Bond Pricing under Knightian Uncertainty: A Short Rate Model with Drift and Volatility Uncertainty

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Abstract

It is shown how to construct an arbitrage-free short rate model under uncertainty about the drift and the volatility. The uncertainty is represented by a set of priors, which naturally leads to a G-Brownian motion. Within this framework, it is shown how to characterize the whole term structure without admitting arbitrage. The pricing of zero-coupon bonds in such a setting differs substantially from the traditional models, since the prices need to be chosen in a different way in order to exclude arbitrage.

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1 Introduction

Nowadays, there is an increasing literature on robust finance, examining financial markets under uncertainty by exploring new ways of describing the randomness influencing those markets. The problem is that most of the traditional financial models fail to represent the uncertainty, since they are based on very strong probabilistic assumptions. This makes it easy to price contracts on financial quantities, but the models neglect the fact that the evolution of those quantities is actually unknown and does not follow any probabilistic law.

In theory, there are several different ways to model a financial market. Many models represent the evolution of asset prices by stochastic processes while the interest rate is assumed to induce a risk-free alternative investment and hence, it is considered to be deterministic or even to be constant. Black and Scholes (1973) is the most famous and fundamental one that goes in this direction. Those models are important for pricing and hedging derivatives on asset prices, for example. Since the market also offers contracts on interest rates, like bonds or interest rate derivatives, there are also models dealing with the fact that the evolution of the interest rate is uncertain. Adapting the methods of Black and Scholes (1973) to the interest rate world, Vasicek (1977) is one of the most important short rate models, i. e., a model where the instantaneous spot rate is represented by a certain diffusion process. That paper shows that the absence of arbitrage implies that the discounted bond prices are martingales with respect to some equivalent martingale measure by introducing the market price of risk. Compared to Black and Scholes (1973), the equivalent martingale measure is not unique and has to be given from the outside.

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Furthermore, an Ornstein-Uhlenbeck process with constant parameters is used in that model to describe the evolution of the short rate. Although it makes the model analytically tractable, it has the undesirable feature of the possibility to obtain a negative interest rate. This is the motivation of Cox, Ingersoll Jr, and Ross (1985) to use different dynamics for the short rate ensuring the positivity of the interest rate. However, it makes the model less tractable. The interesting thing is that those models have an endogenous nature, since the current term structure is meant to be an output of the model. On the other hand, there are also exogenous models including time dependent parameters, which is more realistic and allows the theoretical bond prices to fit the observed prices on the market. Hull and White (1990) is the most famous one, since it generalizes the earlier mentioned models. A much more detailed overview of the main short rate models can be found, for example, in Rogers (1995), in Chapter 3 in Brigo and Mercurio (2007), or in Chapter 5 in Filipovic (2009).

In practice, those models are very popular, since they are simple and handy when it comes to pricing bonds or interest rate derivatives. This is caused by the assumptions on the dynamics of the short rate, which determine the probabilistic law of the interest rate. Although it makes the application of the model very easy, this is problematic, since it is not realistic. Besides, the parameters in the dynamics, that characterize the distribution of the short rate, have to be chosen in a reasonable way. Because of the incompleteness of bond markets, one usually models the dynamics of the short rate directly under the equivalent martingale measure, commonly known as martingale modeling. Hence, not all of the parameters can be obtained by historical data, which can already be quite imprecise, but by inverting the yield curve. This has usually the advantage of an exact fit to the observed data, but is based on some questionable assumptions like differentiability of the yield curve. Indeed, this is not a smooth function. Additionally, it is doubtful to assume that the parameters matching the theoretical to the real prices today are stable enough to describe the evolution of the interest rate in the future. Therefore, the aim is to construct a short rate model with as less as possible assumptions on the parameters of the model. In particular, the parameters should be characterized by uncertainty apart from a probabilistic framework.

In general, this kind of uncertainty is called "model uncertainty" or "Knightian uncertainty". Introduced in Knight (2012), it is considered to be the counterpart of risk, because risk can be measured by a probability in contrast to uncertainty. This uncertainty should be used when events in reality are too complex or the related data is missing to calculate the risk of such an event. So it can be helpful in financial models, since there is usually no reason to believe that the underlying quantities have a certain distribution. On the other hand, this is difficult to model, because it is not amenable to the usual stochastic calculus. Thus, there should be a mathematical theory describing uncertainty about stochastic processes.

Fortunately, there is a recent developement of a whole stochastic calculus describing uncertainty about the drift and the volatility of stochastic processes, which can be found in Peng (2010). Using nonlinear expectations, this theory generalizes the usual probability space to define a generalized normal distribution including uncertainty about the mean and the variance. With this distribution, it is possible to construct processes where the average behaviour and the volatility are uncertain. Furthermore, a stochastic integral and a version of Itô's formula can be obtained within this scope. For the application to finance, it is also worth mentioning that the notion of martingales in this framework has slightly different properties than in the usual one, which are very important for financial markets. Some further results, like the connection to probability theory, a generalized Itô formula, or a Grisanov transform, can be found in Denis, Hu, and Peng (2011), Li and Peng (2011), and Hu, Ji, Peng, and Song (2014).

The general idea of Knightian uncertainty is already applied to asset markets in Avellaneda, Levy, and Parás (1995) and Lyons (1995) or, more recently, in Epstein and Ji (2013) and Vorbrink (2014). All of them are considering volatility uncertainty, but the latter is the only one incorporating a detailed discussion about the concept of arbitrage within this framework. For more general market models and, especially, the concept of no-arbitrage, one may refer to Beißner (2013), Bouchard and Nutz (2015), or Burzoni, Riedel, and Soner (2017). Additionally, there are also models featuring an ambiguous interest rate, like El Karoui and Ravanelli (2009) or Lin and Riedel (2014). However, there are only a few models about pricing bonds or, in general, contracts on the interest rate, which simultaneously allow for an uncertain interest rate. On the one hand, there is Epstein and Wilmott (1999). They recognize the problem of the accurate estimation of parameters in interest rate models and introduce an interest rate model with no underlying probabilistic assumptions. Instead, they use uncertainty to describe the evolution of the short rate. This has the advantage that it is not necessary to characterize the interest rate by any probabilistic law. Apart from that, there is no need to estimate parameters, since there are only bounds for them that have to be known. Hence, it is convenient to use a worst and best case measure to value contracts on the interest rate. This typically leads to a range of values for the contract from the worst to the best, which is similar to the earlier mentioned models about volatility uncertainty. Unfortunately, the discussion about the absence of arbitrage is, like all of the derivations in the model, very intuitive and less mathematical. On the other hand, there is Avellaneda and Lewicki (1996). They adapt the principle of volatility uncertainty to construct an interest rate model that also leads to a range of values. However, the absence of arbitrage is treated in a very intuitive way, too. Hence, the aim of this paper is to set up a short rate model including parameter uncertainty, which also relaxes the probabilistic assumptions of the model, in a mathematically correct way by using the theory of G-Brownian motion from Peng (2010). Within this scope, it is shown how to characterize the whole term structure in order to exclude arbitrage. Indeed, there is a special focus on the concept of arbitrage, since this differs from the classical framework and many models dealing with Knightian uncertainty skip this part.

The paper is organized as follows. The three succeeding sections focus on volatility uncertainty, whereas the uncertainty about the drift of the short rate is incorporated afterwards. First of all, there is a section about the general set up of the model, which includes a construction of the set of beliefs and the characterization of the short rate. After this section, it is possible to deal with the pricing of bonds. This causes some problems, resulting in a change of the dynamics of the short rate. Then one is able to provide bond prices which are not admitting arbitrage. However, the discussion about the market structure and the absence of arbitrage is done separately in a succeding section. Afterwards, there is the just mentioned extension to drift uncertainty to get rid of the choice of parameters. The paper finishes with a conclusion at the end.

2 Model Framework

The fundamental of the model is the short rate, which corresponds to the instantaneous spot interest rate. This is modeled by a diffusion process and is used to determine all of the bond prices, i. e., the whole term structure. However, we additionally want to incorporate volatility uncertainty, which is represented by a set of beliefs. Introduced in Peng (2010), this leads to a sublinear expectation or, more detailed, to a G-expectation space and a G-Brownian motion by the paper Denis, Hu, and Peng (2011), which examines the connection between probability theory and the G-framework. In particular, it enables us to use the G-Brownian motion to describe the behavior of the short rate and, especially, the uncertainty about its volatility.

2.1 Set of Beliefs

Since the agents are uncertain about the volatility of the short rate, we need to consider more than one measure, where each measure should represent a different volatility. The construction of such a set of measures, which is known as the set of beliefs or set of priors, is done similar to Vorbrink (2014). In particular, this set can be used to define a sublinear expectation under which the canonical process becomes a G-Brownian motion.

Let $(\Omega, \mathcal{F}, P)$ be a probability space such that $\Omega := C_0\big([0,T]\big)$, $\mathcal{F} := \mathcal{B}(\Omega)$, and P is the Wiener measure. Furthermore, we denote by $(W_t)_t$ a standard Wiener process on $(\Omega, \mathcal{F}, P)$ and we define the canonical process

$$B_t(\omega) := \omega_t.$$

Additionally, $(\mathcal{F}_t)_t$ is chosen to be the filtration generated by $(B_t)_t$ completed by all P-null sets, i. e.,

$$\mathcal{F}_t := \sigma(\omega_s | 0 \leq s \leq t) \vee \mathcal{N}.$$

The state space for the volatility is given by $\Theta := [\underline{\sigma}, \overline{\sigma}]$, where $\overline{\sigma} \geq \underline{\sigma} > 0$, and the collection of all Θ-valued, progressively measurable processes $\theta = (\sigma_t)_t$ is denoted by $\mathcal{A}^\Theta$, which represents the set of possible volatilities. So we only assume that there is an upper and a lower bound for the volatility without specifying how the volatility process should behave. Now under each measure the canonical process should have a different volatility. Hence, we define the measure

$$P^\theta := P \circ \Big(\int_0^T \sigma_t dW_t \Big)^{-1}$$

for all $\theta \in \mathcal{A}^\Theta$ and denote by $\mathcal{P}$ the closure of $\{P^\theta | \theta \in \mathcal{A}^\Theta\}$ under the topology of weak convergence, which is the set of all beliefs. Finally, we can define a sublinear expectation $\hat{\mathbb{E}}$ as the upper expectation of the set of beliefs $\mathcal{P}$, i. e.,

$$\hat{\mathbb{E}}(X) := \sup_{P^\theta \in \mathcal{P}} \mathbb{E}_{P^\theta}(X)$$

for all measurable X such that $\mathbb{E}_{P^\theta}(X)$ exists for all $P^\theta \in \mathcal{P}$. In particular, $\hat{\mathbb{E}}$ can be understood as a risk measure, since it satisfies the same conditions as a coherent risk measure. For example, if X is some financial loss, then $\hat{\mathbb{E}}$ yields the highest expected loss. However, according to Denis, Hu, and Peng (2011), $\hat{\mathbb{E}}$ corresponds to the G-expectation on $L_G^1(\Omega)$ and the canonical process $(B_t)_t$ is a G-Brownian motion on the G-expectation space $(\Omega, L_G^1(\Omega), \hat{\mathbb{E}})$. The letter G stands for a sublinear function, which is, in this case, given by

$$G(a) = \tfrac{1}{2} \sup_{\sigma \in \Theta} \{\sigma^2 a\}$$

for $a \in \mathbb{R}$. So now we can use $(B_t)_t$ to represent the volatility uncertainty, since the G-Brownian motion has no mean uncertainty but variance uncertainty, i. e.,

$$\hat{\mathbb{E}}(B_t) = 0 = -\hat{\mathbb{E}}(-B_t)$$

and

$$\hat{\mathbb{E}}(B_t^2) = \overline{\sigma}^2 t > \underline{\sigma}^2 t = -\hat{\mathbb{E}}(-B_t^2).$$

Thus, the canonical process can evolve with a volatility being at most $\overline{\sigma}$ and at least $\underline{\sigma}$. Besides, it is important in this framework to introduce the terminology of "quasi surely", which generalizes the notion of "almost surely", since we are dealing with more than one measure. We say that a set $A \in \mathcal{F}$ happens quasi surely if

$$P^\theta(A) = 1 \quad \text{for all } P^\theta \in \mathcal{P}.$$

This means that a set happens quasi surely if it happens almost surely under all beliefs. In particular, all equations in the following should be understood to hold quasi surely. Furthermore, we also use the terminology "$\mathcal{P}$-quasi surely" if we need to indicate under which set of measures the statement holds quasi surely.

2.2 The Short Rate

After the construction of the set of beliefs, we are able to use the G-Brownian motion to characterize the behaviour of the short rate. In general, we want to model it as a diffusion process as it is the common practice in short rate models. Also the structure is chosen to be similar to the classical or traditional models.

In most short rate models, the spot interest rate is modeled as a mean reverting process, since this is reasonable for representing the interest rate. Vasicek (1977), for example, assumes that the short rate evolves as an Ornstein-Uhlenbeck process with constant parameters. This approach is extended by Hull and White (1990) to time dependent parameters, which has the advantage to be more realistic and to admit a better fit to the yield curve. However, the parameters are still deterministic for tractability reasons, which is a quite undesirable feature of the model. Nevertheless, we want to have the same structure, but want to get around the problem of parameter estimation. Hence, the short rate process $(r_t)_t$ is supposed to be given by the stochastic differential equation

$$r_t = r_0 + \int_0^t (\mu_s - \alpha r_s) ds + B_t$$

for some bounded, deterministic function μ and some constant $\alpha > 0$, where $(B_t)_t$ is the G-Brownian motion. So all in all, the model can be seen as a generalization of the Hull and White extended Vasicek model, since the dynamics are the same apart from the uncertain parameters. At this point, the mean reversion level is still time dependent and deterministic, which is generalized in Section 5, and the volatility is time dependent and uncertain. This is an important difference, since it is even more general than assuming a stochastic volatility coefficient. In particular, by Peng (2010), the stochastic differential equation has a unique solution in $\bar{M}_G^2(0,T) \subseteq M_G^2(0,T)$, given by

$$r_t = e^{-\alpha t} r_0 + \int_0^t e^{-\alpha(t-s)} \mu_s ds + \int_0^t e^{-\alpha(t-s)} dB_s.$$

This can be verified by using the Itô formula for G-Brownian motion from Li and Peng (2011), which is done in Appendix A. So after all, we manage to define the short rate without any assumptions on its volatility. As we see, the distribution of the short rate has a deterministic mean but the variance is uncertain. However, we also examine the case of an uncertain mean in the end.

3 Bond Pricing

The next step is to deal with the problem of pricing bonds such that the related market is arbitrage-free, i. e., determining an arbitrage-free term structure. First of all, we show why the usual approach from the traditional models fails, which implies that it is not possible to obtain such a term structure under the given set of beliefs. This leads to an extension of the sublinear expectation space in order to change the dynamics of the short rate. Although it has a more technical reason, there is also a very intuitive interpretation for this. All in all, it enables us to find no-arbitrage prices for bonds.

3.1 Common Approach

A standard result of the theory of financial markets is that a market is arbitrage-free if (and only if) the traded quantities on the market are martingales under a measure equivalent to the real world measure. Indeed, the common practice in short rate models is martingale modeling, since bond markets are incomplete. This means that there is not a unique martingale measure but many of them. Thus, one usually supposes that the short rate satisfies certain dynamics under a given martingale measure. Then the bond prices are chosen such that the discounted bond prices are martingales with respect to the exogenously given measure in order to exclude arbitrage.

In the case of a set of beliefs, we have to choose a different approach. General results on pricing under a set of beliefs can be found, for example, in Bouchard and Nutz (2015). In this framework, the absence of arbitrage is equivalent to the existence of a set of martingale measures which is in some sense equivalent to the set of priors. In fact, it is sufficient to find a set of equivalent measures such that the discounted bond price is a martingale under all equivalent measures. This is actually equivalent to being a symmetric G-martingale under some suitable sublinear expectation. In particular, one may think of the upper expectation of the set of equivalent measures. However, since it is popular to model the short rate directly under the martingale measure, we start with a different approach. We try to find bond prices in such a way that they are martingales with respect to all measures in the initially given set of beliefs, i. e., symmetric G-martingales under the initially given sublinear expectation. So this corresponds to the martingale modeling approach, although we actually want to assume that the set of priors is the set of real world measures. Nevertheless, it shows that we are forced to change the dynamics of the short rate and hence, the set of measures.

Let us consider a bond market where the price of a zero-coupon bond, which yields a payoff of 1 at the maturity τ, at time t is given by $P(t,\tau)$ for $t \leq \tau \leq T$. Furthermore, we define the discounted bond price

$$\tilde{P}(t,\tau) := e^{-\int_0^t r_s ds} P(t,\tau).$$

The aim is to choose the price $P(t,\tau)$ such that the discounted bond price process $\big(\tilde{P}(t,\tau)\big)_t$ is a martingale under all $P^\theta \in \mathcal{P}$, which is equivalent to being a symmetric G-martingale under $\hat{\mathbb{E}}$, and satisfies the terminal condition $P(\tau,\tau) = 1$. Indeed, it turns out that this does not work.

First of all, we like to do the standard approach of reducing the problem to solving a partial differential equation. Let us assume that the bond price is a function with a certain structure and then apply the Itô formula to it. Afterwards, we get some conditions on the bond P if we want to represent it as an integral with respect to the G-Brownian motion, i. e., as a symmetric G-martingale. So we suppose that $P(t,\tau)$ is a function depending on the current time and the current value of the short rate. Hence, it holds

$$P(t,\tau) = f(t, r_t)$$

for some function $f \in C^{1,2}\big((0,\tau) \times \mathbb{R}\big)$. Then, by the Itô formula, we get

$$\begin{aligned} e^{-\int_0^t r_s ds} f(t,r_t) = & f(0,r_0) + \int_0^t e^{-\int_0^s r_u du} \partial_r f(s,r_s) dB_s \\ & + \int_0^t e^{-\int_0^s r_u du} \Big(\partial_t f(s,r_s) + \partial_r f(s,r_s) \big(\mu_s - \alpha r_s\big) - f(s,r_s) r_s \Big) ds \\ & + \tfrac{1}{2} \int_0^t e^{-\int_0^s r_u du} \partial_r^2 f(s,r_s) d\langle B \rangle_s. \end{aligned}$$

Thus, since we want the terms in the last two lines to vanish, we need to find a function solving the partial differential equations

$$\begin{aligned}\partial_t f(t,r) + \big(\mu_t - \alpha r\big)\partial_r f(t,r) - r f(t,r) =& 0\\ \tfrac{1}{2}\partial_r^2 f(t,r) =& 0\end{aligned}$$

for all $(t,r) \in (0,\tau) \times \mathbb{R}$ with respect to the boundary condition

$$f(\tau, r) = 1$$

for all $r \in \mathbb{R}$. From the second partial differential equation, we can deduce that the function f is linear in the second variable, i. e.,

$$f(t,r) = g(t) + h(t)r.$$

Hence, the first partial differential equation now reads as follows:

$$g'(t) + \mu_t h(t) + \big(h'(t) - \alpha h(t) - g(t)\big)r - h(t)r^2 = 0.$$

Since this has to hold for all values of r, we need to have

$$\begin{aligned}g'(t) + \mu_t h(t) =& 0\\ h'(t) - \alpha h(t) - g(t) =& 0\\ h(t) =& 0.\end{aligned}$$

Therefore, f is zero everywhere, which contradicts to the boundary condition. So this approach actually fails.

Henceforth, we have to doubt if there exists a solution to the initial problem. Indeed, we can use the probabilistic approach of pricing zero-coupon bonds to show that there is no solution for the given set of priors at all. So let us assume that $\big(\tilde{P}(t,\tau)\big)_t$ is a symmetric G-martingale or, equivalently, a martingale under all $P^\theta \in \mathcal{P}$. Thus, it holds

$$\tilde{P}(t,\tau) = \mathbb{E}_{P^\theta}\big(\tilde{P}(\tau,\tau)\big|\mathcal{F}_t\big) = \mathbb{E}_{P^\theta}\big(e^{-\int_0^\tau r_s ds}\underbrace{P(\tau,\tau)}_{=1}\big|\mathcal{F}_t\big)$$

for all $P^\theta \in \mathcal{P}$. Hence, we get

$$P(t,\tau) = \mathbb{E}_{P^\theta}(e^{-\int_t^\tau r_s ds}|\mathcal{F}_t)$$

for all $P^\theta \in \mathcal{P}$. However, the right-hand side is the bond price in the Hull and White extended Vasicek model if we consider some deterministic volatility $\theta = (\sigma_t)_t \in \mathcal{A}^\Theta$. Those prices are given by

$$\mathbb{E}_{P^\theta}(e^{-\int_t^\tau r_s ds}|\mathcal{F}_t) = e^{A(t,\tau) - B(t,\tau)r_t},$$

where

$$B(t,\tau) = \tfrac{1}{\alpha}(1 - e^{-\alpha(\tau - t)})$$

and

$$A(t,\tau) = \int_t^\tau \Big(\tfrac{1}{2}\sigma_s^2 B(s,\tau)^2 - \mu_s B(s,\tau)\Big)ds.$$

So in particular, we get

$$P(t,\tau) = e^{A(t,\tau)-B(t,\tau)r_t}$$

for all deterministic volatilities $\theta \in \mathcal{A}^{\Theta}$. Now it is easy to see that this can not hold. The reason is that we can, for example, consider two different constant volatilities. Then it is obvious that the prices are not the same, since the value of $A(t,\tau)$ changes and the rest stays the same. So we get a contradiction. This shows us that there is no possibility to choose the discounted bond price as a symmetric G-martingale. Hence, we need to consider a different set of measures, which actually corresponds to changing the dynamics of the short rate.

Remark 3.1 *The letters A and B are later used to define other functions. Here they are used for the convenience of the reader, since most of the standard literature agrees on using the letters A and B for those functions. To avoid confusion, it is clearly explained on which one we refer when comparing the expressions for the bond prices.*

3.2 Extended $\tilde{G}$-Expectation Space

In the following, we change the dynamics of the short rate process. For this purpose, we extend the sublinear expectation space to use a certain Girsanov transformation. The extension is essentially based on adding another source of randomness to the model to construct a 2-dimensional G-Brownian motion by the same procedure as in the beginning. Then we are able to use the results from Hu, Ji, Peng, and Song (2014), which are including a Girsanov transformation for G-Brownian motion. Afterwards, it is possible to change the dynamics.

Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ be a probability space such that $\tilde{\Omega} := C_0\big([0,T],\mathbb{R}^2\big)$, $\tilde{\mathcal{F}} := \mathcal{B}(\tilde{\Omega})$, and $\tilde{P}$ is the Wiener measure on $\tilde{\Omega}$. Furthermore, we denote by $(\tilde{W}_t)_t$ a 2-dimensional Wiener process on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and we define the canonical process

$$(B_t, \tilde{B}_t)(\tilde{\omega}) := \tilde{\omega}_t.$$

Additionally, let $(\tilde{\mathcal{F}}_t)_t$ be the filtration generated by $(B_t, \tilde{B}_t)_t$ completed by all $\tilde{P}$-null sets. The state space of the volatility is now given by

$$\tilde{\Theta} := \Big\{\gamma \in \mathbb{R}^{2\times 2} \Big| \gamma_{11} = \gamma_{21}^{-1} = \sigma \text{ and } \gamma_{12} = \gamma_{22} = 0 \text{ for } \sigma \in \Theta\Big\},$$

which is a non-empty, bounded and closed subset of $\mathbb{R}^{2\times 2}$. Again, we denote by $\mathcal{A}^{\tilde{\Theta}}$ the collection of all $\tilde{\Theta}$-valued, progressively measurable processes $\tilde{\theta} = (\tilde{\theta}_t)_t$ and define the measure

$$P^{\tilde{\theta}} := \tilde{P} \circ \Big(\int_0^T \tilde{\theta}_t d\tilde{W}_t\Big)^{-1}$$

for all $\tilde{\theta} \in \mathcal{A}^{\tilde{\Theta}}$. Furthermore, we denote by $\tilde{\mathcal{P}}$ the closure of $\{P^{\tilde{\theta}} | \tilde{\theta} \in \mathcal{A}^{\tilde{\Theta}}\}$ under the topology of weak convergence and we define $\hat{\mathbb{E}}^{\tilde{G}}$ as the upper expectation of $\tilde{P}$, i. e.,

$$\hat{\mathbb{E}}^{\tilde{G}}(X) := \sup_{P^{\tilde{\theta}} \in \tilde{\mathcal{P}}} \mathbb{E}_{P^{\tilde{\theta}}}(X)$$

for all $\tilde{\mathcal{F}}$-measurable X such that $\mathbb{E}_{P^{\tilde{\theta}}}(X)$ exists for all $P^{\tilde{\theta}} \in \tilde{\mathcal{P}}$. Then $\hat{\mathbb{E}}^{\tilde{G}}$ corresponds to the $\tilde{G}$-expectation on $L^1_{\tilde{G}}(\tilde{\Omega})$ by Denis, Hu, and Peng (2011) and $(B_t, \tilde{B}_t)_t$ is a $\tilde{G}$-Brownian motion on the $\tilde{G}$-expectation space $(\tilde{\Omega}, L^1_{\tilde{G}}(\tilde{\Omega}), \hat{\mathbb{E}}^{\tilde{G}})$. In this case, the sublinear function $\tilde{G}$ is given by

$$\tilde{G}(A) = \tfrac{1}{2} \sup_{\gamma \in \tilde{\Theta}} \operatorname{tr}(\gamma\gamma^T A) = \tfrac{1}{2} \sup_{\vartheta \in [\underline{\sigma}^2, \overline{\sigma}^2]} \operatorname{tr}\Big(A \begin{pmatrix} \vartheta & 1 \\ 1 & \vartheta^{-1} \end{pmatrix}\Big),$$

where $A \in \mathbb{S}_2$, which is the space of all symmetric 2×2 matrices. Hence, the $\tilde{G}$-expectation space $(\tilde{\Omega}, L^1_{\tilde{G}}(\tilde{\Omega}), \hat{\mathbb{E}}^{\tilde{G}})$ corresponds to the extended $\tilde{G}$-expectation space in Hu, Ji, Peng, and Song (2014), which enables us to use their results on G-backward stochastic differential equations and, especially, the Girsanov transformation for G-Brownian motion. Furthermore, we know that the two sublinear expectations are equal to each other on the initial domain, i. e.,

$$\hat{\mathbb{E}}(\xi) = \hat{\mathbb{E}}^{\tilde{G}}(\xi)$$

for all $\xi \in L^1_G(\Omega)$. Now we consider the G-backward stochastic differential equation

$$Y_t = \xi + \int_t^T \lambda_s Z_s ds - \int_t^T Z_s dB_s - (K_T - K_t),$$

where $\xi \in L^\beta_G(\Omega)$ and $(\lambda_t)_t$ is a bounded process in $M^\beta_G(0, T)$ with $\beta > 1$. For further details on G-backward stochastic differential equations like, for example, the definition of K, the interested reader may refer to Hu, Ji, Peng, and Song (2014). Here we only want to use the Girsanov transformation without giving further insights. In particular, we know that the solution is given by

$$Y_t = \hat{\mathbb{E}}^{\tilde{G}}_t\big(\mathcal{E}^{\tilde{B}}_t \xi\big),$$

where $\hat{\mathbb{E}}^{\tilde{G}}_t$ is the conditional $\tilde{G}$-expectation and

$$\mathcal{E}^{\tilde{B}}_t := \exp\Big(\int_t^T \lambda_s d\tilde{B}_s - \tfrac{1}{2}\int_t^T \lambda_s^2 d\langle \tilde{B}\rangle_s\Big).$$

Besides, we can define a sublinear expectation $\tilde{\mathbb{E}}$ by

$$\tilde{\mathbb{E}}(\xi) := Y_0$$

for all $\xi \in L^\beta_G(\Omega)$ with $\beta > 1$. In the expression for $(Y_t)_t$, we can see that $\tilde{\mathbb{E}}$ basically corresponds to an upper expectation of a set of equivalent measures, since the exponential can be seen as a Radon-Nikodym derivative. Furthermore, we know that

$$\bar{B}_t := B_t - \int_0^t \lambda_s ds$$

is a G-Brownian motion under $\tilde{\mathbb{E}}$. Thus, the dynamics of the short rate can now be written as

$$r_t = r_0 + \int_0^t \big(\mu_s - \alpha r_s + \lambda_s\big) ds + \bar{B}_t$$

for a G-Brownian motion $(\bar{B}_t)_t$. At this point, it is important to mention that such a process $(\lambda_t)_t$ is also used in Vasicek (1977). It is also obtained by a Girsanov transformation and is called the market price of risk, since it measures how much better we are doing with a bond compared to investing in the short rate per one unit of risk. It is also important to mention that actually $(-\lambda_t)_t$ corresponds to the market price of risk, because the process has a different sign in the dynamics compared to the Vasicek model. In particular, we use

$$\lambda_t := \int_0^t e^{-2\alpha(t-s)} d\langle B\rangle_s = \int_0^t e^{-2\alpha(t-s)} d\langle \bar{B}\rangle_s$$

in the following, which corresponds to the variance of the Ornstein-Uhlenbeck process in the usual framework. The reason for this is twofold. On the one hand, this is a technical reason, since this certain expression enables us to find no-arbitrage prices in the following. On the other hand, it is quite reasonable to choose the negative variance as a market price of risk, because the variance measures the risk and actually, it is more risky to invest in the short rate than investing in the bond in this model. Another important observation is that the variance process, as it depends on the quadratic variation of the G-Brownian motion, does not only bear the risk but also the uncertainty. So one may also refer to $(-\lambda_t)_t$ as the market price of uncertainty. However, this is also technically admissible, since the process $\big(\int_0^t e^{-2\alpha(t-s)} d\langle B\rangle_s\big)_t$ is quasi surely bounded. So after all, we get

$$r_t = r_0 + \int_0^t \Big(\mu_s - \alpha r_s + \int_0^s e^{-2\alpha(s-u)} d\langle \bar{B}\rangle_u\Big) ds + \bar{B}_t$$

for the dynamics of the short rate.

3.3 Bond Prices

After changing the dynamics, we can continue searching for bond prices. Now, since the prices should be martingales under a set of equivalent measures, we want the bonds to be symmetric G-martingales under the new sublinear expectation $\tilde{\mathbb{E}}$. In particular, it is sufficient to find a representation as an integral with respect to a G-Brownian motion, which corresponds to the (failing) approach in the beginning of this section. However, this approach is successful under the sublinear expectation $\tilde{\mathbb{E}}$, as we are dealing with a different G-Brownian motion $(\bar{B}_t)_t$. The next lemma shows how to chose the bond price to find such a representation. Hence, we are able to exclude arbitrage, but this is done in the next section. Indeed, the expression for the bond prices are quite similar to the prices from the traditional models, although they have some important different properties.

Lemma 3.1 *Let the price of a bond with maturity τ at time t for $t \leq \tau \leq T$ be given by*

$$P(t,\tau) = \exp\Big(A(t,\tau) - B(t,\tau) r_t - \tfrac{1}{2} B(t,\tau)^2 \lambda_t\Big),$$

where A and B are functions satisfying the ordinary differential equations

$$\begin{aligned} A'(t,\tau) &= \mu_t B(t,\tau) \\ A(\tau,\tau) &= 0 \end{aligned}$$

and

$$\begin{aligned} B'(t,\tau) &= \alpha B(t,\tau) - 1 \\ B(\tau,\tau) &= 0. \end{aligned}$$

Then it holds

$$\tilde{P}(t,\tau) = \tilde{P}(0,\tau) - \int_0^t B(s,\tau)\tilde{P}(s,\tau) d\bar{B}_s$$

and

$$P(\tau,\tau) = 1.$$

The proof is just an application of the Itô formula, since the assumptions on the bond price and the dynamics of the functions A and B are ensuring that the drift terms vanish.

Proof: First of all, define the processes

$$\Lambda_t := \int_0^t e^{2\alpha s} d\langle \bar{B} \rangle_s$$

and

$$R_t := \int_0^t r_s ds.$$

Then the discounted bond price is given by

$$\begin{aligned}\tilde{P}(t,\tau) =& e^{-\int_0^t r_s ds} P(t,\tau) \\ =& \exp\Big(A(t,\tau) - B(t,\tau) r_t - \tfrac{1}{2} B(t,\tau)^2 e^{-2\alpha t} \Lambda_t - R_t \Big),\end{aligned}$$

where $(r_t)_t$, $(\Lambda_t)_t$, and $(R_t)_t$ are processes satisfying the dynamics

$$\begin{aligned} r_t =& r_0 + \int_0^t \Big(\mu_s - \alpha r_s + e^{-2\alpha s} \Lambda_s \Big) ds + \bar{B}_t \\ \Lambda_t =& \Lambda_0 + \int_0^t e^{2\alpha s} d\langle \bar{B} \rangle_s \\ R_t =& R_0 + \int_0^t r_s ds. \end{aligned}$$

Now we can define the function $f : [0,\tau] \times \mathbb{R}^3 \to \mathbb{R}$ by

$$f(t,x) := \exp\Big(A(t,\tau) - B(t,\tau) x_1 - \tfrac{1}{2} B(t,\tau)^2 e^{-2\alpha t} x_2 - x_3 \Big)$$

in order to apply Itô to $f(t, X_t)$, where $(X_t) = (r_t, \Lambda_t, R_t)_t$. Then we get

$$\begin{aligned} f(t,X_t) =& f(0,X_0) + \int_0^t \partial_{x_1} f(s,X_s) d\bar{B}_s \\ &+ \int_0^t \Big(\underbrace{\partial_t f(s,X_s) + \partial_{x_1} f(s,X_s) \Big(\mu_s - \alpha r_s + e^{-2\alpha s} \Lambda_s \Big) + \partial_{x_3} f(s,X_s) r_s}_{=:\Delta^1(s)} \Big) ds \\ &+ \int_0^t \Big(\underbrace{\partial_{x_2} f(s,X_s) e^{2\alpha s} + \tfrac{1}{2} \partial^2_{x_1} f(s,X_s)}_{=:\Delta^2(s)} \Big) d\langle \bar{B} \rangle_s. \end{aligned}$$

Plugging in the derivatives of f yields

$$\begin{aligned}\frac{\Delta^1(t)}{f(t,X_t)} =& A'(t,\tau) - B'(t,\tau)r_t - B(t,\tau)B'(t,\tau)e^{-2\alpha t}\Lambda_t + B(t,\tau)^2 e^{-2\alpha t}\alpha\Lambda_t \\ & - B(t,\tau)\Big(\mu_t - \alpha r_t + e^{-2\alpha t}\Lambda_t\Big) - r_t \\ =& \Big(\underbrace{A'(t,\tau) - \mu_t B(t,\tau)}_{=0}\Big) - \Big(\underbrace{B'(t,\tau) - \alpha B(t,\tau) + 1}_{=0}\Big) r_t \\ & - B(t,\tau)e^{-2\alpha t}\Big(\underbrace{B'(t,\tau) - \alpha B(t,\tau) + 1}_{=0}\Big)\Lambda_t \\ =& 0\end{aligned}$$

and

$$\frac{\Delta^2(t)}{f(t,X_t)} = -\tfrac{1}{2}B(t,\tau)^2 e^{-2\alpha t}e^{2\alpha t} + \tfrac{1}{2}B(t,\tau)^2 = 0.$$

Hence, it holds

$$\tilde{P}(t,\tau) = \tilde{P}(0,\tau) - \int_0^t B(s,\tau)\tilde{P}(s,\tau)d\bar{B}_s,$$

since

$$\tilde{P}(t,\tau) = f(t,X_t).$$

Apart from that, it is easy to see that P satisfies the terminal condition. □

Now we are able to compare the expression for the bond prices with the one from the classical model without volatility uncertainty, which is cited in Subsection 3.1. First of all, the solutions to the ordinary differential equations in Lemma 3.1 are given by

$$B(t,\tau) = \tfrac{1}{\alpha}(1 - e^{-\alpha(\tau - t)})$$

and

$$A(t,\tau) = -\int_t^\tau \mu_s B(s,\tau)ds.$$

Thus, we see that the function B is the same in both cases, but now the squared term is missing in A. However, it corresponds somehow to the additional term in the exponential of the bond price depending on the market price of uncertainty, although they are not the same if we would drop the uncertainty about the volatility.

The most important implications of the prices in this model are as follows. Primarily, we do not have to estimate the volatility for the evolution of the short rate in the future. Admittedly, there is a price we have to pay for that. We have to know the evolution of the G-Brownian motion in the past, since the bond price depends on the quadratic variation of the G-Brownian motion. So this would result in estimating the quadratic variation of the short rate from the past, which, in particular, can be done. Alternatively, one could also estimate the variance of the short rate as a proxy for the market price of uncertainty. Furthermore, there is still the mean reversion level to be chosen, which can be seen either as a good or bad thing. On the one hand, this might be interesting for practical reasons, since it allows to precisely fit the bond prices to the current yield curve. On the other hand, we want to get rid of the parameter estimation. Therefore, we extend the model to uncertainty about the mean reversion level in Section 5.

4 The Bond Market

Before extending the model, we still need to show that the bond market is arbitrage-free with respect to the prices from the previous section, since the possibility to make arbitrage is a sign of mispricing on the market. In order to show this, it is necessary to analyze the corresponding structure of the bond market. Afterwards, it is possible to define the notion of arbitrage and to finally show that there is no arbitrage strategy in the market.

4.1 Market Structure

In this subsection, we want to set up the structure of the bond market. Then we can define the standard notions of the theory of financial markets. Those are including portfolio values of market strategies and the important property of a market strategy being self-financing.

The market in this model basically consists of two investment opportunities. The first one is to invest in the short rate, which is represented by the money-market account

$$M_t = e^{\int_0^t r_s ds}$$

for $t \in [0,T]$. Besides, the market provides bonds for all maturities within the time horizon. In particular, the price of such a bond with maturity $\tau \in [0,T]$ at time t is given by the expression from Section 3. Additionally, we assume that the value of the bond after the maturity grows with the short rate. This can be seen as investing in the money-market account after receiving the payoff. Hence, we have

$$P(t,\tau) = \begin{cases} \exp\left(A(t,\tau) - B(t,\tau)r_t - \frac{1}{2}B(t,\tau)^2\lambda_t\right) & ,t \in [0,\tau] \\ \exp\left(\int_\tau^t r_s ds\right) & ,t \in [\tau,T] \end{cases}$$

However, we want to use the money-market account $(M_t)_t$ as a numéraire in the following, which is the common practice in models of financial markets. So we restrict to the discounted bond prices

$$\tilde{P}(t,\tau) = e^{-\int_0^t r_s ds} P(t,\tau).$$

It is worth to mention that, according to the assumption from above, the value of the discounted bond price is remaining constant after the maturity.

Now the agents can participate in the market by buying and selling a finite number of discounted bonds over the time horizon. This means that they can choose a finite number of bonds and decide on how much of them they want to buy or sell to create a portfolio.

Definition 4.1 *A **market strategy** (π,τ) is a stochastic process*

$$\pi = (\pi_t^1, ..., \pi_t^n)_t$$

with values in $\mathbb{R}^n$ such that $\pi_t^i \tilde{P}(t,\tau_i) \in L_G^2(\Omega_t)$ for all $t \in [0,T]$ for $i = 1,...,n$ and a partition

$$\tau = (\tau_1, ..., \tau_n)$$

*such that $0 < \tau_1 \leq ... \leq \tau_n \leq T$ for $n < \infty$. The corresponding (discounted) **portfolio value** at time $t \in [0,T]$ is given by*

$$\tilde{v}_t(\pi,\tau) = \sum_{i=1}^n \pi_t^i \tilde{P}(t,\tau_i).$$

Furthermore, we want to restrict ourselves to the class of self-financing strategies. This property ensures that no money is added to or taken from the portfolio.

Definition 4.2 *A marktet strategy (π, τ) is* ***self-financing*** *if $\left(\pi_t^i \tilde{P}(t, \tau_i)\right)_t \in M_G^2(0, T)$ for all $i \in \{1, ...n\}$ and*

$$\tilde{v}_t(\pi, \tau) - \tilde{v}_0(\pi, \tau) = \sum_{i=1}^{n} \int_0^t \pi_s^i d\tilde{P}(s, \tau_i)$$

for all $t \in [0, T]$.

Thus, the difference between the portfolio values consists of the so-called gains from trade that can be found on the right-hand side of the equation in the definition. The assumption $\left(\pi_t^i \tilde{P}(t, \tau_i)\right)_t \in M_G^2(0, T)$ is a technical one ensuring that the integrals are well-defined by using the integral representation of the discounted bond price $\tilde{P}$ in Lemma 3.1.

4.2 No-Arbitrage

Now, as the most important concepts regarding the structure of a bond market are defined, we can move on examining if there are any arbitrage opportunities hiding in the market. So first of all, we introduce the notion of arbitrage in this framework and then show that the market is arbitrage-free.

The concept of arbitrage intuitively describes the fact of making something out of nothing or, more detailed, getting the chance to win money without any risk of losing some. Since we are dealing with more than one measure, we need to consider a definition slightly different from the classical one. However, this is the one chosen in other models using a set of priors, like Vorbrink (2014).

Definition 4.3 *A market strategy (π, τ) is an* ***arbitrage*** *if it is self-financing and it holds*

(i)

$$\tilde{v}_0(\pi, \tau) \leq 0,$$

(ii)

$$\tilde{v}_T(\pi, \tau) \geq 0 \quad \mathcal{P}\text{-quasi surely},$$

(iii)

$$P^\theta\Big(\tilde{v}_T(\pi, \tau) > 0\Big) > 0 \quad \text{for at least one } P^\theta \in \mathcal{P}.$$

So this is actually a weaker version than requiring that the strategy has to be an arbitrage in the classical sense with respect to all measures, since the probability of a strictly positive win has not to be strictly positive under each measure. Therefore, the following result should not be surprising, as there are, intuitively speaking, a set of equivalent measures and prices such that the discounted bonds are martingales under all of those measures.

Proposition 4.1 *Let the bond prices be given by the expression from above. Then the bond market is arbitrage-free, i. e., there is no market strategy which is an arbitrage.*

The proof works like the "easy" direction of the fundamental theorem of asset pricing in the single measure setting. We assume that there exists an arbitrage and then use the martingale property of the bond prices to get a contradiction.

Proof: First of all, let us suppose that there is arbitrage. Then there exists a self-financing strategy (π, τ) satisfying

$$\tilde{v}_0(\pi, \tau) \leq 0,$$

$$\tilde{v}_T(\pi, \tau) \geq 0 \quad \mathcal{P}\text{-quasi surely,}$$

and

$$P^\theta\Big(\tilde{v}_T(\pi, \tau) > 0\Big) > 0 \quad \text{for at least one } P^\theta \in \mathcal{P}.$$

Hence, there exists a measure $P^\theta \in \mathcal{P}$ such that

$$0 < \mathbb{E}_{P^\theta}\Big(\tilde{v}_T(\pi, \tau) - \tilde{v}_0(\pi, \tau)\Big) \leq \hat{\mathbb{E}}\Big(\tilde{v}_T(\pi, \tau) - \tilde{v}_0(\pi, \tau)\Big).$$

Indeed, we know that $\hat{\mathbb{E}}$ and $\hat{\mathbb{E}}^{\tilde{G}}$ are the same on $L_G^2(\Omega)$ and, in particular, that the difference of the portfolio values is an element of $L_G^2(\Omega)$. Hence, it holds

$$\hat{\mathbb{E}}^{\tilde{G}}\Big(\tilde{v}_T(\pi, \tau) - \tilde{v}_0(\pi, \tau)\Big) > 0.$$

Furthermore, we can deduce the same inequality for $\tilde{\mathbb{E}}$, since

$$\mathcal{E}_0^{\tilde{B}} = \exp\Big(\int_0^T \lambda_s d\tilde{B}_s - \tfrac{1}{2}\int_0^T \lambda_s d\langle\tilde{B}\rangle_s\Big) > 0 \quad \tilde{\mathcal{P}}\text{-quasi surely.}$$

Thus, we have

$$\tilde{\mathbb{E}}\Big(\tilde{v}_T(\pi, \tau) - \tilde{v}_0(\pi, \tau)\Big) = \hat{\mathbb{E}}^{\tilde{G}}\Big(\mathcal{E}_0^{\tilde{B}}\big(\tilde{v}_T(\pi, \tau) - \tilde{v}_0(\pi, \tau)\big)\Big) > 0.$$

Besides, we know that (π, τ) is self-financing and $\tilde{\mathbb{E}}$ is sublinear. So we get

$$\tilde{\mathbb{E}}\Big(\tilde{v}_T(\pi, \tau) - \tilde{v}_0(\pi, \tau)\Big) = \tilde{\mathbb{E}}\Big(\sum_{i=1}^n \int_0^T \pi_t^i d\tilde{P}(t, \tau_i)\Big) \leq \sum_{i=1}^n \tilde{\mathbb{E}}\Big(\int_0^T \pi_t^i d\tilde{P}(t, \tau_i)\Big).$$

Now we can use the fact that the discounted bond price is constant after the maturity and the representation from Lemma 3.1 to deduce

$$\tilde{\mathbb{E}}\Big(\int_0^T \pi_t^i d\tilde{P}(t, \tau_i)\Big) = \tilde{\mathbb{E}}\Big(-\int_0^{\tau_i} \pi_t^i B(t, \tau_i)\tilde{P}(t, \tau_i) d\bar{B}_t\Big)$$

for all $i \in \{1, ..., n\}$. However, the integrand is an element of the space $M_G^2(0, T)$ by assumption and $(\bar{B}_t)_t$ is a G-Brownian motion under $\tilde{\mathbb{E}}$. Hence, we know that the expression on the right-hand side is zero, which is a standard result of the calculus from Peng (2010). So we can deduce

$$\tilde{\mathbb{E}}\Big(\int_0^T \pi_t^i d\tilde{P}(t, \tau_i)\Big) = 0$$

for all $i \in \{1, ..., n\}$. In the end, this yields the contradiction. □

Now, as the market is arbitrage-free, we formally know that the bond prices from Lemma 3.1 are applicable to the market.

5 Drift Uncertainty

As it is mentioned above, we finally want to extend the model to uncertainty about the mean reversion level of the short rate in addition to the volatility uncertainty, since the mean reversion level is still a parameter that needs to be chosen. For this purpose, we could replace the integral of the mean reversion level by the quadratic variation process of the G-Brownian motion, since it has mean uncertainty. However, this would imply that the evolution of the mean reversion level and the quadratic variation of the short rate are the same, which is not really satisfying. Therefore, we need to construct a new set of beliefs to add a new source of uncertainty. Unlike the previous discussion, we do not show how to change the dynamics of the short rate, but directly suppose that it is evolving under the modified dynamics, which are adjusted by the market price of uncertainty. However, it is only shown how to price the bonds such that the discounted bond can be represented as an integral with respect to a G-Brownian motion without the discussion about the market structure and no-arbitrage, since this is very similar to the previous case.

5.1 Set of Beliefs

In addition to the G-Brownian motion, we need another process representing the uncertainty about the drift or, in this particular case, about the mean reversion level of the short rate. So it is necessary to construct a new set of beliefs in order to add another source of uncertainty to the model.

Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ be the probability space from Subsection 3.2, i. e., $\tilde{\Omega} := C_0\big([0,T], \mathbb{R}^2\big)$, $\tilde{\mathcal{F}} := \mathcal{B}(\tilde{\Omega})$, and $\tilde{P}$ is the Wiener measure on $\tilde{\Omega}$. Furthermore, we denote by $(\tilde{W}_t)_t$ a 2-dimensional Wiener process on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and we define the canonical process

$$(B_t, \tilde{B}_t)(\tilde{\omega}) := \tilde{\omega}_t.$$

Additionally, $(\tilde{\mathcal{F}}_t)_t$ is chosen to be the filtration generated by $(B_t, \tilde{B}_t)_t$ completed by all $\tilde{P}$-null sets. The state space is now given by

$$\Theta^* := \Big\{\gamma \in \mathbb{R}^{2\times 2} \Big| \gamma_{11} = \mu, \gamma_{22} = \sigma \text{ and } \gamma_{12} = \gamma_{21} = 0 \text{ for } (\mu^2, \sigma) \in [\underline{\mu}, \overline{\mu}] \times \Theta\Big\},$$

which is a non-empty, bounded and closed subset of $\mathbb{R}^{2\times 2}$, and the collection of all Θ^*-valued, progressively measurable processes $\theta^* = (\theta^*_t)_t$ is denoted by $\mathcal{A}^{\Theta^*}$. Now we define the measure

$$P^{\theta^*} := \tilde{P} \circ \Big(\int_0^T \theta^*_t \tilde{W}_t\Big)^{-1}$$

for all $\theta^* \in \mathcal{A}^{\Theta^*}$ and denote by $\mathcal{P}^*$ the closure of $\{P^{\theta^*} | \theta^* \in \mathcal{A}^{\Theta^*}\}$ under the topology of weak convergence. Then we can define $\mathbb{E}^*$ as the upper expectation of $\mathcal{P}^*$, i. e.,

$$\mathbb{E}^*(X) := \sup_{P^{\theta^*} \in \mathcal{P}^*} \mathbb{E}_{P^{\theta^*}}(X)$$

for all measurable X such that $\mathbb{E}_{P^{\theta^*}}(X)$ exists for all $P^{\theta^*} \in \mathcal{P}^*$. Now $(B_t, \tilde{B}_t)_t$ becomes a 2-dimensional G^*-Brownian motion under $\mathbb{E}^*$, where

$$G^*(A) = \tfrac{1}{2} \sup_{\gamma \in \Theta^*} \operatorname{tr}(\gamma\gamma^T A).$$

So we can use the G-Brownian motion $(\tilde{B}_t)_t$, as in the previous model, to represent the volatility uncertainty and the quadratic variation process $(\langle B\rangle_t)_t$ to model the mean reversion level of the short rate, since it has mean uncertainty, i. e.,

$$\mathbb{E}^*\big(\langle B\rangle_t\big) = \overline{\mu}t > \underline{\mu}t = -\mathbb{E}^*\big(-\langle B\rangle_t\big).$$

Thus, the process $(\langle B\rangle_t)_t$ grows with a rate that is at most $\overline{\mu}$ and at least $\underline{\mu}$. Besides, one should note that the equations in the following hold $\mathcal{P}^*$-quasi surely.

5.2 Bond Prices

After constructing the general framework, we are able to incorporate the drift uncertainty into the short rate dynamics. Thus, we quickly introduce the short rate dynamics in this subsection and the adjustment with respect to the market price of uncertainty. Furthermore, we show how to price bonds in a manner that is similar to Subsection 3.3.

Now the introduction of the quadratic variation process $(\langle B\rangle_t)_t$ leads to the dynamics

$$r_t = r_0 + \langle B\rangle_t - \int_0^t \alpha r_s ds + \tilde{B}_t$$

for the short rate, since we are uncertain about the mean reversion level. The solution is then given by

$$r_t = e^{-\alpha t} r_0 + \int_0^t e^{-\alpha(t-s)} d\langle B\rangle_s + \int_0^t e^{-\alpha(t-s)} d\tilde{B}_s.$$

This can be verified by the Itô formula, which is shown in Appendix A. However, we directly want to suppose, without changing the dynamics, that the short rate is given by

$$r_t = r_0 + \langle B\rangle_t + \int_0^t \big(-\alpha r_s + \lambda_s\big) ds + \tilde{B}_t,$$

where $(-\lambda_t)_t$ is again representing the market price of uncertainty. Therefore, there is the question on how to choose this quantity. In this case, the uncertainty is not completely characterized by the variance of the short rate,

$$\lambda_t^2 := \int_0^t e^{-2\alpha(t-s)} d\langle \tilde{B}\rangle_s,$$

as in the previous model. Hence, we need to add something representing the uncertainty about the mean, which is given by

$$\lambda_t^1 := \int_0^t e^{-\alpha(t-s)} d\langle B\rangle_s,$$

i. e., the uncertain part of the mean value of the short rate. Thus, we set

$$\lambda_t := \lambda_t^1 + \lambda_t^2$$

and henceforth suppose that the short rate evolves according to the dynamics

$$r_t = r_0 + \langle B\rangle_t + \int_0^t \Big(-\alpha r_s + \int_0^s e^{-\alpha(s-u)} d\langle B\rangle_u + \int_0^s e^{-2\alpha(s-u)} d\langle \tilde{B}\rangle_u\Big) ds + \tilde{B}_t.$$

With this particular choice, we can provide an expression for bond prices ensuring that the bond market is arbitrage-free.

Lemma 5.1 *Let the price of a bond with maturity τ at time t for $t \leq \tau \leq T$ be given by*

$$P(t,\tau) = \exp\Big(B(t,\tau)\lambda_t^1 - B(t,\tau)r_t - \tfrac{1}{2}B(t,\tau)^2\lambda_t^2\Big),$$

where B is a function satisfying the ordinary differential equation

$$\begin{aligned} B'(t,\tau) &= \alpha B(t,\tau) - 1 \\ B(\tau,\tau) &= 0. \end{aligned}$$

Then it holds

$$\tilde{P}(t,\tau) = \tilde{P}(0,\tau) - \int_0^t B(s,\tau)\tilde{P}(s,\tau)d\tilde{B}_s$$

and

$$P(\tau,\tau) = 1.$$

The proof is again based on the application of the Itô formula and some rearrangement, although the expressions are looking a little different as we have the additional source of uncertainty.

Proof: First of all, define the processes

$$\Lambda_t^1 := \int_0^t e^{\alpha s} d\langle B\rangle_s,$$

$$\Lambda_t^2 := \int_0^t e^{2\alpha s} d\langle \tilde{B}\rangle_s,$$

and

$$R_t := \int_0^t r_s ds.$$

Then the discounted bond prices are given by

$$\begin{aligned} \tilde{P}(t,\tau) =& e^{-\int_0^t r_s ds} P(t,\tau) \\ =& \exp\Big(B(t,\tau)e^{-\alpha t}\Lambda_t^1 - B(t,\tau)r_t - \tfrac{1}{2}B(t,\tau)^2 e^{-2\alpha t}\Lambda_t^2 - R_t\Big), \end{aligned}$$

where $(r_t)_t$, $(\Lambda_t^1)_t$, $(\Lambda_t^2)_t$, and $(R_t)_t$ are processes satisfying the dynamics

$$\begin{aligned} r_t =& r_0 + \langle B\rangle_t + \int_0^t \Big(-\alpha r_s + e^{-\alpha s}\Lambda_s^1 + e^{-2\alpha s}\Lambda_s^2\Big)ds + \tilde{B}_t \\ \Lambda_t^1 =& \Lambda_0^1 + \int_0^t e^{\alpha s} d\langle B\rangle_s \\ \Lambda_t^2 =& \Lambda_0^2 + \int_0^t e^{2\alpha s} d\langle \tilde{B}\rangle_s \\ R_t =& R_0 + \int_0^t r_s ds. \end{aligned}$$

So we can define the function $f : [0,T] \times \mathbb{R}^4 \to \mathbb{R}$ by

$$f(t,x) := \exp\Big(B(t,\tau)e^{-\alpha t}x_2 - B(t,\tau)x_1 - \tfrac{1}{2}B(t,\tau)^2 e^{-2\alpha t}x_3 - x_4\Big)$$

in order to apply Itô to $f(t,X_t)$, where $(X_t)_t = (r_t, \Lambda_t^1, \Lambda_t^2, R_t)$. Then we get

$$\begin{aligned}
f(t,X_t) =& f(0,X_0) + \int_0^t \partial_{x_1} f(s,X_s)d\tilde{B}_s \\
&+ \int_0^t \Big(\underbrace{\partial_t f(s,X_s) + \partial_{x_1} f(s,X_s)\Big(-\alpha r_s + e^{-\alpha s}\Lambda_s^1 + e^{-2\alpha s}\Lambda_s^2\Big) + \partial_{x_4} f(s,X_s) r_s}_{=:\Delta^1(s)}\Big)ds \\
&+ \int_0^t \Big(\underbrace{\partial_{x_2} f(s,X_s)e^{\alpha s} + \partial_{x_1} f(s,X_s)}_{=:\Delta^2(s)}\Big)d\langle B\rangle_s \\
&+ \int_0^t \Big(\underbrace{\partial_{x_3} f(s,X_s)e^{2\alpha s} + \tfrac{1}{2}\partial_{x_1}^2 f(s,X_s)}_{=:\Delta^3(s)}\Big)d\langle \tilde{B}\rangle_s .
\end{aligned}$$

Plugging in the derivatives of f yields

$$\begin{aligned}
\frac{\Delta^1(t)}{f(t,X_t)} =& B'(t,\tau)e^{-\alpha t}\Lambda_t^1 - B(t,\tau)e^{-\alpha t}\alpha\Lambda_t^1 - B'(t,\tau)r_t \\
&- B(t,\tau)B'(t,\tau)e^{-2\alpha t}\Lambda_t^2 + B(t,\tau)^2 e^{-2\alpha t}\alpha\Lambda_t^2 \\
&- B(t,\tau)\Big(-\alpha r_t + e^{-\alpha t}\Lambda_t^1 + e^{-2\alpha t}\Lambda_t^2\Big) - r_t \\
=& e^{-\alpha t}\Big(\underbrace{B'(t,\tau) - \alpha B(t,\tau) + 1}_{=0}\Big)\Lambda_t^1 - \Big(\underbrace{B'(t,\tau) - \alpha B(t,\tau) + 1}_{=0}\Big)r_t \\
&- B(t,\tau)e^{-2\alpha t}\Big(\underbrace{B'(t,\tau) - \alpha B(t,\tau) + 1}_{=0}\Big)\Lambda_t^2 \\
=& 0,
\end{aligned}$$

$$\frac{\Delta^2(t)}{f(t,X_t)} = B(t,\tau)e^{-\alpha t}e^{\alpha t} - B(t,\tau) = 0,$$

and

$$\frac{\Delta^3(t)}{f(t,X_t)} = -\tfrac{1}{2}B(t,\tau)^2 e^{-2\alpha t}e^{2\alpha t} + \tfrac{1}{2}B(t,\tau)^2 = 0.$$

Hence, it holds

$$\tilde{P}(t,\tau) = \tilde{P}(0,\tau) - \int_0^t B(s,\tau)\tilde{P}(s,\tau)d\tilde{B}_s,$$

since

$$\tilde{P}(t,\tau) = f(t,X_t).$$

Besides, it is easy to see that P satisfies the terminal condition. $\square$

If we compare the expression for the price of the bond with the one from Subsection 3.3, we see that the function A completely disappears. Instead, we have another term depending on the market price of uncertainty. So compared to the traditional models, we do not have to do any estimations about the evolution of the mean reversion level or the volatility in the future. However, we have to specify the current value of the market price of uncertainty. Therefore, one would have to estimate the evolution of the mean reversion level and the quadratic variation of the short rate in the past. Another important fact is that the extended model has, unlike the previous model, an endogenous nature, since the current term structure is now meant to be an output of the model.

6 Conclusion

This paper shows how to construct a short rate model in a more realistic set up than the traditional one by incorporating Knightian uncertainty and how to characterize the whole term structure without admitting arbitrage.

The model is divided into two parts. The first and main part deals with volatility uncertainty by using a G-Brownian motion. Within this scope, it is shown how to construct the short rate and in particular, how to price bonds. Furthermore, there is a detailed description of the bond market structure and the notion of arbitrage in order to prove that the market is arbitrage-free, since the concept of no-arbitrage under Knightian uncertainty differs from the classical framework. The second part extends the model to drift uncertainty to completely overcome the problem of parameter estimation.

A Appendix

A.1 Verification of the Solution to the SDEs in Subsection 2.2 and 5.2

To verify that the solution of

$$r_t = r_0 + \int_0^t \big(\mu_s - \alpha r_s\big) ds + B_t$$

is given by

$$r_t = e^{-\alpha t} r_0 + \int_0^t e^{-\alpha(t-s)} \mu_s ds + \int_0^t e^{-\alpha(t-s)} dB_s,$$

we define the function

$$f(t,x) := e^{-\alpha t} x$$

and apply Itô to $f(t, X_t)$, where

$$X_t := r_0 + \int_0^t e^{\alpha s} \mu_s ds + \int_0^t e^{\alpha s} dB_s.$$

Then we get

$$\begin{aligned} r_t = f(t, X_t) =& \underbrace{f(0, X_0)}_{=r_0} + \int_0^t \Big(\partial_t f(s, X_s) + \partial_x f(s, X_s) e^{\alpha s} \mu_s\Big) ds + \int_0^t \partial_x f(s, X_s) e^{\alpha s} dB_s \\ =& r_0 + \int_0^t \Big(- \alpha \underbrace{f(s, X_s)}_{=r_s} + \underbrace{e^{-\alpha s} e^{\alpha s}}_{=1} \mu_s\Big) ds + \int_0^t \underbrace{e^{-\alpha s} e^{\alpha s}}_{=1} dB_s \\ =& r_0 + \int_0^t \big(\mu_s - \alpha r_s\big) ds + B_t. \end{aligned}$$

Analogously, one can show that

$$r_t = e^{-\alpha t} r_0 + \int_0^t e^{-\alpha(t-s)} d\langle B\rangle_s + \int_0^t e^{-\alpha(t-s)} d\tilde{B}_s$$

solves

$$r_t = r_0 + \langle B\rangle_t - \int_0^t \alpha r_s ds + \tilde{B}_t.$$

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