# Nearly hyperharmonic functions are infima of excessive functions

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**Abstract**

Let $\mathfrak{X}$ be a Hunt process on a locally compact space $X$ such that the set $\mathcal{E}_{\mathfrak{X}}$ of its Borel measurable excessive functions separates points, every function in $\mathcal{E}_{\mathfrak{X}}$ is the supremum of its continuous minorants in $\mathcal{E}_{\mathfrak{X}}$ and there are strictly positive continuous functions $v, w \in \mathcal{E}_{\mathfrak{X}}$ such that $v/w$ vanishes at infinity.

A numerical function $u \geq 0$ on $X$ is said to be *nearly hyperharmonic*, if $\int^* u \circ X_{\tau_V}\, dP^x \leq u(x)$ for every $x \in X$ and every relatively compact open neighborhood $V$ of $x$, where $\tau_V$ denotes the exit time of $V$. For every such function $u$, its lower semicontinous regularization $\hat{u}$ is excessive.

The main purpose of the paper is to give a short, complete and understandable proof for the statement that $u = \inf\{w \in \mathcal{E}_{\mathfrak{X}} \colon w \geq u\}$ for every Borel measurable nearly hyperharmonic function on $X$. Principal novelties of our approach are the following:

1. A quick reduction to the special case, where starting at points $x \in X$ with $u(x) < \infty$ the process $\mathfrak{X}$ hits the set $\{y \in X \colon \hat{u}(y) < u(y)\}$ $P^x$-a.s. only *finitely* many times.

2. The consequent use of (only) the strong Markov property.

3. The proof of the equality $\int u\, d\mu = \inf\{\int w\, d\mu \colon w \in \mathcal{E}_{\mathfrak{X}},\ w \geq u\}$ not only for measures $\mu$ satisfying $\int w\, d\mu < \infty$ for some excessive majorant $w$ of $u$, but also for all finite measures.

At the end, the measurability assumption on $u$ is weakened considerably.

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## 1 Main result

Let $X$ be a locally compact space with countable base, let $\mathcal{B}$ denote the $\sigma$-algebra of all Borel sets in $X$, and let $\mathcal{B}(X)$, $\mathcal{C}(X)$ respectively be the set of all numerical functions on $X$ which are Borel measurable, continuous and real respectively. As usual, given a set $\mathcal{F}$ of functions on $X$, a superscript "+", a subscript "$b$" respectively will indicate that we consider functions in $\mathcal{F}$ which are positive, bounded respectively. Let $\mathcal{M}(X)$ denote the set of all positive (Radon) measures on $X$.

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Let $\mathfrak{X} = (\Omega, \mathfrak{M}, \mathfrak{M}_t, X_t, \theta_t, P^x)$ be a Hunt process on $X$ (see [4, p. 45]). Let $\mathbb{P} = (P_t)_{t>0}$ denote the transition semigroup of $\mathfrak{X}$, that is, $P_t f(x) = E^x(f \circ X_t)$ for all $t > 0$, $f \in \mathcal{B}^+(X)$ and $x \in X$.

We assume that the Hunt process $\mathfrak{X}$ is *nice* in the following sense. Its set

$$\mathcal{E}_{\mathfrak{X}} := \{w \in \mathcal{B}^+(X) \colon \sup_{t>0} P_t w = w\} \tag{1.1}$$

of (Borel measurable) excessive functions has the following properties:

(C) Continuity: Every $w \in \mathcal{E}_{\mathfrak{X}}$ is the supremum of its minorants in $\mathcal{E}_{\mathfrak{X}} \cap \mathcal{C}(X)$.

(S) Separation: $\mathcal{E}_{\mathfrak{X}}$ is linearly separating, that is, for all $x \neq y$ and $\gamma > 0$, there exists a function $w \in \mathcal{E}_{\mathfrak{X}}$ such that $w(x) \neq \gamma w(y)$.

(T) Transience: There are strictly positive functions $v, w \in \mathcal{E}_{\mathfrak{X}} \cap \mathcal{C}(X)$ such that the quotient $v/w$ tends to 0 at infinity.

Let us observe that (C) trivially holds if the kernels $P_t$, $t > 0$, or at least the corresponding resolvent kernels $V_\lambda := \int_0^\infty e^{-\lambda t} P_t \, dt$, $\lambda > 0$, are strong Feller, that is, map $\mathcal{B}_b(X)$ into $\mathcal{C}_b(X)$.

For every set $A$ in $X$, the first entry time $D_A$ and the first hitting time $T_A$ are defined for $\omega \in \Omega$ by

$$D_A(\omega) := \inf\{s \ge 0 \colon X_s(\omega) \in A\} \quad \text{and} \quad T_A(\omega) := \inf\{s > 0 \colon X_s(\omega) \in A\}.$$

Let $\mathcal{U}_c$ be the set of all relatively compact open sets $V$ in $X$, $V^c := X \setminus V$. A numerical function $u \ge 0$ is called *nearly hyperharmonic* if

$$\int^* u \circ X_{D_{V^c}} \, dP^x \le u(x) \quad \text{for all } x \in X \text{ and neighborhoods } V \in \mathcal{U}_c \text{ of } x. \tag{1.2}$$

Clearly, the set $\mathcal{N}$ (denoted by $\mathcal{N}^+$ in [11]) of such functions is a convex cone which contains $\mathcal{E}_{\mathfrak{X}}$ and is stable under increasing limits and *arbitrary* infima. Moreover, it contains *every* numerical function $u \ge 0$ which vanishes outside a set $E$ which is polar, that is, satisfies $T_E = \infty$ almost surely. For space-time Brownian motion on $\mathbb{R}^d \times \mathbb{R}$, every function $u \colon \mathbb{R}^d \times \mathbb{R} \to [0, \infty]$ satisfying $u(x,t) \le u(x', t')$, whenever $t \le t'$, is nearly hyperharmonic.

The purpose of this paper is to give a short, complete and understandable proof for the following statement (where the implications (3) ⇒ (2) ⇒ (1) hold trivially).

**THEOREM 1.1.** *For every $u \in \mathcal{B}^+(X)$ the following statements are equivalent:*

(1) *The function $u$ is nearly hyperharmonic.*

(2) *The function $u$ is the infimum of its excessive majorants.*

(3) *For all $\mu \in \mathcal{M}(X)$ such that $\mu(A) + \int_{X \setminus A} w \, d\mu < \infty$ for some $A \in \mathcal{B}$ and majorant $w \in \mathcal{E}_{\mathfrak{X}}$ of $u$,*

$$\int u \, d\mu = \inf\{\int w \, d\mu \colon w \in \mathcal{E}_{\mathfrak{X}},\ w \ge u\}. \tag{1.3}$$

*In particular, for every $\varphi \in \mathcal{B}^+(X)$, the function $R_\varphi := \inf\{w \in \mathcal{E}_{\mathfrak{X}} : w \ge \varphi\}$ is the smallest nearly hyperharmonic majorant of $\varphi$.*

**REMARK 1.2.** Of course, (1.3) trivially holds if $\int u\,d\mu = \infty$ (we take $w = \infty$). Since $\mathcal{E}_{\mathfrak{X}}$ is $\wedge$-stable, the set of all $\mu \in \mathcal{M}(X)$ satisfying (1.3) is a convex cone. If $\varepsilon > 0$ and $A \in \mathcal{B}$ such that $\mu(A) + \int_{X\setminus A} w\,d\mu < \infty$, there is a union $A'$ of $A$ with a compact in $X \setminus A$ such that $\mu(A') < \infty$ and $\int_{X\setminus A'} w\,d\mu < \varepsilon$.

In fact, we shall finally prove that, for functions $u\colon X \to [0,\infty]$, the equivalence (1) $\Leftrightarrow$ (3) already holds if $u$ is nearly Borel measurable (Theorem 6.1) and that (1) $\Leftrightarrow$ (2) even holds if $u$ is only supposed to be equal to a Borel measurable function outside a polar set (Theorem 6.2). Moreover, assuming that $u$ is nearly hyperharmonic and equal to a universally measurable function outside a polar set, we characterize the validity of (2) in various ways (Corollary 6.3).

Analogous statements can be found for different settings and functions, which there are called *strongly supermedian*, in [13, 5, 6, 1, 2], but the proofs given therein seem to be either incomprehensible or incomplete (see [13, 5, 6]) or, as in [1] and [2, Section 4], very long and delicate.

The main novelties of our approach are

- the insight that for a proof of inequalities $u \ge \eta \inf\{w \in \mathcal{E}_{\mathfrak{X}} : w \ge u\}$ for $\eta \in (0,1)$ it suffices to consider the special case, where starting in $\{u < \infty\}$ the Hunt process hits the set $\{\hat{u} < u\}$ almost surely only finitely many times,
- the consequent use of (only) the strong Markov property,
- the verification of the equalities in (2) and (3) first for nearly hyperharmonic functions $u \in \mathcal{B}^+(X)$ having a certain finiteness property, which then implies (!) that *every* nearly hyperharmonic $u \in \mathcal{B}^+(X)$ has this property,
- the equality (1.3) not only for measures $\mu$ satisfying $\int w\,d\mu < \infty$ for some excessive majorant $w$ of $u$, but also for all finite measures $\mu$.

Let us observe that the additional statement in Theorem 1.1 is not only of interest in its own right, but also because of the following consequence (see [11, Propositions 2.4, 2.5 and Theorem 3.1]).

**COROLLARY 1.3.** *Let $\varphi \in \mathcal{B}^+(X)$. Then $R_\varphi = \varphi \vee \hat{R}_\varphi \in \mathcal{B}^+(X)$ and*

$$R_\varphi(x) = \sup\{\int \varphi \circ X_{D_{V^c}}\,dP^x : x \in V \in \mathcal{U}_c\}, \qquad x \in X.$$

In Section 2 we discuss the close relationship between nice Hunt processes and balayage spaces and establish a crucial inequality for nearly hyperharmonic functions (Lemma 2.6). In Section 3 we treat the special case indicated above. In Section 4 we shall see very quickly that the equality $R_u = u$ for arbitrary nearly hyperharmonic functions $u \in \mathcal{B}^+(X)$ is a consequence of our result for the special case and yields the additional statement in Theorem 1.1. The implication (1) $\Rightarrow$ (3) is derived in Section 5, and in Section 6 we present our results under weaker measurability assumptions. In Section 7 we briefly indicate the use of our approach in the general setting of standard processes.

## 2 Preliminaries

Let us first recall the following. Let $\mathcal{W}$ be any convex cone of positive numerical functions on $X$ having the properties stated in (C), (S) and (T) for $\mathcal{E}_{\mathfrak{X}}$ (so that every function in $\mathcal{W}$ is lower semicontinuous). The ($\mathcal{W}$-)*fine topology* on $X$ is the coarsest topology on $X$ which is at least as fine as the initial topology and such that every function in $\mathcal{W}$ is continuous. Given $\varphi\colon X \to [0,\infty]$, let $\hat{\varphi}$, $\hat{\varphi}^f$ resp. denote the largest lower semicontinuous, finely lower semicontinuous resp. minorant of $\varphi$.

Then $(X,\mathcal{W})$ is called a *balayage space* provided the following hold (see [3, 9] and [10, Appendix 8.1]):

(i) If $v_n \in \mathcal{W}$, $v_n \uparrow v$, then $v \in \mathcal{W}$.

(ii) If $\mathcal{V} \subset \mathcal{W}$, then $\widehat{\inf \mathcal{V}}^f \in \mathcal{W}$.

(iii) If $u, v', v'' \in \mathcal{W}$, $u \le v' + v''$, then there exist $u', u'' \in \mathcal{W}$ such that $u = u' + u''$ and $u' \le v'$, $u'' \le v''$.

By [3, II.4.9] (see also [9, Corollary 2.3.8]), for our nice Hunt process $\mathfrak{X}$, the pair $(X,\mathcal{E}_{\mathfrak{X}})$ is a balayage space (of course, $\lim_{t\to 0} P_t f = f$ for every $f \in \mathcal{C}_b(X)$ by right continuity of the paths). So we may use results obtained in [3] and in the recent paper [11].

**REMARKS 2.1.** 1. We note that, conversely, for every balayage space $(X,\mathcal{W})$ with $1 \in \mathcal{W}$, there exists a corresponding nice Hunt process (see [3, IV.8.1]). For that matter, the condition $1 \in \mathcal{W}$ is not really restrictive since, given any balayage space $(X,\mathcal{W})$, the standard normalization $\widetilde{\mathcal{W}} := (1/\tilde{w})\mathcal{W}$ with any strictly positive $\tilde{w} \in \mathcal{W} \cap \mathcal{C}(X)$ leads to a balayage space $(X,\widetilde{\mathcal{W}})$ with $1 \in \widetilde{\mathcal{W}}$.

2. A characterization by harmonic kernels reveals that the notion of a balayage space generalizes the notion of a $\mathcal{P}$-harmonic space. Therefore the theory of balayage spaces is known to cover the potential theory for very general partial differential operators of second order (see, for instance, [7]).

Of course, for any numerical function $\varphi \ge 0$ on $X$,

$$R_\varphi := \inf\{w \in \mathcal{E}_{\mathfrak{X}} \colon w \ge \varphi\} \in \mathcal{N},$$

and, by property (ii) of balayage spaces,

$$\hat{R}_\varphi := \widehat{R_\varphi} = \widehat{R_\varphi}^f \in \mathcal{E}_{\mathfrak{X}}. \tag{2.1}$$

By [11, Proposition 2.2 and p. 6], we know even that $\hat{u} = \hat{u}^f \in \mathcal{E}_{\mathfrak{X}}$ for all $u \in \mathcal{N}$.

We recall that, for arbitrary subsets $A$ of $X$ and $w \in \mathcal{E}_{\mathfrak{X}}$,

$$R_w^A := R_{1_A w} = \inf\{v \in \mathcal{E}_{\mathfrak{X}} \colon v \ge w \text{ on } A\} \quad \text{and} \quad \hat{R}_w^A := \widehat{R_w^A},$$

leading to measures $\varepsilon_x^A$ and $\hat{\varepsilon}_x^A$ on $X$ which are characterized by

$$\int w\, d\varepsilon_x^A = R_w^A(x) \quad \text{and} \quad \int w\, d\hat{\varepsilon}_x^A = \hat{R}_w^A(x), \qquad w \in \mathcal{E}_{\mathfrak{X}};$$

see [3, II.4.3, II.5.4 and VI.2.1] (in [3] these measures are denoted by $\overset{\circ}{\varepsilon}{}_x^A$ and $\varepsilon_x^A$).

Given a stopping time $T$, we define as usual[1]

$$P_T f(x) := E^x(f \circ X_T) \quad \text{for all } f \in \mathcal{B}^+(X) \text{ and } x \in X.$$

Suppose for the moment that $A \in \mathcal{B}$. Then, by [3, VI.3.14], both $D_A$ and $T_A$ are stopping times and

$$P_{D_A} w = R_w^A \quad \text{and} \quad P_{T_A} w = \hat{R}_w^A \tag{2.2}$$

for every $w \in \mathcal{E}_{\mathfrak{X}}$ (cf. [4, 6.12]), where obviously $P_{D_A} w = w$ on $A$ and $P_{D_A} w = P_{T_A} w$ on $X \setminus A$. This implies that, for all $f \in \mathcal{B}^+(X)$,

$$P_A f := P_{D_A} f \in \mathcal{B}(X) \quad \text{and} \quad \hat{P}_A f := P_{T_A} f \in \mathcal{B}(X); \tag{2.3}$$

see [3, VI.2.10]. So $P_A$ and $\hat{P}_A$ are kernels on $X$; see [3, Section II].

Of course, (2.2) implies that, for all $x \in X$ and $B \in \mathcal{B}$,

$$P_A(x, B) = \varepsilon_x^A(B) \quad \text{and} \quad \hat{P}_A(x, B) = \hat{\varepsilon}_x^A(B).$$

In particular, our definition of nearly hyperharmonic functions by (1.2) coincides with the definition given by [11, (2.2)].

The following simple stability result will be useful.

**LEMMA 2.2.** *For every $\mu \in \mathcal{M}(X)$, the set $\mathcal{F}$ of all functions $f \in \mathcal{B}^+(X)$ such that $\int f\,d\mu = \inf\{\int w\,d\mu \colon w \in \mathcal{E}_{\mathfrak{X}},\ w \ge f\}$ is a convex cone which is closed under countable sums.*

*Proof.* Of course, $0 \in \mathcal{F}$ and $af \in \mathcal{F}$ for all $a > 0$ and $f \in \mathcal{F}$. Let $(f_n)$ be a sequence in $\mathcal{F}$ and $f := \sum_{n\ge1} f_n$ such that $\int f\,d\mu < \infty$. Given $\varepsilon > 0$, we may choose $w_n \in \mathcal{E}_{\mathfrak{X}}$, $n \in \mathbb{N}$, such that $\int w_n\,d\mu < \int f_n\,d\mu + 2^{-n}\varepsilon$. Then $w := \sum_{n\ge1} w_n \in \mathcal{E}_{\mathfrak{X}}$, $w \ge f$ and $\int w\,d\mu < \int f\,d\mu + \varepsilon$. □

We recall that an arbitrary set $A$ in $X$ is called *thin* at a point $x \in X$ if $\hat{\varepsilon}_x^A \ne \varepsilon_x$. By definition, the *base* $b(A)$ of $A$ is the set of all $x \in X$ such that $A$ is not thin at $x$, that is, $\hat{\varepsilon}_x^A = \varepsilon_x$. By [3, VI.4.8],

$$b(A) = \{x \in X \colon T_A = 0\ P^x\text{-almost surely}\}, \qquad \text{if } A \in \mathcal{B}. \tag{2.4}$$

By [3, VI.4.1 and VI.4.4], the base of every set $A$ in $X$ is a finely closed $G_\delta$-set containing the fine interior of $A$, and $A \cup b(A)$ is the fine closure of $A$. Moreover, for every $x \in X$, the measure $\hat{\varepsilon}_x^A$ is supported by the fine closure of $A$, that is, the inner measure of its complement is zero; see [3, VI.4.6].

A set $F$ in $X$ is called *totally thin* if $b(F) = \emptyset$ so that, in particular, $F$ is finely closed. A *semipolar set* is a countable union of totally thin sets. We know that, for any infimum $u$ of functions in $\mathcal{E}_{\mathfrak{X}}$, the set $\{\hat{u} < u\}$ is semipolar; see [3, VI.5.11].

[1] We tacitly assume that we have an isolated point $\Delta$ added to $X$, that functions on $X$ are identified with functions on $X_\Delta := X \cup \{\Delta\}$ vanishing at $\Delta$ and that $X_t \colon [0,\infty] \to X_\Delta$ with $X_\infty = \Delta$ and $X_t(\omega) = \Delta$, whenever $t \ge s$ and $X_s(\omega) = \Delta$.

**EXAMPLE 2.3.** For space-time Brownian motion on $\mathbb{R}^d \times \mathbb{R}$, every hyperplane $H_t := \mathbb{R}^d \times \{t\}$ is totally thin.

For the remainder of this section let us fix a function $u \in \mathcal{B}^+(X)$ which is nearly hyperharmonic. By [11, Proposition 2.5], for every $A \in \mathcal{B}$,

$$P_A u \le u \quad \text{and} \quad \hat{P}_A u \le u. \tag{2.5}$$

**REMARK 2.4.** Let $S, T$ be stopping times for $\mathfrak{X}$, $S \le T$. Then $P_T w \le P_S w \le w$ for every $w \in \mathcal{E}_{\mathfrak{X}}$; see [3, VI.3.4]. By [11, Corollary 2.6] (which uses that, for $x \in X$, the extreme points in the weak*-compact convex set $\mathcal{M}_x(\mathcal{E}_{\mathfrak{X}})$ of all measures $\mu$ on $X$ satisfying $\int w\, d\mu \le w(x)$ for every $w \in \mathcal{E}_{\mathfrak{X}}$ are the measures $\varepsilon_x^A$, $A \in \mathcal{B}$), this implies that $P_S u \le u$. So the nearly hyperharmonic function $u$ is strongly supermedian in the sense of [1, 2, 5, 6, 14]. By [11, Proposition 2.7], we even get the inequality $P_T u \le P_S u$ (and hence, by a standard argument, $E^x(u \circ X_T | \mathfrak{M}_S) \le u \circ X_S$ $P^x$-a.s. for every $x \in X$).

Since we shall not use these facts in the sequel, they may also be viewed as *consequences* of Theorem 1.1 (to obtain $P_T(u \wedge n) \le P_S(u \wedge n)$, $n \in \mathbb{N}$, we consider the finite measures $\mu := P_T(x, \cdot) + P_S(x, \cdot)$, $x \in X$).

Let us note that (2.5) implies the following.

**LEMMA 2.5.** *The function $u$ is finely upper semicontinuous and, starting in the finely open set $U := \{u < \infty\}$, the process $\mathfrak{X}$ does not leave $U$, that is,*

$$P^x[T_{X\setminus U} < \infty] = 0 \qquad \textit{for every } x \in U. \tag{2.6}$$

*Proof.* Let $a \in [0, \infty]$, $A := \{u \ge a\}$ and $x \in X \setminus A$. By [3, VI.3.14], there exists an increasing sequence $(K_n)$ of compacts in the Borel set $A$ such that $T_{K_n} \downarrow T_A$ $P^x$-a.s. Since $X_{T_{K_n}} \in K_n$ on $[T_{K_n} < \infty]$, the inequalities $P_{K_n} u \le u$ yield that

$$aP^x[T_A < \infty] = \lim_{n\to\infty} aP^x[T_{K_n} < \infty] \le u(x) < a.$$

Hence $P^x[T_A < \infty] < 1$, $x \notin b(A)$. So $A$ is finely closed showing that $u$ is finely upper semicontinuous. Finally, taking $a = \infty$, we see that (2.6) holds. □

For every $V \in \mathcal{U}_c$, due to the lower semicontinuity of $P_{V^c} u$ on $V$, we know that

$$\hat{P}_v u = P_{V^c} u \le \hat{u} \quad \text{on } V \tag{2.7}$$

(see [11, (2.3)]). The following more general estimate will be crucial in Section 3.

**LEMMA 2.6.** *Let $A \in \mathcal{B}$ and $x \in X \setminus b(A)$ such that $x$ is not finely isolated. Then*

$$\hat{P}_A u(x) = E^x(u \circ X_{T_A}) \le \hat{u}(x).$$

*Proof.* Since $u_n := u \wedge n$ is nearly hyperharmonic for every $n \in \mathbb{N}$ and $\hat{u}_n \uparrow \hat{u}$ by [11, Proposition 2.3], we may assume without loss of generality that $u$ is bounded, say $u < M < \infty$.

Let $W_n \in \mathcal{U}_c$ such that $W_n \downarrow \{x\}$ as $n \to \infty$. By assumption, $x \in b(X \setminus \{x\})$, and hence $D_{W_n^c} \downarrow D_{X\setminus\{x\}} = 0$ $P^x$-a.s., whereas $T_A > 0$ $P^x$-a.s. So there exists $n \in \mathbb{N}$ such that $\tau := D_{W_n^c}$ satisfies $P^x[T_A \le \tau] < \varepsilon/M$, and therefore

$$E^x(u \circ X_{T_A}) \le \varepsilon + E^x(u \circ X_{T_A}; T_A > \tau).$$

Let us note that $\tau > 0$ $P^x$-a.s. and that obviously, on the set $[T_A > \tau > 0]$ we have $T_A = \tau + D_A \circ \theta_\tau$ whence $X_{T_A} = X_{D_A} \circ \theta_\tau$. Thus we conclude that

$$\begin{aligned} E^x(u \circ X_{T_A}; T_A > \tau) &\le E^x(u \circ X_{D_A} \circ \theta_\tau) = E^x(E^{X_\tau}(u \circ X_{D_A})) \\ &= P_\tau P_A u(x) \le P_\tau u(x) \le \hat{u}(x), \end{aligned}$$

by the strong Markov property and (2.7). □

Finally, let us recursively define stopping times $S_n^A$, $A \in \mathcal{B}$, $n \ge 0$, by

$$S_0^A := D_A \quad \text{and} \quad S_{n+1}^A := S_n^A + T_A \circ \theta_{S_n^A}. \tag{2.8}$$

So $S_{n+1}^A$ is the time of the first hitting of $A$ after the time $S_n$, $X_{S_{n+1}^A} = X_{T_A} \circ \theta_{S_n^A}$; see [3, Section IV.6].

**PROPOSITION 2.7.** *Let $A \in \mathcal{B}$ and $n \ge 0$. Then*

$$P_A(\hat{P}_A)^n f(x) = E^x(f \circ X_{S_n^A}) \quad \textit{for all } f \in \mathcal{B}^+(X) \textit{ and } x \in X. \tag{2.9}$$

*Proof.* Let $S_n := S_n^A$. For $n = 0$, (2.9) holds by (2.3). Suppose that (2.9) is true for some $n \ge 0$, and let $f \in \mathcal{B}^+(X)$, $x \in X$. Then, by the strong Markov property,

$$\begin{aligned} P_A(\hat{P}_A)^{n+1} f(x) &= E^x((\hat{P}_A f) \circ X_{S_n}) = E^x(E^{X_{S_n}}(f \circ T_A)) \\ &= E^x(f \circ X_{T_A} \circ \theta_{S_n}) = E^x(f \circ X_{S_{n+1}}). \end{aligned}$$

□

# 3 A special case

Throughout this section we fix a nearly hyperharmonic function $u \in \mathcal{B}^+(X)$ and suppose the following.

**ASSUMPTION 3.1.** *Starting in $U := \{u < \infty\}$, the Hunt process $\mathfrak{X}$ hits the set $F := \{\hat{u} < u\}$ a.s. only finitely many times: The stopping times $S_n := S_n^F$ satisfy*

$$P^x\Big(\bigcap_{n \ge 0} [S_n < \infty]\Big) = 0 \quad \textit{for every } x \in U. \tag{3.1}$$

If $x \in b(F)$, then, by (2.4), $S_n = 0$ $P^x$-a.s. for every $n \ge 0$. Hence (3.1) implies that $b(F) \cap U = \emptyset$. So $F$ is in fact totally thin if $u < \infty$. By Lemma 2.5 and a straightforward induction, we see that, for every $x \in U$ and $P^x$-a.e. $\omega \in [S_n < \infty]$,

$$X_{S_n}(\omega) \in F \cap U \quad \textit{and} \quad S_n(\omega) < S_{n+1}(\omega), \tag{3.2}$$

and, for $P^x$-a.e. $\omega \in \Omega$,

$$\{S_n(\omega) \colon n \ge 0,\ S_n(\omega) < \infty\} = \{s \ge 0 \colon X_s(\omega) \in F\}. \tag{3.3}$$

**EXAMPLE 3.2.** Let us consider space-time Brownian motion on $\mathbb{R}^d \times \mathbb{R}$ and fix a sequence $(t_n)$ in $\mathbb{R}$. For $n \in \mathbb{N}$, let $A_n$ be an arbitrary subset of $H_{t_n} = \mathbb{R}^d \times \{t_n\}$ and let

$$v_n := 1_{A_n} + 1_{\mathbb{R}^d \times (t_n, \infty)}.$$

Then $v := \sum_{n \ge 1} 2^{-n} v_n \in \mathcal{N}$, $0 \le v \le 1$, and $\{\hat{v} < v\}$ is the union of all $A_n$, $n \in \mathbb{N}$.

If $\{t_n \colon n \in \mathbb{N}\}$ is dense in $\mathbb{R}$ and $A_n = H_{t_n}$, $n \in \mathbb{N}$, then $\{\hat{v} < v\}$ is finely dense in $\mathbb{R}^d \times \mathbb{R}$ and $S_n = 0$ a.s. for all $n \in \mathbb{N}$. So Assumption 3.1 is very restrictive.

If $t_n = -1/n$ and $A_n = H_{t_n}$, $n \in \mathbb{N}$, then $v := \infty \cdot 1_{\mathbb{R}^d \times [0,\infty]} + \sum_{n \ge 1} 2^{-n} v_n \in \mathcal{N}$ and $v$ satisfies Assumption 3.1 with $b(\{\hat{v} < v\}) = H_0$.

We define functions $g$ and $u_0$ on $X$ by

$$g(x) := E^x\Big(\sum_{n \ge 0} (u - \hat{u}) \circ X_{S_n}\Big) \qquad \text{and} \qquad u_0(x) := u(x) - g(x), \qquad x \in U,$$

and $g(x) = u_0(x) = \infty$, if $x \in X \setminus U$.

**DEFINITION 3.3.** *For $A \in \mathcal{B}$, let $\mathcal{R}(A)$ be the set of sums of a function in $\mathcal{E}_{\mathfrak{X}}$ and countably many functions $P_B w$, where $B \in \mathcal{B}$, $w \in \mathcal{E}_{\mathfrak{X}}$, $B \subset A$ and $w$ is bounded.*

Clearly, $\mathcal{R}(A) \subset \mathcal{R}(X) \subset \mathcal{N} \cap \mathcal{B}^+(X)$ and, by (2.2) and Lemma 2.2,

$$v = R_v = \inf\{w \in \mathcal{E}_{\mathfrak{X}} \colon w \ge v\} \qquad \text{for every } v \in \mathcal{R}(X). \tag{3.4}$$

In this section we shall establish the following result.

**THEOREM 3.4.** *The function $g$ is a minorant of $u$, both $g$ and $u_0$ are nearly hyperharmonic, $g - \hat{g} = u - \hat{u}$ on $U$ and $\hat{u}_0 = u_0$ on $U$.*

*Further, there are functions $g_1, u_1 \in \mathcal{R}(F)$ such that $g_1 = g$ on $U$ and $u_1 = u$ on $U$. In particular, if $u < \infty$, then $u_0 \in \mathcal{E}_{\mathfrak{X}}$ and $u = R_u$.*

We prepare its proof with a lemma leading to estimates by telescoping series.

**LEMMA 3.5.** *Let $V$ be an open set in $X$, $\tau := T_{V^c}$, $n \ge 0$ and $x \in V \cap U$. Then*

$$E^x(u \circ X_{S_n \wedge \tau} - u \circ X_{S_{n+1} \wedge \tau}) \ge E^x(u - \hat{u}) \circ X_{S_n}; S_n < \tau) \ge 0.$$

*Proof.* Let $A_n := [S_n < \tau]$. Of course, $S_n \wedge \tau = S_{n+1} \wedge \tau = \tau$ on $[S_n \ge \tau]$. Hence

$$E^x(u \circ X_{S_n \wedge \tau} - u \circ X_{S_{n+1} \wedge \tau}) = E^x(u \circ X_{S_n}; A_n) - E^x(u \circ X_{S_{n+1} \wedge \tau}; A_n).$$

Let $T := T_{F \cup V^c} = T_F \wedge \tau$. By Lemma 2.6 and the strong Markov property,

$$E^x(\hat{u} \circ X_{S_n}; A_n) \ge E^x(E^{X_{S_n}}(u \circ X_T); A_n) = E^x(u \circ X_T \circ \theta_{S_n}; A_n),$$

where $X_T \circ \theta_{S_n} = X_{S_{n+1} \wedge \tau}$ on $A_n$, since $S_n + T \circ \theta_{S_n} = S_{n+1} \wedge \tau$ on $A_n$. □

**PROPOSITION 3.6.** *The function $g$ is a minorant of $u$.*

*Proof.* Let $x \in U$. By Lemma 3.5 with $V := X$, $\tau = \infty$,

$$g(x) \le \sum_{n \ge 0} \big(E^x(u \circ X_{S_n}) - E^x(u \circ X_{S_{n+1}})\big) \le E^x(u \circ X_{S_0}) = P_F u(x),$$

where $P_F u(x) \le u(x)$, by (2.5). □

Moreover, the following is an important ingredient.

**LEMMA 3.7.** *Let $E$ be a Borel subset of $F$, $T_n := S_n^E$ for $n \geq 0$,*

$$w := \sum_{n\geq 0} (\hat{P}_E)^n 1 \quad \text{and} \quad v(x) := E^x(\sum_{n\geq 0} 1_E \circ X_{T_n}), \quad x \in X.$$

*Then $w \in \mathcal{E}_{\mathfrak{X}}$, $v = P_E w \in \mathcal{R}(F)$ and $P_E w = 1_E + \hat{P}_E w$. Moreover,*

$$P_A v(x) = E^x(\sum_{n\geq 0} 1_E \circ X_{S_n} 1_{[S_n \geq D_A]}) \qquad \text{for all } A \in \mathcal{B} \text{ and } x \in X. \tag{3.5}$$

*In particular, $v(x) = E^x(\sum_{n\geq 0} 1_E \circ X_{S_n})$ for every $x \in X$.*

*Proof.* For every $n \geq 0$, $w_n := (\hat{P}_F)^n 1$ is a bounded function in $\mathcal{E}_{\mathfrak{X}}$, hence $w \in \mathcal{E}_{\mathfrak{X}}$ and, by (2.2), $v = P_E w = \sum_{n\geq 0} P_E w_n \in \mathcal{R}(F)$. Obviously, $1 + \hat{P}_E w = w$. If $x \in E$, then $w(x) = P_E w(x)$. If $x \in X \setminus E$, then trivially $\hat{P}_E w(x) = P_E w(x)$.

Let $A \in \mathcal{B}$ and $x \in U$. By Proposition 2.7 and the strong Markov property,

$$\begin{aligned} P_A v(x) &= E^x(E^{X_{D_A}}(\sum_{n\geq 0} 1_E \circ X_{T_n})) \\ &= E^x(\sum_{n\geq 0} 1_E \circ X_{T_n} \circ \theta_{D_A}) = E^x(\sum_{n\geq 0} 1_E \circ X_{D_A + T_n \circ \theta_{D_A}}), \end{aligned}$$

where, for $P^x$-almost every $\omega \in \Omega$, the last sum is the number of all $s \geq D_A(\omega)$ such that $X_s(\omega) \in E$, which in turn is the sum in (3.5); see (3.3). The proof is completed taking $A = X$. $\square$

*Proof of Theorem 3.4.* There are Borel sets $F_k$ in $F$ and $a_k \in (0, \infty)$, $k \in \mathbb{N}$, with

$$1_U u - 1_U \hat{u} = \sum_{k\geq 1} a_k 1_{F_k}. \tag{3.6}$$

For every $k \in \mathbb{N}$, let $v_k(x) := E^x(\sum_{n\geq 0} 1_{F_k} \circ X_{S_n^{F_k}})$, $x \in X$. Then, by Lemma 3.7,

$$g_1 := \sum_{k\geq 1} a_k v_k \in \mathcal{R}(F) \quad \text{and} \quad g_1 = g \quad \text{on } U. \tag{3.7}$$

So $g \in \mathcal{N} \cap \mathcal{B}(X)$; see (3.2). Further, using [11, (2.4) and Proposition 2.3],

$$g = g_1 = \sum_{k\geq 1} a_k (1_{F_k} + \hat{v}_k^f) = \sum_{k\geq 1} a_k 1_{F_k} + \hat{g}_1^f = (u - \hat{u}) + \hat{g} \quad \text{on } U.$$

Next let $V \in \mathcal{U}_c$ and $x \in V \cap U$. By Lemma 3.7 (with $A := V^c$), we obtain that

$$g(x) - P_\tau g(x) = g_1(x) - P_\tau g_1(x) = \sum_{n\geq 0} E^x((u - \hat{u}) \circ X_{S_n}; S_n < \tau) \geq 0$$

(showing once more that $g$ is nearly hyperharmonic). Hence, by Lemma 3.5,

$$g(x) - P_\tau g(x) \leq \sum_{n\geq 0} E^x(u \circ X_{S_n \wedge \tau} - u \circ X_{S_{n+1} \wedge \tau}),$$

where $E^x(u \circ X_{S_0 \wedge \tau}) = P_{D_{F \cup V^c}} u(x) \le u(x)$. Therefore

$$\begin{aligned} u_0(x) - P_\tau u_0(x) &= u(x) - P_\tau u(x) - (g(x) - P_\tau g(x)) \\ &\ge \lim_{n \to \infty} E^x(u \circ X_{S_n \wedge \tau}) - E^x(u \circ X_\tau). \end{aligned}$$

If $\omega \in \bigcup_{m \ge 0} [S_m = \infty]$ and $\tau(\omega) < \infty$, then trivially $u(X_\tau(\omega)) = \lim_{n \to \infty} u(X_{S_n \wedge \tau}(\omega))$. Hence $E^x(u \circ X_\tau) \le \lim_{n \to \infty} E^x(u \circ X_{S_n \wedge \tau})$, by (3.1) and Fatou's lemma, and we obtain that $u_0(x) - P_\tau u_0(x) \ge 0$. Thus $u_0 \in \mathcal{N}$.

Since $g + u_0 = u$, and hence $\hat{g} + \hat{u}_0 = \hat{u}$, we finally see that $\hat{u}_0 = u_0$ on $U$ and $u_1 := g_1 + \hat{u}_0 \in \mathcal{R}(F)$, $u_1 = g + u_0 = u$ on $U$. □

**COROLLARY 3.8.** *If $w \in \mathcal{E}_{\mathfrak{X}}$ such that $w = \infty$ on $\{u = \infty\}$, then $u + w \in \mathcal{R}(F)$.*

*Proof.* By Theorem 3.4, $u + w = u_1 + g_1 + w \in \mathcal{R}(F)$. □

# 4 The general case

To reduce the general case of a nearly hyperharmonic function $u \in \mathcal{B}^+(X)$ to the special one considered in the previous section we first prove the following.

**LEMMA 4.1.** *Let $F \in \mathcal{B}$, $w \in \mathcal{E}_{\mathfrak{X}}$ and $\eta \in (0,1)$ such that*

$$\inf w(F) > 0 \quad \text{and} \quad \hat{P}_F w \le \eta w \quad \text{on } F. \tag{4.1}$$

*Then $P^x\big(\bigcap_{n \ge 0} [S_n^F < \infty]\big) = 0$ for every $x \in \{w < \infty\}$.*

*Proof.* Obviously, $\hat{P}_F w \le \eta w$ on $F \cup b(F)$ and $a := \inf w(F \cup b(F)) = \inf w(F) > 0$. By induction, $(\hat{P}_F)^n w \le \eta^n w$ on $F \cup b(F)$ for every $n \ge 0$. Let $\Omega_0 := \bigcap_{n \ge 0} [S_n^F < \infty]$. Then, for all $x \in X$ and $n \ge 0$,

$$a P^x(\Omega_0) \le E^x(w \circ X_{S_n^F}) = P_F (\hat{P}_F)^n w(x) \le (\hat{P}_F)^n w(x) \le \eta^n w(x),$$

and hence $P^x(\Omega_0) = 0$ if $w(x) < \infty$. □

**PROPOSITION 4.2.** *Let $u \in \mathcal{N} \cap \mathcal{B}(X)$ with $\inf u(X) > 0$. Further, let $\eta \in (0,1)$,*

$$F := \{\hat{u} < \eta u\} \quad \text{and} \quad v := 1_F u + 1_{X \setminus F} \hat{u}.$$

*Then $\eta u \le v \le u$, $v \in \mathcal{N} \cap \mathcal{B}(X)$ and $v$ satisfies Assumption 3.1.*

*Proof.* Of course, $\hat{u} \le v \le u$, hence $\hat{v} = \hat{u}$ and $v \in \mathcal{N}$, by [11, Proposition 2.2]. Clearly, $F = \{\hat{v} < v\}$ and $\inf \hat{v}(X) = \inf u(X) > 0$.

The set $A := \{\hat{u} \le \eta u\}$ containing $F$ is finely closed, since $u$ is finely upper semicontinuous; see Lemma 2.5. Hence, for every $x \in X$, the measure $P_F(x, \cdot)$ is supported by $A$, and therefore

$$P_F \hat{u}(x) \le \eta P_F u(x) \le \eta u(x).$$

By regularization, $\hat{P}_F \hat{u} \le \eta \hat{u}$, that is, $\hat{P}_F \hat{v} \le \eta \hat{v}$. If $x \in \{v < \infty\}$, then $\hat{v}(x) < \infty$, and $P^x\big(\bigcap_{n \ge 0} [S_n^F < \infty]\big) = 0$, by Proposition 4.1 (applied to $w := \hat{v}$). □

Let us say that a function $u \in \mathcal{N}$ has the *finiteness property* (FP) if, for every $x \in X$ with $u(x) < \infty$, there exists a function $w \in \mathcal{E}_{\mathfrak{X}}$ such that $w = \infty$ on $\{u = \infty\}$ and $w(x) < \infty$. Trivially, every $u \in \mathcal{N}$ with $u < \infty$ has this property (take $w = 0$).

**THEOREM 4.3.** *Let $u \in \mathcal{N}$ be Borel measurable satisfying* (FP). *Then $u = R_u$.*

*Proof.* Let $x \in X$, $\eta \in (0,1)$ and $\varepsilon > 0$. Of course, $u+\varepsilon \in \mathcal{N}$ and $u+\varepsilon$ satisfies (FP). By Proposition 4.2, there exists $v \in \mathcal{N} \cap \mathcal{B}(X)$ satisfying Assumption 3.1 and such that $\eta(u+\varepsilon) \le v \le u+\varepsilon$. We choose $w_1 \in \mathcal{E}_{\mathfrak{X}}$ such that $w_1 = \infty$ on $\{u = \infty\}$ and $w_1(x) < \varepsilon$. Then $v + w_1 \in \mathcal{R}(\{\hat{v} < v\})$, by Corollary 3.8. Thus, by (3.4), $\eta R_u \le R_{v+w_1} = v + w_1 \le u + \varepsilon + w_1$. In particular, $\eta R_u(x) \le u(x) + 2\varepsilon$. □

The following consequence of Theorem 4.3 may be surprising. Its combination with Theorem 4.3 establishes the implication (1) ⇒ (2) in Theorem 1.1.

**COROLLARY 4.4.** *Every Borel measurable $u \in \mathcal{N}$ has the property* (FP).

*Proof.* Let $u \in \mathcal{N}$ be Borel measurable, $x \in X$ with $u(x) < \infty$ and $E := \{u = \infty\}$. Clearly, $1_E = 1 \wedge \inf_{n\in\mathbb{N}}(u/n) \in \mathcal{N}$. By Theorem 4.3, there are $w_n \in \mathcal{E}_{\mathfrak{X}}$, $n \in \mathbb{N}$, such that

$$w_n \ge 1_E \quad \text{and} \quad w_n(x) < 2^{-n}.$$

Then $w := \sum_{n\ge1} w_n \in \mathcal{E}_{\mathfrak{X}}$, $w = \infty$ on $E$ and $w(x) < 1$. □

To prove the additional statement in Theorem 1.1, let us consider $\varphi \in \mathcal{B}^+(X)$ and recall that $N_\varphi := \inf\{u \in \mathcal{N} \colon u \ge \varphi\}$ is the smallest nearly hyperharmonic majorant of $\varphi$, $N_\varphi = \varphi \vee \hat{N}_\varphi \in \mathcal{B}^+(X)$ (see [11, Proposition 2.4]), and hence

$$N_\varphi = \inf\{w \in \mathcal{E}_{\mathfrak{X}} \colon w \ge N_\varphi\} \ge \inf\{w \in \mathcal{E}_{\mathfrak{X}} \colon w \ge \varphi\} = R_\varphi \ge N_\varphi.$$

# 5 The remaining part of Theorem 1.1

For a proof of the implication (1) ⇒ (3) in Theorem 1.1 we note that, for every $u \in \mathcal{N} \cap \mathcal{B}(X)$, the set $\{\hat{u} < u\}$ is semipolar (which, for example, follows from $u = \inf\{w \in \mathcal{E}_{\mathfrak{X}} \colon w \ge u\}$) and that every semipolar Borel set is the union of compacts $K_n$, $n \in \mathbb{N}$, and a polar set $E \in \mathcal{B}$; see [11, Proposition 5.2] in connection with Remark 6.4 below. We start with a lemma on compact sets which will quickly lead to the basic approximation result in Corollary 5.4.

**LEMMA 5.1.** *Let $K$ be a compact in $X$ and let $w$ be a bounded function in $\mathcal{E}_{\mathfrak{X}}$. Then there exists a decreasing sequence $(V_n)$ of finely open Borel sets containing $K$ such that $P_K w = \inf_{n\in\mathbb{N}} P_{V_n} w$.*

*Proof.* By [3, VI.1.2 (and its proof)], $P_K w$ is the infimum of all functions $P_V w$, where $V$ is a finely open Borel set containing $K$. By a topological lemma of Choquet (see [3, I.1.8]), there is a sequence $(V_n)$ of such sets satisfying

$$\hat{P}_K w = \widehat{\inf_{n\in\mathbb{N}} P_{V_n}} w.$$

Fixing a decreasing sequence $(U_n)$ of open sets in $X$ with $\bigcap_{n\in\mathbb{N}} \overline{U}_n = K$, we may assume without loss of generality that $V_{n+1} \subset V_n \subset U_n$ for every $n \in \mathbb{N}$. Then, by

[3, VI.2.6], every function $P_{V_n}w$, $n \in \mathbb{N}$, is harmonic on $X \setminus \overline{U}_n$. Hence $\inf_{n\in\mathbb{N}} P_{V_n}w$ is harmonic on $X \setminus K$, by [3, III.3.1], and we obtain that

$$P_K w = \hat{P}_K w = \inf P_{V_n} w \quad \text{ on } X \setminus K.$$

The proof is completed observing that $P_K w = w = \inf P_{V_n} w$ on $K$. □

**PROPOSITION 5.2.** *Let $A$ be the union of compacts $K_n$ in $X$, $n \in \mathbb{N}$, and a polar set $E \in \mathcal{B}$. Further, let $w \in \mathcal{E}_{\mathfrak{X}}$ and $\mu \in \mathcal{M}(X)$ be such that $w$ is bounded and $\mu(X) < \infty$. Then*

$$\int P_A w \, d\mu = \inf\{\int P_V w \, d\mu \colon A \subset V,\ V \textit{ finely open Borel}\}.$$

*Proof.* Of course, we may suppose that the sequence $(K_n)$ is increasing. Moreover, we may assume that $\mu(A) = 0$, since $P_A w = w = P_V w$, whenever $A \subset V \subset X$.

Let us fix $\varepsilon > 0$. Since $T_E = \infty$ a.s., we have $P_E 1 = 0$ on $X \setminus E$. Hence, by [3, VI.1.9], there exists an open neighborhood $U$ of $E$ such that $\int P_U 1 \, d\mu < \varepsilon$. Moreover, by Lemma 5.1, there exist finely open $V_n \in \mathcal{B}$, $n \in \mathbb{N}$, such that

$$K_n \subset V_n \quad \text{ and } \quad \int P_{V_n} w \, d\mu < \int P_{K_n} w \, d\mu + 2^{-n}\varepsilon.$$

Defining $W_n := V_1 \cup \cdots \cup V_n$ and proceeding as in the proof of [3, VI.1.4] we get

$$\int P_{W_n} w \, d\mu \le \int P_K w \, d\mu + (1 - 2^{-n})\varepsilon \qquad \text{for every } n \in \mathbb{N}.$$

Let $W := \bigcup_{n\ge 1} W_n$ and $V := W \cup U$. Then $P_V w \le P_W w + P_U w$ and $P_{W_n} w \uparrow P_W w$, by [3, VI.1.7]. Thus we finally conclude that $\int P_V w \, d\mu \le \int P_K w \, d\mu + 2\varepsilon$. □

**REMARK 5.3.** In [13, p. 138] such a result is shown for sets $A$ which are *strictly thin*, that is, satisfy $\hat{P}_A 1 < \eta$ on $A$ for some $\eta \in (0, 1)$, proving first the following stunning approximation of the hitting time $T_A$: For every probability measure $\nu$ on $X$ not charging $A$, there exists a decreasing sequence $(V_n)$ of finely open sets containing $A$ such that $\lim_{n\to\infty} P^\nu[T_{V_n} < T_A] = 0$; cf. [12, Propositions 5.1 and 5.2].

**COROLLARY 5.4.** *Let $u \in \mathcal{N} \cap \mathcal{B}(X)$, $v \in \mathcal{R}(\{\hat{u} < u\})$ and let $\mu \in \mathcal{M}(X)$ be a finite measure. Then*

$$\int v \, d\mu := \inf\{\int w \, d\mu \colon w \in \mathcal{E}_{\mathfrak{X}},\ w \ge v\}. \tag{5.1}$$

*Proof.* By (2.1), $P_V w \in \mathcal{E}_{\mathfrak{X}}$ for all finely open sets $V \in \mathcal{B}$ and $w \in \mathcal{E}_{\mathfrak{X}}$. Thus (5.1) follows immediately from Proposition 5.2 and Lemma 2.2. □

**DEFINITION 5.5.** *For every Borel measurable $u \in \mathcal{N}$, let $\mathcal{M}_u(X)$ denote the set of all $\mu \in \mathcal{M}(X)$ such that $u$ is $\mu$-integrable and $\mu(A) + \int_{X\setminus A} w \, d\mu < \infty$ for some Borel set $A$ in $X$ and majorant $w \in \mathcal{E}_{\mathfrak{X}}$ of $u$.*

Let us say that a Borel measurable $u \in \mathcal{N}$ has the *finiteness property* (FP′) if, for every $\mu \in \mathcal{M}_u$, there exists a function $w \in \mathcal{E}_{\mathfrak{X}}$ with $w = \infty$ on the set $\{u = \infty\}$ and $\int w \, d\mu < \infty$. Trivially, every $u \in \mathcal{N}$ with $u < \infty$ has this property (take $w = 0$).

**THEOREM 5.6.** *Let $u \in \mathcal{N}$ be Borel measurable and $\mu \in \mathcal{M}_u(X)$. If $u < \infty$ or, more generally, if $u$ has the property* (FP′), *then*

$$\int u\,d\mu = \inf\{\int w\,d\mu \colon w \in \mathcal{E}_{\mathfrak{X}},\ w \ge u\}.$$

*Proof.* Let $\eta \in (0,1)$ and $\varepsilon > 0$. Assuming that $u$ has the property (FP′), we choose $w_1 \in \mathcal{E}_{\mathfrak{X}}$ such that $w_1 = \infty$ on $\{u = \infty\}$ and $\int w_1\,d\mu < \varepsilon$. By Remark 1.2, there exists $A \in \mathcal{B}$ and $w_0 \in \mathcal{E}_{\mathfrak{X}}$ such that $w_0 \ge u$ and $\mu(A) < \infty$, $\int_{X\setminus A} w_0\,d\mu < \varepsilon$. We fix $\delta > 0$ such that $\delta\mu(A) < \varepsilon$, and define

$$\nu := 1_A\mu \quad\text{ and }\quad u_1 := u + \delta.$$

By Proposition 4.2 and Corollary 3.8, there exists a Borel measurable $v \in \mathcal{N}$ such that $\eta u_1 \le v \le u_1$ and $v + w_1 \in \mathcal{R}(\{\hat{v} < v\})$. Then, by Corollary 5.4,

$$\begin{aligned}\int u\,d\mu + 2\varepsilon &> \int (u_1 + w_1)\,d\nu \ge \int (v + w_1)\,d\nu \\ &= \inf\{\int w\,d\nu \colon w \in \mathcal{E}_{\mathfrak{X}},\ w \ge v + w_1\} \ge \eta \inf\{\int w\,d\nu \colon w \in \mathcal{E}_{\mathfrak{X}},\ w \ge u\}.\end{aligned}$$

So there exists $w \in \mathcal{E}_{\mathfrak{X}}$ such that $w \ge u$ and $\eta \int w\,d\nu < \int u\,d\mu + 2\varepsilon$. We may assume without loss of generality that $w \le w_0$. Then

$$\eta \int w\,d\mu < \eta \int w\,d\nu + \varepsilon < \int u\,d\mu + 3\varepsilon.$$

Letting $\varepsilon$ tend to 0 and $\eta$ tend to 1 the proof is completed. $\square$

**COROLLARY 5.7.** *Every Borel measurable $u \in \mathcal{N}$ has the property* (FP′).

*Proof.* Let $u \in \mathcal{N}$ be Borel measurable, $\mu \in \mathcal{M}_u(X)$ and $E := \{u = \infty\}$. In particular, $\int u\,d\mu < \infty$, and hence $\mu(E) = 0$. Obviously, $1_E = 1 \wedge \inf_{n\in\mathbb{N}}(u/n) \in \mathcal{N}$. Hence, by Theorem 5.6, there exist functions $w_n \in \mathcal{E}_{\mathfrak{X}}$, $n \in \mathbb{N}$, such that

$$w_n \ge 1_E \quad\text{ and }\quad \int w_n\,d\mu < \int 1_E\,d\mu + 2^{-n} = 2^{-n}.$$

Then $w := \sum_{n\ge 1} w_n \in \mathcal{E}_{\mathfrak{X}}$, $w = \infty$ on $E$ and $\int w\,d\mu < 1$. $\square$

Combining Corollary 5.7 with Theorem 5.6 we obtain the implication (1) $\Rightarrow$ (3) in Theorem 1.1.

# 6 Weakening of the measurability assumption

In this section we shall consider nearly hyperharmonic functions which may not be Borel measurable. Let $\mathcal{B}^*$ be the $\sigma$-algebra of all ($\mathcal{B}$-)universally measurable sets and, as in [11], let $\tilde{\mathcal{B}}, \widetilde{\mathcal{B}^*}$ respectively denote the $\sigma$-algebra of all sets $A$ in $X$ for which there exists a set $B$ in $\mathcal{B}, \mathcal{B}^*$ respectively such that the symmetric difference $A \triangle B$ is polar, that is, $\hat{P}_{A\triangle B} 1 = 0$.

Let us observe that, for functions $u \geq 0$ which are $\widetilde{\mathcal{B}^*}$-measurable, the upper integral in (1.2) may be replaced by the integral, since the measures $P_{V^c}(x,\cdot) = \varepsilon_x^{V^c}$, $x \in V \in \mathcal{U}_c$, do not charge polar sets; see [3, VI.5.6].

Let $f$ be a positive function on $X$ which is $\tilde{\mathcal{B}}$-measurable. Since every polar set is contained in a Borel polar set and every countable union of polar sets is polar, there exist $f_1 \in \mathcal{B}^+(X)$ and $f_2 \in \mathcal{B}^+(X)$ such that the set $\{f_2 > 0\}$ is polar and $f_1 \leq f \leq f_1 + f_2$.

The following result extends the implication (1) $\Rightarrow$ (3) of Theorem 1.1 to functions which are nearly Borel measurable, that is, $\tilde{\mathcal{B}} \cap \mathcal{B}^*$-measurable.

**THEOREM 6.1.** *Let $u$ be a nearly Borel measurable function in $\mathcal{N}$. Then*

$$\int u\,d\mu = \inf\{\int w\,d\mu \colon w \in \mathcal{E}_{\mathfrak{X}},\ w \geq u\}$$

*for every $\mu \in \mathcal{M}(X)$ such that $\mu(A) + \int_{X\setminus A} w\,d\mu < \infty$ for some $A \in \mathcal{B}$ and some majorant $w \in \mathcal{E}_{\mathfrak{X}}$ of $u$.*

*Proof.* There exist $u_1, u_2 \in \mathcal{B}^+(X)$ such that $u_1 \leq u \leq u_1 + u_2$ and the set $\{u_2 > 0\}$ is polar. Of course, we may assume that $\hat{u} \leq u_1$. Let us fix $\mu \in \mathcal{M}(X)$ such that $\mu(A) + \int_{X\setminus A} w\,d\mu < \infty$ for some $A \in \mathcal{B}$ and majorant $w \in \mathcal{E}_{\mathfrak{X}}$ of $u$. Choosing $v_1, v_2 \in \mathcal{B}^+(X)$ such that $v_1 \leq u \leq v_1 + v_2$ and $v_2 = 0$ $\mu$-a.e., we may assume that $u_1 \leq v_1$ and $v_2 \leq u_2$. Then $v_1 \in \mathcal{N}$, by [11, Proposition 2.2]. Since $\{v_2 > 0\}$ is polar, we know that $v_2 \in \mathcal{N}$, and hence $v := v_1 + v_2 \in \mathcal{N}$. Thus, by Theorem 1.1,

$$\begin{aligned}\int u\,d\mu = \int v\,d\mu &= \inf\{\int w\,d\mu \colon w \in \mathcal{E}_{\mathfrak{X}}, w \geq v\} \\ &\geq \inf\{\int w\,d\mu \colon w \in \mathcal{E}_{\mathfrak{X}}, w \geq u\} \geq \int u\,d\mu.\end{aligned}$$

□

For the implication (1) $\Rightarrow$ (2) we even have the following.

**THEOREM 6.2.** *Let $u$ be a $\tilde{\mathcal{B}}$-measurable function in $\mathcal{N}$. Then*

$$u = \inf\{w \in \mathcal{E}_{\mathfrak{X}} \colon w \geq u\}.$$

*Proof.* Let us fix $x \in X$. There exist $v_1, v_2 \in \mathcal{B}^+(X)$ such that $v_1 \leq u \leq v_1 + v_2$ and the set $\{v_2 > 0\}$ is polar whence $v_2 \in \mathcal{N}$. Of course, we may assume that $\hat{u} \leq v_1$, $v_1(x) = u(x)$ and $v_2(x) = 0$. By [11, Proposition 2.2], $v_1 \in \mathcal{N}$. So $v := v_1 + v_2 \in \mathcal{N}$ and $R_v = v$, by Theorem 1.1. Thus $R_u(x) \leq R_v(x) = v(x) = u(x) \leq R_u(x)$. □

Using results of [11, Section 4] this leads to a characterization of the equality $R_u = u$ for nearly hyperharmonic $\widetilde{\mathcal{B}^*}$-measurable functions. To this end we recall that the $\sigma$-algebra of all finely Borel subsets of $X$ (that is, the smallest $\sigma$-algebra on $X$ containing all finely open sets) is the smallest $\sigma$-algebra containing $\mathcal{B}$ and all semipolar sets; see [11, Section 5]. In particular, $\tilde{\mathcal{B}} \subset \mathcal{B}^f$.

**THEOREM 6.3.** *Let $u \in \mathcal{N}$ and suppose that $u$ is $\widetilde{\mathcal{B}^*}$-measurable. Then the following statements are equivalent:*

(i) $u = \inf\{w \in \mathcal{E}_{\mathfrak{X}} \colon w \geq u\}$.

(ii) *$u$ is finely upper semicontinuous.*

(iii) *$u$ is finely Borel measurable.*

(iv) *$u$ is $\tilde{\mathcal{B}}$-measurable.*

(v) *The set $\{\hat{u} < u\}$ is semipolar.*

*Proof.* Trivially, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Moreover, (iii) $\Leftrightarrow$ (v) and (iii) $\Rightarrow$ (iv), by [11, Proposition 5.3 and Corollary 5.4]. By Theorem 6.2, (iv) $\Rightarrow$ (i). $\square$

Clearly, previous statements on reduced functions $R_\varphi$ can now be extended to functions $\varphi \geq 0$ on $X$ which are only supposed to be $\mathcal{B}^f \cap \widetilde{\mathcal{B}^*}$-measurable.

**REMARK 6.4.** The result [11, Corollary 5.4] relies on [11, Proposition 5.2] the proof of which uses [8, Theorem 1.5] stating that, given a semipolar set $S$, there exists a measure $\mu$ on $X$ such that $\mu^*(B) > 0$ for every non-polar subset $B$ of $S$. This is correct; its proof, however, is not, since [8, Lemma 1.3] is wrong.

Assuming without loss of generality that $S$ is the union of totally thin Borel sets $F_n$, $n \in \mathbb{N}$, we many obtain a valid proof exhausting each $F_n$ by sets

$$F_{n,m} := F_n \cap K_m \cap \{\hat{P}_{F_n} q < \eta_m q\}, \qquad m \in \mathbb{N},$$

where $q$ is a continuous strict potential on $X$, $(K_m)$ is a sequence of compacts in $X$ and $\eta_m \in (0,1)$ such that $K_m \uparrow X$ and $\eta_m \uparrow 1$ as $m \to \infty$.

Indeed, let us fix $x \in X$ and $m, n \in \mathbb{N}$. Let $F := F_{n,m}$, $\eta := \eta_{n,m}$. We recursively define measures $\mu_k$ on $F$ taking $\mu_0 := \varepsilon_x$ and

$$\mu_k := \hat{\mu}_{k-1}^F := \int \hat{\varepsilon}_y^F \, d\mu_{k-1}(y), \qquad k \in \mathbb{N}.$$

Then $\int q \, d\mu_k = \int \hat{P}_F q \, d\mu_{k-1} \leq \eta \int q \, d\mu_{k-1}$, and hence $\inf q(F) \cdot \mu_k(F) \leq \eta^k q(x)$ for every $k \in \mathbb{N}$. So $\lim_{k\to\infty} \mu_k = 0$ which leads to a proof of [8, Theorem 1.5] not using [8, Lemma 1.3].

# 7 Application of our method to general standard processes

It should be clear to the experts that our approach works as well (at least) for general standard processes as studied in [4] provided we assume that, for the potential kernel $V := \int_0^\infty P_t \, dt$, there exist bounded $\mathcal{B}^*$-measurable functions $h_n$ on $X$ such that $V h_n$ is bounded for every $n \in \mathbb{N}$ and $V h_n \uparrow \infty$; see [4, III.6.12]. A suitable set $\mathcal{N}$ of functions could then be the set of all nearly Borel measurable functions $u \geq 0$ on $X$ satisfying $P_T u \leq u$ for all strong terminal times $T$; see [4, pp. 78 and 124]. The lower semicontinuous regularization $\hat{u}$ of $u$ (which for a nearly hyperharmonic function in our setting of a nice Hunt process is the greatest excessive minorant of $u$) would have to be replaced by $\tilde{u} := \lim_{t\to0} P_t u$, which for supermedian functions $u$ is the greatest excessive minorant.

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