Locally Constant Model Uncertainty Risk Measure

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Abstract

This paper introduces a (coherent) risk measure that describes the uncertainty of the model (represented by a probability measure P_0) by a set $\mathcal{P}_\lambda$ of probability measures each of which has a Radon-Nikodym's derivative (with respect to P_0) that lies within the interval $[\lambda, \frac{1}{\lambda}]$ for some constant $\lambda \in (0, 1]$. Economic considerations are discussed and an explicit representation is obtained that gives a connection to both the expected loss of the financial position and its *average value-at-risk*. Optimal portfolio analysis is performed – different optimization criteria lead to Merton portfolio. Comparison with related problems reveals examples of extreme sensitivity of optimal portfolios to model parameters and the choice of risk measure.

KEYWORDS: Risk measure; Model uncertainty; Value at risk; Average value at risk; Optimal portfolio; Merton portfolio.

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1 Introduction

The theory of (coherent) risk measures[1] allows one to describe the risk of a financial position in monetary terms: the value of a risk measure of a certain position is the amount of numeraire that needs to be added to the position to make it safe. As the future value of a financial position is not deterministic it is classically modelled by a random variable on a probability space that is assumed to be given, the implicit assumption being that one is somehow able to deduce the "correct" probability measure that drives the prices of the underlying assets. In practice, there is always going to be some model uncertainty – one can never be sure that the measure in use is the one that really drives the world.

Ideally, a risk measure should "take into account" both the model uncertainty and the "genuine" uncertainty (due to the randomness of the world). Arguably, coherent risk measures achieve just that: the well known result on robust representations of coherent risk measures proves that each coherent risk measure ρ can be completely characterized by a set $\mathcal{P}$ of probability measures[2]. The characterization allows the (monetary) risk of any financial position X to be calculated as the maximal expected loss of the said position with respect to measures that belong to $\mathcal{P}$:

$$\rho(X) = \max_{P \in \mathcal{P}} E^P[-X]. \tag{1}$$

Unsurprisingly, sets of measures that represent many of the well known coherent risk measures have been characterized explicitly. It is however worth noting that coherent risk measures are usually not defined via the set of probability measures that represents them, but rather via an explicit expression that is somehow economically motivated.

In this paper we take the opposite approach: we introduce a coherent risk measure, the *Locally Constant Model Uncertainty* ($LCMU_\lambda$), via the set of measures that represents it. We will impose conditions on the set of priors so that the risk measure $LCMU_\lambda$ describes the uncertainty of the model (described by a given probability measure P_0) in a way that is, in a sense that will soon become clear, locally constant and quantified by the constant $\lambda \in (0, 1]$[3].

In order to formalize the idea of locally constant model uncertainty we will use ideas that are closely related to *ambiguity theory*, and, in particular, *maxmin expected utility theory*[4]. We assume that we are given a probability measure P_0

[1] The literature begins with Artzner et al. (1999). A serious introduction to coherent and convex risk measures can be found in Föllmer and Schied (2011). Although we will point out some important results, for further details we refer the reader to that text and the references offered therein.

[2] Generally, the probability measures in the set $\mathcal{P}$ are finitely additive, but under reasonable technical assumptions they are sigma-additive. For details see chapters 4.2 and 4.3 in Föllmer and Schied (2011).

[3] This approach somewhat resembles the definition of a risk measure known as the *superhedging price*, but the choice of the set of measures that appear in its robust characterization is completely different, both formally and in motivation.

[4] The theory was introduced in the seminal paper Gilboa and Schmeidler (1989). For a recent review of ambiguity theory and the place of maxmin expected utility theory within it

on the event space that, for now, we assume is a subset of the real line. One might assume that the measure describes the randomness of the world well, but not ideally. More precisely, we assume that for each "small" interval/event $[a, b]$ its probability p prescribed by the measure P_0 could be wrong, but still a good approximation – the "true probability" of the event lies within an interval that contains p. The first approach would be to consider the interval $(p-\varepsilon, p+\varepsilon)$; this resembles the ε-contamination model of Maccheroni et al. (2006) in ambiguity theory. Possible reservations when it comes to this approach would be that we are immediately limited to situations where $\varepsilon < 1$. Arguably even more important than that, when one considers an event's probability one does not necessarily think in terms of whether something is more or less likely for a certain amount of percents; it may be more natural to think in terms of how many more (or less) *times* something is more (or less) likely to happen. For example, for a quite bad model, one may decide that the prescribed probability p could be wrong in either direction: it could be up to twice as likely, or up to two times overestimated.

We choose to describe the model uncertainty by the interval $[p\lambda, p/\lambda]$ for some constant $\lambda \in (0, 1]$. One can think of λ as the *model uncertainty level*: the greater the value, the lower the model uncertainty. In order to be able to use this idea more generally, we would have to consider "infinitesimaly small events". This is the reason why we introduce the set $\mathcal{P}_\lambda$, that defines the risk measure $LCMU_\lambda$, via Radon-Nikodym derivatives: it will contain all the measures P such that $\lambda \leq dP/dP_0 \leq 1/\lambda$.

One of the main results of this paper is a representation theorem: the $LCMU_\lambda$ of a financial position can be represented as a convex combination of its expected future loss (with respect to the given measure P_0) and its *average value-at-risk* calculated at an appropriately chosen level.

Value-at-risk at level λ (VaR_λ) of a financial position is simply a negative value of its λ-quantile; it is a risk measure that is not coherent and has several undesirable properties. Average value-at-risk at level λ ($AVaR_\lambda$) is an average of all the values of value-at-risk at levels between zero and λ. It is a coherent risk measure with technical and economic properties superior to VaR. Given the difference in motivations for introducing $LCMU$ and $AVaR$ it is quite curious that there is a deeper connection between the two measures. The connection is due to the resemblance of the set $\mathcal{P}_\lambda$ to the set that appears in the the representation of $(AVaR_\lambda)$[5]. An agent estimating their risk using $LCMU$ ends up with an estimation that is, in a very precise sense, a mixture of estimations of a risk-neutral agent and an agent utilizing $AVaR$.

Once the $LCMU_\lambda$ risk measure is introduced, optimal portfolio analysis is performed. We consider a continuous-time frictionless financial market with a numeraire the value of which evolves deterministically, and several risky assets. Risky assets are assumed to be a "time dependent version" of geometrical Brownian motion: the drift coefficient and the diffusion matrix are not constants,

see Gilboa and Marinacci (2016).

[5]See, for example, chapter 4.4 in Föllmer and Schied (2011).

but rather deterministic functions. This admittedly simple model has already been studied in the context of risk measures[6].

Problems of choosing the optimal portfolio that minimizes risk (possibly under constraints) or maximizes expected reward under a risk constraint have been solved for $AVaR$. In Gambrah and Pirvu (2014) it was proven that it is optimal to distribute ones wealth between a numeraire and what is essentially a Merton portfolio (Merton (1969)). We prove the same result for $LCMU$, and give an example where the optimal portfolios for $AVaR$ and $LCMU$ coincide.

We also analyze a surprising example where optimizing with respect to the two risk measures leads to completely different optimal portfolios: optimizing with respect to $AVaR$ leads to a portfolio without risky assets, and optimizing with respect to $LCMU$ gives a portfolio with only risky assets! We offer both technical and theoretical explanation as to why this happens[7].

In the next section we formally introduce the risk measure $LCMU_\lambda$ and provide its representation that connects it to $AVaR_\lambda$. We also identify the minimizing measure for each financial position. We conclude the section with some numerical examples and simple comparisons between the two measures. In the third section we introduce the model of the financial market, formulate the results on optimal portfolios and perform the sensitivity analysis. A review of relevant facts about coherent risk measures, a corollary of a generalized version of the Neyman-Pearson lemma, proofs of theorems and additional relevant details are in the appendices.

2 Representation of the LCAN Risk Measure

2.1 Definition

Let $(\Omega, \mathcal{F}, P_0)$ be a given probability space, where P_0 is a probability measure support of which is the whole set Ω. We denote the set of probability measures defined on $(\Omega, \mathcal{F})$ with $\mathcal{M}$. For any $P \in \mathcal{M}$ and a random variable $X \in L^\infty(\Omega, \mathcal{F}, P_0)$ we denote the expectation of the random variable X with respect to probability measure P with $E^P[X]$, and, particularly, we write $E[X]$ for $E^{P_0}[X]$. We define a set of probability measures using Radon-Nikodym derivatives:

$$\mathcal{P}_\lambda = \left\{ P \in \mathcal{M} \,\middle|\, 0 < \lambda \leq \frac{dP}{dP_0} \leq \frac{1}{\lambda} \right\}, \tag{2}$$

where $\lambda \in (0, 1]$ is a given constant. As was mentioned earlier, one can think of λ as the level of model uncertainty: the closer the value of λ is to one there is less model uncertainty, i.e. we have greater confidence that the model is "good". Note that, due to the definition of the set P_λ, all the measures in $\mathcal{P}_\lambda$ are equivalent to P_0.

[6]See Gambrah and Pirvu (2014) and references offered therein.

[7]See the discussion after theorem 3.

We can now define the *locally constant model uncertainty* risk measure:

$$LCMU_\lambda(X) = \sup_{P \in \mathcal{P}_\lambda} E^P[-X]. \tag{3}$$

Clearly[8], this is a coherent risk measure.

2.2 Connection with Average Value-at-Risk

In this subsection we will relate the coherent measure $LCMU$ to the well known *average value-at-risk* ($AVaR$), also known as *expected shortfall*:

$$AVaR_\lambda(X) = \frac{1}{\lambda}\int_0^\lambda VaR_t(X)\,dt = -\frac{1}{\lambda}\int_0^\lambda q_X^+(t)\,dt$$

where $q_X^+(t) = \inf\{x \mid P_0(X \le x) > t\} = -VaR_t(X)$ is the upper quantile function of the random variable X with respect to the measure P_0 that appears in the definition of value-at-risk.

Theorem 1. *The coherent risk measure $LCMU_\lambda$ allows the following representation:*

$$\mathrm{LCMU}_\lambda(X) = \lambda E[-X] + (1-\lambda) AVaR_{\frac{\lambda}{1+\lambda}}(X). \tag{4}$$

The proof of the theorem is along the lines of the proof of robust representation for $AVaR$[9]. It relies on using the generalized version of the well known Neyman-Pearson lemma which we reformulate to fit our context in appendix A. The proof of the theorem can be found in appendix B.

We note that if the distribution of the random variable X has density then $LCMU_\lambda(X)$ can be written as follows:

$$LCMU_\lambda(X) = -E\left[\lambda E[X] + (1-\lambda)X \;\middle|\; X < VaR_{\frac{\lambda}{1+\lambda}}(X)\right];$$

this is due to the representation of $AVaR_\lambda$ for random variables with density ($AVaR_\lambda(X) = E[-X|X < VaR_\lambda(X)]$; Theorem 4.49 in Föllmer and Schied (2011)). Hence, an agent estimating the risk of a financial position using $LCMU$ calculates an expectation of a mixture of the position and its expected value, conditioned on the fact that there will be losses. The value obtained is the amount of numeraire that makes the position safe.

2.3 Maximizing Measure

Careful reading of the proof of the theorem 1 shows that the supremum in the definition (3) of $LCMU_\lambda$ is attained. In particular, if q is a $\frac{\lambda}{1+\lambda}$-quantile of X with respect to P_0 and

$$\psi_X = \mathbb{1}_{\{X<q\}} + k\mathbb{1}_{\{X=q\}},$$

[8] Due to representation theorems for coherent risk measures; see (1).
[9] See also theorem 4.47 in Föllmer and Schied (2011).

one can write:

$$LCMU_\lambda(X) = \int -X\left(\lambda + (1-\lambda)\frac{1+\lambda}{\lambda}\psi_X\right) dP_0.$$

The measure Q_X defined via its Radon-Nikodym's derivative:

$$\frac{dQ_X}{dP_0} = \lambda + (1-\lambda)\frac{1+\lambda}{\lambda}\psi_X \tag{5}$$

belongs to the set $\mathcal{P}_\lambda$ and is the maximizing measure in (3); we record this fact in the following theorem:

Theorem 2. *For any random variable $X \in L^\infty(\Omega, \mathcal{F}, P_0)$, the measure $Q_X \in \mathcal{P}_\lambda$ as defined in* (5) *is the maximizing measure in the defintion* (3) *of the risk measure $LCMU_\lambda$, i.e.:*

$$\mathrm{LCMU}_\lambda(X) = E^{Q_X}[-X].$$

The proof follows from the preceding theorem and we comment on it briefly in appendix B.

2.4 Comparison with Average Value at Risk

Once the connection between *AVaR* and *LCMU* have been established it is worthwhile to explore (numerical) similarities and differences between the two measures. First, one easily notes that both measures satisfy:

$$LCMU_1[X] = AVaR_1[X] = E[-X],$$
$$\lim_{\lambda\to 0+} LCMU_\lambda(X) = \lim_{\lambda\to 0+} AVaR_\lambda(X) = \operatorname{ess\,sup} -X.$$

Furthermore, due to the fact that the set $\mathcal{P}_\lambda$ that represents $LCMU_\lambda$ is clearly a subset of the set that gives the robust representation of $AVaR_\lambda$ the inequality $AVaR_\lambda(X) \geq LCMU_\lambda(X)$ holds for any financial position X. This means that, from a regulatory point of view, $LCMU$ is the less conservative of the two measures.

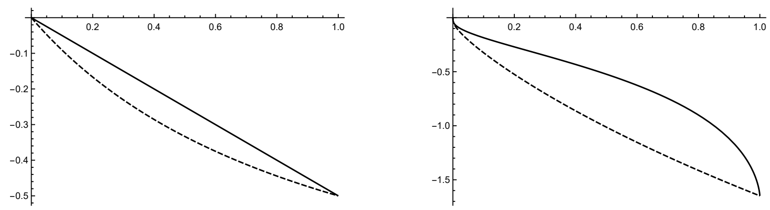
To get a clearer insight into the way that the different risk measures value risk differently we will focus on two simple examples with positions distributed uniformly and log-normally.

2.4.1 Uniform distribution

Suppose a random variable X is uniformly distributed on the interval $[a, b]$. Straightforward computations yield:

$$VaR_\lambda(X) = AVaR_\lambda(X) = -a - \frac{\lambda}{2}(b-a)$$
$$LCMU_\lambda(X) = -a - \frac{\lambda}{1+\lambda}(b-a).$$

Figure 1a contains the graphs of $AVaR_\lambda(X)$ and $LCMU_\lambda(X)$ as functions of λ. The figure confirms that, indeed, $LCMU$ prescribes substantially lower values of numeraire than AVaR.



(a) $X \sim \mathcal{U}([0,1])$ (b) $X \sim \ln \mathcal{N}(0,1)$

Figure 1: The full line represents the function $AVaR_\lambda(X)$, while the dashed line represents $LCMU_\lambda(X)$ (as functions of λ).

2.4.2 Log-normal distribution

Suppose now that the random variable $X \sim \ln \mathcal{N}(\mu, \sigma^2)$ is log-normally distributed. After some computations one can see that:

$$
\begin{aligned}
&VaR_\lambda(X) = -\exp(\mu + \sigma\Phi^{-1}(\lambda)),\\
&AVaR_\lambda(X) = -\frac{1}{\lambda}\int_0^\lambda \exp(\mu + \sigma\Phi^{-1}(t))\,dt = -\exp\left(\mu + \frac{\sigma^2}{2}\right)\frac{\Phi(\Phi^{-1}(\lambda) - \sigma)}{\lambda}\\
&LCMU_\lambda(X) = -\lambda\exp\left(\mu + \frac{\sigma^2}{2}\right) - (1-\lambda)\frac{(\lambda+1)}{\lambda}\int_0^{\frac{\lambda}{\lambda+1}} \exp(\mu + \sigma\Phi^{-1}(t))\,dt\\
&\qquad\qquad\quad = -\exp\left(\mu + \frac{\sigma^2}{2}\right)\left(\lambda + \frac{1-\lambda^2}{\lambda}\Phi\left(\Phi^{-1}\left(\tfrac{\lambda+1}{\lambda}\right) - \sigma\right)\right).
\end{aligned}
$$

The second equality for $AVaR$ is the only one that is slightly more involved; we prove it in appendix C.

Figure 1b contains the graphs of $AVaR_\lambda(X)$ and $LCMU_\lambda(X)$ as functions of λ. As can be seen, the less conservative $LCMU$ can prescribe substantially lower values.

3 Optimal Portfolio Analysis

3.1 Model

Let $(\Omega, \{\mathcal{F}_t\}_{0\le t\le T}, \mathcal{F}, P)$ be a filtered probability space which accommodates a standard m-dimensional Brownian motion $W(t) = (W^j(t))_{j=1,\ldots,m}$. We consider a financial market with a numeraire S_0 and m risky assets S_i which are traded continuously over a finite time horizon $[0,T]$ in a frictionless market. The dynamics of the assets are:

$$
\begin{aligned}
dS_0(t) &= r(t)\,dt,\\
dS_i(t) &= S_i(t)\left(b_i(t)\,dt + \sum_{j=1}^m \sigma_{ij}(t)\,dW^j(t)\right), \quad i = 1,\ldots,m,
\end{aligned}
$$

where $r(t)$ is the deterministic interest rate, the functions $b_i(t)$ are deterministic and denote the drift of the stock, and the volatility matrix $\sigma(t) = (\sigma_{ij}(t))_{i,j=1,\ldots,m}$ is deterministic and invertible. We assume that functions r, b_i and σ_{ij} are square integrable and that the inequalities $0 < r(t) < b(t)$ are satisfied for each t.

Self financing strategies are described by a deterministic vector $\pi(t) = (\pi_1(t), \ldots, \pi_m(t)) \in \mathbb{R}^m$ such that

$$\sum_{i=1}^{m} \pi_i(t) \leq 1, \text{ and } \pi_i(t) \geq 0, \quad i = 1, \ldots, m. \tag{6}$$

An agent following the strategy π invests a fraction π_i of their wealth in the risky stock S_i, while the remainder is invested in the bond (represented by the numeraire S_0). As can be seen, no borrowing or short selling is allowed. Hence, if we denote the wealth at time t by $X^\pi(t)$ and the number of shares of the asset i held in portfolio by $N_i(t)$, we have

$$\pi_i(t) = N_i(t)S_i(t)/X^\pi(t),\ i \geq 1, \text{ and } X^\pi(t) = \sum_{i=0}^{m} N_i(t)S_i(t).$$

Dynamics of S_i imply that the agent's wealth satisfies:

$$dX^\pi(t) = X^\pi(t)\left((r(t) + B(t)'\pi(t))\,dt + \sigma(t)'\pi(t)\,dW(t)\right),$$

where $B(t) = (b_1(t) - r(t), \ldots, b_m(t) - r(t))$ and $'$ is the transposition operator.

Using Ito's lemma, direct calculations yield:

$$X^\pi(T) = X^\pi(0)\exp\left(\int_0^T r(s) + B(s)'\pi(s) - \frac{1}{2}||\sigma(s)'\pi(s)||^2\,ds \right.$$
$$\left. + \int_0^T ||\sigma(s)'\pi(s)||\,dW(s)\right).$$

If we introduce the following notation:

$$R = \exp\left(\int_0^T r(s)\,ds\right),\ x_\pi = X^\pi(0),$$
$$\mu(\pi) = \int_0^T B(s)'\pi(s)\,ds,\ \psi(\pi) = \int_0^T ||\sigma(s)'\pi(s)||^2\,ds$$

then

$$E[X^\pi(T)] = x_\pi R \exp\left(\mu(\pi)\right). \tag{7}$$

3.2 Loss and Risk Measures

We define loss as $L(\pi) = X^\pi(T) - X^\pi(0)$; it is simply a difference between the wealth at the end and at the beginning of the time period. This is the quantity that will be involved in different optimization problems that we solve. In particular, we will consider the quantity $LCMU_\lambda(L(\pi))$ and, for comparison purposes, $AVaR_\lambda(L(\pi))$. Considering a risk measure of a random variable that depends only on the final and, possibly, initial value of the stochastic process is standard in literature (see Schied and Wu (2005)). It is also in the spirit of the classical stochastic control problem in financial mathematics – *Merton's portfolio problem* (Merton (1969)) where the utility of the terminal wealth is considered.

We note that, although the dynamics of the process in question are acknowledged, this approach can be considered static, as we only consider two points in time and do not impose constraints on the financial positions in between the two time endpoints. An alternative would be to consider dynamic versions of risk measures; this is, on a technical level, significantly more involved. Reasons for the complications include having to do with the time consistency of dynamic risk measures and the non-time consistency of AVaR (Cheridito and Stadje (2009)). Considering only deterministic (instead of predictable) trading strategies, as we do here, somewhat offsets the need for dynamic measures as the agent effectively makes a decision about trading throughout the whole period. In any case, when analyzing the results of models that only involve the final time point, one should be aware of the limitations of models of this kind and therefore careful in the interpretations.

It can be shown[10] that:

$$AVaR_\lambda(L(\pi)) = x_\pi \left(1 - \frac{R}{\lambda}\Phi\left(\Phi^{-1}(\lambda) - \sqrt{\psi(\pi)}\right)\exp(\mu(\pi))\right).$$

Combining the expressions for $E[X^\pi(T)]$ and $AVaR_\lambda(L(\pi))$, and using the representation of $LCMU$ from theorem 1, we obtain:

$$\begin{aligned} &LCMU_\lambda(L(\pi)) = \\ &\quad x_\pi \left(1 - R\left(\lambda + \tfrac{1-\lambda^2}{\lambda}\Phi\left(\Phi^{-1}\left(\tfrac{\lambda}{1+\lambda}\right) - \sqrt{\psi(\pi)}\right)\right)\right)\exp(\mu(\pi)).\end{aligned}$$

3.3 Optimization Problems and Merton portfolio

Let $\mathcal{Q}$ be the set of all the trading strategies π which are Borel measurable, deterministic and satisfy the conditions of equation (6). We will consider three problems that lead to different optimal portfolios in $\mathcal{Q}$.

First we consider the unconstrained problem of choosing the portfolio for which the risk measure $LCMU$ prescribes the lowest risk:

$$(P1) \qquad \min_{\pi\in\mathcal{Q}} LCMU_\lambda(L(\pi)).$$

[10] The derivation is quite similar to the derivation in appendix C for $AVaR_\lambda(X)$ for a log-normally distributed position X. For more details see the proof of proposition 3.1.2.1 in Gambrah and Pirvu (2014).

The second problem we consider is finding the lowest risk portfolio among all the portfolios with fixed expected return:

$$(P2) \qquad \min_{\pi \in \mathcal{Q}} LCMU_\lambda(L(\pi)) \text{ such that } E[X^\pi(T)] = M.$$

Finally, we consider the problem of maximizing the expected returns while requiring the risk to be above some positive boundary C:

$$(P3) \qquad \max_{\pi \in \mathcal{Q}} E[X^\pi(T)] \text{ such that } LCMU_\lambda(L(\pi)) \geq C.$$

All three problems have been explicitly solved for risk measures VaR and $AVaR$ in Gambrah and Pirvu (2014). Optimal portfolios for both risk measures and for all three problems are closely related to the trading strategy:

$$\pi_M(t) = (\sigma(t)\sigma(t)')^{-1}B(t); \tag{8}$$

in each case the optimal portfolio is just a multiple of π_M, the well known *Merton portfolio* from Merton (1969) and numerous related problems[11]. Furthermore, due to the strength of the constraint in the problem (P2), the optimal portfolios for both risk measures coincide for that problem. This leads to an interpretation similar to the well known *mutual fund theorem*: if there is a hedge fund with portfolio π_M it is optimal for the agent to distribute their wealth between the hedge fund and the bonds no matter what the optimization criterion is. However, different optimization criteria can lead to different proportions of investments between the hedge fund and the bond.

It turns out that the solutions for problems (P1-P3) are also multiples of π_M.

Theorem 3. *For each of the problems (P1), (P2) and (P3) there are constants c_1, c_2, c_3 such that the solutions to the problems are:*

$$\pi_1^* = c_1\pi_M, \quad \pi_2^* = c_2\pi_M, \quad \pi_3^* = c_3\pi_M.$$

Furthermore, the same portfolio solves the, appropriately reformulated, optimization problem (P2) for the risk measures *LCMU*, *AVaR*, and *VaR*; see theorem 4 in appendix D.

The fact that optimal portfolios when one optimizes with respect to risk measures (as the theorems 3 and 4 show) and with respect to utility functions (as the classical literature Merton (1969) shows) is in some ways surprising. We offer some comments that explain why this is the case in this model, but we also comment on the modelling approach to optimal portfolios in general.

On a technical level this result is driven by strong assumptions: log-normally distributed returns or risky assets, deterministic trading strategies, a frictionless market, and constraints on borrowing and short selling. In a sense, if the market conditions are close to ideal then the conclusions of the classical theory remain valid.

[11] See, for example, Rogers (2013).

However, the theory of risk measures was developed because, among other things, the markets are not ideal: the returns of investments are not distributed log-normally and tails of the “actual distributions” driving the world are heavy. Thus, while the analysis offered in Gambrah and Pirvu (2014) and in our work sheds valuable insight into the optimal portfolio choice with respect to risk measures, it could be considered a mere first step in optimization problems of this kind and further investigation into more robust and realistic models is needed.

The derivations in the proof of theorem 3 are closely related to the ones offered in Gambrah and Pirvu (2014); details are in Appendix D.

3.4 Sensitivity of Optimal Portfolios to the Choice of Risk Measures

Let us consider problems (P1-3) for the risk measures $AVaR_\lambda$ and $LCMU_\lambda$. We have already seen that the solutions of the optimization problems for both risk measures belong to the same class. In this subsection we further explore how does the choice of risk measure influence the optimal allocation between the numeraire and Merton porfolio π_M.

We begin by analyzing optimal portfolios for problem (P1). Due to the close connection between the risk measures $AVaR$ and $LCMU$ (as established by theorem 1) and similarity of results in theorems 3 and 4 it would be expected that the optimal portfolios when optimizing with respect to the two measures behave similarly. Somewhat surprisingly, this is not the case, as we demonstrate below.

If we solve the problem (P1) for $AVaR_\lambda$ and $LCMU_\lambda$ the optimal portfolios are:

$$AVaR_\lambda : \pi_A = c_A\pi_M, \quad LCMU_\lambda : \pi_L = c_L\pi_M,$$

for some constants c_A and c_L; see theorem 4. The analysis in subsection 2.4 shows that $LCMU_\lambda(L(\pi) \geq AVaR_\lambda(L(\pi))$. This implies that $c_A \leq c_B$: constants c_A and c_L determine the amount of numeraire to be kept in the optimal portfolio, hence $AVaR$, being the less risky of the two risk measures, prescribes less risky assets in the optimal portfolio and more numeraire.

The proof reveals how the constants c_A and c_L are calculated. Let us introduce functions:

$$\begin{aligned} g_\lambda(\varepsilon) &:= G^A_\lambda(\Theta\varepsilon, \varepsilon^2) = \Theta\varepsilon + \ln(\varphi_\lambda(\varepsilon^2)), \\ f_\lambda(\varepsilon) &:= G^L_\lambda(\Theta\varepsilon, \varepsilon^2) = \Theta\varepsilon + \ln(\lambda + (1-\lambda)\varphi_{\frac{\lambda}{1+\lambda}}(\varepsilon^2)), \end{aligned}$$

where

$$\varphi_\lambda(y) = \frac{1}{\lambda}\Phi(\Phi^{-1}(\lambda) - \sqrt{y}) \text{ and } \Theta = \sqrt{\int_0^T ||\sigma(s)^{-1}B(s)||^2\, ds}.$$

Let ε_A and ε_L be the solutions of optimization problems

$$(\varepsilon P) \qquad \max_{\varepsilon \in I} g_\lambda(\varepsilon) \text{ and } \max_{\varepsilon \in I} f_\lambda(\varepsilon),$$

where

$$I = \left[0, \int_0^T ||\sigma(s)||^2 \, ds\right].$$

Then $c_A = \varepsilon_A/\Theta$ and $c_L = \varepsilon_L/\Theta$.

For different values of the parameters the solutions c_A and c_L can be on the boundaries of the interval I. In general, the equality $c_A = c_L$ does not hold. Furthermore, there are examples where $c_A = 0$ and $c_L = I$. This means that for certain reasonable values of the parameters of the model it can happen that optimal portfolios with respect to closely related risk measures $AVaR$ and $LCMU$ are completely different: it is optimal with respect to $AVaR$ to not invest in the risky assets, while with respect to $LCMU$ it is optimal to invest *only* in risky assets!

To illustrate that this can indeed be the case we will consider a simple special case of the model we introduced: a market with one risky asset in which the risk rate, the drift coefficient and the diffusion coefficient are all constant. For simplicity we also assume that the time horizon satisfies $T = 1$. Even in this simplified setting, solving optimization problems (εP) is technically cumbersome. We will avoid the complications by making appropriate approximations.

Direct calculations show that under the simplified assumptions the interval I becomes $[0, \sigma^2]$. Thus for $\sigma < 1$ the interval I becomes "small". Once one notices that $f_\lambda(0) = g_\lambda(0) = 0$, we can approximate the functions f_λ and g_λ with their tangents at 0:

$$f_\lambda(\varepsilon) \approx \varepsilon f'_\lambda(0), \qquad g_\lambda(\varepsilon) \approx \varepsilon g'_\lambda(0).$$

As we are demonstrating that the solutions of problems (εP) are on the boundary of the interval I these approximations will suffice. Indeed, it is sufficient to establish "opposite" monotonicities of the functions f_λ and g_λ on the interval I.

Direct calculations show:

$$\begin{aligned}
\varphi'_\lambda(\varepsilon) &= -\frac{1}{\lambda}\Phi'(\Phi(\lambda) - \varepsilon), \\
g'_\lambda(\varepsilon) &= \Theta + \frac{\varphi'_\lambda(\varepsilon)}{\varphi_\lambda(\varepsilon)}, \\
f'_\lambda(\varepsilon) &= \Theta + \frac{(1-\lambda)\varphi'_{\lambda/(1+\lambda)}(\varepsilon)}{\lambda + (1-\lambda)\varphi_{\lambda/(1+\lambda)}(\varepsilon)}.
\end{aligned}$$

It follows that:

$$g'_\lambda(0) = \Theta + \varphi'_\lambda(0) \text{ and } f'_\lambda(0) = \Theta + (1-\lambda)\varphi'_{\lambda/(1+\lambda)}(0).$$

For example, if we choose $\lambda = 0.2$, $\mu - r = 0.4$ and $\sigma = 0.32$ then $\theta = 1.25$, $I = [0, 1.5625]$ and $g'_\lambda(0) < 0$ and $f'_\lambda(0) > 0$. The approximations we introduced are good enough; see figure 2. Indeed, $f_\lambda(\varepsilon)$ achieves its maximum on the right hand side of the interval, and $g_\lambda(\varepsilon)$ achieves its maximum on the left hand side of the interval I; this implies $c_A = 0$ and $c_L = 1.5625$.

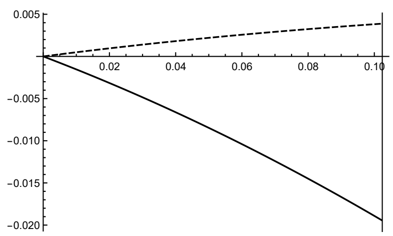


Figure 2: Graphs of functions $f_{0.2}(\varepsilon)$ (dashed) and $g_{0.2}(\varepsilon)$ (full) for $\varepsilon \in I$ and $\mu - r = 0.4$, $\sigma = 0.32$. The vertical line denotes the end of interval I.

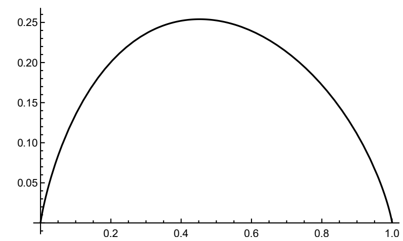


Figure 3: Graph of the function $f'_\lambda(0) - g'_\lambda(0)$ (as a function of λ).

Furthermore, figure 3 shows that the difference $f'_\lambda(0) - g'\lambda(0)$ (as a function of λ) is always positive. Thus, for any value of λ we can always choose a value of Θ such that $g'_\lambda(0) < 0$ and $f'_\lambda(0) > 0$. This can be achieved by choosing the appropriate values of μ, r, and σ such that $\Theta \in [g'_\lambda(0), f'_\lambda(0)]$, and that interval I is "small enough".

We conclude this section by briefly turning to the problems (P2) and (P3).

We note that optimal portfolios with respect to both measures coincide for the problem (P2) (see the solution of (ρP2) in the proof of theorem 4 in appendix D).

As for the problem (P3), the situation is quite similar to sensitivity analysis performed for the problem (P1): there are situations in which optimizing with respect to different measures prescribes radically different optimal behavior. This is due to the similarities of problems (εP) and the optimization problem that the problems (P3) and (ρP3) are reduced to; see the part of the proof of theorem 4 related to the problems (P3) and (ρP3).

4 Conclusion

Motivated by ideas from ambiguity theory we have introduced a new coherent risk measure: locally constant model uncertainty (LCMU). It is explicitly defined via its set of probability measures in a way that makes uncertainty about the probabilities of "small" events constant – the Radon-Nikodym derivative lies within a fixed interval.

We have derived a representation of $LCMU$ as a convex combination of the expected loss of the position and its average value-at-risk (AVaR) calculated at an appropriately chosen interval. We have thus demonstrated a viable connection between ambiguity theory and well established risk measures.

We have considered and solved optimal investment problems in continuous time related to $LCMU$ in a frictionless market with m-assets that evolve following a time dependent version of the multi-dimensional geometric Brownian motion with no-borrowing and no-short-selling constraints. We have proven a version of a *mutual fund theorem*: choosing portfolios that minimize risk or maximize profit with a risk constraint both lead to Merton portfolios; this result was already known for value-at-risk and $AVaR$ in this setting.

We have demonstrated that optimal portfolios can be radically different when optimizing with respect to $LCMU$ and AVaR. This surprising conclusion raises questions about dynamic models of optimal investment in continuous time that deal with risk measures. Our results also demonstrated the fragility of the solutions of optimization problems involving risk measures in dynamic settings, even in mathematically simple contexts.

A Corollary of the Generalized Version of Neyman-Pearson Lemma

Lemma that follows is a direct corollary of the generalized Neyman-Pearson Lemma as formulated in theorem A.30 in Föllmer and Schied (2011).

Lemma 1. *If P and Q are given equivalent measures and $\alpha \in [0,1]$ is a given constant then:*

$$\max\left\{\int \psi\, dQ \,\middle|\, 0 \le \psi \le 1, \int \psi\, dP = \alpha\right\} = \alpha = \int \psi_X\, dQ \tag{9}$$

for

$$\psi_X = \mathbb{1}_{\{\frac{dQ}{dP} > c\}} + k\mathbb{1}_{\{\frac{dQ}{dP} = c\}} \tag{10}$$

where c is a $1-\alpha$-quantile of $\frac{dQ}{dP}$ with respect to P and

$$k = \begin{cases} 0, & P\left(\frac{dQ}{dP} = c\right) = 0 \\ \dfrac{\alpha - P\left(\frac{dQ}{dP} > c\right)}{P\left(\frac{dQ}{dP} = c\right)}, & P\left(\frac{dQ}{dP} = c\right) > 0. \end{cases} \tag{11}$$

B Proofs of theorems 1 and 2

Proof. We begin by rewriting the left hand side of the equation (3) for a fixed random variable $X < 0$:

$$\begin{aligned}\sup_{P\in\mathcal{P}_\lambda} E^P[-X] &= \sup\left\{E^P[-X] \,\middle|\, P \in \mathcal{M}, \lambda \le \frac{dP}{dP_0} \le 1/\lambda\right\} \\ &= \sup\left\{-\int X \frac{dP}{dP_0}\, dP_0 \,\middle|\, P \in \mathcal{M}, \lambda \le \frac{dP}{dP_0} \le 1/\lambda\right\} \\ &= \sup\left\{-\int X\varphi\, dP_0 \,\middle|\, \int \varphi\, dP_0 = 1, \lambda \le \varphi \le 1/\lambda\right\} \\ &= \sup\left\{-E[X]\int \frac{X}{E[X]}\varphi\, dP_0 \,\middle|\, \int \varphi\, dP_0 = 1, \lambda \le \varphi \le 1/\lambda\right\}.\end{aligned}$$

So far we have only used the definitions of the set $\mathcal{P}_\lambda$ and basic properties of the expectation operator and Radon-Nikodym derivatives. We notice that $\displaystyle\int \frac{X}{E[X]}\, dP_0 = 1$, so the random variable $\dfrac{X}{E[X]} > 0$ is a Radon-Nikodym derivative for some measure Q that is equivalent to P_0. Hence, using the last expression above and the inequality $E[-X] > 0$, we have:

$$\begin{aligned}\sup_{P\in\mathcal{P}_\lambda} E^P[-X] =& \sup\left\{-E[X]\int \varphi \frac{dQ}{dP_0}\, dP_0 \,\middle|\, \int \varphi\, dP_0 = 1, \lambda \le \varphi \le 1/\lambda\right\}\\ =& E[-X]\sup\left\{-E[X]\int \varphi\, dQ \,\middle|\, \int \varphi\, dP_0 = 1, \lambda \le \varphi \le 1/\lambda\right\}.\end{aligned} \tag{12}$$

The following equivalence of inequalities:

$$\lambda \le \varphi \le 1/\lambda \Leftrightarrow 0 \le \frac{\lambda}{1-\lambda^2}(\varphi - \lambda) \le 1 \tag{13}$$

allows one to rewrite the right hand side of the equation (12) in terms of a new variable $\psi := \frac{\lambda}{1-\lambda^2}(\varphi - \lambda)$:

$$\begin{aligned}&\sup_{P\in\mathcal{P}_\lambda} E^P[-X] =\\ &\quad= E[-X]\sup\left\{\int \left(\psi\tfrac{1-\lambda^2}{\lambda} + \lambda\right) dQ \,\middle|\, \int \left(\psi\tfrac{1-\lambda^2}{\lambda} + \lambda\right) dP_0 = 1, 0 \le \psi \le 1\right\}\\ &\quad= E[-X]\left(\lambda + \tfrac{1-\lambda^2}{\lambda}\cdot \sup\left\{\int \psi\, dQ \,\middle|\, \int \psi\, dP_0 = \tfrac{\lambda}{1+\lambda}, 0 \le \psi \le 1\right\}\right).\end{aligned} \tag{14}$$

Applying lemma 1 one obtains:

$$\sup\left\{\int \psi\, dQ \,\middle|\, \int \psi\, dP_0 = \tfrac{\lambda}{1+\lambda}, 0 \le \psi \le 1\right\} = \int \psi_X\, dQ, \tag{15}$$

where

$$\psi_X = \mathbb{1}_{\{\frac{dQ}{dP_0} > c\}} + k\mathbb{1}_{\{\frac{dQ}{dP_0} = c\}},$$

for

$$k = \begin{cases} 0, & P\left(\frac{dQ}{dP_0} = c\right) = 0\\ \dfrac{\alpha - P\left(\frac{dQ}{dP_0} > c\right)}{P\left(\frac{dQ}{dP_0} = c\right)}, & P\left(\frac{dQ}{dP_0} = c\right) > 0,\end{cases}$$

and c a $1 - \frac{\lambda}{1+\lambda}$-quantile of $\dfrac{dQ}{dP_0} = \dfrac{X}{E[X]}$ with respect to P_0. Keeping in mind that $X < 0$, the inequality $\dfrac{dQ}{dP_0} > c$ can be written as:

$$\begin{aligned}
X < E[X] \cdot c = & E[X] \cdot \inf \left\{ t \,\middle|\, P_0\left(\frac{dQ}{dP_0} < t\right) > 1 - \frac{\lambda}{1+\lambda} \right\} \\
= & \sup \left\{ tE[X] \mid P_0\left(X > tE[X]\right) - 1 > -\frac{\lambda}{1+\lambda} \right\} \\
= & \sup \left\{ t \mid 1 - P_0\left(X > t\right) < \frac{\lambda}{1+\lambda} \right\} \\
= & \sup \left\{ t \mid P_0\left(X \leq t\right) < \frac{\lambda}{1+\lambda} \right\} =: q.
\end{aligned}$$

We can see that q is a $\frac{\lambda}{1+\lambda}$-quantile of X with respect to P_0 and that inequalities $\frac{dQ}{dP_0} > c$ and $X < q$ are equivalent. Similarly, the equality $\frac{dQ}{dP_0} = c$ holds if and olny if $X = q$ holds. Hence:

$$\psi_X = \mathbb{1}_{\{X<q\}} + k\mathbb{1}_{\{X=q\}}. \tag{16}$$

Combining (14) and (15) one obtains:

$$\begin{aligned}
\sup_{P\in\mathcal{P}_\lambda} E^P[-X] &= E[-X]\left(\lambda + \frac{1-\lambda^2}{\lambda}\int \psi_X \, dQ\right) \\
&= E[-X]\left(\lambda + \frac{1-\lambda^2}{\lambda}\int \psi_X \frac{dQ}{dP_0}\, dP_0\right) \\
&= E[-X]\left(\lambda + \frac{1-\lambda^2}{\lambda}\int \psi_X \frac{X}{E[X]}\, dP_0\right) \\
&= \lambda E[-X] - \frac{1-\lambda^2}{\lambda}\int X\psi_X \, dP_0.
\end{aligned} \tag{17}$$

The integral that appears in the last expression can be rewritten (using 16) as follows:

$$\begin{aligned}
\int X\psi_X \, dP_0 = & \int_{\{X<q\}} X\, dP_0 + k\int_{\{X=q\}} X\, dP_0 \\
& = -\int_{\{X<q\}} (q-X)\, dP_0 + q\int_{\{X<q\}} dP_0 + k\int_{\{X=q\}} q\, dP_0 \\
& = -\int (q-X)^+ \, dP_0 + q\left(P_0(X<q) + kP_0(X=q)\right).
\end{aligned} \tag{18}$$

If $P\left(\frac{dQ}{dP_0} = c\right) = P_0(X=q) = 0$ then clearly

$$P_0(X<q) + kP_0(X=q) = P_0(X<q) = \frac{\lambda}{1+\lambda}; \tag{19}$$

we used the definition of q in the last equality. If $P_0(X=q) > 0$ then, using the definition of k, we have

$$P_0(X<q) + kP_0(X=q) = P_0(X<q) + \left(\frac{\lambda}{1+\lambda} - P_0(X<q)\right) = \frac{\lambda}{1+\lambda}. \tag{20}$$

Thus, in any case, combining (19) and (20) with (18), we obtain:

$$\int X\psi_X \, dP_0 = -\int (q - X)^+ \, dP_0 + q\frac{\lambda}{1+\lambda}.$$

Plugging this into (17), after some simplification, we obtain:

$$\begin{aligned} \sup_{P \in \mathcal{P}_\lambda} E^P[-X] &= \lambda E[-X] + \frac{1-\lambda^2}{\lambda} \int (q - X)^+ \, dP_0 - q(1-\lambda) \\ &= \lambda E[-X] + (1-\lambda) \left(\frac{1+\lambda}{\lambda} E[(q - X)^+] - q \right) \end{aligned} \tag{21}$$

Finally, given that $AVaR_\lambda = \frac{1}{\lambda}E[(q - X)^+] - q$ (lemma 4.46 in Föllmer and Schied 2011), the last expression is equal to the one from the formulation of the theorem.

It remains to note that the case when the inequality $X < 0$ is not satisfied follows directly from the boundedness of X, and cash invariance of $AVaR$, risk measure defined in the theorem, and $E[-X]$.

□

The proof of theorem 2 is a consequence of the preceding proof. Indeed, the assertion is clear for random variables $X < 0$. If, however, the inequality is not satisfied one only has to note that the equality $\psi_X = \psi_Y$ holds for all random variables X and Y such that $X - Y = c \in \mathbb{R}$ a.e. (see equation (16)); and the claim now follows from the cash invariance of $LCMU_\lambda$.

C Calculations for AVaR and LCMU for a Log-Normally Distributed Position

We first introduce a new variable $y = \Phi^{-1}(t)$. We note that:

$$dt = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy,$$

which implies

$$\begin{aligned} AVaR_\lambda(X) &= -\frac{e^\mu}{\lambda} \int_0^\lambda \exp(\sigma\Phi^{-1}(t)) \, dt \\ &= -\frac{e^\mu}{\lambda} \int_{-\infty}^{\Phi^{-1}(\lambda)} \frac{1}{\sqrt{2\pi}} \exp(\sigma y) \exp\left(-\frac{y^2}{2}\right) dy \end{aligned}$$

Now, completing the squares and introducing a new variable $z = y - \sigma$ we get:

$$\begin{aligned}
AVaR_\lambda(X) &= -\frac{\exp\left(\mu + \frac{\sigma^2}{2}\right)}{\lambda} \int_{-\infty}^{\Phi^{-1}(\lambda)} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y-\sigma)^2}{2}\right) dy \\
&= -\frac{\exp\left(\mu + \frac{\sigma^2}{2}\right)}{\lambda} \int_{-\infty}^{\Phi^{-1}(\lambda)-\sigma} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz \\
&= -\frac{\exp\left(\mu + \frac{\sigma^2}{2}\right)}{\lambda} \Phi(\Phi^{-1}(\lambda) - \sigma).
\end{aligned}$$

D Details on Optimal Portfolios

Problems (P1-3) have been solved for risk measures VaR and *AVaR* in theorems 3.2.1, 3.3.1 and 3.4.1 in Gambrah and Pirvu (2014). Careful reading of the proofs reveals technical conditions under which their techniques can be used for other risk measures. We offer slightly more general formulations of the aforementioned theorems from Gambrah and Pirvu (2014) that will allow us to solve problems (P1-3) for $LCMU$.

Let us consider versions of problems (P1-3) where the risk measure $LCMU$ is replaced with a risk measure ρ: we will refer to those problems as (ρP1), (ρP1) and (ρP3). We will give sufficient conditions under which the solutions of the more general problems are multiples of π_M defined in (8). The key assumption is the following:

> *Assumption* (A): There are measurable functions functions G : $\mathbb{R}^2 \to \mathbb{R}$ and $h : \mathbb{R}^2 \to \mathbb{R}$ such that:
>
> 1. $G(\cdot, y)$ is increasing and $G(x, \cdot)$ is decreasing
> 2. $h(x)$ is decreasing.
> 3. $\rho(L(\pi)) = h\left(G\left(\mu(\pi), \psi(\pi)\right)\right).$

Measures $AVaR_\lambda$ and $LCMU_\lambda$ satisfy the assumption. Indeed, if we introduce a function

$$\varphi_\lambda(y) = \frac{1}{\lambda}\Phi(\Phi^{-1}(\lambda) - \sqrt{y})$$

it can easily be confirmed that:

$$\begin{aligned}
AVAR_\lambda(L(\pi)) &= x_\pi - x_\pi R \exp(G^A_\lambda(\mu(\pi), \psi(\pi)), \\
&\qquad \text{where } G^A_\lambda(x, y) = x + \ln(\varphi_\lambda(y)); \\
LCMU_\lambda(L(\pi)) &= x_\pi - x_\pi R \exp(G^L_\lambda(\mu(\pi), \psi(\pi)), \\
&\qquad \text{where } G^L_\lambda(x, y) = x + \ln(\lambda + (1-\lambda)\varphi_{\frac{\lambda}{1+\lambda}}(y)).
\end{aligned}$$

Theorem 4. *If the risk measure ρ satisfies the assumption (A) above, there are constants c_1, c_2 and c_3 such that strategies $\pi_1^* = c_1\pi_M, \pi_2^* = c_2\pi_M, \pi_3^* = c_3\pi_M$ solve problems (ρP1), (ρP2) and (ρP3).*

For the sake of completeness we offer the proof of the theorem; it is essentially the proof offered in Gambrah and Pirvu (2014) with several small imprecisions and errors rectified.

We begin with proving two auxiliary results:

Lemma 2. *For a fixed $\kappa < 0$ the strategy*

$$\pi_\lambda(t) = -\frac{1}{2\kappa}(\sigma(t)\sigma(t)')^{-1}B(t) = -\frac{1}{2\kappa}\pi_M(t)$$

solves the maximization problem:

$$\max_{\pi\in\mathcal{Q}} \mu(\pi) + \kappa\psi(\pi).$$

Proof. Note that for any vectors $\pi, B \in \mathbb{R}^m \backslash \{(0,\dots,0)\}$ and any invertible matrix $\sigma \in \mathbb{R}^{m\times m}$ we have:

$$||\sigma'\pi||^2 + \frac{1}{\kappa}B'\pi = ||\sigma'\pi + \frac{1}{2\kappa}\sigma^{-1}B||^2 - \frac{1}{4\kappa^2}||\sigma^{-1}B||^2.$$

Indeed, by completing the squares:

$$\begin{aligned}
||\sigma'\pi||^2 + \frac{1}{\kappa}B'\pi &= \pi'\sigma\sigma'\pi + \frac{2}{2\kappa}(\sigma^{-1}B)'\sigma'\pi \\
&= \pi'\sigma\sigma'\pi + \frac{1}{2\kappa}\pi'\sigma\sigma^{-1}B + \frac{1}{2\kappa}(\sigma^{-1}B)'\sigma'\pi + \frac{1}{4\kappa^2}(\sigma^{-1}B)'\sigma^{-1}B \\
&\quad - \frac{1}{4\kappa^2}(\sigma^{-1}B)'\sigma^{-1}B \\
&= \pi'\sigma(\sigma'\pi + \frac{1}{2\kappa}\sigma^{-1}B) + \frac{1}{2\kappa}(\sigma^{-1}B)'(\sigma'\pi + \frac{1}{2\kappa}\sigma^{-1}B) \\
&\quad - \frac{1}{4\kappa^2}(\sigma^{-1}B)'\sigma^{-1}B \\
&= (\sigma'\pi + \frac{1}{2\kappa}\sigma^{-1}B)'(\sigma'\pi + \frac{1}{2\kappa}\sigma^{-1}B) - \frac{1}{4\kappa^2}(\sigma^{-1}B)'\sigma^{-1}B.
\end{aligned}$$

Hence:

$$\begin{aligned}
\mu(\pi) + \kappa\psi(\pi) &= \kappa\left(\int_0^T \frac{1}{\kappa}B(s)'\pi(s) + ||\sigma(s)'\pi(s)||^2\, ds\right) \\
&= \kappa\left(\int_0^T ||\sigma(s)'\pi(s) + \frac{1}{2\kappa}\sigma^{-1}(s)B(s)||^2\, ds\right) \\
&\quad - \frac{1}{4\kappa^2}\int_0^T ||\sigma^{-1}(s)B(s)||^2\, ds.
\end{aligned}$$

Note that only the first term in the last expression contains π. Thus, since $\kappa < 0$, the maximization problem from the formulation of the lemma reduces to:

$$\min_{\pi \in \mathcal{Q}} \int_0^T ||\sigma(s)'\pi(s) + \frac{1}{2\kappa}\sigma^{-1}(s)B(s)||^2 \, ds.$$

The last integral is non-negative. Furthermore, direct calculations show that it is equal to zero for π_κ, which proves the claim. $\square$

Lemma 3. *Maximization problem:*

$$\max_{\pi \in \mathcal{Q}} \mu(\pi) \text{ subject to } \psi(\pi) = \varepsilon^2$$

is solved by

$$\pi_\varepsilon = \frac{\varepsilon}{\Theta}(\sigma(t)\sigma(t)')^{-1}B(t) = \frac{\varepsilon}{\Theta}\pi_M(t),$$

where

$$\Theta = \sqrt{\int_0^T ||\sigma(s)^{-1}B(s)||^2 \, ds}.$$

Proof. Direct calculations show that indeed $\psi(\pi_\varepsilon) = \varepsilon^2$.

Previous lemma established a mapping $\kappa \to \pi_\kappa$. Note that, by choosing $\kappa_\varepsilon = -\Theta(2\varepsilon)^{-1} < 0$ we have $\pi_\varepsilon = \pi_{\kappa_\varepsilon}$. The claim now follows directly by considering the Lagrangian: $\mathcal{L}(\pi, \kappa) = \mu(\pi) + \kappa(\psi(\pi) - \varepsilon^2)$. Indeed, for any strategy π satisfying the constraint $\psi(\pi) = \varepsilon^2$ we have:

$$\mu(\pi) = \mathcal{L}(\pi, \kappa_\varepsilon) \leq \mathcal{L}(\pi_\varepsilon, \kappa_\varepsilon) = \mu(\pi_\varepsilon),$$

where the inequality is the consequence of the previous lemma and the fact that, for a fixed $\kappa < 0$, the strategy π_κ maximizes $\mathcal{L}(\pi, \kappa)$.

$\square$

Before we turn to proving the theorem we introduce some notation. For nonnegative ε we denote by $\mathcal{Q}_\varepsilon$ the set of all the strategies $\pi \in \mathcal{Q}$ such that $\psi(\pi) = \varepsilon^2$. Note that, due to the definition of ψ and the assumptions on σ we have $\varepsilon \in I$ where:

$$I = \left[0, \int_0^T ||\sigma(s)||^2 \, ds\right],$$

and for every ε within that interval $\mathcal{Q}_\varepsilon \neq \emptyset$.

Clearly:

$$\bigcup_{\varepsilon \in I} \mathcal{Q}_\varepsilon = \mathcal{Q}.$$

Proof of 4 - $(\rho P1)$. Due to monotonicity of h we can reduce the problem to:

$$\min_{\pi \in \mathcal{Q}} G(\mu(\pi, \psi(\pi)).$$

We first solve the problem for a fixed $\varepsilon \in I$:

$$\min_{\pi \in \mathcal{Q}_\varepsilon} G(\mu(\pi), \varepsilon^2).$$

Due to monotonicity of $G(\cdot, x)$ this reduces to:

$$\min_{\pi \in \mathcal{Q}_\varepsilon} \mu(\pi).$$

By lemma 3 the solution is: $\pi_\varepsilon = \frac{\varepsilon}{\Theta}(\sigma(t)\sigma(t)')^{-1}B(t)$. Clearly, the problem $(\rho P1)$ is now equivalent to:

$$\min_{\varepsilon \in I} G(\mu(\pi_\epsilon), \varepsilon^2).$$

Direct calculations show that $\mu(\pi_M(t)) = \Theta^2$, hence the continuous function:

$$g(\varepsilon) = G(\mu(\pi_\epsilon), \varepsilon^2) = G(\frac{\varepsilon}{\Theta}\mu(\pi_M(t)), \varepsilon^2) = G(\Theta\varepsilon, \varepsilon^2)$$

is defined on a closed interval and thus attains its maximum. This implies that there is a $\varepsilon_1 \in I$ such that $\pi_1 = \pi_{\varepsilon_1} =$ solves the prolem $(\rho P1)$. In that case $c_1 = \varepsilon_1/\Theta$.

□

Proof of 4 - $(\rho P2)$. Similarly as in the proof regarding the problem $(\rho P1)$ the minimization problem immediately reduces to:

$$\max_{\pi \in \mathcal{Q}} G(\mu(\pi, \psi(\pi)) \text{ such that } E[X^\pi(T)] = M$$

The condition $E[X^\pi(T)] = M$ is, due to (7), equivalent to:

$$\mu(\pi) = \ln\left(\frac{M}{x_\pi R}\right) =: \zeta. \tag{22}$$

Hence, the optimization problem can be rewritten as:

$$\max_{\pi \in \mathcal{Q}} G(\zeta, \psi(\pi)) \text{ such that } \mu(\pi) = \zeta,$$

which, due to monotonicity of G, further reduces to:

$$\min_{\pi \in \mathcal{Q}} \psi(\pi) \text{ such that } \mu(\pi) = \zeta.$$

We can now use lemma 2 to solve this problem. Indeed, for a fixed $\kappa < 0$, the maximization problem in the formulation of 2 is equivalent to the problem of minimizing $\psi(\pi) + \frac{1}{\kappa}\mu(\pi)$: the strategy π_κ solves both problems. Hence, for a fixed $\kappa < 0$, $\pi_{1/\kappa}$ solves the problem of minimizing $\psi(\pi) + \kappa\mu(\pi)$, and thus also the equivalent problem:

$$(P_\zeta) \qquad \min_{\pi \in \mathcal{Q}} \psi(\pi) + \kappa(\mu(\pi) - \zeta).$$

Thus, the solution of the initial optimization problem is:

$$\pi_2^* = \frac{\Theta^2}{\zeta}(\sigma(t)\sigma(t)')^{-1}B(t) = \pi_{1/\kappa_2^*} \text{ for } \kappa_2^* = -\Theta^2(2\zeta)^{-1}$$

Indeed, direct calculations show that $\mu(\pi_2^*) = \zeta$ and, for any π that satisfies $\mu(\pi) = \zeta$, we have

$$\psi(\pi) = \psi(\pi) + \kappa_2^*(\mu(\pi) - \zeta) \geq \psi(\pi_2^*) + \kappa_2^*(\mu(\pi_2^*) - \zeta) = \psi(\pi_2^*),$$

where the inequality is due to the fact that π_2^* solves the problem (P_ζ) for $\kappa = \kappa_2^*$.

In this case $c_2 = \Theta^2/\zeta$. □

Proof of 4 - (ρP3) . We introduce the set $\mathcal{Q}' \subset \mathcal{Q}$ of all the strategies π satisfying the condition $\rho(L(\pi)) > C$, where C is the constant related to problems (P3) and (ρP3). We define $\mathcal{Q}'_\varepsilon = \mathcal{Q}_\varepsilon \cap \mathcal{Q}'$ and note that:

$$\mathcal{Q}'_\varepsilon = \left\{\pi \in \mathcal{Q} \mid \psi(\pi) = \varepsilon^2, \rho(L(\pi)) \leq C\right\} \text{ and } \bigcup_{\varepsilon \in I} \mathcal{Q}'_\varepsilon = \mathcal{Q}'.$$

Due to (7) the problem (ρP3) reduces to

$$\max_{\pi \in \mathcal{Q}'} \mu(\pi).$$

Let us consider a simpler problem:

$$(P'_\varepsilon) \qquad \max_{\pi \in \mathcal{Q}'_\varepsilon} \mu(\pi).$$

For a strategy $\pi \in \mathcal{Q}'_\varepsilon$ the constraint $\rho(L(\pi)) > C$ can be rewritten as:

$$\ln \frac{x_\pi - C}{x_\pi R} \leq G(\mu(\pi), \varepsilon^2).$$

The function $G(\cdot, \varepsilon^2)$ is increasing, hence it has an inverse that we denote with G_ε^{-1}. Thus the constraint can be rewritten as:

$$\mu(\pi) \geq G_\varepsilon^{-1}\left(\frac{x_\pi - C}{x_\pi R}\right) =: h(\varepsilon). \tag{23}$$

Let us consider the strategy π_ϵ from 3 that maximizes $\mu(\pi)$ over $\mathcal{Q}_\varepsilon$. Solving the problem P'_ε relies on noticing that the set $\mathcal{Q}'_\varepsilon$ is non-empty if and only if π_ε belongs to it. Indeed, if $\pi \in \mathcal{Q}'_\varepsilon$ then:

$$\mu(\pi_\epsilon) \geq \mu(\pi) \geq h(\varepsilon);$$

the first inequality is due to $\mathcal{Q}'_\varepsilon \subset \mathcal{Q}_\varepsilon$ and the second one is due to 3. This allows us to rewrite the problem (ρP3) as follows:

$$\max_{\varepsilon \in I} \mu(\pi_\epsilon) \text{ such that } \mu(\pi_\varepsilon) \geq h(\varepsilon).$$

Due to the definition of μ, the value $\mu(\pi_\varepsilon)$ is increasing in ε and the problem reduces further to:

$$\max_{\varepsilon \in I} \varepsilon \text{ such that } \mu(\pi_\varepsilon) \geq h(\varepsilon).$$

Continuity of $\mu(\pi_\varepsilon)$ as a function of ε and monotonicity of $g(\varepsilon)$ ensure that the problem has a solution that we denote by ε_3.

This proves that $\pi_3^* = \pi_{\varepsilon_3}$ solves the optimization problem, in which case $c_3 = \varepsilon_3/\Theta$ □

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