Bismut Formula for Lions Derivative of Distribution Dependent SDEs and Applications*

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Abstract

By using Malliavin calculus, Bismut type formulas are established for the Lions derivative of $P_t f(\mu) := \mathbb{E} f(X_t^\mu)$, where $t > 0$, f is a bounded measurable function, and X_t^μ solves a distribution dependent SDE with initial distribution μ. As applications, explicit estimates are derived for the Lions derivative and the total variational distance between distributions of solutions with different initial data. Both degenerate and non-degenerate situations are considered. Due to the lack of the semigroup property and the invalidity of the formula $P_t f(\mu) = \int P_t f(x)\mu(\mathrm{d}x)$, essential difficulties are overcome in the study.

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1 Introduction

The Bismut formula introduced in [3], also called Bismut-Elworthy-Li formula due to [12], is a powerful tool in characterising the regularity of distribution for SDEs and SPDEs. A plenty of results have been derived for this type formulas and applications by using stochastic analysis and coupling methods, see for instance [24] and references therein.

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On the other hand, because of crucial applications in the study of nonlinear PDEs and environment dependent financial systems, the distribution dependent SDEs (also called McKean-Vlasov or mean filed SDEs) have received increasing attentions, see [10, 11, 13, 14, 18, 22, 23] and references therein. Recently, this type SDEs have been applied in [5, 9, 17, 20] to characterize PDEs involving the Lions derivative (L-derivative for short) introduced by P.-L. Lions in his lectures [6]. In this paper, we aim to investigate Bismut type L-derivative formula and applications for distribution dependent SDEs with possibly degenerate noise.

To introduce our main results, we first recall the L-derivative. Let $\mathscr{P}(\mathbb{R}^d)$ be the space of all probability measures on $\mathbb{R}^d$, and let

$$\mathscr{P}_2(\mathbb{R}^d) = \Big\{\mu \in \mathscr{P}(\mathbb{R}^d) : \ \mu(|\cdot|^2) := \int_{\mathbb{R}^d} |x|^2 \mu(\mathrm{d}x) < \infty\Big\}.$$

Then $\mathscr{P}_2(\mathbb{R}^d)$ is a Polish space under the Wasserstein distance

$$\mathbb{W}_2(\mu,\nu) := \inf_{\pi\in\mathscr{C}(\mu,\nu)} \bigg(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 \pi(\mathrm{d}x,\mathrm{d}y)\bigg)^{\frac{1}{2}}, \quad \mu,\nu\in\mathscr{P}_2(\mathbb{R}^d),$$

where $\mathscr{C}(\mu,\nu)$ is the set of couplings for μ and ν; that is, $\pi\in\mathscr{C}(\mu,\nu)$ is a probability measure on $\mathbb{R}^d\times\mathbb{R}^d$ such that $\pi(\cdot\times\mathbb{R}^d)=\mu$ and $\pi(\mathbb{R}^d\times\cdot)=\nu$. We will use $\mathbf{0}$ to denote vectors with components 0, or the constant map taking value $\mathbf{0}$.

Definition 1.1. Let $f : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$, and let $g : M \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ for a differentiable manifold M.

(1) f is called L-differentiable at $\mu\in\mathscr{P}_2(\mathbb{R}^d)$, if the functional

$$L^2(\mathbb{R}^d\to\mathbb{R}^d,\mu) \ni \phi \mapsto f(\mu\circ(\mathrm{Id}+\phi)^{-1})$$

is Fréchet differentiable at $\mathbf{0}\in L^2(\mathbb{R}^d\to\mathbb{R}^d,\mu)$; that is, there exists (hence, unique) $\gamma\in L^2(\mathbb{R}^d\to\mathbb{R}^d,\mu)$ such that

$$\lim_{\mu(|\phi|^2)\to 0} \frac{f(\mu\circ(\mathrm{Id}+\phi)^{-1}) - f(\mu) - \mu(\langle\gamma,\phi\rangle)}{\sqrt{\mu(|\phi|^2)}} = 0. \tag{1.1}$$

In this case, we denote $D^L f(\mu) = \gamma$ and call it the L-derivative of f at μ.

(2) If the L-derivative $D^L f(\mu)$ exists for all $\mu\in\mathscr{P}_2(\mathbb{R}^d)$, then f is called L-differentiable. If, moreover, for every $\mu\in\mathscr{P}_2(\mathbb{R}^d)$ there exists a μ-version $D^L f(\mu)(\cdot)$ such that $D^L f(\mu)(x)$ is jointly continuous in $(x,\mu)\in\mathbb{R}^d\times\mathscr{P}_2(\mathbb{R}^d)$, we denote $f\in C^{(1,0)}(\mathscr{P}_2(\mathbb{R}^d))$.

(3) g is called differentiable on $M\times\mathscr{P}_2(\mathbb{R}^d)$, if for any $(x,\mu)\in M\times\mathscr{P}_2(\mathbb{R}^d)$, $g(\cdot,\mu)$ is differentiable at x and $g(x,\cdot)$ is L-differentiable at μ. If, moreover, $\nabla g(\cdot,\mu)(x)$ and $D^L g(x,\cdot)(\mu)(y)$ are joint continuous in $(x,y,\mu)\in M^2\times\mathscr{P}_2(\mathbb{R}^d)$, where ∇ is the gradient operator on M, we write $g\in C^{1,(1,0)}(M\times\mathscr{P}_2(\mathbb{R}^d))$.

As indicated in [20] that for any $n \ge 1$, $g \in C^1(\mathbb{R}^n)$ and $h_1, \cdots, h_n \in C_b^1(\mathbb{R}^d)$, the cylindrical function

$$\mu \mapsto g(\mu(h_1), \cdots, \mu(h_n))$$

is in $C^{(1,0)}(\mathscr{P}_2(\mathbb{R}^d))$ with

$$D^L g(\mu)(x) = \sum_{i=1}^n \big(\partial_i g(\mu(h_1), \cdots, \mu(h_n))\big) \nabla h_i(x), \quad (x, \mu) \in \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d).$$

Obviously, if f is L-differentiable at μ, then

$$D_\phi^L f(\mu) := \lim_{\varepsilon \downarrow 0} \frac{f(\mu \circ (\mathrm{Id} + \varepsilon\phi)^{-1}) - f(\mu)}{\varepsilon} = \mu\big(\langle D^L f(\mu), \phi\rangle\big), \ \phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu). \tag{1.2}$$

We may call D_ϕ^L the directional L-derivative along ϕ. This directional derivative has been used in earlier references, see for instance [21] for the Wasserstein diffusions constructed using the directional derivative on $\mathscr{P}_2(\mathbb{S}^1)$, where $\mathbb{S}^1$ is the unit circle.

When $D_\phi^L f(\mu)$ is a bounded linear functional of $\phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)$, there exists a unique $\xi \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)$ such that $D_\phi^L f(\mu) = \mu(\langle \xi, \phi\rangle)$ holds for all $\phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)$. In this case, $\phi \mapsto f(\mu \circ (\mathrm{Id} + \phi)^{-1})$ is Gâteaux differentiable at $\mathbf{0}$, and we say that f is weakly L-differentiable at μ, since the Gâteaux differentiability is weaker than the Fréchet one.

By (1.2), for an L-differentiable function f on $\mathscr{P}_2(\mathbb{R}^d)$, we have

$$\|D^L f(\mu)\| := \|D^L f(\mu)(\cdot)\|_{L^2(\mu)} = \sup_{\mu(|\phi|^2) \le 1} |D_\phi^L f(\mu)|. \tag{1.3}$$

For a vector-valued function $f = (f_i)$, or a matrix-valued function $f = (f_{ij})$ with L-differentiable components, we write

$$D_\phi^L f(\mu) = (D_\phi^L f_i(\mu)), \ \text{or} \ D_\phi^L f(\mu) = (D_\phi^L f_{ij}(\mu)), \quad \mu \in \mathscr{P}_2(\mathbb{R}^d).$$

Let W_t be a d-dimensional Brownian motion on the natural filtered probability space $(\Omega^0, \mathscr{F}^0, \{\mathscr{F}_t^0\}_{t \ge 0}, \mathbb{P})$. To ensure that for any $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ there exists a random variable X on $\mathbb{R}^d$ with distribution μ, let μ^0 be a probability measure on $\mathbb{R}^d$ which is equivalent to the Lebesgue measure, and enlarge the probability space as

$$(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t \ge 0}, \mathbb{P}) := (\Omega^0 \times \mathbb{R}^d, \mathscr{F}^0 \times \mathscr{B}(\mathbb{R}^d), \{\mathscr{F}_t^0 \times \mathscr{B}(\mathbb{R}^d)\}_{t \ge 0}, \ \mathbb{P}^0 \times \mu^0).$$

Then

$$W_t(\omega) := W_t(\omega^0), \quad t \ge 0, \omega := (\omega^0, x) \in \Omega$$

is a d-dimensional Brownian motion on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t \ge 0}, \mathbb{P})$. Let $\mathscr{L}_\xi$ denote the distribution of a random variable on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$. In case different probability spaces are concerned, we write $\mathscr{L}_{\xi|\mathbb{P}}$ instead of $\mathscr{L}_\xi$ to emphasize the reference probability measure $\mathbb{P}$.

Consider the following distribution dependent SDE on $\mathbb{R}^d$:

$$\mathrm{d}X_t = b_t(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma_t(X_t, \mathscr{L}_{X_t})\mathrm{d}W_t, \quad X_0 \in L^2(\Omega \to \mathbb{R}^d, \mathscr{F}_0, \mathbb{P}), \tag{1.4}$$

where

$$\sigma : [0, \infty) \times \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}^{d\otimes d}, \quad b : [0, \infty) \times \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$$

are continuous such that for some increasing function $K : [0, \infty) \to [0, \infty)$ there holds

$$\begin{aligned} &|b_t(x, \mu) - b_t(y, \nu)| + \|\sigma_t(x, \mu) - \sigma_t(y, \nu)\| \\ &\leq K(t)\big(|x - y| + \mathbb{W}_2(\mu, \nu)\big), \quad t \geq 0, x, y \in \mathbb{R}^d, \mu, \nu \in \mathscr{P}_2(\mathbb{R}^d) \end{aligned} \tag{1.5}$$

and

$$\|\sigma_t(\mathbf{0}, \delta_{\mathbf{0}})\| + |b_t(\mathbf{0}, \delta_{\mathbf{0}})| \leq K(t), \quad t \geq 0, \tag{1.6}$$

where and in what follows, for $x \in \mathbb{R}^d$ we denote by δ_x the Dirac measure at x, and $\|\cdot\|$ is the operator norm. For any $t \geq 0$, let $L^2(\Omega \to \mathbb{R}^d, \mathscr{F}_t, \mathbb{P})$ be the class of $\mathscr{F}_t$-measurable square integrable random variables on $\mathbb{R}^d$. By (1.5) and (1.6), for any $s \geq 0$ and $X_s \in L^2(\Omega \to \mathbb{R}^d, \mathscr{F}_s, \mathbb{P})$, (1.4) has a unique solution $(X_{s,t})_{t \geq s}$ with $X_{s,s} = X_s$ and

$$\mathbb{E}\Big[\sup_{t \in [s,T]} |X_{s,t}|^2\Big] < \infty, \quad T \geq s, \tag{1.7}$$

see, for instance [27], where gradient estimates and Harnack inequalities are also derived for the associated nonlinear semigroup. See also [16, 18] for weaker conditions ensuring the existence and uniqueness of solutions to (1.4). For any $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ and $s \geq 0$, let $(X^\mu_{s,t})_{t \geq s}$ be the solution to (1.4) with $\mathscr{L}_{X_{s,s}} = \mu$. Denote

$$P^*_{s,t}\mu = \mathscr{L}_{X^\mu_{s,t}}, \quad t \geq s, \mu \in \mathscr{P}_2(\mathbb{R}^d). \tag{1.8}$$

Let

$$(P_{s,t}f)(\mu) = (P^*_{s,t}\mu)(f) := \int_{\mathbb{R}^d} f \mathrm{d}(P^*_{s,t}\mu) = \mathbb{E}f(X^\mu_{s,t}), \quad t \geq s, f \in \mathscr{B}_b(\mathbb{R}^d), \mu \in \mathscr{P}_2(\mathbb{R}^d). \tag{1.9}$$

Then for any $0 \leq s \leq t$, $P_{s,t}$ is a linear operator from $\mathscr{B}_b(\mathbb{R}^d)$ to $\mathscr{B}_b(\mathscr{P}_2(\mathbb{R}^d))$.

In this paper, we aim to establish the Bismut type formula for the L-derivative of $P_{s,t}f$ for $t > s$. By considering the SDE for $\tilde{X}_t := X_{t+s}, t \geq 0$, without loss of generality we may and do assume $s = 0$. So, for simplicity, below we only establish the derivative formula for $P_t f := P_{0,t}f, t > 0$. More precisely, for any $T > 0$, $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ and $\phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)$, we aim to construct an integrable random variable $M^{\mu,\phi}_T$ such that

$$D^L_\phi(P_T f)(\mu) = \mathbb{E}\big[f(X^\mu_T)M^{\mu,\phi}_T\big], \quad f \in \mathscr{B}_b(\mathbb{R}^d), \tag{1.10}$$

which in turn implies the L-differentiability of $P_T f$. Note that the derivative formula for $(P_T f)(x) := (P_T f)(\delta_x)$ along a vector $v \in \mathbb{R}^d$ is derived in [2], which is the special case of (1.10) with $\mu = \delta_x$ and $\phi \equiv v$. Moreover, formulas of the L-derivative and integration by parts have been presented in [8] for the following de-coupled SDE:

$$\mathrm{d}X^{x,\mu}_t = b(t, X^{x,\mu}_t, P^*_t\mu)\mathrm{d}t + \sigma(t, X^{x,\mu}_t, P^*_t\mu)\mathrm{d}W_t, \quad X^{x,\mu}_t = x,$$

which is different from the original SDE (1.4) but has important applications in solving PDEs with Lions' derivatives, see [5, 17, 20] and references within.

When the SDE (1.4) is distribution independent, i.e. $b_t(x,\mu) = b_t(x)$ and $\sigma_t(x,\mu) = \sigma_t(x)$ do not depend on μ, the Bismut type formula

$$\nabla P_T f(x) = \mathbb{E}\big[f(X_T^x) M_T^x\big], \quad x \in \mathbb{R}^d, f \in \mathscr{B}_b(\mathbb{R}^d) \tag{1.11}$$

has been well studied in the literature, where M_T^x is an integrable random variable on $\mathbb{R}^d$, which is measurable in $x \in \mathbb{R}^d$ when it varies, see for instance [1, 15, 25, 26, 28] and references within. Since the coefficients are distribution independent, we have

$$(P_T f)(\mu) = \int_{\mathbb{R}^d} (P_T f)(x)\, \mu(\mathrm{d}x), \tag{1.12}$$

so that $P_T f$ is L-differentiable with $D^L(P_T f)(\mu) = \nabla P_T f$. Hence, by (1.11) and (1.12) we obtain

$$\begin{aligned} D_\phi^L(P_T f)(\mu) &= \mu(\langle D^L P_T f, \phi\rangle) = \int_{\mathbb{R}^d} \mathbb{E}\big[f(X_T^x)\langle M_T^x, \phi(x)\rangle\big]\, \mu(\mathrm{d}x) \\ &= \mathbb{E}\big[f(X_T^\mu)\langle M_T^{X_0^\mu}, \phi(X_0^\mu)\rangle\big]. \end{aligned}$$

Therefore, (1.10) holds for $M_T^{\mu,\phi} = \langle M_T^{X_0^\mu}, \phi(X_0^\mu)\rangle$.

However, when the SDE is distribution dependent, as explained in [27] that in general (1.12) does not hold, so it is non-trivial to establish the Bismut type formula (1.10).

The remainder of the paper is organized as follows. In section 2, we state our main results on Bismut formulas of $D_\phi^L P_T f$ and applications, for both non-degenerate and degenerate distribution dependent SDEs. To establish the Bismut formula using Malliavin calculus, we make necessary preparations in Section 3 concerning partial derivatives in the initial value, and Malliavin derivative for solutions of (1.4). Finally, complete proofs of the main results are addressed in Section 4.

2 Main results

Let $|\cdot|$ denote the norm in $\mathbb{R}^d$, and $\|\cdot\|$ denote the operator norm for matrices or more generally linear operators. We make the following assumption.

(H) For any $t \geq 0$, $b_t, \sigma_t \in C^{1,(1,0)}(\mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d))$. Moreover, there exists a continuous function $K : [0,\infty) \to [0,\infty)$, such that (1.6) holds and

$$\begin{aligned} &\max\Big\{\|\nabla b_t(\cdot,\mu)(x)\|, \|D^L b_t(x,\cdot)(\mu)\|,\ \frac{1}{2}\|\nabla\sigma_t(\cdot,\mu)(x)\|^2, \frac{1}{2}\|D^L\sigma_t(x,\cdot)(\mu)\|^2\Big\} \\ &\leq K(t), \quad t \geq 0, x \in \mathbb{R}^d, \mu \in \mathscr{P}_2(\mathbb{R}^d), \end{aligned}$$

where as in (1.3), $\|D^L f(\mu)\| := \|D^L f(\mu)(\cdot)\|_{L^2(\mu)}$ for an L-differentiable function f at μ.

Obviously, **(H)** implies (1.5) and (1.6), so that the SDE (1.4) has a unique solution for any initial value $X_0 \in L^2(\Omega \to \mathbb{R}^d, \mathscr{F}_0, \mathbb{P})$.

In the following two subsections, we state our main results for non-degenerate and degenerate cases respectively.

2.1 The non-degenerate case

For each $t > 0$, let σ_t be invertible such that

$$\|\sigma_t(x,\mu)^{-1}\| \le \lambda_t, \quad t \ge 0, x \in \mathbb{R}^d, \mu \in \mathscr{P}_2(\mathbb{R}^d) \tag{2.1}$$

holds for some continuous function $\lambda : [0,\infty) \to (0,\infty)$. Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, and let X_t solve (1.4) for $X_0 \in L^2(\Omega \to \mathbb{R}^d, \mathscr{F}_0, \mathbb{P})$ with $\mathscr{L}_{X_0} = \mu$. Given $\phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)$, consider the following SDE for v_t^ϕ on $\mathbb{R}^d$:

$$\begin{aligned} \mathrm{d}v_t^\phi = &\Big\{\nabla_{v_t^\phi} b_t(\cdot, \mathscr{L}_{X_t})(X_t) + \big(\mathbb{E}\langle D^L b_t(y,\cdot)(\mathscr{L}_{X_t})(X_t), v_t^\phi\rangle\big)\big|_{y=X_t}\Big\}\mathrm{d}t \\ &+ \Big\{\nabla_{v_t^\phi} \sigma_t(\cdot, \mathscr{L}_{X_t})(X_t) + \big(\mathbb{E}\langle D^L \sigma_t(y,\cdot)(\mathscr{L}_{X_t})(X_t), v_t^\phi\rangle\big)\big|_{y=X_t}\Big\}\mathrm{d}W_t, \quad v_0^\phi = \phi(X_0). \end{aligned} \tag{2.2}$$

By **(H)**, this linear SDE is well-posed with $\sup_{t\in[0,T]} \mathbb{E}|v_t^\phi|^2 \le C\mu(|\phi|^2)$ for some constant $C = C(T) > 0$, see (4.21) below. Denote $g_s' = \frac{\mathrm{d}}{\mathrm{d}s} g_s$ for a differentiable function g of $s \in \mathbb{R}$.

Theorem 2.1. *Assume* **(H)** *and* (2.1). *Then for any $f \in \mathscr{B}_b(\mathbb{R}^d), \mu \in \mathscr{P}_2(\mathbb{R}^d)$ and $T > 0$, $P_T f$ is L-differentiable at μ such that for any $g \in C^1([0,T])$ with $g_0 = 0$ and $g_T = 1$,*

$$D_\phi^L(P_T f)(\mu) = \mathbb{E}\bigg[f(X_T)\int_0^T \big\langle g_t'\sigma_t(X_t, \mathscr{L}_{X_t})^{-1} v_t^\phi,\ \mathrm{d}W_t\big\rangle\bigg],\ \phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu), \tag{2.3}$$

where X_t solves (1.4) *for $\mathscr{L}_{X_0} = \mu$. Moreover, the limit*

$$D_\phi^L P_T^*\mu := \lim_{\varepsilon \downarrow 0} \frac{P_T^*\mu \circ (\mathrm{Id} + \varepsilon\phi)^{-1} - P_T^*\mu}{\varepsilon} = \psi P_T^*\mu \tag{2.4}$$

exists in the total variational norm, where ψ is the unique element in $L^2(\mathbb{R}^d \to \mathbb{R}, P_T^*\mu)$ such that $\psi(X_T) = \mathbb{E}\big(\int_0^T \langle g_t'\sigma_t(X_t, \mathscr{L}_{X_t})^{-1} v_t^\phi,\ \mathrm{d}W_t\rangle \big| X_T\big)$, and $(\psi P_T^*\mu)(A) := \int_A \psi \mathrm{d}P_T^*\mu,\ A \in \mathscr{B}(\mathbb{R}^d)$.

Remark 2.1. When $f \in C_b^1(\mathbb{R}^d)$, (2.3) can be proved as in the distribution independent case by constructing a proper random variable h on the Cameron-Martin space such that $D_h X_T = \nabla_\phi X_T$. However, for the L-differentiability of $P_T f$, one has to construct $\gamma \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)$ such that (1.1) holds for $P_T f$ replacing f, which is non-trivial.

Moreover, comparing with the classical case where (2.3) for $f \in C_b^1(\mathbb{R}^d)$ can be easily extended to $f \in \mathscr{B}_b(\mathbb{R}^d)$, there is essential difficulty to do this in the distribution dependent setting. More precisely, when b_t and σ_t do not depend on the distribution, we have the semigroup property $P_T f(\mu) = P_{t,T}(P_t f)(\mu)$ for $t \in (0,T)$, where $P_t f(x) := P_t f(\delta_x)$ for the Dirac measure δ_x at point x. In many cases the regularity of P_t ensures that $P_t f \in C_b^1(\mathbb{R}^d)$ for $f \in \mathscr{B}_b(\mathbb{R}^d)$. Then for any $f \in \mathscr{B}_b(\mathbb{R}^d)$, one may apply the derivative formula (2.3) with $(P_{t,T}, P_t f)$ replacing (P_T, f) to derive a derivative formula for $P_T f$. However, in the distribution dependent case, due to the lack of (1.12) we no longer have $P_T f(\mu) = P_{t,T}(P_t f)(\mu)$, so that this argument becomes invalid. To overcome this difficulty we will make a new approximation argument, see step (a) in the proof of Theorem 2.1 for details.

As applications of Theorem 2.1, the following result consists of estimates on the L-derivative and the total variational distance between distributions of solutions with different initial data.

Corollary 2.2. *Assume* **(H)** *and* (2.1) *for some increasing functions K and continuous function λ.*

(1) *For any $f \in \mathscr{B}_b(\mathbb{R}^d)$ and $T > 0$,*

$$\|D^L(P_T f)(\mu)\|^2 := \sup_{\mu(|\phi|^2)\le 1} |D^L_\phi(P_T f)(\mu)|^2 \le \frac{(P_T f^2)(\mu) - (P_t f(\mu))^2}{\int_0^T \lambda_t^{-2} \mathrm{e}^{-8K(t)t}\mathrm{d}t}. \tag{2.5}$$

(2) *For any $T > 0$,*

$$|P_T f(\mu) - P_T f(\nu)|^2 \le \frac{\|f\|_\infty^2 \mathbb{W}_2(\mu,\nu)^2}{\int_0^T \lambda_t^{-2} \mathrm{e}^{-8K(t)t}\mathrm{d}t}, \quad \mu,\nu \in \mathscr{P}_2(\mathbb{R}^d), f \in \mathscr{B}_b(\mathbb{R}^d). \tag{2.6}$$

Consequently, for any $T > 0$ and $\mu, \nu \in \mathscr{P}_2(\mathbb{R}^d)$,

$$\|P_T^*\mu - P_T^*\nu\|_{var}^2 := \sup_{A\in\mathscr{B}(\mathbb{R}^d)} |(P_T^*\mu)(A) - (P_T^*\nu)(A)|^2 \le \frac{\mathbb{W}_2(\mu,\nu)^2}{\int_0^T \lambda_t^{-2} \mathrm{e}^{-8K(t)t}\mathrm{d}t}. \tag{2.7}$$

2.2 Stochastic Hamiltonian systems

Consider the following distribution dependent stochastic Hamiltonian system for $X_t = (X_t^{(1)}, X_t^{(2)})$ on $\mathbb{R}^{m+d} = \mathbb{R}^m \times \mathbb{R}^d$:

$$\begin{cases} \mathrm{d}X_t^{(1)} = b_t^{(1)}(X_t)\mathrm{d}t, \\ \mathrm{d}X_t^{(2)} = b_t^{(2)}(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma_t \mathrm{d}W_t, \end{cases} \tag{2.8}$$

where $(W_t)_{t\ge 0}$ is a d-dimensional Brownian motion as before, and for each $t \ge 0$, σ_t is an invertible $d \times d$-matrix,

$$b_t = (b_t^{(1)}, b_t^{(2)}) : \mathbb{R}^{m+d} \times \mathscr{P}_2(\mathbb{R}^{m+d}) \to \mathbb{R}^{m+d}$$

is measurable with $b_t^{(1)}(x,\mu) = b_t^{(1)}(x)$ independent of the distribution μ. Let $\nabla = (\nabla^{(1)}, \nabla^{(2)})$ be the gradient operator on $\mathbb{R}^{m+d} = \mathbb{R}^m \times \mathbb{R}^d$, where $\nabla^{(i)}$ is the gradient in the i-th component, $i = 1, 2$. Let $\nabla^2 = \nabla\nabla$ denote the Hessian operator on $\mathbb{R}^{m+d}$. We assume

(H1) For every $t \geq 0$, $b_t^{(1)} \in C_b^2(\mathbb{R}^{m+d} \to \mathbb{R}^m)$, $b_t^{(2)} \in C^{1,(1,0)}(\mathbb{R}^{m+d} \times \mathscr{P}_2(\mathbb{R}^{m+d}) \to \mathbb{R}^d)$, and there exists an increasing function $K : [0, \infty) \to [0, \infty)$ such that (1.6) and

$$\|\nabla b_t(\cdot, \mu)(x)\| + \|D^L b_t^{(2)}(x, \cdot)(\mu)\| + \|\nabla^2 b_t^{(1)}(\cdot, \mu)(x)\| \leq K(t)$$

hold for all $t \geq 0, (x, \mu) \in \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d)$.

Obviously, this assumption implies **(H)** for the SDE (2.8). We aim to establish the derivative formula of type (1.10) with P_t and P_t^* being defined by (1.8) and (1.9) for the SDE (2.8). To follow the line of [28] where the distribution independent model was investigated, we need the following assumption **(H2)**.

For any $s \geq 0$, let $\{K_{t,s}\}_{t \geq s}$ solve the following linear random ODE on $\mathbb{R}^{m \otimes m}$:

$$\frac{\mathrm{d}}{\mathrm{d}t} K_{t,s} = (\nabla^{(1)} b^{(1)})(X_t) K_{t,s}, \quad t \geq s, K_{s,s} = I_{m \times m}, \tag{2.9}$$

where $I_{m \times m}$ is the $m \times m$-order identity matrix.

(H2) There exists $B \in \mathscr{B}_b([0, T] \to \mathbb{R}^{m \otimes d})$ such that

$$\langle (\nabla^{(2)} b_t^{(1)} - B_t) B_t^* a, a \rangle \geq -\varepsilon |B_t^* a|^2, \quad \forall a \in \mathbb{R}^m \tag{2.10}$$

holds for some constant $\varepsilon \in [0, 1)$. Moreover, there exists an increasing function $\theta \in C([0, T])$ with $\theta_t > 0$ for $t \in (0, T]$ such that

$$\int_0^t s(T - s) K_{T,s} B_s B_s^* K_{T,s}^* \mathrm{d}s \geq \theta_t I_{m \times m}, \quad t \in (0, T]. \tag{2.11}$$

Example 2.1. Let

$$b_t^{(1)}(x) = Ax^{(1)} + Bx^{(2)}, \quad x = (x^{(1)}, x^{(2)}) \in \mathbb{R}^{m+d}$$

for some $m \times m$-matrix A and $m \times d$-matrix B. If the Kalman's rank condition

$$\mathrm{Rank}[B, AB, \cdots, A^k B] = m$$

holds for some $k \geq 1$, then **(H2)** is satisfied with $\theta_t = c_T t$ for some constant $c_T > 0$, see the proof of [28, Theorem 4.2]. In general, **(H2)** remains true under small perturbations of this $b_t^{(1)}$.

According to the proof of [28, Theorem 1.1], **(H2)** implies that the matrices

$$Q_t := \int_0^t s(T - s) K_{T,s} \nabla^{(2)} b_s^{(1)}(X_s) B_s^* K_{T,s}^* \mathrm{d}s, \quad t \in (0, T]$$

are invertible with

$$\|Q_t^{-1}\| \leq \frac{1}{(1 - \varepsilon)\theta_t}, \quad t \in (0, T]. \tag{2.12}$$

For $(X_t)_{t\in[0,T]}$ solving (2.8) with $\mathscr{L}_{X_0}=\mu$ and $\phi=(\phi^{(1)},\phi^{(2)})\in L^2(\mathbb{R}^{m+d}\to\mathbb{R}^{m+d},\mu)$, let

$$\begin{aligned}\alpha_t^{(2)} =&\frac{T-t}{T}\phi^{(2)}(X_0)-\frac{t(T-t)B_t^*K_{T,t}^*}{\int_0^T\theta_s^2\mathrm{d}s}\int_t^T\theta_s^2Q_s^{-1}K_{T,0}\phi^{(1)}(X_0)\mathrm{d}s\\&-t(T-t)B_t^*K_{T,t}^*Q_T^{-1}\int_0^T\frac{T-s}{T}K_{T,s}\nabla^{(2)}_{\phi^{(2)}(X_0)}b_s^{(1)}(X_s)\mathrm{d}s,\quad t\in[0,T],\end{aligned}\tag{2.13}$$

and

$$\alpha_t^{(1)}=K_{t,0}\phi^{(1)}(X_0)+\int_0^tK_{t,s}\nabla^{(2)}_{\alpha_s^{(2)}}b_s^{(1)}(X_s(x))\,\mathrm{d}s,\quad t\in[0,T].\tag{2.14}$$

Moreover, define

$$\begin{aligned}h_t^\alpha:=\int_0^t\sigma_s^{-1}\Big\{&\big(\mathbb{E}\langle D^Lb_s^{(2)}(y,\cdot)(\mathscr{L}_{X_s})(X_s),\alpha_s\rangle\big)\big|_{y=X_s}\\&+\nabla_{\alpha_s}b_s^{(2)}(\cdot,\mathscr{L}_{X_s})(X_s)-(\alpha_s^{(2)})'\Big\}\mathrm{d}s,\quad t\in[0,T].\end{aligned}\tag{2.15}$$

Let $(D^*,\mathscr{D}(D^*))$ be the Malliavin divergence operator associated with the Brownian motion $(W_t)_{t\in[0,T]}$, see Subsection 3.2 below for details. Then the main result in this part is the following.

Theorem 2.3. *Assume* **(H1)** *and* **(H2)**. *Then $h^\alpha\in\mathscr{D}(D^*)$ with $\mathbb{E}|D^*(h^\alpha)|^p<\infty$ for all $p\in[1,\infty)$. Moreover, for any $f\in\mathscr{B}_b(\mathbb{R}^{m+d})$ and $T>0$, P_Tf is L-differentiable at μ such that*

$$D_\phi^L(P_Tf)(\mu)=\mathbb{E}\big[f(X_T)\,D^*(h^\alpha)\big].\tag{2.16}$$

Consequently:

1. (2.4) *holds for the unique $\psi\in L^2(\mathbb{R}^{m+d}\to\mathbb{R},P_T^*\mu)$ such that $\psi(X_T)=\mathbb{E}(D^*(h^\alpha)|X_T)$.*
2. *There exists a constant $c\ge0$ such that for any $T>0$,*

$$\|D^L(P_Tf)(\mu)\|\le c\sqrt{P_T|f|^2(\mu)-(P_Tf)^2(\mu)}\frac{\sqrt{T}(T^2+\theta_T)}{\int_0^T\theta_s^2\mathrm{d}s},\quad f\in\mathscr{B}_b(\mathbb{R}^{m+d}),\tag{2.17}$$

$$\|P_T^*\mu-P_T^*\nu\|_{var}\le c\mathbb{W}_2(\mu,\nu)\frac{\sqrt{T}(T^2+\theta_T)}{\int_0^T\theta_s^2\mathrm{d}s},\quad\mu,\nu\in\mathscr{P}_2(\mathbb{R}^d).\tag{2.18}$$

3 Preparations

We first introduce a formula of the L-derivative re-organized from [6, Theorem 6.5] and [9, Proposition A.2], then investigate the partial derivatives of X_t in the initial value, and the Malliavin derivatives of X_t with respect to the Brownian motion W_t.

3.1 A formula of L-derivative

The following result is essentially due to [6, Theorem 6.5] for $f \in C^{(1,0)}(\mathscr{P}_2(\mathbb{R}^d))$, and [9, Proposition A.2] for bounded X and Y. We include a complete proof for readers' convenience.

Proposition 3.1. *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be an atomless probability space, and let $X, Y \in L^2(\Omega \to \mathbb{R}^d, \mathbb{P})$ with $\mathscr{L}_X = \mu$. If either X and Y are bounded and f is L-differentiable at μ, or $f \in C^{(1,0)}(\mathscr{P}_2(\mathbb{R}^d))$, then*

$$\lim_{\varepsilon \downarrow 0} \frac{f(\mathscr{L}_{X+\varepsilon Y}) - f(\mu)}{\varepsilon} = \mathbb{E}\langle D^L f(\mu)(X), Y\rangle. \tag{3.1}$$

Consequently,

$$\Big|\lim_{\varepsilon \downarrow 0} \frac{f(\mathscr{L}_{X+\varepsilon Y}) - f(\mu)}{\varepsilon}\Big| = \big|\mathbb{E}\langle D^L f(\mu)(X), Y\rangle\big| \le \|D^L f(\mu)\| \sqrt{\mathbb{E}|Y|^2}. \tag{3.2}$$

Proof. It is easy to see that (3.2) follows from (1.3) and (3.1). Indeed, letting $\phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)$ such that $\phi(X) = \mathbb{E}(Y|X)$, we have

$$\begin{aligned} &\big|\mathbb{E}\langle D^L f(\mu)(X), Y\rangle\big| = \big|\mathbb{E}\langle D^L f(\mu)(X), \phi(X)\rangle\big| = \big|\mu(\langle D^L f(\mu), \phi\rangle)\big| \\ &\le \|D^L f(\mu)\| \cdot \|\phi\|_{L^2(\mu)} = \|D^L f(\mu)\| \big(\mathbb{E}|\mathbb{E}(Y|X)|^2\big)^{\frac{1}{2}} \le \|D^L f(\mu)\| \sqrt{\mathbb{E}|Y|^2}. \end{aligned}$$

Below we prove (3.1) for the stated two situations respectively.

(1) Assume that X and Y are bounded. For any $\mathbb{R}^d$-valued random variable ξ, let $F(\xi) = f(\mathscr{L}_\xi)$. Next, let $(\bar\Omega, \bar{\mathscr{F}}, \bar{\mathbb{P}})$ be an atomless Polish probability space, and let $\bar X \in L^2(\bar\Omega \to \mathbb{R}^d, \bar{\mathbb{P}})$ with $\mathscr{L}_{\bar X|\bar{\mathbb{P}}} = \mu$, where $\mathscr{L}_{\cdot|\bar{\mathbb{P}}}$ denotes the distribution of a random variable under $\bar{\mathbb{P}}$. According to [9, Proposition A.2(iii)], if

$$\bar F(\bar Y) := f(\mathscr{L}_{\bar Y|\bar{\mathbb{P}}}), \quad \bar Y \in L^2(\bar\Omega \to \mathbb{R}^d, \bar{\mathbb{P}})$$

is Fréchet differentiable at $\bar X$ with derivative $D\bar F(\bar X) = D^L f(\mu)(\bar X)$, then

$$\lim_{\varepsilon \downarrow 0} \frac{f(\mathscr{L}_{X+\varepsilon Y}) - f(\mathscr{L}_X) - \varepsilon \mathbb{E}\langle D^L f(\mu)(X), Y\rangle}{\varepsilon} = 0. \tag{3.3}$$

Equivalently, (3.1) holds. Below we construct the desired $\bar X$ and $(\bar\Omega, \bar{\mathscr{F}}, \bar{\mathbb{P}})$ such that $D\bar F(\bar X) = D^L f(\mu)(\bar X)$.

A natural choice of $(\bar\Omega, \bar{\mathscr{F}}, \bar{\mathbb{P}})$ is $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d), \mu)$, but to ensure the atomless property, we take $(\bar\Omega, \bar{\mathscr{F}}, \bar{\mathbb{P}}) = (\mathbb{R}^d \times \mathbb{R}, \mathscr{B}(\mathbb{R}^d \times \mathbb{R}), \mu \times \lambda)$, where λ is the standard Gaussian measure on $\mathbb{R}$. Then $(\bar\Omega, \bar{\mathscr{F}}, \bar{\mathbb{P}})$ is an atomless Polish probability space. Let

$$\bar X(\bar\omega) = x, \quad \bar\omega = (x, r) \in \mathbb{R}^d \times \mathbb{R}.$$

We have $\mathscr{L}_{\bar X} = \mu$. Moreover, let

$$\tilde f(\tilde\mu) = f(\tilde\mu(\cdot \times \mathbb{R})), \quad \tilde\mu \in \mathscr{P}_2(\mathbb{R}^d \times \mathbb{R}).$$

It is easy to see that the L-differentiability of f at μ implies that of $\tilde{f}$ at $\mu\times\delta_0$ with

$$D^L\tilde{f}(\mu\times\delta_0)(x,r)=(D^Lf(\mu)(x),0),\quad (x,r)\in\mathbb{R}^d\times\mathbb{R}. \tag{3.4}$$

Finally, on the probability space $(\Omega,\mathscr{F},\mathbb{P})$ we have

$$F(Y):=f(\mathscr{L}_Y)=\tilde{f}(\mathscr{L}_{\tilde{Y}}),\quad \tilde{Y}:=(Y,0)\in L^2(\Omega\to\mathbb{R}^d\times\mathbb{R},\mathscr{F},\mathbb{P}). \tag{3.5}$$

Letting $\tilde{X}=(X,0)\in L^2(\Omega\to\mathcal{T}^d\times\mathbb{R},\mathscr{F},\mathbb{P})$, by [9, Proposition A.2(iii)], the formula (3.3) holds for $(\tilde{X},\tilde{Y},\tilde{f},\mu\times\delta_0)$ replacing (X,Y,f,μ), i.e.

$$\lim_{\varepsilon\downarrow 0}\frac{\tilde{f}(\mathscr{L}_{\tilde{X}+\varepsilon\tilde{Y}})-\tilde{f}(\mathscr{L}_{\tilde{X}})-\mathbb{E}\langle D^L\tilde{f}(\mu\times\delta_0),\varepsilon\tilde{Y}\rangle}{\varepsilon}=0.$$

Combining this with (3.4) and (3.5), we prove (3.3). Therefore, (3.1) holds.

(2) Let $f\in C^{(1,0)}(\mathscr{P}_2(\mathbb{R}^d))$ and let $\mu\in\mathscr{P}_2(\mathbb{R}^d)$ and $X\in L^2(\Omega\to\mathbb{R}^d,\mathbb{P})$ with $\mathscr{L}_X=\mu$. For any $n\ge 1$, let

$$x_n=\frac{x}{\sqrt{1+n^{-1}|x|^2}},\quad x\in\mathbb{R}^d.$$

By (3.1) for bounded X and Y, for any $n\ge 1$ we have

$$\begin{aligned} f(\mathscr{L}_{X_n+\varepsilon Y_n})-f(\mathscr{L}_{X_n})&=\int_0^\varepsilon\frac{\mathrm{d}}{\mathrm{d}s}f(\mathscr{L}_{X_n+sY_n})\,\mathrm{d}s\\ &=\int_0^\varepsilon\mathbb{E}\langle D^Lf(\mathscr{L}_{X_n+sY_n})(X_n+sY_n),Y_n\rangle\,\mathrm{d}s.\end{aligned} \tag{3.6}$$

Since $f\in C^{(1,0)}(\mathscr{P}_2(\mathbb{R}^d))$, it follows that

$$\sup_{n\ge1,s\in[0,\varepsilon]}\|D^Lf(\mathscr{L}_{X_n+sY_n})\|<\infty,\quad \lim_{n\to\infty}\{f(\mathscr{L}_{X_n+\varepsilon Y_n})-f(\mathscr{L}_{X_n})\}=f(\mathscr{L}_{X+\varepsilon Y})-f(\mathscr{L}_X),$$

and for any $s\in[0,\varepsilon]$,

$$\lim_{n\to\infty}\mathbb{E}\big(|X-X_n|^2+|Y-Y_n|^2+|D^Lf(\mathscr{L}_{X_n+sY_n})(X_n+sY_n)-D^Lf(\mathscr{L}_{X+sY})(X+sY)|^2\big)=0.$$

Then letting $n\to\infty$ in (3.6) we arrive at

$$f(\mathscr{L}_{X+\varepsilon Y})-f(\mathscr{L}_X)=\int_0^\varepsilon\mathbb{E}\langle D^Lf(\mathscr{L}_{X+sY})(X+sY),Y\rangle\,\mathrm{d}s,\quad \varepsilon>0. \tag{3.7}$$

This implies (3.1). More precisely, it is easy to see that $\{\mathscr{L}_{X+sY}\}$ is compact in $\mathscr{P}_2(\mathbb{R}^d)$. So, $f\in C^{(1,0)}(\mathscr{P}_2(\mathbb{R}^d))$ implies

$$A:=\sup_{s\in[0,1]}\sqrt{\mathbb{E}|D^Lf(\mathscr{L}_{X+sY})(X+sY)|^2}=\sup_{s\in[0,1]}\|D^Lf(\mathscr{L}_{X+sY})\|_{L^2(\mathscr{L}_{X+sY})}<\infty. \tag{3.8}$$

Combining this with the continuity property of $D^L f$ on $\mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d)$, we conclude that

$$\lim_{\varepsilon\downarrow 0} D^L f(\mathscr{L}_{X+sY})(X+sY) = D^L f(\mathscr{L}_X)(X) \text{ weakly in } L^2(\Omega\to\mathbb{R}^d,\mathbb{P}).$$

In particular,

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}\langle D^L f(\mathscr{L}_{X+sY})(X+sY), Y\rangle = \mathbb{E}\langle D^L f(\mathscr{L}_X)(X), Y\rangle. \tag{3.9}$$

Moreover, (3.8) implies

$$\sup_{s\in[0,1]} \mathbb{E}\big|\langle D^L f(\mathscr{L}_{X+sY})(X+sY), Y\rangle\big| \le A\sqrt{\mathbb{E}|Y|^2} < \infty.$$

Due to this, (3.7) and (3.9), the dominated convergence theorem gives

$$\begin{aligned}\lim_{\varepsilon\downarrow 0} \frac{f(\mathscr{L}_{X+\varepsilon Y}) - f(\mathscr{L}_X)}{\varepsilon} &= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^\varepsilon \mathbb{E}\langle D^L f(\mathscr{L}_{X+sY})(X+sY), Y\rangle\,\mathrm{d}s\\ &= \mathbb{E}\langle D^L f(\mathscr{L}_X)(X), Y\rangle.\end{aligned}$$

□

3.2 Partial derivative in initial value

For any $T>0$, let $\mathscr{C}_T = C([0,T]\to\mathbb{R}^d)$ be the path space over $\mathbb{R}^d$ with time interval $[0,T]$, and let $X_0, \eta \in L^2(\Omega\to\mathbb{R}^d, \mathscr{F}_0, \mathbb{P})$. For any $\varepsilon\ge 0$, let $(X_t^\varepsilon)_{t\ge0}$ solve the SDE

$$\mathrm{d}X_t^\varepsilon = b_t(X_t^\varepsilon, \mathscr{L}_{X_t^\varepsilon})\mathrm{d}t + \sigma_t(X_t^\varepsilon, \mathscr{L}_{X_t^\varepsilon})\mathrm{d}W_t, \quad X_0^\varepsilon = X_0 + \varepsilon\eta. \tag{3.10}$$

Obviously, $X_t = X_t^0$ solves (1.4) with initial value X_0. Consider the following linear SDE for v_t^η on $\mathbb{R}^d$:

$$\begin{aligned}\mathrm{d}v_t^\eta = &\Big\{\nabla_{v_t^\eta} b_t(\cdot, \mathscr{L}_{X_t})(X_t) + \big(\mathbb{E}\langle D^L b_t(y,\cdot)(\mathscr{L}_{X_t})(X_t), v_t^\eta\rangle\big)\big|_{y=X_t}\Big\}\mathrm{d}t\\ &+ \Big\{\nabla_{v_t^\eta}\sigma_t(\cdot, \mathscr{L}_{X_t})(X_t) + \big(\mathbb{E}\langle D^L \sigma_t(y,\cdot)(\mathscr{L}_{X_t})(X_t), v_t^\eta\rangle\big)\big|_{y=X_t}\Big\}\mathrm{d}W_t, \quad v_0^\eta = \eta.\end{aligned} \tag{3.11}$$

The main result of this part is the following.

Proposition 3.2. *Assume* **(H)**. *Then for any $T>0$, the limit*

$$\nabla_\eta X_t := \lim_{\varepsilon\downarrow 0}\frac{X_t^\varepsilon - X_t}{\varepsilon}, \quad t\in[0,T] \tag{3.12}$$

exists in $L^2(\Omega\to\mathscr{C}_T, \mathbb{P})$. Moreover, $(v_t^\eta := \nabla_\eta X_t)_{t\in[0,T]}$ is the unique solution to the linear SDE (3.11).

To prove the existence of $\nabla_\eta X_t$ in (3.12), it suffices to show that when $\varepsilon \downarrow 0$

$$\xi^\varepsilon(t) := \frac{X_t^\varepsilon - X_t}{\varepsilon}, \quad t \in [0,T] \tag{3.13}$$

is a Cauchy sequence in $L^2(\Omega \to \mathscr{C}_T, \mathbb{P})$, i.e.

$$\lim_{\varepsilon,\delta\downarrow 0} \mathbb{E}\bigg[\sup_{t\in[0,T]} |\xi^\varepsilon(t) - \xi^\delta(t)|^2\bigg] = 0. \tag{3.14}$$

To this end, we need the following two lemmas.

Lemma 3.3. *Assume* **(H)**. *Then*

$$\sup_{\varepsilon\in(0,1]} \mathbb{E}\bigg[\sup_{t\in[0,T]} |\xi^\varepsilon(t)|^2\bigg] < \infty.$$

Proof. By **(H)**, there exists a constant $C_1 > 0$ such that

$$\begin{aligned}
&\mathrm{d}|X_t^\varepsilon - X_t|^2\\
&= \big\{2\langle b_t(X_t^\varepsilon, \mathscr{L}_{X_t^\varepsilon}) - b_t(X_t, \mathscr{L}_{X_t}), X_t^\varepsilon - X_t\rangle + \|\sigma_t(X_t^\varepsilon, \mathscr{L}_{X_t^\varepsilon}) - \sigma_t(X_t, \mathscr{L}_{X_t})\|_{HS}^2\big\}\mathrm{d}t + \mathrm{d}M_t\\
&\le C_1\big\{|X_t^\varepsilon - X_t|^2 + \mathbb{W}_2(\mathscr{L}_{X_t^\varepsilon}, \mathscr{L}_{X_t})^2\big\}\mathrm{d}t + \mathrm{d}M_t,
\end{aligned}$$

where

$$\mathrm{d}M_t := 2\Big\langle X_t^\varepsilon - X_t, (\sigma_t(X_t^\varepsilon, \mathscr{L}_{X_t^\varepsilon}) - \sigma_t(X_t, \mathscr{L}_{X_t}))\mathrm{d}W_t\Big\rangle$$

satisfies

$$\mathrm{d}\langle M\rangle_t \le C_1^2\big\{|X_t^\varepsilon - X_t|^2 + \mathbb{W}_2(\mathscr{L}_{X_t^\varepsilon}, \mathscr{L}_{X_t})^2\big\}^2\mathrm{d}t. \tag{3.15}$$

Then by the BDG inequality, and noting that $\mathbb{W}_2(\mathscr{L}_\xi, \mathscr{L}_\eta)^2 \le \mathbb{E}|\xi - \eta|^2$ for two random variables ξ, η, we may find out a constant $C_2 > 0$ such that

$$\mathbb{E}\bigg[\sup_{s\in[0,t]} |X_s^\varepsilon - X_s|^2\bigg] \le \varepsilon^2|\eta|^2 + 2C_1\int_0^t \mathbb{E}|X_s^\varepsilon - X_s|^2\mathrm{d}s + C_2\mathbb{E}\sqrt{\langle M\rangle_t}. \tag{3.16}$$

Noting that $\mathbb{W}_2(\mathscr{L}_{X_s^\varepsilon}, \mathscr{L}_{X_s})^2 \le \mathbb{E}|X_s^\varepsilon - X_s|^2$, (3.15) yields

$$\begin{aligned}
&C_2\mathbb{E}\sqrt{\langle M\rangle_t} \le C_1C_2\mathbb{E}\bigg(\int_0^t \big\{|X_s^\varepsilon - X_s|^2 + \mathbb{W}_2(\mathscr{L}_{X_s^\varepsilon}, \mathscr{L}_{X_s})^2\big\}^2\mathrm{d}s\bigg)^{\frac{1}{2}}\\
&\le C_1C_2\mathbb{E}\bigg(\sup_{s\in[0,t]}\big\{|X_s^\varepsilon - X_s|^2 + \mathbb{E}|X_s^\varepsilon - X_s|^2\big\}\int_0^t \big\{|X_s^\varepsilon - X_s|^2 + \mathbb{E}|X_s^\varepsilon - X_s|^2\big\}\mathrm{d}s\bigg)^{\frac{1}{2}}\\
&\le \frac{1}{2}\mathbb{E}\Big[\sup_{s\in[0,t]}|X_s^\varepsilon - X_s|^2\Big] + \frac{C_3}{2}\int_0^t \mathbb{E}|X_s^\varepsilon - X_s|^2\,\mathrm{d}s
\end{aligned}$$

for some constant $C_3 > 0$. Combining this with (3.16) and noting that due to (1.7)

$$\mathbb{E}\Big[\sup_{s\in[0,t]}|X_s^\varepsilon - X_s|^2\Big] < \infty,$$

we arrive at

$$\mathbb{E}\bigg[\sup_{s\in[0,t]}|X_s^\varepsilon - X_s|^2\bigg] \le 2\varepsilon^2|\eta|^2 + C_3\int_0^t \mathbb{E}|X_s^\varepsilon - X_s|^2\mathrm{d}s, \quad t\in[0,T], \varepsilon>0.$$

Therefore, Gronwall's inequality gives

$$\sup_{\varepsilon\in(0,1]}\mathbb{E}\bigg[\sup_{t\in[0,T]}|\xi^\varepsilon(t)|^2\bigg] = \sup_{\varepsilon\in(0,1]}\frac{1}{\varepsilon^2}\mathbb{E}\bigg[\sup_{s\in[0,T]}|X_s^\varepsilon - X_s|^2\bigg] \le 2\mathrm{e}^{C_3T}\mathbb{E}|\eta|^2 < \infty.$$

□

For any differentiable (real, vector, or matrix valued) function f on $\mathbb{R}^d\times\mathscr{P}_2(\mathbb{R}^d)$, let

$$\begin{aligned}\Xi_f^\varepsilon(t) = &\frac{f(X_t^\varepsilon,\mathscr{L}_{X_t^\varepsilon}) - f(X_t,\mathscr{L}_{X_t})}{\varepsilon} - \nabla_{\xi^\varepsilon(t)}f(\cdot,\mathscr{L}_{X_t})(X_t)\\ &- \big\{\mathbb{E}\langle D^Lf(y,\cdot)(\mathscr{L}_{X_t})(X_t),\xi^\varepsilon(t)\rangle\big\}\big|_{y=X_t}, \quad t\in[0,T],\varepsilon>0.\end{aligned}\tag{3.17}$$

Lemma 3.4. *Assume* **(H)**. *For any (real, vector, or matrix valued) $C^{1,(1,0)}$-function f on $\mathbb{R}^d\times\mathscr{P}_2(\mathbb{R}^d)$ with*

$$K_f := \sup_{(x,\mu)\in\mathbb{R}^d\times\mathscr{P}_2(\mathbb{R}^d)}\big(|\nabla f(\cdot,\mu)(x)|^2 + \|D^Lf(x,\cdot)(\mu)\|_{L^2(\mu)}^2\big) < \infty,\tag{3.18}$$

there holds

$$\big|\Xi_f^\varepsilon(t)\big|^2 \le 4K_f\big(\mathbb{E}|\xi^\varepsilon(t)|^2 + |\xi^\varepsilon(t)|^2\big) \quad \text{and} \quad \lim_{\varepsilon\downarrow 0}\mathbb{E}\big|\Xi_f^\varepsilon(t)\big|^2 = 0, \quad t\in[0,T].\tag{3.19}$$

Proof. Let $X_t^\varepsilon(s) = X_t + s(X_t^\varepsilon - X_t),\ s\in[0,1]$. By the chain rule and (3.1), we have

$$\begin{aligned}&\frac{f(X_t^\varepsilon,\mathscr{L}_{X_t^\varepsilon}) - f(X_t,\mathscr{L}_{X_t})}{\varepsilon} = \frac{1}{\varepsilon}\int_0^1\Big\{\frac{\mathrm{d}}{\mathrm{d}s}f\big(X_t^\varepsilon(s),\mathscr{L}_{X_t^\varepsilon(s)}\big)\Big\}\mathrm{d}s\\ &= \int_0^1\Big\{\nabla_{\xi^\varepsilon(t)}f(\cdot,\mathscr{L}_{X_t^\varepsilon(s)})(X_t^\varepsilon(s)) + \big(\mathbb{E}\big\langle D^Lf(y,\cdot)(\mathscr{L}_{X_t^\varepsilon(s)})(X_t^\varepsilon(s)),\xi^\varepsilon(t)\big\rangle\big)\big|_{y=X_t^\varepsilon(s)}\Big\}\mathrm{d}s.\end{aligned}$$

Combining this with (3.18) we obtain

$$\begin{aligned}\big|\Xi_f^\varepsilon(t)\big|^2 \le\ & 2\int_0^1\Big|\nabla_{\xi^\varepsilon(t)}\big\{f(\cdot,\mathscr{L}_{X_t^\varepsilon(s)})(X_t^\varepsilon(s)) - f(\cdot,\mathscr{L}_{X_t})(X_t)\big\}\Big|^2\mathrm{d}s\\ &+ 2\int_0^1\Big|\big(\mathbb{E}\big\langle D^Lf(y,\cdot)(\mathscr{L}_{X_t^\varepsilon(s)})(X_t^\varepsilon(s)),\xi^\varepsilon(t)\big\rangle\big)\big|_{y=X_t^\varepsilon(s)}\\ &\qquad - \big(\mathbb{E}\big\langle D^Lf(y,\cdot)(\mathscr{L}_{X_t})(X_t),\xi^\varepsilon(t)\big\rangle\big)\big|_{y=X_t}\Big|^2\mathrm{d}s\\ \le\ & 8K_f\big(|\xi^\varepsilon(t)|^2 + \mathbb{E}|\xi^\varepsilon(t)|^2\big).\end{aligned}\tag{3.20}$$

So, the first inequality in (3.19) holds. Moreover, Lemma 3.3 implies

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}\bigg[\sup_{s\in[0,1]}|X_t^\varepsilon(s)-X_t|^2\bigg]\le\lim_{\varepsilon\downarrow 0}\mathbb{E}|X_t^\varepsilon-X_t|^2=0.$$

Thus, the $C^{1,(1,0)}$-property of f, Lemma 3.3 and the first inequality in (3.20) yield that $\Xi_f^\varepsilon(t)\to 0$ in probability as $\varepsilon\to 0$. Combining this with the first inequality in (3.19), Lemma 3.3, and using the dominated convergence theorem, we derive $\lim_{\varepsilon\downarrow 0}\mathbb{E}\big|\Xi_f^\varepsilon(t)\big|^2=0$. □

Proof of Proposition 3.2. Let $(\Xi_b^\varepsilon(t),K_{b_t})$ and $(\Xi_\sigma^\varepsilon(t),K_{\sigma_t})$ be defined as in (3.17) and (3.18) for b_t and σ_t replacing f respectively. By **(H)**, there exists a constant $C_1>0$ such that

$$\sup_{t\in[0,T]}\big(K_{b_t}+K_{\sigma_t}\big)\le C_1<\infty.$$

Then Lemma 3.4 gives

$$\begin{aligned}&\big|\Xi_b^\varepsilon(t)\big|^2+\big|\Xi_\sigma^\varepsilon(t)\big|^2\le 4C\big(|\xi^\varepsilon(t)|^2+\mathbb{E}|\xi^\varepsilon(t)|^2\big),\\&\lim_{\varepsilon\downarrow 0}\mathbb{E}\Big(\big|\Xi_b^\varepsilon(t)\big|^2+\big|\Xi_\sigma^\varepsilon(t)\big|^2\Big)=0,\quad t\in[0,T].\end{aligned}\tag{3.21}$$

By (3.10), (3.13), and (3.17) for b_t and σ_t replacing f, we have

$$\begin{aligned}\xi^\varepsilon(t)=&\int_0^t\Big\{\Xi_b^\varepsilon(s)+\nabla_{\xi^\varepsilon(s)}b_s(\cdot,\mathscr{L}_{X_s})(X_s)+\big(\mathbb{E}\langle D^Lb_s(y,\cdot)(\mathscr{L}_{X_s})(X_s),\xi^\varepsilon(s)\rangle\big)\big|_{y=X_s}\Big\}\mathrm{d}s\\&+\int_0^t\Big\langle\Xi_\sigma^\varepsilon(s)+\nabla_{\xi^\varepsilon(s)}\sigma_s(\cdot,\mathscr{L}_{X_s})(X_s)+\big(\mathbb{E}\langle D^L\sigma_s(y,\cdot)(\mathscr{L}_{X_s})(X_s),\xi^\varepsilon(s)\rangle\big)\big|_{y=X_s},\ \mathrm{d}W_s\Big\rangle\end{aligned}$$

for $t\in[0,T]$. So, for any $\varepsilon,\delta\in(0,1]$, $\xi^{\varepsilon,\delta}(t):=\xi^\varepsilon(t)-\xi^\delta(t)$ satisfies

$$\begin{aligned}|\xi^{\varepsilon,\delta}(t)|^2\le&\,4\int_0^t\big|\Xi_b^\varepsilon(s)-\Xi_b^\delta(s)\big|^2\mathrm{d}s+4\bigg|\int_0^t\big\langle\Xi_\sigma^\varepsilon(s)-\Xi_\sigma^\delta(s),\mathrm{d}W_s\big\rangle\bigg|^2\\&+4T\int_0^t\Big|\nabla_{\xi^{\varepsilon,\delta}(s)}b_s(\cdot,\mathscr{L}_{X_s})(X_s)+\big(\mathbb{E}\langle D^Lb_s(y,\cdot)(\mathscr{L}_{X_s})(X_s),\xi^{\varepsilon,\delta}(s)\rangle\big)|_{y=X_s}\Big|^2\mathrm{d}s\\&+4\bigg|\int_0^t\Big\langle\nabla_{\xi^{\varepsilon,\delta}(s)}\sigma_s(\cdot,\mathscr{L}_{X_s})(X_s)+\big(\mathbb{E}\langle D^L\sigma_s(y,\cdot)(\mathscr{L}_{X_s})(X_s),\xi^{\varepsilon,\delta}(s)\rangle\big)|_{y=X_s},\ \mathrm{d}W_s\Big\rangle\bigg|^2.\end{aligned}$$

Combining this with **(H)** and using the BDG inequality, we find out a constant $C_2>0$ such that

$$\begin{aligned}\mathbb{E}\bigg[\sup_{s\in[0,t]}\xi^{\varepsilon,\delta}(s)\bigg]\le&\,C_2\int_0^T\mathbb{E}\Big(\big|\Xi_b^\varepsilon(s)-\Xi_b^\delta(s)\big|^2+\big|\Xi_\sigma^\varepsilon(s)-\Xi_\sigma^\delta(s)\big|^2\Big)\mathrm{d}s\\&+C_2\int_0^t\mathbb{E}|\xi^{\varepsilon,\delta}(s)|^2\ \mathrm{d}s,\quad t\in[0,T].\end{aligned}$$

Since Lemma 3.3 ensures that $\mathbb{E}\big[\sup_{s\in[0,t]}\xi^\varepsilon(s)\big]<\infty$, by Gronwall's lemma this yields

$$\mathbb{E}\bigg[\sup_{s\in[0,T]}\xi^{\varepsilon,\delta}(s)\bigg]\le C_2\mathrm{e}^{C_2T}\int_0^T\mathbb{E}\Big(\big|\Xi_b^\varepsilon(s)-\Xi_b^\delta(s)\big|^2+\big|\Xi_\sigma^\varepsilon(s)-\Xi_\sigma^\delta(s)\big|^2\Big)\mathrm{d}s.$$

Combining this with (3.21) and Lemma 3.3, and applying the dominated convergence theorem, we prove the first assertion in Proposition 3.2.

Finally, by (3.10), (3.12), (3.21) and (3.17) for b_t,σ_t replacing f, we conclude that $v_t^\eta:=\nabla_\eta X_t$ solves the SDE (3.11). Since this SDE is linear, the uniqueness is trivial. Then the proof is finished. □

3.3 Malliavin derivative

Consider the Cameron-Martin space

$$\mathbb{H}=\bigg\{h\in C([0,T]\to\mathbb{R}^d):h_0=\mathbf{0},h_t'\text{ exists a.e. }t,\|h\|_{\mathbb{H}}^2:=\int_0^T|h_t'|^2\mathrm{d}t<\infty\bigg\}.$$

Let $\eta\in L^2(\Omega\to\mathbb{R}^d,\mathscr{F}_0,\mathbb{P})$ with $\mathscr{L}_\eta=\mu$, and let μ_T be the distribution of $W_{[0,T]}:=\{W_t\}_{t\in[0,T]}$, which is a probability measure (i.e. Wiener measure) on the path space $\mathscr{C}_T:=C([0,T]\to\mathbb{R}^d)$. For $F\in L^2(\mathbb{R}^d\times\mathscr{C}_T,\mu\times\mu_T)$, $F(\eta,W_{[0,T]})$ is called Malliavin differentiable along direction $h\in\mathbb{H}$, if the directional derivative

$$D_hF(\eta,W_{[0,T]}):=\lim_{\varepsilon\to0}\frac{F(\eta,W_{[0,T]}+\varepsilon h)-F(\eta,W_{[0,T]})}{\varepsilon}$$

exists in $L^2(\Omega,\mathbb{P})$. If the map $\mathbb{H}\ni h\mapsto D_hF\in L^2(\Omega,\mu)$ is bounded, then there exists a unique $DF(\eta,W_{[0,T]})\in L^2(\Omega\to\mathbb{H},\mathbb{P})$ such that $\langle DF(\eta,W_{[0,T]}),h\rangle_{\mathbb{H}}=D_hF(\eta,W_{[0,T]})$ holds in $L^2(\Omega,\mathbb{P})$ for all $h\in\mathbb{H}$. In this case, we write $F(\eta,W_{[0,T]})\in\mathscr{D}(D)$ and call $DF(\eta,W_{[0,T]})$ the Malliavin gradient of $F(\eta,W_{[0,T]})$. It is well known that $(D,\mathscr{D}(D))$ is a closed linear operator from $L^2(\Omega,\mathscr{F}_T,\mathbb{P})$ to $L^2(\Omega\to\mathbb{H},\mathscr{F}_T,\mathbb{P})$. The adjoint operator $(D^*,\mathscr{D}(D^*))$ of $(D,\mathscr{D}(D))$ is called Malliavin divergence. For simplicity, in the sequel we denote $F(\eta,W_{[0,T]})$ by F. Then we have the integration by parts formula

$$\mathbb{E}\big(D_hF\big|\mathscr{F}_0\big)=\mathbb{E}\big(FD^*(h)\big|\mathscr{F}_0\big),\quad F\in\mathscr{D}(D),h\in\mathscr{D}(D^*).\tag{3.22}$$

It is well known that for adapted $h\in L^2(\Omega\to\mathbb{H},\mathbb{P})$, one has $h\in\mathscr{D}(D^*)$ with

$$D^*(h)=\int_0^T\langle h_t',\mathrm{d}W_t\rangle.\tag{3.23}$$

For more details and applications on Malliavin calculus one may refer to [19] and references therein.

For any $\varepsilon\ge0$ and adapted $h\in L^2(\Omega\to\mathbb{H},\mathbb{P})$, let $(X_t^{h,\varepsilon})_{t\ge0}$ solve the SDE

$$\mathrm{d}X_t^{h,\varepsilon}=b_t(X_t^{h,\varepsilon},\mathscr{L}_{X_t^{h,\varepsilon}})\mathrm{d}t+\sigma_t(X_t^{h,\varepsilon},\mathscr{L}_{X_t^{h,\varepsilon}})\mathrm{d}(W_t+\varepsilon h_t),\quad X_0^{h,\varepsilon}=X_0.\tag{3.24}$$

By **(H)** and $h'_\cdot \in L^2(\Omega\times[0,T],\mathbb{P}\times \mathrm{d}t)$, this SDE is well-posed. Obviously, $X_t^{h,0}=X_t$ solves (1.4) with initial value X_0. When $\sigma_t(x,\mu)$ does not depend (x,μ), this SDE reduces to a random ODE for $Y_t^{h,\varepsilon}:=X_t^{h,\varepsilon}-\sigma_t W_t$, which is well-posed also for non-adapted h like h^α in Theorem 2.3. The main result of this part is the following.

Proposition 3.5. *Assume* **(H)**. *Let $h\in L^2(\Omega\to\mathbb{H},\mathbb{P})$, which is adapted if $\sigma_t(x,\mu)$ depends on x or μ. Then the limit*

$$D_h X_t := \lim_{\varepsilon\downarrow 0}\frac{X_t^{h,\varepsilon}-X_t}{\varepsilon},\quad t\in[0,T] \tag{3.25}$$

exists in $L^2(\Omega\to\mathscr{C}_T,\mathbb{P})$. Moreover, $(w_t^h:=D_hX_t)_{t\in[0,T]}$ is the unique solution to the SDE

$$\begin{aligned}\mathrm{d}w_t^h = &\ \Big\{\nabla_{w_t^h}\sigma_t(\cdot,\mathscr{L}_{X_t})(X_t)+\big(\mathbb{E}\langle D^L\sigma_t(y,\cdot)(\mathscr{L}_{X_t})(X_t),w_t^h\rangle\big)\big|_{y=X_t}\Big\}\mathrm{d}W_t\\ &+\Big\{\nabla_{w_t^h}b_t(\cdot,\mathscr{L}_{X_t})(X_t)+\big(\mathbb{E}\langle D^Lb_t(y,\cdot)(\mathscr{L}_{X_t})(X_t),w_t^h\rangle\big)\big|_{y=X_t}+\sigma_t(X_t,\mathscr{L}_{X_t})h_t'\Big\}\mathrm{d}t\end{aligned} \tag{3.26}$$

with $w_0^h=\mathbf{0}$.

Proof. Comparing with the linear SDE (3.11), the additional term $\sigma_t(X_t,\mathscr{L}_{X_t})h_t'$ comes from the derivative with respect to ε at $\varepsilon=0$ of the term $\varepsilon\sigma_t(X_t^{h,\varepsilon},\mathscr{L}_{X_t^{h,\varepsilon}})h_t'$ in (3.24), since

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big\{\varepsilon\sigma_t(X_t^{h,\varepsilon},\mathscr{L}_{X_t^{h,\varepsilon}})\big\}\big|_{\varepsilon=0}=\lim_{\varepsilon\downarrow 0}\sigma_t(X_t^{h,\varepsilon},\mathscr{L}_{X_t^{h,\varepsilon}})=\sigma_t(X_t,\mathscr{L}_{X_t}).$$

Taking this into account, we may prove Proposition 3.5 by repeating the proof of Proposition 3.2. We omit the details to save space. □

4 Proofs of main results

We first present an integration by parts formula for $\nabla_\eta X_T$ with $\eta\in L^2(\Omega\to\mathbb{R}^d,\mathscr{F}_0,\mathbb{P})$, then prove Theorem 2.1, Corollary 2.2 and Theorem 2.3 respectively.

4.1 An integration by parts formula

Theorem 4.1. *Assume* **(H)** *and* (2.1). *Let $f\in C_b^1(\mathbb{R}^d)$ and $\eta\in L^2(\Omega\to\mathbb{R}^d,\mathbb{P})$. Then for any $0\le r<T$ and $g\in C^1([r,T])$ with $g_r=0$ and $g_T=1$,*

$$\mathbb{E}\big(\langle\nabla f(X_T),\nabla_\eta X_T\rangle\big|\mathscr{F}_r\big)=\mathbb{E}\bigg(f(X_T)\int_r^T\langle g_t'\sigma_t(X_t,\mathscr{L}_{X_t})^{-1}v_t^\eta,\ \mathrm{d}W_t\rangle\bigg|\mathscr{F}_r\bigg). \tag{4.1}$$

Proof. Having Propositions 3.2 and 3.5 in hands, the proof is more or less standard. For v_t^η solving (3.11), we take

$$h_t=\int_{t\wedge r}^t g_s'\sigma_s(X_s,\mathscr{L}_{X_s})^{-1}v_s^\eta\,\mathrm{d}s,\quad t\in[0,T]. \tag{4.2}$$

By **(H)**, (2.1), and that $h \in L^2(\Omega \to \mathbb{H}, \mathbb{P})$ is adapted, Proposition 3.5 applies. Let $\tilde{v}_t = g_t v_t^\eta$ for $t \in [r, T]$. Then (3.11) and (4.2) imply

$$\begin{aligned}
\mathrm{d}\tilde{v}_t &= \Big\{\nabla_{\tilde{v}_t} b_t(\cdot, \mathscr{L}_{X_t})(X_t) + \big(\mathbb{E}\langle D^L b_t(y,\cdot)(\mathscr{L}_{X_t})(X_t), \tilde{v}_t\rangle\big)\big|_{y=X_t} + g_t' v_t^\eta\Big\}\mathrm{d}t \\
&\quad + \Big\{\nabla_{\tilde{v}_t} \sigma_t(\cdot, \mathscr{L}_{X_t})(X_t) + \big(\mathbb{E}\langle D^L \sigma_t(y,\cdot)(\mathscr{L}_{X_t})(X_t), \tilde{v}_t\rangle\big)\big|_{y=X_t}\Big\}\mathrm{d}W_t \\
&= \Big\{\nabla_{\tilde{v}_t} b_t(\cdot, \mathscr{L}_{X_t})(X_t) + \big(\mathbb{E}\langle D^L b_t(y,\cdot)(\mathscr{L}_{X_t})(X_t), \tilde{v}_t\rangle\big)\big|_{y=X_t} + \sigma_t(X_t, \mathscr{L}_{X_t}) h_t'\Big\}\mathrm{d}t \\
&\quad + \Big\{\nabla_{\tilde{v}_t} \sigma_t(\cdot, \mathscr{L}_{X_t})(X_t) + \big(\mathbb{E}\langle D^L \sigma_t(y,\cdot)(\mathscr{L}_{X_t})(X_t), \tilde{v}_t\rangle\big)\big|_{y=X_t}\Big\}\mathrm{d}W_t, \quad t \ge r,\ \tilde{v}_r = \mathbf{0}.
\end{aligned}$$

So, $(\tilde{v}_t)_{t\ge r}$ solves the SDE (3.26) with $\tilde{v}_r = \mathbf{0}$. On the other hand, by (4.2) we have $h_t' = 0$ for $t < r$, so that the solution to (3.26) with $w_0^h = 0$ satisfies $w_r^h = 0$. So, the uniqueness of this SDE from time r implies $\tilde{v}_t = w_t^h$ for all $t \ge r$. Combining this with Propositions 3.2 and 3.5, we obtain

$$\nabla_\eta X_T = v_T^\eta = g_T v_T^\eta = \tilde{v}_T = w_T^h = D_h X_T.$$

Thus, by the chain rule and the integration by parts formula (3.22), for any bounded $\mathscr{F}_r$-measurable $G \in \mathscr{D}(D)$, we have

$$\begin{aligned}
&\mathbb{E}\big(G\langle \nabla f(X_T), \nabla_\eta X_T\rangle\big) = \mathbb{E}\big(G\langle \nabla f(X_T), D_h X_T\rangle\ \big) = \mathbb{E}\big(G D_h f(X_T)\big) \\
&= \mathbb{E}\big(D_h\{G f(X_T)\} - f(X_T) D_h G\big) = \mathbb{E}\big(G f(X_T) D^*(h)\big),
\end{aligned}$$

where in the last step we have used $D_h G = 0$ since G is $\mathscr{F}_r$-measurable but $h_t' = 0$ for $t \le r$. Noting that the class of bounded $\mathscr{F}_r$-measurable $G \in \mathscr{D}(D)$ is dense in $L^2(\Omega, \mathscr{F}_r, \mathbb{P})$, this implies

$$\mathbb{E}\big(\langle \nabla f(X_T), \nabla_\eta X_T\rangle \big| \mathscr{F}_r\big) = \mathbb{E}\big(f(X_T) D^*(h) \big| \mathscr{F}_r\big).$$

Combining this with

$$D^*(h) = \int_r^T \langle h_t', \mathrm{d}W_t\rangle = \int_r^T \big\langle g_t' \sigma_t(X_t^\mu, P_t^*\mu)^{-1} v_t, \mathrm{d}W_t\big\rangle$$

due to (3.23) and (4.2), we prove (4.1). □

4.2 Proof of Theorem 2.1

Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$. We first establish (2.3) for $f \in \mathscr{B}_b(\mathbb{R}^d)$, then construct $\gamma \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)$ such that

$$\lim_{\mu(|\phi|^2)\to 0} \frac{|(P_T f)(\mu \circ (\mathrm{Id} + \phi)^{-1}) - (P_T f)(\mu) - \mu(\langle \phi, \gamma\rangle)|}{\sqrt{\mu(|\phi|^2)}} = 0, \tag{4.3}$$

which, by definition, implies that $P_T f$ is L-differentiable at μ with $D^L P_T f(\mu) = \gamma$.

(a) Proof of (2.3) for $f \in \mathscr{B}_b(\mathbb{R}^d)$. When $f \in C_b^1(\mathbb{R}^d)$, (2.3) follows from (4.1) for $\eta = \phi(X_0)$. Below we extend the formula to $f \in \mathscr{B}_b(\mathbb{R}^d)$. For $s \in [0,1]$, let $X_t^{\phi,s}$ solve (1.4) for $X_0^{\phi,s} = X_0 + s\phi(X_0)$. We have $\mu^{\phi,s} := \mathscr{L}_{X_0^{\phi,s}} = \mu \circ (\mathrm{Id} + s\phi)^{-1}$, and by the definition of $\nabla_\eta X_T$ for $\eta = \phi(X_0)$,

$$
\begin{aligned}
(4.4)\qquad &(P_T f)(\mu^{\phi,\varepsilon}) - (P_T f)(\mu) = \mathbb{E}[f(X_T^{\phi,\varepsilon}) - f(X_T)] = \int_0^\varepsilon \frac{\mathrm{d}}{\mathrm{d}s}\mathbb{E}[f(X_T^{\phi,s})]\,\mathrm{d}s \\
&= \int_0^\varepsilon \mathbb{E}\langle (\nabla f)(X_T^{\phi,s}), \nabla_{\phi(X_0)} X_T^{\phi,s}\rangle \,\mathrm{d}s, \quad f \in C_b^1(\mathbb{R}^d).
\end{aligned}
$$

Next, let $(v_t^{\phi,s})_{t\in[0,T]}$ solve (3.11) for $\eta = \phi(X_0)$ and X_t^s replacing X_t, i.e.

$$
\begin{aligned}
(4.5)\qquad \mathrm{d}v_t^{\phi,s} = &\Big\{\nabla_{v_t^{\phi,s}} b_t(\cdot, \mathscr{L}_{X_t^{\phi,s}})(X_t^{\phi,s}) + \big(\mathbb{E}\langle D^L b_t(y,\cdot)(\mathscr{L}_{X_t^{\phi,s}})(X_t^{\phi,s}), v_t^{\phi,s}\rangle\big)\big|_{y=X_t^{\phi,s}}\Big\}\mathrm{d}t \\
&+ \Big\{\nabla_{v_t^{\phi,s}} \sigma_t(\cdot, \mathscr{L}_{X_t^{\phi,s}})(X_t^{\phi,s}) + \big(\mathbb{E}\langle D^L \sigma_t(y,\cdot)(\mathscr{L}_{X_t^{\phi,s}})(X_t^{\phi,s}), v_t^{\phi,s}\rangle\big)\big|_{y=X_t^{\phi,s}}\Big\}\mathrm{d}W_t,
\end{aligned}
$$

for $v_0^{\phi,s} = \phi(X_0)$. Then (4.4) and (4.1) imply

$$
\begin{aligned}
(4.6)\qquad &(P_T f)(\mu^{\phi,\varepsilon}) - (P_T f)(\mu) \\
&= \int_0^\varepsilon \mathbb{E}\bigg[f(X_T^{\phi,s}) \int_0^T \big\langle g_t' \sigma_t(X_t^{\phi,s}, \mathscr{L}_{X_t^{\phi,s}})^{-1} v_t^s,\ \mathrm{d}W_t\big\rangle\bigg]\,\mathrm{d}s, \quad f \in C_b^1(\mathbb{R}^d).
\end{aligned}
$$

By a standard approximation argument, we may extend this formula to all $f \in \mathscr{B}_b(\mathbb{R}^d)$. Indeed, let

$$
\nu_\varepsilon(A) = \int_0^\varepsilon \mathbb{E}\bigg[1_A(X_T^{\phi,s}) \int_0^T \big\langle g_t' \sigma_t(X_t^{\phi,s}, \mathscr{L}_{X_t^s})^{-1} v_t^{\phi,s},\ \mathrm{d}W_t\big\rangle\bigg]\,\mathrm{d}s, \quad A \in \mathscr{B}(\mathbb{R}^d).
$$

Then ν_ε is a finite signed measure on $\mathbb{R}^d$ with

$$
\int_{\mathbb{R}^d} f\mathrm{d}\nu_\varepsilon = \int_0^\varepsilon \mathbb{E}\bigg[f(X_T^{\phi,s}) \int_0^T \big\langle g_t' \sigma_t(X_t^{\phi,s}, \mathscr{L}_{X_t^{\phi,s}})^{-1} v_t^{\phi,s},\ \mathrm{d}W_t\big\rangle\bigg]\,\mathrm{d}s, \quad f \in \mathscr{B}_b(\mathbb{R}^d).
$$

So, (4.6) is equivalent to

$$
(4.7)\qquad \int_{\mathbb{R}^d} f\mathrm{d}P_T^*\mu^{\phi,\varepsilon} - \int_{\mathbb{R}^d} f\mathrm{d}P_T^*\mu = \int_{\mathbb{R}^d} f\mathrm{d}\nu_\varepsilon, \quad f \in C_b^1(\mathbb{R}^d).
$$

Since $\nu_{T,\varepsilon} := P_T^*\mu^{\phi,\varepsilon} + P_T^*\mu + |\nu_\varepsilon|$ is a finite measure on $\mathbb{R}^d$, $C_b^1(\mathbb{R}^d)$ is dense in $L^1(\mathbb{R}^d, \nu_{T,\varepsilon})$. Hence, (4.7) holds for all $f \in \mathscr{B}_b(\mathbb{R}^d) \subset L^1(\mathbb{R}^d, \nu_{T,\varepsilon})$. Consequently, (4.6) holds for all $f \in \mathscr{B}_b(\mathbb{R}^d)$. Thus,

$$
\begin{aligned}
(4.8)\qquad &\frac{(P_T f)(\mu^{\phi,\varepsilon}) - (P_T f)(\mu)}{\varepsilon} \\
&= \frac{1}{\varepsilon}\int_0^\varepsilon \mathbb{E}\bigg[f(X_T^{\phi,s}) \int_0^T \big\langle g_t' \sigma_t(X_t^{\phi,s}, \mathscr{L}_{X_t^{\phi,s}})^{-1} v_t^{\phi,s},\ \mathrm{d}W_t\big\rangle\bigg]\,\mathrm{d}s, \quad f \in \mathscr{B}_b(\mathbb{R}^d).
\end{aligned}
$$

It is easy to see from **(H)** that

$$\lim_{s\to 0}\sup_{t\in[0,T]}\mathbb{E}\big(|X_t^{\phi,s}-X_t|^2+|v_t^{\phi,s}-v_t^{\phi}|^2\big)=0.$$

So,

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^{\varepsilon}\mathbb{E}\bigg|\int_0^T\big\langle g_t'\{\sigma_t(X_t^{\phi,s},\mathscr{L}_{X_t^{\phi,s}})^{-1}v_t^{\phi,s}-\sigma_t(X_t,\mathscr{L}_{X_t})^{-1}v_t^{\phi}\},\mathrm{d}W_t\big\rangle\bigg|=0. \tag{4.9}$$

Combining this with (4.8), we see that (2.3) for $f\in\mathscr{B}_b(\mathbb{R}^d)$ follows from

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}\bigg[\{f(X_T^{\phi,\varepsilon})-f(X_T)\}\int_0^T\big\langle g_t'\sigma_t(X_t,\mathscr{L}_{X_t})^{-1}v_t^{\phi},\mathrm{d}W_t\big\rangle\bigg]=0,\quad f\in\mathscr{B}_b(\mathbb{R}^d). \tag{4.10}$$

To prove this equality, for $r\in(0,T)$ we denote

$$I_r:=\int_0^r\big\langle g_t'\sigma_t(X_t,\mathscr{L}_{X_t})^{-1}v_t^{\phi},\mathrm{d}W_t\big\rangle.$$

Applying (4.1) with $g_t:=\frac{t-r}{T-r}$ for $t\in[r,T]$, we derive

$$\begin{aligned}&\big|\mathbb{E}[I_r\{f(X_T^{\phi,\varepsilon})-f(X_T)\}]\big|=\bigg|\mathbb{E}\bigg[I_r\int_0^{\varepsilon}\langle\nabla f(X_T^{\phi,s}),\nabla_{\phi(X_0)}X_T^{\phi,s}\rangle\mathrm{d}s\bigg]\bigg|\\&\le\mathbb{E}\bigg[|I_r|\cdot\bigg|\int_0^{\varepsilon}\mathbb{E}\big(\langle\nabla f(X_T^{\phi,s}),\nabla_{\phi(X_0)}X_T^{\phi,s}\rangle\big|\mathscr{F}_r\big)\mathrm{d}s\bigg|\bigg]\\&\le\|f\|_\infty\int_0^{\varepsilon}\mathbb{E}\bigg[|I_r|\bigg(\int_r^T\Big|\frac{1}{T-r}\sigma_t(X_t^{\phi,s},\mathscr{L}_{X_t^{\phi,s}})^{-1}v_t^{\phi,s}\Big|^2\mathrm{d}t\bigg)^{\frac12}\bigg]\mathrm{d}s,\quad f\in C_b^1(\mathbb{R}^d).\end{aligned}$$

By the argument extending (4.6) from $f\in C_b^1(\mathbb{R}^d)$ to $f\in\mathscr{B}_b(\mathbb{R}^d)$, we conclude from this that for any $r\in(0,T)$,

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0}\sup_{\|f\|_\infty\le 1}\big|\mathbb{E}[I_r\{f(X_T^{\phi,\varepsilon})-f(X_T)\}]\big|\\&\le\lim_{\varepsilon\downarrow 0}\int_0^{\varepsilon}\mathbb{E}\bigg[|I_r|\bigg(\int_r^T\Big|\frac{1}{T-r}\sigma_t(X_t^{\phi,s},\mathscr{L}_{X_t^{\phi,s}})^{-1}v_t^{\phi,s}\Big|^2\mathrm{d}t\bigg)^{\frac12}\bigg]\mathrm{d}s=0.\end{aligned}$$

Therefore,

$$\begin{aligned}&\limsup_{\varepsilon\downarrow 0}\sup_{\|f\|_\infty\le 1}\bigg|\mathbb{E}\bigg[\{f(X_T^{\phi,\varepsilon})-f(X_T)\}\int_0^T\big\langle g_t'\sigma_t(X_t,\mathscr{L}_{X_t})^{-1}v_t^{\phi}\},\mathrm{d}W_t\big\rangle\bigg]\bigg|\\&=\limsup_{\varepsilon\downarrow 0}\sup_{\|f\|_\infty\le 1}\bigg|\mathbb{E}\bigg[\{f(X_T^{\phi,\varepsilon})-f(X_T)\}\int_r^T\big\langle g_t'\sigma_t(X_t,\mathscr{L}_{X_t})^{-1}v_t^{\phi}\},\mathrm{d}W_t\big\rangle\bigg]\bigg|\\&\le 2\bigg(\mathbb{E}\int_r^T|g_t'\sigma_t(X_t,\mathscr{L}_{X_t})^{-1}v_t^{\phi}|^2\mathrm{d}t\bigg)^{\frac12}\end{aligned} \tag{4.11}$$

holds for $r \in (0,T)$. By letting $r \uparrow T$ we prove (4.10).

(b) For any $f \in \mathscr{B}_b(\mathbb{R}^d)$, we intend to find out $\gamma \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)$ such that

$$\mathbb{E}\bigg[f(X_T)\int_0^T \big\langle g_t' \sigma_t(X_t, \mathscr{L}_{X_t})^{-1} v_t^\phi, \mathrm{d}W_t\big\rangle\bigg] = \mu(\langle \phi, \gamma\rangle), \quad \phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu). \tag{4.12}$$

When $f \in C_b(\mathbb{R}^d)$, in step (c) we will deduce from this and (2.3) that $\gamma = D^L P_T f(\mu)$. To construct the desired γ, consider the SDE

$$\mathrm{d}X_t^\phi = b_t(X_t^\phi, \mathscr{L}_{X_t^\phi})\mathrm{d}t + \sigma_t(X_t^\phi, \mathscr{L}_{X_t^\phi})\mathrm{d}W_t, \quad X_0^\phi = X_0 + \phi(X_0),$$

and let v_t^ϕ solve (2.2). Since (2.2) is a linear equation for v_t^ϕ with initial value $\phi(X_0) \in L^2(\Omega \to \mathbb{R}^d, \mathscr{F}_0, \mathbb{P})$, the functional

$$L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu) \ni \phi \mapsto L\phi := \mathbb{E}\bigg[f(X_T)\int_0^T \big\langle g_t' \sigma_t(X_t, \mathscr{L}_{X_t})^{-1} v_t^\phi, \mathrm{d}W_t\big\rangle\bigg]$$

is linear, and by **(H)** and (2.1), there exists a constant $C > 0$ such that

$$|L\phi|^2 \le \|f\|_\infty^2 \sup_{t\in[0,T]} |g_t'\lambda_t|^2 \mathbb{E}\int_0^T |v_t^\phi|^2 dt \le C\,\mathbb{E}|\phi(X_0)|^2 = C\,\mu(|\phi|^2), \quad \phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu).$$

Then L is a bounded linear functional on the Hilbert space $L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)$. By Riesz's representation theorem, there exists a unique $\gamma \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)$ such that

$$L\phi = \mu(\langle \gamma, \phi\rangle), \quad \phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu).$$

Therefore, (4.12) holds.

(c) Now, for $f \in \mathscr{B}_b(\mathbb{R}^d)$, we intend to verify (4.3) for γ in (4.12), so that $P_T f$ is L-differentiable with $D^L(P_T f)(\mu) = \gamma$. By (4.8) for $\varepsilon = 1$, we have

$$\begin{aligned} &(P_T f)(\mu^1) - (P_T f)(\mu) \\ &= \int_0^1 \mathbb{E}\bigg[f(X_T^{\phi,s})\int_0^T \big\langle g_t' \sigma_t(X_t^{\phi,s}, \mathscr{L}_{X_t^{\phi,s}})^{-1} v_t^{\phi,s},\ \mathrm{d}W_t\big\rangle\bigg], \quad f \in \mathscr{B}_b(\mathbb{R}^d). \end{aligned} \tag{4.13}$$

Combining this with (4.12) and noting that $\mu^1 = \mu \circ (\mathrm{Id} + \phi)^{-1})$, we arrive at

$$\frac{|(P_T f)(\mu \circ (\mathrm{Id} + \phi)^{-1})) - (P_T f)(\mu) - \mu(\langle \phi, \gamma\rangle)|}{\sqrt{\mu(|\phi|^2)}} \le \ \varepsilon_1(\phi) + \varepsilon_2(\phi) + \varepsilon_3(\phi), \tag{4.14}$$

where

$$\begin{aligned} \varepsilon_1(\phi) &:= \frac{1}{\sqrt{\mu(|\phi|^2)}}\int_0^1 \mathbb{E}\bigg|\big(f(X_T^{\phi,s}) - f(X_T)\big)\int_0^T \langle g_t'\sigma_t(X_t^{\phi,s}, \mathscr{L}_{X_t^{\phi,s}})^{-1} v_t^{\phi,s}, \mathrm{d}W_t\rangle\bigg|\mathrm{d}s, \\ \varepsilon_2(\phi) &:= \frac{\|f\|_\infty}{\sqrt{\mu(|\phi|^2)}}\int_0^1 \mathbb{E}\bigg|\int_0^T \langle g_t'\{\sigma_t(X_t^{\phi,s}, \mathscr{L}_{X_t^{\phi,s}})^{-1} - \sigma_t(X_t, \mathscr{L}_{X_t})^{-1}\} v_t^\phi, \mathrm{d}W_t\rangle\bigg|\mathrm{d}s, \end{aligned}$$

$$\varepsilon_3(\phi) := \frac{\|f\|_\infty}{\sqrt{\mu(|\phi|^2)}} \int_0^1 \mathbb{E}\bigg| \int_0^T \big\langle g_t'\{\sigma_t(X_t^{\phi,s}, \mathscr{L}_{X_t^{\phi,s}})^{-1}(v_t^{\phi,s} - v_t^\phi), \mathrm{d}W_t\big\rangle\bigg|\mathrm{d}s.$$

It is easy to deduce from **(H)** that for any $p \geq 2$ there exists a constant $c(p) > 0$ such that

$$\sup_{t\in[0,T],s\in[0,1]} \mathbb{E}\big(|X_t^{\phi,s} - X_t|^p + |v_t^{\phi,s}|^p\big|\mathscr{F}_0\big) \leq c(p)|\phi(X_0)|^p. \tag{4.15}$$

Combining this with the continuity of $\sigma_t(x,\mu)$ in x and μ, we conclude that

$$\lim_{\mu(|\phi|^2)\to 0} \varepsilon_2(\phi) = 0. \tag{4.16}$$

Next, by the argument deducing (2.3) from (4.8), it is easy to see that (4.15) implies

$$\lim_{\mu(|\phi|^2)\to 0} \varepsilon_1(\phi) = 0. \tag{4.17}$$

Moreover, by the SDEs for $v_t^{\phi,s}$ and v_t^ϕ we have

$$\mathrm{d}(v_t^{\phi,s} - v_t^\phi) = \big\{A_t(v_t^{\phi,s} - v_t^\phi) + \tilde{A}_t v_t^{\phi,s}\big\}\mathrm{d}t + \big\{B_t(v_t^{\phi,s} - v_t^\phi) + \tilde{B}_t v_t^\phi\big\}\mathrm{d}W_t,$$

where for a square integrable random variable v on $\mathbb{R}^d$,

$$\begin{aligned}
A_t v &:= \nabla_v b_t(\cdot, \mathscr{L}_{X_t})(X_t) + \big(\mathbb{E}\langle D^L b_t(y,\cdot)(\mathscr{L}_{X_t})(X_t), v\rangle\big)\big|_{y=X_t},\\
\tilde{A}_t v &:= \nabla_v b_t(\cdot, \mathscr{L}_{X_t^{\phi,s}})(X_t^{\phi,s}) + \big(\mathbb{E}\langle D^L b_t(y,\cdot)(\mathscr{L}_{X_t^{\phi,s}})(X_t^{\phi,s}), v\rangle\big)\big|_{y=X_t^{\phi,s}}\\
&\quad - \nabla_v b_t(\cdot, \mathscr{L}_{X_t})(X_t) - \big(\mathbb{E}\langle D^L b_t(y,\cdot)(\mathscr{L}_{X_t})(X_t), v\rangle\big)\big|_{y=X_t},\\
B_t v &:= \nabla_v \sigma_t(\cdot, \mathscr{L}_{X_t})(X_t) + \big(\mathbb{E}\langle D^L \sigma_t(y,\cdot)(\mathscr{L}_{X_t})(X_t), v_t^\phi\rangle\big)\big|_{y=X_t},\\
\tilde{B}_t v &:= \nabla_v\big\{\sigma_t(\cdot, \mathscr{L}_{X_t^{\phi,s}})(X_t^{\phi,s}) + \big(\mathbb{E}\langle D^L \sigma_t(y,\cdot)(\mathscr{L}_{X_t^{\phi,s}})(X_t^{\phi,s}), v\rangle\big)\big|_{y=X_t^{\phi,s}}\\
&\quad - \nabla_v \sigma_t(\cdot, \mathscr{L}_{X_t})(X_t) - \big(\mathbb{E}\langle D^L \sigma_t(y,\cdot)(\mathscr{L}_{X_t})(X_t), v\rangle\big)\big|_{y=X_t}.
\end{aligned}$$

Combining this with (4.15) and **(H)**, there exists a constant $c > 0$ such that

$$\mathrm{d}|v_t^{\phi,s} - v_t^\phi|^2 \leq c|v_t^{\phi,s} - v_t^\phi|^2\mathrm{d}t + c\big(\|\tilde{A}_t\|^2 + \|\tilde{B}_t\|^2\big)\big(|v_t^{\phi,s}|^2 + |v_t^\phi|^2\big)\mathrm{d}t + \mathrm{d}M_t,\ \ |v_0^{\phi,s} - v_0^\phi| = 0 \tag{4.18}$$

holds for some martingale M_t, and that

$$\|\tilde{A}_t\|^2 + \|\tilde{B}_t\|^2 \leq c, \quad \lim_{\mu(|\phi|^2)\to 0} \big(\|\tilde{A}_t\|^2 + \|\tilde{B}_t\|^2\big) = 0, \quad t \in [0,T], s \in [0,1]. \tag{4.19}$$

By (4.18) and (4.15) for $p = 4$, there exists a constant $c' > 0$ such that

$$\begin{aligned}
&\mathbb{E}(|v_t^{\phi,s} - v_t^\phi|^2|\mathscr{F}_0)\\
&\leq c\int_0^t \mathbb{E}(|v_r^{\phi,s} - v_r^\phi|^2|\mathscr{F}_0)\mathrm{d}r + 2c\int_0^T \sqrt{\mathbb{E}(\|\tilde{A}_t\|^4 + \|\tilde{B}_t\|^4|\mathscr{F}_0)} \cdot \sqrt{\mathbb{E}(|v_t^{\phi,s}|^4 + |v_t^\phi|^4|\mathscr{F}_0)}\,\mathrm{d}t
\end{aligned}$$

$$\leq c\int_0^t \mathbb{E}(|v_r^{\phi,s}-v_r^{\phi}|^2|\mathscr{F}_0)\mathrm{d}r + c'\varepsilon(\phi)|\phi(X_0)|^2, \quad s\in[0,1], t\in[0,T],$$

where

$$\varepsilon(\phi) := \int_0^T \sqrt{\mathbb{E}(\|\tilde{A}_t\|^4 + \|\tilde{B}_t\|^4|\mathscr{F}_0)}\,\mathrm{d}t.$$

Then Gronwall's lemma and (4.19) yield

$$\begin{aligned}&\sup_{s\in[0,T]} \mathbb{E}(|v_t^{\phi,s}-v_t^{\phi}|^2|\mathscr{F}_0) \leq c'\mathrm{e}^{cT}\varepsilon(\phi)|\phi(X_0)|^2,\\ &\lim_{\mu(|\phi|^2)\to 0} \mathbb{E}\varepsilon(\phi) = 0.\end{aligned}$$

Combining this with the definition of $\varepsilon_3(\phi)$, **(H)**, and Jensen's inequality for the conditional expectation $\mathbb{E}(\cdot|\mathscr{F}_0)$, we may find out constants $C_1, C_2 > 0$ depending on $\|f\|_\infty$ and T such that

$$\begin{aligned}&\lim_{\mu(|\phi|^2)\to 0}\varepsilon_3(\phi) \leq \lim_{\mu(|\phi|^2)\to 0} \frac{C_1}{\sqrt{\mu(|\phi|^2)}}\int_0^1 \mathbb{E}\bigg(\int_0^T |v_t^{\phi,s}-v_t^{\phi}|^2\mathrm{d}t\bigg)^{\frac{1}{2}}\mathrm{d}s\\ &\leq \lim_{\mu(|\phi|^2)\to 0} \frac{C_1}{\sqrt{\mu(|\phi|^2)}}\int_0^1 \mathbb{E}\bigg(\int_0^T \mathbb{E}(|v_t^{\phi,s}-v_t^{\phi}|^2|\mathscr{F}_0)\mathrm{d}t\bigg)^{\frac{1}{2}}\mathrm{d}s\\ &\leq \lim_{\mu(|\phi|^2)\to 0} \frac{C_2}{\sqrt{\mu(|\phi|^2)}}\int_0^1 \mathbb{E}\big(|\phi(X_0)|\sqrt{\varepsilon(\phi)}\big)\mathrm{d}s\\ &\leq \lim_{\mu(|\phi|^2)\to 0} \frac{C_2\sqrt{(\mathbb{E}|\phi(X_0)|^2)\mathbb{E}\varepsilon(\phi)}}{\sqrt{\mu(|\phi|^2)}} = \lim_{\mu(|\phi|^2)\to 0} C_2\sqrt{\mathbb{E}\varepsilon(\phi)} = 0.\end{aligned}$$

This, together with (4.14), (4.16) and (4.17), implies (4.3). Therefore, $P_T f$ is L-differentiable at μ with $D^L(P_T f)(\mu) = \gamma$.

(d) Finally, (2.3) and (4.8) imply

$$\begin{aligned}&\Big|\frac{P_T^*\mu\circ(\mathrm{Id}+\varepsilon\phi)^{-1} - P_T^*\mu}{\varepsilon}(f) - (\psi P_T^*\mu)(f)\Big|\\ &= \bigg|\frac{(P_T f)(\mu^{\phi,\varepsilon}) - (P_T f)(\mu)}{\varepsilon} - \mathbb{E}\bigg[f(X_T)\int_0^T \langle g_t'\sigma_t(X_t,\mathscr{L}_{X_t})^{-1}v_t^{\phi}, \mathrm{d}W_t\rangle\bigg]\bigg|\\ &\leq \frac{\|f\|_\infty}{\varepsilon}\int_0^\varepsilon \mathbb{E}\bigg|\int_0^T \langle g_t'\{\sigma_t(X_t^{\phi,s},\mathscr{L}_{X_t^{\phi,s}})^{-1}v_t^{\phi,s} - \sigma_t(X_t,\mathscr{L}_{X_t})^{-1}v_t^{\phi}\}, \mathrm{d}W_t\rangle\bigg|\mathrm{d}s\\ &\quad + \frac{1}{\varepsilon}\bigg|\mathbb{E}\bigg[\{f(X_T^{\phi,\varepsilon}) - f(X_T)\}\int_0^T \big\langle g_t'\sigma_t(X_t,\mathscr{L}_{X_t})^{-1}v_t^{\phi}\}, \mathrm{d}W_t\big\rangle\bigg]\bigg|\mathrm{d}s.\end{aligned}$$

Combining this with (4.9) and (4.11) we prove (2.4).

4.3 Proof of Corollary 2.2

Proof of (1). By **(H)** and (2.2), there exists a martingale M_t such that

$$\mathrm{d}|v_t^\phi|^2 \le 4K(t)|v_t^\phi|(|v_t^\phi| + \mathbb{E}|v_t^\phi|)\mathrm{d}t + \mathrm{d}M_t, \quad |v_0^\phi|^2 = |\phi(X_0)|^2, \tag{4.20}$$

where $K(t)$ is increasing in $t \ge 0$. Then

$$\mathbb{E}|v_t^\phi|^2 \le \mathbb{E}|\phi(X_0)|^2 + 4K(t)\int_0^t \big\{\mathbb{E}|v_s^\phi|^2 + (\mathbb{E}|v_s^\phi|)^2\big\}\mathrm{d}s \le \mu(|\phi|^2) + 8K(t)\int_0^t \mathbb{E}|v_s^\phi|^2\mathrm{d}s.$$

By Gronwall's inequality this implies

$$\mathbb{E}|v_t^\phi|^2 \le \mathrm{e}^{8K(t)t}\mu(|\phi|^2), \quad t \in [0,T]. \tag{4.21}$$

Next, since $\mathbb{E}\int_0^T \big\langle g_t'\sigma_t(X_t, \mathscr{L}_{X_t})^{-1}v_t^\phi,\ \mathrm{d}W_t\big\rangle = 0$, (2.3) is equivalent to

$$D_\phi^L(P_Tf)(\mu) = \mathbb{E}\bigg[\big\{f(X_T) - P_Tf(\mu)\big\}\int_0^T \big\langle g_t'\sigma_t(X_t, \mathscr{L}_{X_t})^{-1}v_t^\phi,\ \mathrm{d}W_t\big\rangle\bigg].$$

Combining this with (4.21) and using Jensen's inequality, when $\mu(|\phi|^2) \le 1$ we have

$$\begin{aligned}|D_\phi^L(P_Tf)(\mu)|^2 &\le \big\{(P_Tf^2)(\mu) - (P_Tf(\mu))^2\big\}\int_0^T \mathbb{E}\big|g_t'\sigma_t(X_t, \mathscr{L}_{X_t})^{-1}v_t^\phi\big|^2\mathrm{d}t \\ &\le \big\{(P_Tf^2)(\mu) - (P_Tf(\mu))^2\big\}\int_0^T |g_t'|^2\lambda_t^2\mathrm{e}^{8tK(t)}\mathrm{d}t\end{aligned}$$

for any $g \in C^1([0,T])$ with $g_0 = 0$ and $g_T = 1$. Taking

$$g_t = \frac{\int_0^t \lambda_r^{-2}\mathrm{e}^{-8rK(r)}\mathrm{d}r}{\int_0^T \lambda_r^{-2}\mathrm{e}^{-8rK(r)}\mathrm{d}r}, \quad t \in [0,T],$$

we prove the estimate (2.5). □

Proof of (2). Let $f \in \mathscr{B}_b(\mathbb{R}^d)$ with $\|f\|_\infty \le 1$. By Theorem 2.1, P_Tf is L-differentiable. Moreover, by Theorem 4.1, P_Tf is Lipschitz continuous on $\mathscr{P}_2(\mathbb{R}^d)$. Indeed, for any $\mu_1, \mu_2 \in \mathscr{P}_2(\mathbb{R}^d)$, let $X_1, X_2 \in L^2(\Omega \to \mathbb{R}^d, \mathscr{F}_0, \mathbb{P})$ such that $\mathscr{L}_{X_i} = \mu_i, 1 \le i \le 2$, and $\mathbb{E}|X_1 - X_2|^2 = \mathbb{W}_2(\mu_1, \mu_2)^2$. Let X_t^s be the solution to (1.4) with $X_0 = X_1 + s(X_2 - X_1), s \in [0,1]$. Then Theorem 4.1 implies

$$\begin{aligned}|P_Tf(\mu_1) - P_Tf(\mu_2)|^2 &= |\mathbb{E}f(X_T^0) - \mathbb{E}f(X_T^1)|^2 = \bigg|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\mathbb{E}f(X_T^s)\,\mathrm{d}s\bigg|^2 \\ &= \bigg|\int_0^1 \mathbb{E}\langle\nabla f(X_T^s), \nabla_{X_2-X_1}X_T^s\rangle\mathrm{d}s\bigg|^2 \le c\mathbb{E}|X_2 - X_1|^2 = c\mathbb{W}_2(\mu_1,\mu_2)^2\end{aligned}$$

for some constant $c > 0$.

To apply Proposition 3.1, we take $\{\mu_n,\nu_n\}_{n\ge1}\subset\mathscr{P}_2(\mathbb{R}^d)$ which have compact supports and are absolutely continuous with respect to the Lebesgue measure, such that

$$\lim_{n\to\infty}\big\{\mathbb{W}_2(\mu,\mu_n)+\mathbb{W}_2(\nu,\nu_n)\big\}=0. \tag{4.22}$$

According to [4], see also [6, Theorem 5.8], for any $n\ge1$ there exists a unique map $\phi_n\in L^2(\mathbb{R}^d\to\mathbb{R}^d,\mu)$ such that

$$\nu_n=\mu_n\circ(\mathrm{Id}+\phi_n)^{-1},\quad \mathbb{W}_2(\mu_n,\nu_n)^2=\mu_n(|\phi_n|^2). \tag{4.23}$$

Let $X_n\in L^2(\Omega\to\mathbb{R}^d,\mathscr{F}_0,\mathbb{P})$ such that $\mathscr{L}_{X_n}=\mu_n$. By Proposition 3.1, (2.5) and (4.23), we obtain

$$\begin{aligned}|(P_Tf)(\mu_n)-(P_Tf)(\nu_n)|^2&=\bigg|\int_0^1\frac{\mathrm{d}}{\mathrm{d}s}(P_Tf)(\mathscr{L}_{X_n+s\phi_n(X_n)})\,\mathrm{d}s\bigg|^2\\
&=\bigg|\int_0^1\mathbb{E}\big\langle D^L(P_Tf)(\mathscr{L}_{X_n+s\phi_n(X_n)})(X_n+s\phi_n(X_n)),\phi_n(X_n)\big\rangle\,\mathrm{d}s\bigg|^2\\
&\le\frac{\|f\|_\infty^2\mu_n(|\phi_n|^2)}{\int_0^T\lambda_t^{-2}\mathrm{e}^{-8tK(t)}\mathrm{d}t}=\frac{\|f\|_\infty^2\mathbb{W}_2(\mu_n,\nu_n)^2}{\int_0^T\lambda_t^{-2}\mathrm{e}^{-8tK(t)}\mathrm{d}t}.\end{aligned}$$

By the continuity of P_Tf and (4.22), by letting $n\to\infty$ we prove

$$|(P_Tf)(\mu)-(P_Tf)(\nu)|^2\le\frac{\mathbb{W}_2(\mu,\nu)^2}{\int_0^T\lambda_t^{-2}\mathrm{e}^{-8tK(t)}\mathrm{d}t},\quad \mu,\nu\in\mathscr{P}_2(\mathbb{R}^d),\quad f\in\mathscr{B}_b(\mathbb{R}^d),\|f\|_\infty\le1.$$

Therefore, (2.6) and (2.7) hold. □

4.4 Proof of Theorem 2.3

Let $T>r\ge0,\mu\in\mathscr{P}_2(\mathbb{R}^{m+d})$ and let X_t solve (2.8) with $\mathscr{L}_{X_0}=\mu$. To realize the procedure in the proof of Theorem 2.1 for the present degenerate setting, we first extend Theorem 4.1 using $D^*(h^\alpha_{r,\cdot})$ to replace $\int_r^T\langle g_t'\sigma_t(X_t,\mathscr{L}_{X_t})^{-1}v_t^\eta,\mathrm{d}W_t\rangle$, where for a $C^1([r,T]\to\mathbb{R}^{m+d})$-valued random variable $\alpha_\cdot=(\alpha^{(1)}_\cdot,\alpha^{(2)}_\cdot)$,

$$h^\alpha_{r,t}:=\int_{r\wedge t}^t\sigma_s^{-1}\Big\{\nabla_{\alpha_s}b_s^{(2)}(X_s,\mathscr{L}_{X_s})+\big(\mathbb{E}\langle D^Lb_s^{(2)}(y,\cdot)(\mathscr{L}_{X_s})(X_s),\alpha_s\rangle\big)\big|_{y=X_s}-(\alpha_s^{(2)})'\Big\}\mathrm{d}s \tag{4.24}$$

for $t\in[0,T]$.

Theorem 4.2. *Assume* (**H1**). *Let $T>r\ge0$, $\eta\in L^2(\Omega\to\mathbb{R}^{m+d},\mathscr{F}_0,\mathbb{P})$, and let X_t solve (2.8) with $\mathscr{L}_{X_0}=\mu\in\mathscr{P}_2(\mathbb{R}^{m+d})$. If there exists a $C^1([r,T]\to\mathbb{R}^{m+d})$-valued random variable $\alpha_\cdot=(\alpha^{(1)}_\cdot,\alpha^{(2)}_\cdot)$ such that $\alpha_r=\nabla_\eta X_r,\alpha_T=\mathbf{0}$,*

$$(\alpha_t^{(1)})'=\nabla_{\alpha_t}b_t^{(1)}(X_t),\quad t\in[r,T], \tag{4.25}$$

and $h^\alpha_{r,\cdot}\in\mathscr{D}(D^*)$, then for any $f\in C_b^1(\mathbb{R}^{m+d})$,

$$\mathbb{E}\big(\langle\nabla f(X_T),\nabla_\eta X_T\rangle\big|\mathscr{F}_r\big)=\mathbb{E}\big(f(X_T)\,D^*(h^\alpha_{r,\cdot})\big|\mathscr{F}_r\big). \tag{4.26}$$

Proof. By Proposition 3.5, $w_t := D_{h^\alpha_{r,\cdot}} X_t$ satisfies

$$w_t = \int_0^t \Big\{\nabla_{w_s} b_s(\cdot,\mathscr{L}_{X_s})(X_s) + \Big(\mathbf{0}, \sigma_s (h^\alpha_{r,s})' + \big(\mathbb{E}\langle D^L b^{(2)}_s(y,\cdot)(\mathscr{L}_{X_s})(X_s), w_s\rangle\big)\big|_{y=X_s}\Big)\Big\}\mathrm{d}s.$$

Since $(h^\alpha_{r,\cdot})'(s) = 0$ for $s \le r$, this implies $w_t = 0$ for $t \in [0,r]$ so that

$$w_t = \int_{t\wedge r}^t \Big\{\nabla_{w_s} b_s(\cdot,\mathscr{L}_{X_s})(X_s) + \Big(\mathbf{0}, \sigma_s (h^\alpha_{r,s})' + \big(\mathbb{E}\langle D^L b^{(2)}_s(y,\cdot)(\mathscr{L}_{X_s})(X_s), w_s\rangle\big)\big|_{y=X_s}\Big)\Big\}\mathrm{d}s.$$

Extending α_t with $\alpha_t := \nabla_\eta X_t$ for $t \in [0,r)$, and letting $v_t = w_t + \alpha_t$ for any $t \in [0,T]$, we obtain

$$\begin{aligned} v_t = \alpha_t + \int_{t\wedge r}^t &\Big\{\nabla_{v_s} b_s(\cdot,\mathscr{L}_{X_s})(X_s) + \Big(\mathbf{0}, \big(\mathbb{E}\langle D^L b^{(2)}_s(y,\cdot)(\mathscr{L}_{X_s})(X_s), v_s\rangle\big)\big|_{y=X_s}\Big) \\ &+ (\mathbf{0}, \sigma_s (h^\alpha_s)' - \big(\mathbb{E}\langle D^L b^{(2)}_s(y,\cdot)(\mathscr{L}_{X_s})(X_s), \alpha_s\rangle\big)\big|_{y=X_s}) - \nabla_{\alpha_s} b_s(\cdot,\mathscr{L}_{X_s})(X_s)\Big\}\mathrm{d}s. \end{aligned} \tag{4.27}$$

By (4.25),

$$\int_{t\wedge r}^t \nabla_{\alpha_s} b^{(1)}_s(\cdot,\mathscr{L}_{X_s})(X_s)\,\mathrm{d}s = 1_{\{t>r\}}\big(\alpha^{(1)}_t - \nabla_\eta X^{(1)}_r\big),$$

while the definition of $h^\alpha_{r,s}$ implies

$$\begin{aligned} &\int_{t\wedge r}^t \Big\{\sigma_s (h^\alpha_s)' - \big(\mathbb{E}\langle D^L b^{(2)}_s(y,\cdot)(\mathscr{L}_{X_s})(X_s), \alpha_s\rangle\big)\big|_{y=X_s} - \nabla_{\alpha_s} b^{(2)}_s(\cdot,\mathscr{L}_{X_s})(X_s)\Big\}\mathrm{d}s \\ &= -\int_{t\wedge r}^t (\alpha^{(2)}_s)'\mathrm{d}s = 1_{\{t>r\}}\big(\nabla_\eta X^{(2)}_r - \alpha^{(2)}_t\big). \end{aligned}$$

Combining these with (4.27) and Proposition 3.2 leads to

$$\begin{aligned} v_t &= \nabla_\eta X_r + \int_{t\wedge r}^t \Big\{\nabla_{v_s} b_s(\cdot,\mathscr{L}_{X_s})(X_s) + \Big(\mathbf{0}, \big(\mathbb{E}\langle D^L b^{(2)}_s(y,\cdot)(\mathscr{L}_{X_s})(X_s), v_s\rangle\big)\big|_{y=X_s}\Big)\Big\}\mathrm{d}s \\ &= \eta + \int_0^t \Big\{\nabla_{v_s} b_s(\cdot,\mathscr{L}_{X_s})(X_s) + \Big(\mathbf{0}, \big(\mathbb{E}\langle D^L b^{(2)}_s(y,\cdot)(\mathscr{L}_{X_s})(X_s), v_s\rangle\big)\big|_{y=X_s}\Big)\Big\}\mathrm{d}s, \quad t \in [0,T]. \end{aligned}$$

That is, v_t solves (3.11) so that by Proposition 3.2 we obtain $v_t := w_t + \alpha_t = \nabla_\eta X_t$. Since $\alpha_T = 0$, this implies $D_{h^\alpha_{r,\cdot}} X_T = \nabla_\eta X_T$. Thus, for any bounded $\mathscr{F}_r$-measurable $G \in \mathscr{D}(D)$,

$$\begin{aligned} &\mathbb{E}\big[G\langle \nabla f(X_T), \nabla_\eta X_T\rangle\big] = \mathbb{E}\big[G D_{h^\alpha_{r,\cdot}} f(X_T)\big] \\ &= \mathbb{E}\big[D_{h^\alpha_{r,\cdot}}\{G f(X_T)\} - f(X_T) D_{h^\alpha_{r,\cdot}} G\big] = \mathbb{E}\big[G f(X_T) D^*(h^\alpha_{r,\cdot})\big], \end{aligned} \tag{4.28}$$

where in the last step we have used the integration by parts formula (3.22) and $D_{h^\alpha_{r,\cdot}} G = 0$ since G is $\mathscr{F}_r$-measurable but

$$D_{h^\alpha_{r,\cdot}} G = \int_0^T (h^\alpha_{r,\cdot})'(s) \cdot \{(DG).\}'(s)\mathrm{d}s = 0,$$

$(h^\alpha_{r,\cdot})'(s) = 0$ for $s \le r$. Noting that the class of bounded $\mathscr{F}_r$-measurable functions $G \in \mathscr{D}(D)$ is dense in $L^2(\Omega, \mathscr{F}_r, \mathbb{P})$, (4.28) implies (4.26). □

Proof of Theorem 2.3. With Theorem 4.2 in hands, the proof is completely similar to that of Theorem 2.1. Let

$$v_t^\phi = ((v_t^\phi)^{(1)}, (v_t^\phi)^{(2)}) = (\nabla_{\phi(X_0)} X_t^{(1)}, \nabla_{\phi(X_0)} X_t^{(2)}) = \nabla_{\phi(X_0)} X_t, \quad t \in [0,T].$$

For any $0 \le r < T$, let

$$\begin{aligned}\alpha_{r,t}^{(2)} =& \frac{T-t}{T-r}(v_t^\phi)^{(2)} - \frac{(t-r)(T-t)B_t^* K_{T,t}^*}{\int_0^T \theta_s^2 \mathrm{d}s} \int_t^T \theta_s^2 Q_s^{-1} K_{T,r} (v_t^\phi)^{(1)} \mathrm{d}s \\ & - (t-r)(T-t) B_t^* K_{T,t}^* Q_T^{-1} \int_0^T \frac{T-s}{T} K_{T,s} \nabla^{(2)} b_s^{(1)}(X_s) \phi^{(2)}(X_0) \mathrm{d}s, \quad t \in [r,T],\end{aligned} \tag{4.29}$$

and

$$\alpha_{r,t}^{(1)} = K_{t,r}(v_t^\phi)^{(1)} + \int_r^t K_{t,s} \nabla^{(2)}_{\alpha_s^{(2)}} b_s^{(1)}(X_s(x))\, \mathrm{d}s, \quad t \in [r,T]. \tag{4.30}$$

Then $\alpha_{r,\cdot} := (\alpha_{r,t}^{(1)}, \alpha_{r,t}^{(2)})$ satisfies

$$\alpha_{r,r} = \nabla_{\phi(X_0)} X_r, \quad \alpha_{r,T} = 0,$$

and by (2.9) and Duhamel's formula, (4.30) implies

$$(\alpha_{r,\cdot}^{(1)})'(t) = \nabla_{\alpha_{r,t}} b_t^{(1)}(X_t), \quad t \in [r,T].$$

Moreover, let $h_{r,\cdot}^{\alpha_{r,\cdot}}$ be defined in (4.24) for $\alpha_{r,\cdot}$ replacing α. Noting that **(H1)** and **(H2)** imply [28, (H)] for $l_1 = l_2 = 0$, the proof of [28, Theorem 1.1] with $\phi(s) := (s-r)(T-s)$ for $s \in [r,T]$ ensures that $h_{r,\cdot}^{\alpha_{r,\cdot}} \in \mathscr{D}(D^*)$ with $D^*(h_{r,\cdot}^{\alpha_{r,\cdot}}) \in L^p(\mathbb{P})$ for all $p \in (1,\infty)$. So, by Theorem 2.3 with $\eta = \phi(X_0)$ we obtain

$$\mathbb{E}(\langle \nabla f(X_T), \nabla_{\phi(X_0)} X_T \rangle | \mathscr{F}_r) = \mathbb{E}(f(X_T) D^*(h_{r,\cdot}^{\alpha_{r,\cdot}}) | \mathscr{F}_r), \quad f \in C_b^1(\mathbb{R}^d), r \in [0,T). \tag{4.31}$$

In particular, taking $r = 0$ we obtain $D^*(h) \in L^p(\mathbb{P})$ for all $p \in (1,\infty)$ and

$$D_\phi^L P_T f(\mu) = \mathbb{E}(\langle \nabla f(X_T), \nabla_{\phi(X_0)} X_T \rangle) = \mathbb{E}(f(X_T) D^*(h^\alpha) | \mathscr{F}_r), \quad f \in C_b^1(\mathbb{R}^d). \tag{4.32}$$

Basing on these two formulas, by repeating the proof of Theorem 2.1 with $I_r := \mathbb{E}(D^*(h^\alpha)|\mathscr{F}_r)$, we prove (2.16) and the L-differentiability of $P_T f$ for $f \in \mathscr{B}_b(\mathbb{R}^{m+d})$. Finally, the estimates (2.17) and (2.18) follows from (2.16) as in the proof of Theorem 2.1, together with the corresponding estimate on $\mathbb{E}|D^*(h^\alpha)|^2$ as in the proof of [28, Theorem 1.1]. For instance, below we outline the proof of (2.16).

Firstly, for $s \in (0,1)$ let X_t^s solve (2.8) with $X_0^{\phi,s} = X_0 + s\phi(X_0)$, let $\mu^{\phi,s} = \mathscr{L}_{X_0^{\phi,s}} = \mu \circ (\mathrm{Id} + \phi)^{-1}$, and let $\alpha_{r,t}^{\phi,s}$ be defined as $\alpha_{r,t}$ with $X_t^{\phi,s}$ replacing X_t. Then as in (4.4) and (4.7), (4.32) implies

$$\begin{aligned}(P_T f)(\mu^{\phi,\varepsilon}) - (P_T f)(\mu) &= \int_0^\varepsilon \mathbb{E}\langle (\nabla f)(X_T^{\phi,s}), \nabla_{\phi(X_0)} X_T^{\phi,s} \rangle \,\mathrm{d}s \\ &= \int_0^\varepsilon \mathbb{E}\big[f(X_T^{\phi,s}) D^*(h^{\alpha^{\phi,s}})\big], \quad f \in C_b^1(\mathbb{R}^{m+d}),\end{aligned} \tag{4.33}$$

where $h^{\alpha^{\phi,s}} := h_{0,\cdot}^{\alpha_{0,\cdot}^{\phi,s}}$ satisfies

$$\lim_{s\to 0} \mathbb{E}|D^*(h^{\alpha^{\phi,s}}) - D^*(h)|^2 = 0. \tag{4.34}$$

By the argument leading to (4.8), (4.33) yields

$$\frac{(P_T f)(\mu^{\phi,\varepsilon}) - (P_T f)(\mu)}{\varepsilon} = \frac{1}{\varepsilon}\int_0^\varepsilon \mathbb{E}\big[f(X_T^{\phi,s})D^*(h^{\alpha^{\phi,s}})\big]\,\mathrm{d}s, \quad f \in \mathscr{B}_b(\mathbb{R}^{m+d}).$$

Combining this with (4.34), we prove (2.16) provided

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_0^\varepsilon \mathbb{E}\big[\{f(X_T^{\phi,s}) - f(X_T)\}D^*(h^\alpha)\big]\,\mathrm{d}s = 0. \tag{4.35}$$

For any $r \in (0,T)$, let $I_r = \mathbb{E}(D^*(h^\alpha)|\mathscr{F}_r)$. By (4.33) we obtain

$$\begin{aligned}
&\mathbb{E}\big[\{f(X_T^{\phi,\varepsilon}) - f(X_T)\}I_r\big] = \mathbb{E}\big[I_r \mathbb{E}(f(X_T^{\phi,\varepsilon}) - f(X_T)|\mathscr{F}_r)\big] \\
&= \mathbb{E}\bigg[I_r \int_0^\varepsilon \mathbb{E}\big(\langle \nabla f(X_T^{\phi,s}), \nabla X_T^{\phi,s}\rangle \big|\mathscr{F}_r\big)\,\mathrm{d}s\bigg] = \mathbb{E}\bigg[I_r \int_0^\varepsilon \mathbb{E}\big(f(X_T^{\phi,s})D^*(h_{r,\cdot}^{\alpha_{r,\cdot}})\big|\mathscr{F}_r\big)\,\mathrm{d}s\bigg] \\
&= \int_0^\varepsilon \mathbb{E}\big[I_r f(X_T^{\phi,s})D^*(h_{r,\cdot}^{\alpha_{r,\cdot}})\big]\,\mathrm{d}s, \quad f \in C_b^1(\mathbb{R}^d).
\end{aligned}$$

Combining this with the argument extending (4.8) from $f \in C_b^1(\mathbb{R}^d)$ to $f \in \mathscr{B}_b(\mathbb{R}^d)$, we obtain

$$\mathbb{E}\big[\{f(X_T^{\phi,\varepsilon}) - f(X_T)\}I_r\big] = \int_0^\varepsilon \mathbb{E}\big[I_r f(X_T^{\phi,s})D^*(h_{r,\cdot}^{\alpha_{r,\cdot}})\big]\,\mathrm{d}s, \quad f \in \mathscr{B}_b(\mathbb{R}^d).$$

Consequently,

$$\lim_{\varepsilon\to 0} \mathbb{E}\big[\{f(X_T^{\phi,\varepsilon}) - f(X_T)\}I_r\big] = 0, \quad f \in \mathscr{B}_b(\mathbb{R}^d), r \in (0,T).$$

Then for any $r \in (0,T)$,

$$\begin{aligned}
&\limsup_{\varepsilon\downarrow 0} \bigg|\frac{1}{\varepsilon}\int_0^\varepsilon \mathbb{E}\big[\{f(X_T^{\phi,s}) - f(X_T)\}D^*(h^\alpha)\big]\,\mathrm{d}s\bigg| \\
&= \limsup_{\varepsilon\downarrow 0} \bigg|\frac{1}{\varepsilon}\int_0^\varepsilon \mathbb{E}\big[\{f(X_T^{\phi,s}) - f(X_T)\}\cdot\{D^*(h^\alpha) - I_r\}\big]\,\mathrm{d}s\bigg| \\
&\le 2\|f\|_\infty \mathbb{E}|D^*(h^\alpha) - \mathbb{E}(D^*(h^\alpha)|\mathscr{F}_r)|.
\end{aligned}$$

Letting $r \uparrow T$ we derive (4.35), and hence prove (2.16) as explained above. □

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