TOTAL VARIATION DISTANCE ESTIMATES VIA L^2-NORM FOR POLYNOMIALS IN LOG-CONCAVE RANDOM VECTORS

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Abstract. The paper provides an estimate of the total variation distance between distributions of polynomials defined on a space equipped with a logarithmically concave measure in terms of the L^2-distance between these polynomials.

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1. Introduction

Davydov and Martynova [12] formulated the following interesting property of polynomials on a space with a Gaussian measure.

Theorem A. *Let $d \in \mathbb{N}$ and let g be a non-constant polynomial of degree d on $\mathbb{R}^n$. Then there is a constant $C(d, g)$ depending only on d and g such that for every polynomial f of degree d one has*

$$\|\gamma \circ f^{-1} - \gamma \circ g^{-1}\|_{\mathrm{TV}} \le C(d, g)\|f - g\|_{L^2(\gamma)}^{1/d},$$

where γ is the standard Gaussian measure on $\mathbb{R}^n$ and $\gamma \circ f^{-1}$ and $\gamma \circ g^{-1}$ are the distributions of random variables f and g, respectively.

Note that in [12] the assertion was formulated in terms of multiple stochastic integrals of order d, but the claim above is equivalent to the original one. The cited paper contains no technical details of the proof, which, however, can be found in Martynova's PhD thesis. Nevertheless, since these details are still unpublished and hardly accessible (Martynova's PhD thesis can be only found in some libraries in Saint Petersburg and Moscow), there have been several attempts to give a full proof of the above result. First, Nourdin and Poly [22] obtained the following theorem.

Theorem B. *Let $d \in \mathbb{N}$, $a > 0$, $b > 0$. Then there exists a number $C(d, a, b) > 0$ such that for every pair of polynomials f, g of degree d on $\mathbb{R}^n$ one has*

$$\|\gamma \circ f^{-1} - \gamma \circ g^{-1}\|_{\mathrm{TV}} \le C(d, a, b)\|f - g\|_{L^2(\gamma)}^{1/(2d)},$$

provided that the variance of g is in $[a, b]$.

The above theorem clarifies some dependence of $C(d, g)$ on g: it depends only on the bounds for the variance. However, the power of the L^2-norm in the theorem is worse than in Theorem A. Next, in [9] an intermediate result between Theorem A and Theorem B was obtained. The constant there was worse than in the Nourdin–Poly estimate, but the dependence on the L^2-norm differed from the one in [12] by only a logarithmic factor. Finally, in [23] the following theorem was proved.

Theorem C. *Let $d \in \mathbb{N}$. There is a constant $c(d)$ depending only on d such that, for every pair of polynomials f, g of degree $d \ge 2$ on $\mathbb{R}^n$, one has*

$$\|\gamma \circ f^{-1} - \gamma \circ g^{-1}\|_{\mathrm{TV}} \le c(d)\big(\|\nabla g\|_*^{-1/(d-1)} + 1\big)\|f - g\|_2^{1/d},$$

where

$$\|\nabla g\|_*^2 := \sup_{|e|=1} \int |\partial_e g|^2 \, d\gamma.$$

Note that while Theorem C coincides with the Davydov–Martynove estimate, the constant there is still worse than in the Nourdin–Poly estimate.

This paper generalizes the Davydov–Martynova bound to the case of an arbitrary log-concave measure in place of a Gaussian measure. Recall that a Borel probability measure μ on $\mathbb{R}^n$ is called logarithmically concave (log-concave or convex) if

$$\mu(tA + (1-t)B) \ge \mu(A)^t \mu(B)^{1-t} \quad \forall\, t \in [0,1]$$

for all Borel sets $A, B \subset \mathbb{R}^n$ (see [10] and the discussion in [8, Section 3.10(vi)] and [7, Section 4.3]). This is equivalent to the fact that the measure μ has a density of the form e^{-V} with respect to Lebesgue measure on some affine subspace L, where $V\colon\ L \to (-\infty, +\infty]$ is a convex function. We also recall that the total variation norm of a (signed) measure ν on $\mathbb{R}^n$ is defined by the equality

$$\|\nu\|_{\mathrm{TV}} := \sup\biggl\{\int \varphi\, d\nu,\ \varphi \in C_0^\infty(\mathbb{R}^n),\ \|\varphi\|_\infty \le 1\biggr\},$$

where $\|\varphi\|_\infty := \sup|\varphi(x)|$. The distribution $\mu\circ F^{-1}$ of a measurable function F on a measurable space equipped with a measure μ is the measure on the real line defined by $\mu\circ F^{-1}(A) := \mu(F \in A)$ for all Borel sets A.

The main result of the present paper asserts that, for any $n, d \in \mathbb{N}$ with $d \ge 2$, there is a constant $C(d)$ such that, for every log-concave measure μ and every pair of polynomials f, g of degree d on $\mathbb{R}^n$, one has

$$\sigma_g^{1/d} \|\mu\circ f^{-1} - \mu\circ g^{-1}\|_{\mathrm{TV}} \le C(d)\|f - g\|_2^{1/d},$$

where $\sigma_g^2 := \mathbb{D}g$ is the variance of g. We note that even in the case of a Gaussian measure the obtained result improves the dependence of the constant in comparison to Theorem C. We also note that due to independence of the constant in the inequality of the dimension the same estimate remains valid in the infinite-dimensional case. The proof of the announced inequality develops some ideas from [18], [22], and [23].

2. Preliminaries

This section contains necessary definitions, notation, and several known results which are used further. We mainly consider the finite-dimensional space $\mathbb{R}^n$ equipped with the Borel σ-field and with the standard Euclidian inner product (x, y), $x, y \in \mathbb{R}^n$. Let $|\cdot|$ be the standard norm $|x| := \sqrt{(x,x)}$, $x \in \mathbb{R}^n$. Let $C_0^\infty(\mathbb{R}^n)$ denote the space of all infinitely differentiable functions with compact support.

A log-concave measure μ on $\mathbb{R}^n$ is called isotropic if it is absolutely continuous with respect to Lebesgue measure and

$$\int_{\mathbb{R}^n} (x,\theta)\,\mu(dx) = 0, \quad \int_{\mathbb{R}^n} (x,\theta)^2\,\mu(dx) = |\theta|^2 \quad \forall\, \theta \in \mathbb{R}^n.$$

The Skorohod derivative $D_e\mu$ of a Borel measure μ along a vector $e \in \mathbb{R}^n$ is a bounded signed Borel measure on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} \partial_e\varphi\, d\mu = -\int_{\mathbb{R}^n} \varphi\, d(D_e\mu)$$

for every $\varphi \in C_0^\infty(\mathbb{R}^n)$ (see [7]). It was proved by Krugova [19] (see also [7, Section 4.3]) that every log-concave measure μ with density ρ is Skorohod differentiable along every vector $e \in \mathbb{R}^n$ and for every unit vector e one has

$$\|D_e\mu\|_{\mathrm{TV}} = 2\int_{\langle e\rangle^\perp} \max_t \rho(x + te)dx,$$

where $\langle e\rangle^\perp$ is the orthogonal complement of e.

If the measure μ is fixed, for a μ-measurable function f we set

$$\|f\|_r := \Bigl(\int_{\mathbb{R}^n} |f|^r \, d\mu\Bigr)^{1/r} \text{ for } r > 0, \quad \|f\|_0 := \exp\Bigl(\int_{\mathbb{R}^n} \ln|f| \, d\mu\Bigr) = \lim_{r\to 0} \|f\|_r.$$

An important feature of the 0-"norm" is its multiplicative property, i.e., $\|f\cdot g\|_0 = \|f\|_0 \cdot \|g\|_0$. We also denote the expectation and the variance of the random variable f by the symbols $\mathbb{E}f$ and σ_f^2 respectively, i.e.,

$$\mathbb{E}f := \int_{\mathbb{R}^n} f \, d\mu \quad \sigma_f^2 := \mathbb{D}f = \int_{\mathbb{R}^n} (f - \mathbb{E}f)^2 \, d\mu.$$

Throughout the paper the symbols $c, C, c_1, C_1, \ldots$ denote positive universal constants, the symbols $c(d), C(d), c_1(d), C_1(d), \ldots$ denote positive constants that depend only on one parameter d, and $c(d,n), C(d,n), c_1(d,n), C_1(d,n), \ldots$ denote positive constants that depend only on two parameters d and n. The values of these constants are not necessarily the same in different appearances. Throughout the paper we omit the indication of $\mathbb{R}^n$ in all integrations over the whole space.

We now formulate some key known results which will be applied in the proofs.

The first result is the so-called Carbery–Wright inequality for polynomials on a space with a log-concave measure.

Theorem 2.1 ([11], [21]). *There is an absolute constant c such that for every log-concave measure μ on $\mathbb{R}^n$ and every polynomial f of degree d the following inequality holds true:*

$$\mu(|f| \le t) \Bigl(\int |f| d\mu\Bigr)^{1/d} \le c d\, t^{1/d}.$$

The next result shows that for a log-concave measure all L^p-"norms" on the space of polynomials of a fixed degree are equivalent. These "norms" estimate each other with constants depending only on the degree of polynomials.

Theorem 2.2 ([4], [5]). *There is an absolute constant c such that, whenever μ is a log-concave measure on $\mathbb{R}^n$, $q \ge 1$, for every polynomial f of degree d the following inequalities hold:*

$$\|f\|_q \le (cqd)^d \|f\|_0, \quad \|f\|_q \le (cq)^d \|f\|_1.$$

We also need the following results on the structure of the density of a log-concave measure. The next theorem can be found in [3, Proposition 4.1].

Theorem 2.3 ([3]). *Let μ be a log-concave measure on $\mathbb{R}$ with density ρ. Then*

$$\|\rho\|_\infty^2 \int \Bigl(t - \int \tau \mu(d\tau)\Bigr)^2 \mu(dt) \ge 12^{-1}.$$

Theorem 2.4 ([15], [2]). *For every $n \in \mathbb{N}$, there is a constant $C(n)$ depending only on n such that for every isotropic log-concave measure μ on $\mathbb{R}^n$ with density ρ one has*

$$(\max \rho)^{1/n} \le C(n).$$

There is a conjecture (the so-called hyperplane conjecture) that the constant above can be chosen independent of n, but the best known constant so far is $C_n \sim n^{1/4}$, which is due to Klartag [15].

The following result is Corollary 2.4 in [16].

Theorem 2.5 ([16]). *For every $n \in \mathbb{N}$, there are universal constants $C, c > 0$ such that for every isotropic log-concave measure μ on $\mathbb{R}^n$ with density ρ the following inequality holds:*

$$\rho(x) \le \rho(0) e^{Cn - c|x|}.$$

The next property is a combination of Corollary 5.3 and Lemma 5.4 in [17].

Theorem 2.6 ([17]). *Let $n \in \mathbb{N}$, $n \geq 2$ and let μ be an isotropic log-concave measure on $\mathbb{R}^n$ with density ρ. Let $K = \{x \in \mathbb{R}^n : \rho(x) \geq e^{-20n}\rho(0)\}$. Then*

$$B_{\frac{1}{10}} \subset K.$$

The following theorem states the Poincaré inequality for log-concave measures.

Theorem 2.7 ([3, 14]). *There is an absolute constant M such that for every log-concave measure μ on $\mathbb{R}^n$ and every locally Lipschitz function f one has*

$$\int \Big(f - \int f d\mu\Big)^2 d\mu \leq M \int |x - x_0|^2 d\mu \int |\nabla f|^2 d\mu,$$

where $x_0 = \int x d\mu$.

The following so-called localization lemma from [13] (see also [14] and [20]) plays a crucial role in our proof.

Theorem 2.8 (Localization lemma with p constraints, see [13]). *Let K be a compact convex set in $\mathbb{R}^n$, $F_i \colon K \to \mathbb{R}$, $1 \leq i \leq p$. Assume that all functions F_i are upper semi-continuous. Let $P_{F_1,\ldots,F_p}$ be the set of all log-concave measures with support in K such that*

$$\int F_i \, d\mu \geq 0, \ i = 1, \ldots, p.$$

Let $\Phi \colon P(K) \to \mathbb{R}$ be a convex upper semi-continuous function, where $P(K)$ is the space of all Borel probability measures supported in K equipped with the weak topology. Then $\sup\limits_{\mu \in P_{F_1,\ldots,F_p}} \Phi(\mu)$ is attained on log-concave measures μ such that the smallest affine subspace containing the support of μ is of dimension at most p.

3. Total variation distance estimate

We start with the following reverse Poincaré inequality for polynomials on a space with a log-concave measure. Such estimates are well known for Gaussian measures due to the equivalence of all Sobolev norms on the space of all polynomials of a fixed degree (see [6]).

Theorem 3.1. *Let $n, d \in \mathbb{N}$. There is a constant $C(d)$, which depends only on the degree d, such that, for each log-concave measure μ on $\mathbb{R}^n$, each polynomial f of degree d, and each vector e of unit length, one has*

$$\|\partial_e f\|_2 \leq C(d) \|D_e \mu\|_{\mathrm{TV}} \|f\|_2.$$

Proof. We first consider the one-dimensional case. By homogeneity we can assume that the polynomial f is of the form

$$f(t) = \prod_{i=1}^{d} (t - t_i).$$

Moreover, without loss of generality we can assume that

$$\int t \, \mu(dt) = 0.$$

Using Theorem 2.2 we get

$$\int t^2\,\mu(dt)\int (f'(t))^2\,\mu(dt) \le d\sum_{i=1}^d \int t^2\,\mu(dt)\int\Big|\prod_{j\neq i}(t-t_j)\Big|^2\,\mu(dt)$$

$$\le d(2cd)^{2d}\sum_{i=1}^d\int t^2\,\mu(dt)\prod_{j\neq i}\int|t-t_j|^2\,\mu(dt) = d(2cd)^{2d}\sum_{i=1}^d\int t^2\,\mu(dt)\prod_{j\neq i}\Big(\int t^2\,\mu(dt)+|t_j|^2\Big)$$

$$\le d^2(2cd)^{2d}\prod_{i=1}^d\Big(\int t^2\,\mu(dt)+|t_i|^2\Big) = d^2(2cd)^{2d}\prod_{i=1}^d\int|t-t_i|^2\,\mu(dt)\le d^2(4c^2d)^{2d}\int f^2\,\mu(dt).$$

Thus,

$$\sigma(\mu)\|f'\|_2\le (Cd)^d\|f\|_2,$$

where $\sigma^2(\mu)$ is the variance of μ. The last bound combined with Theorem 2.3 implies that

$$\|f'\|_2\le (Cd)^d\|\rho\|_\infty\|f\|_2,$$

which is equivalent to the inequality

$$\|f'\|_1\le (Cd)^d\|\rho\|_\infty\|f\|_1$$

due to Theorem 2.2.

We now proceed to the general case. Without loss of generality we can assume that $e=e_1$ is the first basis vector. Set $\tilde{x}:=(x_2,\dots,x_n)$ and

$$\tilde{\rho}(x_1,x_2,\dots,x_n):=\frac{\rho(x_1,x_2,\dots,x_n)}{\int\rho(\tau,x_2,\dots,x_n)\,d\tau}.$$

Applying the obtained one-dimensional bound and Theorem 2.2 we get

$$\begin{aligned}\|\partial_{e_1}f\|_1^{1/2} &\le c(d)\int|\partial_{e_1}f|^{1/2}\rho\,dx\\
&= c(d)\int\Big(\int\rho(\tau,\tilde{x})\,d\tau\Big)\int|\partial_{e_1}f|^{1/2}\tilde{\rho}(x_1,\tilde{x})\,dx_1d\tilde{x}\\
&\le c(d)\int\Big(\int\rho(\tau,\tilde{x})\,d\tau\Big)\Big(\int|\partial_{e_1}f|\tilde{\rho}(x_1,\tilde{x})dx_1\Big)^{1/2}d\tilde{x}\\
&\le c_1(d)\int\Big(\int\rho(\tau,\tilde{x})\,d\tau\Big)\Big(\max_t\tilde{\rho}(t,\tilde{x})\int|f|\tilde{\rho}(x_1,\tilde{x})dx_1\Big)^{1/2}d\tilde{x}\\
&\le c_1(d)\Big(\int\max_t\rho(t,\tilde{x})\,d\tilde{x}\Big)^{1/2}\Big(\int|f|\rho\,dx\Big)^{1/2} = c_2(d)\|D_{e_1}\mu\|_{\mathrm{TV}}^{1/2}\|f\|_1^{1/2}.\end{aligned}$$

The theorem is proved. □

We also need the following technical lemma.

Lemma 3.2. *Let $n,d\in\mathbb{N}$, $n\ge 2$. There is a constant $c(d,n)$, depending only on d and n, such that, for every isotropic log-concave measure μ on $\mathbb{R}^n$ with density ρ, every polynomial h of degree d, and every unit vector $e\in\mathbb{R}^n$, the following bound holds:*

$$\int_{\langle e\rangle^\perp}\max_s\big[|h(x+se)|\rho(x+se)\big]\,dx\le c(d,n)\int_{\mathbb{R}^n}|h|\,d\mu.$$

Proof. By Theorem 2.5 there is a bound $\rho(x)\le \rho(0)e^{Cn-c|x|}$ implying

$$\int_{\langle e\rangle^\perp}\max_s\bigl[|h(x+se)|\rho(x+se)\bigr]\,dx \le \rho(0)\int_{\langle e\rangle^\perp}\max_s\bigl[|h(x+se)|e^{Cn-c|x+se|}\bigr]\,dx$$
$$\le \rho(0)e^{Cn}\int_{\langle e\rangle^\perp} e^{-c_1|x|}\max_s\bigl[|h(x+se)|e^{-c_1|s|}\bigr]\,dx.$$

We now note that the function $s\mapsto h(x+se)$ is a polynomial. Thus, $h(x+se)=a_ds^d+a_{d-1}s^{d-1}+\ldots+a_1s+a_0$, where $a_0,\ldots a_d$ are some functions of variable x. Using this representation we can write

$$\max_s\bigl[|h(x+se)|e^{-c_1|s|}\bigr]\le\sum_{j=0}^d|a_j|\max_s[|s|^je^{-c_1|s|}]=\sum_{j=0}^d|a_j|\Bigl(\frac{j}{c_1}\Bigr)^je^{-j}\le c_1(d)\sum_{j=0}^d|a_j|.$$

Since all norms on the space of polynomials of a fixed degree on the real line are equivalent, there is a constant $c_2(d)$ such that

$$\sum_{j=0}^d|a_j|\le c_2(d)\int_{\mathbb{R}}|h(x+se)|e^{-c_1|s|}\,ds.$$

Thus,

$$\int_{\langle e\rangle^\perp}\max_s\bigl[|h(x+se)|\rho(x+se)\bigr]\,dx\le\rho(0)e^{Cn}c_3(d)\int_{\langle e\rangle^\perp}e^{-c_1|x|}\int_{\mathbb{R}}|h(x+se)|e^{-c_1|s|}\,ds\,dx$$
$$\le\rho(0)e^{Cn}c_3(d)\int_{\mathbb{R}^n}|h(y)|e^{-c_1|y|}\,dy.$$

Again, since all norms on the space of polynomials of a fixed degree on $\mathbb{R}^n$ are equivalent, there is a constant $c_4(d,n)$ such that

$$\int_{\mathbb{R}^n}|h(y)|e^{-c_1|y|}\,dy\le c_4(d,n)\int_{B_{\frac{1}{10}}}|h(y)|\,dy.$$

By Theorem 2.6

$$B_{\frac{1}{10}}\subset K,$$

where $K=\{y\in\mathbb{R}^n:\rho(y)\ge e^{-20n}\rho(0)\}$, which implies that

$$\int_{B_{\frac{1}{10}}}|h(y)|\,dy\le\int_K|h(y)|\,dy\le e^{20n}(\rho(0))^{-1}\int_K|h(y)|\rho(y)\,dy\le e^{20n}(\rho(0))^{-1}\int_{\mathbb{R}^n}|h(y)|\rho(y)\,dy.$$

Thus, combining the obtained bounds, we get the announced estimate. $\square$

The following technical lemma provides an estimate similar to the one stated in the introduction, but is not dimension free. However, it is the main step in the proof of the general result.

Lemma 3.3. *Let $n,d\in\mathbb{N}$, $d\ge2$. There is a constant $c(d,n)$, which depends only on d and n, such that, for any isotropic log-concave measure μ on $\mathbb{R}^n$, any polynomials f,g of degree d, any function $\varphi\in C_0^\infty(\mathbb{R}^n)$ with $\|\varphi\|_\infty\le1$, and any vector e of unit length, one has*

$$\|\partial_eg\|_2^{1/d}\int\varphi(f)-\varphi(g)\,d\mu\le c(d,n)\|f-g\|_1^{1/d}.$$

Proof. Let $\rho=e^{-V}$ be the density of μ, where V is a convex function. We first consider the case $\rho\in C^\infty(\mathbb{R}^n)$, $\rho>0$, and $n\ge2$. Let

$$\Phi(t):=\int_{-\infty}^t\varphi(\tau)d\tau.$$

As in [22], [9], and [23], we use the equality

$$\partial_e g(\varphi(f)-\varphi(g)) = \partial_e(\Phi(f)-\Phi(g)) - (\partial_e f - \partial_e g)\varphi(f).$$

Thus,

$$\int \big(\varphi(f)-\varphi(g)\big)\,d\mu = \int \frac{(\partial_e g)^2(\varphi(f)-\varphi(g))}{(\partial_e g)^2+\varepsilon}\,d\mu + \varepsilon\int \frac{\varphi(f)-\varphi(g)}{(\partial_e g)^2+\varepsilon}\,d\mu$$
$$= \int \frac{\partial_e g\partial_e(\Phi(f)-\Phi(g))}{(\partial_e g)^2+\varepsilon}\,d\mu - \int \frac{\partial_e g(\partial_e f-\partial_e g)\varphi(f)}{(\partial_e g)^2+\varepsilon}\,d\mu + \varepsilon\int \frac{\varphi(f)-\varphi(g)}{(\partial_e g)^2+\varepsilon}\,d\mu.$$

We now estimate each term separately starting with the last term. By the Carbery–Wright inequality (Theorem 2.1) one has (see the proof of Lemma 3.1 in [18] or expression (4.4) in [9])

$$\varepsilon\int \frac{\varphi(f)-\varphi(g)}{(\partial_e g)^2+\varepsilon}\,d\mu \le 2c_1 d\Big(\int_0^\infty (s+1)^{-2}s^{1/(2d-2)}\,ds\Big)\|\partial_e g\|_2^{-1/(d-1)}\varepsilon^{1/(2d-2)}$$
$$= C_1(d)\|\partial_e g\|_2^{-1/(d-1)}\varepsilon^{1/(2d-2)}.$$

For the second term we have

$$-\int \frac{\partial_e g(\partial_e f-\partial_e g)\varphi(f)}{(\partial_e g)^2+\varepsilon}\,d\mu \le 2^{-1}\varepsilon^{-1/2}\int |\partial_e f-\partial_e g|d\mu \le C_2(d)\varepsilon^{-1/2}\|D_e\mu\|_{\mathrm{TV}}\|f-g\|_1.$$

Recall that

$$\|D_e\mu\|_{\mathrm{TV}} = 2\int_{\langle e\rangle^\perp}\max_s \rho(x+se)\,dx.$$

Thus, by Theorem 2.5,

$$\|D_e\mu\|_{\mathrm{TV}} \le 2\rho(0)e^{Cn}\int_{\langle e\rangle^\perp}\max_s e^{-c|x+se|}\,dx \le c_1(n)\rho(0)\le c_2(n), \tag{3.1}$$

where Theorem 2.4 was applied in the last inequality. Therefore,

$$-\int \frac{\partial_e g(\partial_e f-\partial_e g)\varphi(f)}{(\partial_e g)^2+\varepsilon}\,d\mu \le c_3(d,n)\varepsilon^{-1/2}\|f-g\|_1.$$

Now, integrating by parts in the first term, we get

$$\int \frac{\partial_e g\partial_e(\Phi(f)-\Phi(g))}{(\partial_e g)^2+\varepsilon}\,d\mu$$
$$= -\int(\Phi(f)-\Phi(g))\Big[\frac{\partial_e^2 g}{(\partial_e g)^2+\varepsilon} - 2\frac{(\partial_e g)^2\partial_e^2 g}{((\partial_e g)^2+\varepsilon)^2}\Big]\,d\mu - \int(\Phi(f)-\Phi(g))\frac{\partial_e g}{(\partial_e g)^2+\varepsilon}\,d(D_e\mu)$$

Up to the factor 3, the first integral above is estimated by

$$\int\Big|\frac{\partial_e^2 g}{(\partial_e g)^2+\varepsilon}\Big||f-g|\,d\mu = \int_{\langle e\rangle^\perp}\int_{\mathbb{R}}\Big|\frac{\partial_e^2 g(x+te)}{(\partial_e g(x+te))^2+\varepsilon}\Big||f(x+te)-g(x+te)|\rho(x+te)\,dtdx$$
$$\le d\varepsilon^{-1/2}\int_{\langle e\rangle^\perp}\Big(\int_{\mathbb{R}}\Big|\frac{1}{\tau^2+1}\Big|\,d\tau\Big)\max_s\big[|f(x+se)-g(x+se)|\rho(x+se)\big]\,dx$$
$$= \pi d\varepsilon^{-1/2}\int_{\langle e\rangle^\perp}\max_s\big[|f(x+se)-g(x+se)|\rho(x+se)\big]\,dx \le c_4(d,n)\varepsilon^{-1/2}\int|f-g|\,d\mu,$$

where Lemma 3.2 was applied in the last inequality. The second integral is not greater than

$$\varepsilon^{-1/2}\int|f-g|\,d|D_e\mu| \le \varepsilon^{-1/2}\sqrt{\|D_e\mu\|_{\mathrm{TV}}}\Big(\int|f-g|^2\,d|D_e\mu|\Big)^{1/2}$$
$$\le \varepsilon^{-1/2}\sqrt{c_2(n)}\Big(\int|f-g|^2\,d|D_e\mu|\Big)^{1/2}.$$

Since $\rho = e^{-V}$, we have $D_e\mu = -V_e' e^{-V}\,dx$ and $|D_e\mu| = |V_e'|e^{-V}$. For a point $x \in \langle e\rangle^\perp$ let $T(x)$ be such that $V_e'(x + T(x)e) = 0$. Then $V_e'(x+te) \le 0$ for $t < T(x)$ and $V_e'(x+te) \ge 0$ for $t > T(x)$ by the convexity of the function $t \mapsto V(x+te)$. Thus,

$$\begin{aligned}
&\int_{\mathbb{R}} |f(x+te) - g(x+te)|^2 |V_e'(x+te)| e^{-V(x+te)}\,dt \\
&\qquad = -\int_{-\infty}^{T(x)} |f(x+te) - g(x+te)|^2 V_e'(x+te) e^{-V(x+te)}\,dt \\
&\qquad\quad + \int_{T(x)}^{\infty} |f(x+te) - g(x+te)|^2 V_e'(x+te) e^{-V(x+te)}\,dt \\
&\qquad = 2|f(x+T(x)e) - g(x+T(x)e)|^2 \rho(x+T(x)e) \\
&\quad - 2\int_{-\infty}^{T(x)} (\partial_e f(x+te) - \partial_e g(x+te))(f(x+te) - g(x+te))\rho(x+te)\,dt \\
&\quad + 2\int_{T(x)}^{\infty} (\partial_e f(x+te) - \partial_e g(x+te))(f(x+te) - g(x+te))\rho(x+te)\,dt \\
&\qquad \le 2\max_s \bigl[|f(x+se) - g(x+se)|^2 \rho(x+se)\bigr] \\
&\qquad\quad + 4\int_{\mathbb{R}} |\partial_e f(x+te) - \partial_e g(x+te)||f(x+te) - g(x+te)|\rho(x+te)\,dt.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\int |f-g|^2\,d|D_e\mu| &= \int_{\langle e\rangle^\perp}\int_{\mathbb{R}} |f(x+te) - g(x+te)|^2 |V_e'(x+te)| e^{-V(x+te)}\,dt\,dx \\
&\le 2\int_{\langle e\rangle^\perp} \max_s \bigl[|f(x+se) - g(x+se)|^2 \rho(x+se)\bigr]\,dx + 4\int |\partial_e f - \partial_e g||f-g|\,d\mu \\
&\le c_5(d,n)\Bigl(\int |f-g|\,d\mu\Bigr)^2,
\end{aligned}$$

where Lemma 3.2, Theorems 2.2 and 3.1, and estimate (3.1) were applied in the last inequality.

Combining the above estimates, we get the bound

$$\int \varphi(f) - \varphi(g)\,d\mu \le c_6(d,n)\Bigl[\|\partial_e g\|_2^{-1/(d-1)} \varepsilon^{1/(2d-2)} + \varepsilon^{-1/2}\|f-g\|_1\Bigr].$$

Taking $\varepsilon = \bigl[\|\partial_e g\|_2^{1/(d-1)}\|f-g\|_1\bigr]^{(2d-2)/d}$, we obtain

$$\int \varphi(f) - \varphi(g)\,d\mu \le 2c_6(d,n)\|\partial_e g\|_2^{-1/d}\|f-g\|_1^{1/d}.$$

In the case of an arbitrary (isotropic log-concave) density ρ on $\mathbb{R}^n$ with $n \ge 2$, the estimate follows from the approximation by the measures with densities ρ_ε, where

$$\rho_\varepsilon(x) := \rho * \psi_\varepsilon\bigl((1+\varepsilon^2)^{\frac{1}{n+2}} \cdot x\bigr),$$

ψ is the density of the standard Gaussian measure on $\mathbb{R}^n$ and $\psi_\varepsilon(x) = \varepsilon^{-n}\psi(\varepsilon^{-1}x)$.

Finally, the one-dimensional case follows from the case $n = 2$ by consideration of the product measure $\mu\otimes\mu$ and polynomials depending only on the first argument. The lemma is proved. ☐

Corollary 3.4. *Let $n, d \in \mathbb{N}$. Then there is a constant $c(d,n)$ depending only on d and n such that, for any isotropic log-concave measure μ on $\mathbb{R}^n$, any pair of polynomials f and g of degree d on $\mathbb{R}^n$, and any function $\varphi \in C_0^\infty(\mathbb{R})$ with $\|\varphi\|_\infty \le 1$ one has*

$$\Bigl(\int |g - \mathbb{E}g|^{1/d} d\mu\Bigr)\int \bigl(\varphi(f) - \varphi(g)\bigr)\,d\mu \le c(d,n)\int |f-g|^{1/d}\,d\mu.$$

Proof. We note that

$$\int |(\nabla g, e)|^{1/d}\, d\mu \le \Big(\int |(\nabla g, e)|^2\, d\mu\Big)^{1/2d} = \|\partial_e g\|_2^{1/d}.$$

Hence, by Lemma 3.3

$$\Big(\int |(\nabla g, e)|^{1/d}\, d\mu\Big) \int \big(\varphi(f) - \varphi(g)\big)\, d\mu \le c(d,n)\|f-g\|_1^{1/d}.$$

Integrating in the above inequality with respect to the normalized surface measure σ_n on the unite sphere, we get

$$\Big(\int_{S^{n-1}} \int |(\nabla g, e)|^{1/d}\, d\mu\, \sigma_n(de)\Big) \int \big(\varphi(f) - \varphi(f)\big)\, d\mu \le c(d,n)\|f-g\|_1^{1/d}.$$

By Fubini's theorem

$$\begin{aligned}\int_{S^{n-1}} \int |(\nabla g, e)|^{1/d}\, d\mu\, \sigma_n(de) &= \int \int_{S^{n-1}} |(\nabla g, e)|^{1/d}\, \sigma_n(de)\, d\mu \\ &= \int |\nabla g|^{1/d} \int_{S^{n-1}} |(e, e_1)|^{1/d}\, \sigma_n(de)\, d\mu = c_1(d,n) \int |\nabla g|^{1/d}\, d\mu.\end{aligned}$$

So, by the above equality and Theorem 2.2, we have

$$\Big(\int |\nabla g|^{1/d}\, d\mu\Big) \int \big(\varphi(f) - \varphi(g)\big)\, d\mu \le c_2(d,n) \int |f-g|^{1/d}\, d\mu.$$

Applying Theorem 2.2 again, we get

$$\||\nabla g|\|_2^{1/d} \le C(d) \int |\nabla g|^{1/d}\, d\mu.$$

Thus, by Theorem 2.7 we get the desired bound. The corollary is proved. ∎

We are now ready to prove the main result of the paper. The key part of the proof is the application of the localization lemma, which enables us to reduce considerations to a space of dimension at most 4.

Theorem 3.5. *Let $d, n \in \mathbb{N}$, $d \ge 2$. Then, there is a constant $C(d)$ depending only on d such that, for any log-concave measure μ on $\mathbb{R}^n$ and any pair of polynomials f and g of degree d on $\mathbb{R}^n$, one has*

$$\sigma_g^{1/d}\|\mu\circ f^{-1} - \mu\circ g^{-1}\|_{\mathrm{TV}} \le C(d)\|f-g\|_2^{1/d},$$

where $\sigma_g^2 := \mathbb{D}g = \int (g - \mathbb{E}g)^2\, d\mu,\ \mathbb{E}g := \int g\, d\mu.$

Proof. Set $R(d) := \max_{n=1,2,3,4} c(d,n)$, where $c(d,n)$ is the constant from Corollary 3.4. Due to Theorem 2.2 it is sufficient to prove that, for any function $\varphi \in C_0^\infty(\mathbb{R})$ with $\|\varphi\|_\infty \le 1$, one has

$$\Big(\int |g - \mathbb{E}g|^{1/d} d\mu\Big) \int \big(\varphi(f) - \varphi(g)\big)\, d\mu \le R(d) \int |f-g|^{1/d}\, d\mu. \tag{3.2}$$

First we consider the case $n \in \{1,2,3,4\}$. Recall that for an arbitrary log-concave measure μ on $\mathbb{R}^n$ there is a nondegenerate linear mapping $T\colon \mathbb{R}^n \to \mathbb{R}^n$ such that measure $\mu\circ T^{-1}$ is isotropic. By Corollary 3.4, for every pair f, g of polynomials of degree d, we have

$$\begin{aligned}\Big(\int |g\circ T^{-1} - \mathbb{E}(g\circ T^{-1})|^{1/d}\, d(\mu\circ T^{-1})\Big) \int \big(\varphi(f\circ T^{-1}) - \varphi(g\circ T^{-1})\big)\, d(\mu\circ T^{-1}) \\ \le R(d) \int |f\circ T^{-1} - g\circ T^{-1}|^{1/d}\, d(\mu\circ T^{-1})\end{aligned}$$

as the functions $f\circ T^{-1}$ and $g\circ T^{-1}$ are also polynomials of degree d. This implies estimate (3.2) for log-concave measures on $\mathbb{R}^n$ with $n\in\{1,2,3,4\}$.

Now let n be an arbitrary positive integer. Fix a convex compact set K, numbers $a,b>0$, and polynomials f,g of degree d. Let

$$F_1=g,\quad F_2=-g,\quad F_3=|g|^{1/d}-a,\quad F_4=b-|f-g|^{1/d}$$

and let $P_{F_1,\dots,F_4}$ be the set of all log-concave measures supported in K such that

$$\int F_i\,d\mu\ge 0,\ i=1,\dots,4.$$

We note that the above conditions are equivalent to the following one:

$$\int g\,d\mu=0,\qquad \int |g|^{1/d}\,d\mu\ge a,\qquad \int |f-g|^{1/d}\,d\mu\le b,$$

Consider the functional

$$\Phi_{f,g}(\mu):=\int\bigl(\varphi(f)-\varphi(g)\bigr)\,d\mu.$$

Note that the restriction of a polynomial to a linear subspace will be again a polynomial (of the same degree) on this subspace. Thus, $\Phi_{f,g}(\mu)\le R(d)ba^{-1}$ for an arbitrary measure $\mu\in P_{F_1,\dots,F_4}$ such that the smallest affine subspace containing the support of μ is of dimension not greater than 4. By Theorem 2.8 we have

$$\Phi_{f,g}(\mu)\le R(d)ba^{-1}$$

for any measure $\mu\in P_{F_1,\dots,F_4}$, which implies bound (3.2) for an arbitrary log-concave measure on $\mathbb{R}^n$ with compact support. The general case follows by approximation. The theorem is proved. □

We now briefly discuss the infinite-dimensional case. Let E be a locally convex space equipped with the Borel σ-field and let E^* be the topological dual space to E. A Radon probability measure μ on E is called log-concave (or convex) if $\mu\circ A^{-1}$ is a log-concave measure on $\mathbb{R}^n$ for every continuous linear operator $A\colon E\to\mathbb{R}^n$. For a Radon probability measure μ on E, denote by $\mathcal{P}^d(\mu)$ the closure in $L^2(\mu)$ of the set of all functions of the form $f(\ell_1,\dots,\ell_n)$, where n is an arbitrary positive integer, $\ell_j\in E^*$ are arbitrary continuous linear functionals, and f is an arbitrary polynomial on $\mathbb{R}^n$ of degree d. It is shown in [1] that every function from $\mathcal{P}^d(\mu)$ has a version that is a polynomial of degree d in the usual algebraic sense, i.e., this version is of the form

$$b_0+b_1(x)+b_2(x,x)+\ldots+b_d(x,\dots,x),$$

where each $b_j(x_1,\dots,x_j)$ is a multilinear function on E^j.

Corollary 3.6. *Let $d,n\in\mathbb{N}$, $d\ge 2$. Then, there is a constant $C(d)$ depending only on d such that, for any log-concave measure μ on a locally convex space E and any functions $f,g\in\mathcal{P}^d(\mu)$, one has*

$$\sigma_g^{1/d}\|\mu\circ f^{-1}-\mu\circ g^{-1}\|_{\rm TV}\le C(d)\|f-g\|_2^{1/d},$$

where $\sigma_g^2:=\mathbb{D}g=\int(g-\mathbb{E}g)^2\,d\mu,\ \mathbb{E}g:=\int g\,d\mu.$

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