Abstract

By using Malliavin calculus, Bismut type formulas are established for the Lions derivative of $P_t f(\mu) := \mathbb{E} f(X_t^\mu)$, where $t > 0$, $f$ is a bounded measurable function, and $X_t^\mu$ solves a distribution dependent SDE with initial distribution $\mu$. As applications, explicit estimates are derived for the Lions derivative and the total variational distance between distributions of solutions with different initial data. Both degenerate and non-degenerate situations are considered. Due to the lack of the semigroup property and the invalidity of the formula $P_t f(\mu) = \int P_t f(x) \mu(dx)$, essential difficulties are overcome in the study.

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1 Introduction

The Bismut formula introduced in [3], also called Bismut-Elworthy-Li formula due to [12], is a powerful tool in characterising the regularity of distribution for SDEs and SPDEs. A plenty of results have been derived for this type formulas and applications by using stochastic analysis and coupling methods, see for instance [24] and references therein.

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On the other hand, because of crucial applications in the study of nonlinear PDEs and environment dependent financial systems, the distribution dependent SDEs (also called McKean-Vlasov or mean filed SDEs) have received increasing attentions, see [10, 11, 13, 14, 18, 22, 23] and references therein. Recently, this type SDEs have been applied in [5, 9, 17, 20] to characterize PDEs involving the Lions derivative (L-derivative for short) introduced by P.-L. Lions in his lectures [6]. In this paper, we aim to investigate Bismut type L-derivative formula and applications for distribution dependent SDEs with possibly degenerate noise.

To introduce our main results, we first recall the L-derivative. Let \( \mathcal{P}(\mathbb{R}^d) \) be the space of all probability measures on \( \mathbb{R}^d \), and let

\[
\mathcal{P}_2(\mathbb{R}^d) = \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) : \mu(\cdot^2) := \int_{\mathbb{R}^d} |x|^2 \mu(dx) < \infty \right\}.
\]

Then \( \mathcal{P}_2(\mathbb{R}^d) \) is a Polish space under the Wasserstein distance

\[
W_2(\mu, \nu) := \inf_{\pi \in \mathcal{C}(\mu, \nu)} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 \pi(dx, dy) \right)^{\frac{1}{2}}, \quad \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d),
\]

where \( \mathcal{C}(\mu, \nu) \) is the set of couplings for \( \mu \) and \( \nu \); that is, \( \pi \in \mathcal{C}(\mu, \nu) \) is a probability measure on \( \mathbb{R}^d \times \mathbb{R}^d \) such that \( \pi(\cdot \times \mathbb{R}^d) = \mu \) and \( \pi(\mathbb{R}^d \times \cdot) = \nu \). We will use \( 0 \) to denote vectors with components 0, or the constant map taking value 0.

**Definition 1.1.** Let \( f : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \), and let \( g : M \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \) for a differentiable manifold \( M \).

1. \( f \) is called \( L \)-differentiable at \( \mu \in \mathcal{P}_2(\mathbb{R}^d) \), if the functional

\[
L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu) \ni \phi \mapsto f(\mu \circ (\mathrm{Id} + \phi)^{-1})
\]

is Fréchet differentiable at \( 0 \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu) \); that is, there exists (hence, unique) \( \gamma \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu) \) such that

\[
\lim_{\mu(\phi) \to 0} \frac{f(\mu \circ (\mathrm{Id} + \phi)^{-1}) - f(\mu) - \mu(\gamma, \phi)}{\sqrt{\mu(\phi^2)}} = 0. \tag{1.1}
\]

In this case, we denote \( D^Lf(\mu) = \gamma \) and call it the \( L \)-derivative of \( f \) at \( \mu \).

2. If the \( L \)-derivative \( D^Lf(\mu) \) exists for all \( \mu \in \mathcal{P}_2(\mathbb{R}^d) \), then \( f \) is called \( L \)-differentiable.

If, moreover, for every \( \mu \in \mathcal{P}_2(\mathbb{R}^d) \) there exists a \( \mu \)-version \( D^L f(\mu)(\cdot) \) such that \( D^L f(\mu)(x) \) is jointly continuous in \( (x, \mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \), we denote \( f \in C^{(1,0)}(\mathcal{P}_2(\mathbb{R}^d)) \).

3. \( g \) is called differentiable on \( M \times \mathcal{P}_2(\mathbb{R}^d) \), if for any \( (x, \mu) \in M \times \mathcal{P}_2(\mathbb{R}^d) \), \( g(\cdot, \mu) \) is differentiable at \( x \) and \( g(x, \cdot) \) is \( L \)-differentiable at \( \mu \). If, moreover, \( \nabla g(\cdot, \mu)(x) \) and \( D^L g(x, \cdot)(\mu)(y) \) are joint continuous in \( (x, y, \mu) \in M^2 \times \mathcal{P}_2(\mathbb{R}^d) \), where \( \nabla \) is the gradient operator on \( M \), we write \( g \in C^{1,(1,0)}(M \times \mathcal{P}_2(\mathbb{R}^d)) \).
As indicated in [20] that for any \( n \geq 1, g \in C^1(\mathbb{R}^n) \) and \( h_1, \ldots, h_n \in C^1_0(\mathbb{R}^d) \), the cylindrical function
\[
\mu \mapsto g(\mu(h_1), \ldots, \mu(h_n))
\]
is in \( C^{(1,0)}(\mathcal{P}_2(\mathbb{R}^d)) \) with
\[
D^L g(\mu)(x) = \sum_{i=1}^n \left( \partial_i g(\mu(h_1), \ldots, \mu(h_n)) \right) \nabla h_i(x), \quad (x, \mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d).
\]

Obviously, if \( f \) is \( L \)-differentiable at \( \mu \), then
\[
D^L_\phi f(\mu) := \lim_{\varepsilon \downarrow 0} \frac{f(\mu \circ (\text{Id} + \varepsilon \phi)^{-1}) - f(\mu)}{\varepsilon} = \mu((D^L f(\mu), \phi)), \quad \phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu).
\]

We may call \( D^L_\phi \) the directional \( L \)-derivative along \( \phi \). This directional derivative has been used in earlier references, see for instance [21] for the Wasserstein diffusions constructed using the directional derivative on \( \mathcal{P}_2(\mathbb{S}^1) \), where \( \mathbb{S}^1 \) is the unit circle.

When \( D^L_\phi f(\mu) \) is a bounded linear functional of \( \phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu) \), there exists a unique \( \xi \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu) \) such that \( D^L_\phi f(\mu) = \mu((\xi, \phi)) \) holds for all \( \phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu) \). In this case, \( \phi \mapsto f(\mu \circ (\text{Id} + \phi)^{-1}) \) is Gâteaux differentiable at \( 0 \), and we say that \( f \) is weakly \( L \)-differentiable at \( \mu \), since the Gâteaux differentiability is weaker than the Fréchet one.

By (1.2), for an \( L \)-differentiable function \( f \) on \( \mathcal{P}_2(\mathbb{R}^d) \), we have
\[
\|D^L f(\mu)\| := \|D^L f(\mu)(\cdot)\|_{\mathcal{L}(\mu)} = \sup_{\|\phi\|_1 \leq 1} |D^L_\phi f(\mu)|.
\]

For a vector-valued function \( f = (f_i) \), or a matrix-valued function \( f = (f_{ij}) \) with \( L \)-differentiable components, we write
\[
D^L_\phi f(\mu) = (D^L_\phi f_i(\mu)), \quad D^L_\phi f(\mu) = (D^L_\phi f_{ij}(\mu)), \quad \mu \in \mathcal{P}_2(\mathbb{R}^d).
\]

Let \( W_t \) be a \( d \)-dimensional Brownian motion on the natural filtered probability space \( (\Omega^0, \mathcal{F}^0, \{\mathcal{F}_t^0\}_{t \geq 0}, \mathbb{P}) \). To ensure that for any \( \mu \in \mathcal{P}_2(\mathbb{R}^d) \) there exists a random variable \( X \) on \( \mathbb{R}^d \) with distribution \( \mu \), let \( \mu^0 \) be a probability measure on \( \mathbb{R}^d \) which is equivalent to the Lebesgue measure, and enlarge the probability space as
\[
(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) := (\Omega^0 \times \mathbb{R}^d, \mathcal{F}^0 \times \mathcal{B}(\mathbb{R}^d), \{\mathcal{F}_t^0 \times \mathcal{B}(\mathbb{R}^d)\}_{t \geq 0}, \mathbb{P}^0 \times \mu^0).
\]

Then
\[
W_t(\omega) := W_t(\omega^0), \quad t \geq 0, \omega := (\omega^0, x) \in \Omega
\]
is a \( d \)-dimensional Brownian motion on \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \). Let \( \mathcal{L}_\xi \) denote the distribution of a random variable on the probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). In case different probability spaces are concerned, we write \( \mathcal{L}_\xi|\mathcal{F} \) instead of \( \mathcal{L}_\xi \) to emphasize the reference probability measure \( \mathbb{P} \).

Consider the following distribution dependent SDE on \( \mathbb{R}^d \):
\[
dX_t = b_t(X_t, \mathcal{L}_X_t)dt + \sigma_t(X_t, \mathcal{L}_X_t)dW_t, \quad X_0 \in L^2(\Omega \to \mathbb{R}^d, \mathcal{F}_0, \mathbb{P}),
\]
are continuous such that for some increasing function $K : [0, \infty) \to [0, \infty)$ there holds

$$
\|b_t(x, \mu) - b_t(y, \nu)\| + \|\sigma_t(x, \mu) - \sigma_t(y, \nu)\| \\
\leq K(t)\left(\|x - y\| + \mathbb{W}_2(\mu, \nu)\right), \quad t \geq 0, x, y \in \mathbb{R}^d, \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)
$$

(1.5)

and

$$
\|\sigma_t(0, \delta_0)\| + \|b_t(0, \delta_0)\| \leq K(t), \quad t \geq 0,
$$

(1.6)

where and in what follows, for $x \in \mathbb{R}^d$ we denote by $\delta_x$ the Dirac measure at $x$, and $\| \cdot \|$ is the operator norm. For any $t \geq 0$, let $L^2(\Omega \to \mathbb{R}^d, \mathcal{F}_t, \mathbb{P})$ be the class of $\mathcal{F}_t$-measurable square integrable random variables on $\mathbb{R}^d$. By (1.5) and (1.6), for any $s \geq 0$ and $X_s \in L^2(\Omega \to \mathbb{R}^d, \mathcal{F}_s, \mathbb{P})$, (1.4) has a unique solution $(X_{s,t})_{t \geq s}$ with $X_{s,s} = X_s$ and

$$
\mathbb{E} \left[ \sup_{t \in [s,T]} |X_{s,t}|^2 \right] < \infty, \quad T \geq s,
$$

(1.7)

see, for instance [27], where gradient estimates and Harnack inequalities are also derived for the associated nonlinear semigroup. See also [16, 18] for weaker conditions ensuring the existence and uniqueness of solutions to (1.4). For any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ and $s \geq 0$, let $(X^\mu_{s,t})_{t \geq s}$ be the solution to (1.4) with $\mathcal{L}_{X^\mu_{s,s}} = \mu$. Denote

$$
P^\mu_{s,t} = \mathcal{L}_{X^\mu_{s,t}}, \quad t \geq s, \mu \in \mathcal{P}_2(\mathbb{R}^d).
$$

Let

$$
(P_{s,t}f)(\mu) = (P^\mu_{s,t}f)(f) := \int_{\mathbb{R}^d} f d(P^\mu_{s,t}\mu) = \mathbb{E}f(X^\mu_{s,t}), \quad t \geq s, \mu \in \mathcal{P}_2(\mathbb{R}^d),
$$

(1.8)

Then for any $0 \leq s \leq t$, $P_{s,t}$ is a linear operator from $\mathcal{B}_b(\mathbb{R}^d)$ to $\mathcal{B}_b(\mathcal{P}_2(\mathbb{R}^d))$.

In this paper, we aim to establish the Bismut type formula for the $L$-derivative of $P_{s,t}f$ for $t > s$. By considering the SDE for $X_t := X_{t+s}, t \geq 0$, without loss of generality we may and do assume $s = 0$. So, for simplicity, below we only establish the derivative formula for $P_t f := P_{0,t}f, t > 0$. More precisely, for any $T > 0$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ and $\phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)$, we aim to construct an integrable random variable $M_{T,\phi}^\mu$ such that

$$
D^L_T(P_T f)(\mu) = \mathbb{E}\left[ f(X_T^\mu) M_{T,\phi}^\mu \right], \quad f \in \mathcal{B}_b(\mathbb{R}^d),
$$

(1.10)

which in turn implies the $L$-differentiability of $P_T f$. Note that the derivative formula for $(P_T f)(x) := (P_T f)(\delta_x)$ along a vector $v \in \mathbb{R}^d$ is derived in [2], which is the special case of (1.10) with $\mu = \delta_x$ and $\phi \equiv v$. Moreover, formulas of the $L$-derivative and integration by parts have been presented in [8] for the following de-coupled SDE:

$$
dX_t^x = b(t, X_t^x, P_t^x\mu)dt + \sigma(t, X_t^x, P_t^x\mu)dW_t, \quad X^x_0 = x,
$$
which is different from the original SDE (1.4) but has important applications in solving PDEs with Lions’ derivatives, see [5, 17, 20] and references within.

When the SDE (1.4) is distribution independent, i.e. \( b_t(x, \mu) = b_t(x) \) and \( \sigma_t(x, \mu) = \sigma_t(x) \) do not depend on \( \mu \), the Bismut type formula

\[
\nabla P_T f(x) = \mathbb{E}[f(X_T^x)M_T^x], \quad x \in \mathbb{R}^d, f \in \mathcal{B}_b(\mathbb{R}^d)
\]

has been well studied in the literature, where \( M_T^x \) is an integrable random variable on \( \mathbb{R}^d \), which is measurable in \( x \in \mathbb{R}^d \) when it varies, see for instance [1, 15, 25, 26, 28] and references within. Since the coefficients are distribution independent, we have

\[
(P_T f)(\mu) = \int_{\mathbb{R}^d} (P_T f)(x) \mu(dx),
\]

so that \( P_T f \) is \( L \)-differentiable with \( D^L (P_T f)(\mu) = \nabla P_T f \). Hence, by (1.11) and (1.12) we obtain

\[
D^L_\phi(P_T f)(\mu) = \mu(\langle D^L P_T f, \phi \rangle) = \int_{\mathbb{R}^d} \mathbb{E}[f(X_T^x)(M_T^x, \phi(x))] \mu(dx)
= \mathbb{E}[f(X_T^x)(M_T^{X_0^x}, \phi(X_0^x))].
\]

Therefore, (1.10) holds for \( M_T^{X_\phi^x} = \langle M_T^{X_0^x}, \phi(X_0^x) \rangle \).

However, when the SDE is distribution dependent, as explained in [27] that in general (1.12) does not hold, so it is non-trivial to establish the Bismut type formula (1.10).

The remainder of the paper is organized as follows. In section 2, we state our main results on Bismut formulas of \( D^L_\phi P_T f \) and applications, for both non-degenerate and degenerate distribution dependent SDEs. To establish the Bismut formula using Malliavin calculus, we make necessary preparations in Section 3 concerning partial derivatives in the initial value, and Malliavin derivative for solutions of (1.4). Finally, complete proofs of the main results are addressed in Section 4.

2 Main results

Let \( | \cdot | \) denote the norm in \( \mathbb{R}^d \), and \( \| \cdot \| \) denote the operator norm for matrices or more generally linear operators. We make the following assumption.

\[ \textbf{(H)} \] For any \( t \geq 0, b_t, \sigma_t \in C^{1,(1,0)}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)) \). Moreover, there exists a continuous function \( K : [0, \infty) \rightarrow [0, \infty) \), such that (1.6) holds and

\[
\max \left\{ |\nabla b_t(\cdot, \mu)(x)|, \|D^L b_t(x, \cdot)(\mu)\|, \frac{1}{2} \|\nabla \sigma_t(\cdot, \mu)(x)\|^2, \frac{1}{2} \|D^L \sigma_t(x, \cdot)(\mu)\|^2 \right\}
\leq K(t), \quad t \geq 0, x \in \mathbb{R}^d, \mu \in \mathcal{P}_2(\mathbb{R}^d),
\]

where as in (1.3), \( \|D^L f(\mu)\| := \|D^L f(\mu)(\cdot)\|_{L^2(\mu)} \) for an \( L \)-differentiable function \( f \) at \( \mu \).
Obviously, (H) implies (1.5) and (1.6), so that the SDE (1.4) has a unique solution for any initial value $X_0 \in L^2(\Omega \to \mathbb{R}^d, \mathcal{F}_0, \mathbb{P})$.

In the following two subsections, we state our main results for non-degenerate and degenerate cases respectively.

### 2.1 The non-degenerate case

For each $t > 0$, let $\sigma_t$ be invertible such that

$$\|\sigma_t(x, \mu)^{-1}\| \leq \lambda_t, \quad t \geq 0, x \in \mathbb{R}^d, \mu \in \mathcal{P}_2(\mathbb{R}^d)$$

holds for some continuous function $\lambda : [0, \infty) \to (0, \infty)$. Let $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, and let $X_t$ solve (1.4) for $X_0 \in L^2(\Omega \to \mathbb{R}^d, \mathcal{F}_0, \mathbb{P})$ with $\mathcal{L}X_0 = \mu$. Given $\phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)$, consider the following SDE for $v^\phi_t$ on $\mathbb{R}^d$:

$$
\begin{align*}
\text{d}v^\phi_t &= \left\{ \nabla_{v^\phi_t} b_t(\cdot, \mathcal{L}X_t)(X_t) + \left( \mathbb{E}(D^2 b_t(y, \cdot)(\mathcal{L}X_t)(X_t), v^\phi_t) \right) \bigg|_{y = X_t} \right\} \text{d}t \\
& \quad + \left\{ \nabla_{v^\phi_t} \sigma_t(\cdot, \mathcal{L}X_t)(X_t) + \left( \mathbb{E}(D^2 \sigma_t(y, \cdot)(\mathcal{L}X_t)(X_t), v^\phi_t) \right) \bigg|_{y = X_t} \right\} \text{d}W_t, \quad v^\phi_0 = \phi(X_0).
\end{align*}
$$

By (H), this linear SDE is well-posed with $\sup_{t \in [0, T]} \mathbb{E}\|v^\phi_t\|^2 \leq C \mu(|\phi|^2)$ for some constant $C = C(T) > 0$, see (4.21) below. Denote $g_s = \frac{d}{dt}g_s$ for a differentiable function $g$ of $s \in \mathbb{R}$.

**Theorem 2.1.** Assume (H) and (2.1). Then for any $f \in \mathcal{B}_h(\mathbb{R}^d), \mu \in \mathcal{P}_2(\mathbb{R}^d)$ and $T > 0$, $P_T f$ is $L$-differentiable at $\mu$ such that for any $g \in C^1([0, T])$ with $g_0 = 0$ and $g_T = 1$,

$$
D^L_{\phi}(P_T f)(\mu) = \mathbb{E}\left[ f(X_T) \int_0^T \langle g'_s \sigma_t(X_t, \mathcal{L}X_t)^{-1} v^\phi_t, \text{d}W_t \rangle \right], \quad \phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu),
$$

where $X_t$ solves (1.4) for $\mathcal{L}X_0 = \mu$. Moreover, the limit

$$
D^L_{\phi}P^*_T \mu := \lim_{\varepsilon \to 0} \frac{P^*_T \mu \circ (\text{Id} + \varepsilon \phi)^{-1} - P^*_T \mu}{\varepsilon} = \psi P^*_T \mu
$$

exists in the total variation norm, where $\psi$ is the unique element in $L^2(\mathbb{R}^d \to \mathbb{R}, P^*_T \mu)$ such that $\psi(X_T) = \mathbb{E}\left( \int_0^T \langle g'_s \sigma_t(X_t, \mathcal{L}X_t)^{-1} v^\phi_t, \text{d}W_t \rangle | X_T \right)$, and $(\psi P^*_T)\mu(A) := \int_A \psi \text{d}P^*_T \mu$, $A \in \mathcal{B}(\mathbb{R}^d)$.

**Remark 2.1.** When $f \in C^1_b(\mathbb{R}^d)$, (2.3) can be proved as in the distribution independent case by constructing a proper random variable $h$ on the Cameron-Martin space such that $D_hX_T = \nabla_{\phi}X_T$. However, for the $L$-differentiability of $P_T f$, one has to construct $\gamma \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)$ such that (1.1) holds for $P_T f$ replacing $f$, which is non-trivial.

Moreover, comparing with the classical case where (2.3) for $f \in C^1_b(\mathbb{R}^d)$ can be easily extended to $f \in \mathcal{B}_h(\mathbb{R}^d)$, there is essential difficulty to do this in the distribution dependent setting. More precisely, when $b_t$ and $\sigma_t$ do not depend on the distribution, we have the semigroup property $P_T f(\mu) = Pt,T(P_t f)(\mu)$ for $t \in (0, T)$, where $P_t f(x) := P_t f(\delta_x)$ for
the Dirac measure \( \delta_x \) at point \( x \). In many cases the regularity of \( P_t \) ensures that \( P_t f \in C^1_b(\mathbb{R}^d) \) for \( f \in \mathcal{B}_b(\mathbb{R}^d) \). Then for any \( f \in \mathcal{B}_b(\mathbb{R}^d) \), one may apply the derivative formula (2.3) with \((P_tT, P_t f)\) replacing \((P_t, f)\) to derive a derivative formula for \( P_t f \). However, in the distribution dependent case, due to the lack of (1.12) we no longer have \( P_T f(\mu) = P_{tT}(P_t f)(\mu) \), so that this argument becomes invalid. To overcome this difficulty we will make a new approximation argument, see step (a) in the proof of Theorem 2.1 for details.

As applications of Theorem 2.1, the following result consists of estimates on the \( L\)-derivative and the total variational distance between distributions of solutions with different initial data.

**Corollary 2.2.** Assume (H) and (2.1) for some increasing functions \( K \) and continuous function \( \lambda \).

1. For any \( f \in \mathcal{B}_b(\mathbb{R}^d) \) and \( T > 0 \),
   \[ \| D^L (P_T f)(\mu) \|^2 := \sup_{\mu(|\phi|^2) \leq 1} \| D^L_\phi (P_T f)(\mu) \|^2 \leq \frac{(P_T f^2)(\mu) - (P_T f)(\mu)^2}{\int_0^T \lambda_t^{-2} e^{-8K(t)} t dt}. \]

2. For any \( T > 0 \),
   \[ \| P_T f(\mu) - P_T f(\nu) \|^2 \leq \frac{\| f \|^2_{\infty} \mathcal{W}_2(\mu, \nu)^2}{\int_0^T \lambda_t^{-2} e^{-8K(t)} t dt}, \quad \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d), f \in \mathcal{B}_b(\mathbb{R}^d). \]

Consequently, for any \( T > 0 \) and \( \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d) \),

\[ \| P^*_T \mu - P^*_T \nu \|^2_{\text{var}} := \sup_{\Lambda \in \mathcal{M}(\mathbb{R}^d)} \| (P^*_T \mu)(\Lambda) - (P^*_T \nu)(\Lambda) \|^2 \leq \frac{\mathcal{W}_2(\mu, \nu)^2}{\int_0^T \lambda_t^{-2} e^{-8K(t)} t dt}. \]

### 2.2 Stochastic Hamiltonian systems

Consider the following distribution dependent stochastic Hamiltonian system for \( X_t = (X_t^{(1)}, X_t^{(2)}) \) on \( \mathbb{R}^{m+d} = \mathbb{R}^m \times \mathbb{R}^d \):

\[ \begin{align*}
    dX_t^{(1)} &= b_t^{(1)}(X_t) dt, \\
    dX_t^{(2)} &= b_t^{(2)}(X_t, \mathcal{L}X_t) dt + \sigma_t dW_t,
\end{align*} \]

where \((W_t)_{t \geq 0}\) is a \( d\)-dimensional Brownian motion as before, and for each \( t \geq 0 \), \( \sigma_t \) is an invertible \( d \times d\)-matrix,

\[ b_t = (b_t^{(1)}, b_t^{(2)}): \mathbb{R}^{m+d} \times \mathcal{P}_2(\mathbb{R}^{m+d}) \to \mathbb{R}^{m+d} \]

is measurable with \( b_t^{(1)}(x, \mu) = b_t^{(1)}(x) \) independent of the distribution \( \mu \). Let \( \nabla = (\nabla^{(1)}, \nabla^{(2)}) \) be the gradient operator on \( \mathbb{R}^{m+d} = \mathbb{R}^m \times \mathbb{R}^d \), where \( \nabla^{(i)} \) is the gradient in the \( i \)-th component, \( i = 1, 2 \). Let \( \nabla^2 = \nabla \nabla \) denote the Hessian operator on \( \mathbb{R}^{m+d} \). We assume
(H1) For every $t \geq 0$, $b_t^{(1)} \in C^2_b(\mathbb{R}^{m+d} \to \mathbb{R}^m)$, $b_t^{(2)} \in C^{1,1,0}(\mathbb{R}^{m+d} \times \mathcal{P}_2(\mathbb{R}^{m+d}) \to \mathbb{R}^d)$, and there exists an increasing function $K : [0, \infty) \to [0, \infty)$ such that (1.6) and

$$
\|\nabla b_t(\cdot, \mu)(x)\| + \|D^2 b_t^{(2)}(x, \cdot)(\mu)\| + \|\nabla^2 b_t^{(1)}(\cdot, \mu)(x)\| \leq K(t)
$$

hold for all $t \geq 0$, $(x, \mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$.

Obviously, this assumption implies (H) for the SDE (2.8). We aim to establish the derivative formula of type (1.10) with $P_t$ and $P_t^*$ being defined by (1.8) and (1.9) for the SDE (2.8). To follow the line of [28] where the distribution independent model was investigated, we need the following assumption (H2).

For any $s \geq 0$, let $\{K_{t,s}\}_{t \geq s}$ solve the following linear random ODE on $\mathbb{R}^{m \otimes m}$:

$$
\frac{d}{dt} K_{t,s} = (\nabla^{(1)} b_t^{(1)})(X_t) K_{t,s}, \quad t \geq s, \quad K_{s,s} = I_{m \times m},
$$

where $I_{m \times m}$ is the $m \times m$-order identity matrix.

(H2) There exists $B \in \mathcal{B}_b([0,T] \to \mathbb{R}^{m \otimes d})$ such that

$$
\langle (\nabla^{(2)} b_t^{(1)} - B_t^* a) K_t^* a, a \rangle \geq -\varepsilon |B_t^* a|^2, \quad \forall a \in \mathbb{R}^m
$$

holds for some constant $\varepsilon \in [0, 1)$. Moreover, there exists an increasing function $\theta \in C([0,T])$ with $\theta_t > 0$ for $t \in (0, T]$ such that

$$
\int_0^t s(T - s) K_{T,s} B_t^* K_{T,s}^* ds \geq \theta_t I_{m \times m}, \quad t \in (0, T].
$$

Example 2.1. Let

$$
b_t^{(1)}(x) = Ax^{(1)} + Bx^{(2)}, \quad x = (x^{(1)}, x^{(2)}) \in \mathbb{R}^{m+d}
$$

for some $m \times m$-matrix $A$ and $m \times d$-matrix $B$. If the Kalman’s rank condition

$$\text{Rank}[B, AB, \ldots, A^k B] = m
$$

holds for some $k \geq 1$, then (H2) is satisfied with $\theta_t = c_T t$ for some constant $c_T > 0$, see the proof of [28, Theorem 4.2]. In general, (H2) remains true under small perturbations of this $b_t^{(1)}$.

According to the proof of [28, Theorem 1.1], (H2) implies that the matrices

$$
Q_t := \int_0^t s(T - s) K_{T,s} \nabla^{(2)} b_t^{(1)}(X_s) B_t^* K_{T,s}^* ds, \quad t \in (0, T]
$$

are invertible with

$$
\|Q_t^{-1}\| \leq \frac{1}{(1 - \varepsilon) \theta_t}, \quad t \in (0, T].
$$
For \((X_t)_{t \in [0,T]}\) solving (2.8) with \(\mathcal{L}X_0 = \mu\) and \(\phi = (\phi^{(1)}, \phi^{(2)}) \in L^2(\mathbb{R}^{m+d} \to \mathbb{R}^{m+d}, \mu)\), let
\[
\alpha^{(2)}_t = \frac{T - t}{T} \phi^{(2)}(X_0) - \frac{t(T - t)}{T} B_t^* K_{T,t}^* \int_0^T \theta_2^s Q_s^{-1} K_{T,0} \phi^{(1)}(X_0) ds
\]
\[
- t(T - t) B_t^* K_{T,t}^* Q_T^{-1} \int_0^T \frac{1}{T} - \frac{s}{T} K_{T,s} \nabla^{(2)}_{\alpha^{(2)}_s} (X_s) ds, \quad t \in [0,T],
\]
and
\[
\alpha^{(1)}_t = K_{t,0} \phi^{(1)}(X_0) + \int_0^T K_{t,s} \nabla^{(2)}_{\alpha^{(2)}_s} (X_s(x)) ds, \quad t \in [0,T].
\]
Moreover, define
\[
h_t^\alpha := \int_0^t \sigma_s^{-1} \left\{ \left( \mathbb{E}(D^2 b_s^{(2)}(y, \cdot, (\mathcal{L}X_s)(X_s), \alpha_s)) \right)_{y = X_s} \right. \\
- \nabla_{\alpha^{(2)}_s} (\cdot, \mathcal{L}X_s)(X_s) - (\alpha^{(2)}_s) \right\} ds, \quad t \in [0,T].
\]
Let \((D^*, \mathcal{D}(D^*))\) be the Malliavin divergence operator associated with the Brownian motion \((W_t)_{t \in [0,T]}\); see Subsection 3.2 below for details. Then the main result in this part is the following.

**Theorem 2.3.** Assume (H1) and (H2). Then \(h^\alpha \in \mathcal{D}(D^*)\) with \(\mathbb{E}|D^*(h^\alpha)|^p < \infty\) for all \(p \in [1, \infty)\). Moreover, for any \(f \in \mathcal{B}_b(\mathbb{R}^{m+d})\) and \(T > 0\), \(P_T f\) is \(L\)-differentiable at \(\mu\) such that
\[
D^L_{\phi}(P_T f)(\mu) = \mathbb{E} \left[ f(X_T) D^*(h^\alpha) \right].
\]
Consequently:

1. (2.4) holds for the unique \(\psi \in L^2(\mathbb{R}^{m+d} \to \mathbb{R}, P_T^{*}\mu)\) such that \(\psi(X_T) = \mathbb{E}(D^*(h^\alpha)|X_T)\).
2. There exists a constant \(c \geq 0\) such that for any \(T > 0\),
\[
\|D^L(P_T f)(\mu)\| \leq c \sqrt{T P_T |f|^2(\mu) - (P_T f)^2(\mu)} \frac{\sqrt{T}(T^2 + \theta_T)}{\int_0^T \theta_2^s ds}, \quad f \in \mathcal{B}_b(\mathbb{R}^{m+d}),
\]
\[
\|P_T^{*}\mu - P_T^{*}\nu\|_{\text{var}} \leq c \mathbb{W}_2(\mu, \nu) \frac{\sqrt{T}(T^2 + \theta_T)}{\int_0^T \theta_2^s ds}, \quad \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d).
\]

### 3 Preparations

We first introduce a formula of the \(L\)-derivative re-organized from [6, Theorem 6.5] and [9, Proposition A.2], then investigate the partial derivatives of \(X_t\) in the initial value, and the Malliavin derivatives of \(X_t\) with respect to the Brownian motion \(W_t\).
3.1 A formula of $L$-derivative

The following result is essentially due to [6, Theorem 6.5] for $f \in C^{(1,0)}(\mathcal{P}_2(\mathbb{R}^d))$, and [9, Proposition A.2] for bounded $X$ and $Y$. We include a complete proof for readers’ convenience.

**Proposition 3.1.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an atomless probability space, and let $X, Y \in L^2(\Omega \to \mathbb{R}^d, \mathbb{P})$ with $\mathcal{L}_X = \mu$. If either $X$ and $Y$ are bounded and $f$ is $L$-differentiable at $\mu$, or $f \in C^{(1,0)}(\mathcal{P}_2(\mathbb{R}^d))$, then

\[
\lim_{\varepsilon \downarrow 0} \frac{f(\mathcal{L}_{X+\varepsilon Y}) - f(\mu)}{\varepsilon} = \mathbb{E}\langle D^L f(\mu)(X), Y \rangle.
\]

Consequently,

\[
\lim_{\varepsilon \downarrow 0} \frac{f(\mathcal{L}_{X+\varepsilon Y}) - f(\mu)}{\varepsilon} = \mathbb{E}\langle D^L f(\mu)(X), Y \rangle \leq \|D^L f(\mu)\| \sqrt{\mathbb{E}\langle Y^2 \rangle}.
\]

**Proof.** It is easy to see that (3.2) follows from (1.3) and (3.1). Indeed, letting $\phi \in L^2(\mathbb{R}^d \to \mathbb{R}, \mu)$ such that $\phi(X) = \mathbb{E}(Y|X)$, we have

\[
\left| \mathbb{E}\langle D^L f(\mu)(X), Y \rangle \right| = \left| \mathbb{E}\langle D^L f(\mu)(X), \phi(X) \rangle \right| = \left| \mu(\langle D^L f(\mu), \phi \rangle) \right|
\]

\[
\leq \|D^L f(\mu)\| \cdot \|\phi\|_{L^2(\mu)} = \|D^L f(\mu)\| \left( \mathbb{E}\langle \mathbb{E}(Y|X)^2 \rangle \right)^{\frac{1}{2}} \leq \|D^L f(\mu)\| \sqrt{\mathbb{E}\langle Y^2 \rangle}.
\]

Below we prove (3.1) for the stated two situations respectively.

(1) Assume that $X$ and $Y$ are bounded. For any $\mathbb{R}^d$-valued random variable $\xi$, let $F(\xi) = f(\mathcal{L}_\xi)$. Next, let $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ be an atomless Polish probability space, and let $\bar{X} \in L^2(\bar{\Omega} \to \mathbb{R}^d, \bar{\mathbb{P}})$ with $\mathcal{L}_{\bar{X}|\bar{\mathbb{P}}} = \mu$, where $\mathcal{L}_{\bar{X}}$ denotes the distribution of a random variable under $\bar{\mathbb{P}}$. According to [9, Proposition A.2(iii)], if

\[
\bar{F}(\bar{Y}) := f(\mathcal{L}_{\bar{Y}|\bar{\mathbb{P}}}), \quad \bar{Y} \in L^2(\bar{\Omega} \to \mathbb{R}^d, \bar{\mathbb{P}})
\]

is Fréchet differentiable at $\bar{X}$ with derivative $D\bar{F}(\bar{X}) = D^L f(\mu)(\bar{X})$, then

\[
\lim_{\varepsilon \downarrow 0} \frac{f(\mathcal{L}_{X+\varepsilon Y}) - f(X) - \varepsilon \mathbb{E}\langle D^L f(\mu)(X), Y \rangle}{\varepsilon} = 0.
\]

Equivalently, (3.1) holds. Below we construct the desired $\bar{X}$ and $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ such that $D\bar{F}(\bar{X}) = D^L f(\mu)(\bar{X})$.

A natural choice of $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ is $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu)$, but to ensure the atomless property, we take $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}) = (\mathbb{R}^d \times \mathbb{R}, \mathcal{B}(\mathbb{R}^d \times \mathbb{R}), \mu \times \lambda)$, where $\lambda$ is the standard Gaussian measure on $\mathbb{R}$. Then $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ is an atomless Polish probability space. Let

\[
\bar{X}(\bar{\omega}) = x, \quad \bar{\omega} = (x, r) \in \mathbb{R}^d \times \mathbb{R}.
\]

We have $\mathcal{L}_{\bar{X}} = \mu$. Moreover, let

\[
f(\bar{\mu}) = f(\mu(\cdot \times \mathbb{R})), \quad \bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}).
\]
It is easy to see that the $L$-differentiability of $f$ at $\mu$ implies that of $\bar{f}$ at $\mu \times \delta_0$ with
\begin{equation}
D^L \bar{f}(\mu \times \delta_0)(x, r) = (D^L f(\mu)(x), 0), \quad (x, r) \in \mathbb{R}^d \times \mathbb{R}.
\end{equation}

Finally, on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we have
\begin{equation}
F(Y) := f(\mathcal{L}_Y) = \bar{f}(\mathcal{L}_{\bar{Y}}), \quad \bar{Y} := (Y, 0) \in L^2(\Omega \to \mathbb{R}^d \times \mathbb{R}, \mathcal{F}, \mathbb{P}).
\end{equation}

Letting $X = (X, 0) \in L^2(\Omega \to T^d \times \mathbb{R}, \mathcal{F}, \mathbb{P})$, by [9, Proposition A.2(iii)], the formula (3.3) holds for $(X, \bar{Y}, \bar{f}, \mu \times \delta_0)$ replacing $(X, Y, f, \mu)$, i.e.
\begin{equation}
\lim_{\varepsilon \to 0} \frac{\bar{f}(\mathcal{L}_{X+\varepsilon Y}) - \bar{f}(\mathcal{L}_{\bar{X}}) - \mathbb{E}(D^L \bar{f}(\mu \times \delta_0), \varepsilon \bar{Y})}{\varepsilon} = 0.
\end{equation}

Combining this with (3.4) and (3.5), we prove (3.3). Therefore, (3.1) holds.

(2) Let $f \in C^{(1,0)}(\mathcal{P}_2(\mathbb{R}^d))$ and let $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ and $X \in L^2(\Omega \to \mathbb{R}^d, \mathbb{P})$ with $\mathcal{L}_X = \mu$. For any $n \geq 1$, let
\[ x_n = \frac{x}{\sqrt{1 + n^{-1}|x|^2}}, \quad x \in \mathbb{R}^d. \]

By (3.1) for bounded $X$ and $Y$, for any $n \geq 1$ we have
\begin{equation}
f(\mathcal{L}_{X_n + sY_n}) - f(\mathcal{L}_{X_n}) = \int_0^\varepsilon \frac{d}{ds} f(\mathcal{L}_{X_n + sY_n}) \, ds
\end{equation}
\begin{equation}
= \int_0^\varepsilon \mathbb{E}(D^L f(\mathcal{L}_{X_n + sY_n})(X_n + sY_n), Y_n) \, ds.
\end{equation}

Since $f \in C^{(1,0)}(\mathcal{P}_2(\mathbb{R}^d))$, it follows that
\begin{equation}
\sup_{n \geq 1, s \in [0, \varepsilon]} \|D^L f(\mathcal{L}_{X_n + sY_n})\| < \infty, \quad \lim_{n \to \infty} f(\mathcal{L}_{X_n + sY_n}) - f(\mathcal{L}_{X_n}) = f(\mathcal{L}_{X + \varepsilon Y}) - f(\mathcal{L}_X),
\end{equation}
and for any $s \in [0, \varepsilon]$,
\begin{equation}
\lim_{n \to \infty} \mathbb{E}(|X - X_n|^2 + |Y - Y_n|^2 + |D^L f(\mathcal{L}_{X_n + sY_n})(X_n + sY_n) - D^L f(\mathcal{L}_{X + sY})(X + sY)|^2) = 0.
\end{equation}

Then letting $n \to \infty$ in (3.6) we arrive at
\begin{equation}
f(\mathcal{L}_{X + \varepsilon Y}) - f(\mathcal{L}_X) = \int_0^\varepsilon \mathbb{E}(D^L f(\mathcal{L}_{X + sY})(X + sY), Y) \, ds, \quad \varepsilon > 0.
\end{equation}

This implies (3.1). More precisely, it is easy to see that $\{\mathcal{L}_{X + sY}\}$ is compact in $\mathcal{P}_2(\mathbb{R}^d)$. So, $f \in C^{(1,0)}(\mathcal{P}_2(\mathbb{R}^d))$ implies
\begin{equation}
A := \sup_{s \in [0, 1]} \sqrt{\mathbb{E}|D^L f(\mathcal{L}_{X + sY})(X + sY)|^2} = \sup_{s \in [0, 1]} \|D^L f(\mathcal{L}_{X + sY})\|_{L^2(\mathcal{L}_{X + sY})} < \infty.
\end{equation}
Combining this with the continuity property of $D^L f$ on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, we conclude that

$$\lim_{\varepsilon \to 0} D^L f(\mathcal{L}_{X+sY})(X + sY) = D^L f(\mathcal{L}_X)(X)$$ weakly in $L^2(\Omega \to \mathbb{R}^d, \mathbb{P})$.

In particular,

$$\lim_{\varepsilon \to 0} \mathbb{E} \langle D^L f(\mathcal{L}_{X+sY})(X + sY), Y \rangle = \mathbb{E} \langle D^L f(\mathcal{L}_X)(X), Y \rangle.$$

Moreover, (3.8) implies

$$\sup_{s \in [0,1]} \mathbb{E} \left| \langle D^L f(\mathcal{L}_{X+sY})(X + sY), Y \rangle \right| \leq A \sqrt{\mathbb{E} |Y|^2} < \infty.$$

Due to this, (3.7) and (3.9), the dominated convergence theorem gives

$$\lim_{\varepsilon \to 0} \frac{f(\mathcal{L}_{X+\varepsilon Y}) - f(\mathcal{L}_X)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^{\varepsilon} \mathbb{E} \langle D^L f(\mathcal{L}_{X+sY})(X + sY), Y \rangle \, ds$$

$$= \mathbb{E} \langle D^L f(\mathcal{L}_X)(X), Y \rangle.$$ 

\[ \square \]

### 3.2 Partial derivative in initial value

For any $T > 0$, let $\mathcal{C}_T = C([0, T] \to \mathbb{R}^d)$ be the path space over $\mathbb{R}^d$ with time interval $[0, T]$, and let $X_0, \eta \in L^2(\Omega \to \mathbb{R}^d, \mathcal{F}_0, \mathbb{P})$. For any $\varepsilon \geq 0$, let $(X_t^\varepsilon)_{t \geq 0}$ solve the SDE

$$dX_t^\varepsilon = b_t(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon})dt + \sigma_t(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon})dW_t, \quad X_0^\varepsilon = X_0 + \varepsilon \eta.$$

Obviously, $X_t = X_t^0$ solves (1.4) with initial value $X_0$. Consider the following linear SDE for $v_t^\eta$ on $\mathbb{R}^d$:

$$dv_t^\eta = \left\{ \nabla_{v_t^\eta} b_t(\cdot, \mathcal{L}_{X_t})(X_t) + \left( \mathbb{E} \langle D^L b_t(y, \cdot) (\mathcal{L}_{X_t})(X_t), v_t^\eta \rangle \right) \bigg|_{y=X_t} \right\} dt$$

$$+ \left\{ \nabla_{v_t^\eta} \sigma_t(\cdot, \mathcal{L}_{X_t})(X_t) + \left( \mathbb{E} \langle D^L \sigma_t(y, \cdot) (\mathcal{L}_{X_t})(X_t), v_t^\eta \rangle \right) \bigg|_{y=X_t} \right\} dW_t, \quad v_0^\eta = \eta.$$

The main result of this part is the following.

**Proposition 3.2.** Assume (H). Then for any $T > 0$, the limit

$$\nabla_{\eta} X_t := \lim_{\varepsilon \downarrow 0} \frac{X_t^\varepsilon - X_t}{\varepsilon}, \quad t \in [0, T]$$

exists in $L^2(\Omega \to \mathcal{C}_T, \mathbb{P})$. Moreover, $(v_t^\eta := \nabla_{\eta} X_t)_{t \in [0, T]}$ is the unique solution to the linear SDE (3.11).
To prove the existence of $\nabla_\eta X_t$ in (3.12), it suffices to show that when $\varepsilon \downarrow 0$

\[(3.13) \quad \xi^\varepsilon(t) := \frac{X^\varepsilon_t - X_t}{\varepsilon}, \quad t \in [0, T]\]

is a Cauchy sequence in $L^2(\Omega \to \mathcal{C}_T, \mathbb{P})$, i.e.

\[(3.14) \quad \lim_{\varepsilon, \delta \downarrow 0} \mathbb{E} \left[ \sup_{t \in [0, T]} |\xi^\varepsilon(t) - \xi^\delta(t)|^2 \right] = 0.\]

To this end, we need the following two lemmas.

**Lemma 3.3.** Assume (H). Then

\[\sup_{\varepsilon \in [0, 1]} \mathbb{E} \left[ \sup_{t \in [0, T]} |\xi^\varepsilon(t)|^2 \right] < \infty.\]

**Proof.** By (H), there exists a constant $C_1 > 0$ such that

\[
d|X^\varepsilon_t - X_t|^2 \\
= 2b_t(X^\varepsilon_t, \mathcal{L}_X), X^\varepsilon_t - X_t) + \|\sigma_t(X^\varepsilon_t, \mathcal{L}_X) - \sigma_t(X_t, \mathcal{L}_X)\|_{H^S}^2 \right) dt + dM_t \\
\leq C_1 \left\{ |X^\varepsilon_t - X_t|^2 + \mathbb{W}_2(\mathcal{L}_X, \mathcal{L}_X)^2 \right\} dt + dM_t,
\]

where

\[dM_t := 2 \langle X^\varepsilon_t - X_t, (\sigma_t(X^\varepsilon_t, \mathcal{L}_X) - \sigma_t(X_t, \mathcal{L}_X))dW_t \rangle\]

satisfies

\[(3.15) \quad d\langle M \rangle_t \leq C_1^2 \left\{ |X^\varepsilon_t - X_t|^2 + \mathbb{W}_2(\mathcal{L}_X, \mathcal{L}_X)^2 \right\}^2 dt.\]

Then by the BDG inequality, and noting that $\mathbb{W}_2(\mathcal{L}_x, \mathcal{L}_y)^2 \leq \mathbb{E}|\xi - \eta|^2$ for two random variables $\xi, \eta$, we may find out a constant $C_2 > 0$ such that

\[(3.16) \quad \mathbb{E} \left[ \sup_{s \in [0, t]} |X^\varepsilon_s - X_s|^2 \right] \leq \varepsilon^2 |\eta|^2 + 2C_1 \int_0^t \mathbb{E}|X^\varepsilon_s - X_s|^2 ds + C_2 \mathbb{E}\sqrt{\langle M \rangle_t}.\]

Noting that $\mathbb{W}_2(\mathcal{L}_{X^\varepsilon}, \mathcal{L}_{X_s})^2 \leq \mathbb{E}|X^\varepsilon_s - X_s|^2$, (3.15) yields

\[
C_2 \mathbb{E}\sqrt{\langle M \rangle_t} \leq C_1 C_2 \mathbb{E} \left( \int_0^t \left\{ |X^\varepsilon_s - X_s|^2 + \mathbb{W}_2(\mathcal{L}_{X^\varepsilon}, \mathcal{L}_{X_s})^2 \right\}^2 ds \right)^{1/2} \\
\leq C_1 C_2 \mathbb{E} \left( \sup_{s \in [0, t]} \left\{ |X^\varepsilon_s - X_s|^2 + \mathbb{E}|X^\varepsilon_s - X_s|^2 \right\} \int_0^t \left\{ |X^\varepsilon_s - X_s|^2 + \mathbb{E}|X^\varepsilon_s - X_s|^2 \right\} ds \right)^{1/2} \\
\leq \frac{1}{2} \mathbb{E} \left[ \sup_{s \in [0, t]} |X^\varepsilon_s - X_s|^2 \right] + \frac{C_3}{2} \int_0^t \mathbb{E}|X^\varepsilon_s - X_s|^2 ds.
\]
for some constant $C_3 > 0$. Combining this with (3.16) and noting that due to (1.7)

$$\mathbb{E} \left[ \sup_{s \in [0,t]} |X^\varepsilon_s - X_s|^2 \right] < \infty,$$

we arrive at

$$\mathbb{E} \left[ \sup_{s \in [0,t]} |X^\varepsilon_s - X_s|^2 \right] \leq 2\varepsilon^2 \mathbb{E} |\eta|^2 + C_3 \int_0^t \mathbb{E} |X^\varepsilon_s - X_s|^2 ds, \quad t \in [0,T], \varepsilon > 0.$$

Therefore, Gronwall’s inequality gives

$$\sup_{\varepsilon \in (0,1]} \mathbb{E} \left[ \sup_{t \in [0,T]} |\xi^\varepsilon(t)|^2 \right] = \sup_{\varepsilon \in (0,1]} \frac{1}{\varepsilon^2} \mathbb{E} \left[ \sup_{s \in [0,T]} |X^\varepsilon_s - X_s|^2 \right] \leq 2e^{C_3T} \mathbb{E} |\eta|^2 < \infty. \quad \square$$

For any differentiable (real, vector, or matrix valued) function $f$ on $\mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d)$, let

$$\Xi_f(t) = \frac{f(X^\varepsilon_t, \mathcal{L}_X^\varepsilon) - f(X_t, \mathcal{L}_X)}{\varepsilon} - \nabla \xi^\varepsilon(t) f(\cdot, \mathcal{L}_X)(X_t) - \left\{ \mathbb{E} \langle D^L f(y, \cdot) (\mathcal{L}_X^\varepsilon)(X_t), \xi^\varepsilon(t) \rangle \right\}_{y=X_t}, \quad t \in [0,T], \varepsilon > 0.$$

**Lemma 3.4.** Assume (H). For any (real, vector, or matrix valued) $C^{1,(1,0)}$-function $f$ on $\mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d)$ with

$$K_f := \sup_{(x,\mu) \in \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d)} \left( |\nabla f(\cdot, \mu)(x)|^2 + \|D^L f(x, \cdot)(\mu)\|^2_{L^2(\mu)} \right) < \infty,$$

there holds

$$\left| \Xi^\varepsilon_f(t) \right|^2 \leq 4K_f \left( \mathbb{E} |\xi^\varepsilon(t)|^2 + |\xi^\varepsilon(t)|^2 \right) \quad \text{and} \quad \lim_{\varepsilon \downarrow 0} \mathbb{E} \left| \Xi^\varepsilon_f(t) \right|^2 = 0, \quad t \in [0,T].$$

**Proof.** Let $X^\varepsilon_t(s) = X_t + s(X^\varepsilon_t - X_t)$, $s \in [0,1]$. By the chain rule and (3.1), we have

$$\begin{aligned}
\frac{f(X^\varepsilon_t, \mathcal{L}_X^\varepsilon) - f(X_t, \mathcal{L}_X)}{\varepsilon} &= \frac{1}{\varepsilon} \int_0^1 \left\{ \frac{d}{ds} f(X^\varepsilon_t(s), \mathcal{L}_X^\varepsilon(s)) \right\} ds \\
&= \int_0^1 \left\{ \nabla \xi^\varepsilon(t) f(\cdot, \mathcal{L}_X^\varepsilon(s))(X^\varepsilon_t(s)) + \mathbb{E} \langle D^L f(y, \cdot) (\mathcal{L}_X^\varepsilon(s))(X^\varepsilon_t(s)), \xi^\varepsilon(t) \rangle \right\}_{y=X^\varepsilon_t(s)} ds.
\end{aligned}$$

Combining this with (3.18) we obtain

$$\left| \Xi^\varepsilon_f(t) \right|^2 \leq 2 \int_0^1 \left| \nabla \xi^\varepsilon(t) \left\{ f(\cdot, \mathcal{L}_X^\varepsilon(s))(X^\varepsilon_t(s)) - f(\cdot, \mathcal{L}_X)(X_t) \right\} \right|^2 ds \\
+ 2 \int_0^1 \left| \mathbb{E} \langle D^L f(y, \cdot) (\mathcal{L}_X^\varepsilon(s))(X^\varepsilon_t(s)), \xi^\varepsilon(t) \rangle \right|_{y=X^\varepsilon_t(s)} ds \\
- \left( \mathbb{E} \langle D^L f(y, \cdot) (\mathcal{L}_X)(X_t), \xi^\varepsilon(t) \rangle \right)_{y=X_t}^2 ds \leq 8K_f \left( |\xi^\varepsilon(t)|^2 + \mathbb{E} |\xi^\varepsilon(t)|^2 \right).$$

(3.20)
Then Lemma 3.4 gives
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left[ \sup_{s \in [0,1]} |X_t^\varepsilon(s) - X_t|^2 \right] \leq \lim_{\varepsilon \to 0} \mathbb{E}|X_t^\varepsilon - X_t|^2 = 0.
\]
Thus, the $C^1(0,\varepsilon)$-property of $f$, Lemma 3.3 and the first inequality in (3.20) yield that $\Xi_j^\varepsilon(t) \to 0$ in probability as $\varepsilon \to 0$. Combining this with the first inequality in (3.19), Lemma 3.3, and using the dominated convergence theorem, we derive $\lim_{\varepsilon \to 0} \mathbb{E}|\Xi_j^\varepsilon(t)|^2 = 0$. \qed

**Proof of Proposition 3.2.** Let $(\Xi^\varepsilon(t), K_{b_t})$ and $(\Xi^\varepsilon(t), K_{\sigma_t})$ be defined as in (3.17) and (3.18) for $b_t$ and $\sigma_t$ replacing $f$ respectively. By (H), there exists a constant $C_1 > 0$ such that
\[
\sup_{t \in [0,T]} (K_{b_t} + K_{\sigma_t}) \leq C_1 < \infty.
\]
Then Lemma 3.4 gives
\[
(3.21) \quad \left| \Xi^\varepsilon_b(t) \right|^2 + \left| \Xi^\varepsilon_\sigma(t) \right|^2 \leq 4C \left( \|\xi^\varepsilon(t)\|_1^2 + \mathbb{E}|\xi^\varepsilon(t)|^2 \right),
\]
\[
\lim_{\varepsilon \to 0} \mathbb{E}(\|\Xi^\varepsilon_b(t)\|^2 + \|\Xi^\varepsilon_\sigma(t)\|^2) = 0, \quad t \in [0,T].
\]
By (3.10), (3.13), and (3.17) for $b_t$ and $\sigma_t$ replacing $f$, we have
\[
\xi^\varepsilon(t) = \int_0^t \left\{ \Xi^\varepsilon_b(s) + \nabla \xi^\varepsilon(s)b_s(\cdot, \mathcal{L}X_s)(X_s) + (\mathbb{E}(D^L b_s(y, \cdot)(\mathcal{L}X_s)(X_s), \xi^\varepsilon(s))) \right\} ds
\]
\[
+ \int_0^t \left( \Xi^\varepsilon_\sigma(s) + \nabla \xi^\varepsilon(s)\sigma_s(\cdot, \mathcal{L}X_s)(X_s) + (\mathbb{E}(D^L \sigma_s(y, \cdot)(\mathcal{L}X_s)(X_s), \xi^\varepsilon(s))) \right)_{y = X_s} dW_s
\]
for $t \in [0,T]$. So, for any $\varepsilon, \delta \in (0,1]$, $\xi^{\varepsilon,\delta}(t) := \xi^\varepsilon(t) - \xi^\delta(t)$ satisfies
\[
\left| \xi^{\varepsilon,\delta}(t) \right|^2 \leq 4 \int_0^t \left| \Xi^\varepsilon_b(s) - \Xi^\delta_b(s) \right|^2 ds + 4 \int_0^t \left| \Xi^\varepsilon_\sigma(s) - \Xi^\delta_\sigma(s) \right| ds
\]
\[
+ 4T \int_0^t \left| \nabla \xi^{\varepsilon,\delta}(s)b_s(\cdot, \mathcal{L}X_s)(X_s) + (\mathbb{E}(D^L b_s(y, \cdot)(\mathcal{L}X_s)(X_s), \xi^{\varepsilon,\delta}(s))) \right|_{y = X_s}^2 ds
\]
\[
+ 4 \int_0^t \left( \nabla \xi^{\varepsilon,\delta}(s)\sigma_s(\cdot, \mathcal{L}X_s)(X_s) + (\mathbb{E}(D^L \sigma_s(y, \cdot)(\mathcal{L}X_s)(X_s), \xi^{\varepsilon,\delta}(s))) \right)_{y = X_s} dW_s \right|^2.
\]
Combining this with (H) and using the BDG inequality, we find out a constant $C_2 > 0$ such that
\[
\mathbb{E} \left[ \sup_{s \in [0,t]} \xi^{\varepsilon,\delta}(s) \right] \leq C_2 \int_0^t \mathbb{E} \left( \left| \Xi^\varepsilon_b(s) - \Xi^\delta_b(s) \right|^2 + \left| \Xi^\varepsilon_\sigma(s) - \Xi^\delta_\sigma(s) \right|^2 \right) ds
\]
\[
+ C_2 \int_0^t \mathbb{E}\left| \xi^{\varepsilon,\delta}(s) \right|^2 ds, \quad t \in [0,T].
\]
Since Lemma 3.3 ensures that $\mathbb{E}\left[\sup_{s \in [0,t]} \xi^\varepsilon(s)\right] < \infty$, by Gronwall’s lemma this yields

$$
\mathbb{E}\left[\sup_{s \in [0,T]} \xi^{x,\varepsilon}(s)\right] \leq C_2 e^{C_2 T} \int_0^T \mathbb{E}\left(\left|\xi^{x,\varepsilon}_\tau(s) - \xi^{x,\varepsilon}_\tau(s)\right|^2 + \left|\xi^{x,\varepsilon}_\sigma(s) - \xi^{x,\varepsilon}_\sigma(s)\right|^2\right) ds.
$$

Combining this with (3.21) and Lemma 3.3, and applying the dominated convergence theorem, we prove the first assertion in Proposition 3.2.

Finally, by (3.10), (3.12), (3.21) and (3.17) for $b_t, \sigma_t$ replacing $f$, we conclude that $v_t^\eta := \nabla \eta X_t$ solves the SDE (3.11). Since this SDE is linear, the uniqueness is trivial. Then the proof is finished. \hfill \Box

### 3.3 Malliavin derivative

Consider the Cameron-Martin space

$$
\mathbb{H} = \left\{ h \in C([0,T] \to \mathbb{R}^d) : h_0 = 0, h_t' \text{ exists a.e. } t, \|h\|_H^2 := \int_0^T \|h_t\|^2 dt < \infty \right\}.
$$

Let $\eta \in L^2(\Omega \to \mathbb{R}^d, \mathcal{F}_0, \mathbb{P})$ with $\mathcal{L}_\eta = \mu$, and let $\mu_T$ be the distribution of $W_{[0,T]} := \{W_t\}_{t \in [0,T]}$, which is a probability measure (i.e., Wiener measure) on the path space $\mathcal{C}_T := C([0,T] \to \mathbb{R}^d)$. For $F \in L^2(\mathbb{R}^d \times \mathcal{C}_T, \mu \times \mu_T)$, $F(\eta, W_{[0,T]})$ is called Malliavin differentiable along direction $h \in \mathbb{H}$, if the directional derivative

$$
D_h F(\eta, W_{[0,T]}) := \lim_{\varepsilon \to 0} \frac{F(\eta, W_{[0,T]} + \varepsilon h) - F(\eta, W_{[0,T]})}{\varepsilon}
$$

exists in $L^2(\Omega, \mathbb{P})$. If the map $\mathbb{H} \ni h \mapsto D_h F \in L^2(\Omega, \mu)$ is bounded, then there exists a unique $DF(\eta, W_{[0,T]} \in L^2(\Omega \to \mathbb{H}, \mathbb{P})$ such that $\langle DF(\eta, W_{[0,T]}), h \rangle_\mathbb{H} = D_h F(\eta, W_{[0,T]}$ holds in $L^2(\Omega, \mathbb{P})$ for all $h \in \mathbb{H}$. In this case, we write $F(\eta, W_{[0,T]} \in \mathcal{D}(D)$ and call $DF(\eta, W_{[0,T]}$ the Malliavin gradient of $F(\eta, W_{[0,T]}$. It is well known that $(D, \mathcal{D}(D)$ is a closed linear operator from $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ to $L^2(\Omega \to \mathbb{H}, \mathcal{F}_T, \mathbb{P})$. The adjoint operator $(D^*, \mathcal{D}(D^*))$ of $(D, \mathcal{D}(D)$ is called Malliavin divergence. For simplicity, in the sequel we denote $F(\eta, W_{[0,T]}$ by $F$. Then we have the integration by parts formula

$$
(3.22) \quad \mathbb{E}(D_h F|\mathcal{F}_0) = \mathbb{E}(FD^* h|\mathcal{F}_0), \quad F \in \mathcal{D}(D), h \in \mathcal{D}(D^*).
$$

It is well known that for adapted $h \in L^2(\Omega \to \mathbb{H}, \mathbb{P})$, one has $h \in \mathcal{D}(D^*)$ with

$$
(3.23) \quad D^* h = \int_0^T \langle h'_t, dW_t \rangle.
$$

For more details and applications on Malliavin calculus one may refer to [19] and references therein.

For any $\varepsilon \geq 0$ and adapted $h \in L^2(\Omega \to \mathbb{H}, \mathbb{P})$, let $(X_t^{h,\varepsilon})_{t \geq 0}$ solve the SDE

$$
(3.24) \quad dX_t^{h,\varepsilon} = b_t(X_t^{h,\varepsilon}, \mathcal{L}_t^{h,\varepsilon}) dt + \sigma_t(X_t^{h,\varepsilon}, \mathcal{L}_t^{h,\varepsilon}) d(W_t + \varepsilon h_t), \quad X_0^{h,\varepsilon} = X_0.
$$
By (H) and \( h' \in L^2(\Omega \times [0, T], \mathbb{P} \times dt) \), this SDE is well-posed. Obviously, \( X_t^{h,0} = X_t \) solves (1.4) with initial value \( X_0 \). When \( \sigma_t(x, \mu) \) does not depend \((x, \mu)\), this SDE reduces to a random ODE for \( Y_t^{h,\varepsilon} := X_t^{h,\varepsilon} - \sigma_t W_t \), which is well-posed also for non-adapted \( h \) like \( h^\alpha \) in Theorem 2.3. The main result of this part is the following.

**Proposition 3.5.** Assume (H). Let \( h \in L^2(\Omega \to \mathbb{H}, \mathbb{P}) \), which is adapted if \( \sigma_t(x, \mu) \) depends on \( x \) or \( \mu \). Then the limit

\[
D_h X_t := \lim_{\varepsilon \downarrow 0} \frac{X_t^{h,\varepsilon} - X_t}{\varepsilon}, \quad t \in [0, T]
\]

exists in \( L^2(\Omega \to \mathcal{C}_T, \mathbb{P}) \). Moreover, \((w_t^h := D_h X_t)_{t \in [0, T]}\) is the unique solution to the SDE

\[
dw_t^h = \left\{ \nabla w_t^h \sigma_t(\cdot, \mathcal{L}_X)(X_t) + \left( \mathbb{E}(D^2 \sigma_t(y, \cdot)(\mathcal{L}_X)(X_t), w_t^h) \right) \big|_{y=X_t} \right\} dW_t
\]

\[
+ \left\{ \nabla w_t^h b_t(\cdot, \mathcal{L}_X)(X_t) + \left( \mathbb{E}(D b_t(y, \cdot)(\mathcal{L}_X)(X_t), w_t^h) \right) \big|_{y=X_t} + \sigma_t(X_t, \mathcal{L}_X) h_t^t \right\} dt
\]

with \( w_0^h = 0 \).

**Proof.** Comparing with the linear SDE (3.11), the additional term \( \varepsilon \sigma_t(X_t, \mathcal{L}_X) h_t^t \) comes from the derivative with respect to \( \varepsilon \) in \( 0 \) of the term \( \varepsilon \sigma_t(X_t^{h,\varepsilon}, \mathcal{L}_X^{h,\varepsilon}) h_t^t \) in (3.24), since

\[
\frac{d}{d\varepsilon} \left( \varepsilon \sigma_t(X_t^{h,\varepsilon}, \mathcal{L}_X^{h,\varepsilon}) \right) \big|_{\varepsilon=0} = \lim_{\varepsilon \downarrow 0} \varepsilon \sigma_t(X_t^{h,\varepsilon}, \mathcal{L}_X^{h,\varepsilon}) = \sigma_t(X_t, \mathcal{L}_X).
\]

Taking this into account, we may prove Proposition 3.5 by repeating the proof of Proposition 3.2. We omit the details to save space. \( \square \)

## 4 Proofs of main results

We first present an integration by parts formula for \( \nabla \eta X_T \) with \( \eta \in L^2(\Omega \to \mathbb{R}^d, \mathcal{F}_0, \mathbb{P}) \), then prove Theorem 2.1, Corollary 2.2 and Theorem 2.3 respectively.

### 4.1 An integration by parts formula

**Theorem 4.1.** Assume (H) and (2.1). Let \( f \in C^1_b(\mathbb{R}^d) \) and \( \eta \in L^2(\Omega \to \mathbb{R}^d, \mathbb{P}) \). Then for any \( 0 \leq r < T \) and \( g \in C^1([r, T], \mathbb{P}) \) with \( g_r = 0 \) and \( g_T = 1 \),

\[
\mathbb{E}(\langle \nabla f(X_T), \nabla \eta X_T \rangle | \mathcal{F}_r) = \mathbb{E} \left( f(X_T) \int_r^T \langle g'_t \sigma_t(X_t, \mathcal{L}_X)^{-1} v_t^n, dW_t \rangle | \mathcal{F}_r \right).
\]

**Proof.** Having Propositions 3.2 and 3.5 in hands, the proof is more or less standard. For \( v_t^n \) solving (3.11), we take

\[
h_t = \int_{t \wedge r}^t g_s' \sigma_s(X_s, \mathcal{L}_X) v_t^n ds, \quad t \in [0, T].
\]
By (H), (2.1), and that $h \in L^2(\Omega \to \mathbb{H}, \mathbb{P})$ is adapted, Proposition 3.5 applies. Let $\tilde{v}_t = g_t v^\eta_t$ for $t \in \lbrack r, T \rbrack$. Then (3.11) and (4.2) imply

$$
d\tilde{v}_t = \left\{ \nabla \tilde{v}_t b(\cdot, \mathcal{L}_X_t)(X_t) + \left( \mathbb{E}(D^L b_t(y, \cdot)(\mathcal{L}_X_t)(X_t), \tilde{v}_t) \right) \bigg|_{y = X_t} + g_t' v^\eta_t \right\} dt
$$

$$+

\left\{ \nabla \tilde{v}_t \sigma_t(\cdot, \mathcal{L}_X_t)(X_t) + \left( \mathbb{E}(D^L \sigma_t(y, \cdot)(\mathcal{L}_X_t)(X_t), \tilde{v}_t) \right) \bigg|_{y = X_t} + \sigma_t(X_t, \mathcal{L}_X_t) h'_t \right\} dt
$$

$$= \left\{ \nabla \tilde{v}_t b(\cdot, \mathcal{L}_X_t)(X_t) + \left( \mathbb{E}(D^L b_t(y, \cdot)(\mathcal{L}_X_t)(X_t), \tilde{v}_t) \right) \bigg|_{y = X_t} + \sigma_t(X_t, \mathcal{L}_X_t) h'_t \right\} dt
$$

$$+

\left\{ \nabla \tilde{v}_t \sigma_t(\cdot, \mathcal{L}_X_t)(X_t) + \left( \mathbb{E}(D^L \sigma_t(y, \cdot)(\mathcal{L}_X_t)(X_t), \tilde{v}_t) \right) \bigg|_{y = X_t} \right\} dW_t, \quad t \geq r, \quad \tilde{v}_r = 0.
$$

Therefore, $(\tilde{v}_t)_{t \geq r}$ solves the SDE (3.26) with $\tilde{v}_r = 0$. On the other hand, by (4.2) we have $h'_t = 0$ for $t < r$, so that the solution to (3.26) with $u^0_0 = 0$ satisfies $u^0_0 = 0$. So, the uniqueness of this SDE from time $r$ implies $\tilde{v}_t = u_t$ for all $t \geq r$. Combining this with Propositions 3.2 and 3.5, we obtain

$$\nabla \eta X_T = v^\eta_T = g_T v^\eta_T = \tilde{v}_T = u_T = D_h X_T.
$$

Thus, by the chain rule and the integration by parts formula (3.22), for any bounded $\mathcal{F}_r$-measurable $G \in \mathcal{D}(D)$, we have

$$
\mathbb{E}(G(\nabla f(X_T), \nabla \eta X_T)) = \mathbb{E}(G(\nabla f(X_T), D_h X_T)) = \mathbb{E}(G D_h f(X_T))
$$

$$= \mathbb{E}(D_h \{ G f(X_T) \} - f(X_T) D_h G) = \mathbb{E}(G f(X_T) D^*(h)),
$$

where in the last step we have used $D_h G = 0$ since $G$ is $\mathcal{F}_r$-measurable but $h'_t = 0$ for $t \leq r$. Noting that the class of bounded $\mathcal{F}_r$-measurable $G \in \mathcal{D}(D)$ is dense in $L^2(\Omega, \mathcal{F}_r, \mathbb{P})$, this implies

$$\mathbb{E}(\langle \nabla f(X_T), \nabla \eta X_T \rangle | \mathcal{F}_r) = \mathbb{E}(f(X_T) D^*(h) | \mathcal{F}_r).
$$

Combining this with

$$D^*(h) = \int_r^T \langle h'_t, dW_t \rangle = \int_r^T \langle g_t' \sigma_t(X_t, h_t, P_t \mu)^{-1} v_t, dW_t \rangle
$$

due to (3.23) and (4.2), we prove (4.1).

\[ \square \]

### 4.2 Proof of Theorem 2.1

Let $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. We first establish (2.3) for $f \in \mathcal{B}_b(\mathbb{R}^d)$, then construct $\gamma \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)$ such that

$$(4.3) \quad \lim_{\mu(\mathcal{A}) \to 0} \frac{|(P_T f)(\mu \circ (\text{Id} + \phi)^{-1}) - (P_T f)(\mu) - \mu(\langle \phi, \gamma \rangle)|}{\sqrt{\mu(\mathcal{A})}} = 0,
$$

which, by definition, implies that $P_T f$ is $L$-differentiable at $\mu$ with $D^L P_T f(\mu) = \gamma$.

(a) Proof of (2.3) for $f \in \mathcal{B}_b(\mathbb{R}^d)$. When $f \in C^1_b(\mathbb{R}^d)$, (2.3) follows from (4.1) for $\eta = \phi(X_0)$. Below we extend the formula to $f \in \mathcal{B}_b(\mathbb{R}^d)$. For $s \in [0, 1]$, let $X^{\phi,s}_t$ solve (1.4)
for $X_0^\phi = X_0 + s\phi(X_0)$. We have $\mu^\phi := \mathcal{L}_{X_0^\phi} = \mu \circ (\text{Id} + s\phi)^{-1}$, and by the definition of $\nabla_\eta X_T$ for $\eta = \phi(X_0)$,

\begin{equation}
(P_T f) (\mu^\phi_e) - (P_T f)(\mu) = \mathbb{E}[f(X_{T_T}^\phi) - f(X_T)] = \int_0^\varepsilon \frac{d}{ds} \mathbb{E}[f(X_T^\phi)] \, ds
\end{equation}

(4.4)

Next, let $(v^\phi_t)_{t \in [0,T]}$ solve (3.11) for $\eta = \phi(X_0)$ and $X_t$ replacing $X_t$, i.e.

\begin{equation}
\mathbb{d}v^\phi_t = \left\{ \begin{array}{l}
\nabla v^\phi_t \cdot b_t(\cdot, \mathcal{L}_{X_t^\phi})(X_t^\phi) + (\mathbb{E} \langle D^s b_t(y, \cdot)(\mathcal{L}_{X_t^\phi})(X_t^\phi), v^\phi_t \rangle) \big|_{y = X_t^\phi} \big) dt \\
+ \nabla v^\phi_t \sigma_t(\cdot, \mathcal{L}_{X_t^\phi})(X_t^\phi) + (\mathbb{E} \langle D^s \sigma_t(y, \cdot)(\mathcal{L}_{X_t^\phi})(X_t^\phi), v^\phi_t \rangle) \big|_{y = X_t^\phi} \big) \mathbb{d}W_t,
\end{array} \right.
\end{equation}

(4.5)

for $v_0^\phi = \phi(X_0)$. Then (4.4) and (4.1) imply

\begin{equation}
(P_T f) (\mu^\phi_e) - (P_T f)(\mu) = \int_0^\varepsilon \mathbb{E} \left[ f(X_T^\phi) \int_0^T \langle g^s \sigma_t(X_t^\phi, \mathcal{L}_{X_t^\phi})^{-1} v_t^\phi, \mathbb{d}W_t \rangle \right] \, ds, \quad f \in C^1_b(\mathbb{R}^d).
\end{equation}

(4.6)

By a standard approximation argument, we may extend this formula to all $f \in \mathcal{B}(\mathbb{R}^d)$. Indeed, let

$$
\nu_\varepsilon(A) = \int_0^\varepsilon \mathbb{E} \left[ 1_A(X_T^\phi) \int_0^T \langle g^s \sigma_t(X_t^\phi, \mathcal{L}_{X_t^\phi})^{-1} v_t^\phi, \mathbb{d}W_t \rangle \right] \, ds, \quad A \in \mathcal{B}(\mathbb{R}^d).
$$

Then $\nu_\varepsilon$ is a finite signed measure on $\mathbb{R}^d$ with

$$
\int_{\mathbb{R}^d} f \, d\nu_\varepsilon = \int_0^\varepsilon \mathbb{E} \left[ f(X_T^\phi) \int_0^T \langle g^s \sigma_t(X_t^\phi, \mathcal{L}_{X_t^\phi})^{-1} v_t^\phi, \mathbb{d}W_t \rangle \right] \, ds, \quad f \in \mathcal{B}(\mathbb{R}^d).
$$

So, (4.6) is equivalent to

\begin{equation}
\int_{\mathbb{R}^d} f \, dP_T^\varepsilon \mu^\phi_e - \int_{\mathbb{R}^d} f \, dP_T \mu = \int_{\mathbb{R}^d} f \, d\nu_\varepsilon, \quad f \in C^1_b(\mathbb{R}^d).
\end{equation}

(4.7)

Since $\nu_{T,e} := P_T^\varepsilon \mu^\phi_e + P_T \mu + |\nu_\varepsilon|$ is a finite measure on $\mathbb{R}^d$, $C^1_b(\mathbb{R}^d)$ is dense in $L^1(\mathbb{R}^d, \nu_{T,e})$. Hence, (4.7) holds for all $f \in \mathcal{B}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d, \nu_{T,e})$. Consequently, (4.6) holds for all $f \in \mathcal{B}(\mathbb{R}^d)$. Thus,

\begin{equation}
(P_T f)(\mu^\phi_e) - (P_T f)(\mu) = \frac{1}{\varepsilon} \int_0^\varepsilon \mathbb{E} \left[ f(X_T^\phi) \int_0^T \langle g^s \sigma_t(X_t^\phi, \mathcal{L}_{X_t^\phi})^{-1} v_t^\phi, \mathbb{d}W_t \rangle \right] \, ds, \quad f \in \mathcal{B}(\mathbb{R}^d).
\end{equation}

(4.8)
It is easy to see from (H) that
\[
\lim_{s \to 0} \sup_{t \in [0, T]} \mathbb{E}\left( |X_t^{\phi, s} - X_t|^2 + |v_t^{\phi, s} - v_t^{\phi}|^2 \right) = 0.
\]

So,
\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\varepsilon \mathbb{E} \left[ \int_0^T \langle g_t' \{ \sigma_t(X_t^{\phi, s}, \mathcal{L}_{X_t^{\phi, s}})^{-1} v_t^{\phi, s} - \sigma_t(X_t, \mathcal{L}_t)^{-1} v_t^{\phi} \}, dW_t \rangle \right] = 0. \tag{4.9}
\]

Combining this with (4.8), we see that (2.3) for \( f \in \mathcal{B}_b(\mathbb{R}^d) \) follows from
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left[ \left\{ f(X_T^{\phi, \varepsilon}) - f(X_T) \right\} \int_0^T \langle g_t' \sigma_t(X_t, \mathcal{L}_t)^{-1} v_t^{\phi}, dW_t \rangle \right] = 0, \quad f \in \mathcal{B}_b(\mathbb{R}^d). \tag{4.10}
\]

To prove this equality, for \( r \in (0, T) \) we denote
\[
I_r := \int_0^r \langle g_t' \sigma_t(X_t, \mathcal{L}_t)^{-1} v_t^{\phi}, dW_t \rangle.
\]

Applying (4.1) with \( g_t := \frac{t-r}{T-r} \) for \( t \in [r, T] \), we derive
\[
\left| \mathbb{E}[I_r \{ f(X_T^{\phi, \varepsilon}) - f(X_T) \}] \right| = \left| \mathbb{E} \left[ I_r \int_0^\varepsilon \langle \nabla f(X_T^{\phi, s}), \nabla \phi(X_0) X_T^{\phi, s} \rangle ds \right] \right|
\leq \mathbb{E} \left[ |I_r| \cdot \int_0^\varepsilon \mathbb{E}(\langle \nabla f(X_T^{\phi, s}), \nabla \phi(X_0) X_T^{\phi, s} \rangle |\mathcal{F}_r| ds \right]
\leq \|f\|_\infty \int_0^\varepsilon \mathbb{E} \left[ |I_r| \left( \int_r^T \left| \frac{1}{T-r} \sigma_t(X_t^{\phi, s}, \mathcal{L}_{X_t^{\phi, s}})^{-1} v_t^{\phi, s} \right|^2 dt \right)^{\frac{1}{2}} \right] ds, \quad f \in C^1_b(\mathbb{R}^d).
\]

By the argument extending (4.6) from \( f \in C^1_b(\mathbb{R}^d) \) to \( f \in \mathcal{B}_b(\mathbb{R}^d) \), we conclude from this that for any \( r \in (0, T) \),
\[
\lim_{\varepsilon \to 0} \sup_{\|f\|_{\infty} \leq 1} \left| \mathbb{E}[I_r \{ f(X_T^{\phi, \varepsilon}) - f(X_T) \}] \right|
\leq \lim_{\varepsilon \to 0} \int_0^\varepsilon \mathbb{E} \left[ |I_r| \left( \int_r^T \left| \frac{1}{T-r} \sigma_t(X_t^{\phi, s}, \mathcal{L}_{X_t^{\phi, s}})^{-1} v_t^{\phi, s} \right|^2 dt \right)^{\frac{1}{2}} \right] ds = 0.
\]

Therefore,
\[
\lim_{\varepsilon \to 0} \sup_{\|f\|_{\infty} \leq 1} \left| \mathbb{E} \left[ \left\{ f(X_T^{\phi, \varepsilon}) - f(X_T) \right\} \int_0^T \langle g_t' \sigma_t(X_t, \mathcal{L}_t)^{-1} v_t^{\phi}, dW_t \rangle \right] \right|
\leq 2 \left( \mathbb{E} \int_r^T \left| g_t' \sigma_t(X_t, \mathcal{L}_t)^{-1} v_t^{\phi} \right|^2 dt \right)^{\frac{1}{2}} \tag{4.11}
\]

\[\text{20}\]
holds for \( r \in (0, T) \). By letting \( r \uparrow T \) we prove (4.10).

(b) For any \( f \in \mathcal{B}_b(\mathbb{R}^d) \), we intend to find out \( \gamma \in L^2(\mathbb{R}^d \rightarrow \mathbb{R}^d, \mu) \) such that

\[
(4.12) \quad \mathbb{E} \left[ f(X_T) \int_0^T \langle g'_t \sigma_t(X_t, \mathcal{L}_{X_t})^{-1} v^p_t, dW_t \rangle \right] = \mu(\langle \phi, \gamma \rangle), \quad \phi \in L^2(\mathbb{R}^d \rightarrow \mathbb{R}^d, \mu).
\]

When \( f \in C_b(\mathbb{R}^d) \), in step (c) we will deduce from this and (2.3) that \( \gamma = DL_P f(\mu) \). To construct the desired \( \gamma \), consider the SDE

\[
dX^\phi_t = b_t(X^\phi_t, \mathcal{L}_{X^\phi_t}) dt + \sigma_t(X^\phi_t, \mathcal{L}_{X^\phi_t}) dW_t, \quad X^\phi_0 = X_0 + \phi(X_0),
\]

and let \( v^\phi_t \) solve (2.2). Since (2.2) is a linear equation for \( v^\phi_t \) with initial value \( \phi(X_0) \in L^2(\Omega \rightarrow \mathbb{R}^d, \mathcal{F}_0, \mathbb{P}) \), the functional

\[
L^2(\mathbb{R}^d \rightarrow \mathbb{R}^d, \mu) \ni \phi \mapsto L\phi := \mathbb{E} \left[ f(X_T) \int_0^T \langle g'_t \sigma_t(X_t, \mathcal{L}_{X_t})^{-1} v^\phi_t, dW_t \rangle \right]
\]

is linear, and by (H) and (2.1), there exists a constant \( C > 0 \) such that

\[
|L\phi|^2 \leq \|f\|_\infty^2 \sup_{t \in [0, T]} |g'_t \lambda_t|^2 \mathbb{E} \int_0^T |v^\phi_t|^2 dt \leq C \mathbb{E} |\phi(X_0)|^2 = C \mu(|\phi|^2), \quad \phi \in L^2(\mathbb{R}^d \rightarrow \mathbb{R}^d, \mu).
\]

Then \( L \) is a bounded linear functional on the Hilbert space \( L^2(\mathbb{R}^d \rightarrow \mathbb{R}^d, \mu) \). By Riesz’s representation theorem, there exists a unique \( \gamma \in L^2(\mathbb{R}^d \rightarrow \mathbb{R}^d, \mu) \) such that

\[
L\phi = \mu(\langle \gamma, \phi \rangle), \quad \phi \in L^2(\mathbb{R}^d \rightarrow \mathbb{R}^d, \mu).
\]

Therefore, (4.12) holds.

(c) Now, for \( f \in \mathcal{B}_b(\mathbb{R}^d) \), we intend to verify (4.3) for \( \gamma \) in (4.12), so that \( P_T f \) is \( L \)-differentiable with \( DL_P f(\mu) = \gamma \). By (4.8) for \( \varepsilon = 1 \), we have

\[
(4.13) \quad (P_T f)(\mu^1) - (P_T f)(\mu) = \int_0^1 \mathbb{E} \left[ f(X^\phi_t) \int_0^T \langle g'_t \sigma_t(X_t, \mathcal{L}_{X_t})^{-1} v^\phi_t, dW_t \rangle dt \right], \quad f \in \mathcal{B}_b(\mathbb{R}^d).
\]

Combining this with (4.12) and noting that \( \mu^1 = \mu \circ (\text{Id} + \phi)^{-1} \), we arrive at

\[
(4.14) \quad \frac{|(P_T f)(\mu \circ (\text{Id} + \phi)^{-1}) - (P_T f)(\mu) - \mu(\langle \phi, \gamma \rangle)|}{\sqrt{\mu(|\phi|^2)}} \leq \varepsilon_1(\phi) + \varepsilon_2(\phi) + \varepsilon_3(\phi),
\]

where

\[
\varepsilon_1(\phi) := \frac{1}{\sqrt{\mu(|\phi|^2)}} \int_0^1 \mathbb{E} \left[ \left( f(X^\phi_t) - f(X_T) \right) \int_0^T \langle g'_t \sigma_t(X_t, \mathcal{L}_{X_t})^{-1} v^\phi_t, dW_t \rangle ds \right],
\]

\[
\varepsilon_2(\phi) := \frac{\|f\|_\infty}{\sqrt{\mu(|\phi|^2)}} \int_0^1 \mathbb{E} \left[ \int_0^T \left\langle g'_t \sigma_t(X_t, \mathcal{L}_{X_t})^{-1} - \sigma_t(X_t, \mathcal{L}_{X_t})^{-1} \right\rangle v^\phi_t, dW_t \right] ds,
\]

and

\[
\varepsilon_3(\phi) := \frac{\|f\|_\infty}{\sqrt{\mu(|\phi|^2)}} \int_0^1 \mathbb{E} \left[ \int_0^T \left\langle g'_t \sigma_t(X_t, \mathcal{L}_{X_t})^{-1} \right\rangle v^\phi_t, dW_t \right] ds.
\]
\[ \varepsilon_3(\phi) := \frac{\|f\|_\infty}{\sqrt{\mu(\|\phi\|^2)}} \int_0^1 \mathbb{E} \left[ \int_0^T g_t^{\phi} \{\sigma_t(X_t^{\phi,s}, \mathcal{L}_{X_t^{\phi,s}})^{-1}(v_t^{\phi,s} - v_t^{\phi})\} dW_t \right] ds. \]

It is easy to deduce from (H) that for any \( p \geq 2 \) there exists a constant \( c(p) > 0 \) such that
\[(4.15) \quad \sup_{t \in [0,T], s \in [0,1]} \mathbb{E} \left( |X_t^{\phi,s} - X_t|^p + |v_t^{\phi,s}|^p \right) \leq c(p) |\phi(X_0)|^p. \]

Combining this with the continuity of \( \sigma_t(x, \mu) \) in \( x \) and \( \mu \), we conclude that
\[(4.16) \quad \lim_{\mu(\|\phi\|) \to 0} \varepsilon_2(\phi) = 0. \]

Next, by the argument deducing (2.3) from (4.8), it is easy to see that (4.15) implies
\[(4.17) \quad \lim_{\mu(\|\phi\|) \to 0} \varepsilon_1(\phi) = 0. \]

Moreover, by the SDEs for \( v_t^{\phi,s} \) and \( v_t^{\phi} \) we have
\[ d(v_t^{\phi,s} - v_t^{\phi}) = \left\{ A_t(v_t^{\phi,s} - v_t^{\phi}) + \tilde{A}_t v_t^{\phi,s} \right\} dt + \left\{ B_t(v_t^{\phi,s} - v_t^{\phi}) + \tilde{B}_t v_t^{\phi} \right\} dW_t, \]
where for a square integrable random variable \( v \) on \( \mathbb{R}^d \),
\[
A_t v := \nabla v b_t(\cdot, \mathcal{L}_{X_t})(X_t) + \left( \mathbb{E} \langle D^L b_t(y, \cdot) (\mathcal{L}_{X_t})(X_t), v \rangle \right)_{y = X_t},
\]
\[
\tilde{A}_t v := \nabla v b_t(\cdot, \mathcal{L}_{X_t^{\phi,s}})(X_t^{\phi,s}) + \left( \mathbb{E} \langle D^L b_t(y, \cdot) (\mathcal{L}_{X_t^{\phi,s}})(X_t^{\phi,s}), v \rangle \right)_{y = X_t^{\phi,s}} - \nabla v b_t(\cdot, \mathcal{L}_{X_t})(X_t) + \left( \mathbb{E} \langle D^L b_t(y, \cdot) (\mathcal{L}_{X_t})(X_t), v \rangle \right)_{y = X_t},
\]
\[
B_t v := \nabla v \sigma_t(\cdot, \mathcal{L}_{X_t})(X_t) + \left( \mathbb{E} \langle D^L \sigma_t(y, \cdot) (\mathcal{L}_{X_t})(X_t), v \rangle \right)_{y = X_t},
\]
\[
\tilde{B}_t v := \nabla v \sigma_t(\cdot, \mathcal{L}_{X_t^{\phi,s}})(X_t^{\phi,s}) + \left( \mathbb{E} \langle D^L \sigma_t(y, \cdot) (\mathcal{L}_{X_t^{\phi,s}})(X_t^{\phi,s}), v \rangle \right)_{y = X_t^{\phi,s}} - \nabla v \sigma_t(\cdot, \mathcal{L}_{X_t})(X_t) + \left( \mathbb{E} \langle D^L \sigma_t(y, \cdot) (\mathcal{L}_{X_t})(X_t), v \rangle \right)_{y = X_t}.
\]

Combining this with (4.15) and (H), there exists a constant \( c > 0 \) such that
\[(4.18) \quad d|v_t^{\phi,s} - v_t^{\phi}|^2 \leq c|v_t^{\phi,s} - v_t^{\phi}|^2 dt + c(\|\tilde{A}_t\|^2 + \|\tilde{B}_t\|^2) (|v_t^{\phi,s}|^2 + |v_t^{\phi}|^2) dt + dM_t, \quad |v_0^{\phi,s} - v_0^{\phi}| = 0 \]
holds for some martingale \( M_t \), and that
\[(4.19) \quad \|\tilde{A}_t\|^2 + \|\tilde{B}_t\|^2 \leq c, \quad \lim_{\mu(\|\phi\|) \to 0} (\|\tilde{A}_t\|^2 + \|\tilde{B}_t\|^2) = 0, \quad t \in [0, T], s \in [0,1]. \]

By (4.18) and (4.15) for \( p = 4 \), there exists a constant \( c' > 0 \) such that
\[
\mathbb{E}(|v_t^{\phi,s} - v_t^{\phi}|^2 |\mathcal{F}_0) \leq c \int_0^t \mathbb{E}(|v_r^{\phi,s} - v_r^{\phi}|^2 |\mathcal{F}_0) dr + 2c \int_0^T \sqrt{\mathbb{E}(\|\tilde{A}_t\|^4 + \|\tilde{B}_t\|^4 |\mathcal{F}_0) \cdot \sqrt{\mathbb{E}(|v_t^{\phi,s}|^4 + |v_t^{\phi}|^4 |\mathcal{F}_0) dt}}.
\]
\[
\leq c \int_0^t \mathbb{E}(|\mu_t(\phi) - \phi_t|^p |\mathcal{F}_0)dr + c'\varepsilon(\phi)|\phi(X_0)|^2, \quad s \in [0,1], t \in [0,T],
\]

where
\[
\varepsilon(\phi) := \int_0^T \sqrt{\mathbb{E}(|\tilde{A}_t|^{4} + |\tilde{B}_t|^{4}|\mathcal{F}_0)dt}.
\]

Then Gronwall’s lemma and (4.19) yield
\[
\sup_{s \in [0,T]} \mathbb{E}(|\mu_t^{\phi,s} - \phi_t|^p |\mathcal{F}_0) \leq c' e^{cT\varepsilon(\phi)}|\phi(X_0)|^2,
\]

\[
\lim_{\mu(\phi^2) \to 0} \mathbb{E}\varepsilon(\phi) = 0.
\]

Combining this with the definition of \(\varepsilon_3(\phi)\), \((H)\), and Jensen’s inequality for the conditional expectation \(\mathbb{E}(\cdot |\mathcal{F}_0)\), we may find out constants \(C_1, C_2 > 0\) depending on \(\|f\|_\infty\) and \(T\) such that

\[
\lim_{\mu(\phi^2) \to 0} \varepsilon_3(\phi) \leq \lim_{\mu(\phi^2) \to 0} \frac{C_1}{\sqrt{\mu(\phi^2)}} \int_0^1 \mathbb{E}\left(\int_0^T |\mu_t^{\phi,s} - \phi_t|^2 dt\right)^{\frac{1}{2}} ds
\]

\[
\leq \lim_{\mu(\phi^2) \to 0} \frac{C_1}{\sqrt{\mu(\phi^2)}} \int_0^1 \mathbb{E}\left(\int_0^T \mathbb{E}(|\mu_t^{\phi,s} - \phi_t|^2 |\mathcal{F}_0)dt\right)^{\frac{1}{2}} ds
\]

\[
\leq \lim_{\mu(\phi^2) \to 0} \frac{C_2}{\sqrt{\mu(\phi^2)}} \int_0^1 \mathbb{E}(|\phi(X_0)|\sqrt{\varepsilon(\phi)}) ds
\]

\[
\leq \lim_{\mu(\phi^2) \to 0} \frac{C_2\mathbb{E}(\phi(X_0)^2)\mathbb{E}_\varepsilon(\phi)}{\sqrt{\mu(\phi^2)}} = \lim_{\mu(\phi^2) \to 0} C_2\sqrt{\mathbb{E}(\phi)} = 0.
\]

This, together with (4.14), (4.16) and (4.17), implies (4.3). Therefore, \(P_Tf\) is \(L\)-differentiable at \(\mu\) with \(D^L(P_Tf)(\mu) = \gamma\).

(d) Finally, (2.3) and (4.8) imply

\[
\left| \frac{P_T^\ast \mu \circ (\text{Id} + \varepsilon \phi)^{-1} - P_T^\ast \mu}{\varepsilon} - (\psi P_T^\ast \mu)(f) \mid \left\langle \frac{(P_Tf)(\mu) - (P_Tf)(\mu)}{\varepsilon} - \mathbb{E} \left[ f(X_T) \int_0^T \langle g_t^f \sigma_t(X_t, \mathcal{L}_{X_t})^{-1} v_t^\phi, dW_t \rangle \right] \right\rangle \right| \\
\leq \|f\|_\infty \int_0^\varepsilon \left| \mathbb{E} \left[ \int_0^T \langle g_t^f \sigma_t(X_t, \mathcal{L}_{X_t})^{-1}v_t^\phi, dW_t \rangle \right] \right| ds \\
+ \frac{1}{\varepsilon} \mathbb{E} \left[ \left\{ f(X_T^{\phi,\varepsilon}) - f(X_T) \right\} \int_0^T \langle g_t^f \sigma_t(X_t, \mathcal{L}_{X_t})^{-1} v_t^\phi \rangle, dW_t \right] \right| ds.
\]

Combining this with (4.9) and (4.11) we prove (2.4).
4.3 Proof of Corollary 2.2

Proof of (1). By (H) and (2.2), there exists a martingale $M_t$ such that
\begin{equation}
|v^\phi_t|^2 \leq 4K(t)|v^\phi_t|(|v^\phi_t| + E|v^\phi_t|)dt + dM_t, \quad |v^\phi_0|^2 = |\phi(X_0)|^2,
\end{equation}
where $K(t)$ is increasing in $t \geq 0$. Then
\begin{equation}
E|v^\phi_t|^2 \leq E|\phi(X_0)|^2 + 4K(t) \int_0^t \{E|v^\phi_s|^2 + (E|v^\phi_t|^2)^2\} ds \leq \mu(|\phi|^2) + 8K(t) \int_0^t E|v^\phi_s|^2 ds.
\end{equation}
By Gronwall’s inequality this implies
\begin{equation}
E|v^\phi_t|^2 \leq e^{8K(t)t} \mu(|\phi|^2), \quad t \in [0,T].
\end{equation}
Next, since $E \int_0^T \langle g^f_t \sigma_t(X_t, \mathcal{L}_{X_t})^{-1} v^\phi_t, \, dW_t \rangle = 0$, (2.3) is equivalent to
\begin{equation}
D^f_\phi(P_Tf)(\mu) = E \left[ \left\{ f(X_T) - P_Tf(\mu) \right\} \int_0^T \langle g^f_t \sigma_t(X_t, \mathcal{L}_{X_t})^{-1} v^\phi_t, \, dW_t \rangle \right].
\end{equation}
Combining this with (4.21) and using Jensen’s inequality, when $\mu(|\phi|^2) \leq 1$ we have
\begin{align*}
|D^f_\phi(P_Tf)(\mu)|^2 &\leq \left\{ (P_Tf^2)(\mu) - (P_Tf(\mu))^2 \right\} \int_0^T E|g^f_t \sigma_t(X_t, \mathcal{L}_{X_t})^{-1} v^\phi_t|^2 dt \\
&\leq \left\{ (P_Tf^2)(\mu) - (P_Tf(\mu))^2 \right\} \int_0^T |g^f_t|^2 \lambda_t^2 e^{8K(t)} dt
\end{align*}
for any $g \in C^1([0,T])$ with $g_0 = 0$ and $g_T = 1$. Taking
\begin{equation}
g_t = \frac{\int_0^t \lambda^{-2} e^{-8rK(r)} dr}{\int_0^T \lambda^{-2} e^{-8rK(r)} dr}, \quad t \in [0,T],
\end{equation}
we prove the estimate (2.5). \hfill \Box

Proof of (2). Let $f \in \mathcal{B}_0(\mathbb{R}^d)$ with $\|f\|_{\infty} \leq 1$. By Theorem 2.1, $P_Tf$ is $L$-differentiable. Moreover, by Theorem 4.1, $P_Tf$ is Lipschitz continuous on $\mathcal{B}_2(\mathbb{R}^d)$. Indeed, for any $\mu_1, \mu_2 \in \mathcal{B}_2(\mathbb{R}^d)$, let $X_1, X_2 \in L^2(\Omega \rightarrow \mathbb{R}^d, \mathcal{F}_0, \mathbb{P})$ such that $\mathcal{L}_{X_i} = \mu_i, 1 \leq i \leq 2$, and $E|X_1 - X_2|^2 = \mathbb{W}_2(\mu_1, \mu_2)^2$. Let $X_T^i$ be the solution to (1.4) with $X_0 = X_1 + s(X_2 - X_1), s \in [0,1]$. Then Theorem 4.1 implies
\begin{align*}
|P_Tf(\mu_1) - P_Tf(\mu_2)|^2 &= |E f(X_0^0) - E f(X_1^0)|^2 = \left| \int_0^1 \frac{d}{ds} E f(X_T^s) \, ds \right|^2 \\
&= \left| \int_0^1 E \langle \nabla f(X_T^s), \nabla(X_2 - X_1) \rangle ds \right|^2 \leq cE|X_2 - X_1|^2 = c\mathbb{W}_2(\mu_1, \mu_2)^2
\end{align*}
for some constant $c > 0$. 

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To apply Proposition 3.1, we take \( \{\mu_n, \nu_n\}_{n \geq 1} \subset \mathcal{P}_2(\mathbb{R}^d) \) which have compact supports and are absolutely continuous with respect to the Lebesgue measure, such that

\[
\lim_{n \to \infty} \left\{ \mathbb{W}_2(\mu_n, \mu_n) + \mathbb{W}_2(\nu_n, \nu_n) \right\} = 0.
\]

According to \cite{4}, see also \cite[Theorem 5.8]{6}, for any \( n \geq 1 \) there exists a unique map \( \phi_n \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu) \) such that

\[
\nu_n = \mu_n \circ (\text{Id} + \phi_n)^{-1}, \quad \mathbb{W}_2(\mu_n, \nu_n)^2 = \mu_n(|\phi_n|^2).
\]

Let \( X_n \in L^2(\Omega \to \mathbb{R}^d, \mathcal{F}_0, \mathbb{P}) \) such that \( \mathcal{L}_{X_n} = \mu_n \). By Proposition 3.1, \( (2.5) \) and \( (4.23) \), we obtain

\[
\left\| (P_T f)(\mu_n) - (P_T f)(\nu_n) \right\|^2 = \left\| \int_0^1 \frac{d}{ds}(P_T f)(\mathcal{L}_{X_n+\phi_n}(X_n)) \right\|_{L^2(\mathcal{L}_{X_n+\phi_n}(X_n))}^2
\]

\[
\leq \int_0^1 E\langle D^L(P_T f)(\mathcal{L}_{X_n+\phi_n}(X_n)) \rangle(X_n, \phi_n(X_n)) \right\|_{L^2(\mathcal{L}_{X_n+\phi_n}(X_n))}^2
\]

By the continuity of \( P_T f \) and \( (4.22) \), by letting \( n \to \infty \) we prove

\[
\left\| (P_T f)(\mu) - (P_T f)(\mu) \right\|^2 \leq \frac{\mathbb{W}_2(\mu, \nu)}{\int_0^T \lambda_t^{-1} e^{-8K(t)} dt}, \quad \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d), \quad f \in \mathcal{B}_b(\mathbb{R}^d), \quad \|f\|_\infty \leq 1.
\]

Therefore, \( (2.6) \) and \( (2.7) \) hold.

\[\square\]

### 4.4 Proof of Theorem 2.3

Let \( T > r \geq 0, \mu \in \mathcal{P}_2(\mathbb{R}^{n+m}) \) and let \( X_t \) solve \( (2.8) \) with \( \mathcal{L}_{X_0} = \mu \). To realize the procedure in the proof of Theorem 2.1 for the present degenerate setting, we first extend Theorem 4.1 using \( D^*(h^{\alpha}_{r,t}) \) to replace \( \int_T^T \langle \eta_t^2 \sigma_t(X_t, \mathcal{L}_{X_t})^{-1} \nu_t^2, dW_t \rangle \), where for a \( C^1([r, T] \to \mathbb{R}^{n+m}) \)-valued random variable \( \alpha = (\alpha^{(1)}, \alpha^{(2)}) \),

\[
h^{\alpha}_{r,t} := \int_{t,T} \sigma_s^{-1} \left\{ \nabla_{\alpha_s} h^{\alpha}_{s,t} \right\} ds
\]

for \( t \in [0, T] \).

**Theorem 4.2.** Assume \( (H1) \). Let \( T > r \geq 0, \eta \in L^2(\Omega \to \mathbb{R}^{n+m}, \mathcal{F}_0, \mathbb{P}) \), and let \( X_t \) solve \( (2.8) \) with \( \mathcal{L}_{X_0} = \mu \in \mathcal{P}_2(\mathbb{R}^{n+m}) \). If there exists a \( C^1([r, T] \to \mathbb{R}^{n+m}) \)-valued random variable \( \alpha = (\alpha^{(1)}, \alpha^{(2)}) \) such that \( \alpha_r = \nabla_{\eta} X_r, \alpha_T = 0 \),

\[
(\alpha^{(1)}_t)' = \nabla_{\alpha_t} h^{(1)}_{r,t}(X_t), \quad t \in [r, T],
\]

and \( h^{\alpha}_{r,t} \in \mathcal{P}(D^*) \), then for any \( f \in C_b^1(\mathbb{R}^{m+n}) \),

\[
\mathbb{E}(\langle \nabla f(X_T), \nabla_{\eta} X_T \rangle | \mathcal{F}_r) = \mathbb{E}\left( f(X_T) D^*(h^{\alpha}_{r,T}) | \mathcal{F}_r \right).
\]

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Proof. By Proposition 3.5, \( w_t := D_{h^o_{r,T}} X_t \) satisfies
\[
w_t = \int_0^t \left\{ \nabla_{w_s} b_s \left( \cdot, \mathcal{L}_{X_s} \right)(X_s) + \left( 0, \sigma_s h_s^o \right)' + \left( \mathbb{E} \langle D^2 b_s \rangle(y, \cdot) \left( \mathcal{L}_{X_s} \right)(X_s), w_s \right) \right\} y = X_s ds.
\]
Since \((h_{r,T}^o)'(s) = 0\) for \( s \leq r \), this implies \( w_t = 0 \) for \( t \in [0, r] \) so that
\[
w_t = \int_r^T \left\{ \nabla_{w_s} b_s \left( \cdot, \mathcal{L}_{X_s} \right)(X_s) + \left( 0, \sigma_s h_s^o \right)' + \left( \mathbb{E} \langle D^2 b_s \rangle(y, \cdot) \left( \mathcal{L}_{X_s} \right)(X_s), w_s \right) \right\} y = X_s ds.
\]
Extending \( \alpha_t \) with \( \alpha_t := \nabla_\eta X_t \) for \( t \in [0, r] \), and letting \( v_t = w_t + \alpha_t \) for any \( t \in [0, T] \), we obtain
\[
v_t = \alpha_t + \int_0^t \left\{ \nabla_{v_s} b_s \left( \cdot, \mathcal{L}_{X_s} \right)(X_s) + \left( 0, \left( \mathbb{E} \langle D^2 b_s \rangle(y, \cdot) \left( \mathcal{L}_{X_s} \right)(X_s), v_s \right) \right\} y = X_s ds
\]
\[= \left( 0, \sigma_s h_s^o \right)' - \left( \mathbb{E} \langle D^2 b_s \rangle(y, \cdot) \left( \mathcal{L}_{X_s} \right)(X_s), \sigma_s \right) - \nabla_{\alpha_s} b_s \left( \cdot, \mathcal{L}_{X_s} \right)(X_s) \right\} ds \]
By (4.25),
\[
\int_0^t \nabla_{\alpha_s} b_s \left( \cdot, \mathcal{L}_{X_s} \right)(X_s) \right\} ds = 1_{\{t > r\}} \left( \alpha^{(1)}_t - \nabla_\eta X^{(2)}_t \right), \]
while the definition of \( h^o_{r,s} \) implies
\[
\int_0^t \left\{ \sigma_s \left( h^o_s \right)' - \left( \mathbb{E} \langle D^2 b_s \rangle(y, \cdot) \left( \mathcal{L}_{X_s} \right)(X_s), \sigma_s \right) \right\} y = X_s ds
\]
\[= 1_{\{t > r\}} \left( \alpha^{(2)}_r - \alpha^{(2)}_t \right).
\]
Combining these with (4.27) and Proposition 3.2 leads to
\[
v_t = \nabla_\eta X_T + \int_0^t \left\{ \nabla_{v_s} b_s \left( \cdot, \mathcal{L}_{X_s} \right)(X_s) + \left( 0, \left( \mathbb{E} \langle D^2 b_s \rangle(y, \cdot) \left( \mathcal{L}_{X_s} \right)(X_s), v_s \right) \right\} y = X_s ds
\]
\[= \alpha_t + \int_r^T \left\{ \nabla_{\alpha_s} b_s \left( \cdot, \mathcal{L}_{X_s} \right)(X_s) + \left( 0, \left( \mathbb{E} \langle D^2 b_s \rangle(y, \cdot) \left( \mathcal{L}_{X_s} \right)(X_s), v_s \right) \right\} y = X_s ds \]
\[= \eta + \int_0^t \left\{ \nabla_{v_s} b_s \left( \cdot, \mathcal{L}_{X_s} \right)(X_s) + \left( 0, \left( \mathbb{E} \langle D^2 b_s \rangle(y, \cdot) \left( \mathcal{L}_{X_s} \right)(X_s), v_s \right) \right\} y = X_s dsight\} ds, \quad t \in [0, T].
\]
That is, \( v_t \) solves (3.11) so that by Proposition 3.2 we obtain \( v_t := w_t + \alpha_t = \nabla_\eta X_t. \) Since \( \alpha_T = 0 \), this implies \( D_{h^o_{r,T}} X_T = \nabla_\eta X_T \). Thus, for any bounded \( \mathcal{F}_r \)-measurable \( G \in \mathcal{B}(D), \)
\[
\mathbb{E} \left[ G \langle \nabla f(X_T), \nabla_{\eta} X_T \rangle \right] = \mathbb{E} \left[ G \langle D_{h^o_{r_T}} f(X_T) \rangle \right] = \mathbb{E} \left[ D_{h^o_{r,T}} \langle G f(X_T) \rangle - f(X_T) D_{h^o_{r,T}} G \right] = \mathbb{E} \left[ G f(X_T) D^*(h^o_{r,T}) \right],
\]
where in the last step we have used the integration by parts formula (3.22) and \( D_{h^o_{r,T}} G = 0 \) since \( G \) is \( \mathcal{F}_r \)-measurable but
\[
D_{h^o_{r,T}} G = \int_0^T (h^o_{r,T})'(s) \cdot \langle (DG)'Y(s) \rangle ds = 0,
\]
\((h^o_{r,T})'(s) = 0 \) for \( s \leq r \). Noting that the class of bounded \( \mathcal{F}_r \)-measurable functions \( G \in \mathcal{B}(D) \) is dense in \( L^2(\Omega, \mathcal{F}_r, \mathbb{P}) \), (4.28) implies (4.26). \( \square \)
Proof of Theorem 2.3. With Theorem 4.2 in hands, the proof is completely similar to that of Theorem 2.1. Let

$$v_t^\phi = ((v_t^\phi)^{(1)}, (v_t^\phi)^{(2)}) = (\nabla_{\phi(X_0)}X_t^{(1)}, \nabla_{\phi(X_0)}X_t^{(2)}) = \nabla_{\phi(X_0)}X_t, \ t \in [0, T].$$

For any $0 \leq r < T$, let

$$\alpha_{r,t}^{(2)} = \frac{T - t}{T - r}(v_t^\phi)^{(2)} - \frac{(t - r)(T - t)}{T - r}B_t^sK_{T,t}(v_t^\phi)^{(1)} - \frac{(t - r)}{(T - r)}\int_0^T \frac{\theta_s^2Q_r^{-1}K_{T,r}(v_t^\phi)^{(1)}}{\theta_s^2}ds,$$

(4.29)

$$- (t - r)(T - t)B_t^sK_{T,t}Q_s^{-1} \int_0^T \frac{\theta_s^2Q_r^{-1}K_{T,r}(v_t^\phi)^{(1)}}{\theta_s^2}ds, \ t \in [r, T],$$

and

$$\alpha_{r,t}^{(1)} = K_{t,r}(v_t^\phi)^{(1)} + \int_r^t K_{t,s} \nabla_{\alpha_s^b}b_s^{(1)}(X_s(x))ds, \ t \in [r, T].$$

(4.30)

Then $\alpha_r := (\alpha_{r,t}^{(1)}, \alpha_{r,t}^{(2)})$ satisfies

$$\alpha_{r,r} = \nabla_{\phi(X_0)}X_r, \ \alpha_{r,T} = 0,$$

and by (2.9) and Duhamel’s formula, (4.30) implies

$$(\alpha_{r,t}^{(1)})'(t) = \nabla_{\alpha_s^b}b_t^{(1)}(X_t), \ t \in [r, T].$$

Moreover, let $h_{r;r}^{\alpha_{r;}}$ be defined in (4.24) for $\alpha_r$, replacing $\alpha$. Noting that (H1) and (H2) imply [28, (H)] for $L_1 = L_2 = 0$, the proof of [28, Theorem 1.1] with $\phi(s) := (s - r)(T - s)$ for $s \in [r, T]$ ensures that $h_{r;r}^{\alpha_{r;}} \in \mathcal{D}(D^s)$ with $D^s(h_{r;r}^{\alpha_{r;}}) \in L^p(\mathbb{P})$ for all $p \in (1, \infty)$. So, by Theorem 2.3 with $\eta = \phi(X_0)$ we obtain

$$\mathbb{E}(\langle \nabla f(X_T), \nabla_{\phi(X_0)}X_T \rangle | \mathcal{F}_r) = \mathbb{E}(f(X_T)D^s(h_{r;r}^{\alpha_{r;}}) | \mathcal{F}_r), \ f \in C_b^1(\mathbb{R}^d), r \in [0, T).$$

(4.31)

In particular, taking $r = 0$ we obtain $D^s(h) \in L^p(\mathbb{P})$ for all $p \in (1, \infty)$ and

$$D^s \phi P_T f(\mu) = \mathbb{E}(\langle \nabla f(X_T), \nabla_{\phi(X_0)}X_T \rangle) = \mathbb{E}(f(X_T)D^s(h^\alpha) | \mathcal{F}_r), \ f \in C_b^1(\mathbb{R}^d).$$

(4.32)

Basing on these two formulas, by repeating the proof of Theorem 2.1 with $I_r := \mathbb{E}(D^s(h^\alpha) | \mathcal{F}_r)$, we prove (2.16) and the $L$-differentiability of $P_T f$ for $f \in \mathcal{B}_b(\mathbb{R}^{m+d})$. Finally, the estimates (2.17) and (2.18) follows from (2.16) as in the proof of Theorem 2.1, together with the corresponding estimate on $\mathbb{E}|D^s(h^\alpha)|^2$ as in the proof of [28, Theorem 1.1]. For instance, below we outline the proof of (2.16).

Firstly, for $s \in (0, 1)$ let $X_t^s$ solve (2.8) with $X_0^s = X_0 + s\phi(X_0)$, let $\mu^\phi,s := \mathcal{L}_{X_0^s} = \mu \circ (\text{Id} + \phi)^{-1}$, and let $\alpha_{r,t}^\phi,s$ be defined as $\alpha_{r,t}$ with $X_t^s$ replacing $X_t$. Then as in (4.4) and (4.7), (4.32) implies

$$(P_T f)(\mu^\phi,e) - (P_T f)(\mu) = \int_0^e \mathbb{E}(\langle \nabla f(X_t^s), \nabla_{\phi(X_0)}X_t^s \rangle) ds$$

(4.33)

$$= \int_0^e \mathbb{E}[f(X_t^s)D^s(h_{r;r}^{\alpha_{r;}})], \ f \in C^1_b(\mathbb{R}^{m+d}),$$
where $h^{\alpha_{\phi \epsilon}} := h_{0.\epsilon}^{\alpha_{\phi \epsilon}}$ satisfies
\begin{equation}
(4.34) \quad \lim_{\epsilon \to 0} \mathbb{E}[D^*(h^{\alpha_{\phi \epsilon}}) - D^*(h)]^2 = 0.
\end{equation}

By the argument leading to (4.8), (4.33) yields
\[
\lim_{\epsilon \to 0} \mathbb{E}[D^*(h^{\alpha_{\phi \epsilon}}) - D^*(h)]^2 = 0.
\]

Combining this with (4.34), we prove (2.16) provided
\begin{equation}
(4.35) \quad \lim_{\epsilon \to 0} \mathbb{E}[\{f(X_T^{\phi, \epsilon}) - f(X_T)\} D^*(h^\alpha)] ds = 0.
\end{equation}

For any $r \in (0, T)$, let $I_r = \mathbb{E}(D^*(h^\alpha)|\mathcal{F}_r)$. By (4.33) we obtain
\[
\mathbb{E}[\{f(X_T^{\phi, \epsilon}) - f(X_T)\} I_r] = \mathbb{E}[I_r \mathbb{E}(f(X_T^{\phi, \epsilon}) - f(X_T)|\mathcal{F}_r)]
\]
\[
= \mathbb{E}\left[I_r \int_0^\epsilon \mathbb{E}((\nabla f(X_T^{\phi, \epsilon}), \nabla X_T^{\phi, \epsilon})|\mathcal{F}_r) ds\right]
\]
\[
= \mathbb{E}\left[I_r \int_0^\epsilon \mathbb{E}(f(X_T^{\phi, \epsilon}) D^*(h^\alpha_{\phi \epsilon})|\mathcal{F}_r) ds\right]
\]
\[
= \int_0^\epsilon \mathbb{E}[I_r f(X_T^{\phi, \epsilon}) D^*(h^\alpha_{\phi \epsilon})] ds, \quad f \in C_b^1(\mathbb{R}^d).
\]

Combining this with the argument extending (4.8) from $f \in C_b^1(\mathbb{R}^d)$ to $f \in \mathcal{B}_b(\mathbb{R}^d)$, we obtain
\[
\mathbb{E}[\{f(X_T^{\phi, \epsilon}) - f(X_T)\} I_r] = \int_0^\epsilon \mathbb{E}[I_r f(X_T^{\phi, \epsilon}) D^*(h^\alpha_{\phi \epsilon})] ds, \quad f \in \mathcal{B}_b(\mathbb{R}^d).
\]

Consequently,
\[
\lim_{\epsilon \to 0} \mathbb{E}[\{f(X_T^{\phi, \epsilon}) - f(X_T)\} I_r] = 0, \quad f \in \mathcal{B}_b(\mathbb{R}^d), r \in (0, T).
\]

Then for any $r \in (0, T),$
\[
\limsup_{\epsilon \downarrow 0} \left| \frac{1}{\epsilon} \int_0^\epsilon \mathbb{E}[\{f(X_T^{\phi, \epsilon}) - f(X_T)\} D^*(h^\alpha)] ds \right|
\]
\[
= \limsup_{\epsilon \downarrow 0} \left| \frac{1}{\epsilon} \int_0^\epsilon \mathbb{E}[\{f(X_T^{\phi, \epsilon}) - f(X_T)\} \cdot \{D^*(h^\alpha) - I_r\}] ds \right|
\]
\[
\leq 2 \|f\|_\infty \mathbb{E}[D^*(h^\alpha) - \mathbb{E}(D^*(h^\alpha)|\mathcal{F}_r)].
\]

Letting $r \uparrow T$ we derive (4.35), and hence prove (2.16) as explained above. \hfill \Box

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References


