mathbb{P}} - a.s.$

$$\|\langle \tilde{u}_n(t), l \rangle - \langle \tilde{u}(t), l \rangle\|_{L^\infty(0, T)} \rightarrow 0$$

Moreover, since \tilde{u}_n is bounded in $L^4(\Omega, L^\infty([0, T], L^2))$, $\langle \tilde{u}_n(t), l \rangle$ is bounded in $L^4(\Omega)$ for any t , we have

$$\lim_{n \rightarrow \infty} E^{\tilde{\mathbb{P}}} |G^{(1)}(t, \tilde{u}_n) - G^{(1)}(t, \tilde{u})| = 0$$

For $G^{(2)}$, since

$$\tilde{u}_n \rightarrow \tilde{u} \text{ in } L^2([0, T], L^2) \text{ } \tilde{\mathbb{P}} - a.s.,$$

we have

$$\int_0^t \langle F(\tilde{u}_n(s)), l \rangle ds \rightarrow \int_0^t \langle F(\tilde{u}(s)), l \rangle ds \text{ } \tilde{\mathbb{P}} - a.s., l \in C^1(\mathbb{T}^2).$$

Moreover, since \tilde{u}_n is bounded in $L^4(\Omega, L^\infty([0, T], L^2)) \cap L^2(\Omega, L^2([0, T], H^1))$ and $l \in C^1(\mathbb{T}^2)$, we have

$$E^{\tilde{\mathbb{P}}} \left(\int_0^T | \langle F(\tilde{u}_n(s)), l \rangle | ds \right)^2 \leq C.$$

Therefore, we have

$$\lim_{n \rightarrow \infty} E^{\tilde{\mathbb{P}}} | G^{(2)}(t, \tilde{u}_n) - G^{(2)}(t, \tilde{u}) | = 0.$$

By the definition of M_l in (M2), we have

$$(4.25) \quad \lim_{n \rightarrow \infty} E^{\tilde{\mathbb{P}}} | M_l(t, \tilde{u}_n) - M_l(t, \tilde{u}) | = 0.$$

Let $t > s$ and g be any bounded and real-valued \mathcal{F}_s -measurable continuous function on \mathcal{X} . Using (4.25) we have:

$$\begin{aligned} E^{\hat{P}}((M_l(t, u) - M_l(s, u))g(u)) &= E^{\tilde{\mathbb{P}}}((M_l(t, \tilde{u}) - M_l(s, \tilde{u}))g(\tilde{u})) \\ &= \lim_{n \rightarrow \infty} E^{\tilde{\mathbb{P}}}((M_l(t, \tilde{u}_n) - M_l(t, \tilde{u}_n))g(\tilde{u}_n)) \\ &= \lim_{n \rightarrow \infty} E^{\hat{P}_n}((M_l(t, u) - M_l(s, u))g(u)) \\ &= 0, \end{aligned}$$

where the last step is due to (M2) for \hat{P}_n .

Then we have

$$(4.26) \quad E^{\hat{P}}(M_l(t, u) | \mathcal{F}_s) = M_l(s, u).$$

On the other hand, by Burkholder-Davis-Gundy's inequality, growth condition of σ and Lemma 4.1, Lemma 4.2 we have:

$$\begin{aligned} \sup_n E^{\tilde{\mathbb{P}}} | M_l(t, \tilde{u}_n) |^4 &\leq C \sup_n E^{\tilde{\mathbb{P}}} \left(\int_0^t \|\sigma^*(\tilde{u}_n(s))(l)\|_{\ell^2}^2 ds \right)^2 \\ &\leq C \sup_n \int_0^t E^{\tilde{\mathbb{P}}} (\|\sigma^*(\tilde{u}_n(s))(l)\|_{\ell^2}^4) ds \\ &\leq C \sup_n E^{\tilde{\mathbb{P}}} \int_0^t \|\sigma^*(s, \tilde{u}_n)\|_{L^2(H^1, \ell^2)}^4 ds \|l\|_{H^1}^4 \\ &= C \sup_n E^{\tilde{\mathbb{P}}} \int_0^t \|\sigma(s, \tilde{u}_n)\|_{L^2(\ell^2, H^{-1})}^4 ds \|l\|_{H^1}^4 \\ &\leq C \sup_n E^{\tilde{\mathbb{P}}} \int_0^t (K'_0 + K'_1 \|\tilde{u}_n\|_{L^2}^2)^2 ds \\ &< +\infty, \end{aligned}$$

where the third inequality is due to the reason that the normal norm of the operator is smaller than the Hilbert-Schmidt norm of the operator, the fourth inequality is a result of $\|\sigma\|_{L^2(\ell^2, H^{-1})} = \|\sigma^*\|_{L^2(H^1, \ell^2)}$.

Then by (4.25) we obtain

$$\lim_{n \rightarrow \infty} E^{\tilde{\mathbb{P}}} | M_l(t, \tilde{u}_n) - M_l(t, \tilde{u}) |^2 = 0.$$

On the other hand, by Lipchitz condition of σ ,

$$\lim_{n \rightarrow \infty} E^{\tilde{\mathbb{P}}} \int_0^t \|\sigma^*(s, \tilde{u}_n(s))(l) - \sigma^*(s, \tilde{u}(s))(l)\|_{\ell^2}^2 ds = 0.$$

Thus, using the same method used for proving $E^{\hat{P}}(M_l(t, u) \mid \mathcal{F}_s) = M_l(s, u)$, we obtain

$$E^{\hat{P}}(M_l^2(t, u) - \int_0^t \|\sigma^*(s, u(s))(l)\|_{l^2}^2 ds \mid \mathcal{F}_s) = M_l^2(s, u) - \int_0^s \|\sigma^*(r, u(r))(l)\|_{l^2}^2 dr.$$

(M2) holds.

The results follow. \square

Finally let us turn to the proof of the pathwise uniqueness.

Proof of Theorem 4.2. Set

$$\tilde{w} := u - v.$$

Then we have

$$(4.27) \quad \langle u(t), e_k \rangle = \langle u(0), e_k \rangle + \int_0^t \langle -u \cdot \nabla u + \partial_1^2 u, e_k \rangle ds + \int_0^t \langle \sigma(s, u(s)) dW(s), e_k \rangle,$$

and

$$(4.28) \quad \langle v(t), e_k \rangle = \langle v(0), e_k \rangle + \int_0^t \langle -v \cdot \nabla v + \partial_1^2 v, e_k \rangle ds + \int_0^t \langle \sigma(s, v(s)) dW(s), e_k \rangle.$$

(4.27)-(4.28) ensures that

$$(4.29) \quad \langle \tilde{w}(t), e_k \rangle = \int_0^t \langle -\tilde{w} \cdot \nabla u + v \cdot \nabla \tilde{w} + \partial_1^2 \tilde{w}, e_k \rangle ds + \int_0^t \langle \sigma(s, u(s)) - \sigma(s, v(s)) dW(s), e_k \rangle.$$

Set $\varphi_k := \langle \tilde{w}(s), e_k \rangle$. Itô's formula and (4.29) yield:

$$\begin{aligned} d\varphi_k^2 &= 2\varphi_k d\varphi_k + \|(\sigma(s, u(s)) - \sigma(s, v(s)))^* e_k\|_{l^2}^2 ds \\ &= 2\langle \tilde{w}(s), e_k \rangle \langle -\tilde{w} \cdot \nabla u + v \cdot \nabla \tilde{w} + \partial_1^2 \tilde{w}, e_k \rangle ds \\ &\quad + 2\langle \tilde{w}(s), e_k \rangle \langle (\sigma(s, u(s)) - \sigma(s, v(s))) dW(s), e_k \rangle \\ &\quad + \|(\sigma(s, u(s)) - \sigma(s, v(s)))^* e_k\|_{l^2}^2 ds. \end{aligned}$$

Since $L_2 < \frac{2}{5}$, we can choose $0 < \hat{\alpha} < 1$ and $0 < \hat{\beta} < 1$, such that $L_2 + 2\hat{\alpha} + \frac{4}{\hat{\beta}}L_2 < 2$. From the calculation in the uniqueness part of deterministic cases (the proof of Thm. 3.1), we know that

$$\begin{aligned} (4.30) \quad & |(\tilde{w} \cdot \nabla u \mid \tilde{w})| \\ & \leq \|\tilde{w} \cdot \nabla u\|_{L^2} \|\tilde{w}\|_{L^2} \\ & \leq \hat{\alpha} \|\partial_1 \tilde{w}\|_{L^2}^2 + C(\hat{\alpha}) (\|\partial_1 u\|_{L^2}^{\frac{2}{3}} \|\partial_1 \partial_2 u\|_{L^2}^{\frac{2}{3}} + \|\partial_2 u\|_{L^2}^{\frac{2}{3}} \|\partial_1 \partial_2 u\|_{L^2}^{\frac{2}{3}}) \|\tilde{w}\|_{L^2}^2. \end{aligned}$$

Set $q(t) := \int_0^t 2C(\hat{\alpha}) (\|\partial_1 u\|_{L^2}^{\frac{2}{3}} + \|\partial_2 u\|_{L^2}^{\frac{2}{3}}) \|\partial_1 \partial_2 u\|_{L^2}^{\frac{2}{3}} ds$

By Itô's formula:

$$\begin{aligned} (4.31) \quad e^{-q(t)} \varphi_k^2 &= \varphi(0)^2 + 2 \int_0^t e^{-q(s)} \langle \tilde{w}(s), e_k \rangle \langle -\tilde{w} \cdot \nabla u + v \cdot \nabla \tilde{w} + \partial_1^2 \tilde{w}, e_k \rangle ds \\ &\quad + 2 \int_0^t e^{-q(s)} \langle \tilde{w}(s), e_k \rangle \langle (\sigma(s, u(s)) - \sigma(s, v(s))) dW(s), e_k \rangle \\ &\quad + \int_0^t e^{-q(s)} \|(\sigma(s, u(s)) - \sigma(s, v(s)))^* e_k\|_{l^2}^2 ds - \int_0^t q'(s) e^{-q(s)} \varphi_k^2 ds. \end{aligned}$$

Notice that

$$(4.32) \quad \sum_{k=1}^{\infty} \varphi_k^2 = \|\tilde{w}(t)\|_{L^2}^2.$$

The dominated convergence theorem imply when $N \rightarrow \infty$:

$$\begin{aligned} & 2 \left| \sum_{k \leq N} \int_0^t e^{-q(s)} \langle \tilde{w}(s), e_k \rangle \langle -\tilde{w} \cdot \nabla u, e_k \rangle ds \right| \\ & \longrightarrow 2 \left| \int_0^t e^{-q(s)} \langle \tilde{w} \cdot \nabla \tilde{w}, u \rangle ds \right| \end{aligned}$$

since by (4.30),

$$\begin{aligned} (4.33) \quad & \int_0^t e^{-q(s)} \|\tilde{w} \cdot \nabla u\|_{L^2} \|\tilde{w}\|_{L^2} ds \\ & \leq \int_0^t 2\hat{\alpha} e^{-q(s)} \|\partial_1 \tilde{w}\|_{L^2}^2 + 2C(\hat{\alpha}) e^{-q(s)} (\|\partial_1 u\|_{L^2}^{\frac{2}{3}} \|\partial_1 \partial_2 u\|_{L^2}^{\frac{2}{3}} + \|\partial_2 u\|_{L^2}^{\frac{2}{3}} \|\partial_1 \partial_2 u\|_{L^2}^{\frac{2}{3}}) \|\tilde{w}\|_{L^2}^2 ds \\ & = \int_0^t [2\hat{\alpha} e^{-q(s)} \|\partial_1 \tilde{w}\|_{L^2}^2 + 2C(\hat{\alpha}) e^{-q(s)} q'(s) \|\tilde{w}\|_{L^2}^2] ds. \end{aligned}$$

Notice that

$$\left\| \left(\sum_{k \leq N} \langle \tilde{w}(s), e_k \rangle e_k \right) \right\|_{L^2} \leq \|\tilde{w}(s)\|_{L^2}.$$

Now we follow the same calculation of (3.13) and (3.14):

$$\begin{aligned} & \left| e^{-q(s)} \langle v \cdot \nabla \tilde{w}, \sum_{k \leq N} \langle \tilde{w}(s), e_k \rangle e_k \rangle \right| \\ & \leq e^{-q(s)} \left(\|v\|_{L^2}^{\frac{1}{2}} \|\partial_1 v\|_{L^2}^{\frac{1}{2}} \|\partial_1 \tilde{w}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 \tilde{w}\|_{L^2}^{\frac{1}{2}} + \|v\|_{L^2}^{\frac{1}{2}} \|\partial_1 v\|_{L^2}^{\frac{1}{2}} \|\partial_2 \tilde{w}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 \tilde{w}\|_{L^2}^{\frac{1}{2}} \right) \|\tilde{w}\|_{L^2}. \end{aligned}$$

Since the latter belongs to $L^1([0, T])$, we use the dominated convergence theorem again and obtain:

$$(4.34) \quad 2 \sum_{k \leq N} \int_0^t e^{-q(s)} \langle \tilde{w}(s), e_k \rangle \langle v \cdot \nabla \tilde{w}, e_k \rangle ds \longrightarrow 0 \text{ as } N \rightarrow \infty.$$

Similarly, by the dominated convergence theorem, we have

$$(4.35) \quad 2 \sum_{k \leq N} \int_0^t e^{-q(s)} \langle \tilde{w}(s), e_k \rangle \langle \partial_1^2 \tilde{w}, e_k \rangle ds \longrightarrow -2 \int_0^t e^{-q(s)} \|\partial_1 \tilde{w}\|_{L^2}^2 ds \text{ as } N \rightarrow \infty,$$

and

$$\begin{aligned} (4.36) \quad & \sum_{k \leq N} \int_0^t e^{-q(s)} \|(\sigma(s, u(s)) - \sigma(s, v(s)))^* e_k\|_{\ell^2}^2 ds \\ & \longrightarrow \int_0^t e^{-q(s)} \|\sigma(s, u(s)) - \sigma(s, v(s))\|_{L^2(\ell^2, H)}^2 ds \\ & \leq \int_0^t e^{-q(s)} (L_1 \|\tilde{w}\|_{L^2}^2 + L_2 \|\partial_1 \tilde{w}\|_{L^2}^2) ds. \end{aligned}$$

Since $L_2 < \frac{2}{\beta}$, we can choose $0 < \hat{\beta} < 1$, such that $\frac{4}{\hat{\beta}}L_2 + L_2 < 2$. By Burkholder-Davis-Gundy's inequality as well as the dominated convergence theorem, we deduce

$$\begin{aligned}
(4.37) \quad & 2 \mid E \left(\sup_{0 \leq s \leq t} \int_0^s \sum_{k \leq N} e^{-q(r)} \langle \tilde{w}(r), e_k \rangle \langle (\sigma(r, u(r)) - \sigma(r, v(r))) dW(r), e_k \rangle \right) \mid \\
& \leq 4E \left(\int_0^t e^{-2q(s)} \|\sigma(s, u(s)) - \sigma(s, v(s))\|_{L^2(\ell^2, H)}^2 \|\tilde{w}\|_{L^2}^2 ds \right)^{\frac{1}{2}} \\
& \leq \hat{\beta} E \left(\sup_{s \leq t} (e^{-q(s)} \|\tilde{w}(s)\|_{L^2}^2) \right) \\
& \quad + \frac{4}{\hat{\beta}} E \int_0^t e^{-q(s)} (L_1 \|\tilde{w}(s)\|_{L^2}^2 + L_2 \|\partial_1 \tilde{w}(s)\|_{L^2}^2) ds.
\end{aligned}$$

Combining (4.31)-(4.37) and dropping some negative terms, we obtain:

$$(1 - \hat{\beta}) E \left(\sup_{0 \leq s \leq t} e^{-q(s)} \|\tilde{w}\|_{L^2}^2 \right) \leq E \int_0^t \left(1 + \frac{4}{\hat{\beta}}\right) L_1 e^{-q(s)} \|\tilde{w}(s)\|_{L^2}^2 ds.$$

By Gronwall's inequality we obtain $\tilde{w} = 0$ $\tilde{\mathbb{P}} - a.s.$ □

Acknowledgments. We would like to thank Professor Zhiming Ma for constant encouragement and profitable guidance. P. Zhang is partially supported by NSF of China under Grants 11371347 and 11688101, and innovation grant from National Center for Mathematics and Interdisciplinary Sciences of the Chinese Academy of Sciences. R. Zhu is supported in part by NSFC (11771037, 11671035, 11371099). Financial support by the DFG through the CRC 1283 "Taming uncertainty and profiting from randomness and low regularity in analysis, stochastics and their applications" is acknowledged.

REFERENCES

- [1] H. Bahouri, J.-Y. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren der mathematischen Wissenschaften, **343**, Springer-Verlag Berlin Heidelberg, 2011.
- [2] H. Bessaih and A. Millet, On stochastic modified 3D Navier-Stokes equations with anisotropic viscosity, *Journal of Mathematical Analysis and Applications*, **462** (2018), 915-956.
- [3] Z. Brzeźniak, and E. Motyl, Existence of a martingale solution of the stochastic Navier-Stokes equations in unbounded 2D and 3D domains, *Journal of Differential Equations*, **254** (2013), 1627-1685.
- [4] Z. Brzeźniak, M. Ondreját et al, Stochastic geometric wave equations with values in compact Riemannian homogeneous spaces, *The Annals of Probability*, **41** (2013), 1938-1977.
- [5] J. -Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, Fluids with anisotropic viscosity, *ESAIM: Mathematical Modelling and Numerical Analysis*, **34** (2000), 315-335.
- [6] J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, *Mathematical geophysics: An introduction to rotating fluids and the Navier-Stokes equations*, **32**, Oxford University Press on Demand, 2006.
- [7] J. -Y. Chemin and P. Zhang, On the global wellposedness to the 3-D incompressible anisotropic Navier-Stokes equations, *Communications in Mathematical Physics*, **272** (2007), 529-566.
- [8] G. Da Prato and J. Zabczyk, *Stochastic equations in infinite dimensions*, Cambridge University Press, 1992.
- [9] V. W. Ekman and others, *On the influence of the earth's rotation on ocean-currents.*, Almqvist & Wiksells boktryckeri, A.-B., 1905.
- [10] N. E. Glatt-Holtz and V. C. Vicol, Local and global existence of smooth solutions for the stochastic Euler equations with multiplicative noise, *The Annals of Probability*, **42** (2014), 80-145.
- [11] B. Goldys, M. Röckner and X. Zhang, Martingale solutions and Markov selections for stochastic partial differential equations, *Stochastic Processes and their Applications*, **119** (2009), 1725-1764.
- [12] E. Grenier and N. Masmoudi, Ekman layers of rotating fluids, the case of well prepared initial data, *Communications in Partial Differential Equations*, **22** (1997), 213-218.
- [13] A. Jakubowski, Short Communication: The Almost Sure Skorokhod Representation for Subsequences in Nonmetric Spaces, *Theory of Probability & Its Applications*, **42** (1998), 209-216.
- [14] J. Leray, Essai sur le mouvement d'un liquide visqueux emplissant l'espace, *Acta Mathematica*, **63** (1933), 193-248.

- [15] W. Liu and M. Röckner, *Stochastic partial differential equations: an introduction*, Springer 2015.
- [16] M. Paicu, Équation anisotrope de Navier-Stokes dans des espaces critiques, *Revista Matemática Iberoamericana*, **21** (2005), 179–235.
- [17] J. Pedlovsky, *Geophysical Fluid Dynamics*, Springer 1979.
- [18] M. Röckner, R. C. Zhu and X. C. Zhu, Sub- and supercritical stochastic quasi-geostrophic equation, *Annals of Probability*, **43** (2015), 1202–1273.
- [19] H. Triebel, *Theory of Function Spaces*, Birkhäuser, Basel, 1983