NONLOCAL OPERATORS WITH SINGULAR ANISOTROPIC KERNELS

JAMIL CHAKER AND MORITZ KASSMANN

Abstract. We study nonlocal operators acting on functions in the Euclidean space. The operators under consideration generate anisotropic jump processes, e.g., a jump process that behaves like a stable process in each direction but with a different index of stability. Its generator is the sum of one-dimensional fractional Laplace operators with different orders of differentiability. We study such operators in the general framework of bounded measurable coefficients. We prove a weak Harnack inequality and Hölder regularity results for solutions to corresponding integro-differential equations.

1. Introduction

In this article we study regularity estimates of weak solutions to integro-differential equations driven by nonlocal operators with anisotropic singular kernels. Since the formulation of the main results involves various technical definitions, let us first look at a simple case.

For $\alpha \in (0,2)$, the fractional Laplace operator $-(-\Delta)^{\alpha/2}$ can be represented as an integro-differential operator $L : C_c^\infty(\mathbb{R}^d) \to C(\mathbb{R}^d)$ in the following form

$$Lv(x) = \int_{\mathbb{R}^d} \big(v(x+h) - 2v(x) + v(x-h)\big)\, \pi(\mathrm{d}h) \qquad (x \in \mathbb{R}^d)\,, \tag{1.1}$$

where the Borel measure $\pi(\mathrm{d}h)$ on $\mathbb{R}^d \setminus \{0\}$ is defined by $\pi(\mathrm{d}h) = c_{d,\alpha} \frac{\mathrm{d}h}{|h|^{d+\alpha}}$ and $c_{d,\alpha}$ is an appropriate positive constant. Due to its behavior with respect to integration and scaling, π is a stable Lévy measure. The fractional Laplace operator generates a strongly continuous contraction semigroup, which corresponds to a stochastic jump process $(X_t)_{t\geq 0}$ in $\mathbb{R}^d$. Given $A \subset \mathbb{R}^d$, the quantity $\pi(A)$ describes the expected number of jumps $(X_t - X_{t-}) \in A$ within the interval $0 \leq t \leq 1$. A second representation of $-(-\Delta)^{\alpha/2}$ is given with the help of Fourier analysis because $-\mathcal{F}((-\Delta)^{\alpha/2}u)(\xi) = |\xi|^\alpha \mathcal{F}(u)(\xi)$. The function $\xi \mapsto \psi(\xi) = |\xi|^\alpha$ is called multiplier of the fractional Laplace operator or symbol of the corresponding stable Lévy process.

In this article we study a rather general class of anisotropic nonlocal operators, which contains as a simple example an operator $L^{\alpha_1,\alpha_2} : C_c^\infty(\mathbb{R}^2) \to C(\mathbb{R})$ as in (1.1) with the measure π being a singular measure defined by

$$\pi^{\alpha_1,\alpha_2}(\mathrm{d}h) = c_{1,\alpha_1}|h_1|^{-1-\alpha_1}\mathrm{d}h_1\delta_{\{0\}}(\mathrm{d}h_2) + c_{1,\alpha_2}|h_2|^{-1-\alpha_2}\mathrm{d}h_2\delta_{\{0\}}(\mathrm{d}h_1)\,, \tag{1.2}$$

where $h = (h_1, h_2)$ and $\alpha_1, \alpha_2 \in (0,2)$. Note that $\mathcal{F}(L^{\alpha_1,\alpha_2}u)(\xi) = \big(|\xi_1|^{\alpha_1} + |\xi_2|^{\alpha_2}\big)\mathcal{F}(u)(\xi)$ for smooth functions u and $\xi \in \mathbb{R}^2$. Since the multiplier equals $|\xi_1|^{\alpha_1} + |\xi_2|^{\alpha_2}$, one can identify the operator L^{α_1,α_2} with $-(-\partial_{11})^{\alpha_1} - (-\partial_{22})^{\alpha_2}$. The aim of this article is to study such operators

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with bounded measurable coefficients and to establish local regularity results such as Hölder regularity results. Our main auxiliary result is a weak Harnack inequality.

Let us briefly explain why the weak Harnack inequality is a suitable tool. The (strong) Harnack inequality states that there is a positive constant c such that for every positive function $u : \mathbb{R}^d \to \mathbb{R}$ satisfying $Lu = 0$ in B_2 the estimate $u(x) \le cu(y)$ holds true for all $x, y \in B_1$. The Harnack inequality is known to hold true for $L = -(-\Delta)^{\alpha/2}$, the proof follows from the explicit computations in [21]. It is known to fail in the case of $L^{\alpha,\alpha} : C_c^\infty(\mathbb{R}^2) \to C(\mathbb{R})$ as in (1.1) with the measure π being a singular measure defined by (1.2) with $\alpha_1 = \alpha_2 = \alpha$, cf. [3] for a proof using techniques from Analysis and [1] for a proof using the corresponding jump process. As a consequence of the main result in [12], the weak Harnack inequality holds true in this setting. The main aim of the present work implies that it holds true even in the case $\alpha_1 \neq \alpha_2$.

We study regularity of solutions $u : \Omega \to \mathbb{R}$ to nonlocal equations of the form $-\mathcal{L}u = f$ in Ω, where $\mathcal{L}$ is a nonlocal operator of the form

$$\mathcal{L}u(x) = \lim_{\epsilon \to 0} \int_{\mathbb{R}^d \setminus B_\epsilon(x)} (u(y) - u(x))\, \mu(x, \mathrm{d}y) \tag{1.3}$$

and $\Omega \subset \mathbb{R}^d$ is an open and bounded set. The operator is determined by a family of measures $(\mu(x, \cdot))_{x \in \mathbb{R}^d}$, which play the role of variable coefficients. Note that we will not assume any further regularity of $\mu(x, \mathrm{d}y)$ in the first variable than measurability and boundedness. Before discussing the precise assumptions on $\mu(x, \mathrm{d}y)$, let us define a family of reference measures $\mu_{\text{axes}}(x, \mathrm{d}y)$. Given $\alpha_1, \dots, \alpha_d \in (0, 2)$, we consider a family of measures $(\mu_{\text{axes}}(x, \cdot))_{x \in \mathbb{R}^d}$ on $\mathbb{R}^d$ defined by

$$\mu_{\text{axes}}(x, \mathrm{d}y) = \sum_{k=1}^{d} \Big(\alpha_k (2 - \alpha_k) |x_k - y_k|^{-1-\alpha_k} \, \mathrm{d}y_k \prod_{i \neq k} \delta_{\{x_i\}}(\mathrm{d}y_i) \Big). \tag{1.4}$$

The family $(\mu_{\text{axes}}(x, \cdot))_{x \in \mathbb{R}^d}$ is stationary in the sense that there is a measure $\nu_{\text{axes}}(\mathrm{d}h)$ with $\mu_{\text{axes}}(x, A) = \nu_{\text{axes}}(A - \{x\})$ for every $x \in \mathbb{R}^d$ and every measurable set $A \subset \mathbb{R}^d$. In other words, if one defines an operator $\mathcal{L}$ as in (1.3) with μ being replaced by μ_{axes}, then the operator is translation invariant. The measure $\mu_{\text{axes}}(x, \cdot)$ charges only those sets that intersect one of the lines $\{x + te_k \,|\, t \in \mathbb{R}\}$, where $k \in \{1, \dots, d\}$. In order to compensate for the anisotropy of the jumping measures, we replace the Euclidean metric by a new metric. Squares are replaced by rectangles defined as follows. Set $\alpha_{\max} = \max\{\alpha_i |\, i \in \{1, \dots, d\}\}$.

Definition 1.1. For $r > 0$ and $x \in \mathbb{R}^d$ we define

$$M_r(x) = \bigtimes_{k=1}^{d} \left(x_k - r^{\frac{\alpha_{\max}}{\alpha_k}}, x_k + r^{\frac{\alpha_{\max}}{\alpha_k}} \right) \qquad \text{and } M_r = M_r(0)\,.$$

Moreover, we define a metric on $\mathbb{R}^d$ by

$$\mathbb{d}(x, y) = \sup_{k \in \{1, \dots, d\}} \left\{ |x_k - y_k|^{\frac{\alpha_k}{\alpha_{\max}}} \mathbb{1}_{\{|x_k - y_k| \le 1\}}(x, y) + \mathbb{1}_{\{|x_k - y_k| > 1\}}(x, y) \right\}. \tag{1.5}$$

For $0 < r \le 1$, the rectangle $M_r(x)$ equals the ball $\{y \in \mathbb{R}^d |\, \mathbb{d}(x, y) < r\}$ in the metric space $(\mathbb{R}^d, \mathbb{d})$. Note that for every $x \in \mathbb{R}^d$ and $r > 0$

$$\int\limits_{\mathbb{R}^d \setminus M_r(x)} \mu_{\text{axes}}(x, \mathrm{d}y) = c\, r^{-\alpha_{\max}}$$

for some constant $c > 0$. In the definition of $M_r(x)$ we could replace $r^{\frac{\alpha_{\max}}{\alpha_k}}$ by any term of the form $r^{\frac{a}{\alpha_k}}$ with an arbitrary choice of $a \geq \alpha_{\max}$ in order to guarantee that $\mathbb{d}$ is a metric. Our choice of $a = \alpha_{\max}$ has the nice effect that for $\alpha_1 = \ldots = \alpha_d$ the rectangles $M_r(x)$ become squares.

With regard to (1.3), let us formulate and explain our main assumptions on $(\mu(x,\cdot))_{x\in\mathbb{R}^d}$.

Assumption 1. *We assume*

$$\sup_{x\in M_3} \int_{\mathbb{R}^d} (|x-y|^2 \wedge 1)\mu(x,\mathrm{d}y) < \infty\,, \tag{A1-a}$$

and for all measurable sets $A, B \subset \mathbb{R}^d$

$$\int_A \int_B \mu(x,\mathrm{d}y)\,\mathrm{d}x = \int_B \int_A \mu(x,\mathrm{d}y)\,\mathrm{d}x\,. \tag{A1-b}$$

Note that (A1-a) is nothing but an uniform Lévy-integrability condition. It allows $\mu(x, A)$ to have a singularity for $x \in \overline{A}$. We do not impose any condition on $\mu(x,\mathrm{d}y)$ for $x \in \mathbb{R}^d \setminus M_3$ because we study nonlocal equations only in the set M_1. Condition (A1-b) requires symmetry of the family $(\mu(x,\cdot))_{x\in\mathbb{R}^d}$. Examples of $\mu(x,\mathrm{d}y)$ satisfying these two conditions are given by μ_{axes} as in (1.4) and by

$$\mu_1(x,\mathrm{d}y) = a(x,y)|x-y|^{-d-\alpha}\mathrm{d}y\,,$$

where $\alpha \in (0, 2)$ and $a(x, y) \in [1, 2]$ is a measurable symmetric function.

The following assumption is our main assumption. It relates $\mu(x,\mathrm{d}y)$ to the reference family $\mu_{\text{axes}}(x,\mathrm{d}y)$. The easiest way to do this would be to assume that there is a constant $\Lambda \geq 1$ such that for every $x \in \mathbb{R}^d$ and every nonnegative measurable function $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$

$$\Lambda^{-1} \int f(x,y)\mu_{\text{axes}}(x,\mathrm{d}y) \leq \int f(x,y)\mu(x,\mathrm{d}y) \leq \Lambda \int f(x,y)\mu_{\text{axes}}(x,\mathrm{d}y)\,. \tag{1.6}$$

We will work under a weaker condition, which appears naturally in our framework. For $u, v \in L^2_{loc}(\mathbb{R}^d)$ and $\Omega \subset \mathbb{R}^d$ open and bounded, we define

$$\mathcal{E}^{\mu}_{\Omega}(u,v) = \int_\Omega \int_\Omega (u(y)-u(x))(v(y)-v(x))\,\mu(x,\mathrm{d}y)\,\mathrm{d}x$$

and $\mathcal{E}^{\mu}(u, v) = \mathcal{E}^{\mu}_{\mathbb{R}^d}(u, v)$ whenever the quantities are finite.

Assumption 2. *There is a constant* $\Lambda \geq 1$ *such that for* $0 < \rho \leq 1$, $x_0 \in M_1$ *and* $w \in L^2_{loc}(\mathbb{R}^d)$

$$\Lambda^{-1}\mathcal{E}^{\mu_{axes}}_{M_\rho(x_0)}(w,w) \leq \mathcal{E}^{\mu}_{M_\rho(x_0)}(w,w) \leq \Lambda\mathcal{E}^{\mu_{axes}}_{M_\rho(x_0)}(w,w). \tag{A2}$$

Let us briefly discuss this assumption. Assume $a(x, y) \in [1, 2]$ is symmetric and μ_{axes} is defined as in (1.4) with respect to some $\alpha_1, \ldots, \alpha_d \in (0, 2)$. If we define μ_2 by $\mu_2(x, A) = \int_A a(x,y)\mu_{\text{axes}}(x,\mathrm{d}y)$, then μ_2 obviously satisfies Assumption 2. If $\alpha_1 = \alpha_2 = \ldots = \alpha_d = \alpha$, then it is proved in [12] that μ_1 satisfies (A2). Note that comparability of the quadratic forms $\mathcal{E}^{\mu_{\text{axes}}}(w, w)$ and $\mathcal{E}^{\mu_1}(w, w)$ follows from comparability of the respective multipliers.

In general, studying Assumption 2 is a research project in itself. Let us mention one curiosity. Given $x \in \mathbb{R}^d$, Assumption 2 does not require $\mu(x,\mathrm{d}y)$ to be singular with respect to the Lebesgue measure. One can construct an absolutely continuous measure ν_{cusp} on $\mathbb{R}^d$ such that for μ_3 given by $\mu_3(x, A) = \nu_{\text{cusp}}(A - \{x\})$, Assumption 2 is satisfied. Since computations are rather lengthy, they will be carried out elsewhere.

We need one more assumption related to cut-off functions, Assumption 3 resp. (A3). Since this assumption is not restrictive at all but rather technical, we provide it in Subsection 2.1.

The quadratic forms introduced above relate to integro-differential operators in the following way. Given a sufficiently nice family of measures μ (any of μ_a, μ_1, μ_2, would do) and sufficiently regular functions $u, v : \mathbb{R}^d \to \mathbb{R}$, one has $\mathcal{E}^{\mu}(u,v) = -2\int_{\mathbb{R}^d} \mathcal{L}u(x)v(x)\mathrm{d}x$ with $\mathcal{L}$ as in (1.3). That is why we will study solutions u to $-\mathcal{L}u = f$ with the help of bilinear forms like $\mathcal{E}^{\mu}$. In order to do this, we need to define appropriate Sobolev-type function spaces.

Definition 1.2. Let $\Omega \subset \mathbb{R}^d$ open. We define the function spaces

$$V^{\mu}(\Omega|\mathbb{R}^d) = \Big\{ u : \mathbb{R}^d \to \mathbb{R} \text{ meas.} \,|\, u\big|_{\Omega} \in L^2(\Omega), (u,u)_{V^{\mu}(\Omega|\mathbb{R}^d)} < \infty \Big\}, \tag{1.7}$$

$$H^{\mu}_{\Omega}(\mathbb{R}^d) = \Big\{ u : \mathbb{R}^d \to \mathbb{R} \text{ meas.} \,|\, u \equiv 0 \text{ on } \mathbb{R}^d \setminus \Omega, \|u\|_{H^{\mu}_{\Omega}(\mathbb{R}^d)} < \infty \Big\}, \tag{1.8}$$

where

$$(u,v)_{V^{\mu}(\Omega|\mathbb{R}^d)} = \int_{\Omega}\int_{\mathbb{R}^d} (u(x)-u(y))(v(x)-v(y))\,\mu(x,\mathrm{d}y)\,\mathrm{d}x\,,$$

$$\|u\|^2_{H^{\mu}_{\Omega}(\mathbb{R}^d)} = \|u\|^2_{L^2(\Omega)} + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (u(y)-u(x))^2 \mu(x,\mathrm{d}y)\,\mathrm{d}x\,.$$

The space $V^{\mu}(\Omega|\mathbb{R}^d)$ is a nonlocal analog of the space $H^1(\Omega)$. Fractional regularity is required inside of Ω whereas inside of $\mathbb{R}^d \setminus \Omega$ integrability suffices. The space $H^{\mu}_{\Omega}(\mathbb{R}^d)$ can be seen as a nonlocal analog of $H^1_0(\Omega)$. See [14] and [11] for further studies of these spaces.

We are now in a position to formulate our main results:

Theorem 1.3. *Assume (A1-a), (A1-b), (A2) and (A3). Let $f \in L^q(M_1)$ for some $q > \max\{2, \sum_{k=1}^d \frac{1}{\alpha_k}\}$. Assume $u \in V^{\mu}(M_1|\mathbb{R}^d)$, $u \geq 0$ in M_1 satisfies*

$$\mathcal{E}^{\mu}(u,\varphi) \geq (f,\varphi) \quad \textit{for every non-negative } \varphi \in H^{\mu}_{M_1}(\mathbb{R}^d). \tag{1.9}$$

Then there exist $p_0 \in (0,1)$, $c_1 > 0$, independent of u, such that

$$\inf_{M_{\frac14}} u \geq c_1 \left(\fint_{M_{\frac12}} u(x)^{p_0}\,\mathrm{d}x \right)^{1/p_0} - \sup_{x \in M_{\frac{15}{16}}} 2\int_{\mathbb{R}^d \setminus M_1} u^-(z)\mu(x,\mathrm{d}z) - \|f\|_{L^q(M_{\frac{15}{16}})}.$$

Note that, throughout this article, sup resp. inf denote the essential supremum resp. the essential infimum. In the case $\alpha = \alpha_1 = \ldots = \alpha_d$, the condition $q > \max\{2, \sum_{k=1}^d \frac{1}{\alpha_k}\}$ becomes $q > \max\{2, d/\alpha\}$, which is natural. As is well known, the weak Harnack inequality implies a decay of oscillation-result and Hölder regularity estimates for weak solutions:

Theorem 1.4. *Assume (A1-a), (A1-b), (A2) and (A3). Let $f \in L^q(M_1)$ for some $q > \max\{2, \sum_{k=1}^d \frac{1}{\alpha_k}\}$. Assume $u \in V^{\mu}(M_1|\mathbb{R}^d)$ satisfies*

$$\mathcal{E}^{\mu}(u,\varphi) = (f,\varphi) \quad \textit{for every non-negative } \varphi \in H^{\mu}_{M_1}(\mathbb{R}^d).$$

Then there are $c_1 \geq 1$ and $\delta \in (0,1)$, independent of u, such that for almost every $x, y \in M_{\frac12}$

$$|u(x)-u(y)| \leq c_1 |x-y|^{\delta} \left(\|u\|_{\infty} + \|f\|_{L^q(M_{\frac{15}{16}})} \right). \tag{1.10}$$

Let us discuss selected related results in the literature.

The research questions of this article are strongly influenced by the fundamental contributions of [9, 20, 19] on Hölder estimates for weak solutions u to second order equations of the form

$$\operatorname{div}(A(x)\nabla u(x)) = f \tag{1.11}$$

for uniformly positive definite and measurable coefficients $A(\cdot)$. [19] establishes the Harnack inequality as an important tool for weak solutions to this equation. Similarly to the present work, (1.11) is interpreted in the weak sense, i.e., instead of (1.11) one considers

$$\mathcal{E}^{local}(u,v) := \int A(x)\nabla u(x)\nabla v(x)\mathrm{d}x = 0$$

for every test function v. Analogous results for integro-differential equations in variational form resp. for nonlocal bilinear forms with differentiability order $\alpha \in (0,2)$ have been studied by several authors with different methods. Important contributions include [2, 6, 16, 10, 4, 13, 12, 7, 8, 25, 26]. These articles include operators of the form (1.3) with $\mu = \mu_1$ where no further regularity assumption of $a(x,y)$ apart from boundedness is imposed. Formally speaking, Hölder regularity estimates for fractional equations are stronger than the ones for local equations if the results are robust with respect to $\alpha \to 2-$ as in [16, 8]. Hölder regularity results have also been obtained for linear and nonlinear nonlocal equations in non-divergence form, i.e., for operators not generating quadratic forms. We do not discuss these results here.

We comment on related regularity results if the measures are singular with respect to the Lebesgue measure. [1] and [27] study regularity of solutions to systems of stochastic differential equations which lead to nonlocal operators in nondivergence form with singular measures including versions of $L^{\alpha,\alpha}$ with continuous bounded coefficients. These results have been extended to the case of operators with possibly different values for α_i in [5]. Assuming that the systems studied in [1] are diagonal, [18] establishes sharp two-sided heat kernel estimates. It is very interesting that operators of the form L^{α_1,α_2} appear also in the study of random walks on groups driven by anisotropic measures. Results on the potential theory can be found in [22], [23], [24].

The closest to our article are [12] where from we borrow several ideas. [12] establishes results similar to Theorem 1.3 and Theorem 1.4 in a general framework which includes operators (1.3) with μ_{axes} and μ_2. The assumption $\alpha_1 = \alpha_2 = \ldots = \alpha_d$ is essential for the main results in [12]. The main aim of the present work is to remove this restriction. This makes it necessary to study the anisotropic setting in detail and to develop new functional inequalities resp. embedding results. Luckily, the John-Nirenberg embedding has been established in the context of general metric measure spaces. Another related article is [17], which deals with nonlocal parabolic equations involving singular jumping measures. Note that, different from [17, 12], we allow the functions u to be (super-)solutions for inhomogeneous equations.

The article is organized as follows. Section 2 contains auxiliary results like functional inequalities, embedding results, and technical results regarding cut-off functions. In Section 3 we establish several intermediate results for functions u satisfying (1.9) and prove Theorem 1.3. In Section 4 we deduce Theorem 1.4.

2. Auxiliary results

The aim of this section is to provide more or less technical results needed later. In particular, we introduce appropriate cut-off functions, establish Sobolev-type embeddings and prove a Poincaré inequality in our anisotropic setting.

2.1. **Cut-off functions.** As mentioned above, we need to impose one condition further to Assumption 1 and Assumption 2. We need to make sure that the nonlocal operator $\mathcal{L}$ resp. the corresponding quadratic form behaves nicely with respect to cut-off functions. Let us first explain a simple example. If $r > 0$ and $\tau \in C_c^2(\overline{B_{2r}})$ with $\tau \equiv 1$ on B_r and τ linear on $B_{2r} \setminus B_r$, then $|\nabla \tau| \leq c r^{-1}$ in $\mathbb{R}^d$ with a constant $c > 0$ independent of r. As we shall explain below, there is a similar relation in our nonlocal anisotropic setting. Note that, in general, the nonlocal analog of $|\nabla \tau(x)|^2$ is given by $\frac{1}{2}\int_{\mathbb{R}^d} (\tau(y) - \tau(x))^2 \mu(x, \mathrm{d}y)$. In the framework of Dirichlet forms, both objects are the corresponding carré du champ operator of τ.

We make the following assumption on the family $(\mu(x, \cdot))_{x \in \mathbb{R}^d}$.

Assumption 3. *There is a constant $C \geq 1$ such that for every $x \in M_3$ and $r > 0$*

$$\mu(x, \mathbb{R}^d \setminus M_r(x)) \leq C \mu_{axes}(x, \mathbb{R}^d \setminus M_r(x)). \tag{A3}$$

One can easily show that for any $r > 0$ and $x \in \mathbb{R}^d$

$$\mu_{\text{axes}}(x, \mathbb{R}^d \setminus M_r(x)) \asymp r^{-\alpha_{\max}}, \tag{2.1}$$

where the implicit constant depends only on $\alpha_{\min}, \alpha_{\max}$ and d. Assumption 3 implies bounds on $\mu(x, \mathbb{R}^d \setminus M_r(x))$ for $r \to 0$ as well as for $r \to \infty$. Note that one could relax the assumption further by imposing a weaker bound for $r \geq 1$. Theorem 1.4 remains true if one assumes *some* polynomial decay as $r \to \infty$.

We show that Assumption 3 implies a nice behavior of the family μ with respect to cut-off functions. We say that $\tau = (\tau_{x_0,r,\lambda})_{x_0,r,\lambda}$ is an admissable family of cut-off functions $\tau_{x_0,r,\lambda} \in C^{0,1}(\mathbb{R}^d)$ if for some constant $c \geq 1$ the following is true for all $x_0 \in \mathbb{R}^d$, $0 < r \leq 1$ and $1 < \lambda \leq 2$:

$$\begin{cases} \operatorname{supp}(\tau) \subset M_{\lambda r}(x_0), \quad \|\tau\|_\infty \leq 1, \quad \tau \equiv 1 \text{ on } M_r(x_0), \\ \|\partial_k \tau\|_\infty \leq \frac{c}{(\lambda^{\frac{\alpha_{\max}}{\alpha_k}} - 1) r^{\frac{\alpha_{\max}}{\alpha_k}}} \text{ for all } k \in \{1, \dots, d\}. \end{cases} \tag{2.2}$$

The existence of such a family is standard.

Lemma 2.1. *Assume μ satisfies Assumption 1 and Assumption 3. There is $c_1 \geq 1$ such that for all $x_0 \in M_1$, $r \in (0, 1], 1 < \lambda \leq 2$ and every admissible cut-off function in the sense of (2.2) the following is true:*

$$\sup_{x \in M_3} \int_{\mathbb{R}^d} (\tau(y) - \tau(x))^2 \mu(x, \mathrm{d}y) \leq c_1 r^{-\alpha_{\max}} \left(\sum_{k=1}^d (\lambda^{\frac{\alpha_{\max}}{\alpha_k}} - 1)^{-\alpha_k} \right).$$

The constant c_1 can be chosen independently of $\alpha_1, \dots, \alpha_d$.

Proof. As explained above, for every $r > 0$ and $x \in \mathbb{R}^d$ we have $\mu_{\text{axes}}(x, \mathbb{R}^d \setminus M_r(x)) \leq 4d r^{-\alpha_{\max}}$. Note that the assertion of Lemma 2.1 can be proved directly in the case where μ equals μ_{axes}.

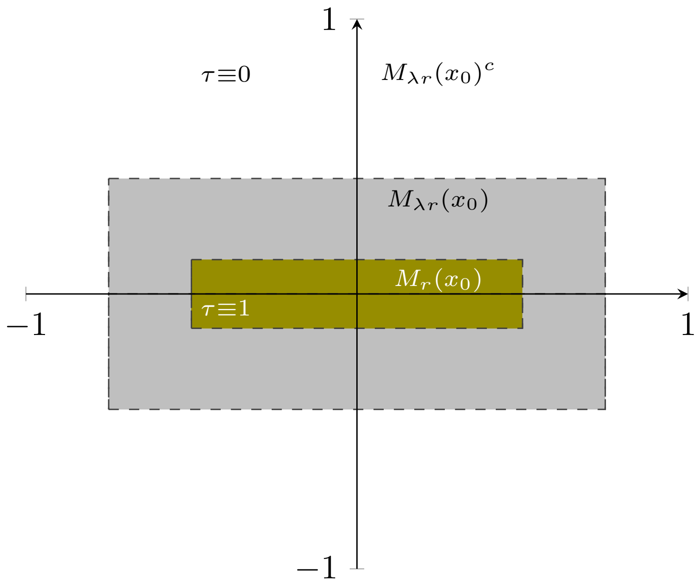

FIGURE 1. Example of τ for $x_0 = 0$, $\alpha_1 = \frac{3}{2}$, $\alpha_2 = \frac{1}{2}$, $r = \frac{1}{2}$, $\lambda = \frac{3}{2}$

Given $x \in M_3$ and $y \in \mathbb{R}^d$, let $\ell = (\ell_0(x,y), \dots, \ell_d(x,y)) \in \mathbb{R}^{d(d+1)}$ be a polygonal chain connecting x and y with

$$\ell_k(x,y) = (l_1^k, \dots, l_d^k), \quad \text{where } \begin{cases} l_j^k = y_j, & \text{if } j \le k, \\ l_j^k = x_j, & \text{if } j > k. \end{cases}$$

Then $x = \ell_0(x,y)$, $y = \ell_d(x,y)$ and $|\ell_{k-1}(x,y) - \ell_k(x,y)| = |x_k - y_k|$ for all $k \in \{1, \dots, d\}$. We observe

$$\begin{aligned} \int_{\mathbb{R}^d} (\tau(x) - \tau(y))^2 \mu(x, \mathrm{d}y) &= \int_{\mathbb{R}^d} \left(\sum_{k=1}^d (\tau(\ell_{k-1}(x,y)) - \tau(\ell_k(x,y))) \right)^2 \mu(x, \mathrm{d}y) \\ &\le d \sum_{k=1}^d \int_{\mathbb{R}^d} (\tau(\ell_{k-1}(x,y)) - \tau(\ell_k(x,y)))^2 \mu(x, \mathrm{d}y) =: d \sum_{k=1}^d I_k. \end{aligned}$$

For $k \in \{1, \dots, d\}$, set $\eta_k = (\lambda^{\frac{\alpha_{\max}}{\alpha_k}} - 1)^{\frac{\alpha_k}{\alpha_{\max}}}$. Then

$$\begin{aligned} I_k = &\int_{M_{\eta_k r}(x)} (\tau(\ell_{k-1}(x,y)) - \tau(\ell_k(x,y)))^2 \mu(x, \mathrm{d}y) \\ &+ \int_{\mathbb{R}^d \setminus M_{\eta_k r}(x)} (\tau(\ell_{k-1}(x,y)) - \tau(\ell_k(x,y)))^2 \mu(x, \mathrm{d}y) =: A + B. \end{aligned}$$

Using Assumption 3, (2.1) and the definition of η_k,

$$B \le \mu(x, \mathbb{R}^d \setminus M_{\eta_k r}(x)) \le c_2 (\lambda^{\frac{\alpha_{\max}}{\alpha_k}} - 1)^{-\alpha_k} r^{-\alpha_{\max}}.$$

By the mean value theorem, Assumption 3 and the definition η_k,

$$A \le \|\partial_k \tau\|_\infty^2 \int_{M_{\eta_k r}(x)} |x_k - y_k|^2 \mu(x, \mathrm{d}y)$$

$$
\begin{aligned}
&= \|\partial_k \tau\|_\infty^2 \sum_{n=0}^{\infty} \int_{M_{\eta_k r 2^{-n}}(x) \setminus M_{\eta_k r 2^{-n-1}}(x)} |x_k - y_k|^2 \mu(x, \mathrm{d}y) \\
&\le \|\partial_k \tau\|_\infty^2 \sum_{n=0}^{\infty} \int_{M_{\eta_k r 2^{-n}}(x) \setminus M_{\eta_k r 2^{-n-1}}(x)} (\eta_k r 2^{-n})^{2\frac{\alpha_{\max}}{\alpha_k}} \mu(x, \mathrm{d}y) \\
&\le \|\partial_k \tau\|_\infty^2 \sum_{n=0}^{\infty} (\eta_k r 2^{-n})^{2\frac{\alpha_{\max}}{\alpha_k}} \mu(x, \mathbb{R}^d \setminus M_{\eta_k r 2^{-n-1}}(x)) \\
&\le 4d \|\partial_k \tau\|_\infty^2 \sum_{n=0}^{\infty} (\eta_k r 2^{-n})^{2\frac{\alpha_{\max}}{\alpha_k}} (\eta_k r 2^{-n-1})^{-\alpha_{\max}} \\
&\le \frac{c_3}{(\lambda^{\frac{\alpha_{\max}}{\alpha_k}} - 1)^2 r^{2\frac{\alpha_{\max}}{\alpha_k}}} \sum_{n=0}^{\infty} (\eta_k r 2^{-n})^{2\frac{\alpha_{\max}}{\alpha_k}} (\eta_k r 2^{-n-1})^{-\alpha_{\max}} \\
&\le c_4 \left(\sum_{n=0}^{\infty} 2^{n(\alpha_{\max} - 2\frac{\alpha_{\max}}{\alpha_k})} \right) (\lambda^{\frac{\alpha_{\max}}{\alpha_k}} - 1)^{-\alpha_k} r^{-\alpha_{\max}} \\
&= c_5 (\lambda^{\frac{\alpha_{\max}}{\alpha_k}} - 1)^{-\alpha_k} r^{-\alpha_{\max}}.
\end{aligned}
$$

Summing up over all k leads to the claim. □

For future purposes, let us deduce a helpful observation.

Corollary 2.2. *Assume μ satisfies Assumption 1 and Assumption 3. Let $x_0 \in M_1$, $r \in (0,1]$, $1 < \lambda \le 2$ and $\tau \in C^1(\mathbb{R}^d)$. Assume τ satisfies (2.2). There is a constant $c_1 > 0$, independent of $u, x_0, \lambda, r, \alpha_1, \dots, \alpha_d$, such that for every $u \in V^\mu(M_{\lambda r}(x_0)|\mathbb{R}^d)$*

$$
\int_{M_{\lambda r}(x_0)} \int_{\mathbb{R}^d \setminus M_{\lambda r}(x_0)} u(x)^2 \tau(x)^2 \, \mu(x, \mathrm{d}y) \, \mathrm{d}x \le c_1 r^{-\alpha_{\max}} \left(\sum_{k=1}^{d} (\lambda^{\frac{\alpha_{\max}}{\alpha_k}} - 1)^{-\alpha_k} \right) \|u\|_{L^2(M_{\lambda r}(x_0))}^2.
$$

2.2. **Sobolev-type inequalities.** One important tool in our studies are Sobolev-type inequalities. We begin with a comparability result, which gives a representation of $(u,u)_{V^{\mu_{\text{axes}}}(\mathbb{R}^d|\mathbb{R}^d)}$ in terms of the Fourier transform of u.

Lemma 2.3. *There is a constant $C \ge 1$ such that for every $u \in V^{\mu_{axes}}(\mathbb{R}^d|\mathbb{R}^d)$*

$$
C^{-1} \Big\| \widehat{u}(\xi) \Big(\sum_{k=1}^{d} |\xi_k|^{\alpha_k} \Big)^{\frac{1}{2}} \Big\|_{L^2_\xi(\mathbb{R}^d)}^2 \le \mathcal{E}^{\mu_{axes}}(u,u) \le C \Big\| \widehat{u}(\xi) \Big(\sum_{k=1}^{d} |\xi_k|^{\alpha_k} \Big)^{\frac{1}{2}} \Big\|_{L^2_\xi(\mathbb{R}^d)}^2.
$$

Proof. By Fubini's and Plancherel's theorem,

$$
\mathcal{E}^{\mu_{\text{axes}}}(u,u) = \sum_{k=1}^{d} \alpha_k (2 - \alpha_k) \int_{\mathbb{R}^d} |\widehat{u}(\xi)|^2 \int_{\mathbb{R}} \frac{(1 - e^{i\xi_k h_k})^2}{|h_k|^{1+\alpha_k}} \, \mathrm{d}h_k \, \mathrm{d}\xi.
$$

Furthermore, there is a constant $c_1 \ge 1$, independent of $\alpha_1, \dots, \alpha_d$, such that for any $k \in \{1, \dots, d\}$

$$
c_1^{-1} |\xi_k|^{\alpha_k} \le \alpha_k (2 - \alpha_k) \int_{\mathbb{R}} \frac{(1 - e^{i\xi_k h_k})^2}{|h_k|^{1+\alpha_k}} \, \mathrm{d}h_k \le c_1 |\xi_k|^{\alpha_k}.
$$

Hence the assertion follows. □

A crucial tool in our approach is the following Sobolev-type inequality. We define the quantity

$$\beta = \sum_{j=1}^{d} \frac{1}{\alpha_j}. \tag{2.3}$$

Theorem 2.4. *There is a constant* $c_1 = c_1(d, 2\beta/(\beta-1)) > 0$ *such that for every compactly supported* $u \in V^{\mu_{axes}}(\mathbb{R}^d|\mathbb{R}^d)$

$$\|u\|^2_{L^{\frac{2\beta}{\beta-1}}(\mathbb{R}^d)} \leq c_1 \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x)-u(y))^2 \, \mu_{axes}(x, \mathrm{d}y) \, \mathrm{d}x \right).$$

We thank A. Schikorra for discussing this result and its proof with us. We believe that this result has been established several times in the literature but we were not able to find a reference. Note that the case $\alpha_1 = \dots = \alpha_d = \alpha \in (0,2)$ implies $\beta = d/\alpha$ and

$$\Theta := \frac{2\beta}{\beta-1} = \frac{2d}{d-\alpha}$$

in Theorem 2.4. Hence, in this case, Theorem 2.4 is a direct application of the Sobolev embedding $H^{\alpha/2}(\mathbb{R}^d) \subset L^{\Theta}(\mathbb{R}^d)$.

Proof. Let $\Theta := 2\beta/(\beta-1)$. We denote the Hölder conjugate of Θ by Θ'. Note

$$\begin{aligned} \|u\|_{L^{\Theta}(\mathbb{R}^d)} &= \|u\|_{L^{\Theta,\Theta}(\mathbb{R}^d)} \leq c_2 \|u\|_{L^{\Theta,2}(\mathbb{R}^d)} \leq c_3 \|\widehat{u}\|_{L^{\Theta',2}(\mathbb{R}^d)} \\ &\leq c_3 \left\| \left(\sum_{k=1}^{d} |\xi_k|^{\alpha_k} \right)^{-\frac{1}{2}} \right\|_{L_\xi^{2\Theta'/(2-\Theta'),\infty}(\mathbb{R}^d)} \left\| \left(\sum_{k=1}^{d} |\xi_k|^{\alpha_k} \right)^{\frac{1}{2}} \widehat{u}(\xi) \right\|_{L_\xi^2(\mathbb{R}^d)}. \end{aligned} \tag{2.4}$$

Our aim is to show

$$K(\xi) = \left(\sum_{k=1}^{d} |\xi_k|^{\alpha_k} \right)^{-\frac{1}{2}} \in L^{2\Theta'/(2-\Theta'),\infty}(\mathbb{R}^d),$$

which implies the assertion by Lemma 2.3.

Let $\xi \in \mathbb{R}^d$. Then there is obviously an index $i \in \{1, \dots, d\}$ such that

$$|\xi_i|^{\alpha_i} \geq |\xi_j|^{\alpha_j} \quad \text{for all } j \neq i.$$

Thus there is a $c_4 \geq 1$, depending only on d, such that

$$c_4^{-1} |\xi_i|^{-\alpha_i/2} \leq \left(\sum_{j=1}^{d} |\xi_j|^{\alpha_i} \right)^{-1/2} = \left(|\xi_i|^{\alpha_k} \left(1 + \sum_{j \neq i} \frac{|\xi_j|^{\alpha_j}}{|\xi_i|^{\alpha_i}} \right) \right)^{-1/2} \leq c_4 |\xi_i|^{-\alpha_i/2}.$$

Hence

$$\begin{aligned} |\{|K(\xi) \geq t\}| &= \left| \left\{ \left| \left(\sum_{k=1}^{d} |\xi_k|^{\alpha_k} \right)^{-1/2} \right| \geq t \right\} \right| \\ &\leq \sum_{i=1}^{d} \left| \{ (|\xi_i|^{-\alpha_i/2} \geq t) \wedge (|\xi_i|^{\alpha_i} \geq |\xi_j|^{\alpha_j}) \text{ for all } j \neq i \} \right| \\ &= \sum_{i=1}^{d} \left| \{ (|\xi_i| \leq t^{-2/\alpha_i}) \wedge (|\xi_j| \leq |\xi_i|^{\alpha_i/\alpha_j}) \text{ for all } j \neq i \} \right| =: c_4 \sum_{i=1}^{d} \eta_i. \end{aligned}$$

For each $i \in \{1, \dots, d\}$, we have

$$\eta_i = 2^d \int_0^{t^{-2/\alpha_i}} \left(\prod_{j\neq i} \int_0^{\xi_i^{\alpha_i/\alpha_j}} \mathrm{d}\xi_j \right) \mathrm{d}\xi_i = 2^d \int_0^{t^{-2/\alpha_i}} \xi_i^{\sum_{j\neq i} \frac{\alpha_i}{\alpha_j}} \mathrm{d}\xi_i$$
$$= \frac{2^d}{\sum_{j\neq i} \frac{\alpha_i+\alpha_j}{\alpha_j}} t^{-\frac{2}{\alpha_i}\left(\sum_{j\neq i} \frac{\alpha_i}{\alpha_j}+1\right)} \leq \frac{2^d}{d-1} t^{-2\left(\sum_{j=1}^d \frac{1}{\alpha_j}\right)} = c_5 t^{-2\beta}.$$

Hence, we have $K \in L^{2\beta,\infty}$, if

$$\frac{2\Theta'}{2-\Theta'} = 2\beta \iff \frac{2-\Theta'}{\Theta'} = \frac{1}{\beta} \iff \frac{1}{\Theta} = \frac{1}{2} - \frac{1}{2\beta} = \frac{1}{2}\left(\frac{\beta-1}{\beta}\right) \iff \Theta = \frac{2\beta}{\beta-1},$$

from which the assertion follows. ☐

Theorem 2.5. *Assume μ satisfies Assumption 1 and Assumption 3. Let $x_0 \in M_1$, $r \in (0,1]$ and $1 < \lambda \leq 2$. Then there is a constant $c_1 = c_1(d, 2\beta/(\beta-1)) > 0$, independent of $x_0, \lambda, r, \alpha_1, \dots, \alpha_d$, such that for $u \in V^{\mu_{axes}}(M_{\lambda r}(x_0)|\mathbb{R}^d)$*

$$\|u\|^2_{L^{\frac{2\beta}{\beta-1}}(M_r(x_0))} \leq c_1 \Bigg(\int_{M_{\lambda r}(x_0)} \int_{M_{\lambda r}(x_0)} (u(x)-u(y))^2 \, \mu_{axes}(x, \mathrm{d}y) \, \mathrm{d}x$$
$$+ r^{-\alpha_{\max}} \left(\sum_{k=1}^d (\lambda^{\frac{\alpha_{\max}}{\alpha_k}} - 1)^{-\alpha_k} \right) \|u\|^2_{L^2(M_{\lambda r}(x_0))} \Bigg). \tag{2.5}$$

Proof. Let $\tau : \mathbb{R}^d \to \mathbb{R}$ be as in (2.2). For simplicity of notation we write $M_r = M_r(x_0)$. Let $v \in L^2(\mathbb{R}^d)$ such that $v \equiv u$ on $M_{\lambda r}$ and $\mathcal{E}^\mu(v, v) < \infty$.

By Theorem 2.4 there is a $c_2 = c_2(d, \Theta) > 0$ such that

$$\|v\tau\|^2_{L^\Theta(\mathbb{R}^d)} \leq c_2 \Bigg(\int_{M_{\lambda r}} \int_{M_{\lambda r}} (v(x)\tau(x) - v(x)\tau(y))^2 \, \mu_{\text{axes}}(x, \mathrm{d}y) \, \mathrm{d}x$$
$$+ 2 \int_{M_{\lambda r}} \int_{(M_{\lambda r})^c} (v(x)\tau(x) - v(x)\tau(y))^2 \, \mu_{\text{axes}}(x, \mathrm{d}y) \, \mathrm{d}x \Bigg)$$
$$=: c_2(I_1 + 2I_2).$$

We have

$$I_1 \leq \frac{1}{4} \Bigg(\int_{M_{\lambda r}} \int_{M_{\lambda r}} 2[(v(y)-v(x))(\tau(x)+\tau(y))]^2 \, \mu_{\text{axes}}(x, \mathrm{d}y) \, \mathrm{d}x$$
$$+ \int_{M_{\lambda r}} \int_{M_{\lambda r}} 2[(v(x)+v(y))(\tau(x)-\tau(y))]^2 \, \mu_{\text{axes}}(x, \mathrm{d}y) \, \mathrm{d}x \Bigg)$$
$$= \frac{1}{2}(J_1 + J_2),$$

Using $(\tau(x) + \tau(y)) \leq 2$ for all $x, y \in M_{\lambda r}$ leads to

$$J_1 \leq 4 \int_{M_{\lambda r}} \int_{M_{\lambda r}} (u(y)-u(x))^2 \, \mu_{\text{axes}}(x, \mathrm{d}y) \, \mathrm{d}x.$$

By $(v(x)+v(y))^2(\tau(x)-\tau(x))^2 \leq 2v(x)^2(\tau(x)-\tau(x))^2+2v(y)^2(\tau(x)-\tau(x))^2$ and Lemma 2.1, we have

$$\begin{aligned} J_2 &\leq 4\|v\|^2_{L^2(M_{\lambda r})} \sup_{x\in M_3} \int_{\mathbb{R}^d} (\tau(y)-\tau(x))^2 \mu_{\text{axes}}(x,\mathrm{d}y) \\ &\leq c_3 r^{-\alpha_{\max}} \left(\sum_{k=1}^d (\lambda^{\frac{\alpha_{\max}}{\alpha_k}}-1)^{-\alpha_k} \right) \|u\|^2_{L^2(M_{\lambda r})}. \end{aligned}$$

Moreover, by Corollary 2.2

$$I_2 \leq c_4 r^{-\alpha_{\max}} \left(\sum_{k=1}^d (\lambda^{\frac{\alpha_{\max}}{\alpha_k}}-1)^{-\alpha_k} \right) \|u\|^2_{L^2(M_{\lambda r})}.$$

Hence there is a constant c_1, independent of $x_0, \lambda, r, \alpha_1, \dots, \alpha_d$ and u, such that

$$\begin{aligned} &\|u\|^2_{L^\Theta(M_r)} = \|v\|^2_{L^\Theta(M_r)} = \|v\tau\|^2_{L^\Theta(M_r)} \leq \|v\tau\|^2_{L^\Theta(\mathbb{R}^d)} \\ &\leq c_1 \left(\int_{M_{\lambda r}} \int_{M_{\lambda r}} (u(x)-u(y))^2 \, \mu_{\text{axes}}(x,\mathrm{d}y) \, \mathrm{d}x + r^{-\alpha_{\max}} \left(\sum_{k=1}^d (\lambda^{\frac{\alpha_{\max}}{\alpha_k}}-1)^{-\alpha_k} \right) \|u\|^2_{L^2(M_{\lambda r})} \right). \end{aligned}$$

□

We deduce the following corollary.

Corollary 2.6. *Assume μ satisfies Assumption 1, Assumption 2 and Assumption 3. Let $x_0 \in M_1$ and $r \in (0,1)$. Let $1 < \lambda \leq \min(r^{-1}, 2)$. Let $\Theta = 2\beta/(\beta-1)$. Then there is $c_1 \geq 1$, independent of $x_0, \lambda, r, \alpha_1, \dots, \alpha_d$, but depending on d, Θ, such that for $u \in V^\mu(M_{\lambda r}(x_0)|\mathbb{R}^d)$.*

$$\begin{aligned} \|u\|^2_{L^{\frac{2\beta}{\beta-1}}(M_r(x_0))} \leq c_1 \Bigg(&\int_{M_{\lambda r}(x_0)} \int_{M_{\lambda r}(x_0)} (u(x)-u(y))^2 \, \mu(x,\mathrm{d}y) \, \mathrm{d}x \\ &+ r^{-\alpha_{\max}} \left(\sum_{k=1}^d (\lambda^{\frac{\alpha_{\max}}{\alpha_k}}-1)^{-\alpha_k} \right) \|u\|^2_{L^2(M_{\lambda r}(x_0))} \Bigg). \end{aligned}$$

Proof. Since by assumption $\rho := \lambda r \leq 1$, the assertion follows immediately by Theorem 2.5 and Assumption 2. □

2.3. **Poincaré inequality.** Finally, we establish a Poincaré inequality in our setting. Let $\Omega \subset \mathbb{R}^d$ be an open and bounded set. For $f \in L^1(\Omega)$, set

$$[f]_\Omega := \fint_\Omega f(x)\,\mathrm{d}x = \frac{1}{|\Omega|} \int_\Omega f(x)\,\mathrm{d}x.$$

Lemma 2.7. *Assume μ satisfies Assumption 1 and Assumption 2. There exists a constant $c_1 \geq 1$ such that for every $r \in (0,1]$, $x_0 \in M_1$ and $v \in V^\mu(M_r(x_0)|\mathbb{R}^d)$*

$$\|v-[v]_{M_r(x_0)}\|^2_{L^2(M_r(x_0))} \leq c_1 r^{\alpha_{\max}} \mathcal{E}^\mu_{M_r(x_0)}(v,v).$$

Proof. To simplify notation, we assume $x_0 = 0$. Via translation, the assertion follows for general $x_0 \in \mathbb{R}^d$. Let

$$\gamma = \max\left\{ (\alpha_k(2-\alpha_k))^{-1} \,|\, k \in \{1,\dots,d\} \right\}.$$

By Jensen's inequality

$$\begin{aligned}\|v-[v]_{M_r}\|^2_{L^2(M_r)} &= \int_{M_r}\left(\frac{1}{|M_r|}\int_{M_r}(v(x)-v(y))\,\mathrm{d}y\right)^2\,\mathrm{d}x\\ &\leq \frac{1}{|M_r|}\int_{M_r}\int_{M_r}(v(x)-v(y))^2\,\mathrm{d}y\,\mathrm{d}x := J.\end{aligned}$$

Let $\ell = (\ell_0(x,y),\dots,\ell_d(x,y))$ be a polygonal chain connecting x and y with

$$\ell_k(x,y) = (l_1^k,\dots,l_d^k), \quad \text{where } \begin{cases} l_j^k = y_j, & \text{if } j\leq k,\\ l_j^k = x_j, & \text{if } j>k.\end{cases}$$

Then

$$J \leq \frac{d}{|M_r|}\sum_{k=1}^d \int_{M_r}\int_{M_r}(v(\ell_{k-1}(x,y)) - v(\ell_k(x,y)))^2\,\mathrm{d}y\,\mathrm{d}x := \frac{d}{|M_r|}\sum_{k=1}^d I_k.$$

We fix $k\in\{1,\dots,d\}$ and set $w=\ell_{k-1}(x,y) = (y_1,\dots,y_{k-1},x_k,\dots,x_d)$. Let $z := x+y-w = (x_1,\dots,x_{k-1},y_k,\dots,y_d)$. Then $\ell_k(x,y) = w+e_k(z_k-w_k) = (y_1,\dots,y_k,x_{k+1},\dots,x_d)$. By Fubini's Theorem

$$\begin{aligned} I_k &= \left(\prod_{j\neq k}\int_{-r^{\alpha_{\max}/\alpha_j}}^{r^{\alpha_{\max}/\alpha_j}}\mathrm{d}z_j\right)\int_{-r^{\alpha_{\max}/\alpha_1}}^{r^{\alpha_{\max}/\alpha_1}}\cdots\int_{-r^{\alpha_{\max}/\alpha_d}}^{r^{\alpha_{\max}/\alpha_d}}\int_{-r^{\frac{\alpha_{\max}}{\alpha_k}}}^{r^{\frac{\alpha_{\max}}{\alpha_k}}}(v(w)-v(w+e_k(z_k-w_k)))^2\\ &\qquad\qquad \mathrm{d}z_k\,\mathrm{d}w_d\cdots\mathrm{d}w_1\\ &\leq 2^{d-1}r^{\sum_{j\neq k}\alpha_{\max}/\alpha_j}\int_{M_r}\int_{-r^{\frac{\alpha_{\max}}{\alpha_k}}}^{r^{\frac{\alpha_{\max}}{\alpha_k}}}(v(w)-v(w+e_k(z_k-w_k)))^2\frac{2^{1+\alpha_k}r^{\frac{\alpha_{\max}}{\alpha_k}+\alpha_{\max}}}{|x_k-z_k|^{1+\alpha_k}}\,\mathrm{d}z_k\,\mathrm{d}w\\ &\leq 42^d r^{\alpha_{\max}\beta}r^{\alpha_{\max}}\int_{M_r}\int_{-r^{\frac{\alpha_{\max}}{\alpha_k}}}^{r^{\frac{\alpha_{\max}}{\alpha_k}}}(v(w)-v(w+e_k(z_k-w_k)))^2\frac{1}{|x_k-z_k|^{1+\alpha_k}}\,\mathrm{d}z_k\,\mathrm{d}w\\ &\leq \frac{4r^{\alpha_{\max}}}{\alpha_k(2-\alpha_k)}|M_r|\int_{M_r}\int_{-r^{\frac{\alpha_{\max}}{\alpha_k}}}^{r^{\frac{\alpha_{\max}}{\alpha_k}}}(v(w)-v(w+e_k(z_k-w_k)))^2\frac{\alpha_k(2-\alpha_k)}{|x_k-z_k|^{1+\alpha_k}}\,\mathrm{d}z_k\,\mathrm{d}w\\ &\leq 4r^{\alpha_{\max}}\gamma|M_r|\int_{M_r}\int_{-r^{\frac{\alpha_{\max}}{\alpha_k}}}^{r^{\frac{\alpha_{\max}}{\alpha_k}}}(v(w)-v(w+e_k(z_k-w_k)))^2\frac{\alpha_k(2-\alpha_k)}{|x_k-z_k|^{1+\alpha_k}}\,\mathrm{d}z_k\,\mathrm{d}w.\end{aligned}$$

Hence there are $c_1,c_2>0$, independent of ρ, v and x_0, but depending on d and γ, such that

$$\begin{aligned}&\|v-[v]_{M_r}\|^2_{L^2(M_r)}\\ &\leq c_2 r^{\alpha_{\max}}\sum_{k=1}^d\int_{M_r}\int_{-r^{\frac{\alpha_{\max}}{\alpha_k}}}^{r^{\frac{\alpha_{\max}}{\alpha_k}}}(v(w)-v(w+e_k(z_k-w_k)))^2\frac{\alpha_k(2-\alpha_k)}{|x_k-z_k|^{1+\alpha_k}}\,\mathrm{d}z_k\,\mathrm{d}w\\ &= c_2 r^{\alpha_{\max}}\mathcal{E}^{\mu_{\text{axes}}}_{M_r}(v,v) \leq c_1 r^{\alpha_{\max}}\mathcal{E}^{\mu}_{M_r}(v,v),\end{aligned}$$

where we used Assumption 2 in the last inequality. □

3. Properties of weak supersolutions

In this section we prove our main auxiliary result, that is a weak Harnack inequality for weak supersolutions using the Moser iteration technique. For this purpose we establish a Poincaré inequality and show that the logarithm of weak supersolutions are functions of bounded mean oscillation. Note, that throughout this section we assume Assumption 1, Assumption 2 and Assumption 3.

Let $\lambda > 0$, $\Omega \subset \mathbb{R}^d$ be open, $u \in V^{\mu_{\text{axes}}}(\Omega|\mathbb{R}^d)$ and $\Psi : \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism defined by

$$\Psi(x) = \begin{pmatrix} \lambda^{\frac{\alpha_{\max}}{\alpha_1}} & \cdots & 0 \\ \vdots & \ddots & 0 \\ 0 & 0 & \lambda^{\frac{\alpha_{\max}}{\alpha_d}} \end{pmatrix} x.$$

Then by change of variables, the energy form $\mathcal{E}^{\mu_{\text{axes}}}_{\Omega}$ behaves as follows

$$\mathcal{E}^{\mu_{\text{axes}}}_{\Omega}(u \circ \Psi, u \circ \Psi) = \lambda^{\alpha_{\max} - \alpha_{\max}\beta} \mathcal{E}^{\mu_{\text{axes}}}_{\Psi(\Omega)}(u, u).$$

The next lemma provides a key estimate for $\log u$.

Lemma 3.1. *Let $x_0 \in M_1$, $r \in (0, 1]$ and $1 < \lambda \leq 2$. Assume $f \in L^q(M_{\lambda r}(x_0))$ for some $q > 2$. Assume $u \in V^{\mu}(M_{\lambda r}(x_0)\big|\mathbb{R}^d)$ is nonnegative in $\mathbb{R}^d$ and satisfies*

$$\begin{aligned} \mathcal{E}^{\mu}(u, \varphi) &\geq (f, \varphi) \quad \text{for any nonnegative } \varphi \in H^{\mu}_{M_{\lambda r}(x_0)}(\mathbb{R}^d), \\ u(x) &\geq \epsilon \qquad\quad \text{for almost all } x \in M_{\lambda r}(x_0) \text{ and some } \epsilon > 0. \end{aligned} \tag{3.1}$$

There exists a constant $c \geq 1$, independent of $x_0, \lambda, r, \alpha_1, \dots, \alpha_d$ and u, such that

$$\begin{aligned} &\int_{M_r(x_0)} \int_{M_r(x_0)} \left(\sum_{k=1}^{\infty} \frac{(\log u(y) - \log u(x))^{2k}}{(2k)!} \right) \mu(x, \mathrm{d}y)\, \mathrm{d}x \\ &\qquad \leq c \left(\sum_{k=1}^{d} \left(\lambda^{\frac{\alpha_{\max}}{\alpha_k}} - 1 \right)^{-\alpha_k} \right) r^{-\alpha_{\max}} |M_{\lambda r}(x_0)| + \epsilon^{-1} \|f\|_{L^q(M_{\lambda r}(x_0))} |M_{\lambda r}(x_0)|^{\frac{q-1}{q}}. \end{aligned}$$

Proof. We follow the lines of [12, Lemma 4.4].

Let $\tau : \mathbb{R}^d \to \mathbb{R}$ be as in (2.2). Then by Lemma 2.1 and (A3), there is $c_2 > 0$, such that

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(y) - \tau(x))^2 \mu(x, \mathrm{d}y) \leq c_2 r^{-\alpha_{\max}} \left(\sum_{k=1}^{d} (\lambda^{\frac{\alpha_{\max}}{\alpha_k}} - 1)^{-\alpha_k} \right).$$

For brevity, we write $M_{\lambda r}(x_0) = M_{\lambda r}$ and $M_r(x_0) = M_r$ within this proof. By definition of τ and (A1-b),

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (\tau(y)-\tau(x))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\qquad= \int_{M_{\lambda r}}\int_{M_{\lambda r}} (\tau(y)-\tau(x))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x + 2\int_{M_{\lambda r}}\int_{M_{\lambda r}^c} (\tau(y)-\tau(x))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\qquad\le 2\int_{M_{\lambda r}}\int_{\mathbb{R}^d} (\tau(y)-\tau(x))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\qquad\le 2|M_{\lambda r}| \sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d} (\tau(y)-\tau(x))^2\,\mu(x,\mathrm{d}y) \\
&\qquad\le c_3\left(\sum_{k=1}^d (\lambda^{\frac{\alpha_{\max}}{\alpha_k}}-1)^{-\alpha_k}\right) r^{-\alpha_{\max}}|M_{\lambda r}|.
\end{aligned} \tag{3.2}$$

Let $-\varphi(x) = -\tau^2(x)u^{-1}(x) \le 0$. By (3.1), we deduce as in the proof of [16, Lemma 3.3]

$$\begin{aligned}
(f,-\varphi) &\ge \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (u(y)-u(x))\left(\tau^2(x)u^{-1}(x)-\tau^2(y)u^{-1}(y)\right)\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&= \int_{M_{\lambda r}}\int_{M_{\lambda r}} \tau(x)\tau(y)\left(\frac{\tau(x)u(y)}{\tau(y)u(x)}+\frac{\tau(y)u(x)}{\tau(x)u(y)}-\frac{\tau(y)}{\tau(x)}-\frac{\tau(x)}{\tau(y)}\right)\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\quad + 2\int_{M_{\lambda r}}\int_{M_{\lambda r}^c} (u(y)-u(x))\left(\tau^2(x)u^{-1}(x)-\tau^2(y)u^{-1}(y)\right)\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\ge \int_{M_r}\int_{M_r}\left(2\sum_{k=1}^\infty \frac{(\log u(y)-\log u(x))^{2k}}{(2k)!}\right)\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\quad - \int_{M_{\lambda r}}\int_{M_{\lambda r}} (\tau(x)-\tau(y))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\quad + 2\int_{M_{\lambda r}}\int_{M_{\lambda r}^c} (u(y)-u(x))\left(\tau^2(x)u^{-1}(x)-\tau^2(y)u^{-1}(y)\right)\,\mu(x,\mathrm{d}y)\,\mathrm{d}x,
\end{aligned}$$

Using the nonnegativity of u in $\mathbb{R}^d$, the third term on the right-hand side can be estimated as follows:

$$\begin{aligned}
&2\int_{M_{\lambda r}}\int_{M_{\lambda r}^c} (u(y)-u(x))\left(\tau^2(x)u^{-1}(x)-\tau^2(y)u^{-1}(y)\right)\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&= 2\int_{M_{\lambda r}}\int_{M_{\lambda r}^c} (u(y)-u(x))\left(-\tau^2(x)u^{-1}(x)\right)\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&= 2\int_{M_{\lambda r}}\int_{M_{\lambda r}^c} \frac{\tau^2(x)}{u(x)}u(y)\,\mu(x,\mathrm{d}y)\,\mathrm{d}x - 2\int_{M_{\lambda r}}\int_{M_{\lambda r}^c} \tau^2(x)\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\ge -2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (\tau(y)-\tau(x))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x,
\end{aligned}$$

Therefore, by the Hölder inequality and $|u^{-1}| \leq \epsilon^{-1}$

$$\begin{aligned} &\int_{M_r}\int_{M_r}\left(2\sum_{k=1}^{\infty}\frac{(\log u(y)-\log u(x))^{2k}}{(2k)!}\right)\mu(x,\mathrm{d}y)\,\mathrm{d}x \\ &\leq 3\int_{M_{\lambda r}}\int_{\mathbb{R}^d}(\tau(x)-\tau(y))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x+(f,-\tau^2u^{-1}) \\ &\leq c_1\left(\sum_{k=1}^{d}\left(\lambda^{\frac{\alpha_{\max}}{\alpha_k}}-1\right)^{-\alpha_k}\right)r^{-2}|M_{\lambda r}|+\|f\|_{L^q(M_{\lambda r})}\|u^{-1}\|_{L^{q/(q-1)}(M_{\lambda r})} \\ &\leq c_1\left(\sum_{k=1}^{d}\left(\lambda^{\frac{\alpha_{\max}}{\alpha_k}}-1\right)^{-\alpha_k}\right)r^{-2}|M_{\lambda r}|+\epsilon^{-1}\|f\|_{L^q(M_{\lambda r})}|M_{\lambda r}|^{(q-1)/q}. \end{aligned} \tag{3.3}$$

□

The following corollary is a direct consequence of Lemma 2.7 for $v=\log(u)$ and Lemma 3.1.

Corollary 3.2. *Let $x_0\in M_1$, $r\in(0,1]$ and $\frac{5}{4}\leq\lambda\leq 2$. Let $f\in L^q(M_{2r}(x_0))$ for some $q>2$. Assume $u\in V^{\mu}(M_{\lambda r}(x_0)|\mathbb{R}^d)$ is nonnegative in $\mathbb{R}^d$ and satisfies*

$$\begin{aligned} \mathcal{E}^{\mu}(u,\varphi)&\geq(f,\varphi) \quad \text{for any nonnegative } \varphi\in H^{\mu}_{M_{\lambda r}(x_0)}(\mathbb{R}^d), \\ u(x)&\geq\epsilon \qquad \text{for almost all } x\in M_{2r} \text{ and } \epsilon>r^{\alpha_{\max}(q-\beta)/q}\|f\|_{L^q(M_{\lambda r}(x_0))}. \end{aligned} \tag{3.4}$$

Then there exists a constant $c\geq 1$, independent of x_0, r and u, such that

$$\|\log u-[\log u]_{M_r(x_0)}\|^2_{L^2(M_r(x_0))}\leq c|M_r(x_0)|. \tag{3.5}$$

Proof. Set $M_r=M_r(x_0)$ and $M_{\lambda r}=M_{\lambda r}(x_0)$. Note

$$|M_{\lambda r}|=\left(\prod_{k=1}^{d}2(\lambda r)^{\frac{\alpha_{\max}}{\alpha_k}}\right)=\lambda^{\alpha_{\max}\beta}2^d r^{\alpha_{\max}\beta}=\lambda^{\alpha_{\max}\beta}|M_r|. \tag{3.6}$$

By Lemma 2.7 for $v:=\log(u)$, Lemma 3.1 and (3.6), we observe

$$\begin{aligned} &\|\log u-[\log u]_{M_r}\|^2_{L^2(M_r)}\leq c_1 r^{\alpha_{\max}}\mathcal{E}^{\mu}_{M_r(x_0)}(\log u,\log u) \\ &\leq 2c_1r^{\alpha_{\max}}\int_{M_r}\int_{M_r}\left(\sum_{k=1}^{\infty}\frac{(\log u(y)-\log u(x))^{2k}}{(2k)!}\right)\mu(x,\mathrm{d}y)\,\mathrm{d}x \\ &\leq 2c_1r^{\alpha_{\max}}\left(c_3\left(\sum_{k=1}^{d}\left(\lambda^{\frac{\alpha_{\max}}{\alpha_k}}-1\right)^{-\alpha_k}\right)r^{-\alpha_{\max}}|M_{\lambda r}|+\epsilon^{-1}\|f\|_{L^q(M_{\lambda r})}|M_{\lambda r}|^{(q-1)/q}\right) \\ &\leq 2c_1r^{\alpha_{\max}}\left(c_3\left(\sum_{k=1}^{d}\left(\left(\frac{5}{4}\right)^{\frac{\alpha_{\max}}{\alpha_k}}-1\right)^{-\alpha_k}\right)r^{-\alpha_{\max}}|M_{\lambda r}|+r^{-\alpha_{\max}(q-\beta)/q}|M_{\lambda r}|^{(q-1)/q}\right) \\ &=2c_1r^{\alpha_{\max}}\left(c_3\left(\sum_{k=1}^{d}\left(\left(\frac{5}{4}\right)^{\frac{\alpha_{\max}}{\alpha_k}}-1\right)^{-\alpha_k}\right)r^{-\alpha_{\max}}|M_{\lambda r}|+2^d\lambda^{\alpha_{\max}(q-1)/q}r^{-\alpha_{\max}+\alpha_{\max}\beta}\right) \\ &\leq 2c_1\left(c_3c_4d|M_{\lambda r}|+4|M_r|\right) \\ &=2c_1\left(c_3c_4d2^{2\beta}|M_r|+4|M_r|\right)=c_1(d,\beta)|M_r|. \end{aligned}$$

Here we have used the fact, that there is a $c_4 = c_4(\alpha_{\max}) > 0$ such that $\max\{(5/4)^{\alpha_{\max}/x} - 1)^{-x} | \, x \in (0, \alpha_{\max}]\} \le c_4$. $\square$

A consequence of the foregoing results is the following theorem.

Theorem 3.3. *Assume $x_0 \in M_1$, $r \in (0,1]$ and $f \in L^q(M_{\frac{5}{4}r}(x_0))$ for some $q > 2$. Assume $u \in V^\mu(M_{\frac{5}{4}r}(x_0)|\mathbb{R}^d)$ is nonnegative in $\mathbb{R}^d$ and satisfies*

$$\begin{aligned} &\mathcal{E}^\mu(u,\varphi) \ge (f,\varphi) \quad \text{for any nonnegative } \varphi \in H^\mu_{M_{\frac{5}{4}r}(x_0)}(\mathbb{R}^d), \\ &u(x) \ge \epsilon \qquad \text{for almost all } x \in M_{\frac{5}{4}r} \text{ and some } \epsilon > r^{\alpha_{\max}(q-\beta)/q} \|f\|_{L^q(M_{\frac{5}{4}r}(x_0))}. \end{aligned}$$

Then there exist $\overline{p} \in (0,1)$ and $c \ge 1$, independent of x_0, r, u and ϵ, such that

$$\left(\fint_{M_r(x_0)} u(x)^{\overline{p}} \, \mathrm{d}x \right)^{1/\overline{p}} \mathrm{d}x \le c \left(\fint_{M_r(x_0)} u(x)^{-\overline{p}} \, \mathrm{d}x \right)^{-1/\overline{p}}. \tag{3.7}$$

Proof. This proof follows the proof of [12, Lemma 4.5].

The main idea is to prove $\log u \in \mathrm{BMO}(M_r(x_0))$ and use the John-Nirenberg inequality for doubling metric measure spaces. Let $x_0 \in M_1$ and $r \in (0,1]$. Endowed with the Lebesgue measure, the metric measure space $(M_r(x_0), d, \mathrm{d}x)$ is a doubling space. Let $z_0 \in M_r(x_0)$ and $\rho > 0$ such that $M_{2\rho}(z_0) \subset M_r(x_0)$. Note that by (3.6) $|M_{2\rho}|^{\frac{q}{q-1}} \le 2^{4\beta+d}|M_\rho|$. Corollary 3.2 and the Hölder inequality imply

$$\begin{aligned} \int_{M_\rho(z_0)} \left| \log u(x) - [\log u]_{M_\rho(z_0)} \right| \mathrm{d}x &\le \| \log u - [\log u]_{M_\rho(z_0)} \|_{L^2(M_\rho(z_0))} \sqrt{|M_\rho|} \\ &\le c_2 |M_\rho|. \end{aligned}$$

This proves $\log u \in \mathrm{BMO}(M_r(x_0))$. The John-Nirenberg inequality [15, Theorem 19.5] states, that $\log u \in \mathrm{BMO}(M_r(x_0))$, iff for each $M_\rho \Subset M_r(x_0)$ and $\kappa > 0$

$$|\{x \in M_\rho | \, |\log u(x) - [\log u]_{M_\rho}| > \kappa\}| \le c_3 e^{-c_4 \kappa} |M_\rho|, \tag{3.8}$$

where the positive constants c_3, c_4 and the BMO norm depend only on each other, the dimension d and the doubling constant.

By Cavalieri's principle, we have for $h : M_R(x_0) \to [0,\infty]$, using the change of variable $t = e^\kappa$, that

$$\begin{aligned} \fint_{M_r(x_0)} e^{h(x)} \, \mathrm{d}x &= \frac{1}{|M_r|} \Bigg(\int_0^1 |\{x \in M_r(x_0) | \, e^{h(x)} > t\}| \, \mathrm{d}t \\ &\qquad + \int_1^\infty |\{x \in M_r(x_0) | \, e^{h(x)} > t\}| \, \mathrm{d}t \Bigg) \\ &\le 1 + \frac{1}{|M_r|} \int_0^\infty e^\kappa |\{x \in M_r(x_0) | \, h(x) > \kappa\}| \, \mathrm{d}\kappa. \end{aligned}$$

Let $\overline{p} \in (0,1)$ be chosen such that $\overline{p} < c_4$. The application of (3.8) implies

$$\begin{aligned} &\fint_{M_r(x_0)} \exp\left(\overline{p} | \log u(y) - [\log u]_{M_r(x_0)}| \right) \, \mathrm{d}y \\ &\qquad \le 1 + \int_0^\infty e^\kappa \frac{|\{x \in M_r(x_0) | \, |\log u(x) - [\log u]_{M_r(x_0)}| > \kappa/\overline{p}\}|}{|M_r|} \, \mathrm{d}\kappa \end{aligned}$$

$$\begin{aligned}
&\leq 1+\int_0^\infty e^\kappa \frac{c_3 e^{-c_4\kappa/\overline{p}}|M_r(x_0)|}{|M_r|}\,\mathrm{d}\kappa \\
&\leq 1+c_3\int_0^\infty e^{(1-c_4/\overline{p})\kappa}\,\mathrm{d}\kappa \\
&= 1+\frac{c_3}{c_4/\overline{p}-1} = \frac{c_4-\overline{p}+c_3\overline{p}}{c_4-\overline{p}} =: c_5<\infty.
\end{aligned}$$

Hence

$$\begin{aligned}
&\left(\fint_{M_r(x_0)} u(y)^{\overline{p}}\,\mathrm{d}y\right)\left(\fint_{M_r(x_0)} u(y)^{-\overline{p}}\,\mathrm{d}y\right) \\
&= \left(\fint_{M_r(x_0)} e^{\overline{p}(\log u(y)-[\log u]_{M_r})}\,\mathrm{d}y\right)\left(\fint_{M_r(x_0)} e^{-\overline{p}(\log u(y)-[\log u]_{M_r})}\,\mathrm{d}y\right) \leq c_5^2 = c_1.
\end{aligned}$$

□

3.1. **The weak Harnack inequality.** In this subsection we prove the weak Harnack inequality Theorem 1.3, using the Moser iteration technique for negative exponents.

Let $\Omega\subset\mathbb{R}^d$ be open and bounded. Let $q>\beta$ and $f\in L^{\frac{\beta}{\beta-1}}(\Omega)$. Then Lyapunov's inequality implies for any $a>0$

$$\|f\|_{L^{\frac{q}{q-1}}(\Omega)} \leq \frac{\beta}{q}a\,\|f\|_{L^{\frac{\beta}{\beta-1}}(\Omega)} + \frac{q-\beta}{q}a^{-\beta/(q-\beta)}\,\|f\|_{L^1(\Omega)}\,. \tag{3.9}$$

Lemma 3.4. *There exist positive constants $c_1,c_2>0$ such that for every $a,b>0$, $p>1$ and $0\leq\tau_1,\tau_2\leq 1$ the following is true:*

$$(b-a)(\tau_1^2a^{-p}-\tau_2^2b^{-p}) \geq c_1\left(\tau_1 a^{\frac{-p+1}{2}}-\tau_2 b^{\frac{-p+1}{2}}\right)^2 - \frac{c_2 p}{p-1}(\tau_1-\tau_2)^2(b^{-p+1}+a^{-p+1})\,.$$

A proof of Lemma 3.4 can be found in the published version of [12].

Lemma 3.5. *Assume $x_0\in M_1$ and $r\in[0,1)$. Moreover, let $1<\lambda\leq\min(r^{-1},\sqrt{2})$ and $f\in L^q(M_{\lambda r}(x_0))$ for some $q>\max\{2,\beta\}$. Assume $u\in V^\mu(M_{\lambda r}(x_0)|\mathbb{R}^d)$ satisfies*

$$\begin{aligned}
\mathcal{E}^\mu(u,\varphi)&\geq(f,\varphi) \quad \text{for any nonnegative } \varphi\in H^\mu_{M_{\lambda r}(x_0)}(\mathbb{R}^d),\\
u(x)&\geq\epsilon \qquad\quad \text{for a.a. } x\in M_{\lambda r}(x_0) \text{ and some } \epsilon>\|f\|_{L^q(M_{\lambda r}(x_0))}r^{\alpha_{\max}(q-\beta)/q}.
\end{aligned}$$

Then for any $p>1$, there is a $c\geq1$ independent of $u,x_0,r,p,\alpha_1,\ldots,\alpha_d$ and ϵ, such that

$$\|u^{-1}\|^{p-1}_{L^{(p-1)\frac{\beta}{\beta-1}}(M_r(x_0))} \leq c\frac{p}{p-1}\left(\sum_{k=1}^d(\lambda^{\frac{\alpha_{\max}}{\alpha_k}}-1)^{-\alpha_k}\right)r^{-\alpha_{\max}}\,\|u^{-1}\|^{p-1}_{L^{p-1}(M_{\lambda r}(x_0))}\,.$$

Proof. Let $\tau:\mathbb{R}^d\to\mathbb{R}$ be as in (2.2). We follow the idea of the proof of [12, Lemma 4.6]. For brevity let $M_r=M_r(x_0)$. Since $\mathcal{E}^\mu(u,\varphi)\geq(f,\varphi)$ for any nonnegative $\varphi\in H_{M_r}(\mathbb{R}^d)$ we get

$$\mathcal{E}^\mu(u,-\tau^2u^{-p})\leq(f,-\tau^2u^{-p}).$$

Furthermore

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(u(y)-u(x))\left(\tau(x)^2u(x)^{-p}-\tau(y)^2u(y)^{-p}\right)\mu(x,\mathrm{d}y)\,\mathrm{d}x$$

$$
\begin{aligned}
&= \int_{M_{\lambda r}} \int_{M_{\lambda r}} (u(y)-u(x)) \left(\tau(x)^2 u(x)^{-p} - \tau(y)^2 u(y)^{-p}\right) \mu(x, \mathrm{d}y) \, \mathrm{d}x \\
&\qquad + 2 \int_{M_{\lambda r}} \int_{M_{\lambda r}^c} (u(y)-u(x)) \left(\tau(x)^2 u(x)^{-p} - \tau(y)^2 u(y)^{-p}\right) \mu(x, \mathrm{d}y) \, \mathrm{d}x \\
&=: J_1 + 2J_2.
\end{aligned}
$$

We first study J_2. By Lemma 2.1 and (A3),

$$
\begin{aligned}
J_2 &= \int_{M_{\lambda r}} \int_{M_{\lambda r}^c} u(y)\tau(x)^2 u(x)^{-p} \mu(x, \mathrm{d}y) \, \mathrm{d}x - \int_{M_{\lambda r}} \int_{M_{\lambda r}^c} \tau(x)^2 u(x)^{-p+1} \mu(x, \mathrm{d}y) \, \mathrm{d}x \\
&\geq - \int_{M_{\lambda r}} \int_{M_{\lambda r}^c} \tau(x)^2 u(x)^{-p+1} \mu(x, \mathrm{d}y) \, \mathrm{d}x \\
&\geq - \|u^{-p+1}\|_{L^1(M_{\lambda r})} \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(y)-\tau(x))^2 \mu(x, \mathrm{d}y) \\
&\geq - \|u^{-1}\|_{L^{p-1}(M_{\lambda r})}^{p-1} c_2 r^{-\alpha_{\max}} \left(\sum_{k=1}^d \left(\lambda^{\frac{\alpha_{\max}}{\alpha_k}} - 1 \right)^{-\alpha_k} \right).
\end{aligned}
$$

Applying Lemma 3.4 for $a = u(x), b = u(y), \tau_1 = \tau(x), \tau_2 = \tau(y)$ on J_1, there exist $c_3, c_4 > 0$ such that

$$
\begin{aligned}
J_1 &= \int_{M_{\lambda r}} \int_{M_{\lambda r}} (u(y)-u(x)) \left(\tau(x)^2 u(x)^{-p} - \tau(y)^2 u(y)^{-p}\right) \mu(x, \mathrm{d}y) \, \mathrm{d}x \\
&\geq c_3 \int_{M_{\lambda r}} \int_{M_{\lambda r}} \left(\tau(x)u(x)^{\frac{-p+1}{2}} - \tau(x)u(x)^{\frac{-p+1}{2}}\right)^2 \mu(x, \mathrm{d}y) \, \mathrm{d}x \\
&\qquad - c_4 \frac{p}{p-1} \int_{M_{\lambda r}} \int_{M_{\lambda r}} (\tau(y)-\tau(x))^2 (u(y)^{-p+1} + u(x)^{-p+1}) \mu(x, \mathrm{d}y) \, \mathrm{d}x.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\int_{M_{\lambda r}} \int_{M_{\lambda r}} \left(\tau(x)u(x)^{\frac{-p+1}{2}} - \tau(x)u(x)^{\frac{-p+1}{2}}\right)^2 \mu(x, \mathrm{d}y) \, \mathrm{d}x \\
&\quad \leq \frac{1}{c_3} J_1 + c_4 \frac{p}{p-1} \int_{M_{\lambda r}} \int_{M_{\lambda r}} (\tau(y)-\tau(x))^2 (u(y)^{-p+1} + u(x)^{-p+1}) \mu(x, \mathrm{d}y) \, \mathrm{d}x \\
&\quad = \frac{1}{c_3} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(y)-u(x)) \left(\tau(x)^2 u(x)^{-p} - \tau(y)^2 u(y)^{-p}\right) \mu(x, \mathrm{d}y) \, \mathrm{d}x - \frac{2}{c_3} J_2 \\
&\qquad + c_4 \frac{p}{p-1} \int_{M_{\lambda r}} \int_{M_{\lambda r}} (\tau(y)-\tau(x))^2 (u(y)^{-p+1} + u(x)^{-p+1}) \mu(x, \mathrm{d}y) \, \mathrm{d}x \\
&\quad \leq \frac{1}{c_3} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(y)-u(x)) \left(\tau(x)^2 u(x)^{-p} - \tau(y)^2 u(y)^{-p}\right) \mu(x, \mathrm{d}y) \, \mathrm{d}x \\
&\qquad + \frac{16}{c_3} \|u^{-p+1}\|_{L^1(M_{\lambda r})} r^{-\alpha_{\max}} \left(\sum_{k=1}^d \left(\lambda^{\frac{\alpha_{\max}}{\alpha_k}} - 1 \right)^{-\alpha_k} \right) \\
&\qquad + c_4 \frac{p}{p-1} \int_{M_{\lambda r}} \int_{M_{\lambda r}} (\tau(y)-\tau(x))^2 (u(y)^{-p+1} + u(x)^{-p+1}) \mu(x, \mathrm{d}y) \, \mathrm{d}x.
\end{aligned} \tag{3.10}
$$

We derive the assertion from (3.10).

The first expression of the right-hand-side of (3.10) can be estimated with the help of (3.9) as follows:

$$\begin{aligned}
\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} &(u(y)-u(x))\left(\tau(x)^2u(x)^{-p}-\tau(y)^2u(y)^{-p}\right)\mu(x,\mathrm{d}y)\,\mathrm{d}x = \mathcal{E}^{\mu}(u,-\tau^2u^{-p})\\
&\leq (f,-\tau^2u^{-p}) \leq \epsilon^{-1}|(f,-\tau^2u^{-p+1})| = \epsilon^{-1}|(\tau f,\tau u^{-p+1})|\\
&\leq \epsilon^{-1}\|\tau f\|_{L^q(\mathbb{R}^d)}\|\tau u^{-p+1}\|_{L^{\frac{q}{q-1}}(\mathbb{R}^d)}\\
&\leq \epsilon^{-1}\|\tau f\|_{L^q(\mathbb{R}^d)}\left(\frac{\beta}{q}a\|\tau u^{-p+1}\|_{L^{\frac{\beta}{\beta-1}}(\mathbb{R}^d)}+\frac{q-\beta}{q}a^{\frac{-\beta}{q-\beta}}\|\tau u^{-p+1}\|_{L^1(\mathbb{R}^d)}\right)\\
&\leq \epsilon^{-1}\|f\|_{L^q(M_{\lambda r})}\left(\frac{\beta}{q}a\|\tau u^{-p+1}\|_{L^{\frac{\beta}{\beta-1}}(\mathbb{R}^d)}+\frac{q-\beta}{q}a^{\frac{-\beta}{q-\beta}}\|\tau u^{-p+1}\|_{L^1(\mathbb{R}^d)}\right)\\
&\leq r^{\alpha_{\max}(\beta-q)/q}\left(\frac{\beta}{q}a\|\tau u^{-p+1}\|_{L^{\frac{\beta}{\beta-1}}(\mathbb{R}^d)}+\frac{q-\beta}{q}a^{\frac{-\beta}{q-\beta}}\|\tau u^{-p+1}\|_{L^1(\mathbb{R}^d)}\right),
\end{aligned}$$

where $a>0$ can be chosen arbitrarily. Set

$$a = r^{\alpha_{\max}(q-\beta)/q}\omega$$

for some $\omega>0$. Since $\lambda\leq\sqrt{2}$, for all $k\in\{1,\dots,d\}$

$$\lambda \leq (2^{1/\alpha_k}+1)^{\alpha_k/2} \iff (\lambda^{2/\alpha_k}-1)^{-\alpha_k}\geq\frac{1}{2}.$$

Using $(\lambda^{\frac{\alpha_{\max}}{\alpha_k}}-1)^{-\alpha_k}\geq(\lambda^{2/\alpha_k}-1)^{-\alpha_k}$, leads to

$$\left(\sum_{k=1}^{d}(\lambda^{\alpha_{\max}/\alpha_k}-1)^{-\alpha_k}\right)\geq 1.$$

Altogether, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} &(u(y)-u(x))\left(\tau(x)^2u(x)^{-p}-\tau(y)^2u(y)^{-p}\right)\mu(x,\mathrm{d}y)\,\mathrm{d}x\\
&\leq \frac{\beta}{q}\omega\|u^{-p+1}\|_{L^{\frac{\beta}{\beta-1}}(M_{\lambda r})}+\frac{q-\beta}{\beta}r^{-\alpha_{\max}}\omega^{\frac{-\beta}{q-\beta}}\|\tau u^{-p+1}\|_{L^1(M_{\lambda r})}\\
&\leq \frac{\beta}{q}\omega\|u^{-1}\|^{p-1}_{L^{(p-1)\frac{\beta}{\beta-1}}(M_{\lambda r})}\\
&\quad+\frac{q-\beta}{\beta}r^{-\alpha_{\max}}\left(\sum_{k=1}^{d}(\lambda^{\frac{\alpha_{\max}}{\alpha_k}}-1)^{-\alpha_k}\right)\omega^{\frac{-\beta}{q-\beta}}\|u^{-1}\|^{p-1}_{L^{p-1}(M_{\lambda r})}.
\end{aligned}\tag{3.11}$$

The third expression of the right-hand-side of (3.10) can be estimated as follows:

$$\begin{aligned}
\int_{M_{\lambda r}}\int_{M_{\lambda r}} &(\tau(y)-\tau(x))^2(u(y)^{-p+1}+u(x)^{-p+1})\mu(x,\mathrm{d}y)\,\mathrm{d}x\\
&= 2\int_{M_{\lambda r}}\int_{M_{\lambda r}}(\tau(y)-\tau(x))^2(u(x)^{-p+1})\mu(x,\mathrm{d}y)\,\mathrm{d}x\\
&\leq 2\|u^{-p+1}\|_{L^1(M_{\lambda r})}\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}(\tau(y)-\tau(x))^2\mu(x,\mathrm{d}y)\\
&\leq c_5r^{-\alpha_{\max}}\left(\sum_{k=1}^{d}\left(\lambda^{\frac{\alpha_{\max}}{\alpha_k}}-1\right)^{-\alpha_k}\right)\|u^{-1}\|^{p-1}_{L^{p-1}(M_{\lambda r})}.
\end{aligned}$$

By Corollary 2.6, we can estimate the left-hand-side of (3.10) from below

$$\int_{M_{\lambda r}}\int_{M_{\lambda r}} \left(\tau(x)u(x)^{\frac{-p+1}{2}} - \tau(x)u(x)^{\frac{-p+1}{2}}\right)^2 \mu(x, \mathrm{d}y)\,\mathrm{d}x$$
$$\geq c_6 \|\tau u^{\frac{-p+1}{2}}\|^2_{L^{\frac{2\beta}{\beta-1}}(M_r)} - r^{-\alpha_{\max}} \left(\sum_{k=1}^{d} (\lambda^{\frac{\alpha_{\max}}{\alpha_k}} - 1)^{-\alpha_k}\right) \|\tau u^{\frac{-p+1}{2}}\|^2_{L^2(M_{\lambda r})}$$
$$\geq c_6 \|u^{\frac{-p+1}{2}}\|^2_{L^{\frac{2\beta}{\beta-1}}(M_r)} - r^{-\alpha_{\max}} \left(\sum_{k=1}^{d} (\lambda^{\frac{\alpha_{\max}}{\alpha_k}} - 1)^{-\alpha_k}\right) \|u^{\frac{-p+1}{2}}\|^2_{L^2(M_{\lambda r})}$$
$$= c_6 \|u^{-1}\|^{p-1}_{L^{(p-1)\frac{\beta}{\beta-1}}(M_r)} - r^{-\alpha_{\max}} \left(\sum_{k=1}^{d} (\lambda^{\frac{\alpha_{\max}}{\alpha_k}} - 1)^{-\alpha_k}\right) \|u^{-1}\|^{p-1}_{L^{p-1}(M_{\lambda r})}.$$

Combining these estimates there exists a constant $c_1 > 0$, independent of $x_0, r, \lambda, \alpha_1, \ldots, \alpha$ and u, but depending on d and $2\beta/(\beta-1)$, such that

$$\|u^{-1}\|^{p-1}_{L^{(p-1)\frac{\beta}{\beta-1}}(M_r)} \leq c_1(\omega^{\frac{-\beta}{q-\beta}} + \frac{p}{p-1}) \left(\sum_{k=1}^{d} (\lambda^{\frac{\alpha_{\max}}{\alpha_k}} - 1)^{-\alpha_k}\right) r^{-\alpha_{\max}} \|u^{-1}\|^{p-1}_{L^{p-1}(M_{\lambda r})}$$
$$+ \frac{1}{c_2 c_4}\omega \|u^{-1}\|_{L^{(p-1)\frac{\beta}{\beta-1}}(M_{\lambda r})}.$$

Choosing ω small enough proves the assertion. □

Lemma 3.6. *Assume $x_0 \in M_1$ and $r \in [0,1)$. Let $1 < \lambda \leq \min(r^{-1}, \sqrt{2})$. Assume $f \in L^q(M_{\lambda r}(x_0))$ for some $q > \max\{2, \beta\}$ and let $u \in V^{\mu}(M_{\lambda r}(x_0)|\mathbb{R}^d)$ satisfy*

$$\mathcal{E}^{\mu}(u, \varphi) \geq (f, \varphi) \quad \text{for any nonnegative } \varphi \in H^{\mu}_{M_{\lambda r}}(\mathbb{R}^d),$$
$$u(x) \geq \epsilon \qquad \text{for almost all } x \in M_{\lambda r} \text{ and some } \epsilon > \|f\|_{L^q(M_{\lambda r}(x_0))} r^{\alpha_{\max}(q-\beta)/q}.$$

Then for any $p_0 > 0$, there is a constant $c_1 > 0$, independent of $u, x_0, \lambda, r, \epsilon$ and $\alpha_1, \ldots, \alpha_d$, such that

$$\inf_{x \in M_r(x_0)} u(x) \geq c_1 \left(⨍_{M_{2r}(x_0)} u(x)^{-p_0}\,\mathrm{d}x\right)^{-1/p_0}. \tag{3.12}$$

Proof. We set $M_r = M_r(x_0)$. For $n \in \mathbb{N}_0$ we define the sequences

$$r_n = \left(\frac{n+2}{n+1}\right) r \quad \text{and} \quad p_n = p_0 \left(\frac{\beta}{\beta-1}\right)^n.$$

Then $r_0 = 2r$, $r_k > r_{k+1}$ for all $k \in \mathbb{N}_0$ and $r_n \searrow r$ as $n \to \infty$. Note

$$r_n = \frac{(n+2)^2}{(n+1)(n+3)} r_{n+1} =: \lambda_n r_{n+1}.$$

Moreover $p_0 = p_0$, $p_k < p_{k+1}$ for all $k \in \mathbb{N}_0$ and $p_n \nearrow +\infty$ as $n \to \infty$.
Using

$$\frac{-\alpha_{\max}}{p_n} - \frac{\alpha_{\max}\beta}{p_{n+1}} = \frac{-\alpha_{\max}\beta}{p_n},$$

we have

$$\frac{r_{n+1}^{-\alpha_{\max}/p_n}}{|M_{r_{n+1}}|^{1/p_{n+1}}} = \frac{2^{d/(\beta p_n)} \lambda_n^{\alpha_{\max}\beta/p_n}}{|M_{r_n}|^{1/p_n}}.$$

Moreover, by Lemma 3.5, we have for $p = p_n + 1$

$$\|u^{-1}\|_{L^{p_{n+1}}(M_{r_{n+1}})} = \|u^{-1}\|_{L^{p_n \frac{\beta}{\beta-1}}(M_{r_{n+1}})}$$
$$\leq c_2^{1/p_n} \left(\frac{p_n+1}{p_n}\right)^{1/p_n} \left(\sum_{k=1}^d (\lambda_n^{\frac{\alpha_{\max}}{\alpha_k}} - 1)^{-\alpha_k}\right)^{1/p_n} r_{n+1}^{-\alpha_{\max}/p_n} \|u^{-1}\|_{L^{p_n}(M_{r_n})}.$$

This yields

$$\left(\fint_{M_{r_{n+1}}} (u^{-1})^{p_{n+1}}\right)^{1/p_{n+1}}$$
$$\leq 2^{d/(\beta p_n)} \lambda_n^{\alpha_{\max}\beta/p_n} c_2^{1/p_n} \left(\frac{p_n+1}{p_n}\right)^{1/p_n} \left(\sum_{k=1}^d (\lambda_n^{\frac{\alpha_{\max}}{\alpha_k}} - 1)^{-\alpha_k}\right)^{1/p_n} \left(\fint_{M_{r_n}} (u^{-1})^{p_n}\right)^{1/p_n}.$$

which is equivalent to

$$\left(\fint_{M_{r_n}} u^{-p_n}\right)^{-1/p_n} \leq 2^{d/(\beta p_n)} \lambda_n^{\alpha_{\max}\beta/p_n} c_2^{1/p_n} \left(\frac{p_n+1}{p_n}\right)^{1/p_n}$$
$$\times \left(\sum_{k=1}^d (\lambda_n^{\frac{\alpha_{\max}}{\alpha_k}} - 1)^{-\alpha_k}\right)^{1/p_n} \left(\fint_{M_{r_{n+1}}} u^{-p_{n+1}}\right)^{-1/p_{n+1}}. \tag{3.13}$$

Iterating (3.13) leads to

$$\left(\fint_{M_{r_0}} u^{-p_0}\right)^{-1/p_0} \leq \left(\prod_{j=0}^n 2^{d/(\beta p_j)}\right) \left(\prod_{j=0}^n \lambda_j^{\alpha_{\max}\beta/p_j}\right) \left(\prod_{j=0}^n c_2^{1/p_j}\right) \left(\prod_{j=0}^n \left(\frac{p_j+1}{p_j}\right)^{1/p_j}\right)$$
$$\times \left(\prod_{j=0}^n \left(\sum_{k=1}^d (\lambda_j^{\frac{\alpha_{\max}}{\alpha_k}} - 1)^{-\alpha_k}\right)^{1/p_j}\right) \left(\fint_{M_{r_{n+1}}} u^{-p_{n+1}}\right)^{-1/p_{n+1}}. \tag{3.14}$$

One can easily show that the expressions on the right-hand-side of (3.14) are bounded for $n \to \infty$. Since

$$\lim_{n\to\infty} \left(\fint_{M_{r_n}} u^{-p_n}\right)^{-1/p_n} = \inf_{x \in M_r} u(x),$$

taking the limit $n \to \infty$ in (3.14), proves the assertion. □

From Lemma 3.6 and Theorem 3.3 we immediately conclude the following result.

Corollary 3.7. *Let $f \in L^q(M_1)$ for some $q > \max\{2, \beta\}$. There are $p_0, c > 0$ such that for every $u \in V^\mu(M_1|\mathbb{R}^d)$ with $u \geq 0$ in $\mathbb{R}^d$ and*

$$\mathcal{E}^\mu(u,\varphi) \geq (f,\varphi) \quad \text{for every nonnegative } \varphi \in H^\mu_{M_1}(\mathbb{R}^d),$$

the following holds

$$\inf_{M_{\frac14}} u \geq c \left(\fint_{M_{\frac12}} u(x)^{p_0} \, dx\right)^{1/p_0} - \|f\|_{L^q(M_{\frac{15}{16}})}.$$

Proof. This proof follows the lines of [12, Theorem 4.1]. Define $v = u + \|f\|_{L^q(M_{\frac{15}{16}})}$. Then for any nonnegative $\varphi \in H^{\mu}_{M_1}(\mathbb{R}^d)$, one obviously has

$$\mathcal{E}^{\mu}(u,\varphi) = \mathcal{E}^{\mu}(v,\varphi).$$

By Theorem 3.3 there are a $c_2 > 0$, $p_0 \in (0,1)$ such that

$$\left(\fint_{M_{\frac{1}{2}}} v(x)^{p_0} \, \mathrm{d}x \right)^{1/p_0} \mathrm{d}x \le c_2 \left(\fint_{M_{\frac{1}{2}}} v(x)^{-p_0} \, \mathrm{d}x \right)^{-1/p_0}. \tag{3.15}$$

Moreover, by Lemma 3.6 there is a $c_3 > 0$ such that for $r = \frac{1}{2}$ and p_0 as in (3.15)

$$\inf_{x \in M_{\frac{1}{4}}} v(x) \ge c_3 \left(\fint_{M_{\frac{1}{2}}} v(x)^{-p_0} \, \mathrm{d}x \right)^{-1/p_0} \ge \frac{c_3}{c_2} \left(\fint_{M_{\frac{1}{2}}} u(x)^{p_0} \, \mathrm{d}x \right)^{1/p_0}.$$

which is equivalent to

$$\inf_{M_{\frac{1}{4}}} u \ge c_1 \left(\fint_{M_{\frac{1}{2}}} u(x)^{p_0} \, \mathrm{d}x \right)^{1/p_0} - \|f\|_{L^q(M_{\frac{15}{16}})}.$$

□

Given $g : \mathbb{R}^d \to \mathbb{R}$, let $g^+(x) := \max\{g(x), 0\}, g^-(x) := -\min\{g(x), 0\}$.

We have all ingredients in order to prove Theorem 1.3.

Proof of Theorem 1.3. For any nonnegative $\varphi \in H^{\mu}_{M_1}(\mathbb{R}^d)$

$$\mathcal{E}^{\mu}(u^+,\varphi) = \mathcal{E}^{\mu}(u,\varphi) + \mathcal{E}^{\mu}(u^-,\varphi) \ge (f,\varphi) + \mathcal{E}^{\mu}(u^-,\varphi). \tag{3.16}$$

Since $\varphi \in H^{\mu}_{M_1}(\mathbb{R}^d)$ and $u^- \equiv 0$ on M_1 , we have

$$(f,\varphi) = \int_{\mathbb{R}^d} f(x)\varphi(x) \, \mathrm{d}x = \int_{M_1} f(x)\varphi(x) \, \mathrm{d}x$$

and

$$\begin{aligned} \mathcal{E}^{\mu}(u^-,\varphi) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u^-(y) - u^-(x))(\varphi(y) - \varphi(x)) \, \mu(x, \mathrm{d}y) \, \mathrm{d}x \\ &= -2 \int_{M_1} \int_{(M_1)^c} u^-(y)\varphi(x) \, \mu(x, \mathrm{d}y) \, \mathrm{d}x. \end{aligned}$$

Hence, we get from (3.16)

$$\mathcal{E}^{\mu}(u^+,\varphi) \ge \int_{M_1} \varphi(x) \left(f(x) - 2 \int_{(M_1)^c} u^-(y) \, \mu(x, \mathrm{d}y) \right) \mathrm{d}x.$$

Therefore, u^+ satisfies all assumptions of Corollary 3.7 with $q = +\infty$ and $\widetilde{f} : M_1 \to \mathbb{R}$, defined by

$$\widetilde{f}(x) = f(x) - 2 \int_{\mathbb{R}^d \setminus M_1} u^-(y) \, \mu(x, \mathrm{d}y).$$

If $\sup\limits_{x\in M_{\frac{15}{16}}} \int_{\mathbb{R}^d\setminus M_1} u^-(z)\mu(x,\mathrm{d}z) = \infty$, then the assertion of the theorem is obviously true. Thus we can assume this quantity to be finite. Applying Corollary 3.7 and Hölder's inequality

$$\begin{aligned}
\inf_{M_{\frac14}} u &\geq c_1 \left(\fint_{M_{\frac12}} u(x)^{p_0}\,\mathrm{d}x \right)^{1/p_0} - \|\widetilde{f}\|_{L^q(M_{\frac{15}{16}})} \\
&= c_1 \left(\fint_{M_{\frac12}} u(x)^{p_0}\,\mathrm{d}x \right)^{1/p_0} - \|f\|_{L^q(M_{\frac{15}{16}})} - 2 \left\| \int_{\mathbb{R}^d\setminus M_1} u^-(y)\,\mu(x,\mathrm{d}y) \right\|_{L^q(M_{\frac{15}{16}})} \\
&\geq c_1 \left(\fint_{M_{\frac12}} u(x)^{p_0}\,\mathrm{d}x \right)^{1/p_0} - \|f\|_{L^q(M_{\frac{15}{16}})} - \sup_{x\in M_{\frac{15}{16}}} 2 \int_{\mathbb{R}^d\setminus M_1} u^-(y)\,\mu(x,\mathrm{d}y).
\end{aligned}$$

□

An immediate consequence of Theorem 1.3 is the following result, which follows via scaling and translation.

Corollary 3.8. *Let $x_0 \in M_1$, $r \in (0,1]$. Let $f \in L^q(M_1(x_0))$ for some $q > \max\{2,\beta\}$. Assume $u \in V^\mu(M_r(x_0)|\mathbb{R}^d)$ satisfies $u \geq 0$ in $M_r(x_0)$ and $\mathcal{E}^\mu(u,\varphi) \geq (f,\varphi)$ for every $\varphi \in H^\mu_{M_r(x_0)}(\mathbb{R}^d)$. Then there exists $p_0 \in (0,1)$, $c > 0$, independent of u, x_0 and r, such that*

$$\begin{aligned}
\inf_{M_{\frac14 r}(x_0)} u \geq c \left(\fint_{M_{\frac12 r}(x_0)} u(x)^{p_0}\,\mathrm{d}x \right)^{1/p_0} &- r^{\alpha_{\max}} \sup_{x\in M_{\frac{15}{16}r}(x_0)} 2 \int_{\mathbb{R}^d\setminus M_r(x_0)} u^-(z)\mu(x,\mathrm{d}z) \\
&- r^{\alpha_{\max}(1-\frac{\beta}{q})} \|f\|_{L^q}(M_{\frac{15}{16}r}(x_0)).
\end{aligned}$$

4. Hölder regularity estimates for weak solutions

In this section we prove the main result of this article, i.e., an a priori Hölder estimate for weak solutions to $\mathcal{L}u = f$ in M_1. For this purpose, we first prove a decay of oscillation result. Given a real-valued function f, $\operatorname{osc}\limits_S f$ denotes the essential oscillation on S, i.e., $\operatorname{osc}\limits_S f = \sup\limits_S f - \inf\limits_S f$, where sup resp. inf denote the essential supremum resp. infimum. We extend the scheme for the derivation of a priori Hölder estimates developed in [12]. We adapt the proof of [12, Theorem 1.4] to the anisotropic setting. We also include a right-hand side function f.

Theorem 4.1. *Assume Assumption 1, Assumption 2 and Assumption 3. Let $x_0 \in \mathbb{R}^d$, $r_0 \in (0,1]$. Let $c_a \geq 1$, $p > 0$ and $\Theta > \lambda > \sigma > 1$. Let $f \in L^q(M_1(x_0))$ for some $q > \max\{2,\beta\}$. We assume that the weak Harnack inequality holds true in $M_r(x_0)$, i.e.,*

For every $0 < r \leq r_0$ and $u \in V^\mu(M_r(x_0)|\mathbb{R}^d)$ satisfying $u \geq 0$ in $M_r(x_0)$ and $\mathcal{E}^\mu(u,\varphi) = (f,\varphi)$ for every $\varphi \in H^\mu_{M_r(x_0)}(\mathbb{R}^d)$,

$$\left(\fint_{M_{\frac{r}{\lambda}}(x_0)} u(x)^p\,\mathrm{d}x \right)^{1/p} \leq c_a \left(\inf_{M_{\frac{r}{\Theta}}(x_0)} u + r^{\alpha_{\max}} \sup_{x\in M_{\frac{r}{\sigma}}(x_0)} \int_{\mathbb{R}^d} u^-(z)\mu(x,\mathrm{d}z) + r^{\alpha_{\max}(1-\frac{\beta}{q})} \|f\|_{L^q(M_{\frac{r}{\sigma}}(x_0))} \right). \tag{4.1}$$

Then there exists $\delta \in (0,1)$, $c \geq 1$ such that for $r \in (0, r_0]$, $u \in V^\mu(M_r(x_0)|\mathbb{R}^d)$ satisfying $\mathcal{E}^\mu(u,\varphi) = (f,\varphi)$ for every $\varphi \in H^\mu_{M_r(x_0)}(\mathbb{R}^d)$,

$$\operatorname*{osc}_{M_\rho(x_0)} u \leq 2\Theta^\delta \|u\|_\infty \left(\frac{\rho}{r}\right)^\delta + c\Theta^\delta \left(\frac{\rho}{r}\right)^\delta r^{\alpha_{\max}(1-\frac{\beta}{q})} \|f\|_{L^q(M_{\frac{r}{\sigma}}(x_0))}, \qquad (0 < \rho \leq r)\,. \tag{4.2}$$

Proof. The strategy of the proof is well-known and can be traced back to G. A. Harnack himself. In the following, we write M_r instead of $M_r(x_0)$ for $r > 0$.

Let c_a and p be the constants from (4.1). Set $\kappa = (2c_a 2^{1/p})^{-1}$ and $\delta = \frac{\log(\frac{2}{2-\kappa})}{\log(\Theta)}$, which implies

$$1 - \frac{\kappa}{2} \leq \Theta^{-\delta}\,. \tag{4.3}$$

This estimate appears to be important later. Note that the inequality remains true if we choose δ even smaller.

Assume $0 < r \leq r_0$ and $u \in V^\mu(M_r(x_0)|\mathbb{R}^d)$ satisfies $\mathcal{E}^\mu(u,\varphi) = (f,\varphi)$ for every $\varphi \in H^\mu_{M_r(x_0)}(\mathbb{R}^d)$. Set

$$\widetilde{u}(x) = u(x)\Big[\|u\|_\infty + \tfrac{2}{\kappa} r^{\alpha_{\max}(1-\frac{\beta}{q})} \|f\|_{L^q(M_{\frac{r}{\sigma}})}\Big]^{-1}$$

Set $b_0 = \|\widetilde{u}\|_\infty$, $a_0 = \inf\{\widetilde{u}(x)|\, x \in \mathbb{R}^d\}$ and $b_{-n} = b_0, a_{-n} = a_0$ for $n \in \mathbb{N}$. Our aim is to construct an increasing sequence $(a_n)_{n\in\mathbb{Z}}$ and a decreasing sequence $(b_n)_{n\in\mathbb{Z}}$ such that for all $n \in \mathbb{Z}$

$$\begin{cases} a_n \leq \widetilde{u}(z) \leq b_n \\ b_n - a_n \leq 2\Theta^{-n\delta} \end{cases} \tag{4.4}$$

for almost all $z \in M_{r\Theta^{-n}}$. Before we prove (4.4), we show that (4.4) implies the assertion. Let $\rho \in (0, r]$. There is $j \in \mathbb{N}_0$ such that $r\Theta^{-j-1} \leq \rho \leq r\Theta^{-j}$. Note, that this implies in particular $\Theta^{-j} \leq \rho\Theta/r$. From (4.4), we deduce

$$\operatorname*{osc}_{M_\rho} \widetilde{u} \leq \operatorname*{osc}_{M_{r\Theta^{-j}}} \widetilde{u} \leq b_j - a_j \leq 2\Theta^{-\delta j} \leq 2\Theta^\delta \left(\frac{\rho}{r}\right)^\delta,$$

where from the assertion follows. It remains to show (4.4).

Assume there is $k \in \mathbb{N}$ and there are b_n, a_n, such that (4.4) holds true for $n \leq k-1$. We need to choose b_k, a_k such that (4.4) still holds for $n = k$. For $z \in \mathbb{R}^d$ set

$$v(z) = \left(\widetilde{u}(z) - \frac{b_{k-1} + a_{k-1}}{2}\right)\Theta^{(k-1)\delta}\,.$$

Then $|v(z)| \leq 1$ for almost every $z \in M_{r\Theta^{-(k-1)}}$ and $\mathcal{E}^\mu(v,\varphi) = (\widetilde{f},\varphi)$ for every function $\varphi \in H^\mu_{M_{r\Theta^{-(k-1)}}}(\mathbb{R}^d)$, where

$$\widetilde{f}(x) = \frac{\Theta^{(k-1)\delta}}{\|u\|_\infty + \frac{2}{\kappa} r^{\alpha_{\max}(1-\frac{\beta}{q})} \|f\|_{L^q(M_{\frac{r}{\sigma}})}} f(x). \tag{4.5}$$

Let $z \in \mathbb{R}^d$ be such that $z \notin M_{r\Theta^{-k+1}}$. Choose $j \in \mathbb{N}$ such that $z \in M_{r\Theta^{-k+j+1}} \setminus M_{r\Theta^{-k+j}}$. For such z and j, we conclude

$$\frac{v(z)}{\Theta^{(k-1)\delta}} \geq a_{k-j-1} - \frac{b_{k-1} + a_{k-1}}{2}$$

$$\geq -(b_{k-j-1} - a_{k-j-1}) + \frac{b_{k-1} - a_{k-1}}{2}$$
$$\geq -2\Theta^{-(k-j-1)\delta} + \frac{b_{k-1} - a_{k-1}}{2}.$$

Thus

$$v(z) \geq 1 - 2\Theta^{j\delta} \tag{4.6}$$

and similarly

$$v(z) \leq 2\Theta^{j\delta} - 1 \tag{4.7}$$

for $z \in M_{r\Theta^{-k+j+1}} \setminus M_{r\Theta^{-k+j}}$. We will distinguish two cases.

(1) First assume

$$|\{x \in M_{\frac{r\Theta^{-k+1}}{\lambda}} \,|\, v(x) \leq 0\}| \geq \frac{1}{2}|M_{\frac{r\Theta^{-k+1}}{\lambda}}|. \tag{4.8}$$

Our aim is to show that in this case

$$v(z) \leq 1 - \kappa \quad \text{ for almost every } z \in M_{r\Theta^{-k}}. \tag{4.9}$$

We will first show that this implies (4.4). Recall, that (4.4) holds true for $n \leq k-1$. Hence we need to find a_k, b_k satisfying (4.4). Assume (4.9) holds.

Then for almost any $z \in M_{r\Theta^{-k}}$

$$\begin{aligned}\widetilde{u}(z) &= \frac{1}{\Theta^{(k-1)\delta}} v(z) + \frac{b_{k-1} + a_{k-1}}{2} \\ &\leq \frac{1}{\Theta^{(k-1)\delta}}(1-\kappa) + \frac{b_{k-1} + a_{k-1}}{2} \\ &\leq a_{k-1} + \left(1 - \frac{\kappa}{2}\right) 2\Theta^{-(k-1)\delta} \\ &\leq a_{k-1} + 2\Theta^{-k\delta}.\end{aligned}$$

Here, we have used (4.3). If we now set $a_k = a_{k-1}$ and $b_k = b_k + 2\Theta^{-k\delta}$, then by the induction hypothesis $u(z) \geq a_{k-1} = a_k$ and by the previous calculation $u(z) \leq b_k$. Hence (4.4) follows.

It remains to prove (4.9), i.e., $v(z) \leq 1-\kappa$ for almost every $z \in M_{r\Theta^{-k}}$. Consider $w = 1 - v$ and note $w \geq 0$ in $M_{r\Theta^{-(k-1)}}$ and $\mathcal{E}^{\mu}(w, \varphi) = (\widetilde{f}, \varphi)$ for every $\varphi \in H^{\mu}_{M_{r\Theta^{-(k-1)}}}(\mathbb{R}^d)$, where $\widetilde{f}$ is defined as in (4.5). We apply the weak Harnack inequality (4.1) to the function w for $r_1 = r\Theta^{-k+1} \in (0, r]$. Then

$$\left(\fint_{M_{\frac{r_1}{\lambda}}} w(x)^p \,\mathrm{d}x \right)^{1/p} \leq c_a \Bigg(\inf_{M_{\frac{r_1}{\Theta}}} w + r_1^{\alpha_{\max}} \sup_{x \in M_{\frac{r_1}{\sigma}}} \int_{\mathbb{R}^d} w^-(z) \mu(x, \mathrm{d}z) + \frac{r_1^{\alpha_{\max}(1-\frac{\beta}{q})} \|f\|_{L^q(M_{\frac{r_1}{\sigma}})} \Theta^{(k-1)\delta}}{\|u\|_\infty + \frac{2}{\kappa} r^{\alpha_{\max}(1-\frac{\beta}{q})} \|f\|_{L^q(M_{\frac{r}{\sigma}})}} \Bigg).$$

We assume $\delta \leq \alpha_{\max}(1 - \frac{\beta}{q})$. Then

$$\frac{r_1^{\alpha_{\max}(1-\frac{\beta}{q})}\|f\|_{L^q(M_{\frac{r_1}{\sigma}})}\Theta^{(k-1)\delta}}{\|u\|_\infty + \frac{2}{\kappa}r^{\alpha_{\max}(1-\frac{\beta}{q})}\|f\|_{L^q(M_{\frac{r}{\sigma}})}} \leq \frac{\|f\|_{L^q(M_{\frac{r_1}{\sigma}})}\Theta^{(k-1)(\delta-\alpha_{\max}(1-\frac{\beta}{q}))}}{\frac{2}{\kappa}\|f\|_{L^q(M_{\frac{r_1}{\sigma}})}} \leq \frac{\kappa}{2}.$$

Using assumption (4.8) the left hand side can be estimated as follows

$$\begin{aligned}\left(\fint_{M_{\frac{r\Theta^{-(k-1)}}{\lambda}}} w(x)^p \,\mathrm{d}x\right)^{1/p} &\geq \left(\fint_{M_{\frac{r\Theta^{-(k-1)}}{\lambda}}} w(x)^p \mathbb{1}_{\{v(x)\leq 0\}} \,\mathrm{d}x\right)^{1/p} \\ &= \left(\frac{|\{x\in M_{\frac{r\Theta^{-k+1}}{\lambda}} \,|\, v(x)\leq 0\}|}{|M_{\frac{r\Theta^{-k+1}}{\lambda}}|}\right)^{1/p} \\ &\geq \left(\frac{\frac{1}{2}|M_{\frac{r\Theta^{-k+1}}{\lambda}}|}{|M_{\frac{r\Theta^{-k+1}}{\lambda}}|}\right)^{1/p} = \frac{1}{2^{1/p}}.\end{aligned}$$

Moreover by (4.7)

$$(1-v(z))^- \leq (1-2\Theta^{j\delta}+1)^- = 2\Theta^{j\delta}-2. \tag{4.10}$$

Consequently

$$\inf_{M_{r\Theta^{-k}}} w \geq 2\kappa - \frac{\kappa}{2} - (r\Theta^{-(k-1)})^{\alpha_{\max}} \sup_{x\in M_{\frac{r\Theta^{-(k-1)}}{\sigma}}} \int_{\mathbb{R}^d} w^-(z)\mu(x,\mathrm{d}z)$$

Let us show that the last term depends continuously on δ and can be made arbitrarily small. Note, that $w\geq 0$ in $M_{r\Theta^{-(k-1)}}$. Let $x\in M_{\frac{r\Theta^{-(k-1)}}{\sigma}}$ and $j\in\mathbb{N}$. From (4.10) we deduce,

$$\begin{aligned}&\int_{\mathbb{R}^d} w^-(z)\mu(x,\mathrm{d}z) = \int_{\mathbb{R}^d\setminus M_{r\Theta^{-(k-1)}}} w^-(z)\mu(x,\mathrm{d}z) \\ &= \sum_{j=1}^{\infty}\int_{M_{r\Theta^{-k+j+1}}\setminus M_{r\Theta^{-k+j}}} w^-(z)\mu(x,\mathrm{d}z) \leq \sum_{j=1}^{\infty}(2\Theta^{j\delta}-2)\int_{\mathbb{R}^d\setminus M_{r\Theta^{-k+j}}} \mu(x,\mathrm{d}z).\end{aligned}$$

Since

$$r\Theta^{-k+j} - r\frac{\Theta^{-k+1}}{\sigma} \geq r\Theta^{-k+j}\frac{\sigma-1}{\sigma},$$

we know $M_{r\Theta^{-k+j}(\sigma-1)/\sigma}(x) \subset M_{r\Theta^{-k+j}}(x_0) = M_{r\Theta^{-k+j}}$ and thus $\mathbb{R}^d\setminus M_{r\Theta^{-k+j}} \subset \mathbb{R}^d\setminus M_{r\Theta^{-k+j}(\sigma-1)/\sigma}(x)$. Applying Assumption 3 and (2.1) leads to

$$\begin{aligned}&(r\Theta^{-(k-1)})^{\alpha_{\max}}\mu(x,\mathbb{R}^d\setminus M_{r\Theta^{-k+j}}) \\ &\qquad\leq (r\Theta^{-(k-1)})^{\alpha_{\max}}\mu(x,\mathbb{R}^d\setminus M_{r\Theta^{-k+j}(\sigma-1)/\sigma}(x)) \\ &\qquad\leq c_1(r\Theta^{-(k-1)})^{\alpha_{\max}}\mu_{\mathrm{axes}}(x,\mathbb{R}^d\setminus M_{r\Theta^{-k+j}(\sigma-1)/\sigma}(x)) \\ &\qquad\leq c_1 4d(r\Theta^{-(k-1)})^{\alpha_{\max}}(r\Theta^{-k+j}(\sigma-1)/\sigma)^{-\alpha_{\max}} \\ &\qquad= c_1 4d(\sigma/(\sigma-1))^{\alpha_{\max}}(\Theta^{-j+1})^{\alpha_{\max}}.\end{aligned}$$

Thus for every $l \in \mathbb{N}$,

$$r_1^{\alpha_{\max}} \int_{\mathbb{R}^d} w^-(z)\mu(x, \mathrm{d}z) \le c_2 \sum_{j=1}^{l} (\Theta^{j\delta} - 1)(\Theta^{-j+1})^{\alpha_{\max}} + c_2 \sum_{j=l+1}^{\infty} \Theta^{j\delta}(\Theta^{-j+1})^{\alpha_{\max}}$$
$$=: I_1 + I_2 \,.$$

with a positive constant c_2 depending only on σ and d. From now on, we assume $\delta \le \frac{\alpha}{2}$. First, we choose $l \in \mathbb{N}$ sufficiently large in dependence of $\alpha_{\max}$ such that $I_2 \le \frac{\kappa}{4}$. Second, we choose δ sufficiently small such that $I_1 \le \frac{\kappa}{4}$. Since these choices are independent of x and k, we have proved

$$r_1^{\alpha_{\max}} \sup_{x \in M_{\frac{r_1}{\sigma}}} \int_{\mathbb{R}^d} w^-(z)\mu(x, \mathrm{d}z) \le \frac{\kappa}{2} \,.$$

Thus

$$w \ge \inf_{M_{r\Theta^{-k}}} w \ge \kappa \quad \text{on } M_{r\Theta^{-k}},$$

or equivalently $v \le 1 - \kappa$ on $M_{r\Theta^{-k}}$.

(2) Next, we assume

$$|\{x \in M_{\frac{r\Theta^{-k+1}}{\lambda}} \,|\, v(x) > 0\}| \ge \frac{1}{2}|M_{\frac{r\Theta^{-k+1}}{\lambda}}|. \tag{4.11}$$

Our aim is to show that in this case

$$v(z) \ge -1 + \kappa \qquad \text{for almost every } M_{r\Theta^{-k}}.$$

Similar to the first case, this estimate together with (4.4) and (4.3) implies for almost every $z \in M_{r\Theta^{-k}}$

$$\widetilde{u}(z) \ge b_{k-1} - 2\Theta^{-k\delta}.$$

Choosing $b_k = b_{k-1}$ and $a_k = a_{k-1} - 2\Theta^{-k\delta}$, then by the induction hypothesis $\widetilde{u}(z) \le b_{k-1} = b_k$ and by the previous calculation $\widetilde{u}(z) \ge a_k$. Hence (4.4) follows.

It remains to show in this case $v(z) \le -1 + \kappa$ for almost every $z \in M_{r\Theta^{-k}}$.

Consider $w = 1 + v$ and note $\mathcal{E}^\mu(v, \varphi) = (\widetilde{f}, \varphi)$ for every $\varphi \in H^\mu_{M_{r\Theta^{-(k-1)}}}(\mathbb{R}^d)$ and $w \ge 0$ in $M_{r\Theta^{-(k-1)}}$. Then the desired statement follows analogously to the first case 1.

□

Finally, we can prove our main result concerning Hölder regularity estimates.

Proof of Theorem 1.4. If $\mathrm{d}(x, y) \ge \frac{1}{4}$, then (1.10) follows from the observation

$$\mathrm{d}(x, y) = \max_{k \in \{1,\dots,d\}} \left\{ |x_k - y_k|^{\frac{\alpha_k}{\alpha_{\max}}} \right\} \le |x - y|^{\alpha_{\min}/\alpha_{\max}} \qquad (x, y \in M_{\frac{1}{2}}) \,.$$

Let $\rho \in (0, \frac{1}{4})$. We cover $M_{1-4\rho}$ by a countable family of balls $(M^i)_i$ with respect to the metric space $(\mathbb{R}^d, \mathrm{d})$ with radii ρ. Denote by $2M^j$ the ball with the same center as M^j but with radius 2ρ. By Theorem 4.1 there is a $\delta_1 \in (0, 1)$ and $c_2 > 0$, independent of ρ, such that

$$\operatorname*{osc}_{2M^j} u \le c_2 \rho^{\delta_1} (\|u\|_\infty + \|f\|_{L^q(M_{\frac{15}{16}})}) \,.$$

Given $\rho \in (0, \frac{1}{4})$, almost every pair $(x, y) \in M_{\frac{1}{2}} \times M_{\frac{1}{2}}$ satisfying $\frac{\rho}{2} < \mathbb{d}(x, y) \leq \rho$, has the property $|u(x) - u(y)| \leq \operatorname{osc}_{2M^j} u$ for some j and therefore satisfies

$$|u(x) - u(y)| \leq \operatorname*{osc}_{2M^j} u \leq c_3 |x - y|^{\delta_1 \alpha_{\min}/\alpha_{\max}} (\|u\|_\infty + \|f\|_{L^q(M_{\frac{15}{16}})}).$$

Altogether, there are $\delta_2 \in (0, 1)$ and $c_4 > 0$ such that for almost every pair $(x, y) \in M_{\frac{1}{2}} \times M_{\frac{1}{2}}$

$$|u(x) - u(y)| \leq c_4 |x - y|^{\delta_2} (\|u\|_\infty + \|f\|_{L^q(M_{\frac{15}{16}})}).$$

□

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