Superhedging prices of European and American options in a non-linear incomplete market with default

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Abstract

This paper studies the superhedging prices and the associated superhedging strategies for European and American options in a non-linear incomplete market with default. We present the seller's and the buyer's point of view. The underlying market model consists of a risk-free asset and a risky asset driven by a Brownian motion and a compensated default martingale. The portfolio process follows non-linear dynamics with a non-linear driver f. By using a dynamic programming approach, we first provide a dual formulation of the seller's (superhedging) price for the European option as the supremum over a suitable set of equivalent probability measures $Q \in \mathcal{Q}$ of the f-evaluation/expectation under Q of the payoff. We also provide an infinitesimal characterization of this price as the minimal supersolution of a constrained BSDE with default. By a form of symmetry, we derive corresponding results for the buyer. We also give a dual representation of the seller's (superhedging) price for the American option associated with an irregular payoff (ξ_t) (not necessarily càdlàg) in terms of the value of a non-linear mixed control/stopping problem. We also provide an infinitesimal characterization of this price in terms of a constrained reflected BSDE. When ξ is càdlàg, we show a duality result for the buyer's price. These results rely on first establishing a non-linear optional decomposition for processes which are $\mathcal{E}^f$-strong supermartingales under Q, for all $Q \in \mathcal{Q}$.

Key-words: European options, American options, incomplete markets, non-linear pricing, BSDEs with constraints, constrained reflected BSDEs, f-expectation, control problems with non-linear expectation, optimal stopping with non-linear expectation, non-linear optional decomposition, pricing-hedging duality

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1 Introduction

We consider a financial market with a default time ϑ, which contains one risky asset whose price dynamics are driven by a one-dimensional Brownian motion and a compensated default martingale. We study the case of a market with imperfections which are encoded in the non-linearity of the portfolio dynamics. We note that our market is *incomplete* in the sense that not every European contingent claim can be replicated by a portfolio. In this framework, we are interested in the problem of pricing and hedging of European and American options, from the point of view of the seller, and of the buyer.

We recall that in the case of a *non-linear complete* market, the (hedging) price of the European option for the seller is given by the non-linear f-evaluation (expectation) of the terminal payoff, where f is the non-linear driver of the replicating portfolio (cf. [17] and [20] in the Brownian case, and the recent works [16], [12] and [15] in the default case).

In our framework, since all contingent claims are not necessarily replicable, we define the (superhedging) price for the seller of the European option as the minimal initial capital which allows the seller to build a (non-linear) portfolio whose terminal value dominates the payoff of the option. We provide a dual formulation of this price as the supremum, over a suitable set of equivalent probability measures $Q \in \mathcal{Q}$, of the (f, Q)-evaluation [1] of the payoff. The set $\mathcal{Q}$ is related to the set of the so-called martingale probability measures. In the case when f is linear, our result reduces to the well-known dual representation from the literature on linear incomplete markets (cf. [19] and [21]). We also provide an infinitesimal characterization of the (superhedging) price of the European option for the seller as the minimal supersolution of a *constrained BSDE* with default. By a form of symmetry, we derive corresponding results for the buyer's superhedging price.

We then turn to the pricing problem for the American option with payoff (ξ_t). We recall that in the case of a *non-linear complete* market, the seller's (superhedging) price of the American option with a càdlàg payoff (ξ_t) is equal to the value of the optimal stopping problem with non-linear f-evaluation/expectation, associated with the given payoff (ξ_t) (cf. [16]). Moreover, the price process admits an infinitesimal characterization as the solution of the reflected BSDE associated with driver f and obstacle (ξ_t) (cf. [20] in the Brownian case and for a continuous payoff (ξ_t), and [42] (resp. [16]) in the case of Poisson jumps (resp. default jump) and a càdlàg process (ξ_t)). More recently, these results have been generalized to the case of an irregular payoff (ξ_t) (not necessarily càdlàg) in [24] and [25].

In the *non-linear incomplete* market of the present paper, we provide a dual formulation of the seller's (superhedging) price u_0 of the American option associated with an irregular payoff (ξ_t) (not necessarily càdlàg) in terms of the value of a non-linear mixed control/stopping problem. More precisely, we show that u_0 is equal to the supremum over all probability measures $Q \in \mathcal{Q}$ and all stopping times τ of the (f, Q)-evaluation of ξ_τ. In the case when f is linear and (ξ_t) is càdlàg, our result reduces to the well-known dual representation from the literature on linear incomplete markets (cf. [35]). We also provide an infinitesimal characterization of the (superhedging) price u_0 for the seller as the minimal supersolution

[1] or, in other terms, the f-evaluation/expectation under the probability measure Q.

of a *constrained reflected BSDE* with default (associated with the driver f and the obstacle (ξ_t)). We note that, even in the linear case (f linear), these results are new, since in the literature, only the càdlàg case has been studied. The treatment of the non càdlàg case requires the introduction of an additional non decreasing process corresponding to the right-hand jumps of the price process. Using some specific techniques of the control theory and the general theory of processes, we show that this process only increases when the price is equal to the payoff.

A crucial step in the proof of these results is to establish a non-linear optional decomposition for optional (not necessarily càdlàg) processes which are (f, Q)-strong supermartingales for all $Q \in \mathcal{Q}$. This generalizes the well-known result shown in the literature when f is linear and (ξ_t) is càdlàg (cf. [19] and [21]).

The paper is organized as follows: in Section 2, we introduce some notation and definitions. In Section 3, we first present our market model (subsection 3.1). In Subsection 3.2 (resp. 3.3), we define the buyer's and seller's superhedging prices of the European (resp. American) option, we discuss no-arbitrage issues, and state the main duality results for this type of options. In the subsequent sections, we prove these results and provide the infinitesimal characterizations of the superhedging price processes. In Section 4, some useful preliminaries results on strong $\mathcal{E}$-supermartingale families and processes are given. Section 5 is devoted to the study of processes which are (f, Q)-strong supermartingales, for all $Q \in \mathcal{Q}$. For this class of processes, we establish a non-linear predictable and a non-linear optional decomposition. In Section 6, we consider the case of the European option: using the results from Section 5, we sketch the proof of the duality and give the infinitesimal characterizations of the seller's and the buyer's (superhedging) prices in terms of *constrained BSDEs* with default. Section 7 is devoted to the case of the American option from the point of view of the seller. We first study the value Y of the associated non-linear mixed control/stopping problem, which we write as the (essential) supremum of a family of reflected BSDEs. We show in particular that Y is the smallest optional process which is an (f, Q)-strong supermartingale for all $Q \in \mathcal{Q}$, and which dominates the payoff process. We also study the strict value Y^+ of our non-linear mixed control/stopping problem. We show, in particular, that Y^+ can be aggregated by a càdlàg adapted process. We then give the detailed proof of the dual representation of the seller's superhedging price. Finally, we provide and show the infinitesimal characterization of the seller's (superhedging) price process. Section 8 is devoted to the case of the American option from the point of view of the buyer. We first study the value $\tilde{Y}$ of the associated dual problem, which we write as the (essential) infimum of a family of reflected BSDEs. Then, under the additional assumption that the payoff is càdlàg, we prove in detail the dual representation of the buyer's superhedging price. The Appendix is devoted to some useful technical results and to a discussion on reflected BSDEs with a non positive jump at the default time.

2 Notation and definitions

Let $(\Omega, \mathcal{G}, P)$ be a complete probability space equipped with two stochastic processes: a unidimensional standard Brownian motion W and a jump process N defined by $N_t = \mathbf{1}_{\vartheta \leq t}$ for all $t \in [0, T]$, where ϑ is a random variable which models a default time. We assume that this default can appear after any fixed time, that is $P(\vartheta \geq t) > 0$ for all $t \geq 0$. We denote by $\mathbb{G} = \{\mathcal{G}_t, t \geq 0\}$ the *augmented filtration* generated by W and N. We denote by $\mathcal{P}$ the predictable σ-algebra. We suppose that W is a $\mathbb{G}$-Brownian motion. Let (Λ_t) be the predictable compensator of the nondecreasing process (N_t). Note that $(\Lambda_{t\wedge\vartheta})$ is then the predictable compensator of $(N_{t\wedge\vartheta}) = (N_t)$. By uniqueness of the predictable compensator, $\Lambda_{t\wedge\vartheta} = \Lambda_t$, $t \geq 0$ a.s. We assume that Λ is absolutely continuous w.r.t. Lebesgue's measure, so that there exists a nonnegative process λ, called the intensity process, such that $\Lambda_t = \int_0^t \lambda_s ds$, $t \geq 0$. To simplify the presentation, we suppose that λ is bounded. Since $\Lambda_{t\wedge\vartheta} = \Lambda_t$, λ vanishes after ϑ. Let M be the compensated martingale given by

$$M_t := N_t - \int_0^t \lambda_s ds \,.$$

Let $T > 0$ be the terminal time. We define the following sets:

- S^2 is the set of $\mathbb{G}$-adapted RCLL processes φ such that $\mathbb{E}[\sup_{0\leq t\leq T} |\varphi_t|^2] < +\infty$.
- $\mathcal{A}^2$ is the set of real-valued non decreasing RCLL $\mathbb{G}$-predictable processes A with $A_0 = 0$ and $\mathbb{E}(A_T^2) < \infty$.
- $\mathbb{H}^2$ is the set of $\mathbb{G}$-predictable processes Z such that $\|Z\|^2 := \mathbb{E}\Big[\int_0^T |Z_t|^2 dt\Big] < \infty\,.$
- $\mathbb{H}^2_\lambda := L^2(\Omega \times [0,T], \mathcal{P}, \lambda_t\, dP \otimes dt)$, equipped with the norm $\|U\|^2_\lambda := \mathbb{E}\Big[\int_0^T |U_t|^2 \lambda_t dt\Big] < \infty\,.$

Note that, without loss of generality, we may assume that if $U \in \mathbb{H}^2_\lambda$, it vanishes after ϑ.

- We denote by $\mathcal{T}$ the set of stopping times τ such that $\tau \in [0, T]$ a.s.
- For S in $\mathcal{T}$, we denote by $\mathcal{T}_S$ the set of stopping times τ such that $S \leq \tau \leq T$ a.s.

As in [25], the notation $\mathbb{S}^2$ stands for the vector space of $\mathbb{R}$-valued optional (not necessarily cadlag) processes ϕ such that $|||\phi|||^2_{\mathbb{S}^2} := E[ess \sup_{\tau\in\mathcal{T}_0} |\phi_\tau|^2] < \infty$. By Proposition 2.1 in [25], the space $\mathbb{S}^2$ endowed with the norm $|||\cdot|||_{\mathbb{S}^2}$ is a Banach space. We note that the space S^2 is the sub-space of RCLL processes of $\mathbb{S}^2$.

Recall that in this setup, we have a martingale representation theorem with respect to W and M (see [29], [34]).

We give the definition of a λ-admissible driver:

Definition 2.1 (Driver, λ-*admissible* driver)**.** *A function g is said to be a* driver *if $g : \Omega \times [0,T] \times \mathbb{R}^3 \to \mathbb{R}$; $(\omega, t, y, z, k) \mapsto g(\omega, t, y, z, k)$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^3)-$ measurable, and such that $g(.,0,0,0) \in \mathbb{H}^2$. A driver g is called a λ-*admissible driver *if moreover there exists a constant $C \geq 0$ such that $dP \otimes dt$-a.s. , for each (y_1, z_1, k_1), (y_2, z_2, k_2),*

$$|g(t, y_1, z_1, k_1) - g(t, y_2, z_2, k_2)| \leq C(|y_1 - y_2| + |z_1 - z_2| + \sqrt{\lambda_t}|k_1 - k_2|). \qquad (2.1)$$

A nonnegative constant C which satisfies this inequality is called a λ-constant *associated with driver g.*

By condition (2.1) and since $\lambda_t = 0$ on $]\vartheta, T]$, g does not depend on k on $]\vartheta, T]$.

Let g be a λ-*admissible driver*. For all $\eta \in L^2(\mathcal{G}_T)$, there exists a unique solution $(X(T,\eta), Z(T,\eta), K(T,\eta))$ (denoted simply by (X, Z, K)) in $S^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ of the following BSDE (see [12]):

$$-dX_t = g(t, X_t, Z_t, K_t)dt - Z_t dW_t - K_t dM_t; \quad X_T = \eta. \qquad (2.2)$$

We call g-conditional expectation, denoted by $\mathcal{E}^g$, the operator defined for each $T' \in [0,T]$ and for each $\eta \in L^2(\mathcal{G}_{T'})$ by $\mathcal{E}^g_{t,T'}(\eta) := X_t(T', \eta)$ a.s. for all $t \in [0, T']$.

We introduce the following assumption :

Assumption 2.2. *Assume that there exists a bounded map*

$$\gamma : \Omega \times [0,T] \times \mathbb{R}^4 \to \mathbb{R}\,;\, (\omega, t, y, z, k_1, k_2) \mapsto \gamma_t^{y,z,k_1,k_2}(\omega)$$

$\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^4)$-measurable and satisfying $dP \otimes dt$-a.s. , for all $(y, z, k_1, k_2) \in \mathbb{R}^4$,

$$g(t, y, z, k_1) - g(t, y, z, k_2) \geq \gamma_t^{y,z,k_1,k_2}(k_1 - k_2)\lambda_t, \qquad (2.3)$$

and

$$\gamma_t^{y,z,k_1,k_2} > -1. \qquad (2.4)$$

Assumption 2.2 ensures the strict monotonicity of the operator $\mathcal{E}^g$ with respect to terminal condition (see [12, Section 3.3]).

Definition 2.3. *Let $Y \in \mathbb{S}^2$. The process (Y_t) is said to be a strong $\mathcal{E}^g$-supermartingale* [2] *(resp. martingale) if $\mathcal{E}^g_{\sigma,\tau}(Y_\tau) \leq Y_\sigma$ (resp. $= Y_\sigma$) a.s. on $\sigma \leq \tau$, for all $\sigma, \tau \in \mathcal{T}_0$.*

Note that, by the flow property of BSDEs, for each $\tau \in \mathcal{T}_0$ and for each $\eta \in L^2(\mathcal{G}_\tau)$, the process $\mathcal{E}^g_{\cdot,\tau}(\eta)$ is an $\mathcal{E}^g$-martingale.

[2] In the case where Y is moreover RCLL (that is, $Y \in S^2$), we often omit the term "strong".

3 Market model and main duality results

3.1 Market model $\mathcal{M}^f$

We now consider a financial market which consists of one risk-free asset, whose price process $S^0 = (S_t^0)_{0\leq t\leq T}$ satisfies $dS_t^0 = S_t^0 r_t dt$, and one risky asset with price process S which admits a discontinuity at time ϑ. Throughout the sequel, we consider that the price process $S = (S_t)_{0\leq t\leq T}$ evolves according to the equation

$$dS_t = S_{t^-}(\mu_t dt + \sigma_t dW_t + \beta_t dM_t). \tag{3.1}$$

All the processes σ r, μ, β are supposed to be predictable (that is $\mathcal{P}$-measurable), satisfying $\sigma_t > 0$ $dP \otimes dt$ a.s. and $\beta_\vartheta > -1$ a.s., and such that σ, λ, σ^{-1}, β are bounded.

We consider an investor, endowed with an initial wealth equal to x, who can invest his/her wealth in the two assets of the market. At each time t, the investor chooses the amount φ_t of wealth invested in the risky asset. A process $\varphi_. = (\varphi_t)_{0\leq t\leq T}$ is called a *portfolio strategy* if it belongs to $\mathbb{H}^2$.

The value of the associated portfolio (also called *wealth*) at time t is denoted by $V_t^{x,\varphi}$ (or simply V_t).

In the classical linear case, the wealth process satisfies the self financing condition:

$$dV_t = (r_t V_t + \varphi_t(\mu_t - r_t))dt + \varphi_t \sigma_t dW_t + \varphi_t \beta_t dM_t. \tag{3.2}$$

Setting $Z_t := \varphi_t \sigma_t$, we get

$$dV_t = (r_t V_t + Z_t \theta_t)dt + Z_t dW_t + Z_t \sigma_t^{-1} \beta_t dM_t, \tag{3.3}$$

where $\theta_t := \dfrac{\mu_t - r_t}{\sigma_t}$.

We assume now that the dynamics of the wealth is *nonlinear*. More precisely, let $x \in \mathbb{R}$ be an initial wealth and let φ in $\mathbb{H}^2$ be a portfolio strategy. We suppose that the associated *wealth* process $V_t^{x,\varphi}$ (or simply V_t) satisfies the following (forward) dynamics:

$$-dV_t = f(t, V_t, \varphi_t\sigma_t)dt - \varphi_t\sigma_t dW_t - \varphi_t \beta_t dM_t, \tag{3.4}$$

with $V_0 = x$, where f is a *nonlinear* λ-admissible driver which does not depend on k, such that $f(t, 0, 0) = 0$. [3] In the classical linear case (see (3.3)), we have $f(t, y, z) = -r_t y - z\theta_t$ (which is a λ-admissible driver).

We have the following lemma.

Lemma 3.1. *For each $x \in \mathbb{R}$ and each φ in $\mathbb{H}^2$, the associated wealth process $(V_t^{x,\varphi})$ is an $\mathcal{E}^f$-martingale.*

[3] so that $\mathcal{E}^f_{\cdot,T'}(0) = 0$ for all $T' \in [0, T]$.

Proof. Let $x \in \mathbb{R}$ and φ in $\mathbb{H}^2$ be given. We note that the process $(V_t^{x,\varphi}, \varphi_t\sigma_t, \varphi_t\beta_t)$ is the solution of the BSDE with default jump associated with driver f and the terminal condition $\eta := V_T^{x,\varphi}$. The result then follows from the flow property of BSDEs. $\square$

Remark 3.2. *We note that for an arbitrary random variable $\eta \in L^2$, there does not necessarily exist a pair of processes (X, φ) such that $(X_t, \varphi_t\sigma_t, \varphi_t\beta_t)$ is solution of the BSDE with default jump associated with driver f and terminal condition η, that is, such that (X, φ) satisfies the dynamics* (3.4) *with $X_T = \eta$. In other terms, the market is incomplete.*

In the sequel, we will often use the following change of variables which maps a process $\varphi \in \mathbb{H}^2$ to $Z \in \mathbb{H}^2$ defined by $Z_t = \varphi_t\sigma_t$. With this change of variables, the wealth process $V = V_t^{x,\varphi}$ (for a given $x \in \mathbb{R}$) is the unique process satisfying

$$-dV_t = f(t, V_t, Z_t)dt - Z_t dW_t - Z_t\sigma_t^{-1}\beta_t dM_t,\ V_0 = x. \tag{3.5}$$

In the following, our non-linear incomplete market model is denoted by $\mathcal{M}^f$.

3.2 Superhedging prices and dual representations for European options

Let η in $L^2(\mathcal{G}_T)$ be the payoff of the European option (with maturity T). It is called replicable if there exists $x \in \mathbb{R}$ and $\varphi \in \mathbb{H}^2$ such that $\eta = V_T^{x,\varphi}$ a.s. This is equivalent to the existence of $(X, Z) \in \mathcal{S}^2 \times \mathbb{H}^2$ such that

$$-dX_t = f(t, X_t, Z_t)dt - Z_t dW_t - Z_t\sigma_t^{-1}\beta_t dM_t,\ X_T = \eta \text{ a.s.}$$

It is clear that all European contingent claims are not necessarily replicable and so the market is incomplete (cf. also Remark 3.2). We introduce the *superhedging price for the seller* of the claim with payoff η defined as the minimal initial capital which allows the seller to build a superhedging strategy for the claim, that is

$$v_0 := \inf\{x \in \mathbb{R} :\ \exists \varphi \in \mathbb{H}^2 \ \text{ s.t. } V_T^{x,\varphi} \geq \eta \text{ a.s.}\}.$$

We introduce the *superhedging price for the buyer* of the claim with payoff η defined as the maximal initial price which allows the buyer to build a superhedging strategy for the claim, that is

$$\tilde{v}_0 := \sup\{x \in \mathbb{R} :\ \exists \varphi \in \mathbb{H}^2 \ \text{ s.t. } V_T^{-x,\varphi} + \eta \geq 0 \text{ a.s.}\}.$$

Note that $\tilde{v}_0$ is equal to the opposite of the superhedging price for the seller of the option with payoff $-\eta$. [4]

[4] Recall that when $\beta = 0$ and when the filtration is the natural filtration associated with the Brownian motion, the market is complete and we have $v_0 = \mathcal{E}^f_{0,T}(\eta)$ (cf. [17]) and $\tilde{v}_0 = -\mathcal{E}^f_{0,T}(-\eta)$.

Definition 3.3. *Let $x \in \mathbb{R}$. Let $y \in \mathbb{R}$ and φ in $\mathbb{H}^2$. We say that (y, φ) is an* arbitrage opportunity for the seller[5] (resp. for the buyer [6]) of the European option with initial price *x if*

$$y < x \quad \textit{and} \quad V_T^{y,\varphi} - \eta \geq 0 \;\; \textit{a.s.} \;(\textit{ resp. } y > x \quad \textit{and} \quad V_T^{-y,\varphi} + \eta \geq 0) \quad \textit{a .s.} \tag{3.6}$$

Proposition 3.4. *Let $x \in \mathbb{R}$. There exists an arbitrage opportunity for the seller (resp. for the buyer) of the European option with price x if and only if $x > v_0$ (resp. $x < \tilde{v}_0$)*

Sketch of the proof : The proof relies on quite similar arguments as those used in the proof of Proposition 5.11 in [16].

Definition 3.5. *A real number x is called an* arbitrage-free price for the European option *if there exists no arbitrage opportunity, neither for the seller nor for the buyer.*

By Propositions 3.4, we get

Proposition 3.6. *If $v_0 < \tilde{v}_0$, there does not exist any arbitrage-free price for the European option. If $v_0 \geq \tilde{v}_0$, the interval $[\tilde{v}_0, v_0]$ is the set of all arbitrage-free prices. We call it the* arbitrage-free *interval for the European option.*

As mentioned before, the market $\mathcal{M}^f$ is incomplete. We recall that in the *linear* case, that is, when $f(t, y, z) = -r_t y - \theta_t z$, a dual representation of v_0 and $\tilde{v}_0$ can be achieved via a martingale approach which is based on the notion of martingale probability measures defined as follows: a probability measure R equivalent to P is called a *martingale probability measure* if the discounted risky-asset price $(e^{-\int_0^t r_s ds} S_t)$ is a martingale under R. This is equivalent to the following definition given in [43]: a probability measure R is a *martingale probability measure* if the discounted (linear) wealth processes are R-martingales, that is, for all $x \in \mathbb{R}, \varphi \in \mathbb{H}^2$, the process $(e^{-\int_0^t r_s ds} \bar{V}_t^{x,\varphi})$ (where $\bar{V}^{x,\varphi}$ follows the linear dynamics (3.2)) is a martingale under R. For example, the probability R^0 which admits ζ_T^0 as density with respect to P on $\mathcal{G}_T$, where (ζ_t^0) satisfies

$$d\zeta_t^0 = -\zeta_t^0 \, \theta_t dW_t; \quad \zeta_0^0 = 1, \tag{3.7}$$

is a martingale probability measure. It corresponds to the so-called "minimal martingale probability measure" in the sense of Föllmer-Schweizer. However, there exist more than one martingale probability measures (cf. (3.11)); they can be characterized via their densities with respect to P.

In our non linear framework, by analogy with the linear case, we are thus naturally led to introduce the notion of $\mathcal{E}^f$-martingale property under a given probability measure Q. To this aim, we first introduce the notion of f-evaluation under Q.

[5]This means that the seller sells the European option at the price x strictly greater than the amount y which is enough to be hedged (by using the strategy φ). He/she thus makes the profit $x - y > 0$ at time 0.

[6]This means that the buyer buys the European option at the price x, stricly smaller than the amount y, which, borrowed at time 0, allows him/her to recover his/her debt at time T (by using the strategy φ). He/she thus makes the profit $y - x > 0$ at time 0.

Let Q be a probability measure, equivalent to P. From the $\mathbb{G}$-martingale representation theorem (cf. [34], [30]), its density process (ζ_t) satisfies

$$d\zeta_t = \zeta_{t^-}(\alpha_t dW_t + \nu_t dM_t); \zeta_0 = 1, \tag{3.8}$$

where (α_t) and (ν_t) are predictable processes with $\nu_{\vartheta\wedge T} > -1$ a.s. By Girsanov's theorem, the process $W_t^Q := W_t - \int_0^t \alpha_s ds$ is a Brownian motion under Q, and the process $M_t^Q := M_t - \int_0^t \nu_s \lambda_s ds$ is a martingale under Q.

We define the spaces S_Q^2, $\mathbb{H}_Q^2$ and $\mathbb{H}_{Q,\lambda}^2$ similarly to $S^2, \mathbb{H}^2$ and $\mathbb{H}_\lambda^2$, but under probability Q instead of P.

Definition 3.7. *We call f-evaluation under Q, or (f,Q)-evaluation in short, denoted by $\mathcal{E}_Q^f$, the operator defined for each $T' \in [0,T]$ and for each $\eta \in L_Q^2(\mathcal{G}_{T'})$ by $\mathcal{E}_{Q,t,T'}^f(\eta) := X_t$ for all $t \in [0,T']$, where (X,Z,K) is the solution in $S_Q^2 \times \mathbb{H}_Q^2 \times \mathbb{H}_{Q,\lambda}^2$ of the BSDE under Q associated with driver f, terminal time T' and terminal condition η, and driven by W^Q and M^Q, that is* [7]

$$-dX_t = f(t, X_t, Z_t)dt - Z_t dW_t^Q - K_t dM_t^Q; \quad X_{T'} = \eta.$$

We note that $\mathcal{E}_P^f = \mathcal{E}^f$.

Definition 3.8. *Let $Y \in S_Q^2$. The process (Y_t) is said to be a (strong) $\mathcal{E}_Q^f$-martingale, or an (f,Q)-martingale, if $\mathcal{E}_{Q,\sigma,\tau}^f(Y_\tau) = Y_\sigma$ a.s. on $\sigma \le \tau$, for all $\sigma, \tau \in \mathcal{T}_0$.*

We now introduce the concept of f-martingale probability measure.

Definition 3.9. *A probability measure Q equivalent to P is called an f-*martingale probability measure *if for all $x \in \mathbb{R}$ and for all $\varphi \in \mathbb{H}^2 \cap \mathbb{H}_Q^2$, the wealth process $V^{x,\varphi}$ is a strong $\mathcal{E}_Q^f$-martingale, or in other terms an (f,Q)-martingale.*

We note that P is an *f-martingale probability measure* (cf. Lemma 3.1).

Remark 3.10. *(linear case) Let R^0 be the minimal martingale probability measure, with density ζ^0 defined by (3.7). Suppose $f(t,y,z) = -r_t y - \theta_t z$. We note that, in this case, the (f,P)-martingale property of the (linear) wealth processes (cf. Lemma 3.1) is equivalent to the well-known R^0-martingale property of the discounted wealth processes. In other terms, the f-martingale probability property of P corresponds to the well-known martingale probability property of R^0 (see also Remark 3.14 concerning this correspondance between P and R^0).*

We denote by $\mathcal{Q}$ the set of f-martingale probability measures Q such that the coefficients (α_t) and (ν_t) associated with its density with respect to P (see equation (3.8)) are bounded. We note that $P \in \mathcal{Q}$.

Let $\mathcal{V}$ be the set of bounded predictable processes ν such that $\nu_{\vartheta\wedge T} > -1$ a.s., which is equivalent to $\nu_t > -1$ for all $t \in [0,T]$ $\lambda_t dP \otimes dt$-a.s. (cf. Remark 9 in [12]).

[7] We note that since we have a representation theorem for $(Q,\mathbb{G})$-martingales with respect to W^Q and M^Q (see e.g. Proposition 6 in the appendix of [12]), this BSDE admits a unique solution (X,Z,K) in $S_Q^2 \times \mathbb{H}_Q^2 \times \mathbb{H}_{Q,\lambda}^2$.

Proposition 3.11. *(Characterization of $\mathcal{Q}$) Let Q be a probability measure equivalent to P, such that the coefficients α and ν of its density* (3.8) *with respect to P are bounded. The two following assertions are equivalent:*
(i) $Q \in \mathcal{Q}$, that is, Q is an f-martingale probability measure.
(ii) there exists $\nu \in \mathcal{V}$ such that $Q = Q^\nu$, where Q^ν is the probability measure which admits ζ_T^ν as density with respect to P on $\mathcal{G}_T$, where ζ^ν satisfies

$$d\zeta_t^\nu = \zeta_{t^-}^\nu(-\nu_t\lambda_t\beta_t\sigma_t^{-1}dW_t + \nu_t dM_t); \zeta_0^\nu = 1. \tag{3.9}$$

We note that the mapping $\nu \mapsto Q^\nu$ is a one-to-one mapping that carries $\mathcal{V}$ onto $\mathcal{Q}$. In particular, for $\nu = 0$, we have $Q^\nu = Q^0 = P$. We also note that the set $\mathcal{Q}$ does not depend on the driver f.

Proof. Let Q be a probability measure equivalent to P, such that the coefficients α and ν of its density (3.8) with respect to P are bounded. Note that (ν_t) belongs to $\mathcal{V}$. Let $x \in \mathbb{R}$ and let $\varphi \in \mathbb{H}^2 \cap \mathbb{H}^2_Q$. The associated wealth process $V = V^{x,\varphi}$ satisfies (3.5) with $Z_t = \varphi_t\sigma_t$. Expressing W and M in terms of W^Q and M^Q, we get

$$-dV_t = f(t, V_t, Z_t)dt - Z_t(\alpha_t + \nu_t\lambda_t\beta_t\sigma_t^{-1})dt - Z_t dW_t^Q - Z_t\sigma_t^{-1}\beta_t dM_t^Q. \tag{3.10}$$

Suppose that $\alpha_t = -\nu_t\lambda_t\beta_t\sigma_t^{-1}$ for all $t \in [0, T]$ $dP \otimes dt$-a.e. Then, V is a $\mathcal{E}^f_Q$-martingale. Conversely, if V is a $\mathcal{E}^f_Q$-martingale, then $\alpha_t = -\nu_t\lambda_t\beta_t\sigma_t^{-1}$. This follows from the following lemma.

Lemma 3.12. *Let g be a λ-admissible driver. Let (A_t) be a RCLL predictable process with square integrable total variation and $A_0 = 0$. Suppose (Y, Z, K) is the solution of both the BSDE with generalized driver $g(\cdot, y, z, k)dt + dA_t$ and the BSDE with driver g (with the same terminal time T and the terminal condition). We then have $A_T = 0$ a.s.*

Sketch of the proof : The proof relies on the uniqueness of the decomposition of a special semi-martingale. $\square$

We now provide a connection between f-martingale probabilities and martingale probabilities. Let R be a probability measure, equivalent to P such that the coefficients α and ν of its density with respect to P (cf. (3.8)) are bounded. By similar arguments as in the proof of Proposition 3.11, we derive that R is a martingale probability measure if and only if there exists $\nu \in \mathcal{V}$ such that $R = R^\nu$, where R^ν is the probability measure with density process $\tilde{\zeta}^\nu$ (with respect to P) satisfying

$$d\tilde{\zeta}_t^\nu = \tilde{\zeta}_{t^-}^\nu((-\theta_t - \nu_t\lambda_t\beta_t\sigma_t^{-1})dW_t + \nu_t dM_t); \tilde{\zeta}_0^\nu = 1. \tag{3.11}$$

We denote by $\mathcal{P}$ the set of all such probability measures.

By this observation together with Proposition 3.11, we derive the following result.

Proposition 3.13. *There exists a one to one mapping from $\mathcal{Q}$ on $\mathcal{P}$. More precisely, the mapping T_θ, which, for each $\nu \in \mathcal{V}$, maps the f-martingale probability Q^ν (with density ζ^ν given by* (3.9)*) onto the martingale probability measure R^ν (with density $\tilde{\zeta}^\nu$) is a one to one correspondance from $\mathcal{Q}$ on $\mathcal{P}$.*

Remark 3.14. *Loosely speaking, the mapping T_θ translates the "Brownian coefficient" (of the density) by $-\theta$. We note that $T_\theta(P) = R^0$ (which completes the observation made in Remark 3.10 on P and R^0).*

For each $\nu \in \mathcal{V}$, the (f, Q^ν)-evaluation can be seen as a nonlinear pricing system:

$$\mathcal{E}^f_{Q^\nu} : (T', \eta) \mapsto \mathcal{E}^f_{Q^\nu,\cdot,T'}(\eta)$$

which, to each European option with maturity $T' \in [0,T]$ and payoff $\eta \in L^2_{Q^\nu}(\mathcal{G}_{T'})$, associates the price process $\mathcal{E}^f_{Q^\nu,t,T'}(\eta)$, $t \in [0,T']$.

Remark 3.15. *(linear case) When $f(t,y,z) = -r_t y - \theta_t z$, we have for each ν in $\mathcal{V}$, $\mathcal{E}^f_{Q^\nu,0,S}(\eta) = E_{R^\nu}(e^{-\int_0^T r_s ds}\eta)$. In this case, the operator $\mathcal{E}^f_{Q^\nu}$ thus reduces to the linear price system associated by duality with the martingale probability measure Q^ν (for more details see [28], [19] section 1.7).*

We now consider a European option with maturity T and payoff η such that there exist $x \in \mathbb{R}$ and $\varphi \in \cap_{\nu\in\mathcal{V}}\mathbb{H}^2_{Q^\nu}$ such that

$$|\eta| \;\leq\; V_T^{x,\varphi} = x - \int_0^T f(s, V_s^{x,\varphi}, \sigma_s\varphi_s)ds + \int_0^T \varphi_s\sigma_s dW_s + \int_0^T \beta_s\varphi_s dM_s \quad \text{a.s.} \tag{3.12}$$

Note that for all $\nu \in \mathcal{V}$, the wealth process $V^{x,\varphi}$ is an $\mathcal{E}^f_{Q^\nu}$-martingale since $(V^{x,\varphi}, \sigma\varphi, \beta\varphi)$ is the solution of the BSDE under Q^ν (driven by W^{Q^ν} and M^{Q^ν}) associated with driver f, terminal time T and terminal condition $V_T^{x,\varphi}$ (cf. the proof of Proposition 3.11).

Since $|\eta| \leq V_T^{x,\varphi}$ a.s. it follows that for all $\nu \in \mathcal{V}$, $\mathcal{E}^f_{Q^\nu,0,T}(|\eta|) \leq \mathcal{E}^f_{Q^\nu,0,T}(V_T^{x,\varphi}) = x$.

One of the main results of this paper is the following representation:

Theorem 3.16 (*Seller's superhedging price for the European option*)**.** *The superhedging price for the seller v_0 of the European option with payoff η satisfies the equality*

$$v_0 = \sup_{\nu\in\mathcal{V}} \mathcal{E}^f_{Q^\nu,0,T}(\eta).$$

For the proof of the above dual representation result, we refer to Section 6.

Remark 3.17. *In the linear case, by Remark 3.15, this result reduces to the well-known dual representation of the superhedging price for the seller of the European option in an incomplete (linear) market (cf. [19] and [21]).*

Since the superhedging price of the option for the buyer $\tilde{v}_0$ is equal to the opposite of the superhedging price for the seller of the option with payoff $-\eta$, we derive from Theorem 3.16 the following dual representation result for $\tilde{v}_0$:

$$\tilde{v}_0 = -\sup_{\nu\in\mathcal{V}} \mathcal{E}^f_{Q^\nu,0,T}(-\eta).$$

Remark 3.18. *Note that it is possible that $v_0 < \tilde{v}_0$, and hence, that there does not exist an arbitrage-free price for the European option with payoff η. A simple example is given by $f(t, y, z) = -|y|$ and $\eta = 1$. In this case, we have $v_0 = e^{-T}$ and $\tilde{v}_0 = e^T$.*

Remark 3.19. *We note that when $f(t, y, z) \geq -f(t, -y, -z)$ (which is satisfied for example when f is convex with respect to (y, z)) then, for all $\nu \in \mathcal{V}$, we have $\mathcal{E}^f_{Q^\nu,0,T}(\eta) \geq -\mathcal{E}^f_{Q^\nu,0,T}(-\eta)$. By taking the supremum over $\nu \in \mathcal{V}$, using the above dual representations of v_0 and $\tilde{v}_0$, we get $v_0 \geq \tilde{v}_0$.*

Moreover, when $f(t, y, z) = -f(t, -y, -z)$ (which is satisfied for example when f is linear), then $v_0 = \tilde{v}_0$, and this constant is the unique arbitrage-free price for the European option *with payoff η.*

3.3 Superhedging prices and dual representations for American options

We recall that $\mathbb{S}^2$ denotes the vector space of $\mathbb{R}$-valued optional (not necessarily cadlag) processes ϕ such that $|||\phi|||^2_{\mathbb{S}^2} := E[ess \sup_{\tau \in \mathcal{T}_0} |\phi_\tau|^2] < \infty$.

Let us consider an American option associated with maturity T and a payoff given by a process $(\xi_t) \in \mathbb{S}^2$.

The *superhedging price for the seller* of the American option at time 0, denoted by u_0, is classically defined as the minimal initial capital which enables the seller to be superhedged no matter what the exercise time chosen by the buyer is. More precisely, we have the following definition.

Definition 3.20. *A* super-hedge for the seller *against the American option with initial price $x \in \mathbb{R}$ is a portfolio strategy $\varphi \in \mathbb{H}^2$ such that $V^{x,\varphi}_t \geq \xi_t$, $0 \leq t \leq T$ a.s.*

For each $x \in \mathbb{R}$, we denote by $\mathcal{A}(x)$ the set of all *super-hedges for the seller* associated with initial price x.

The *superhedging price for the seller* of the American option at time 0 is thus defined by

$$u_0 := \inf\{x \in \mathbb{R},\ \exists \varphi \in \mathcal{A}(x)\}. \tag{3.13}$$

We now consider the American option from the point of view of the buyer.

Definition 3.21. *A* super-hedge for the buyer *against the American option with initial price $z \in \mathbb{R}$ is a pair (τ, φ) of a stopping time $\tau \in \mathcal{T}$ and a portfolio strategy $\varphi \in \mathbb{H}^2$ such that $V^{-z,\varphi}_\tau + \xi_\tau \geq 0$ a.s.*

For each $z \in \mathbb{R}$, we denote by $\mathcal{B}(z)$ the set of all *super-hedges for the buyer* associated with initial price z.

We now define the *buyer's price* $\tilde{u}_0$ of the American option as the supremum of the initial prices which allow the buyer to be super-hedged, that is [8]

$$\tilde{u}_0 = \sup\{z \in \mathbb{R},\ \exists (\tau, \varphi) \in \mathcal{B}(z)\}. \tag{3.14}$$

[8] Note that u_0, $\tilde{u}_0 \in \overline{\mathbb{R}}$. We shall see below that, under an appropriate assumption on the process (ξ_t) (cf. (7.1)), u_0 and $\tilde{u}_0$ are finite.

We now introduce the definitions of an *arbitrage opportunity for the seller and for the buyer* of the American option.

Definition 3.22. *Let $x \in \mathbb{R}$. Let $y \in \mathbb{R}$, and let φ in $\mathbb{H}^2$. We say that (y, φ) is an* arbitrage opportunity for the seller *of the American option with initial price x if*

$$y < x \quad \textit{and} \quad V_\tau^{y,\varphi} - \xi_\tau \geq 0 \quad \textit{a.s.} \quad \textit{for all } \tau \in \mathcal{T}.$$

Definition 3.23. *Let $x \in \mathbb{R}$. Let $y \in \mathbb{R}$, let $\tau \in \mathcal{T}$ and let $\varphi \in \mathbb{H}^2$. We say that (y, τ, φ) is an* arbitrage opportunity for the buyer *of the American option with initial price x, if*

$$y > x \quad \textit{and} \quad V_\tau^{-y,\varphi} + \xi_\tau \geq 0 \quad \textit{a.s.}$$

Proposition 3.24. *Let $x \in \mathbb{R}$. There exists an arbitrage opportunity for the seller (resp. for the buyer) of the American option with price x if and only if $x > u_0$ (resp. $x < \tilde{u}_0$).*

The proof, which relies on the same arguments as those of the proof of Proposition 5.11 in [16] (see also [31]) is omitted.

Definition 3.25. *A real number x is called an* arbitrage-free price for the American option *if there exists no arbitrage opportunity, neither for the seller nor for the buyer.*

By Propositions 3.24, we get

Proposition 3.26. *If $u_0 < \tilde{u}_0$, there does not exist any arbitrage-free price for the American option. If $u_0 \geq \tilde{u}_0$, the interval $[\tilde{u}_0, u_0]$ is the set of all arbitrage-free prices. We call it the* arbitrage-free *interval for the American option.*

We now assume that the payoff ξ is such that there exist $x \in \mathbb{R}$ and $\varphi \in \cap_{\nu \in \mathcal{V}} \mathbb{H}^2_{Q^\nu}$ such that

$$|\xi_t| \;\leq\; V_t^{x,\varphi} = x - \int_0^t f(s, V_s^{x,\varphi}, \sigma_s \varphi_s) ds + \int_0^t \varphi_s \sigma_s dW_s + \int_0^t \beta_s \varphi_s dM_s, \quad 0 \leq t \leq T, \quad \text{a.s.} \tag{3.15}$$

The following theorems are two of the main results of this paper. The proofs will be given in the following sections.

Theorem 3.27 (*Seller's superhedging price for the American option*)**.** *The superhedging price for the seller u_0 of the American option with payoff ξ satisfies the equality*

$$u_0 = \sup_{(\tau,\nu) \in \mathcal{T} \times \mathcal{V}} \mathcal{E}^f_{Q^\nu,0,\tau}(\xi_\tau).$$

Remark 3.28. *In the linear case, by Remark 3.15, this result reduces to the well-known dual representation of the superhedging price for the seller of the American option in an incomplete (linear) market (cf. [35]).*

Theorem 3.29 (*Buyer's superhedging price for the American option*)**.** *Let $(\xi_t) \in \mathbb{S}^2$. Suppose that (ξ_t) is right-continuous and left-uppersemicontinuous along stopping times. The* superhedging price for the buyer $\tilde{u}_0$ *of the American option satisfies*

$$\tilde{u}_0 = - \sup_{\nu \in \mathcal{V}} \inf_{\tau \in \mathcal{T}} \mathcal{E}^f_{Q^\nu,0,\tau}(-\xi_\tau).$$

4 Preliminary results on $\mathcal{E}^g$-supermartingale families (resp. processes)

In this section, we give some general preliminary properties of $\mathcal{E}^g$-supermartingale families and $\mathcal{E}^g$-supermartingale processes, which will be useful in the sequel.

Let us recall the definition of an admissible family of random variables indexed by stopping times in $\mathcal{T}$ (or $\mathcal{T}$-system in the vocabulary of Dellacherie and Lenglart [10]).

Definition 4.1. *We say that a family $X = (X(S),\, S \in \mathcal{T})$ is* admissible *if it satisfies the following conditions*

1. For all $S \in \mathcal{T}$, $X(S)$ is a real-valued $\mathcal{F}_S$-measurable random variable.

2. For all $S, S' \in \mathcal{T}$, $X(S) = X(S')$ a.s. on $\{S = S'\}$.

Moreover, we say that an admissible family X is uniformly square-integrable *if* $\mathbb{E}[ess \sup_{S \in \mathcal{T}} (X(S))^2] < \infty$.

Let g be a λ-admissible driver satisfying Assumption 2.2.
We give the definition of an $\mathcal{E}^g$-*supermartingale* (resp. $\mathcal{E}^g$- *submartingale*, $\mathcal{E}^g$-*martingale*) family. [9]

Definition 4.2. *A uniformly square integrable admissible family $(X(S),\ S \in \mathcal{T})$ is said to be an $\mathcal{E}^g$*-supermartingale *(resp. $\mathcal{E}^g$*- submartingale, $\mathcal{E}^g$-martingale*)* family *if for all $S, S' \in \mathcal{T}$ such that $S \geq S'$ a.s., $\mathcal{E}^g_{S',S}(X(S)) \leq$ (resp. $\geq, =$) $X(S')$ a.s.*

Lemma 4.3. *Let $(X(S),\ S \in \mathcal{T})$ be an $\mathcal{E}^g$-supermartingale family. Then, there exists a r.u.s.c. optional process (X_t) such that $\mathbb{E}[ess \sup_{S \in \mathcal{T}} (X(S))^2] < \infty$ which aggregates the family $(X(S),\ S \in \mathcal{T})$, that is, such that $X(S) = X_S$ a.s. for all $S \in \mathcal{T}$. Moreover, the process (X_t) is a strong $\mathcal{E}^g$-supermartingale, that is, for all $S, S' \in \mathcal{T}$ such that $S \geq S'$ a.s., $\mathcal{E}^g_{S',S}(X_S) \leq X_{S'}$ a.s.*

Proof. By Lemma 4.6 in [25], the $\mathcal{E}^g$-supermartingale family $(X(S),\ S \in \mathcal{T})$ is right-upper semicontinuous (along stopping times). It follows from Theorem 4 in [10] that there exists an r.u.s.c. optional process (X_t) which aggregates the family $(X(S),\ S \in \mathcal{T})$. The process (X_t) is clearly a strong $\mathcal{E}^g$-supermartingale. □

Remark 4.4. *Note that, as a consequence of the above lemma, we recover a result of [24] (Lemma 5.1 in [24]), namely, a strong $\mathcal{E}^g$-supermartingale is necessarily r.u.s.c.*

Lemma 4.5. *If $(X_t)_{t \in [0,T]}$ be a strong $\mathcal{E}^g$-supermartingale, then the process of right-limits $(X_{t^+})_{t \in [0,T]}$ (where, by convention, $X_{T^+} := X_T$) is a strong $\mathcal{E}^g$-supermartingale.*

[9] When $g = 0$, it reduces to the notion of *supermartingale family*, or *supermartingale $\mathcal{T}$-system* in the terminology of Dellacherie-Lenglart [10].

Proof. Since (X_t) is a strong $\mathcal{E}^g$-supermartingale, (X_t) has right limits (cf. the $\mathcal{E}^g$-Mertens decomposition of strong $\mathcal{E}^g$-supermartingales provided in [24]). Let us show that the process (X_{t^+}) is a strong $\mathcal{E}^g$-supermartingale. Let S, θ be two stopping times belonging to $\mathcal{T}$ with $S \leq \theta$ a.s. There exist two nondecreasing sequences of stopping times (S_n) and (θ_n) such that for each n, $S_n > S$ a.s. on $\{S < T\}$, and $\theta_n > \theta$ a.s. on $\{\theta < T\}$. Replacing if necessary S_n by $S_n \wedge \theta_n$, we can suppose that for each n, $S_n \leq \theta_n$ a.s. Let $\nu \in \mathcal{V}$. Since the process (X_t) is a strong $\mathcal{E}^g$-supermartingale, it follows that for each n, $\mathcal{E}^g_{S_n,\theta_n}(X_{\theta_n}) \leq X_{S_n}$ a.s. By the monotonicity property of $\mathcal{E}^g$, we derive that, for each n, $\mathcal{E}^g_{S,S_n}(\mathcal{E}^g_{S_n,\theta_n}(X_{\theta_n})) \leq \mathcal{E}^g_{S,S_n}(X_{S_n})$ a.s., which, by the consistency property of $\mathcal{E}^g$ implies

$$\mathcal{E}^g_{S,\theta_n}(X_{\theta_n}) \leq \mathcal{E}^g_{S,S_n}(X_{S_n}) a.s.$$

By letting n tend to ∞ in the above inequality and by applying the continuity property (with respect to terminal time and terminal condition) of BSDEs with default (cf. [12]), we obtain

$$\mathcal{E}^g_{S,\theta}(X_{\theta^+}) \leq \mathcal{E}^g_{S,S}(X_{S^+}) = X_{S^+} \text{ a.s.}$$

Hence, the process (X_{t^+}) is a strong $\mathcal{E}^g$-supermartingale. □

Let $\mathcal{V}$ be a non-empty set. Let $(f^\nu,\ \nu \in \mathcal{V})$ be a family of λ-admissible drivers satisfying Assumption 2.2.

Proposition 4.6. *Let $(\xi_t) \in \mathbb{S}^2$ be* right lower-semicontinuous.
Let $(X_t)_{t\in[0,T]}$ be an optional process such that (X_t) is a strong $\mathcal{E}^{f^\nu}$-supermartingale for all $\nu \in \mathcal{V}$ and such that $X_t \geq \xi_t$ for all $t \in [0,T]$ a.s. Assume moreover that (X_t) is minimal, *that is, (X_t) is the smallest optional process satisfying these properties.*

Then, the process (X_t) is right-continuous.

Remark 4.7. *This property still holds in the case when the constraint $X_t \geq \xi_t$ for all $t \in [0,T]$ a.s. is replaced by the (terminal) constraint $X_T \geq \eta$ a.s., where η is a given random variable belonging to $L^2(\mathcal{G}_T)$.*

Proof. Since (X_t) is r.u.s.c. (cf. Remark 4.4), we have $X_{t^+} \leq X_t$, for all $t \in [0,T]$ a.s. Let us prove the converse inequality. We first show that $X_{t^+} \geq \xi_t$, for all $t \in [0,T]$ a.s. Let $\theta \in \mathcal{T}$. Let $(\theta^n)_{n\in\mathbb{N}}$ be a non increasing sequence of stopping times such that $\theta_n \in \mathcal{T}_{\theta^+}$ for all n, and such that $\theta = \lim_{n\to\infty} \theta_n$ a.s. As $X_{\theta^n} \geq \xi_{\theta^n}$ for all n a.s., we get $X_{\theta^+} \geq \liminf_{n\to\infty} \xi_{\theta^n}$ a.s. Now, by the right lower-semicontinuity property of (ξ_t), we have $\liminf_{n\to\infty} \xi_{\theta^n} \geq \xi_\theta$ a.s. We deduce $X_{\theta^+} \geq \xi_\theta$ a.s. Since this holds for all $\theta \in \mathcal{T}$, it follows that $X_{t^+} \geq \xi_t$, for all $t \in [0,T]$ a.s. On the other hand, since (X_t) is a strong $\mathcal{E}^{f^\nu}$-supermartingale for all $\nu \in \mathcal{V}$, it follows by Lemma 4.5 that (X_{t^+}) is a strong $\mathcal{E}^{f^\nu}$-supermartingale for all $\nu \in \mathcal{V}$. Hence, using the *minimality* property of (X_t), we derive that $X_t \leq X_{t^+}$, for all $t \in [0,T]$ a.s. We conclude that $X_{t^+} = X_t$, for all $t \in [0,T]$ a.s. The proof is thus complete. □

Recall that $f(t,y,z)$ is the non-linear Lipschitz driver given in the beginning (cf. Section 3), and that $\mathcal{V}$ denotes the set of bounded predictable processes ν such that $\nu_{\vartheta\wedge T} > -1$ a.s. We now introduce a family of drivers $(f^\nu, \nu \in \mathcal{V})$, which will be used in the sequel.

Definition 4.8 (Driver f^ν and $\mathcal{E}^\nu$-expectation)**.** *For $\nu \in \mathcal{V}$, we define*

$$f^\nu(\omega, t, y, z, k) := f(\omega, t, y, z) + \nu_t(\omega)\lambda_t(\omega)\big(k - \beta_t(\omega)\sigma_t^{-1}(\omega)z\big).$$

The mapping f^ν is a λ-admissible driver [10].
The associated non-linear family of operators, denoted by $\mathcal{E}^{f^\nu}$ or, simply, $\mathcal{E}^\nu$, is defined as follows: for each $T' \leq T$ and each $\eta \in L^2(\mathcal{G}_{T'})$, we have $\mathcal{E}^\nu_{\cdot,T'}(\eta) := X^\nu_\cdot$, where (X^ν, Z^ν, K^ν) is the unique solution in $S^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ of the BSDE

$$-dX^\nu_t = \big(f(t, X^\nu_t, Z^\nu_t) + \nu_t\lambda_t(K^\nu_t - \beta_t\sigma_t^{-1}Z^\nu_t)\big)\, dt - Z^\nu_t dW_t - K^\nu_t dM_t;\ X^\nu_{T'} = \eta. \tag{4.1}$$

Remark 4.9. *By Proposition 3.11, for each $\nu \in \mathcal{V}$, for all $T' \leq T$ and $\eta \in L^2(\mathcal{G}_{T'}) \cap L^2_{Q^\nu}(\mathcal{G}_{T'})$, we derive that the (f^ν, P)-evaluation of η is equal to its (f, Q^ν)-evaluation, that is,*

$$\mathcal{E}^\nu_{\cdot,T'}(\eta) = \mathcal{E}^f_{Q^\nu,\cdot,T'}(\eta).$$

5 Processes which are strong $\mathcal{E}^\nu$-supermartingales for all $\nu \in \mathcal{V}$. Non-linear predictable and optional decompositions

Proposition 5.1 (Predictable $\mathcal{E}^f$-decomposition)**.** *Let $(X_t) \in \mathbb{S}^2$ be a strong $\mathcal{E}^\nu$-supermartingale for all $\nu \in \mathcal{V}$. There exists a unique process $(Z, K, A, C) \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2 \times \mathbb{C}^2$ such that*

$$-dX_t = f(t, X_t, Z_t)dt - Z_t dW_t - K_t dM_t + dA_t + dC_{t-} \tag{5.1}$$

and

$$A_\cdot + \int_0^\cdot (K_s - \beta_s\sigma_s^{-1}Z_s)\lambda_s ds \in \mathcal{A}^2 \quad \text{and} \quad (K_t - \beta_t\sigma_t^{-1}Z_t)\lambda_t \leq 0,\ t \in [0, T],\ dP \otimes dt - \text{a.e.} \tag{5.2}$$

Remark 5.2. *Recall that by Remark 5 in [12], the condition $(K_t - \beta_t\sigma_t^{-1}Z_t)\lambda_t \leq 0,\ t \in [0, T],\ dP \otimes dt - \text{a.e.}$ is equivalent to $K_\vartheta - \beta_\vartheta\sigma_\vartheta^{-1}Z_\vartheta \leq 0$, P-a.s.*

Remark 5.3. *Note that excepting the default time ϑ, the left-side jumps of X are predictable and correspond to the ones of the predictable non decreasing process A.*

Proof. As (X_t) is a strong $\mathcal{E}^0$-supermartingale, by the $\mathcal{E}^0$-Mertens decomposition (see Theorem 9.1 in Appendix), there exists a unique process (Z, K, A, C) in $\mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2 \times \mathbb{C}^2$ such that equation (5.1) holds. Let $\nu \in \mathcal{V}$. Since (X_t) is a strong $\mathcal{E}^\nu$- supermartingale,

[10] Since ν is a predictable process, f^ν is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^3)-$ measurable. As, moreover, ν is bounded, f^ν is a λ-admissible driver.

by the $\mathcal{E}^\nu$-Mertens decomposition (cf. , there exists a unique process $(Z^\nu, K^\nu, A^\nu, C^\nu)$ in $\mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2 \times \mathbb{C}^2$ such that, such that

$$-dX_t = \big(f(t, X_t, Z^\nu_t) + (K^\nu_t - \beta_t \sigma_t^{-1} Z^\nu_t)\nu_t \lambda_t\big)\, dt - Z^\nu_t dW_t - K^\nu_t dM_t + dA^\nu_t + dC^\nu_{t-}. \quad (5.3)$$

By applying the uniqueness of the canonical decomposition of a special optional semimartingale (cf. Lemma 9.3 in the Appendix), together with the uniqueness of the representation of the martingale part as the sum of two stochastic integrals (with respect to W and M), we have $Z_t = Z^\nu_t$ $dP \otimes dt$-a.e. and $K_t = K^\nu_t$ $dP \otimes dt$-a.e., $C_{t-} = C^\nu_{t-}$, for all t a.s. and $f(t, X_t, Z_t)dt + dA_t = f(t, Y_t, Z^\nu_t)dt + (K^\nu_t - \beta_t \sigma_t^{-1} Z^\nu_t)\nu_t \lambda_t dt + dA^\nu_t$ for all t a.s.

Using the above equalities, we derive that

$$dA^\nu_t = dA_t - (K_t - \beta_t \sigma_t^{-1} Z_t)\nu_t \lambda_t dt. \quad (5.4)$$

Let us show that this implies that $(K_t - \beta_t \sigma_t^{-1} Z_t)\lambda_t \leq 0,\ t \in [0, T],\ dP \otimes dt -$ a.e. Suppose by contradiction that there exists a predictable set $A \subset [0, T] \times \Omega$ such that $(dP \otimes dt)(A) > 0$ and $(K_t - \beta_t \sigma_t^{-1} Z_t)\lambda_t > 0,\ t \in [0, T],\ dP \otimes dt -$ a.e. on A. For each $n \in \mathbb{N}$, set $\nu^n_t := n\mathbf{1}_A$. Note that (ν^n_t) is a bounded predictable process with $\nu^n_t > -1$. Hence, $\nu^n \in \mathcal{V}$. Using equality (5.4), we derive that for n sufficiently large, we have $E[A^{\nu^n}_T] = E[A_T - n\int_0^T (K_t - \beta_t \sigma_t^{-1} Z_t)\lambda_t \mathbf{1}_A dt] < 0$. We thus get a contradiction with the non decreasing property of A^{ν^n}. Hence, $(K_t - \beta_t \sigma_t^{-1} Z_t)\lambda_t \leq 0$ $dP \otimes dt$-a.s.

Let us show that condition (5.4) implies that the process $A_\cdot + \int_0^\cdot (K_s - \beta_s \sigma_s^{-1} Z_s)\lambda_s ds$ is nondecreasing. Suppose by contradiction that there exist $B \in \mathcal{G}_T$ with $P(B) > 0$, as well as $\varepsilon > 0$ and $(t, s) \in [0, T]^2$ with $t < s$, such that $\int_t^s (dA_r + (K_r - \beta_r \sigma_r^{-1} Z_r)\lambda_r dr) \leq -\varepsilon$ a.s. on B. For each $n \in \mathbb{N}^*$, set $\nu^n := -1 + \dfrac{1}{n}$. Note that $\nu^n \in \mathcal{V}$. From (5.4), we derive that $\int_t^s (dA_r + (K_r - \beta_r \sigma_r^{-1} Z_r)(-1 + \dfrac{1}{n})\lambda_r dr) \geq 0$ a.s. We thus get that for all $n \in \mathbb{N}^*$,

$$-\varepsilon \geq \int_t^s (dA_r + (K_r - \beta_r \sigma_r^{-1} Z_r)\lambda_r dr) \geq \frac{1}{n} \int_t^s (K_r - \beta_r \sigma_r^{-1} Z_r)\lambda_r dr \text{ a.s. on B.}$$

By letting n tend to $+\infty$ in this inequality, we obtain a contradiction. Hence, the process $A_\cdot + \int_0^\cdot (K_s - \beta_s \sigma_s^{-1} Z_s)\lambda_s ds$ is nondecreasing.
Moreover, the uniqueness of the decomposition follows by Lemma 9.3.

□

Theorem 5.4 (Optional $\mathcal{E}^f$-decomposition)**.** *Let (Y_t) be an optional process belonging to $\mathbb{S}^2$. Suppose that it is an $\mathcal{E}^\nu$-strong supermartingale for each $\nu \in \mathcal{V}$. Then, there exists a unique $Z \in \mathbb{H}^2$, a unique $C \in \mathbb{C}^2$ and a unique nondecreasing optional RCLL process h, with $h_0 = 0$ and $E[h_T^2] < \infty$ such that*

$$-dY_t = f(t, Y_t, Z_t)dt - Z_t \sigma_t^{-1}(\sigma_t dW_t + \beta_t dM_t) + dC_{t^-} + dh_t. \quad (5.5)$$

Proof. By Proposition 5.1, there exist $(Z, K, A, C) \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2 \times \mathbb{C}^2$ such that (5.1) and (5.2) hold. Set $h_t := A_t - \int_0^t (K_s - \beta_s \sigma_s^{-1} Z_s) dM_s$. Since $dM_t = dN_t - \lambda_t dt$, we have

$$h_t = A_t + \int_0^t (K_s - \beta_s \sigma_s^{-1} Z_s)\lambda_s ds - \int_0^t (K_s - \beta_s \sigma_s^{-1} Z_s) dN_s. \tag{5.6}$$

Now, by property (5.2), the process $A_\cdot + \int_0^\cdot (K_s - \beta_s \sigma_s^{-1} Z_s)\lambda_s ds$ is non decreasing. Moroever, the process $\int_0^\cdot (K_s - \beta_s \sigma_s^{-1} Z_s) dN_s$ is a purely discontinuous process which admits a unique jump, given by $K_\vartheta - \beta_\vartheta \sigma_\vartheta^{-1} Z_\vartheta$ (at time ϑ). Now by Remark 5.2, we have $K_\vartheta - \beta_\vartheta \sigma_\vartheta^{-1} Z_\vartheta \leq 0$ a.s. We thus derive that the process $\int_0^\cdot (K_s - \beta_s \sigma_s^{-1} Z_s) dN_s$ is non increasing. Hence, by the equality (5.6), we derive that the process (h_t) is non decreasing. Using (5.1), we thus get the equation (5.5).

It remains to show the uniqueness of the processes Z, C, and h in (5.5). To show this, we first show that if Y is decomposable as in (5.5), then the process Y' defined by $Y'_t = Y_t - \Delta Y_\vartheta \mathbb{I}_{t \geq \vartheta}$ is a *special* optional semimartingale (cf. Appendix). By equation (5.5), we have

$$\Delta Y_\vartheta = Z_\vartheta \sigma_\vartheta^{-1} \beta_\vartheta - \Delta h_\vartheta. \tag{5.7}$$

Subtracting $\Delta Y_\vartheta \mathbb{I}_{t \geq \vartheta}$ on both sides of the equation (5.5), we get

$$Y_t - \Delta Y_\vartheta \mathbb{I}_{t \geq \vartheta} = Y_0 - \int_0^t f(s, Y_s, Z_s) ds + \int_0^t Z_s \sigma_s^{-1}(\sigma_s dW_s + \beta_s dM_s) - C_{t^-} - h_t - \Delta Y_\vartheta \mathbb{I}_{t \geq \vartheta}. \tag{5.8}$$

Using this and the expression (5.7) for ΔY_ϑ, we get

$$Y_t - \Delta Y_\vartheta \mathbb{I}_{t \geq \vartheta} = Y_0 - \int_0^t f(s, Y_s, Z_s) ds + \int_0^t Z_s \sigma_s^{-1}(\sigma_s dW_s + \beta_s dM_s) - C_{t^-} - h_t - Z_\vartheta \sigma_\vartheta^{-1} \beta_\vartheta \mathbb{I}_{t \geq \vartheta} + \Delta h_\vartheta \mathbb{I}_{t \geq \vartheta}. \tag{5.9}$$

We set $B_t := h_t - \Delta h_\vartheta \mathbb{I}_{t \geq \vartheta}$. By Lemma 9.4, the process B is a (predictable) process in $\mathcal{A}^2$. Recall that we have also set $Y'_t = Y_t - \Delta Y_\vartheta \mathbb{I}_{t \geq \vartheta}$. With this notation, equation (5.9) becomes

$$Y'_t = Y'_0 - \int_0^t f(s, Y_s, Z_s) ds + \int_0^t Z_s \sigma_s^{-1}(\sigma_s dW_s + \beta_s dM_s) - C_{t^-} - B_t - Z_\vartheta \sigma_\vartheta^{-1} \beta_\vartheta \mathbb{I}_{t \geq \vartheta}. \tag{5.10}$$

Since $dM_t = dN_t - \lambda_t dt$, we get

$$Y'_t = Y'_0 - \int_0^t f(s, Y_s, Z_s) ds + \int_0^t Z_s dW_s - C_{t^-} - B_t - \int_0^t Z_s \sigma_s^{-1} \beta_s \lambda_s ds. \tag{5.11}$$

We conclude that Y' is a special optional semimartingale.
Let now $\tilde{Z}$, $\tilde{C}$, and $\tilde{h}$ be such that $\tilde{Z} \in \mathbb{H}^2$, $\tilde{C} \in \mathbb{C}^2$ and $\tilde{h}$ is a nondecreasing optional RCLL process with $\tilde{h}_0 = 0$ and $E[\tilde{h}_T^2] < \infty$, and such that the decomposition (5.5) holds with $\tilde{Z}$, $\tilde{C}$, and $\tilde{h}$ (in place of Z, C, h). We show that $\tilde{Z} = Z$ in $\mathbb{H}^2$, $\tilde{C}_t = C_t$, for all t a.s. and $\tilde{h}_t = h_t$, for all t a.s. By the same reasoning as above, we have that (5.11) holds also with Z, C, and B replaced by $\tilde{Z}$, $\tilde{C}$, and $\tilde{B}$, where $\tilde{B}$ is defined by $\tilde{B}_t := \tilde{h}_t - \Delta \tilde{h}_\vartheta \mathbb{I}_{t \geq \vartheta}$. We note that, due

to (5.7), $\Delta\tilde{h}_\vartheta = \Delta h_\vartheta$. Hence, showing the equality $\tilde{h}_t = h_t$, for all t a.s. is equivalent to showing that $\tilde{B}_t = B_t$, for all t a.s.
Now, as Y' is a special optional semimartingale admitting the decomposition (5.11) with Z, C, and B, on one hand, and with $\tilde{Z}$, $\tilde{C}$, and $\tilde{B}$, on the other hand, we have, by the uniqueness of the special optional semimartingale decomposition (cf. Lemma 9.2 in the Appendix), that $C = \tilde{C}$, $f(t, Y_t, Z_t)dt + dB_t + Z_t\sigma_t^{-1}\beta_t\lambda_t dt = f(t, Y_t, \tilde{Z}_t)dt + d\tilde{B}_t + \tilde{Z}_t\sigma_t^{-1}\beta_t\lambda_t dt$, and $Z_t dW_t = \tilde{Z}_t dW_t$. From the last equality, using the uniqueness of the martingale representation, we get $Z = \tilde{Z}$ in $\mathbb{H}^2$. This, together with the second equality, gives the equality of B and $\tilde{B}$. The proof is thus complete.

□

Remark 5.5. *Note that in the classical linear case when f is given by $f(t, y, z) = -r_t y - z\theta_t$ (see [19], [21] and [35]) and when the process Y is moreover RCLL, the above $\mathcal{E}^f$-decomposition corresponds to the well known optional decomposition of an RCLL process, which is a supermartingale under each martingale probability measure, up to a discounting and a change of probability measure.*

Proposition 5.6. *Let $(X_t) \in \mathbb{S}^2$. The process (X_t) admits an optional decomposition of the form* (5.5) *if and only if it admits a decomposition of the form* (5.1) *with the conditions* (5.2)*.*

Proof. By the proof of Theorem 5.4, we derive that if (X_t) admits a decomposition of the form (5.1) with conditions (5.2), then it admits an optional decomposition of the form (5.5).

It remains to show the converse. Suppose that there exists $(Z, C) \in \mathbb{H}^2 \times \mathbb{C}^2$ and a nondecreasing optional RCLL process h, with $h_0 = 0$ and $E[h_T^2] < \infty$ such that the equation (5.5) holds. By Lemma 9.4, h has the following decomposition $h_t = B_t + \int_0^t \psi_s dN_s$, where B is a (predictable) process in $\mathcal{A}^2$ and $\psi \in \mathbb{H}^2_\lambda$ with $\psi_t\lambda_t \geq 0$ $dP \otimes dt$-a.s. Let (A_t) be the process defined for all $t \in [0, T]$ by

$$A_t := B_t + \int_0^t \psi_s\lambda_s ds. \tag{5.12}$$

We have $A \in \mathcal{A}^2$. Let (K_t) be the process defined for all $t \in [0, T]$ by

$$K_t := \beta_t\sigma_t^{-1}Z_t - \psi_t. \tag{5.13}$$

Note that $K \in \mathbb{H}^2_\lambda$. Now, since $\psi_t\lambda_t \geq 0$ $dP \otimes dt$-a.s., we have $(K_t - \beta_t\sigma_t^{-1}Z_t)\lambda_t \leq 0$ $dP \otimes dt$-a.s. Moreover, by (5.12) and (5.13), we have $B_t = A_t + \int_0^t (K_s - \beta_s\sigma_s^{-1}Z_s)\lambda_s ds$. Since $B \in \mathcal{A}^2$, we derive that the conditions (5.2) hold.
Moreover, since $N_t = M_t + \int_0^t \lambda_s ds$, by (5.12), we get $h_t = B_t + \int_0^t \psi_s dN_s = A_t + \int_0^t \psi_s dM_s$ a.s. Hence, using (5.5) and (5.13), we derive that the process (Z, K, A, C) satisfies the equation (5.1). The proof is thus complete. □

6 Superhedging price for the European option: a sketch of the proof of the duality and infinitesimal characterization

We now consider a European option with maturity T and payoff η such that there exist $x \in \mathbb{R}$ and $\varphi \in \mathbb{H}^2$ such that

$$|\eta| \; \leq \; V_T^{x,\varphi} = x - \int_0^T f(s, V_s^{x,\varphi}, \sigma_s \varphi_s) ds + \int_0^T \varphi_s \sigma_s dW_s + \int_0^T \beta_s \varphi_s dM_s \quad \text{a.s.} \tag{6.1}$$

For each $S \in \mathcal{T}$, we define the $\mathcal{F}_S$-measurable random variable:

$$X(S) := ess \sup_{\nu \in \mathcal{V}_S} \mathcal{E}^{\nu}_{S,T}(\eta). \tag{6.2}$$

Lemma 6.1. *The family $(X(S),\ S \in \mathcal{T})$ is the smallest admissible family such that for each $\nu \in \mathcal{V}$, it is an $\mathcal{E}^{\nu}$-supermartingale family satisfying for all $S \in \mathcal{T}_0$, $X(T) = \eta$ a.s.*

Sketch of the proof : The proof, which is not detailed in this version, relies on properties of the $\mathcal{E}^{\nu}$-evaluations.

Using the above Lemma 6.1, we get the following result.

Lemma 6.2. *There exists an RCLL adapted process $(X_t) \in S^2$ which aggregates the value family $(X(S))$. The process (X_t) is a strong $\mathcal{E}^{\nu}$-supermartingale for all $\nu \in \mathcal{V}$ and $X_T = \eta$ a.s. Moreover, the process (X_t) is the smallest process in S^2 satisfying these properties.*

Proof. Lemma 6.1 implies in particular that the value family $(X(S))$ is a strong $\mathcal{E}^0$-supermartingale family. By Lemma 4.3 together with Remark 4.7, we derive that there exists an RCLL process $(X_t) \in S^2$ aggregating the family $(X(S))$. The other properties of the aggregating process (X_t) follow directly from Lemma 6.1. □

From this property and Proposition 5.1, we derive that the right-continuous process (X_t) admits the *predictable $\mathcal{E}^f$-decomposition* from Proposition 5.1 with $C = 0$. Moreover, by Theorem 5.4, we get the following result.

Lemma 6.3. *The right-continuous process (X_t) admits the* optional $\mathcal{E}^f$-decomposition *from Theorem 5.4 (with $C = 0$).*

We will now give a dual representation for the seller's superhedging price v_0 in terms of the value process (X_t) (at time 0) . We also give a superhedging strategy for the seller. From this result, we will deduce the dual representation (in terms of the f-martingale probability measures) stated in Theorem 3.27.

Theorem 6.4 (Dual representation)**.** *The* seller's superhedging price v_0 *of the European option is equal to the value X_0 (at time 0) of the non-linear control problem* (6.2)*, that is*

$$v_0 = \sup_{\nu \in \mathcal{V}} \mathcal{E}^{\nu}_{0,T}(\eta). \tag{6.3}$$

Moreover, the portfolio strategy $\varphi^* := \sigma^{-1}Z$, where the process Z is the one from the $\mathcal{E}^f$-optional decomposition of the value process X from Theorem 5.4, is a superhedging strategy for the seller, that is, $V_T^{x,\varphi} \geq \eta$ a.s.

Sketch of the proof : The proof, which is not detailed in this version, relies on Lemma 6.3 and quite similar arguments to those used in Theorem 7.12.

Remark 6.5. *Some related results are given in [2] for European options in a Brownian framework.*

Proof of Theorem 3.16: The proof follows from the previous Theorem 6.4 and from Remark 4.9. Indeed, under an additional integrability condition $\varphi \in \cap_{\nu \in \mathcal{V}} \mathbb{H}^2_{Q^\nu}$ on the process φ from Assumption (7.1), by Remark 4.9, the above dual representation of the *superhedging price* can be written in terms of the f-martingale probability measures, that is

$$v_0 = \sup_{\nu \in \mathcal{V}} \mathcal{E}^f_{Q^\nu,0,T}(\eta),$$

which ends the proof of Theorem 3.16.

We now introduce the notion of a *supersolution* of the *constrained BSDE* with driver f and terminal condition η.

Definition 6.6. *Let $\eta \in L^2(\mathcal{G}_T)$. A process $X' \in \mathbb{S}^2$ is said to be a* supersolution *of the* constrained BSDE *with driver f and terminal condition η if there exists a process $(Z', K', A') \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2$ such that*

$$-\, dX'_t = f(t, X'_t, Z'_t)dt + dA'_t - Z'_t dW_t - K'_t dM_t; \quad X'_T = \eta \text{ a.s.}; \tag{6.4}$$

$$A'_\cdot + \int_0^\cdot (K'_s - \beta_s \sigma_s^{-1} Z'_s)\lambda_s ds \in \mathcal{A}^2 \quad \text{and} \quad (K'_t - \beta_t \sigma_t^{-1} Z'_t)\lambda_t \leq 0,\ t \in [0, T],\ dP \otimes dt - \text{a.e.}; \tag{6.5}$$

We give the following infinitesimal characterization of the seller's superhedging price of the European option.

Theorem 6.7. *The* seller's superhedging price *process (X_t) of the European option is the minimal supersolution of the* constrained BSDE *associated with driver f and terminal condition η from Definition 6.6.*

Let (Z, K, A) be the unique process in $\mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2$ such that (X, Z, K, A) satisfies (6.4) *and* (6.5). *The process $\varphi := \sigma^{-1}Z$ is a superhedging strategy for the seller, that is, $V_T^{X_0,\varphi} \geq \eta$ a.s.*

Sketch of the proof: The proof, which is not detailed in this version, relies on Lemma 6.2, Lemma 6.3 and Proposition 5.1.

Remark 6.8. *Recall that the* buyer's superhedging price $\tilde{v}_0$ *for the European option with payoff η is equal to the opposite of the seller's superhedging price for the European option with payoff $-\eta$ (cf. Section 3.2). From this and from the results on the seller's superhedging price, we derive the corresponding results for the buyer's superhedging price for the European option.*

7 Seller's superhedging price for the American option: proof of the duality and infinitesimal characterization

Let now (ξ_t) be an optional process such that there exist $x \in \mathbb{R}$ and $\varphi \in \mathbb{H}^2$ with

$$|\xi_t| \;\leq\; V_t^{x,\varphi} = x - \int_0^t f(s, V_s^{x,\varphi}, \sigma_s\varphi_s)ds + \int_0^t \varphi_s\sigma_s dW_s + \int_0^t \beta_s\varphi_s dM_s, \quad 0 \leq t \leq T, \quad \text{a.s.} \tag{7.1}$$

Note that for all $\nu \in \mathcal{V}$, the process $V^{x,\varphi}$ is an $\mathcal{E}^\nu$-martingale since $(V^{x,\varphi}, \sigma\varphi, \beta\varphi)$ is the solution of the BSDE associated with driver f^ν, terminal time T and terminal condition $V_T^{x,\varphi}$. As the process $V^{x,\varphi}$ belongs to S^2, we have $\xi \in \mathbb{S}^2$.

7.1 Non-linear problem of control and stopping. The value family (Y(S)).

Establishing the dual representation for the seller's superheding price is based on the study of the following non-linear problem of control and stopping.
For each $S \in \mathcal{T}$, let $Y(S)$ be the $\mathcal{G}_S$-measurable random variable defined by

$$Y(S) := ess \sup_{(\tau,\nu)\in\mathcal{T}_S\times\mathcal{V}} \mathcal{E}^\nu_{S,\tau}(\xi_\tau). \tag{7.2}$$

Remark 7.1. *We note that for each $S \in \mathcal{T}$, $\tau \in \mathcal{T}_S$ and each $\nu \in \mathcal{V}$, the random variable $\mathcal{E}^\nu_{S,\tau}(\xi_\tau)$ depends on the control ν only through the values of ν on the interval $[S,\tau]$. For each $S \in \mathcal{T}$, let $\mathcal{V}_S$ be the set of bounded predictable processes ν defined on $[S,T]$, such that $\nu_t > -1$, $S \leq t \leq T$, $dP \otimes dt$-a.s. We thus have*

$$Y(S) = ess \sup_{(\tau,\nu)\in\mathcal{T}_S\times\mathcal{V}_S} \mathcal{E}^\nu_{S,\tau}(\xi_\tau) \quad \text{a.s.}$$

In order to facilitate the study of the *non-linear problem of control and stopping* (7.2), we introduce the following *auxiliary non-linear optimal stopping problem*: for $\nu \in \mathcal{V}$, for $S \in \mathcal{T}$,

$$Y^\nu(S) = ess \sup_{\tau\in\mathcal{T}_S} \mathcal{E}^\nu_{S,\tau}(\xi_\tau) \tag{7.3}$$

We know from [25] that the value family $(Y^\nu(S))_{S\in\mathcal{T}}$ of the auxiliary optimal stopping problem can be aggregated by an optional process $(Y_t^\nu)_{t\in[0,T]} \in \mathbb{S}^2$ which is a strong $\mathcal{E}^\nu$-supermartingale.
From the definitions and Remark 7.1, we have, for all $S \in \mathcal{T}$,

$$Y(S) = ess \sup_{\nu\in\mathcal{V}} Y_S^\nu = ess \sup_{\nu\in\mathcal{V}_S} Y_S^\nu \text{ a.s.} \tag{7.4}$$

Let us note also that $Y(S) \geq Y_S^0$ a.s., as $0 \in \mathcal{V}$. Moreover, since $|\xi_t| \leq V_t^{x,\varphi}$, $0 \leq t \leq T$ a.s. it follows that for all $S \in \mathcal{T}$, $\tau \in \mathcal{T}_S$ and $\nu \in \mathcal{V}$, $\mathcal{E}^\nu_{S,\tau}(\xi_\tau) \leq \mathcal{E}^\nu_{S,\tau}(|\xi_\tau|) \leq \mathcal{E}^\nu_{S,\tau}(V_\tau^{x,\varphi}) = V_S^{x,\varphi}$ a.s. Hence, taking the essential supremum over $\tau \in \mathcal{T}_S$ and $\nu \in \mathcal{V}$ in this inequality, we derive that $Y(S) \leq V_S^{x,\varphi}$ a.s.
Since $Y^0 \in \mathbb{S}^2$ and $V^{x,\varphi} \in \mathbb{S}^2$, it follows that $E[ess\sup_{S\in\mathcal{T}} Y(S)^2] < +\infty$.

Lemma 7.2. *The value family $(Y(S))_{S\in\mathcal{T}}$ of the non-linear problem of control and stopping is an admissible family.*

Proof. The result is an easy consequence of the representation (7.4).
For each $S \in \mathcal{T}$, $Y(S)$ is $\mathcal{G}_S$-measurable as the essential supremum of $\mathcal{G}_S$-measurable random variables. Let $S, S' \in \mathcal{T}$ such that $S = S'$ a.s. We have $Y_S^\nu = Y_{S'}^\nu$ a.s. for all $\nu \in \mathcal{V}$. Hence, $ess\sup_{\nu\in\mathcal{V}} Y_S^\nu = ess\sup_{\nu\in\mathcal{V}} Y_{S'}^\nu$ a.s. From this, together with (7.4), we get $Y(S) = Y(S')$ a.s. The admissibility of the value family is thus proven. □

Proposition 7.3. (Maximizing sequence) *Let $S \in \mathcal{T}$. There exists a sequence of controls $(\nu^n)_{n\in\mathbb{N}}$ with $\nu^n \in \mathcal{V}_S$, for all n, such that the sequence $(Y_S^{\nu^n})_{n\in\mathbb{N}}$ is non- decreasing and satisfies:*

$$Y(S) \quad = \lim_{n\to\infty} \uparrow Y_S^{\nu^n} \quad \text{a.s.} \tag{7.5}$$

Proof. We show that the set $\{Y_S^\nu, \nu \in \mathcal{V}_S\}$ is stable under pairwise maximization. Indeed, let $\nu, \nu' \in \mathcal{V}_S$. Set $A := \{\, Y_S^{\nu'} \leq Y_S^\nu \,\}$. We have $A \in \mathcal{F}_S$. Set $\tilde{\nu} := \nu\mathbf{1}_A + \nu'\mathbf{1}_{A^c}$. Then $\tilde{\nu} \in \mathcal{V}_S$. We have $Y_S^{\tilde{\nu}}\mathbf{1}_A = ess\sup_{\tau\in\mathcal{T}_S} \mathcal{E}^{\tilde{\nu}}_{S,\tau}(\xi_\tau)\mathbf{1}_A = ess\sup_{\tau\in\mathcal{T}_S} \mathcal{E}^{f^{\tilde{\nu}}\mathbf{1}_A}_{S,\tilde{\tau}}(\xi_{\tilde{\tau}}\mathbf{1}_A) = ess\sup_{\tau\in\mathcal{T}_S} \mathcal{E}^{f^{\nu}\mathbf{1}_A}_{S,\tau}(\xi_\tau\mathbf{1}_A) = ess\sup_{\tau\in\mathcal{T}_S} \mathcal{E}^{\nu}_{S,\tau}(\xi_\tau)\mathbf{1}_A = Y_S^\nu\mathbf{1}_A$ a.s. and similarly on A^c. It follows that $Y_S^{\tilde{\nu}} = Y_S^\nu\mathbf{1}_A + Y_S^{\nu'}\mathbf{1}_{A^c} = Y_S^\nu \vee Y_S^{\nu'}$ a.s. The result of the proposition follows by a classical result on essential suprema (cf. Neveu (1975)). □

Proposition 7.4. *The family $(Y(S))$ satisfies the following properties: $(Y(S))$ is an $\mathcal{E}^\nu$-supermartingale family for all $\nu \in \mathcal{V}$ and $Y(S) \geq \xi_S$ a.s. for all $S \in \mathcal{T}$. Moreover, $(Y(S))$ is the smallest family satisfying these properties.*

Proof. For all $S \in \mathcal{T}$, for all $\nu \in \mathcal{V}_S$, $Y_S^\nu \geq \xi_S$ a.s. Hence, for all $S \in \mathcal{T}$, $Y(S) \geq \xi_S$ a.s. Let $S, S' \in \mathcal{T}$ be such that $S \geq S'$ a.s. By Proposition 7.3, there exists a sequence of controls $(\nu^n)_{n\in\mathbb{N}}$, with ν_n in $\mathcal{V}_S$ for all n, such that $Y(S) \quad = \lim_{n\to\infty} \uparrow Y_S^{\nu^n}$ a.s. Let $\nu \in \mathcal{V}$. By the continuity property of $\mathcal{E}^\nu$ with respect to terminal condition, we have $\mathcal{E}^\nu_{S',S}(Y(S)) = \lim_{n\to\infty} \mathcal{E}^\nu_{S',S}(Y_S^{\nu_n})$ a.s. For each n, we set $\tilde{\nu}_t^n := \nu_t\mathbf{1}_{]S',S]}(t) + \nu_t^n\mathbf{1}_{]S,T]}(t)$. We note that $\tilde{\nu}^n \in \mathcal{V}_{S'}$; hence, $f^{\tilde{\nu}^n}$ is λ-admissible. We have $f^{\tilde{\nu}^n} = f^\nu\mathbf{1}_{]S',S]} + f^{\nu^n}\mathbf{1}_{]S,T]}$. Moreover, $Y_S^{\nu_n} = Y_S^{\tilde{\nu}_n}$ (as $f^{\tilde{\nu}^n} = f^{\nu^n}$ on $[S,T]$, $dt \otimes dP$-a.e.). From these observations, we deduce

$$\mathcal{E}^\nu_{S',S}(Y_S^{\nu_n}) = \mathcal{E}^{\tilde{\nu}^n}_{S',S}(Y_S^{\tilde{\nu}_n}) \leq Y_{S'}^{\tilde{\nu}_n},$$

where the (last) inequality is due to the fact that $Y^{\tilde{\nu}_n}$ is a strong $\mathcal{E}^{\tilde{\nu}^n}$-supermartingale. We thus get $\mathcal{E}^\nu_{S',S}(Y(S)) = \lim\limits_{n\to\infty} \mathcal{E}^\nu_{S',S}(Y_S^{\nu_n}) \leq \liminf\limits_{n\to\infty} Y_{S'}^{\tilde{\nu}_n} \leq Y(S')$ a.s., where the last inequality follows from (7.4). As $\nu \in \mathcal{V}$ is arbitrary, we conclude that the family $(Y(S))$ is an $\mathcal{E}^\nu$-supermartingale family for all $\nu \in \mathcal{V}$.

Let us prove the second statement. Let $(Y'(S), S \in \mathcal{T})$ be an admissible family satisfying the properties: $(Y'(S))$ is an $\mathcal{E}^\nu$-supermartingale family for all $\nu \in \mathcal{V}$ and $Y'(S) \geq \xi_S$ a.s. for all $S \in \mathcal{T}$. Let $\nu \in \mathcal{V}$. By the properties of Y', for all $S \in \mathcal{T}$, for all $\tau \in \mathcal{T}_S$, $Y'(S) \geq \mathcal{E}^\nu_{S,\tau}(Y'(\tau)) \geq \mathcal{E}^\nu_{S,\tau}(\xi_\tau)$ a.s. By taking the essential supremum over $\tau \in \mathcal{T}_S$ and $\nu \in \mathcal{V}$, we get $Y'(S) \geq Y(S)$ a.s. □

Corollary 7.5. *There exists an r.u.s.c. process $(Y_t) \in \mathbb{S}^2$ which aggregates the value family $(Y(S))$ of the problem of control and stopping* (7.2). *The process (Y_t) is a strong $\mathcal{E}^\nu$-supermartingale for all $\nu \in \mathcal{V}$ and $Y_t \geq \xi_t$, for all $t \in [0,T]$, a.s. Moreover, the process (Y_t) is the smallest process in $\mathbb{S}^2$ satisfying these properties.*

Proof. The above Proposition 7.4 implies in particular that the value family $(Y(S))$ is a strong $\mathcal{E}^0$-supermartingale family. By Lemma 4.3, there exists an r.u.s.c. process (Y_t) in $\mathbb{S}^2$ aggregating the family $(Y(S))$. The other properties of the aggregating process (Y_t) follow directly from Proposition 7.4. □

Corollary 7.6 (The right-continuous case)**.** *Assume moreover that the process (ξ_t) in problem* (7.2) *is* RCLL. *Then, the process (Y_t) is RCLL. Moreover, (Y_t) is the smallest RCLL process in $\mathbb{S}^2$ satisfying the properties: for each $\nu \in \mathcal{V}$, (Y_t) is a (strong) RCLL $\mathcal{E}^\nu$-supermartingale greater than or equal to (ξ_t).*

7.2 The strict value family $(Y^+(S))$

Let S be a stopping time in $\mathcal{T}_0$. We denote by $\mathcal{T}_{S^+}$ the set of stopping times $\theta \in \mathcal{T}_0$ with $\theta > S$ a.s. on $\{S < T\}$ and $\theta = T$ a.s. on $\{S = T\}$. The strict value $Y^+(S)$ (at time S) is defined by

$$Y^+(S) := ess \sup_{(\tau,\nu) \in \mathcal{T}_{S^+} \times \mathcal{V}} \mathcal{E}^\nu_{S,\tau}(\xi_\tau). \tag{7.6}$$

We note that (as for $Y(S)$) the set $\mathcal{V}$ in the above problem can be replaced with the set $\mathcal{V}_S$ without changing the value of the problem.
We note also that $Y^+(S) = \xi_T$ a.s. on $\{S = T\}$.
Let S be a stopping time in $\mathcal{T}_0$ and let $\nu \in \mathcal{V}$. We introduce the following *auxiliary (strict) optimal stopping problem* (to be compared with (7.3)):

$$Y^{\nu,+}(S) := ess \sup_{\tau \in \mathcal{T}_{S^+}} \mathcal{E}^\nu_{S,\tau}(\xi_\tau). \tag{7.7}$$

We know from [25] (cf. Proposition 9.1) that there exists a strong $\mathcal{E}^\nu$-supermartingale, denoted by $(Y^{\nu,+}_t)$, which aggregates the value family $(Y^{\nu,+}(S))$ of the above (strict) optimal stopping problem. Note that we have

$$Y^+(S) = ess \sup_{\nu \in \mathcal{V}} Y^{\nu,+}_S = ess \sup_{\nu \in \mathcal{V}_S} Y^{\nu,+}_S \quad \text{a.s.} \tag{7.8}$$

Using the above representation and the same type of arguments as those used above for the value family $(Y(S))_{S\in\mathcal{T}_0}$, we show that the strict value family $(Y^+(S))_{S\in\mathcal{T}_0}$ is an admissible family, satisfying the integrability condition $\mathbb{E}[ess\sup_{S\in\mathcal{T}_0}(Y^+(S))^2] < \infty$ and the following properties:

Proposition 7.7. *For each $S \in \mathcal{T}_0$, there exists a maximizing sequence $(\nu^n) = (\nu^n(S)) \in \mathcal{V}_S^n$ for the optimal control problem from equation (7.8), that is, $Y_S^+ = \lim_{n\to\infty} \uparrow Y_S^{\nu_n,+}$ a.s.*

Proposition 7.8. *The family $(Y^+(S))_{S\in\mathcal{T}_0}$ is an $\mathcal{E}^\nu$-supermartingale family for each $\nu \in \mathcal{V}$.*

As above, we deduce the following

Corollary 7.9. *There exists a process $(Y_t^+) \in \mathbb{S}^2$ which aggregates the strict value family $(Y^+(S))_{S\in\mathcal{T}_0}$. The process (Y_t^+) is a strong $\mathcal{E}^\nu$-supermartingale for all $\nu \in \mathcal{V}$.*

Moreover, the following result holds true. The result is based on the representation (7.8) and on properties of the strict value process $(Y_t^{\nu,+})$ of the auxiliary optimal stopping problem (7.7).
We recall that (Y_{t+}) denotes the process of right limits of the process (Y_t). We recall also that (Y_{t+}) is well-defined as (Y_t) is a strong $\mathcal{E}^\nu$-supermartingale, and hence, has right (and left) limits.
We recall that (Y_{t+}^ν) denotes the process of right limits of the process (Y_t^ν).

Theorem 7.10. **(i)** *The strict value process (Y_t^+) is right-continuous.*

(ii) *For all $S \in \mathcal{T}_0$, $Y_S^+ = Y_{S+}$ a.s. (in other words, the strict value process (Y_t^+) coincides with the process of right limits (Y_{t+})).*

(iii) *For all $S \in \mathcal{T}_0$, $Y_S = Y_{S+} \vee \xi_S$ a.s.*

We have the following intermediary result:

Proposition 7.11. *For all $S \in \mathcal{T}_0$,*

$$\mathbb{E}[Y_S^+] = \sup_{\nu\in\mathcal{V}} \mathbb{E}[Y_S^{\nu,+}].$$

Proof. From the representation (7.8), we deduce $\mathbb{E}[Y_S^+] = \mathbb{E}[ess\sup_{\nu\in\mathcal{V}} Y_S^{\nu,+}] \geq \sup_{\nu\in\mathcal{V}} \mathbb{E}[Y_S^{\nu,+}]$. We now show the converse inequality. By Proposition 7.7, there exists a sequence $(\nu_n) = (\nu_n(S))$ in $\mathcal{V}_S^n$ such that $Y_S^+ = \lim_{n\to\infty} \uparrow Y_S^{\nu_n,+}$. We thus have $\mathbb{E}[Y_S^+] = \mathbb{E}[\lim_{n\to\infty} \uparrow Y_S^{\nu_n,+}] = \lim_{n\to\infty} \uparrow \mathbb{E}[Y_S^{\nu_n,+}]$, where we have used dominated convergence to exchange limit and expectation. For all n, we have $\mathbb{E}[Y_S^{\nu_n,+}] \leq \sup_{\nu\in\mathcal{V}} \mathbb{E}[Y_S^{\nu,+}]$.[11] We conclude that $\mathbb{E}[Y_S^+] \leq \sup_{\nu\in\mathcal{V}} \mathbb{E}[Y_S^{\nu,+}]$. The proposition is thus proved. □

We are now ready to prove Theorem 7.10.

[11] Indeed, each process $\nu \in \mathcal{V}_S$ can be seen as a process $\tilde{\nu}$ in $\mathcal{V}$ by setting $\tilde{\nu} = \nu$ on $[S,T]$ and $\tilde{\nu} = 0$ on $[0,S)$.

Proof of Theorem 7.10. To prove statement (i), we first show that the process (Y_t^+) is right-lowersemicontinuous along stopping times in expectation. Let $S \in \mathcal{T}_0$, let (S_n) be a non-increasing sequence of stopping times in $\mathcal{T}_S$ with $\lim \downarrow S_n = S$ a.s. By Proposition 7.11, we have $\mathbb{E}[Y_{S_n}^+] = \sup_{\nu\in\mathcal{V}} \mathbb{E}[Y_{S_n}^{\nu,+}]$, for all $n \in \mathbb{N}$. Hence, $\liminf_{n\to\infty} \mathbb{E}[Y_{S_n}^+] = \liminf_{n\to\infty} \sup_{\nu\in\mathcal{V}} \mathbb{E}[Y_{S_n}^{\nu,+}] \geq \sup_{\nu\in\mathcal{V}} \liminf_{n\to\infty} \mathbb{E}[Y_{S_n}^{\nu,+}]$. Now, for all $\nu \in \mathcal{V}$, the process $(Y^{\nu,+})$ is right-continuous (cf. Theorem 9.2 in [25]), hence right-continuous along stopping times (cf. [10]); by dominated convergence, we thus have $\liminf_{n\to\infty} \mathbb{E}[Y_{S_n}^{\nu,+}] = \lim_{n\to\infty} \mathbb{E}[Y_{S_n}^{\nu,+}] = \mathbb{E}[Y_S^{\nu,+}]$. This, together with the above computation, gives $\liminf_{n\to\infty} \mathbb{E}[Y_{S_n}^+] \geq \sup_{\nu\in\mathcal{V}} \mathbb{E}[Y_S^{\nu,+}] = \mathbb{E}[Y_S^+]$, where the last equality holds due to Proposition 7.11. We conclude that the process (Y_t^+) is right-lowersemicontinuous along stopping times in expectation. On the other hand, we know already that the process (Y_t^+) is right-uppersemicontinuous along stopping times, and hence right-uppersemicontinuous along stopping times in expectation (due to its integrability). Hence, (Y_t^+) is right-continuous along stopping times in expectation. We deduce that (Y_t^+) is right-continuous (cf. e.g. [11]). We now show (ii). Let $S \in \mathcal{T}_0$. One inequality, namely the inequality $Y_{S+} \geq Y_S^+$ a.s., follows from the right-continuity of (Y_t^+), established in (i). Indeed, let (S_n) be a non-increasing sequence of stopping times in $\mathcal{T}_{S^+}$ with $\lim \downarrow S_n = S$ a.s. We know that $Y_\tau \geq Y_\tau^+$ a.s., for all $\tau \in \mathcal{T}_0$. Hence, $Y_{S_n} \geq Y_{S_n}^+$ a.s., for all n. By taking the limit when $n \to \infty$ and by using the right-continuity of (Y_t^+), we get $Y_{S+} \geq Y_S^+$ a.s. For the converse inequality, we first show

$$\mathcal{E}^0_{S,S^n}(Y_{S_n}) \leq Y_S^+ \text{ a.s. for all } n. \tag{7.9}$$

To prove this, we fix n and we take $(\tau^p, \nu^p) \in \mathcal{T}_{S_n} \times \mathcal{V}$ an optimizing sequence for the problem with value Y_{S_n}, i.e. $Y_{S_n} = \lim_{p\to\infty} \mathcal{E}^{\nu^p}_{S_n,\tau_p}(\xi_{\tau_p})$. We have

$$\mathcal{E}^0_{S,S_n}(Y_{S_n}) = \mathcal{E}^0_{S,S_n}(\lim_{p\to\infty} \mathcal{E}^{\nu^p}_{S_n,\tau_p}(\xi_{\tau_p})) = \lim_{p\to\infty} \mathcal{E}^0_{S,S_n}(\mathcal{E}^{\nu^p}_{S_n,\tau_p}(\xi_{\tau_p})) \text{ a.s.,} \tag{7.10}$$

where we have used the continuity property of $\mathcal{E}^0_{S,S^n}(\cdot)$ with respect to the terminal condition (recall that here n is fixed). We set $\bar{\nu}_t^p := \nu_t^p \mathbb{I}_{\{t>S_n\}}$ (hence, $\bar{\nu}_t^p = 0$ on $\{t \leq S_n\}$). We note that $\bar{\nu} \in \mathcal{V}$. Using the definition of $\bar{\nu}$ and the consistency property of $\mathcal{E}$-expectations, we get $\mathcal{E}^0_{S,S_n}(\mathcal{E}^{\nu^p}_{S_n,\tau_p}(\xi_{\tau_p})) = \mathcal{E}^{\bar{\nu}}_{S,\tau_p}(\xi_{\tau_p}) \leq Y_S^+$ a.s. (where for the inequality we have used that $\tau_p \in \mathcal{T}_{S^+}$). From this, together with equation (7.10), we derive the desired inequality (7.9). From (7.9) and using the continuity of $\mathcal{E}$-expectations with respect to the terminal time and the terminal condition, we derive $Y_S^+ \geq \lim_{n\to\infty} \mathcal{E}^0_{S,S^n}(Y_{S_n}) = \mathcal{E}^0_{S,S}(Y_{S+}) = Y_{S+}$ a.s. Hence, $Y_S^+ \geq Y_{S+}$ a.s., which, together with the previously shown converse inequality, proves the equality.

We now show (iii). Using successively statement (ii), relation (7.8), Theorem 9.2 (iii) in [25] , and relation (7.4), we get

$$Y_S^+ \vee \xi_S = Y_{S+} \vee \xi_S = ess\sup_{\nu\in\mathcal{V}} \left(Y_S^{\nu,+} \vee \xi_S \right) = ess\sup_{\nu\in\mathcal{V}} Y_S^\nu = Y_S \quad \text{a.s.}$$

□

7.3 Proof of the dual representation

We will now give a dual representation for the seller's superhedging price u_0 in terms of the value (at time 0) of the non-linear problem of control and stopping studied above. We also give a superhedging strategy for the seller. From this result, we will deduce the dual representation (in terms of the f-martingale probability measures) stated in Theorem 3.27.

Theorem 7.12. *The* superhedging price u_0 *of the American option is equal to the value* Y_0 *(at time* 0*) of the non-linear problem of control and stopping* (7.2)*, that is*

$$u_0 = \sup_{(\tau,\nu)\in\mathcal{T}\times\mathcal{V}} \mathcal{E}^{\nu}_{0,\tau}(\xi_\tau). \tag{7.11}$$

Moreover, the portfolio strategy $\varphi^* := \sigma^{-1}Z$, where the process Z is the one from the $\mathcal{E}^f$-optional decomposition of the value process Y from Theorem 5.4, is a superhedging strategy for the seller, that is, $\varphi^* \in \mathcal{A}(u_0)$.

Remark 7.13. *Some related results are given in [2] for European options in a Brownian framework.*

Proof. In order to prove the results of the above theorem, it is sufficient to show that $u_0 = Y_0$ and $\varphi^* \in \mathcal{A}(Y_0)$.
Let $\mathcal{H}$ be the set of initial capitals which allow the seller to be "super-hedged", that is $\mathcal{H} = \{x \in \mathbb{R} : \exists \varphi \in \mathcal{A}(x)\}$. Note that $u_0 = \inf \mathcal{H}$.

Let us first show that $Y_0 \geq u_0$. To this aim, we prove that

$$\varphi^* \in \mathcal{A}(Y_0). \tag{7.12}$$

We consider the portfolio associated with the initial capital Y_0 and the strategy φ^*. By (3.4)-(3.5), the value of this portfolio $(V_t^{Y_0,\varphi^*})$ satisfies the following forward equation:

$$V_t^{Y_0,\varphi^*} = Y_0 - \int_0^t f(s, V_s^{Y_0,\varphi^*}, Z_s)ds + \int_0^t \sigma_s^{-1} Z_s(\sigma_s dW_s + \beta_s dM_s),\ 0 \leq t \leq T \quad \text{a.s.} \tag{7.13}$$

Moreover, since Y satisfies the $\mathcal{E}^f$-optional decomposition from Theorem 5.4, we have

$$Y_t = Y_0 - \int_0^t f(s, Y_s, Z_s)ds + \int_0^t \sigma_s^{-1} Z_s(\sigma_s dW_s + \beta_s dM_s) - h_t - C_{t^-},\ 0 \leq t \leq T \quad \text{a.s.} \tag{7.14}$$

Since (h_t) and (C_{t^-}) are nondecreasing, by the comparison result for forward differential equations, we thus derive $V_t^{Y_0,\varphi^*} \geq Y_t$, $0 \leq t \leq T$ a.s. Hence, since $Y_t \geq \xi_t$, we get $V_t^{Y_0,\varphi^*} \geq \xi_t$, $0 \leq t \leq T$ a.s., which implies the desired property (7.12). We thus derive that $Y_0 \in \mathcal{H}$, and hence that $Y_0 \geq u_0$.

Let us show the converse inequality. Let $x \in \mathcal{H}$. There exists $\varphi \in \mathcal{A}(x)$ such that $V_t^{x,\varphi} \geq \xi_t$, $0 \leq t \leq T$ a.s. For each $\tau \in \mathcal{T}$ we thus have $V_\tau^{x,\varphi} \geq \xi_\tau$ a.s. Let $\nu \in \mathcal{V}$. By taking the $\mathcal{E}^\nu$-evaluation in the above inequality, using the monotonicity of $\mathcal{E}^\nu$ and the $\mathcal{E}^\nu$-martingale property of the wealth process $V^{x,\varphi}$, we obtain $x = \mathcal{E}^\nu_{0,\tau}(V_\tau^{x,\varphi}) \geq \mathcal{E}^\nu_{0,\tau}(\xi_\tau)$. By arbitrariness of $\tau \in \mathcal{T}$ and $\nu \in \mathcal{V}$, we get $x \geq \sup_{(\tau,\nu)\in\mathcal{T}\times\mathcal{V}} \mathcal{E}^\nu_{0,\tau}(\xi_\tau) = Y_0$, which holds for all $x \in \mathcal{H}$. By taking the infimum over $x \in \mathcal{H}$, we obtain $u_0 \geq Y_0$. We derive that $u_0 = Y_0$, which completes the proof. □

Proof of Theorem 3.27: The proof follows from the previous theorem 7.12 and from Remark 4.9. Indeed, under an additional integrability condition $\varphi \in \cap_{\nu \in \mathcal{V}} \mathbb{H}^2_{Q^\nu}$ on the process φ from Assumption (7.1), by Remark 4.9, the above dual representation of the *superhedging price* can be written in terms of the f-martingale probability measures (characterized in Proposition 3.11), that is

$$u_0 = \sup_{(\tau,\nu) \in \mathcal{T} \times \mathcal{V}} \mathcal{E}^f_{Q^\nu,0,\tau}(\xi_\tau),$$

which ends the proof of Theorem 3.27.

Remark 7.14. *From a financial point of view, the process (h_t) can be interpreted as the cumulative amount the seller withdraws from the hedging portfolio up to time t. More precisely, for each time t, the seller can withdraw the amount dh_t from his/her portfolio between t and $t + dt$. In particular, at time ϑ, the seller can withdraw the amount Δh_ϑ from his/her portfolio, which, by equation* (5.5), *is equal to*

$$\Delta h_\vartheta = \beta_\vartheta \sigma_\vartheta^{-1} Z_\vartheta - \Delta Y_\vartheta \quad \text{a.s.}$$

The term $\beta_\vartheta \sigma_\vartheta^{-1} Z_\vartheta = \beta_\vartheta \varphi^*_\vartheta$ represents the jump at the default time ϑ of the amount invested in the risky asset S (which is equal to the jump of the value of the portfolio). Note that in this case, the value of the hedging portfolio, denoted by $(V_t^{Y_0,\varphi^*,h})$, taking into account these withdrawals, satisfies

$$dV_t^{Y_0,\varphi^*,h} = -f(t, V_t^{Y_0,\varphi^*,h}, \sigma_t \varphi^*_t)dt + \varphi^*_t(\sigma_t dW_t + \beta_t dM_t) - dh_t; \; V_0^{Y_0,\varphi^*,h} = Y_0.$$

We thus have $V_t^{Y_0,\varphi^*} = V_t^{Y_0,\varphi^*,0}$.

7.4 Characterization of the seller's superhedging price process as the minimal supersolution of a constrained reflected BSDE

Definition 7.15. *Let $\xi \in \mathbb{S}^2$. A process $Y' \in \mathbb{S}^2$ is said to be a* supersolution *of the* constrained reflected BSDE *with driver f and obstacle ξ if there exists a process $(Z', K', A', C') \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2 \times \mathbb{C}^2$ such that*

$$-\,dY'_t = f(t, Y'_t, Z'_t)dt + dA'_t + dC'_{t-} - Z'_t dW_t - K'_t dM_t; \tag{7.15}$$

$$Y'_T = \xi_T \; a.s. \quad \text{and} \quad Y'_t \geq \xi_t \text{ for all } t \in [0,T] \quad a.s.; \tag{7.16}$$

$$(Y'_\tau - \xi_\tau)(C'_\tau - C'_{\tau-}) = 0 \; a.s. \text{ for all } \tau \in \mathcal{T}_0; \tag{7.17}$$

$$A'_\cdot + \int_0^\cdot (K'_s - \beta_s \sigma_s^{-1} Z'_s)\lambda_s ds \in \mathcal{A}^2 \quad \text{and} \quad (K'_t - \beta_t \sigma_t^{-1} Z'_t)\lambda_t \leq 0, \; t \in [0,T], \; dP \otimes dt - \text{a.e.}; \tag{7.18}$$

Remark 7.16. *This definition can be extended to the case of a general driver g (which maybe depend also on k).*

Equation (7.17) is referred to as *Skorokhod condition* for the process C'.

Remark 7.17. *The process A' can be uniquely decomposed as the sum of two nondecreasing processes B' and $\hat{B}$ belonging to $\mathcal{A}^2$ with $dB'_t \perp d\hat{B}_t$,*[12] *such that B' satisfies the* Skorokhod

[12] in the sense of Definition 2.3 from [13]

condition, *that is*

$$\int_0^t (Y'_{s^-} - \xi_{s^-})dB'_s = 0 \quad \text{a.s.} \tag{7.19}$$

Note that the processes B' and $\hat{B}$ are given by $B'_t = \int_0^t \mathbf{1}_{\{Y'_{s^-}=\xi_{s^-}\}} dA'_s$ and $\hat{B}_t = \int_0^t \mathbf{1}_{\{Y'_{s^-}>\xi_{s^-}\}} dA'_s$ for all $t \in [0,T]$. It follows that $Y' \in \mathbb{S}^2$ is a supersolution *of the* constrained reflected BSDE *with driver f and obstacle ξ if and only if there exists a process $(Z', K', B', \hat{B}, C') \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2 \times \mathcal{A}^2 \times \mathbb{C}^2$ such that*

$$-\,dY'_t = f(t, Y'_t, Z'_t)dt + dB'_t + d\hat{B}_t + dC'_{t-} - Z'_t dW_t - K'_t dM_t; \tag{7.20}$$

$$Y'_T = \xi_T \ \textit{a.s.} \quad \textit{and} \quad Y'_t \geq \xi_t \textit{ for all } t \in [0,T] \ \textit{a.s.}; \tag{7.21}$$

$$(Y'_\tau - \xi_\tau)(C'_\tau - C'_{\tau-}) = 0 \ \textit{a.s. for all } \tau \in \mathcal{T}_0; \tag{7.22}$$

$$\int_0^t (Y'_{s^-} - \xi_{s^-})dB'_s = 0 \ \textit{a.s.} \quad \textit{and} \quad d\hat{B}_t \perp dB'_t, \tag{7.23}$$

and such that the constraints (7.18) *hold, with A' replaced by $B' + \hat{B}$.*
In the particular case when $\hat{B} = 0$, since B' satisfies the Skorokhod condition*, the process (Y', Z', K', B', C') is thus a* solution [13] *of the reflected BSDE (with irregular obstacle (ξ_t)), here with the additional constraints* (7.18)*. Thus, when passing from the notion of a* solution *of the reflected BSDE to the notion of a* supersolution *of the reflected BSDE, there appears an additional nondecreasing predictable process $\hat{B}$, which increases only when $Y'_{t^-} > \xi_{t^-}$.*

Theorem 7.18. *The seller's price process (Y_t) is a supersolution of the* constrained reflected BSDE *associated with driver f and obstacle ξ from Definition 7.15, that is, there exists a unique process $(Z, K, A, C) \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2 \times \mathbb{C}^2$ such that (Y, Z, K, A, C) satisfies Definition 7.15. Moreover, it is the minimal one, that is, if (Y'_t) is another supersolution, then $Y'_t \geq Y_t$ for all $t \in [0,T]$ a.s.*

Moreover, the portfolio strategy $\varphi^* := \sigma^{-1} Z$ is a superhedging strategy for the seller, that is, $\varphi^* \in \mathcal{A}(u_0)$.

Remark 7.19. *Suppose here that there is no default in the market. In this case, the filtration $\mathbb{G}$ is the one associated with the Brownian motion W, and in the dynamics of the price process (S_t) and of the wealth process (V_t), $M = 0$ and $\beta = 0$. Hence, the market is complete, and we have $\mathcal{V} = \{0\}$. From this observation, we derive that for each $S \in \mathcal{T}$, $Y_S = Y^0_S = ess\sup_{\tau \in \mathcal{T}_S} \mathcal{E}^0_{S,\tau}(\xi_\tau)$ a.s. By Theorem 6.7 in [26], (Y_t) is thus the* solution *of the reflected BSDE associated with driver f and irregular obstacle (ξ_t). In other words, there exists $(Z, K, B, C) \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2 \times \mathbb{C}^2$ such that equations* (7.20) *to* (7.23) *hold with $\hat{B} = 0$.*

Proof. Since Y is the value process, we have $Y_T = \xi_T$ a.s. and $Y_t \geq \xi_t$ for all $t \in [0,T]$ a.s. Moreover, by Corollary 7.5, the value process Y is a strong $\mathcal{E}^\nu$-supermartingale for all $\nu \in \mathcal{V}$. Hence, by Proposition 5.1, there exists a unique process $(Z, K, A, C) \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2 \times \mathbb{C}^2$ such

[13] in the sense from Definition 2.3 in [26], which, in the case of a right-continuous obstacle, corresponds to the well-known notion of a solution of a reflected BSDE)

that equation (5.1) and the conditions (5.2) hold. We now show that the process C satisfies the *Skorokhod condition* (7.17). Let $\tau \in \mathcal{T}_0$. By Theorem 7.10 (iii), we have $Y_\tau = Y_{\tau+} \vee \xi_\tau$ a.s.. Hence, $\Delta_+ Y_\tau = \mathbf{1}_{\{Y_\tau = \xi_\tau\}} \Delta_+ Y_\tau$ a.s. On the other hand, since (Y, Z, K, A, C) satisfies equation (5.1), we have $\Delta C_\tau = -\Delta_+ Y_\tau$ a.s. We conclude that $\Delta C_\tau = \mathbf{1}_{\{Y_\tau = \xi_\tau\}} \Delta C_\tau$ a.s. Hence, the Skorokhod condition (7.17) is satisfied.

It remains to show that (Y_t) is the *minimal* supersolution of the constrained reflected BSDE from Definition 7.15. Let Y' be another supersolution of this constrained reflected BSDE and let (Z', K', A', C') be the associated process (from the definition of a supersolution). We have $Y'_t \geq \xi_t$ for all $t \in [0, T]$ a.s. Let now $\nu \in \mathcal{V}$. Let A'^{ν} be the process defined by

$$A'^{\nu}_t := A'_t - \int_0^t (K'_s - \beta_s \sigma_s^{-1} Z'_s) \nu_s \lambda_s ds, \quad 0 \leq t \leq T.$$

Since $\nu \in \mathcal{V}$, we have $\nu_t + 1 > 0$ $dP \otimes dt$-a.s. This together with the second condition from (7.18) imply that $(K'_t - \beta_t \sigma_t^{-1} Z'_t) \lambda_t (1 + \nu_t) \leq 0$ $dP \otimes dt$-a.s. Then, using the first condition from (7.18) (and the definition of A'^{ν}), we obtain that the process A'^{ν} is nondecreasing. On the other hand, since (Y', Z', K', A', C') satisfies the dynamics from Definition 7.15, we have

$$-dY'_t = \left(f(t, Y'_t, Z'_t) + (K'_t - \beta_t \sigma_t^{-1} Z'_t) \nu_t \lambda_t \right) dt + dA'^{\nu}_t + dC'_{t-} - Z'_t dW_t - K'_t dM_t.$$

Hence, by the $\mathcal{E}^g$-Mertens decomposition of strong $\mathcal{E}^g$-supermartingales (recalled in Theorem 9.1 of the Appendix) applied to the driver $g := f^{\nu}$, we derive that the process Y' is a strong $\mathcal{E}^{\nu}$- supermartingale. Since this holds for all $\nu \in \mathcal{V}$, we derive from Corollary 7.5 that $Y'_t \geq Y_t$, for all $t \in [0, T]$ a.s. $\square$

Remark 7.20. *This result can be extended to any λ-admissible driver (depending also on k).*

Definition 7.21. *Let $\xi \in \mathbb{S}^2$. A process $Y' \in \mathbb{S}^2$ is called a* supersolution *of the* optional reflected BSDE *associated with driver f and obstacle ξ if there exist $Z' \in \mathbb{H}^2$, $C' \in \mathbb{C}^2$ and a nondecreasing optional RCLL process h', with $h'_0 = 0$ and $E[(h'_T)^2] < \infty$ such that*

$$\begin{aligned}
&-dY'_t = f(t, Y'_t, Z'_t) dt - Z'_t \sigma_t^{-1} (\sigma_t dW_t + \beta_t dM_t) + dC'_{t-} + dh'_t; \\
&Y'_T = \xi_T \quad \text{and} \quad Y'_t \geq \xi_t \textit{ for all } t \in [0, T] \quad \textit{a.s.}; \\
&(Y'_\tau - \xi_\tau)(C'_\tau - C'_{\tau-}) = 0 \textit{ a.s. for all } \tau \in \mathcal{T}_0.
\end{aligned}$$

Remark 7.22. *We call the above equation an* optional *reflected BSDE because the associated non decreasing right-continuous process is optional but not necessarily predictable contrary the reflected BSDEs considered in the literature.*

Note also that when the obstacle ξ is right-continuous, the purely discontinuous non decreasing process C' (corresponding to the right-jumps of Y') is equal to 0.

From Theorem 5.4 together with Proposition 5.6, we derive the following result:

Theorem 7.23. *The seller's superhedging price (Y_t) of the American option is a supersolution of the* optional reflected BSDE *from Definition 7.21. Moreover, it is the minimal one, that is, if (Y'_t) is another supersolution, then $Y'_t \geq Y_t$ for all $t \in [0, T]$ a.s.*

8 Buyer's superhedging price for the American option: proof of the duality result

We define $\tilde{f}(t,\omega,y,z) := -f(t,\omega,-y,-z)$.

Let $\nu \in \mathcal{V}$. We denote by $\mathcal{E}^{\tilde{f}^\nu}$ or $\tilde{\mathcal{E}}^\nu$ the nonlinear conditional expectation associated with the λ-admissible driver $\tilde{f}^\nu(t,y,z,k) := \tilde{f}(t,y,z) + \nu_t\lambda_t(k - \beta_t\sigma_t^{-1}z)$. Hence, for each $T' \le T$ and each $\eta \in L^2(\mathcal{G}_{T'})$, we have $\tilde{\mathcal{E}}^\nu_{\cdot,T'}(\eta) = \tilde{X}^\nu$ a.s., where $(\tilde{X}^\nu, \tilde{Z}^\nu, \tilde{K}^\nu)$ be the unique solution in $S^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ of the BSDE associated with driver $\tilde{f}^\nu$, terminal time T' and terminal condition η.

Remark 8.1. *Let $\nu \in \mathcal{V}$ and $T' \le T$. Note that for all $\eta \in L^2(\mathcal{G}_{T'})$, we have $\tilde{\mathcal{E}}^\nu_{\cdot,T'}(\eta) = -\mathcal{E}^\nu_{\cdot,T'}(-\eta)$, since $\tilde{f}^\nu(t,y,z,k) = -f^\nu(t,-y,-z,-k)$.*

Let $\eta \in L^2(\mathcal{G}_{T'}) \cap L^2_{Q^\nu}(\mathcal{G}_{T'})$. By Remark 4.9, $\mathcal{E}^\nu_{\cdot,T'}(\eta) = \mathcal{E}^f_{Q^\nu,\cdot,T'}(\eta)$. We thus have $\tilde{\mathcal{E}}^\nu_{\cdot,T'}(\eta) = -\mathcal{E}^f_{Q^\nu,\cdot,T'}(-\eta)$.

For each $S \in \mathcal{T}$, we define the $\mathcal{F}_S$-measurable random variable $\tilde{Y}(S)$ as follows:

$$\tilde{Y}(S) := ess \inf_{\nu\in\mathcal{V}_S} ess \sup_{\tau\in\mathcal{T}_S} \tilde{\mathcal{E}}^\nu_{S,\tau}(\xi_\tau) \quad \text{a.s.} \tag{8.1}$$

8.1 First properties of the value family $\tilde{Y}$

Let us first show that $E[ess \sup_{\tau\in\mathcal{T}} \tilde{Y}^2(\tau)] < \infty$.

As $0 \in \mathcal{V}$, we have $\tilde{Y}(S) \ge ess \sup_{\tau\in\mathcal{T}_S} \mathcal{E}^0_{S,\tau}(\xi_\tau) = \tilde{Y}^0_S$ a.s., where $(\tilde{Y}^0_t)$ is the first coordinate of the solution of the reflected BSDE associated with driver $\tilde{f}$ and lower obstacle (ξ_t). Now, since $|\xi_t| \le V^{x,\varphi}_t$, $0 \le t \le T$ a.s., we get that for all $S \in \mathcal{T}$, $\tau \in \mathcal{T}_S$ and $\nu \in \mathcal{V}$, $\tilde{\mathcal{E}}^\nu_{S,\tau}(\xi_\tau) = -\mathcal{E}^\nu_{S,\tau}(-\xi_\tau) \ge -\mathcal{E}^\nu_{S,\tau}(|\xi_\tau|) \ge -\mathcal{E}^\nu_{S,\tau}(V^{x,\varphi}_\tau) = -V^{x,\varphi}_S$ a.s. Hence, taking the essential supremum over $\tau \in \mathcal{T}_S$ and then the essential infimum over $\nu \in \mathcal{V}$ in this inequality, we obtain $\tilde{Y}(S) \ge -V^{x,\varphi}_S$ a.s.
Since $\tilde{Y}^0 \in S^2$ and $V^{x,\varphi} \in S^2$, it follows that $E[ess \sup_{S\in\mathcal{T}} \tilde{Y}(S)^2] < +\infty$.

Using the characterization of the solution of a reflected BSDE with lower obstacle in terms of an optimal stopping problem with g-expectations (see Theorem 4.2 in [24] when (ξ_t) is right-u.s.c. payoff), we can rewrite the value function of our problem as follows

$$\tilde{Y}(S) = ess \inf_{\nu\in\mathcal{V}_S} \tilde{Y}^\nu_S = ess \inf_{\nu\in\mathcal{V}} \tilde{Y}^\nu_S, \tag{8.2}$$

where $\tilde{Y}^\nu$ is the solution of the reflected BSDE associated with driver $\tilde{f}^\nu$, obstacle $(\xi_t)_{0\le t<T}$ and terminal condition ξ_T.

Proposition 8.2. (Minimizing sequence) *Let $S \in \mathcal{T}$. There exists a sequence of controls $(\nu^n)_{n\in\mathbb{N}}$ with $\nu^n \in \mathcal{V}_S$, for all n, such that the sequence $(\tilde{Y}^{\nu^n}_S)_{n\in\mathbb{N}}$ is non-increasing and satisfies:*

$$\tilde{Y}(S) \quad = \lim_{n\to\infty} \downarrow \tilde{Y}^{\nu^n}_S \quad \text{a.s.} \tag{8.3}$$

Proof. Same proof as for Y. We show that the set $\{\tilde{Y}_S^\nu, \nu \in \mathcal{V}_S\}$ is stable under pairwise maximization. The result of the proposition follows by a classical result on essential suprema/infima (cf. Neveu (1975)). $\square$

Proposition 8.3. (Aggregation) *Let $(\xi_t) \in \mathbb{S}^2$ (without any regularity assumption). There exists an r.u.s.c. process $(\tilde{Y}_t) \in \mathbb{S}^2$ which aggregates the value family $(\tilde{Y}(S))$ of the problem of control and stopping* (8.1).

The proof of the proposition uses the following

Lemma 8.4. *For all $S \in \mathcal{T}_0$,*

$$\mathbb{E}[\tilde{Y}(S)] = \inf_{\nu \in \mathcal{V}} \mathbb{E}[\tilde{Y}_S^\nu].$$

Proof. From the representation (8.2), we deduce $\mathbb{E}[\tilde{Y}(S)] = \mathbb{E}[ess \inf_{\nu \in \mathcal{V}} \tilde{Y}_S^\nu] \leq \inf_{\nu \in \mathcal{V}} \mathbb{E}[Y_S^\nu]$. We now show the converse inequality. By Proposition 8.2, there exists a sequence of controls $(\nu_n) = (\nu_n(S))$ in $\mathcal{V}_S^n$ such that $\tilde{Y}(S) = \lim_{n\to\infty} \downarrow \tilde{Y}_S^{\nu_n}$. We thus have $\mathbb{E}[\tilde{Y}(S)] = \mathbb{E}[\lim_{n\to\infty} \downarrow \tilde{Y}_S^{\nu_n}] = \lim_{n\to\infty} \downarrow \mathbb{E}[\tilde{Y}_S^{\nu_n}]$, where we have used dominated convergence to exchange limit and expectation. For all n, we have $\mathbb{E}[\tilde{Y}_S^{\nu_n}] \geq \inf_{\nu \in \mathcal{V}} \mathbb{E}[\tilde{Y}_S^\nu]$.We conclude that $\mathbb{E}[\tilde{Y}(S)] \geq \inf_{\nu \in \mathcal{V}} \mathbb{E}[\tilde{Y}_S^\nu]$. The proposition is thus proved. $\square$

We now prove Proposition 8.3.

Proof. To prove the result, we first show that the family $(\tilde{Y}(S))$ is right-uppersemicontinuous along stopping times in expectation. Let $S \in \mathcal{T}_0$, let (S_n) be a non-increasing sequence of stopping times in $\mathcal{T}_S$ with $\lim \downarrow S_n = S$ a.s. By the previous Lemma 8.4, we have $\mathbb{E}[\tilde{Y}(S_n)] = \inf_{\nu \in \mathcal{V}} \mathbb{E}[\tilde{Y}_{S_n}^\nu]$, for all $n \in \mathbb{N}$. Hence, $\limsup_{n\to\infty} \mathbb{E}[\tilde{Y}(S_n)] = \limsup_{n\to\infty} \inf_{\nu \in \mathcal{V}} \mathbb{E}[\tilde{Y}_{S_n}^\nu] \leq \inf_{\nu \in \mathcal{V}} \limsup_{n\to\infty} \mathbb{E}[\tilde{Y}_{S_n}^\nu] \leq \inf_{\nu \in \mathcal{V}} \mathbb{E}[\limsup_{n\to\infty} \tilde{Y}_{S_n}^\nu]$, where we have used Fatou's lemma to obtain the last inequality. Now, for all $\nu \in \mathcal{V}$, the process $(\tilde{Y}_t^\nu)$ right-uppersemicontinuous along stopping times, so $\limsup_{n\to\infty} \tilde{Y}_{S_n}^\nu \leq \tilde{Y}_S^\nu$. Using this and the above computations, we get $\limsup_{n\to\infty} \mathbb{E}[\tilde{Y}(S_n)] \leq \inf_{\nu \in \mathcal{V}} \mathbb{E}[\limsup_{n\to\infty} \tilde{Y}_{S_n}^\nu] \leq \inf_{\nu \in \mathcal{V}} \mathbb{E}[\tilde{Y}_S^\nu] = \mathbb{E}[\tilde{Y}(S)]$, where the (last) equality is due to Lemma 8.4. We conclude that the family $(\tilde{Y}(S))$ is right-uppersemicontinuous along stopping times in expectation. Hence, the family $(\tilde{Y}(S))$ is right-uppersemicontinuous along stopping times (cf. Theorem 12 in [10]). By Corollary 11 in [10], there exists a unique r.u.s.c. optional process $(\tilde{Y}_t)$ which aggregates the family. The process $(\tilde{Y}_t)$ is in $\mathbb{S}^2$, due to the fact that $E[ess \sup_{S \in \mathcal{T}} \tilde{Y}(S)^2] < +\infty$.

$\square$

Remark 8.5. *Due to the above aggregation result (Proposition 8.3), we can replace $\tilde{Y}(S)$ by $\tilde{Y}_S$ in the representation* (8.2) *and in Proposition 8.2.*

8.2 Proof of the dual representation for the buyer's superhedging price

We now define the *backward semigroup* of operators $\mathbf{Y}^{g,\xi} = \left(\mathbf{Y}^{g,\xi}_{t,T'}\right)_{0 \leq t \leq T' \leq T}$ associated with a reflected BSDE with driver g and obstacle ξ (see e.g. [4] and [14]). Recall that this notion of stochastic backward semigroup was first introduced by Peng [**?**] and applied to study the dynamic programming principle for stochastic control problems.

Let g be a λ-*admissible* driver. Let $(\xi_t) \in \mathbb{S}^2$.
For each $T' \in [0,T]$ and each $\eta \in L^2(\mathcal{F}_{T'})$, we define

$$\mathbf{Y}^{g,\xi}_{t,T'}(\eta) := Y_t, \quad 0 \leq t \leq T', \tag{8.4}$$

where $(Y_t)_{0\leq t\leq T'}$ corresponds to the first component of the solution of the reflected BSDE associated with terminal time T', driver g and (lower) obstacle $(\xi_t \mathbf{1}_{t<T'} + \eta \mathbf{1}_{t=T'})$. Note that (Y_t) can be extended to the whole interval $[0,T]$ by setting $Y_t = \eta$ for all $t \in [T',T]$. [14]

More generally, for each stopping time $\theta \in \mathcal{T}$ and each $\eta \in L^2(\mathcal{F}_\theta)$, we define $\mathbf{Y}^{g,\xi}_{\cdot,\theta}(\eta) := Y_\cdot$, where $Y_\cdot$ is the first component of the solution of the reflected BSDE associated with terminal time T, terminal condition η, driver $g\mathbf{1}_{t\leq\theta}$, and obstacle $(\xi_t \mathbf{1}_{t<\theta} + \eta \mathbf{1}_{t\geq\theta})$.

For each $\nu \in \mathcal{V}$, we consider the *backward semigroup* of operators $\mathbf{Y}^{\tilde{f}^\nu,\xi} = \left(\mathbf{Y}^{\tilde{f}^\nu,\xi}_{t,T'}\right)$. To abbreviate the notation, we denote it by $\mathbf{Y}^{\nu,\xi} = \left(\mathbf{Y}^{\nu,\xi}_{t,T'}\right)$.
Note that $\tilde{Y}^\nu_t = \mathbf{Y}^{\nu,\xi}_{\mathbf{t},\mathbf{T}}(\xi_{\mathbf{T}})$, for all $t \in [0,T]$, a.s.

Proposition 8.6. (Dynamic Programming Principle) *Suppose that the payoff process (ξ_t) is right-uppersemicontinuous. The value process $(\tilde{Y}_t)$ satisfies the following Dynamic Programming Principle: for all S, S' in $\mathcal{T}_0$ such that $S \leq S'$ a.s., we have*

$$\tilde{Y}_S = ess \inf_{\nu \in \mathcal{V}_S} \mathbf{Y}^{\nu,\xi}_{S,S'}(\tilde{Y}_{S'}) \;\; a.s. \tag{8.5}$$

Proof. Let $S, S' \in \mathcal{T}$ be such that $S \leq S'$ a.s. By Proposition 8.2, there exists a sequence of controls $(\nu^n)_{n\in\mathbb{N}}$, with ν_n in $\mathcal{V}_{S'}$ for all n, such that $\tilde{Y}_{S'} \;\; = \lim_{n\to\infty} \downarrow \tilde{Y}^{\nu^n}_{S'}$ a.s. Let $\nu \in \mathcal{V}_S$. By the continuity property of Reflected BSDEs with respect to the terminal condition (cf. Lemma 9.5), we have $\mathbf{Y}^{\nu,\xi}_{S,S'}(\tilde{Y}_{S'}) = \mathbf{Y}^{\nu,\xi}_{S,S'}(\lim_{n\to\infty} \tilde{Y}^{\nu^n}_{S'}) = \lim_{n\to\infty} \mathbf{Y}^{\nu,\xi}_{S,S'}(\tilde{Y}^{\nu_n}_{S'})$ a.s. For each n, we set $\bar{\nu}^n_t := \nu_t \mathbf{1}_{]S,S']}(t) + \nu^n_t \mathbf{1}_{]S',T]}(t)$. We have $\tilde{f}^{\bar{\nu}^n} = \tilde{f}^\nu \mathbf{1}_{]S,S']} + \tilde{f}^{\nu^n} \mathbf{1}_{]S',T]}$ and $\tilde{Y}^{\nu_n}_{S'} = \tilde{Y}^{\bar{\nu}_n}_{S'}$. We deduce

$$\mathbf{Y}^{\nu,\xi}_{S,S'}(\tilde{Y}^{\nu_n}_{S'}) = \mathbf{Y}^{\bar{\nu}^n,\xi}_{S,S'}(\tilde{Y}^{\bar{\nu}_n}_{S'}) = \tilde{Y}^{\bar{\nu}_n}_S \text{ a.s.},$$

where the last equality follows the flow (or semi-group) property of reflected BSDEs. We thus get
$\mathbf{Y}^{\nu,\xi}_{S,S'}(\tilde{Y}_{S'}) = \lim\limits_{n\to\infty} \mathbf{Y}^{\nu,\xi}_{S,S'}(\tilde{Y}^{\nu_n}_{S'}) \;=\; \lim\limits_{n\to\infty} \tilde{Y}^{\bar{\nu}_n}_S \geq \tilde{Y}_S$ a.s., where the (last) inequality follows from (8.2). As $\nu \in \mathcal{V}_S$ is arbitrary, we derive $ess \inf_{\nu\in\mathcal{V}_S} \mathbf{Y}^{\nu,\xi}_{S,S'}(\tilde{Y}_{S'}) \geq \tilde{Y}_S$ a.s.

We now prove the converse inequality. Let $\nu \in \mathcal{V}_S$. By the flow property of reflected BSDEs, we have $\tilde{Y}^\nu_S = \mathbf{Y}^{\nu,\xi}_{S,S'}(\tilde{Y}^\nu_{S'})$ a.s. On the other hand, $\tilde{Y}^\nu_{S'} \geq \tilde{Y}_{S'}$ a.s. (cf. property (8.2)). From this, by the comparison theorem for reflected BSDEs (cf.), we deduce $\mathbf{Y}^{\nu,\xi}_{S,S'}(\tilde{Y}^\nu_{S'}) \geq \mathbf{Y}^{\nu,\xi}_{S,S'}(\tilde{Y}_{S'})$ a.s. Hence, $\tilde{Y}^\nu_S = \mathbf{Y}^{\nu,\xi}_{S,S'}(\tilde{Y}^\nu_{S'}) \geq \mathbf{Y}^{\nu,\xi}_{S,S'}(\tilde{Y}_{S'})$ a.s. By taking the essential infimum over $\nu \in \mathcal{V}_S$, we get $ess \inf_{\nu\in\mathcal{V}_S} \tilde{Y}^\nu_S \geq ess \inf_{\nu\in\mathcal{V}_S} \mathbf{Y}^{\nu,\xi}_{S,S'}(\tilde{Y}_{S'})$ a.s. But, $\tilde{Y}_S = ess \inf_{\nu\in\mathcal{V}_S} \tilde{Y}^\nu_S$ a.s.

[14]Recall that, by the flow property for reflected BSDEs, the family of operators $\mathbf{Y}^{g,\xi} = \left(\mathbf{Y}^{g,\xi}_{t,T'}\right)_{0\leq t\leq T'\leq T}$ satisfies a *semi-group* property.

(cf. (8.2)). Hence, $\tilde{Y}_S \geq ess\inf_{\nu\in\mathcal{V}_S} \mathbf{Y}^{\nu,\xi}_{S,S'}(\tilde{Y}_{S'})$ a.s., which is the desired inequality. As both inequalities hold, we have the equality (8.5). The proof is complete. $\square$

Proposition 8.7. (The case of a right-continuous pay-off process (ξ_t)). *Let (ξ_t) be a process in $\mathbb{S}^2$ assumed to be* right-continuous. *The value process $(\tilde{Y}_t)$ of the problem of control and stopping* (8.1) *is* right-continuous.

Proof. We already know from Proposition 8.3 that the value process $(\tilde{Y}_t)$ is r.u.s.c. We now show that $(\tilde{Y}_t)$ is right-lowersemicontinuous. Let $S \in \mathcal{T}_0$, let (S_n) be a non-increasing sequence of stopping times in $\mathcal{T}_S$ with $\lim_{n\to+\infty} S_n = S$ a.s. and for all $n \in \mathbb{N}$, $S_n > S$ a.s. on $\{S < T\}$, and such that $\lim_{n\to+\infty} \tilde{Y}_{S_n}$ exists a.s. Since $0 \in \mathcal{V}_S$, by the dynamic programming principle, we have $\tilde{Y}_S \leq \mathbf{Y}^{0,\xi}_{S,S_n}(\tilde{Y}_{S_n})$ a.s. Hence, by the continuity property of Reflected BSDEs with respect to the pair terminal time-terminal condition[15] (cf. Lemma 9.5 or [14, Lemma A.6]), we thus get

$$\tilde{Y}_S \leq \lim_{n\to\infty} \mathbf{Y}^{0,\xi}_{S,S_n}(\tilde{Y}_{S_n}) = \mathbf{Y}^{0,\xi}_{S,S}(\lim_{n\to\infty} \tilde{Y}_{S_n}) = \lim_{n\to+\infty} \tilde{Y}_{S_n} \quad \text{a.s.}$$

By Lemma 5 of Dellacherie and Lenglart [10] [16], the process $\tilde{Y}$ is thus right-lowersemicontinuous. The proof is thus complete. $\square$

Lemma 8.8. *Let (ξ_t) be a process in $\mathbb{S}^2$. We define the following stopping times:*

$$\tilde{\tau} := \inf\{t \in [0,T] : \tilde{Y}_t = \xi_t\} \tag{8.6}$$

$$\textit{For } \varepsilon > 0, \qquad \tilde{\tau}_\varepsilon := \inf\{t \in [0,T] : \tilde{Y}_t \leq \xi_t + \varepsilon\} \tag{8.7}$$

We note that $\tilde{\tau}^\varepsilon \leq \tilde{\tau}$ a.s.

(i) *If (ξ_t) is right-uppersemicontinuous and also left-uppersemicontinuous along stopping times, then, for all $\nu \in \mathcal{V}$, the value process $(\tilde{Y}_t)$ is a strong $\tilde{\mathcal{E}}^\nu$-submartingale on $[0,\tilde{\tau}]$.*

(ii) *If (ξ_t) is only right-uppersemicontinuous, then then, for all $\epsilon > 0$, for all $\nu \in \mathcal{V}$, the value process $(\tilde{Y}_t)$ is a strong $\tilde{\mathcal{E}}^\nu$-submartingale on $[0,\tilde{\tau}_\varepsilon]$.*

Proof. We show (i). Let $\nu \in \mathcal{V}$. Let S,τ in $\mathcal{T}$ be such that $0 \leq S \leq \tau \leq \tilde{\tau}$ a.s. We show that $\tilde{\mathcal{E}}^\nu_{S,\tau}(\tilde{Y}_\tau) \geq \tilde{Y}_S$. By the representation (8.2) and Proposition 8.2, there exists a minimizing sequence for $\tilde{Y}_\tau$, that is, there exists $\nu^p := \nu^p(\tau) \in \mathcal{V}_\tau$ such that $\tilde{Y}_\tau = \lim_{p\to\infty} \downarrow \tilde{Y}^{\nu^p}_\tau$. Hence, $\tilde{\mathcal{E}}^\nu_{S,\tau}(\tilde{Y}_\tau) = \tilde{\mathcal{E}}^\nu_{S,\tau}(\lim_{p\to\infty} \tilde{Y}^{\nu^p}_\tau) = \lim_{p\to\infty} \tilde{\mathcal{E}}^\nu_{S,\tau}(\tilde{Y}^{\nu^p}_\tau)$, where we have used the continuity property of the non-linear expectation $\tilde{\mathcal{E}}^\nu(\cdot)$ with respect to terminal condition. For all

[15] We note that the condition of applicability of the continuity property in the case of a right-continuous obstacle ξ, namely the condition $\lim_{n\to\infty} \tilde{Y}_{S_n} \geq \xi_S$, is satisfied here: indeed, $\tilde{Y}_{S_n} \geq \xi_{S_n}$ a.s. for all n; hence, $\lim_{n\to\infty} \tilde{Y}_{S_n} \geq \lim_{n\to\infty} \xi_{S_n} = \xi_S$, where we have used the assumption of right-continuity of ξ for the last equality.

[16] The chronology Θ (in the vocabulary and notation of [10]) which we work with here is the chronology of all stopping times, that is, $\Theta = \mathcal{T}_0$; hence $[\Theta] = \Theta = \mathcal{T}_0$.

$p \in \mathbb{N}$, we set $\bar{\nu}_t^p := \nu_t^p \mathbb{I}_{\{t>\tau\}} + \nu_t \mathbb{I}_{\{t \leq \tau\}}$. We have $\bar{\nu}^p \in \mathcal{V}$. We thus get $\lim_{p\to\infty} \tilde{\mathcal{E}}^{\nu}_{S,\tau}(\tilde{Y}^{\nu^p}_\tau) = \lim_{p\to\infty} \tilde{\mathcal{E}}^{\bar{\nu}^p}_{S,\tau}(\tilde{Y}^{\bar{\nu}^p}_\tau) \geq ess \inf_{\mu\in\mathcal{V}} \tilde{\mathcal{E}}^{\mu}_{S,\tau}(\tilde{Y}^{\mu}_\tau)$. Putting together the above computations gives

$$\tilde{\mathcal{E}}^{\nu}_{S,\tau}(\tilde{Y}_\tau) \geq ess \inf_{\mu\in\mathcal{V}} \tilde{\mathcal{E}}^{\mu}_{S,\tau}(\tilde{Y}^{\mu}_\tau). \tag{8.8}$$

For all $\mu \in \mathcal{V}$, we set $\tau^\mu := \inf\{t \in [0,T] : \tilde{Y}^{\mu}_t = \xi_t\}$. We notice that, for all $\mu \in \mathcal{V}$, $\tilde{\tau} \leq \tau^\mu$ a.s.; this follows from the definitions of $\tilde{\tau}$ and τ^μ and from the fact that $\xi_t \leq \tilde{Y}_t \leq \tilde{Y}^{\mu}_t$ for all t a.s. By Lemma 4.2 in [24], for all $\mu \in \mathcal{V}$, the process $(\tilde{Y}^{\mu}_t)$ is a strong $\tilde{\mathcal{E}}^{\mu}$-martingale on $[0,\tau^\mu]$; hence, also a strong $\tilde{\mathcal{E}}^{\mu}$-martingale on $[0,\tilde{\tau}]$ (as $\tilde{\tau} \leq \tau^\mu$ a.s.). Hence, for all $\mu \in \mathcal{V}$, $\tilde{\mathcal{E}}^{\mu}_{S,\tau}(\tilde{Y}^{\mu}_\tau) = \tilde{Y}^{\mu}_S$ (recall that $0 \leq S \leq \tau \leq \tilde{\tau}$ a.s.) Using this and (8.8), we get $\tilde{\mathcal{E}}^{\nu}_{S,\tau}(\tilde{Y}_\tau) \geq ess \inf_{\mu\in\mathcal{V}} \tilde{Y}^{\mu}_S = \tilde{Y}_S$, where the (last) equality is due to the representation (8.2). Property (i) is this proved.
Let us show (ii). Let Let $\varepsilon > 0$. Let S, τ in $\mathcal{T}$ be such that $0 \leq S \leq \tau \leq \tilde{\tau}_\varepsilon$ a.s. By exactly the same arguments as in part (i), we get

$$\tilde{\mathcal{E}}^{\nu}_{S,\tau}(\tilde{Y}_\tau) \geq ess \inf_{\mu\in\mathcal{V}} \tilde{\mathcal{E}}^{\mu}_{S,\tau}(\tilde{Y}^{\mu}_\tau). \tag{8.9}$$

For all $\mu \in \mathcal{V}$, we set $\tau^\mu_\varepsilon := \inf\{t \in [0,T] : \tilde{Y}^{\mu}_t \leq \xi_t + \varepsilon\}$. We note that, for all $\mu \in \mathcal{V}$, $\tilde{\tau}_\varepsilon \leq \tau^\mu_\varepsilon$ a.s. By Lemma 4.1 in [24], for all $\mu \in \mathcal{V}$, the process $(\tilde{Y}^{\mu}_t)$ is a strong $\tilde{\mathcal{E}}^{\mu}$-martingale on $[0,\tau^\mu_\varepsilon]$; hence, also a strong $\tilde{\mathcal{E}}^{\mu}$-martingale on $[0,\tilde{\tau}_\varepsilon]$ (as $\tilde{\tau}_\varepsilon \leq \tau^\mu_\varepsilon$ a.s.). From this and (8.9), we conclude as in part (i). $\square$

We will now give a dual representation for the buyer's superhedging price $\tilde{u}_0$ in terms of the value (at time 0) of the non-linear problem of control and stopping studied above. We also give a super-hedge for the buyer. From this result, we will deduce the dual representation (in terms of the f-martingale probability measures) stated in Theorem 3.29.

Theorem 8.9 (*Buyer's superhedging price of the American option*)**.** *Let $(\xi_t) \in \mathbb{S}^2$. Suppose that (ξ_t) is right-continuous and left-uppersemicontinuous along stopping times. The* buyer's price $\tilde{u}_0$ *of the American option satisfies*

$$\tilde{u}_0 = \inf_{\nu\in\mathcal{V}} \sup_{\tau\in\mathcal{T}} \tilde{\mathcal{E}}^{\nu}_{0,\tau}(\xi_\tau). \tag{8.10}$$

Let $\tilde{\tau} = \inf\{t \geq 0 : \ \tilde{Y}_t = \xi_t\}$. There exists a portfolio strategy $\tilde{\varphi} \in \mathbb{H}^2$ such that $(\tilde{\tau}, \tilde{\varphi})$ is a super-hedge for the buyer, that is, such that $(\tilde{\tau}, \tilde{\varphi}) \in \mathcal{B}(\tilde{u}_0)$.

Proof. In order to prove the results of the theorem, it is sufficient to show that $\tilde{u}_0 = \tilde{Y}_0$ and that there exists $(\tilde{\tau}, \tilde{\varphi}) \in \mathcal{B}(\tilde{Y}_0)$.
Let $\mathcal{S}$ be the set of initial prices which allow the buyer to be "super-hedged", that is $\mathcal{S} = \{x \in \mathbb{R} : \exists(\tau,\varphi) \in \mathcal{B}(x)\}$. Note that $\tilde{u}_0 = \sup \mathcal{S}$.

Let us first show that $\tilde{Y}_0 \leq \tilde{u}_0$. To this aim, we prove that $\tilde{Y}_0 \in \mathcal{S}$, that is, there exists a portfolio strategy $\tilde{\varphi} \in \mathbb{H}^2$ such that

$$(\tilde{\tau}, \tilde{\varphi}) \in \mathcal{B}(\tilde{Y}_0). \tag{8.11}$$

By the first assertion of Lemma 8.8, the process $(\tilde{Y}_{t\wedge\tilde{\tau}})$ is an strong $\tilde{\mathcal{E}}^\nu$-submartingale for all $\nu \in \mathcal{V}$. This together with the first assertion from Remark 8.1 implies that $(-\tilde{Y}_{t\wedge\tilde{\tau}})$ is an strong $\mathcal{E}^\nu$-supermartingale for all $\nu \in \mathcal{V}$. Now, since ξ is right-continuous, by Proposition 8.7, we derive that $\tilde{Y}$ is right-continuous.

Hence, by the optional $\mathcal{E}^f$-decomposition of strong $\mathcal{E}^\nu$-supermartingale for each $\nu \in \mathcal{V}$ (cf. Theorem 5.4), there exists a unique pair $(\tilde{Z}, \tilde{C}) \in \mathbb{H}^2 \times \mathbb{C}^2$ and a unique nondecreasing optional RCLL process $\tilde{h}$, with $\tilde{h}_0 = 0$ and $E[\tilde{h}_T^2] < \infty$ such that

$$-\tilde{Y}_t = -\tilde{Y}_0 - \int_0^t f(s, -\tilde{Y}_s, \tilde{Z}_s)ds + \int_0^t \tilde{Z}_s\sigma_s^{-1}(\sigma_s dW_s + \beta_s dM_s) - \tilde{h}_t - \tilde{C}_{t^-},\ 0 \le t \le \tilde{\tau} \text{ a.s.} \tag{8.12}$$

Since ξ is right-continuous, by Proposition 8.7, we derive that $\tilde{Y}$ is right-continuous. Hence, the process $\tilde{C}$ in the above decomposition is equal to 0. We now consider the portfolio associated with the initial capital $-\tilde{Y}_0$ and the strategy

$$\tilde{\varphi} := \sigma^{-1}\tilde{Z}. \tag{8.13}$$

By (3.4)-(3.5), the value of the portfolio process $(V_t^{-\tilde{Y}_0,\tilde{\varphi}})$ satisfies:

$$V_t^{-\tilde{Y}_0,\tilde{\varphi}} = -\tilde{Y}_0 - \int_0^t f(s, V_s^{-\tilde{Y}_0,\tilde{\varphi}}, \tilde{Z}_s)ds + \int_0^t \tilde{Z}_s\sigma_s^{-1}(\sigma_s dW_s + \beta_s dM_s),\ 0 \le t \le T. \tag{8.14}$$

By (8.12) and (8.14) and the comparison result for forward differential equations, we get $-\tilde{Y}_t \le V_t^{-\tilde{Y}_0,\tilde{\varphi}}$, $0 \le t \le \tilde{\tau}$ a.s. We thus have $V_{\tilde{\tau}}^{-\tilde{Y}_0,\tilde{\varphi}} + \tilde{Y}_{\tilde{\tau}} \ge 0$ a.s.
Hence, by the definition of $\tilde{\tau}$ and the right-continuity of $\tilde{Y}$ and ξ, we get $\tilde{Y}_{\tilde{\tau}} = \xi_{\tilde{\tau}}$ a.s. We thus conclude that

$$V_{\tilde{\tau}}^{-\tilde{Y}_0,\tilde{\varphi}} + \xi_{\tilde{\tau}} \ge 0 \quad \text{a.s.},$$

which implies the desired property (8.11). We thus have $\tilde{Y}_0 \le \tilde{u}_0$.

Let us show the converse inequality.

Let $x \in \mathcal{S}$. By definition of $\mathcal{S}$, there exists $(\theta, \varphi) \in \mathcal{B}(x)$, that is, such that $V_\theta^{-x,\varphi} \ge -\xi_\theta$ a.s. Let $\nu \in \mathcal{V}$. By taking the $\mathcal{E}^\nu$-evaluation in the above inequality, using the monotonicity of $\mathcal{E}^\nu$ and the $\mathcal{E}^\nu$-martingale property of the process $V^{-x,\varphi}$, we derive that $-x = \mathcal{E}_{0,\theta}^\nu(V_\theta^{-x,\varphi}) \ge \mathcal{E}_{0,\theta}^\nu(-\xi_\theta) = -\tilde{\mathcal{E}}_{0,\theta}^\nu(\xi_\theta)$, where the last equality follows from the first assertion of Remark 8.1. We deduce $x \le \sup_{\tau\in\mathcal{T}} \tilde{\mathcal{E}}_{0,\tau}^\nu(\xi_\tau)$. Since $\nu \in \mathcal{V}$ is arbitrary, we get

$$x \le \inf_{\nu\in\mathcal{V}} \sup_{\tau\in\mathcal{T}} \tilde{\mathcal{E}}_{0,\tau}^\nu(\xi_\tau) = \tilde{Y}_0,$$

which holds for any $x \in \mathcal{S}$. By taking the supremum over $x \in \mathcal{S}$, we get $\tilde{u}_0 \le \tilde{Y}_0$. It follows that $\tilde{u}_0 = \tilde{Y}_0$. By (8.11), we get $(\tilde{\tau}, \tilde{\varphi}) \in \mathcal{B}(\tilde{u}_0)$, which completes the proof. □

Remark 8.10. *We emphasize that the superhedging portfolio strategy* $\tilde{\varphi}$ *is given by* (8.13) *via the optional decomposition* (8.12) *of* $\tilde{Y}$ *on* $[0, \tilde{\tau}]$.

Proof of Theorem 3.29: The proof follows from the previous theorem 8.9 and from Remark 8.1. Indeed, under the additional integrability condition $\varphi \in \cap_{\nu\in\mathcal{V}}\mathbb{H}^2_{Q^\nu}$ on the process φ from Assumption (7.1), by Remark 8.1, the above dual representation can be written in terms of the f-martingale probability measures, that is

$$\tilde{u}_0 = -\sup_{\nu\in\mathcal{V}}\inf_{\tau\in\mathcal{T}} \mathcal{E}^f_{Q^\nu,0,\tau}(-\xi_\tau), \tag{8.15}$$

which ends the proof of Theorem 3.29.

Remark 8.11. *The above dual representations* (8.10) *and* (8.15) *still hold when* (ξ_t) *is only right-continuous (without being left-uppersemicontinuous along stopping times). However, in this case, there does not necessarily exist a super-hedge for the buyer.*

9 Appendix

In this Appendix, we first recall the $\mathcal{E}^g$-Mertens decomposition of $\mathcal{E}^g$- supermartingales proved in [24], and then provide some useful results.

Theorem 9.1 ($\mathcal{E}^g$-Mertens decomposition of $\mathcal{E}^g$-supermartingales)**.** *Let* (Y_t) *be an optional process in* $\mathbb{S}^2$. *Then* (Y_t) *is a* $\mathcal{E}^g$*-submartingale if and only if there exists a non decreasing right continuous and predictable processes* A *in* $\mathcal{A}^2$*, a non decreasing adapted right continuous and purely discontinuous processes* C *in* $\mathbb{C}^2$ *and* $(Z,K) \in \mathbb{H}^2 \times \mathbb{H}^2_\nu$ *such that*

$$-\,dY_s = g(s, Y_s, Z_s, K_s)ds - Z_s dW_s - K_t dM_t + dA_s + dC_{s^-}. \tag{9.1}$$

Moreover, this decomposition is unique.

Remark 9.2. *Using the above decomposition, we deduce that a* $\mathcal{E}^g$*-supermartingale admits left and right limits.*

Lemma 9.3. *(Uniqueness of the canonical decomposition of a special optional semimartingale) Let* X *be an optional semimartingale with decomposition*[17]

$$X_t = X_0 + m_t - a_t - b_t, \text{ for all } t \in [0,T] \text{ a.s.} \tag{9.2}$$

with (m_t) *a (right-continuous) local martingale,* (a_t) *a* predictable *right-continuous process of finite variation, such that* $a_0 = 0$*,* (b_t) *a* predictable *left-continuous process of finite variation, purely discontinuous and such that* $b_{0-} = 0$*. Then, the decomposition* (9.2) *is unique and will be called* the canonical decomposition of a special optional semimartingale.

[17] An optional semimartingale with a decomposition of this from (with (a_t) and (b_t) *predictable* processes) can be seen as a generalisation of the notion of *special* semimartingale from the right-continuous to the general case.

Proof. Let $X_t = X_0 + m'_t - a'_t - b'_t$, for all $t \in [0,T]$ a.s., be (another) decomposition with (m'_t), (a'_t) and (b'_t) as in the lemma. From this decomposition, it follows that $X_{t+} - X_t = -(b'_{t+} - b'_t)$ for all t a.s. From (9.2), it follows that $X_{t+} - X_t = -(b_{t+} - b_t)$ for all t a.s. Hence, $b'_{t+} - b'_t = b_{t+} - b_t$ for all t a.s. As b and b' are purely discontinuous with the same initial value, we get $b'_t = b_t$, for all t a.s. and the uniqueness of b is proven. We now note that $(X_t + b_t)_t$ is a special right-continuous semimartingale (this follows from (9.2)). Hence, by Theorem 30, Chapter III in [40] the processes (m_t) and (a_t) are unique. □

Lemma 9.4. *Let h be a nondecreasing optional RCLL process h, with $h_0 = 0$ and $E[h_T^2] < \infty$. Then, h has at most one totally inaccessible jump and this jump is at ϑ. All the other jumps of h are predictable. Moreover, h can be uniquely decomposed as follows:*

$$h_t = B_t + \Delta h_\vartheta \mathbb{I}_{t \geq \vartheta} = B_t + \int_0^t \psi_s dN_s,$$

where B is a (predictable) process in $\mathcal{A}^2$ and ψ is a process in $\mathbb{H}^2_\lambda$ such that $\psi_\theta \geq 0$ a.s. on $\{\theta \leq T\}$.

Proof. As h is a square-integrable nondecreasing optional RCLL process, h is a square-integrable RCLL submartingale. So, by the classical Doob-Meyer decomposition, h can be uniquely decomposed as $h_t = a_t + m_t$, with (a_t) a (predictable) process in $\mathcal{A}^2$ and (m_t) a square-integrable martingale such that $m_0 = 0$. Now, by the martingale representation of $\mathbb{G}$-martingales and as $dM_s = dN_s - \lambda_s ds$, we get $m_t = \int_0^t \varphi_s dW_s - \int_0^t \psi_s \lambda_s ds + \int_0^t \psi_s dN_s$, where $\varphi \in \mathbb{H}^2$ and $\psi \in \mathbb{H}^2_\lambda$. Hence, $h_t = a_t + m_t = B_t + \int_0^t \psi_s dN_s = B_t + \psi_\vartheta \mathbb{I}_{t \geq \vartheta}$, where we have set $B_t := a_t + \int_0^t \varphi_s dW_s - \int_0^t \psi_s \lambda_s ds$. The process (B_t) is clearly predictable (as the sum of three predictable processes). The equality $h_t = B_t + \psi_\vartheta \mathbb{I}_{t \geq \vartheta}$, together with the predictability of B and the non-decreasingness of h, implies that $\Delta h_\vartheta = \psi_\vartheta \geq 0$ a.s. on $\{\theta \leq T\}$ and that B is non-decreasing. The proof is thus complete. □

We now show that the non-linear operator $\mathbf{Y}^{g,\xi}$ induced by the reflected BSDE with driver g and obstacle $(\xi_t)_{t<T}$, defined by (8.4), simply denoted by $\mathbf{Y}^g$, is continuous with respect to terminal condition. Moreover, for each $\theta \in \mathcal{T}_0$ and each $\eta \in L^2(\mathcal{G}_\theta)$, some additional assumptions on (ξ_t) and η, $\mathbf{Y}^g$ is continuous with respect to the pair terminal time-terminal condition at the point (θ, η).

Lemma 9.5. *Let g be a λ-admissible driver satisfying Assumption 2.3. Let $(\xi_t) \in \mathbb{S}^2$, supposed to be right-u.s.c. Let $(\theta^n)_{n\in\mathbb{N}}$ be a non increasing sequence of stopping times in $\mathcal{T}_0$, converging a.s. to θ. Let $(\eta^n)_{n\in\mathbb{N}}$ be a sequence of random variables such that $E[\sup_n (\eta^n)^2] < +\infty$, and for each n, η^n is $\mathcal{G}_{\theta^n}$-measurable. Assume that the sequence (η^n) converges a.s. to an $\mathcal{G}_\theta$-measurable random variable η.*

We assume the following condition: for all sequence $(\tau^n)_{n\in\mathbb{N}}$ of stopping times in $\mathcal{T}_\theta$, such that $\tau^n \to \theta$ a.s. as n tends to ∞, we have

$$\limsup_{n\to\infty} \xi_{\tau_n} \leq \eta \quad \text{a.s.} \tag{9.3}$$

Then, for each $S \in \mathcal{T}_0$, $\lim_{n\to+\infty} \mathbf{Y}^g_{S,\theta^n}(\eta^n) = \mathbf{Y}^g_{S,\theta}(\eta)$ a.s.

When for each n, $\theta_n = \theta$ a.s., the result still holds without any assumption on (ξ_t).

Proof. In the particular case when for each n, $\theta_n = \theta$ a.s., the result follows from the a priori estimates for reflected BSDEs with irregular obstacles (cf. Theorem 5.5 in [25]), which do not require any additional assumption on (ξ_t).
Let us now consider the general case. Using the same arguments as those used in the proof of Lemma A.6 in [14], we show that $\liminf_{n\to\infty} \mathbf{Y}^g_{\theta,\theta_n}(\eta^n) \geq \eta$ a.s. It thus remains to show that $\limsup_{n\to\infty} \mathbf{Y}^g_{\theta,\theta_n}(\eta^n) \leq \eta$ a.s. Let $\varepsilon > 0$. By Theorem 4.2 in [24] (which holds since (ξ_t) is right-u.s.c.), there exists $\tau_n^\varepsilon \in \mathcal{T}_\theta$ such that

$$\mathbf{Y}^g_{\theta,\theta_n}(\eta^n) \leq \mathcal{E}_{\theta,\tau_n^\varepsilon \wedge \theta_n}(\xi_{\tau_n^\varepsilon} \mathbf{1}_{\tau_n^\varepsilon < \theta_n} + \eta^n \mathbf{1}_{\tau_n^\varepsilon \geq \theta_n}) + \varepsilon \quad \text{a.s.} \tag{9.4}$$

Now, by condition (9.3), we have $\limsup_{n\to\infty} \xi_{\tau_n^\varepsilon \wedge \theta_n} \leq \eta$ a.s., which implies that $\limsup_{n\to\infty}(\xi_{\tau_n^\varepsilon} \mathbf{1}_{\tau_n^\varepsilon < \theta_n} + \eta^n \mathbf{1}_{\tau_n^\varepsilon \geq \theta_n}) \leq \eta$ a.s. Hence, using the Fatou property for BSDEs both with respect to the pair terminal time-terminal condition (cf. e.g. Lemma A.5 in [14]) together with (9.4), we derive that $\limsup_{n\to\infty} \mathbf{Y}^g_{\theta,\theta_n}(\eta^n) \leq \eta + \varepsilon$ a.s. The desired result follows. □

Remark 9.6. *When the obstacle (ξ_t) is right-continuous, the condition* (9.3) *reduces to $\xi_\theta \leq \eta$ a.s. In this case, we thus recover the continuity result shown in [14] (cf. [14, Lemma A.6]).*

A result on reflected BSDEs with a non positive jump at the default time ϑ:
Let $\mathcal{V}$ be the set of bounded predictable processes ν such that $\nu_t \geq 0$ $dP \otimes dt$-a.e.
Let g be a λ-admissible driver and let (δ_t) be a bounded predictable process.

For each $\nu \in \mathcal{V}$, we define

$$g^\nu(\omega, t, y, z, k) := g(\omega, t, y, z, k) + \nu_t(\omega)\lambda_t(\omega)\big(k - \delta_t(\omega) z\big)$$

Note that g^ν is a λ-admissible driver. For each $S \in \mathcal{T}$, the value $Y(S)$ at time S is defined by

$$Y(S) := ess \sup_{(\tau,\nu)\in\mathcal{T}_S\times\mathcal{V}} \mathcal{E}^\nu_{S,\tau}(\xi_\tau), \tag{9.5}$$

where $\mathcal{E}^\nu = \mathcal{E}^{g^\nu}$. By similar arguments as in the previous case (cf. the proof of Corollary 7.5), there exists an r.u.s.c. process $(Y_t) \in \mathbb{S}^2$ which aggregates the value family $(Y(S))$, which is a strong $\mathcal{E}^\nu$-supermartingale for all $\nu \in \mathcal{V}$ and $Y_t \geq \xi_t$, for all $t \in [0, T]$, a.s. Moreover, the process (Y_t) is the smallest process in $\mathbb{S}^2$ satisfying these properties.

Now, by similar arguments as those used in the proof of Proposition 5.1, it can be shown that

Proposition 9.7. *Let $(X_t) \in \mathbb{S}^2$. If the process (X_t) is a strong $\mathcal{E}^\nu$-supermartingale for all $\nu \in \mathcal{V}$, then there exists a unique process $(Z, K, A, C) \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2 \times \mathbb{C}^2$ such that*

$$-dX_t = g(t, X_t, Z_t, K_t)dt - Z_t dW_t - K_t dM_t + dA_t + dC_{t-} \tag{9.6}$$

and

$$(K_t - \delta_t Z_t)\lambda_t \leq 0, \; t \in [0,T], \; dP \otimes dt - \text{a.e.} \tag{9.7}$$

Moreover, the converse statement holds.

Note that when $\delta = 0$, the constraint (9.7) means that the jump of the process (X_t) at the default time ϑ is non positive.

Remark 9.8. *The constraint* (9.7) *is equivalent to* $K_\vartheta \leq \delta_\vartheta Z_\vartheta$ *a.s. Note that this constraint corresponds to the second constraint from* (5.2)*. There is here only one constraint* (9.7) *while in the previous case, we had two constraints (see* (5.2)*). This comes from the fact that here* $\mathcal{V}$ *is the set of bounded predictable processes* ν *with* $\nu_t \geq 0$ $dP \otimes dt$*-a.e. (while in the previous case, we had* $\nu_t > -1$ $dP \otimes dt$*-a.e.).*

By similar arguments as those used in the proof of Theorem 7.18, it can be shown that the value process (Y_t) is a supersolution of the *constrained reflected BSDE* from Definition 7.15 with f replaced by g and the constraints (5.2) replaced by the constraint (9.7). We thus have the following result.

Proposition 9.9. *There exists a unique process* $(Z, K, A, C) \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2 \times \mathbb{C}^2$ *such that*

$$\begin{aligned}
&- dY_t = g(t, Y_t, Z_t, K_t)dt + dA_t + dC_{t-} - Z_t dW_t - K_t dM_t; && (9.8)\\
&Y_T = \xi_T \text{ a.s.} \quad \text{and} \quad Y_t \geq \xi_t \text{ for all } t \in [0,T] \quad \text{a.s.};\\
&(Y_\tau - \xi_\tau)(C_\tau - C_{\tau-}) = 0 \text{ a.s. for all } \tau \in \mathcal{T}_0;\\
&(K_t - \delta_t Z_t)\lambda_t \leq 0, \; t \in [0,T], \; dP \otimes dt - \text{a.e.} && (9.9)
\end{aligned}$$

In other words, the value process (Y_t) *is a supersolution of the above* constrained reflected BSDE. *Moreover, it is the minimal one, that is, if* (Y'_t) *is another supersolution, then* $Y'_t \geq Y_t$ *for all* $t \in [0,T]$ *a.s.*

Note that when $\delta = 0$ and the obstacle is right-continuous, our result gives the existence of a minimal supersolution of the reflected BSDE with driver g, obstacle ξ and with non positive jumps, which correponds to a result shown in [5] by using a penalization approach. Moreover, our result provides a dual representation (with non linear expectation) of this minimal supersolution.

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