A SEMIGROUP APPROACH TO NONLINEAR LÉVY PROCESSES

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Abstract. We study the relation between Lévy processes under nonlinear expectations, nonlinear semigroups and fully nonlinear PDEs. First, we establish a one-to-one relation between nonlinear Lévy processes and nonlinear Markovian convolution semigroups. Second, we provide a condition on a family of infinitesimal generators $(A_\lambda)_{\lambda\in\Lambda}$ of linear Lévy processes which guarantees the existence of a nonlinear Lévy process such that the corresponding nonlinear Markovian convolution semigroup is a viscosity solution of the fully nonlinear PDE $\partial_t u = \sup_{\lambda\in\Lambda} A_\lambda u$. The results are illustrated with several examples.

Key words: Lévy process, convex expectation space, Markovian convolution semigroup, fully nonlinear PDE, Nisio semigroup

AMS 2010 Subject Classification: 60G51; 49L25; 47H20

1. Introduction

In this paper we study the relation between Lévy processes under nonlinear expectations, nonlinear semigroups and fully nonlinear PDEs. Let $(X_t)_{t\geq 0}$ be an $\mathbb{R}^d$-valued Lévy process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then

$$(S(t)f)(x) := \mathbb{E}(f(x + X_t))$$

for $f \in \mathrm{BUC}(\mathbb{R}^d)$, $t \geq 0$ and $x \in \mathbb{R}^d$, defines a strongly continuous semigroup on the space of bounded and uniformly continuous functions $\mathrm{BUC}(\mathbb{R}^d)$ whose infinitesimal generator $A\colon D(A) \subset \mathrm{BUC}(\mathbb{R}^d) \to \mathrm{BUC}(\mathbb{R}^d)$ is given by an integro-differential operator which is uniquely determined by a Lévy triplet, see Applebaum [1] or Sato [26]. Moreover, $(S(t)f)_{t\geq 0}$ is the solution of the abstract Cauchy problem

$$\partial_t u(t) = Au(t), \quad t > 0,$$

with $u(0) = f \in \mathrm{BUC}(\mathbb{R}^d)$. For a detailed discussion on operator semigroups we refer to Pazy [22] or Engel and Nagel [8].

We first extend the well-known relation between Lévy processes and Markovian convolution semigroups of probability measures to a nonlinear setting. Nonlinear Lévy processes were introduced in [17] as càdlàg processes with stationary and independent increments under a sublinear expectation. The G-Brownian motion due to [23, 24] is a special case of a nonlinear Lévy process, see also Dolinsky et al. [6] or Denis et al. [4]. For an introduction to nonlinear expectations we refer to [16]. Since we do not require any path regularity we call a process $(X_t)_{t\geq 0}$ an $\mathcal{E}$-Lévy process with values in an abelian group G, if it has stationary and independent increments and $X_t \to X_0$ in distribution as $t \searrow 0$ with respect to a convex expectation $\mathcal{E}$. We then provide a relation between $\mathcal{E}$-Lévy processes and convex Markovian convolution semigroups, i.e. strongly

Date: March 7, 2019.

We thank two anonymous referees for their helpful comments. Financial support through the German Research Foundation via CRC 1283 “Taming Uncertainty” is gratefully acknowledged.

continuous semigroups $(\mathscr{S}(t))_{t\geq 0}$ on $\mathrm{BUC}(G)$ such that each $\mathscr{S}(t)$ is a translation invariant convex kernel. The proof relies on a version of Kolmogorov's extension theorem for nonlinear expectations elaborated in [5].

Our main focus lies on the construction of $\mathcal{E}$-Lévy processes with nonlinear generators. We start with an arbitrary family Λ of generators

$$A_\lambda \colon D(A_\lambda) \subset \mathrm{BUC}(G) \to \mathrm{BUC}(G)$$

of Markovian convolution semigroups $S_\lambda = (S_\lambda(t))_{t\geq 0}$ of linear operators. We then construct the smallest Markovian convolution semigroup $(\mathscr{S}(t))_{t\geq 0}$ which dominates each $(S_\lambda(t))_{t\geq 0}$. The corresponding $\mathcal{E}$-Lévy process can be viewed as a process with independent increments whose distribution is uncertain, i.e. any distrubution of the increments associated to $(S_\lambda)_{\lambda\in\Lambda}$ is taken into account. We basically follow an idea by Nisio [21] in order to construct a sublinear Markovian convolution semigroup which results from a given family of linear Markovian convolution semigroups by constant optimization. In [21] Nisio considers strongly continuous semigroups on the space of all bounded measurable functions. However, by a theorem of Lotz (see e.g. [2, Corollary 4.3.19]), any strongly continuous semigroup on the space of bounded measurable functions already has a bounded generator, which is not suitable for most applications. We therefore modify Nisio's construction to the space $\mathrm{BUC}(G)$. Under the condition that the subspace

$$\left\{ f \in \bigcap_{\lambda\in\Lambda} D(A_\lambda) : \big\{ A_\lambda f \colon \lambda \in \Lambda \big\} \text{ is bounded and uniformly equicontinuous} \right\} \tag{1.1}$$

is dense in $\mathrm{BUC}(G)$ we construct a (strongly continuous) Markovian convolution semigroup $(\mathscr{S}(t))_{t\geq 0}$ on $\mathrm{BUC}(G)$ with corresponding $\mathcal{E}$-Lévy process $(X_t)_{t\geq 0}$ on a sublinear expectation space $(\Omega, \mathcal{F}, \mathcal{E})$ such that

$$u(t,x) := (\mathscr{S}(t)f)(x) = \mathcal{E}(f(x+X_t)), \quad t \geq 0,\, x \in G,$$

is a viscosity solution of the fully nonlinear PDE

$$\partial_t u = \sup_{\lambda\in\Lambda} A_\lambda u \quad \text{on } (0,\infty)\times G$$

with $u(0) = f$ for all $f \in \mathrm{BUC}(G)$. In particular, the generator of the $\mathcal{E}$-Lévy process $(X_t)_{t\geq 0}$ is given by $\sup_{\lambda\in\Lambda} A_\lambda$. Here, we use a slightly different notion of viscosity solution which fits to the semigroup setting. However, in many cases, particularly for the classical case $G = \mathbb{R}^d$, this leads to the same class or an even larger class of test functions. We refer to Crandall et al. [3] for the classical definition and a detailed discussion of viscosity solutions. Moreover, we give a condition on the generators $(A_\lambda)_{\lambda\in\Lambda}$ which guarantees that the corresponding $\mathcal{E}$-Lévy process is tight, or equivalently each $\mathscr{S}(t)$ is continuous from above. Throughout, the state space is an abelian group, which gives the opportunity to consider certain classes of cylindrical G-Wiener Processes as an infinite dimensional extension of the G-Brownian motion, or nonlinear Lévy processes on the d-dimensional Torus.

Nonlinear $\mathbb{R}^d$-valued Lévy processes have first been introduced in Hu and Peng [17], where G-Lévy processes with a decomposition $X = X^c + X^d$ into a continuous and a jump part are considered. Under the assumption that X^c is a G-Brownian motion and $\mathcal{E}(|X_t^d|) \leq ct$ for some constant c, it is shown that $u(t,x) = \mathcal{E}(f(x+X_t))$ is a viscosity solution of $\partial_t u(t,x) - G(u(t,x+\cdot) - u(t,x)) = 0$ and $u(0) = f$, where $G(\varphi(\cdot)) := \lim_{h\downarrow 0} \frac{1}{h}\mathcal{E}(\varphi(X_h))$. The function G is shown to have a Lévy-Khinchine representation in terms of a set Λ of Lévy triplets (b,Σ,μ) satisfying an integrability condition, i.e. $u(t,x)$ is the solution of

$$\partial_t u = \sup_{(b,\Sigma,\mu)\in\Lambda} A_{b,\Sigma,\mu}u, \quad \text{and} \quad u(0)=f, \tag{1.2}$$

where $A_{b,\Sigma,\mu}$ is the generator with the Lévy triplet (b,Σ,μ), see also Example 3.2. Conversely, starting from the unique solution of (1.2), Hu and Peng [17] give a construction of the respective nonlinear Lévy process. In Nutz and Neufeld [19] the authors consider upper expectations $\mathcal{E}(\cdot)=\sup_{\mathbb{P}}\mathbb{E}_{\mathbb{P}}(\cdot)$ over a class of all semimartingales with given differential characteristics in a set Λ of Lévy triplets, which in [20] is shown to be analytic. This allows them to construct conditional nonlinear expectations and nonlinear Lévy processes with general characteristics whose distributions are defined for all measurable functions. Under the conditions

$$\sup_{(b,\Sigma,\mu)\in\Lambda}\Big(|b|+|\Sigma|+\int_{\mathbb{R}^d}|y|\wedge|y|^2\,\mathrm{d}\mu(y)\Big)<\infty \quad \text{and} \quad \lim_{\varepsilon\searrow 0}\sup_{(b,\Sigma,\mu)\in\Lambda}\int_{|z|\le\varepsilon}|z|^2\,\mathrm{d}\mu(z)=0 \tag{1.3}$$

it is shown that $u(t,x)=\mathcal{E}(f(x+X_t))$ is the unique viscosity solution of (1.2). The conditions in (1.3) are weaker than the integrability condition in [17] and allow for instance to consider classes of Lévy processes with infinite variation jumps. Our main condition (1.1) in the context of $\mathbb{R}^d$-valued processes is guaranteed under

$$\sup_{(b,\Sigma,\mu)\in\Lambda}\Big(|b|+|\Sigma|+\int_{\mathbb{R}^d}1\wedge|y|^2\,\mathrm{d}\mu(y)\Big)<\infty, \tag{1.4}$$

which does not exclude any Lévy triplet at all. In particular, Lévy processes with non-integrable jumps can be considered, see e.g. Example 3.6, and for finite Λ the condition (1.4) is always satisfied. In order to obtain uniqueness for the viscosity solution of (1.2) one additionally needs the second condition in (1.3) and tightness of the family of Lévy measures $\{\mu:(b,\Sigma,\mu)\in\Lambda\}$, which is due to [17]. In Hollender [12] the results of [19] are generalized to upper expectations over state-dependent Lévy triplets, see also Kühn [15] for existence results on the respective integro-differential equations under fairly general conditions. A related concept to nonlinear Lévy processes are second order backward stochastic differential equation with jumps, see Kazi-Tani et al. [13], [14] and also Soner et al. [27].

The paper is organized as follows. In Section 2 we introduce the notation and discuss our main results which are illustrated with several examples in Section 3. The relation between $\mathcal{E}$-Lévy processes and Markovian convolution semigroups is given in Section 4. Finally, in Section 5 we prove the main result by constructing a version of Nisio semigroups on $\mathrm{BUC}(G)$.

2. Main results

We say that $(\Omega,\mathcal{F},\mathcal{E})$ is a *convex expectation space* if $(\Omega,\mathcal{F})$ is a measurable space and $\mathcal{E}\colon\mathcal{L}^\infty(\Omega,\mathcal{F})\to\mathbb{R}$ is a convex expectation which is continuous from below. As usual $\mathcal{L}^\infty(\Omega,\mathcal{F})$ denotes the space of all bounded measurable functions $\Omega\to\mathbb{R}$. Recall that a convex expectation on a convex set M with $\mathbb{R}\subset M$ is a functional $\mathcal{E}\colon M\to\mathbb{R}$ which satisfies

$\mathcal{E}(X)\le\mathcal{E}(Y)$ whenever $X\le Y$,
$\mathcal{E}(\alpha)=\alpha$ for all $\alpha\in\mathbb{R}$, and
$\mathcal{E}\big(\lambda X+(1-\lambda)Y\big)\le\lambda\mathcal{E}(X)+(1-\lambda)\mathcal{E}(Y)$ for all $\lambda\in[0,1]$.

If in addition $\mathcal{E}$ is positive homogeneous, i.e. $\mathcal{E}(\lambda X) = \lambda\mathcal{E}(X)$ for all $\lambda > 0$, then $\mathcal{E}$ is called a *sublinear* expectation and $(\Omega, \mathcal{F}, \mathcal{E})$ is a *sublinear* expectation space. A convex expectation is said to be *continuous from below* if $\mathcal{E}(X_n) \nearrow \mathcal{E}(X)$ for every increasing sequence (X_n) in $\mathcal{L}^\infty(\Omega, \mathcal{F})$ which converges pointwise to $X \in \mathcal{L}^\infty(\Omega, \mathcal{F})$.

Let $(\Omega, \mathcal{F}, \mathcal{E})$ be a convex expectation space, and let G be an abelian group with a translation invariant metric d such that (G, d) is separable and complete. We denote by $C_b(G)$ and $\mathrm{BUC}(G)$ the spaces of all bounded functions $f\colon G \to \mathbb{R}$ which are continuous and uniformly continuous, respectively. For an $\mathcal{F}$-$\mathcal{B}$-measurable random variable $X\colon \Omega \to G$ with values in G endowed with the Borel σ-algebra $\mathcal{B}$, the functional

$$\mathcal{E} \circ X^{-1}\colon C_b(G) \to \mathbb{R}, \quad f \mapsto \mathcal{E}(f(X))$$

defines a convex expectation which is called the *distribution* of X under $\mathcal{E}$. Given another random variable $Y\colon \Omega \to S$ with values in a Polish space S, for $f \in C_b(S \times G)$ the function

$$S \to \mathbb{R}, \quad y \mapsto \mathcal{E}(f(y, X))$$

is bounded and lower semicontinuous. In fact, for $g \in \mathrm{BUC}(S \times G)$, it follows that

$$|\mathcal{E}(g(y, X)) - \mathcal{E}(g(z, X))| \leq \|g(y, \cdot\,) - g(z, \cdot\,)\|_\infty$$

for $y, z \in G$ and therefore, $y \mapsto \mathcal{E}(g(y, X))$ is uniformly continuous. Approximating $f \in C_b(S \times G)$ from below by a sequence $(g_n)_{n\in\mathbb{N}} \subset \mathrm{BUC}(S \times G)$, see [5, Remark 5.4 a)], we obtain that $y \mapsto \mathcal{E}(f(y, X))$ is lower semicontinuous. Hence, $\mathcal{E}(f(y, X))|_{y=Y}$ is in $\mathcal{L}^\infty(\Omega, \mathcal{F})$, which shows that $\mathcal{E}\big(\mathcal{E}(f(y, X))|_{y=Y}\big)$ is well-defined. Then, X is called *independent* of Y if

$$\mathcal{E}(f(Y, X)) = \mathcal{E}\big(\mathcal{E}(f(y, X))|_{y=Y}\big)$$

for all $f \in C_b(S \times G)$.

Definition 2.1. a) We say that $\mathscr{S}\colon \mathrm{BUC}(G) \to \mathrm{BUC}(G)$ is a convex *kernel* if $(\mathscr{S}\,\cdot\,)(x)$ is a convex expectation on $\mathrm{BUC}(G)$ for all $x \in G$. It is called *continuous from above* if $\mathscr{S} f_n \searrow \mathscr{S} f$ (pointwise convergence) for every decreasing sequence (f_n) in $\mathrm{BUC}(G)$ which converges pointwise to $f \in \mathrm{BUC}(G)$.

b) A convex kernel $\mathscr{S}\colon \mathrm{BUC}(G) \to \mathrm{BUC}(G)$ is called a convex *Markovian convolution* if $\mathscr{S} f_n \nearrow \mathscr{S} f$ for every increasing sequence (f_n) in $\mathrm{BUC}(G)$ which converges pointwise to $f \in \mathrm{BUC}(G)$ and

$$\big(\mathscr{S} f\big)(x) = \big(\mathscr{S} f_x\big)(0)$$

for all $f \in \mathrm{BUC}(G)$ and $x \in G$, where $f_x\colon G \to \mathbb{R}$ is given by $y \mapsto f(x + y)$.

c) We say that $(\mathscr{S}(t))_{t\geq 0}$ is a convex *Markovian convolution semigroup* on $\mathrm{BUC}(G)$ if
   (i) $\mathscr{S}(t)$ is a convex Markovian convolution for all $t \geq 0$,
   (ii) $\mathscr{S}(0)f = f$ for all $f \in \mathrm{BUC}(G)$,
   (iii) $\mathscr{S}(s + t) = \mathscr{S}(s)\mathscr{S}(t)$ for all $s, t \geq 0$,
   (iv) $\lim_{t\searrow 0} \|\mathscr{S}(t)f - f\|_\infty = 0$ for all $f \in \mathrm{BUC}(G)$.

   In this case, we say that $(\mathscr{S}(t))_{t\geq 0}$ is continuous from above if each $\mathscr{S}(t)$ is so.

d) Let $(\Omega, \mathcal{F}, \mathcal{E})$ be a convex expectation space. Then, $(X_t)_{t\geq 0}$ is called an *$\mathcal{E}$-Lévy process* if
   (i) $X_t\colon \Omega \to G$ is measurable for all $t \geq 0$,
   (ii) $\mathcal{E}(f(X_0)) = f(0)$ for all $f \in C_b(G)$,
   (iii) $\mathcal{E} \circ (X_{s+t} - X_s)^{-1} = \mathcal{E} \circ X_t^{-1}$ for all $s, t \geq 0$,
   (iv) $X_{s+t} - X_s$ is independent of $(X_{t_1}, \ldots, X_{t_n})$ for all $s, t \geq 0$, $n \in \mathbb{N}$, $0 \leq t_1 < \ldots < t_n \leq s$,
   (v) $\mathcal{E}(f(X_t)) \to f(0)$ for all $f \in C_b(G)$, i.e. $X_t \to X_0$ in distribution as $t \searrow 0$.

Remark 2.2. Let $\mathscr{S} : \mathrm{BUC}(G) \to \mathrm{BUC}(G)$ be a convex kernel which is continuous from above. Then, the mapping

$$\mathcal{E}_0 \colon \mathrm{BUC}(G) \to \mathbb{R}, \quad f \mapsto \big(\mathscr{S} f\big)(0)$$

is a convex expectation which is continuous from above. If $\mathscr{S}$ is in addition a Markovian convolution then, by [5, Theorem 3.10] with $\Omega := G$, $\mathcal{F}$ the Borel σ-algebra on G and $M = \mathrm{BUC}(G)$, there exists a convex expectation space $(\Omega, \mathcal{F}, \mathcal{E})$ and a random variable X (here the identity on G) such that

$$\big(\mathscr{S} f\big)(x) = \mathcal{E}(f(x + X))$$

for all $f \in C_b(G)$ and $x \in G$. By [5, Remark 5.4 c)] the convex kernel $\mathscr{S}$ has a unique extension to a convex kernel $\mathscr{S} : C_b(G) \to C_b(G)$ which is continuous from above.

Our first result connects convex Markovian convolution semigroups and $\mathcal{E}$-Lévy processes. The proof is given in Section 4.

Theorem 2.3. *For every convex Markovian convolution semigroup $(\mathscr{S}(t))_{t\geq 0}$ which is continuous from above, there exists a convex expectation space $(\Omega, \mathcal{F}, \mathcal{E})$ and an $\mathcal{E}$-Lévy process $(X_t)_{t\geq 0}$ such that*

$$\big(\mathscr{S}(t) f\big)(x) = \mathcal{E}(f(x + X_t)) \tag{2.1}$$

for every $f \in \mathrm{BUC}(G)$ and $x \in G$.

Conversely, for every $\mathcal{E}$-Lévy process $(X_t)_{t\geq 0}$ on a convex expectation space $(\Omega, \mathcal{F}, \mathcal{E})$, the family $(\mathscr{S}(t))_{t\geq 0}$ defined by (2.1) is a convex Markovian convolution semigroup.

In this paper, we use the following definition of a viscosity solution.

Definition 2.4. Let $D \subset \mathrm{BUC}(G)$ and $\mathcal{A} \colon D \subset \mathrm{BUC}(G) \to \mathrm{BUC}(G)$. Then, we say that $u \colon [0, \infty) \to \mathrm{BUC}(G)$ is a *D-viscosity subsolution* of the PDE

$$u_t = \mathcal{A} u \tag{2.2}$$

if $u \colon [0, \infty) \to \mathrm{BUC}(G)$ is continuous and for every $t > 0$ and $x \in G$ we have

$$\partial_t \psi(t, x) \leq \big(\mathcal{A}\psi(t)\big)(x)$$

for every differentiable $\psi \colon (0, \infty) \to \mathrm{BUC}(G)$ which satisfies $(\psi(t))(x) = (u(t))(x)$, $\psi(s) \geq u(s)$ and $\psi(s) \in D$ for all $s > 0$. Here and in the following, we use the notation $\psi(t, x) := (\psi(t))(x)$.

Analogously, u is called a *D-viscosity supersolution* of (2.2) if $u \colon [0, \infty) \to \mathrm{BUC}(G)$ is continuous and for every $t > 0$ and $x \in G$ we have

$$\partial_t \psi(t, x) \geq \big(\mathcal{A}\psi(t)\big)(x)$$

for every differentiable $\psi \colon (0, \infty) \to \mathrm{BUC}(G)$ which satisfies $(\psi(t))(x) = (u(t))(x)$, $\psi(s) \leq u(s)$ and $\psi(s) \in D$ for all $s > 0$.

We say that u is a *D-viscosity solution* of (2.2) if u is a D-viscosity subsolution and a D-viscosity supersolution.

Now we are ready to state our main result. Given a family $(A_\lambda)_{\lambda\in\Lambda}$ of generators of Lévy processes we provide the existence of an $\mathcal{E}$-Lévy process on a sublinear expectation space with generator $\sup_{\lambda\in\Lambda} A_\lambda$. The corresponding sublinear Markovian convolution semigroup is a viscosity solution of the fully nonlinear PDE $u_t = \sup_{\lambda\in\Lambda} A_\lambda u$ with $u(0, \cdot) = f \in \mathrm{BUC}(G)$. The proof is postponed to Section 5.

Theorem 2.5. *Let $\Lambda \neq \emptyset$ be an index set. Assume that the following holds:*

(A1) For each $\lambda \in \Lambda$ let $A_\lambda \colon D(A_\lambda) \subset \mathrm{BUC}(G) \to \mathrm{BUC}(G)$ be the generator of a Markovian convolution semigroup $(S_\lambda(t))_{t\geq 0}$ of linear operators.

(A2) The subspace

$$D := \left\{ f \in \bigcap_{\lambda\in\Lambda} D(A_\lambda) : \big\{A_\lambda f \colon \lambda \in \Lambda\big\} \text{ is bounded and uniformly equicontinuous} \right\}$$

is dense in $\mathrm{BUC}(G)$.

Then, there exists a sublinear expectation space $(\Omega, \mathcal{F}, \mathcal{E})$ and an $\mathcal{E}$-Lévy process $(X_t)_{t\geq 0}$ such that for each $f \in \mathrm{BUC}(G)$ the function

$$(u(t))\,(x) := \mathcal{E}(f(x + X_t)), \quad t \geq 0,\, x \in G \tag{2.3}$$

is a D-viscosity solution of the fully nonlinear PDE

$$u_t(t,x) = \sup_{\lambda\in\Lambda} \big(A_\lambda u(t)\big)(x), \quad (t,x) \in (0,\infty) \times G, \tag{2.4}$$

$$u(0,x) = f(x), \quad x \in G. \tag{2.5}$$

Moreover, there exists a set $\mathcal{P}$ of probability measures on $(\Omega, \mathcal{F})$ such that $\mathcal{E}(Y) = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}(Y)$ for all $Y \in \mathcal{L}^\infty(\Omega, \mathcal{F})$.

Remark 2.6. In the situation of Theorem 2.5, if each $A_\lambda \colon \mathrm{BUC}(G) \to \mathrm{BUC}(G)$ is a bounded linear operator and $\sup_{\lambda\in\Lambda} \|A_\lambda\| < \infty$, then the mapping

$$\mathrm{BUC}(G) \to \mathrm{BUC}(G), \quad u \mapsto \sup_{\lambda\in\Lambda} A_\lambda u$$

is Lipschitz continuous. Therefore, by the Picard-Lindelöf theorem, the function u in (2.3) is a classical solution of the fully nonlinear PDE (2.4)-(2.5), which satisfies $u \in C^1([0,\infty); \mathrm{BUC}(G))$.

In most applications, the conditions (A1) and (A2) in Theorem 2.5 can be easily verified as shown in Section 3. For the sake of illustration, we consider the case $G = \mathbb{R}^d$, where the Lévy-Khintchine formula characterizes generators of Markovian convolution semigroups of linear operators by means of so-called Lévy triplets, see e.g. Applebaum [1] or Sato [26]. Given a set Λ of Lévy triplets (b, Σ, μ), i.e. $b \in \mathbb{R}^d$, $\Sigma \in \mathbb{R}^{d\times d}$ is a symmetric positive semidefinite matrix and μ is a Lévy measure, the condition (1.4) is sufficient to guarantee (A2) with $\mathrm{BUC}^2(\mathbb{R}^d) \subset D$. Here, $\mathrm{BUC}^2(\mathbb{R}^d)$ denotes the space of all functions which are twice differentiable with bounded uniformly continuous derivatives up to order 2. For more details, we refer to Example 3.2 which contains $G = \mathbb{R}^d$ as a special case.

Remark 2.7. Let $\psi \in C_b^{2,3}((0,\infty) \times \mathbb{R}^d)$, where $C_b^{2,3}((0,\infty) \times \mathbb{R}^d)$ stands for the space of all functions of $(t,x) \in (0,\infty) \times \mathbb{R}^d$ for which all partial derivatives up to order 2 in t and up to order 3 in x exist, are continuous and bounded. Then,

$$\lim_{h\to 0} \sup_{x\in\mathbb{R}^d} \left| \frac{\psi(t+h,x) - \psi(t,x)}{h} - \partial_t \psi(t,x) \right| = 0$$

for all $t > 0$ and therefore, $\psi \colon (0,\infty) \to \mathrm{BUC}(\mathbb{R}^d)$ is differentiable with $\psi(s) \in \mathrm{BUC}^2(\mathbb{R}^d)$ for all $s > 0$ using the identification $(\psi(s))(x) := \psi(s,x)$. Therefore, the class of test functions considered in the framework of $\mathrm{BUC}^2(\mathbb{R}^d)$-viscosity solutions includes the class $C_b^{2,3}((0,\infty) \times \mathbb{R}^d)$ of test functions, which is often considered in classical viscosity theory, see e.g. Denis et al. [4] or Hu and Peng [17]. Assuming in addition to (1.4) that for every $\varepsilon > 0$ there exists some $M > 0$ such that

$$\sup_{(b,\Sigma,\mu)\in\Lambda} \mu\big(\mathbb{R}^d \setminus B(0,M)\big) < \varepsilon, \tag{2.6}$$

where $B(0,M) := \{x \in \mathbb{R}^d : |x| \leq M\}$, and the second condition in (1.3) one obtains from [17, Corollary 53] the uniqueness of the viscosity solution of the PDE (1.2).

Let $C_0(G)$ be the closure of the space $C_c(G)$ of all continuous functions with compact support w.r.t. the supremum norm $\|\cdot\|_\infty$. Note that the existence of a function in $C_0(G) \setminus \{0\}$ already implies that G is locally compact and vice versa since G is a topological abelian group, which is metrizable. The following additional condition on the generators $(A_\lambda)_{\lambda\in\Lambda}$ implies that the related Markovian convolution semigroup is continuous from above.

Proposition 2.8. *In addition to the assumptions in Theorem 2.5, suppose:*

(A3) For every $\varepsilon > 0$ there exists $\varphi \in \bigcap_{\lambda\in\Lambda} \big(D(A_\lambda) \cap C_0(G)\big)$ with $0 \leq \varphi \leq 1$, $\varphi(0) = 1$ and $\sup_{\lambda\in\Lambda} \|A_\lambda\varphi\|_\infty \leq \varepsilon$.

Then, the related Markovian convolution semigroup $(\mathscr{S}(t)f)(x) = \mathcal{E}(f(x + X_t))$, $t \geq 0$, is continuous from above on $\mathrm{BUC}(G)$.

The proof is given at the end of Section 5. In line with Remark 2.2, under (A3) the Markovian convolution semigroup $(\mathscr{S}(t))_{t\geq0}$ has a unique extension from $\mathrm{BUC}(G)$ to $C_b(G)$ which is continuous from above. Moreover, continuity from above implies a dual max-representation for the sublinear expectation $\mathcal{E}$ in terms of probability measures, see [5, Lemma 2.4, Lemma 3.2]. For instance, in the case $G = \mathbb{R}^d$, where the generators are given by Lévy triplets $(b,\Sigma,\mu) \in \Lambda$, the condition (A3) holds if (1.4) is satisfied and the set of Lévy measures $\{\mu : (b,\Sigma,\mu) \in \Lambda\}$ is tight.

3. Examples

Let $\mathrm{ca}_+^1(G)$ be the set of all Borel probability measures on G.

Example 3.1 (Compound Poisson processes)**.** For $\lambda \geq 0$ and $\mu \in \mathrm{ca}_+^1(G)$, let

$$\big(A_{\lambda,\mu}f\big)(x) := \lambda \int_G f(x+y) - f(x)\,\mathrm{d}\mu(y), \quad f \in \mathrm{BUC}(G),\ x \in G.$$

Then, $A_{\lambda,\mu}\colon \mathrm{BUC}(G) \to \mathrm{BUC}(G)$ is a bounded linear operator which satisfies the positive maximum principle (cf. [11, Definition 4.5.1]), i.e. for $f \in \mathrm{BUC}(G)$ and $x_0 \in G$ with $f(x_0) = \max_{x\in G} f(x) \geq 0$ one has $\big(A_{\lambda,\mu}f\big)(x_0) \leq 0$. Further, since $A_{\lambda,\mu}$ is bounded and linear, it generates the linear uniformly continuous semigroup $(e^{tA_{\lambda,\mu}})_{t\geq0}$. Recall that for a bounded linear operator $B\colon \mathrm{BUC}(G) \to \mathrm{BUC}(G)$ the exponential $e^B := \sum_{k=0}^\infty \frac{1}{k!}B^k$ of B is again a bounded linear operator $\mathrm{BUC}(G) \to \mathrm{BUC}(G)$. We first show that $S_{\lambda,\mu}(t) := e^{tA_{\lambda,\mu}}$ satisfies

$$\big(S_{\lambda,\mu}(t)f\big)(x) = \mathbb{E}(f(x+J_t)) = \int_G f(x+y)\,\mathrm{d}\big(\mathbb{P}\circ J_t^{-1}\big)(y), \quad f \in \mathrm{BUC}(G)$$

for all $t \geq 0$, where $(J_t)_{t\geq0}$ is a compound Poisson process with rate λ and jump size distribution μ on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. In particular, $\big(S_{\lambda,\mu}(t)\big)_{t\geq0}$ is a linear Markovian convolution semigroup. Indeed, let $J_t = \sum_{i=1}^{N_t} Y_i$ for an i.i.d. sequence $(Y_i)_{i\in\mathbb{N}}$ of random variables $Y_i\colon \Omega \to G$ such that $\mathbb{P}\circ Y_i^{-1} = \mu$, and a random variable

$N_t\colon \Omega \to \mathbb{N}_0$ which is independent of $(Y_i)_{i\in\mathbb{N}}$ and satisfies $\mathbb{P}(N_t = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$ for all $n \in \mathbb{N}$. Then, for $f \in \mathrm{BUC}(G)$ and $x \in G$ we have

$$\begin{aligned}
\mathbb{E}(f(x+J_t)) &= \sum_{n=0}^{\infty} \mathbb{E}(f(x+Y_1+\ldots+Y_n))e^{-\lambda t}\frac{(\lambda t)^n}{n!} \\
&= e^{-\lambda t}\sum_{n=0}^{\infty} \frac{t^n\lambda^n}{n!}\int_G \cdots \int_G f(x+y_1+\ldots+y_n)\,\mathrm{d}\mu(y_1)\cdots \mathrm{d}\mu(y_n) \\
&= e^{-\lambda t}\sum_{n=0}^{\infty} \frac{\big(t^n(A_{\lambda,\mu}+\lambda)^n f\big)(x)}{n!} = e^{-\lambda t}\big(e^{t(A_{\lambda,\mu}+\lambda)}f\big)(x) \\
&= \big(e^{tA_{\lambda,\mu}}f\big)(x) = \big(S_{\lambda,\mu}(t)f\big)(x),
\end{aligned}$$

where we used $\big((A_{\lambda,\mu}+\lambda)f\big)(x) = \lambda\int_G f(x+y)\,\mathrm{d}\mu(y)$.

Now, assume that $\Lambda \subset [0,\infty)$ is bounded and $\mathcal{Q} \subset \mathrm{ca}^1_+(G)$. Since

$$\left\{ f \in \mathrm{BUC}(G) : \begin{array}{l} \{A_{\lambda,\mu}f\colon (\lambda,\mu)\in\Lambda\times\mathcal{Q}\} \text{ is bounded} \\ \text{and uniformly equicontinuous} \end{array} \right\} = \mathrm{BUC}(G),$$

the assumptions (A1) and (A2) are satisfied with $D = \mathrm{BUC}(G)$. Hence, by Theorem 2.5 and Remark 2.6 there exists a nonlinear expectation space $(\Omega,\mathcal{F},\mathcal{E})$ and an $\mathcal{E}$-Lévy process $(X_t)_{t\geq 0}$ such that for all $f \in \mathrm{BUC}(G)$ the function

$$u(t,x) := \big(u(t)\big)(x) := \mathcal{E}(f(x+X_t)), \quad t \geq 0,\ x \in G,$$

is the unique classical solution of the fully nonlinear PIDE

$$\begin{aligned}
u_t(t,x) &= \sup_{(\lambda,\mu)\in\Lambda\times\mathcal{Q}} \big(A_{\lambda,\mu}u(t)\big)(x), \quad (t,x)\in(0,\infty)\times G, \\
u(0,x) &= f(x), \quad x \in G,
\end{aligned}$$

with $u \in C^1([0,\infty);\mathrm{BUC}(G))$.

Example 3.2 (Lévy processes on real separable Hilbert spaces)**.** Let $G = H$ be a real separable Hilbert space, and consider a set Λ of Lévy triplets (b,Σ,μ), i.e. $b \in H$, $\Sigma\colon H \to H$ is a symmetric positive semidefinite trace class operator and μ is a Lévy measure on H, i.e. a Borel measure satisfying

$$\mu(\{0\}) = 0 \quad \text{and} \quad \int_H 1 \wedge \|y\|^2\,\mathrm{d}\mu(y) < \infty.$$

By the Lévy-Khintchine formula (see e.g. [18, Theorem 5.7.3]), for each Lévy triplet (b,Σ,μ) there exists a Markovian convolution semigroup $(S_{b,\Sigma,\mu}(t))_{t\geq 0}$ of linear operators on $\mathrm{BUC}(H)$ with generator $A_{b,\Sigma,\mu}\colon D(A_{b,\Sigma,\mu}) \subset \mathrm{BUC}(H) \to \mathrm{BUC}(H)$. Moreover, $\mathrm{BUC}^2(H) \subset D(A_{b,\Sigma,\mu})$, where $\mathrm{BUC}^2(H)$ denotes the space of all functions $H \to \mathbb{R}$ which are twice differentiable with bounded uniformly continuous derivatives up to order 2. For $f \in \mathrm{BUC}^2(H)$, the function $A_{b,\Sigma,\mu}f$ is given by

$$\big(A_{b,\Sigma,\mu}f\big)(x) = \langle b, \nabla f(x)\rangle + \frac{1}{2}\mathrm{tr}\big(\Sigma\nabla^2 f(x)\big) + \int_H f(x+y) - f(x) - \langle \nabla f(x), h(y)\rangle\,\mathrm{d}\mu(y)$$

for $x \in H$. Here, the function $h\colon H \to H$ is defined by $h(y) = y$ for $\|y\| \leq 1$, and $h(y) = 0$ whenever $\|y\| > 1$. In particular, (A1) is satisfied. Assume that

$$C := \sup_{(b,\Sigma,\mu)\in\Lambda} \left(\|b\| + \|\Sigma\|_{\mathrm{tr}} + \int_H 1 \wedge \|y\|^2\,\mathrm{d}\mu(y) \right) < \infty, \tag{3.1}$$

where $\|\cdot\|_{\mathrm{tr}}$ denotes the trace norm. We will verify that (A2) is satisfied under (3.1). Let

$$D := \left\{ f \in \bigcap_{(b,\Sigma,\mu)\in\Lambda} D(A_{b,\Sigma,\mu}) \, : \, \begin{array}{l} \{A_{b,\Sigma,\mu} f \, : \, (b,\Sigma,\mu) \in \Lambda\} \text{ is bounded} \\ \text{and uniformly equicontinuous} \end{array} \right\}.$$

Since $\mathrm{BUC}^2(H)$ is dense in $\mathrm{BUC}(H)$, it suffices to show that $\mathrm{BUC}^2(H) \subset D$. Let $f \in \mathrm{BUC}^2(H)$. Then, $f \in D(A_{b,\Sigma,\mu})$ for any Lévy triplet $(b, \Sigma, \mu) \in \Lambda$. In the sequel, we denote by $\|\nabla^2 f(x)\|$ the operator norm of the bounded linear operator $\nabla^2 f(x) \colon H \to H$ for all $x \in H$ and by $\|\nabla^2 f\|_\infty := \sup_{x\in H} \|\nabla^2 f(x)\|$. Then, by Taylor's theorem we have

$$\begin{aligned}
\big|\big(A_{b,\Sigma,\mu} f\big)(x)\big| &\le \|b\|\|\nabla f(x)\| + \tfrac{1}{2}\|\Sigma\|_{\mathrm{tr}}\|\nabla^2 f(x)\| \\
&\quad + \int_H \big|f(x+y) - f(x) - \langle \nabla f(x), h(y)\rangle\big| \,\mathrm{d}\mu(y) \\
&\le \|b\|\|\nabla f\|_\infty + \tfrac{1}{2}\|\Sigma\|_{\mathrm{tr}}\|\nabla^2 f\|_\infty + \max\left\{2\|f\|_\infty, \frac{1}{2}\|\nabla^2 f\|_\infty\right\} \int_H 1 \wedge \|y\|^2 \,\mathrm{d}\mu(y) \\
&\le 2C \max\big\{\|f\|_\infty, \|\nabla f\|_\infty, \|\nabla^2 f\|_\infty\big\}
\end{aligned}$$

for all $x \in H$ and all $(b, \Sigma, \mu) \in \Lambda$, so that

$$\sup_{(b,\Sigma,\mu)\in\Lambda} \|A_{b,\Sigma,\mu} f\|_\infty \le 2C \max\big\{\|f\|_\infty, \|\nabla f\|_\infty, \|\nabla^2 f\|_\infty\big\}. \tag{3.2}$$

Let $\varepsilon > 0$. Since $f \in \mathrm{BUC}^2(H)$ there exists $\delta > 0$ such that for all $x, z \in H$ with $\|x - z\| \le \delta$ and $\theta \in \{f, \nabla f, \nabla^2 f\}$ it holds

$$2C \sup_{y\in H} \|\theta(x+y) - \theta(z+y)\| \le \varepsilon.$$

Let $x, z \in H$ be fixed with $\|x - z\| \le \delta$ and $g \colon H \to \mathbb{R}$ be defined by

$$g(y) := f(x+y) - f(z+y)$$

for all $y \in H$, so that $g \in \mathrm{BUC}^2(H)$ with $2C \max\big\{\|g\|_\infty, \|\nabla g\|_\infty, \|\nabla^2 g\|_\infty\big\} \le \varepsilon$. By (3.2) we get

$$\big|\big(A_{b,\Sigma,\mu} f\big)(x) - \big(A_{b,\Sigma,\mu} f\big)(z)\big| = \big|\big(A_{b,\Sigma,\mu} g\big)(0)\big| \le 2C \max\big\{\|g\|_\infty, \|\nabla g\|_\infty, \|\nabla^2 g\|_\infty\big\} \le \varepsilon$$

for all $(b, \Sigma, \mu) \in \Lambda$, which shows that (A2) is satisfied. Therefore, by Theorem 2.5 there exists a sublinear expectation space $(\Omega, \mathcal{F}, \mathcal{E})$ and an $\mathcal{E}$-Lévy process $(X_t)_{t\ge0}$ such that for all $f \in \mathrm{BUC}(H)$ the function

$$u(t,x) := \big(u(t)\big)(x) := \mathcal{E}(f(x + X_t)), \quad t \ge 0,\, x \in H,$$

is a $\mathrm{BUC}^2(H)$-viscosity solution of the fully nonlinear PIDE

$$\begin{aligned}
u_t(t,x) &= \sup_{(b,\Sigma,\mu)\in\Lambda} \big(A_{b,\Sigma,\mu} u(t)\big)(x), \quad (t,x) \in (0,\infty) \times H, \\
u(0,x) &= f(x), \quad x \in H.
\end{aligned}$$

Example 3.3 (Lévy processes on the d-dimensional Torus $\mathbb{T}^d$)**.** Let $G = \mathbb{T}^d$, where $\mathbb{T}^d$ denotes the d-dimensional Torus represented by $(-\pi, \pi]^d$. We say that (b, Σ, μ, ν) is a *Lévy quadruple* if $b \in \mathbb{R}^d$, $\Sigma \in \mathbb{R}^{d\times d}$ is a symmetric positive semidefinite matrix, μ is a positive finite measure on $\mathbb{T}^d$ and ν is a positive measure on $\mathbb{T}^d$ with $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{T}^d} |y|^2 \,\mathrm{d}\nu(y) < \infty.$$

This definition is motivated by the Lévy-Itô decomposition of a Lévy process in $\mathbb{R}^d$, where the Lévy measure is further decomposed into a measure describing the large jumps (here: μ) and another measure describing the small jumps (here: ν). For each Lévy quadruple (b,Σ,μ,ν) we consider

$$\big(A_{b,\Sigma,\mu,\nu}f\big)(x) := b\cdot\nabla f(x) + \frac{1}{2}\mathrm{tr}\big(\Sigma\nabla^2 f(x)\big) + \int_{\mathbb{T}^d} f(x+y)-f(x)\,\mathrm{d}\mu(y)$$
$$+\int_{\mathbb{T}^d} f(x+y)-f(x)-\nabla f(x)\cdot y\,\mathrm{d}\nu(y)$$

for $x\in\mathbb{T}^d$ and $f\in D(A_{b,\Sigma,\mu,\nu}) := \big\{g\in C(\mathbb{T}^d)\,:\, A_{b,\Sigma,\mu,\nu}g\in C(\mathbb{T}^d)\big\}$. Recall that $C(\mathbb{T}^d)$ can be identified with the set of all 2π-periodic continuous functions on $\mathbb{R}^d$.

We start by proving that for any Lévy quadruple (b,Σ,μ,ν) the operator $A_{b,\Sigma,\mu,\nu}$ generates a Markovian convolution semigroup $(S_{b,\Sigma,\mu,\nu}(t))_{t\geq 0}$ of linear operators on $\mathbb{T}^d$. In order to do so, let (b,Σ,μ,ν) be a Lévy quadruple and define

$$\eta f := \int_{(-\pi,\pi]^d} f(y)\,\mathrm{d}\nu(y) + \sum_{k\in\mathbb{Z}^d\setminus\{0\}}\lambda_k\int_{(-\pi,\pi]^d} f(x+2\pi k)\,\mathrm{d}\mu(y)$$

for $f\in\mathcal{L}^\infty(\mathbb{R}^d)$, where $\lambda_k\geq 0$ for all $k\in\mathbb{Z}^d\setminus\{0\}$ and $\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\lambda_k = 1$. Then, η is a Lévy measure on $\mathbb{R}^d$, see e.g. Example 3.2 with $H=\mathbb{R}^d$, so that (b,Σ,η) is a Lévy triplet. Hence, there exists a Markovian convolution semigroup $(S_{b,\Sigma,\eta}(t))_{t\geq 0}$ of linear operators on $\mathbb{R}^d$ with generator $A_{b,\Sigma,\eta}$. As the space $C(\mathbb{T}^d)$ of all 2π-periodic continuous functions is a closed subspace of $\mathrm{BUC}(\mathbb{R}^d)$, which is invariant under $S_{b,\Sigma,\eta}(t)$ for all $t\geq 0$, we obtain that

$$S(t) := \big(S_{b,\Sigma,\eta}(t)\big)|_{C(\mathbb{T}^d)},\quad t\geq 0$$

defines a Markovian convolution semigroup of linear operators on $C(\mathbb{T}^d)$. Let A denote the generator of the semigroup $(S(t))_{t\geq 0}$. As $C(\mathbb{T}^d)$ is a closed subspace of $\mathrm{BUC}(\mathbb{R}^d)$ and $\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\lambda_k = 1$, we get that

$$Af = A_{b,\Sigma,\eta}f = A_{b,\Sigma,\mu,\nu}f$$

for all $f\in D(A)$. In particular, $A_{b,\Sigma,\mu,\nu}$ is the generator of $(S(t))_{t\geq 0}$.

Now, let Λ be a set of Lévy quadruples with

$$\sup_{(b,\Sigma,\mu,\nu)\in\Lambda}\left(|b|+|\Sigma|+\mu(\mathbb{T}^d)+\int_{\mathbb{T}^d}|y|^2\,\mathrm{d}\nu(y)\right)<\infty. \tag{3.3}$$

Then, in a similar way as in Example 3.2, one can show that for every $f\in C^2(\mathbb{T}^d)=\mathrm{BUC}^2(\mathbb{T}^d)$, the set $\{A_{b,\Sigma,\mu,\nu}f\colon (b,\Sigma,\mu,\nu)\in\Lambda\}$ is bounded and equicontinuous. Therefore, the assumptions (A1) and (A2) are satisfied. Hence, there exists a nonlinear expectation space $(\Omega,\mathcal{F},\mathcal{E})$ and an $\mathcal{E}$-Lévy process $(X_t)_{t\geq 0}$ such that for all $f\in C(\mathbb{T}^d)$ the function

$$u(t,x) := \big(u(t)\big)(x) := \mathcal{E}(f(x+X_t)),\quad t\geq 0, x\in\mathbb{T}^d,$$

is a $C^2(\mathbb{T}^d)$-viscosity solution of the fully nonlinear PIDE

$$\begin{aligned} u_t(t,x) &= \sup_{(b,\Sigma,\mu,\nu)\in\Lambda}\big(A_{b,\Sigma,\mu,\nu}u(t)\big)(x),\quad (t,x)\in(0,\infty)\times\mathbb{T}^d,\\ u(0,x) &= f(x),\quad x\in\mathbb{T}^d. \end{aligned}$$

We also refer to Hunt [10] for a Lévy-Khintchine formula on compact Lie groups.

Example 3.4. Let $A\colon D(A)\subset \mathrm{BUC}(G)\to \mathrm{BUC}(G)$ be the generator of a Markovian convolution semigroup $(S(t))_{t\geq 0}$ and $\Lambda\subset[0,\infty)$ bounded. For all $\lambda\in\Lambda$ let $A_\lambda:=\lambda A$ and $S_\lambda(t):=S(\lambda t)$ for $t\geq 0$. Then, the assumptions (A1) and (A2) are satisfied with $D=D(A)$. Hence, there exists a nonlinear expectation space $(\Omega,\mathcal{F},\mathcal{E})$ and an $\mathcal{E}$-Lévy process $(X_t)_{t\geq 0}$ such that for all $u_0\in \mathrm{BUC}(G)$ the function

$$u(t,x):=\mathcal{E}(f(x+X_t)),\quad t\geq 0, x\in G$$

is a $D(A)$-viscosity solution of the fully nonlinear PDE

$$\begin{aligned} u_t(t,x) &= \sup_{\lambda\in\Lambda}\big(\lambda A u(t)\big)(x), \quad (t,x)\in(0,\infty)\times G,\\ u(0,x) &= f(x), \quad x\in G. \end{aligned}$$

For instance, if A is the generator of a cylindrical Wiener process on a separable Hilbert space, one obtains an $\mathcal{E}$-Lévy process which can be viewed as a cylindrical G-Wiener process.

Example 3.5. For all $h>0$ let $\mu_h:=\frac{1}{h^2}\delta_h$ and consider $\Lambda:=\big\{(0,0,\mu_h)\,:\,h>0\big\}$ in Example 3.2 with $d=1$. Then, we have that

$$\sup_{h>0}\int_{\mathbb{R}} 1\wedge|y|^2\,\mathrm{d}\mu_h(y)=\sup_{h>0}\frac{1}{h^2}\int_{\mathbb{R}}|y|^2\,\mathrm{d}\delta_h(y)=1,$$

so that the assumptions (A1) and (A2) are satisfied. However, the second condition in (1.3) does not hold. By Example 3.2, there exists a sublinear expectation space $(\Omega,\mathcal{F},\mathcal{E})$ and an $\mathcal{E}$-Lévy process $(X_t)_{t\geq 0}$ such that for all $f\in\mathrm{BUC}(\mathbb{R})$ the function

$$u(t,x):=\mathcal{E}(f(x+X_t)),\quad t\geq 0, x\in\mathbb{R},$$

is a $\mathrm{BUC}^2(\mathbb{R})$-viscosity solution of the fully nonlinear PIDE

$$\begin{aligned} u_t(t,x) &= \sup_{h>0}\big(A_{0,0,\mu_h}u(t)\big)(x), \quad (t,x)\in(0,\infty)\times\mathbb{R},\\ u(0,x) &= f(x), \quad x\in\mathbb{R}. \end{aligned}$$

Note that $\big\|A_{(0,0,\mu_h)}f-\frac12 f''\big\|_\infty=\sup_{x\in\mathbb{R}}\big|\frac{f(x+h)-f(x)-f'(x)h}{h^2}-\frac12 f''(x)\big|\to 0$ as $h\searrow 0$ for all $f\in\mathrm{BUC}^2(\mathbb{R})$.

Example 3.6 (Cauchy distributed jumps)**.** For $\gamma>0$ let δ_γ be given by

$$\mu_\gamma\big((-\infty,b)\big):=\frac{\gamma}{\pi}\int_{-\infty}^b\frac{1}{y^2+\gamma^2}\,\mathrm{d}y=\frac12+\arctan\left(\frac{b}{\gamma}\right)$$

for $b\in\mathbb{R}$. Let $\Gamma\subset(0,\infty)$. Then, we have that

$$\sup_{\gamma\in\Gamma}\int_{\mathbb{R}}1\wedge|y|^2\,\mathrm{d}\mu_\gamma(y)\leq\sup_{\gamma\in\Gamma}\mu_\gamma(\mathbb{R})=1,$$

so that (A1) and (A2) are satisfied, but the first condition in (1.3) is violated. By Example 3.1 with $G=\mathbb{R}$ (using the notation from Example 3.2), there exists a sublinear expectation space $(\Omega,\mathcal{F},\mathcal{E})$ and an $\mathcal{E}$-Lévy process $(X_t)_{t\geq 0}$ such that for all $f\in\mathrm{BUC}(\mathbb{R})$ the function

$$u(t,x):=\mathcal{E}(f(x+X_t)),\quad t\geq 0, x\in\mathbb{R},$$

is the unique classical solution of the fully nonlinear PIDE

$$\begin{aligned} u_t(t,x) &= \sup_{\gamma\in\Gamma}\big(A_{0,0,\mu_\gamma}u(t)\big)(x), \quad (t,x)\in(0,\infty)\times\mathbb{R},\\ u(0,x) &= f(x), \quad x\in\mathbb{R} \end{aligned}$$

with $u\in C^1([0,\infty);\mathrm{BUC}(\mathbb{R}))$.

4. Proof of Theorem 2.3

Proof. Let $(\mathscr{S}(t))_{t\geq 0}$ be a convex Markovian convolution semigroup which is continuous from above. By Remark 2.2, every $\mathscr{S}(t)$ has a unique extension to a convex kernel $\mathscr{S}(t): C_b(G)\to C_b(G)$ which is continuous from above. Then the family

$$\mathcal{E}_{s,t}(x,f) := (\mathscr{S}(t-s)f)(x), \quad 0\leq s<t$$

of convex kernels on $C_b(G)$ is continuous from above and satisfies the Chapman-Kolmogorov equations, i.e.

$$\mathcal{E}_{s,t}(x,\mathcal{E}_{t,u}(\cdot,f)) = \mathcal{E}_{s,u}(x,f)$$

for all $0\leq s<t<u$, $f\in C_b(G)$ and $x\in G$. Hence, it follows from [5, Theorem 5.6] that there exists a convex expectation space $(\Omega,\mathcal{F},\mathcal{E})$ and a family of random variables $X_t:\Omega\to G$, $t\geq 0$, such that

$$\mathcal{E}(f(X_0)) = f(0)$$

for all $f\in C_b(G)$ and

$$\mathcal{E}\big(f(X_{t_1},\ldots,X_{t_n},X_t)\big) = \mathcal{E}\big(\mathcal{E}_{s,t}\left(X_s, f(X_{t_1},\ldots,X_{t_n},\,\cdot\,)\right)\big)$$

for all $0\leq s<t$, $n\in\mathbb{N}$, $0\leq t_1<\ldots<t_n\leq s$, and $f\in C_b(G^{n+1})$. Recall that $(x_{t_1},\ldots x_{t_n},x_s)\mapsto \mathcal{E}_{s,t}\left(x_s,f(x_{t_1},\ldots,x_{t_n},\,\cdot\,)\right)$ is continuous, see e.g. [5, Proposition 5.5] with $S=G$ and $T=G^n$. Next, we verify that $(X_t)_{t\geq 0}$ is an $\mathcal{E}$-Lévy process. For $f\in C_b(G)$ and $s,t\geq 0$ one has

$$\mathcal{E}\big(f(X_t)\big) = \mathcal{E}\big(\mathcal{E}_{0,t}(X_0,f)\big) = \mathcal{E}\big((\mathscr{S}(t)f)(X_0)\big) = (\mathscr{S}(t)f)(0)$$

and

$$\begin{aligned}\mathcal{E}\big(f(X_{s+t}-X_s)\big) &= \mathcal{E}\big(\tilde f(X_s,X_{s+t})\big) = \mathcal{E}\big(\mathcal{E}_{s,s+t}(X_s,\tilde f(X_s,\,\cdot\,)\big)\\ &= \mathcal{E}\big((\mathscr{S}(t)\tilde f(X_s,\,\cdot\,))(X_s)\big) = \mathcal{E}\big((\mathscr{S}(t)f)(0)\big) = (\mathscr{S}(t)f)(0),\end{aligned} \tag{4.1}$$

where $\tilde f(x,y):=f(y-x)$ and $(\mathscr{S}(t)\tilde f(x,\,\cdot\,))(x) = (\mathscr{S}(t)f_{-x})(x) = (\mathscr{S}(t)f)(0)$ because $\mathscr{S}(t)$ is a Markovian convolution. This shows that the random variables $X_{s+t}-X_s$ and X_t have the same distribution under $\mathcal{E}$. Moreover, for $s,t\geq 0$, $0\leq t_1<\ldots<t_n\leq s$ and $f\in C_b(G^{n+1})$, it follows by (4.1) that

$$(\mathscr{S}(t)f(x_{t_1},\ldots,x_{t_n},\,\cdot\,))(0) = \mathcal{E}\big(f(x_{t_1},\ldots,x_{t_n},X_{s+t}-X_s)\big)$$

for all $x_{t_1},\ldots,x_{t_n}\in G$ and therefore,

$$\begin{aligned}&\mathcal{E}\big(f(X_{t_1},\ldots,X_{t_n},X_{s+t}-X_s)\big) = \mathcal{E}\big(f_{-X_s}(X_{t_1},\ldots,X_{t_n},X_{s+t})\big)\\ &= \mathcal{E}\big(\mathcal{E}_{s,s+t}\big(X_s,f_{-X_s}(X_{t_1},\ldots,X_{t_n},\,\cdot\,)\big)\big)\\ &= \mathcal{E}\big(\big(\mathscr{S}(t)f_{-X_s}(X_{t_1},\ldots,X_{t_n},\,\cdot\,)\big)(X_s)\big)\\ &= \mathcal{E}\big(\big(\mathscr{S}(t)f(X_{t_1},\ldots,X_{t_n},\,\cdot\,)\big)(0)\big)\\ &= \mathcal{E}\big(\mathcal{E}\big(f(x_{t_1},\ldots,x_{t_n},X_{s+t}-X_s)\big)|_{x_{t_1}=X_{t_1},\ldots,x_{t_n}=X_{t_n}}\big)\end{aligned}$$

which shows that $X_{s+t}-X_s$ is independent of $(X_{t_1},\ldots,X_{t_n})$. It remains to verify that $X_t\to X_0$ in distribution as $t\searrow 0$. To that end, fix $f\in C_b(G)$ and notice that there exists an increasing sequence (f_n) in $\mathrm{BUC}(G)$ which converges pointwise to f. By Dini's theorem it follows that the continuous functions $t\mapsto \big(\mathscr{S}(t)f_n\big)(0)$ converge uniformly on $[0,1]$ to the continuous function $t \mapsto \big(\mathscr{S}(t)f\big)(0)$. Hence, for every $\varepsilon > 0$ there exists some $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}|\mathcal{E}(f(X_t)) - f(0)| &\le \big|\big(\mathscr{S}(t)f\big)(0) - \big(\mathscr{S}(t)f_{n_0}\big)(0)\big| + \|\mathscr{S}(t)f_{n_0} - f_{n_0}\|_\infty \\ &\quad + |f_{n_0}(0) - f(0)| \\ &\le \varepsilon, \quad \text{as } t \searrow 0.\end{aligned}$$

Conversely, let $(X_t)_{t\ge 0}$ be an $\mathcal{E}$-Lévy process on a convex expectation space $(\Omega, \mathcal{F}, \mathcal{E})$. Then, the family $(\mathscr{S}(t))_{t\ge 0}$ defined by

$$\big(\mathscr{S}(t)f\big)(x) := \mathcal{E}(f(x + X_t))$$

for $f \in \mathrm{BUC}(G)$ and $x \in G$ is a convex Markovian convolution semigroup. Indeed, for $f \in \mathrm{BUC}(G)$, $x \in G$ and $t \ge 0$, it holds that

$$\big(\mathscr{S}(t)f_x\big)(0) = \mathcal{E}(f(x + X_t)) = \big(\mathscr{S}(t)f\big)(x),$$

so that $\mathscr{S}(t)$ is a convex Markovian convolution which satisfies

$$\big(\mathscr{S}(0)f\big)(x) = \mathcal{E}(f(x + X_0)) = \mathcal{E}(f_x(X_0)) = f_x(0) = f(x).$$

Moreover, since $\mathcal{E}(f(x + (X_{t+s} - X_s))) = \mathcal{E}(f(x + X_t)) = \big(\mathscr{S}(t)f\big)(x)$ for all $s, t \ge 0$, $f \in \mathrm{BUC}(G)$, $x \in G$, and $X_{t+s} - X_s$ is independent of X_s we obtain

$$\begin{aligned}\big(\mathscr{S}(t+s)f\big)(0) = \mathcal{E}(f(X_{t+s})) &= \mathcal{E}(f(X_s + (X_{t+s} - X_s))) \\ &= \mathcal{E}\big(\big(\mathscr{S}(t)f\big)(X_s)\big) = \big(\mathscr{S}(s)\mathscr{S}(t)f\big)(0).\end{aligned}$$

Since $\mathscr{S}(s)$ and $\mathscr{S}(t)$ are Markovian convolutions we conclude the semigroup property $\mathscr{S}(t+s) = \mathscr{S}(s)\mathscr{S}(t)$. It remains to show that $\lim_{t\searrow 0} \|\mathscr{S}(t)f - f\|_\infty = 0$ for all $f \in \mathrm{BUC}(G)$. To do so, we first show that for every $c \ge 0$ and $\delta > 0$ we have that

$$\mathcal{E}\big(c1_{G\setminus B(0,\delta)}(X_t)\big) \to 0 \tag{4.2}$$

as $t \searrow 0$. Let $\varphi\colon G \to \mathbb{R}$ be defined by

$$\varphi(y) := \frac{c}{\delta}\big(d(y,0) \wedge \delta\big)$$

for $y \in G$. Then, $\varphi \in C_b(G)$ with $\varphi \ge 0$, $\varphi(0) = 0$ and $\varphi(y) = c$ for all $y \in G \setminus B(0,\delta)$, so that

$$0 \le \mathcal{E}\big(c1_{G\setminus B(0,\delta)}(X_t)\big) = \mathcal{E}\big(\varphi(X_t)1_{G\setminus B(0,\delta)}(X_t)\big) \le \mathcal{E}(\varphi(X_t)) \to 0$$

as $t \searrow 0$. Now, fix $\varepsilon > 0$ and $f \in \mathrm{BUC}(G)$. Then there exists some $\delta > 0$ such that $|f(x+y) - f(x)| < \frac{\varepsilon}{2}$ for all $x, y \in G$ with $d(y,0) < \delta$. For each $x \in G$ let $g_x \in \mathcal{L}^\infty(G)$ be given by $y \mapsto 1_{B(0,\delta)}(y)(f(x+y) - f(x))$. Then $\|g_x\|_\infty \le \frac{\varepsilon}{2}$ for all $x \in G$, and by (4.2) there exists some $t_0 > 0$ such that

$$\mathcal{E}\big(2\|f\|_\infty 1_{G\setminus B(0,\delta)}(X_t)\big) \le \frac{\varepsilon}{2}$$

for all $0 < t < t_0$. Since $\mathcal{E}\colon \mathcal{L}^\infty(\Omega, \mathcal{F}) \to \mathbb{R}$ is 1-Lipschitz continuous, we get

$$\begin{aligned}\big|\big(\mathscr{S}(t)f\big)(x) - f(x)\big| = \big|\mathcal{E}\big(f(x+X_t) - f(x)\big)\big| &\le \mathcal{E}\big(\,|f(x+X_t) - f(x)|\,\big) \\ &\le \mathcal{E}\big(g_x(X_t) + 2\|f\|_\infty 1_{G\setminus B(0,\delta)}(X_t)\big) \\ &\le \|g_x\|_\infty + \mathcal{E}\big(2\|f\|_\infty 1_{G\setminus B(0,\delta)}(X_t)\big) \le \varepsilon\end{aligned}$$

for all $0 < t < t_0$, which shows that $\|\mathscr{S}(t)f - f\|_\infty \le \varepsilon$ for all $0 < t < t_0$.

□

5. Randomizing linear semigroups and the proof of Theorem 2.5

Throughout this section all Markovian convolutions are defined on $\mathrm{BUC}(G)$. We assume that $\{A_\lambda : \lambda \in \Lambda\}$ is a given non-empty family of operators which satisfies the assumptions (A1) and (A2). Recall that A_λ generates a Markovian convolution semigroup $(S_\lambda(t))_{t\geq 0}$ of linear operators, and the domain

$$D = \left\{ f \in \bigcap_{\lambda\in\Lambda} D(A_\lambda) : \big\{A_\lambda f \colon \lambda \in \Lambda\big\} \text{ is bounded and uniformly equicontinuous} \right\}$$

is dense in $\mathrm{BUC}(G)$.

We consider finite partitions $P := \{\pi \subset [0,\infty) : 0 \in \pi,\, \pi \text{ finite}\}$. For a partition $\pi = \{t_0, t_1, \ldots, t_m\} \in P$ with $0 = t_0 < t_1 < \ldots < t_m$ we set $|\pi|_\infty := \max_{j=1,\ldots,m}(t_j - t_{j-1})$. The set of partitions with end-point t is denoted by P_t, i.e. $P_t := \{\pi \in P : \max \pi = t\}$.

For $f \in BUC(G)$ and $t \geq 0$, we define

$$\big(J_t f\big)(x) := \sup_{\lambda\in\Lambda} \big(S_\lambda(t) f\big)(x), \quad x \in G,$$

and for a partition $\pi \in P$, we set

$$J_\pi f := J_{t_1 - t_0} \cdots J_{t_m - t_{m-1}} f,$$

where we assume that $\pi = \{t_0, t_1, \ldots, t_m\}$ with $0 = t_0 < t_1 < \ldots < t_m$, and $J_{\{0\}} f := f$. Note that $J_t = J_{\{0,t\}}$ for $t > 0$. Moreover, since $S_\lambda(t)$ is continuous from below for all $\lambda \in \Lambda$ and $t \geq 0$, it follows that J_π is continuous from below for all $\pi \in P$.

Let $f \in D$. Then, by definition of D, the family $(A_\lambda f)_{\lambda\in\Lambda}$ is bounded and, throughout the rest of this section, we denote

$$L_f := \sup_{\lambda\in\Lambda} \|A_\lambda f\|_\infty < \infty.$$

Lemma 5.1. *a) J_π is a sublinear Markovian convolution for all $\pi \in P$.*
b) Let $f \in D$. Then,

$$\|J_{t_1} f - J_{t_2} f\|_\infty \leq L_f |t_1 - t_2|, \quad t_1, t_2 \geq 0, \tag{5.1}$$

$$\|J_\pi f - f\|_\infty \leq L_f\, t, \quad \pi \in P_t,\, t > 0. \tag{5.2}$$

Proof. a) Since $S_\lambda(t)$ is a linear Markovian convolution for all $\lambda \in \Lambda$, J_t is a sublinear Markovian convolution for all $t \geq 0$. As this property is preserved under compositions, the same holds for J_π.

b) Let $f \in D$. For $t_1, t_2 \geq 0$, $x \in G$, and $\lambda_0 \in \Lambda$ we have that

$$\begin{aligned} \big(S_{\lambda_0}(t_1) f\big)(x) - \big(J_{t_2} f\big)(x) &\leq \big(S_{\lambda_0}(t_1) f\big)(x) - \big(S_{\lambda_0}(t_2) f\big)(x) \\ &\leq \|S_{\lambda_0}(t_1) f - S_{\lambda_0}(t_2) f\|_\infty \\ &\leq \sup_{\lambda\in\Lambda} \|S_\lambda(t_1) f - S_\lambda(t_2) f\|_\infty \end{aligned}$$

and therefore, taking the supremum over $\lambda_0 \in \Lambda$,

$$\big(J_{t_1} f\big)(x) - \big(J_{t_2} f\big)(x) \leq \sup_{\lambda\in\Lambda} \|S_\lambda(t_1) f - S_\lambda(t_2) f\|_\infty.$$

By symmetry and taking the supremum over all $x \in G$, we thus get that

$$\|J_{t_1} f - J_{t_2} f\|_\infty \leq \sup_{\lambda\in\Lambda} \|S_\lambda(t_1) f - S_\lambda(t_2) f\|_\infty.$$

Let $\lambda \in \Lambda$ and w.l.o.g. let $t_1 < t_2$. Then, as $f \in D(A_\lambda)$, it follows that (see, e.g. [7, Lemma II.1.3])

$$\begin{aligned}\|S_\lambda(t_1)f - S_\lambda(t_2)f\|_\infty &= \left\| \int_{t_1}^{t_2} S_\lambda(s)A_\lambda f \,\mathrm{d}s \right\|_\infty \le \int_{t_1}^{t_2} \|S_\lambda(s)A_\lambda f\|_\infty \,\mathrm{d}s \\ &\le (t_2 - t_1) \sup_{0\le s\le t} \|S_\lambda(s)A_\lambda f\|_\infty \le (t_2 - t_1)\|A_\lambda f\|_\infty \\ &\le L_f|t_1 - t_2|\end{aligned}$$

for all $\lambda \in \Lambda$. Here, we used the fact that $\|S_\lambda(s)g\|_\infty \le \|g\|_\infty$ for all $\lambda \in \Lambda$, $s \ge 0$ and $g \in \mathrm{BUC}(G)$. Taking the supremum over all $\lambda \in \Lambda$, we obtain (5.1).

To show (5.2), let $\pi = \{0, t_1, \ldots, t_{m-1}, t\} \in P_t$ with $0 < t_1 < \ldots < t_{m-1} < t$ and $m \in \mathbb{N}$. Note that the case $\pi = \{0\}$ is trivial. For $m = 1$, we have $\pi = \{0, t\}$, and by (5.1),

$$\|J_\pi f - f\|_\infty = \|J_t f - J_0 f\|_\infty \le L_f\, t.$$

Now let $m \ge 2$, and set $\pi' := \{0, t_1, \ldots, t_{m-1}\}$. By induction, we have $\|J_{\pi'} f - f\|_\infty \le L_f\, t_{m-1}$. As $J_{\pi'}$ is a sublinear Markovian convolution and therefore 1-Lipschitz continuous, we obtain

$$\begin{aligned}\|J_\pi f - f\|_\infty &= \|J_{\pi'} J_{t-t_{m-1}} f - f\|_\infty \\ &\le \|J_{\pi'} J_{t-t_{m-1}} f - J_{\pi'} f\|_\infty + \|J_{\pi'} f - f\|_\infty \\ &\le \|J_{t-t_{m-1}} f - f\|_\infty + \|J_{\pi'} f - f\|_\infty \\ &\le L_f(t - t_{m-1}) + L_f\, t_{m-1} = L_f\, t.\end{aligned}$$

□

The following result shows that $J_\pi f$ depends continuously on the partition π.

Lemma 5.2. *Let $m \in \mathbb{N}$, $\pi = \{0, t_1, \ldots, t_m\} \in P$ with $0 < t_1 < \ldots < t_m$, and for $n \in \mathbb{N}$ let $\pi_n = \{0, t_1^n, \ldots, t_m^n\} \in P$ with $\lim_{n\to\infty} t_j^n = t_j$ for $j = 1, \ldots, m$. Then*

$$\|J_\pi f - J_{\pi_n} f\|_\infty \to 0, \quad \text{as } n \to \infty,$$

for all $f \in BUC(G)$.

Proof. Note that $0 < t_1^n < \ldots < t_m^n$ for sufficiently large n. Fix $f \in \mathrm{BUC}(G)$. We have to show the continuity of the map $(t_1, \ldots, t_m) \mapsto J_{\{0,t_1,\ldots,t_m\}} f$. By definition of J_π, it is sufficient to show that the map

$$[0, \infty) \to \mathrm{BUC}(G), \quad t \mapsto J_t f$$

is continuous. Let $\varepsilon > 0$ and $t \ge 0$, and let (t^n) be a sequence in $[0, \infty)$ such that $t^n \to t$. By assumption (A2) there exists $f_0 \in D$ with $\|f - f_0\|_\infty \le \frac{\varepsilon}{3}$. Since $J_s = J_{0,s}$ is a sublinear Markovian convolution by Lemma 5.1 a), it is 1-Lipschitz and it follows that

$$\|J_s f - J_s f_0\|_\infty \le \|f - f_0\|_\infty < \frac{\varepsilon}{3}$$

for all $s \ge 0$. Hence, it follows from Lemma 5.1 that

$$\begin{aligned}\|J_t f - J_{t^n} f\|_\infty &\le \|J_t f - J_t f_0\|_\infty + \|J_t f_0 - J_{t^n} f_0\|_\infty + \|J_{t^n} f - J_{t^n} f_0\|_\infty \\ &\le \tfrac{2}{3}\varepsilon + L_{f_0}|t - t^n| < \varepsilon\end{aligned}$$

for sufficiently large $n \in \mathbb{N}$. □

The above results allow to consider the limit of $J_\pi f$ when the mesh size of the partition tends to zero. To that end, we first note that for $s,t\geq 0$, $f\in \mathrm{BUC}(G)$ and $x\in G$, it holds

$$\begin{aligned}\big(J_{t+s}f\big)(x)&=\sup_{\lambda\in\Lambda}\big(S_\lambda(t+s)f\big)(x)=\sup_{\lambda\in\Lambda}\big(S_\lambda(t)S_\lambda(s)f\big)(x)\\&\leq\sup_{\lambda\in\Lambda}\big(S_\lambda(t)J_sf\big)(x)=\big(J_tJ_sf\big)(x).\end{aligned}$$

From this, we obtain for $\pi_1,\pi_2\in P_t$ with $\pi_1\subset\pi_2$ the pointwise inequality

$$J_{\pi_1}f\leq J_{\pi_2}f. \tag{5.3}$$

In particular, for $\pi_1,\pi_2\in P_t$ we have that

$$\big(J_{\pi_1}f\big)\vee\big(J_{\pi_2}f\big)\leq J_\pi f, \tag{5.4}$$

where $\pi:=\pi_1\cup\pi_2\in P_t$. Therefore, we define

$$\big(\mathscr{S}(t)f\big)(x):=\sup_{\pi\in P_t}\big(J_\pi f\big)(x) \tag{5.5}$$

for all $t\geq 0$, $x\in G$ and $f\in\mathrm{BUC}(G)$. Note that $\mathscr{S}(0)f=f$ for all $f\in\mathrm{BUC}(G)$.

Lemma 5.3. *$\mathscr{S}(t)$ is a sublinear Markovian convolution for all $t\geq 0$, which satisfies*

$$\|\mathscr{S}(t)f-f\|_\infty\leq L_f\,t,\quad f\in D.$$

Moreover,

$$\lim_{t\searrow 0}\|\mathscr{S}(t)f-f\|_\infty=0$$

for all $f\in BUC(G)$.

Proof. As J_π is a sublinear Markovian convolution for all $\pi\in P_t$, the same holds for $\mathscr{S}(t)$. For $f\in D$, $x\in G$ and $\varepsilon>0$ there exists $\pi_0\in P_t$ such that

$$\big(\mathscr{S}(t)f\big)(x)-f(x)\leq\big(J_{\pi_0}f\big)(x)-f(x)+\varepsilon\leq\sup_{\pi\in P_t}\|J_\pi f-f\|_\infty+\varepsilon$$

and

$$f(x)-\big(\mathscr{S}(t)f\big)(x)\leq f(x)-\big(J_{\pi_0}f\big)(x)\leq\sup_{\pi\in P_t}\|J_\pi f-f\|_\infty.$$

By Lemma 5.1 it follows that

$$\|\mathscr{S}(t)f-f\|_\infty\leq\sup_{\pi\in P_t}\|J_\pi f-f\|_\infty\leq L_f\,t.$$

From this, we obtain that $\lim_{t\searrow 0}\|\mathscr{S}(t)f-f\|_\infty=0$ for $f\in\mathrm{BUC}(G)$ with the same density argument as in the proof of Lemma 5.2. □

Lemma 5.4. *For $t\geq 0$, let $(\pi_n)_{n\in\mathbb{N}}$ be a sequence in P_t such that $\pi_n\subset\pi_{n+1}$ for all $n\in\mathbb{N}$, and $\lim_{n\to\infty}|\pi_n|_\infty=0$. Then*

$$J_{\pi_n}f\nearrow\mathscr{S}(t)f,\quad\text{as } n\to\infty,$$

for all $f\in\mathrm{BUC}(G)$.

Proof. Fix $f\in\mathrm{BUC}(G)$. For $t=0$, the statement is trivial. For $t>0$ and $x\in G$ we define

$$\big(J_\infty f\big)(x):=\sup_{n\in\mathbb{N}}\big(J_{\pi_n}f\big)(x)$$

Then, J_∞ is a sublinear Markovian convolution. Since $\pi_n\subset\pi_{n+1}$, it follows from (5.3) that

$$J_{\pi_n}f\nearrow J_\infty f,\quad\text{as } n\to\infty.$$

By definition of $\mathscr{S}(t)$, it clearly holds $J_\infty f \leq \mathscr{S}(t)f$. As for the other inequality, let $\pi = \{t_0, t_1, \ldots, t_m\} \in P_t$ with $m \in \mathbb{N}$ and $0 = t_0 < t_1 < \ldots < t_m = t$. Since $|\pi_n|_\infty \searrow 0$ as $n \to \infty$, we may w.l.o.g. assume that $\#\pi_n \geq m+1$ for all $n \in \mathbb{N}$. Moreover, we can choose $0 = t_0^n < t_1^n < \ldots < t_m^n = t$ with $\pi_n' := \{t_0^n, t_1^n, \ldots, t_m^n\} \subset \pi_n$ and $\lim_{n\to\infty} t_i^n = t_i$ for all $i = 1, \ldots, m-1$. Then, by Lemma 5.2, we have that

$$\|J_\pi f - J_{\pi_n'} f\|_\infty \to 0, \quad \text{as } n \to \infty.$$

Since

$$J_\infty f \geq J_{\pi_n} f \geq J_{\pi_n'} f \geq J_\pi f - \|J_\pi f - J_{\pi_n'} f\|_\infty$$

we obtain that $J_\infty f \geq J_\pi f$. Taking the supremum over all $\pi \in P_t$, we thus get that $J_\infty f = \mathscr{S}(t)f$. □

Corollary 5.5. *Let $t \geq 0$. Then, there exists a sequence (π_n) in P_t such that*

$$J_{\pi_n} f \nearrow \mathscr{S}(t)f, \quad \text{as } n \to \infty.$$

Moreover,

$$\mathscr{S}(t)f = \sup_{n\in\mathbb{N}} \big(J_{\frac{t}{n}}\big)^n f = \sup_{n\in\mathbb{N}} \big(J_{2^{-n}t}\big)^{2^n} f = \lim_{n\to\infty} \big(J_{2^{-n}t}\big)^{2^n} f, \tag{5.6}$$

for all $f \in \mathrm{BUC}(G)$, where the supremum is understood pointwise.

Proof. Choose $\pi_n := \big\{\frac{kt}{2^n} \,:\, k \in \{0, \ldots, 2^n\}\big\}$ in Lemma 5.4 to obtain the first statement. In particular,

$$\mathscr{S}(t)f = \mathscr{S}(t)f = \sup_{n\in\mathbb{N}} J_{\pi_n} f = \sup_{n\in\mathbb{N}} \big(J_{2^{-n}t}\big)^{2^n} f = \lim_{n\to\infty} \big(J_{2^{-n}t}\big)^{2^n} f. \tag{5.7}$$

For $\tilde{\pi}_n := \big\{\frac{kt}{n} \,:\, k \in \{0, \ldots, n\}\big\}$ it holds $\tilde{\pi}_{2^n} = \pi_n$ for all $n \in \mathbb{N}$. Therefore, by (5.7), it follows that

$$\mathscr{S}(t)f = \sup_{n\in\mathbb{N}} J_{\pi_n} f = \sup_{n\in\mathbb{N}} J_{\tilde{\pi}_{2^n}} f \leq \sup_{n\in\mathbb{N}} J_{\tilde{\pi}_n} f \leq \mathscr{S}(t)f.$$

Hence,

$$\mathscr{S}(t)f = \sup_{n\in\mathbb{N}} J_{\tilde{\pi}_n} f = \sup_{n\in\mathbb{N}} \big(J_{\frac{t}{n}}\big)^n f,$$

which yields the first equality in (5.6) and therefore the assertion. □

Proposition 5.6 (Dynamic programming principle)**.** *$(\mathscr{S}(t))_{t\geq 0}$ is a Markovian convolution semigroup of sublinear operators. In particular, for every $s, t \geq 0$ one has*

$$\mathscr{S}(s+t)f = \mathscr{S}(s)\mathscr{S}(t)f \tag{5.8}$$

for every $f \in BUC(G)$.

Proof. Let $\pi_0 \in P_{s+t}$ and $\pi := \pi_0 \cup \{s\}$. Then, $\pi \in P_{s+t}$ with $\pi_0 \subset \pi$, and by (5.3) we get that

$$J_{\pi_0} f \leq J_\pi f.$$

We have already shown all properties of a Markovian convolution semigroup except the semigroup property (5.8), i.e. the dynamic programming principle.

If $s = 0$ or $t = 0$ the statement is trivial. Therefore, let $s, t > 0$. Let $m \in \mathbb{N}$ and $0 = t_0 < t_1 < \ldots < t_m = s + t$ with $t_i = s$ for some $i \in \{1, \ldots, m\}$, and define $\pi := \{t_0, \ldots, t_m\} \in P_{s+t}$. Then, for $\pi_1 := \{t_0, \ldots, t_i\} \in P_s$ and $\pi_2 := \{t_i - s, \ldots, t_m - s\} \in P_t$ we have

$$J_{\pi_1} = J_{t_1 - t_0} \cdots J_{t_i - t_{i-1}}, \quad \text{and} \quad J_{\pi_2} = J_{t_{i+1} - t_i} \cdots J_{t_m - t_{m-1}},$$

and therefore

$$\begin{aligned} J_\pi f &= J_{t_1-t_0}\cdots J_{t_m-t_{m-1}}f = \big(J_{t_1-t_0}\cdots J_{t_i-t_{i-1}}\big)\big(J_{t_{i+1}-t_i}\cdots J_{t_m-t_{m-1}}f\big)\\ &= J_{\pi_1}J_{\pi_2}f \le J_{\pi_1}\mathscr{S}(t)f \le \mathscr{S}(s)\mathscr{S}(t)f. \end{aligned}$$

Taking the supremum over all $\pi \in P_{s+t}$, we get $\mathscr{S}(s+t)f \le \mathscr{S}(s)\mathscr{S}(t)f$. Conversely, by Corollary 5.5 there exists a sequence (π_n) in P_t such that $J_{\pi_n}f \nearrow \mathscr{S}(t)f$ as $n \to \infty$. For $\pi_0 \in P_s$ and $\pi_n' := \pi_0 \cup \{r+s : r \in \pi_n\} \in P_{s+t}$ it holds $J_{\pi_n'} = J_{\pi_0}J_{\pi_n}$. Since J_{π_0} is continuous from below we have

$$J_{\pi_0}\mathscr{S}(t)f = \lim_{n\to\infty} J_{\pi_0}J_{\pi_n}f = \lim_{n\to\infty} J_{\pi_n'}f \le \mathscr{S}(s+t)f.$$

Taking the supremum over all $\pi_0 \in P_s$, we get that $\mathscr{S}(s)\mathscr{S}(t)f \le \mathscr{S}(s+t)f$. □

The following lemma is a special case of Jensen's inequality for vector valued functions. For the reader's convenience we provide a short proof and refer to [9, Section 1.2.] for an introduction to Bochner integration.

Lemma 5.7. *Let $\mathscr{S}\colon \mathrm{BUC}(G) \to \mathrm{BUC}(G)$ be convex and continuous. Let $(\Omega, \mathcal{F}, \nu)$ be a finite measure space with $\nu \neq 0$. Further, let $g\colon \Omega \to \mathrm{BUC}(G)$ be bounded and $\mathcal{F}$-$\mathcal{B}(\mathrm{BUC}(G))$-measurable with separable range $g(\Omega)$, i.e. g is Bochner integrable, such that $\mathscr{S}g\colon \Omega \to \mathrm{BUC}(G)$, $\omega \mapsto \mathscr{S}\big(g(\omega)\big)$ is again bounded. Then, $\mathscr{S}g$ is Bochner integrable and we have that*

$$\mathscr{S}\left(\frac{1}{\nu(\Omega)}\int_\Omega g\,\mathrm{d}\nu\right) \le \frac{1}{\nu(\Omega)}\int_\Omega \mathscr{S}g\,\mathrm{d}\nu.$$

Proof. Since $\mathscr{S}$ is continuous, we obtain that $\mathscr{S}g : \Omega \to \mathrm{BUC}(G)$ is $\mathcal{F}$-$\mathcal{B}(\mathrm{BUC}(G))$-measurable with separable range $(\mathscr{S}g)(\Omega)$ and thus Bochner integrable. If g is a simple function, $\mathscr{S}g$ is a simple function and the assertion follows by convexity of $\mathscr{S}$. Since g is $\mathcal{F}$-$\mathcal{B}(\mathrm{BUC}(G))$-measurable with separable range $g(\Omega)$, there exists a sequence of simple functions $(g_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty}\|g_n(\omega)-g(\omega)\|_\infty = 0$ for all $\omega \in \Omega$. By continuity of $\mathscr{S}$, we obtain $\lim_{n\to\infty}\|\mathscr{S}g_n(\omega)-\mathscr{S}g(\omega)\|_\infty = 0$ for all $\omega \in \Omega$. Hence, by definition of Bochner's integral it follows that

$$\begin{aligned} \mathscr{S}\left(\frac{1}{\nu(\Omega)}\int_\Omega g\,\mathrm{d}\nu\right) &= \lim_{n\to\infty}\mathscr{S}\left(\frac{1}{\nu(\Omega)}\int_\Omega g_n\,\mathrm{d}\nu\right)\\ &\le \lim_{n\to\infty}\frac{1}{\nu(\Omega)}\int_\Omega \mathscr{S}g_n\,\mathrm{d}\nu = \frac{1}{\nu(\Omega)}\int_\Omega \mathscr{S}g\,\mathrm{d}\nu. \end{aligned}$$

□

Fix $f \in D$. Since $\{A_\lambda f\colon \lambda \in \Lambda\} \subset \mathrm{BUC}(G)$ is bounded and uniformly equicontinuous, it follows that

$$\mathcal{A}f := \sup_{\lambda\in\Lambda} A_\lambda f \in \mathrm{BUC}(G),$$

where the supremum is understood pointwise.

Lemma 5.8. *Fix $f \in D$. Then, for $\pi \in P$ and $t > 0$ it holds*

$$J_\pi f - f \le \int_0^{\max\pi} \mathscr{S}(s)\mathcal{A}f\,\mathrm{d}s \quad \text{and} \quad \mathscr{S}(t)f - f \le \int_0^t \mathscr{S}(s)\mathcal{A}f\,\mathrm{d}s.$$

Proof. Since $\mathcal{A}f \in \mathrm{BUC}(G)$, the mapping $[0,\infty) \to \mathrm{BUC}(G)$, $s \mapsto \mathscr{S}(s)\mathcal{A}f$ is continuous. Therefore, the Bochner integrals are well-defined. Then, for all $t, h > 0$ we have

$$\begin{aligned} J_h f - f &= \sup_{\lambda\in\Lambda} S_\lambda(h)f - f = \sup_{\lambda\in\Lambda} \int_0^h S_\lambda(s)A_\lambda f \,\mathrm{d}s \\ &\leq \int_0^h \mathscr{S}(s)\mathcal{A}f\,\mathrm{d}s = \int_t^{t+h} \mathscr{S}(s-t)\mathcal{A}f\,\mathrm{d}s. \end{aligned} \tag{5.9}$$

We prove the first inequality by induction on $m = \#\pi$. If $m = 1$, i.e. if $\pi = \{0\}$, the statement is trivial. Hence, assume that

$$J_{\pi'} f - f \leq \int_0^{\max \pi'} \mathscr{S}(s)\mathcal{A}f\,\mathrm{d}s$$

for all $\pi' \in P$ with $\#\pi' = m$ for some $m \in \mathbb{N}$. Let $\pi = \{t_0, t_1, \ldots, t_m\}$ with $0 = t_0 < t_1 < \ldots < t_m$ and $\pi' := \pi \setminus \{t_m\}$. Then, it follows from (5.9) and Lemma 5.7 that

$$\begin{aligned} J_\pi f - J_{\pi'} f &\leq J_{\pi'}\big(J_{t_m - t_{m-1}} f - f\big) \leq J_{\pi'}\Big(\int_{t_{m-1}}^{t_m} \mathscr{S}(s-t_{m-1})\mathcal{A}f\,\mathrm{d}s\Big) \\ &\leq \mathscr{S}(t_{m-1})\Big(\int_{t_{m-1}}^{t_m} \mathscr{S}(s-t_{m-1})\mathcal{A}f\,\mathrm{d}s\Big) \leq \int_{t_{m-1}}^{t_m} \mathscr{S}(s)\mathcal{A}f\,\mathrm{d}s, \end{aligned}$$

where the first inequality follows from the sublinearity of $J_{\pi'}$. By induction, we thus get

$$\begin{aligned} J_\pi f - f &= \big(J_\pi f - J_{\pi'} f\big) + \big(J_{\pi'} f - f\big) \leq \int_{t_{m-1}}^{t_m} \mathscr{S}(s)\mathcal{A}f\,\mathrm{d}s + \int_0^{t_{m-1}} \mathscr{S}(s)\mathcal{A}f\,\mathrm{d}s \\ &= \int_0^{\max\pi} \mathscr{S}(s)\mathcal{A}f\,\mathrm{d}s. \end{aligned}$$

In particular, for every $\pi \in P_t$ it holds

$$J_\pi f - f \leq \int_0^t \mathscr{S}(s)\mathcal{A}f\,\mathrm{d}s.$$

Taking the supremum over all $\pi \in P_t$ yields the second assertion. $\square$

Lemma 5.9. *Let $M \subset \mathrm{BUC}(G)$ be bounded and uniformly equicontinuous. Then,*

$$\sup_{\lambda\in\Lambda}\sup_{g\in M} \|S_\lambda(t)g - g\|_\infty \to 0, \quad \textit{as } t \searrow 0.$$

Proof. Let $\varepsilon > 0$ and $C := \sup_{g\in M}\|g\|_\infty$. Then, there exists some $\delta > 0$ such that

$$\sup_{g\in M} |g(x) - g(y)| \leq \varepsilon$$

for all $x, y \in G$ with $d(x,y) \leq \delta$. Let $\varphi(y) := \frac{1}{\delta}\big(d(y,0)\wedge\delta\big)$ for all $y \in G$. Then, $\varphi \in \mathrm{BUC}(G)$, $0 \leq \varphi \leq 1$, $\varphi(0) = 0$ and $\varphi(y) = 1$ for all $y \in G \setminus B(0,\delta)$. Fix $\lambda \in \Lambda$, $g \in M$ and $x \in G$. Since $g(x+\cdot) - g(x) \leq \varepsilon + 2C\varphi$, we get

$$\big|\big(S_\lambda(t)g\big)(x) - g(x)\big| = \big|\big[S_\lambda(t)\big(g(x+\cdot) - g(x)\big)\big](0)\big| \leq \varepsilon + 2C\big(\mathscr{S}(t)\varphi\big)(0),$$

so that

$$\sup_{\lambda\in\Lambda}\sup_{g\in M}\|S_\lambda(t)g - g\|_\infty \leq \varepsilon + 2C\big(\mathscr{S}(t)\varphi\big)(0).$$

Since $\big(\mathscr{S}(t)\varphi\big)(0) \to 0$ as $t \searrow 0$ by Lemma 5.3, we obtain the assertion. $\square$

Lemma 5.10. *For every $f\in D$ one has*
$$\lim_{h\searrow 0}\left\|\frac{\mathscr{S}(h)f-f}{h}-\mathcal{A}f\right\|_\infty=0.$$

Proof. Let $\varepsilon>0$. Then, by Lemma 5.9 and Lemma 5.3 there exists $h_0>0$ such that
$$S_\lambda(h)A_\lambda f-A_\lambda f\geq -\varepsilon\quad\text{for all }\lambda\in\Lambda$$
and
$$\mathscr{S}(h)\mathcal{A}f-\mathcal{A}f\leq\varepsilon$$
for all $0<h\leq h_0$. Then, for every $0<h\leq h_0$ and $\lambda\in\Lambda$ we get
$$\mathscr{S}(h)f-f\geq S_\lambda(h)f-f=\int_0^h S_\lambda(s)A_\lambda f\,\mathrm{d}s\geq\big(A_\lambda f-\varepsilon\big)h$$
so that
$$\frac{\mathscr{S}(h)f-f}{h}\geq\mathcal{A}f-\varepsilon. \tag{5.10}$$
On the other hand, it follows from Lemma 5.8 that for $0<h\leq h_0$
$$\mathscr{S}(h)f-f-h\mathcal{A}f\leq\int_0^h\mathscr{S}(s)\mathcal{A}f\,\mathrm{d}s-h\mathcal{A}f=\int_0^h\mathscr{S}(s)\mathcal{A}f-\mathcal{A}f\,\mathrm{d}s\leq h\varepsilon,$$
which yields
$$\frac{\mathscr{S}(h)f-f}{h}-\mathcal{A}f\leq\varepsilon.$$
Together with (5.10), we obtain
$$\left\|\frac{\mathscr{S}(h)f-f}{h}-\mathcal{A}f\right\|_\infty\leq\varepsilon$$
for all $0<h\leq h_0$. $\square$

Proposition 5.11. *For $f\in\mathrm{BUC}(G)$ the function*
$$u(t,x)=\big(u(t)\big)(x)=\big(\mathscr{S}(t)f\big)(x),\quad t\geq 0,\,x\in G,$$
is a D-viscosity solution of the fully nonlinear PDE
$$\begin{aligned}u_t(t,x)&=\sup_{\lambda\in\Lambda}\big(A_\lambda u(t)\big)(x),\quad (t,x)\in(0,\infty)\times G,\\ u(0,x)&=f(x),\quad x\in G.\end{aligned}$$

Proof. Fix $t>0$ and $x\in G$. We first show that u is a D-viscosity subsolution. Let $\psi\colon(0,\infty)\to\mathrm{BUC}(G)$ differentiable with $\big(\psi(t)\big)(x)=\big(u(t)\big)(x)$, $\psi(s)\geq u(s)$ and $\psi(s)\in D$ for all $s>0$. Then, for every $h\in(0,t)$, it follows from Proposition 5.6 that
$$\begin{aligned}0&=\frac{\mathscr{S}(h)\mathscr{S}(t-h)f-\mathscr{S}(t)f}{h}=\frac{\mathscr{S}(h)u(t-h)-u(t)}{h}\\&\leq\frac{\mathscr{S}(h)\psi(t-h)-u(t)}{h}\leq\frac{\mathscr{S}(h)\big(\psi(t-h)-\psi(t)\big)+\mathscr{S}(h)\psi(t)-u(t)}{h}\\&=\mathscr{S}(h)\left(\frac{\psi(t-h)-\psi(t)}{h}\right)+\frac{\mathscr{S}(h)\psi(t)-\psi(t)}{h}+\frac{\psi(t)-u(t)}{h}.\end{aligned}$$
Let $\varepsilon>0$. Then, by Lemma 5.10 and Lemma 5.3, there exists $0<h_0<t$ such that for all $0<h<h_0$ one has
$$\frac{\mathscr{S}(h)\psi(t)-\psi(t)}{h}\leq\mathcal{A}\psi(t)+\frac{\varepsilon}{3},\quad\frac{\psi(t-h)-\psi(t)}{h}\leq-\psi_t(t)+\frac{\varepsilon}{3},$$

and

$$\mathscr{S}(h)\big(-\psi_t(t)\big) \leq -\psi_t(t) + \frac{\varepsilon}{3}.$$

Hence, we get

$$\begin{aligned} 0 &\leq \mathscr{S}(h)\big(-\psi_t(t)\big) + \mathcal{A}\psi(t) + \frac{2\varepsilon}{3} + \frac{\psi(t) - u(t)}{h} \\ &\leq -\psi_t(t) + \mathcal{A}\psi(t) + \varepsilon + \frac{\psi(t) - u(t)}{h} \end{aligned}$$

for all $0 < h < h_0$. Since $\big(\psi(t)\big)(x) = \big(u(t)\big)(x)$ we obtain that

$$0 \leq -\big(\psi_t(t)\big)(x) + \big(\mathcal{A}\psi(t)\big)(x) + \varepsilon.$$

Letting $\varepsilon \searrow 0$ yields $\big(\psi_t(t)\big)(x) \leq \big(\mathcal{A}\psi(t)\big)(x)$.

To show that u is a D-viscosity supersolution, let $\psi\colon (0,\infty) \to \mathrm{BUC}(G)$ differentiable with $\big(\psi(t)\big)(x) = \big(u(t)\big)(x)$, $\psi(s) \in D$ and $\psi(s) \leq u(s)$ for all $s > 0$. By Proposition 5.6, for all $h > 0$ with $0 < h < t$ we get

$$\begin{aligned} 0 &= \frac{\mathscr{S}(t)f - \mathscr{S}(h)\mathscr{S}(t-h)f}{h} = \frac{u(t) - \mathscr{S}(h)u(t-h)}{h} \leq \frac{u(t) - \mathscr{S}(h)\psi(t-h)}{h} \\ &= \frac{u(t) - \psi(t)}{h} + \frac{\psi(t) - \mathscr{S}(h)\psi(t)}{h} + \frac{\mathscr{S}(h)\psi(t) - \mathscr{S}(h)\psi(t-h)}{h} \\ &\leq \frac{u(t) - \psi(t)}{h} + \frac{\psi(t) - \mathscr{S}(h)\psi(t)}{h} + \mathscr{S}(h)\left(\frac{\psi(t) - \psi(t-h)}{h}\right) \end{aligned}$$

Let $\varepsilon > 0$. Then, by Lemma 5.10 and Lemma 5.3 there exists $0 < h_0 < t$ such that

$$\frac{\psi(t) - \mathscr{S}(h)\psi(t)}{h} \leq -\mathcal{A}\psi(t) + \frac{\varepsilon}{3},$$

$$\frac{\psi(t) - \psi(t-h)}{h} \leq \psi_t(t) + \frac{\varepsilon}{3},$$

and

$$\mathscr{S}(h)\big(\psi_t(t)\big) \leq \psi_t(t) + \frac{\varepsilon}{3}$$

for all $0 < h < h_0$. We thus get

$$0 \leq \frac{u(t) - \psi(t)}{h} - \mathcal{A}\psi(t) + \mathscr{S}(h)\psi_t(t) + \frac{2\varepsilon}{3} \leq \frac{u(t) - \psi(t)}{h} - \mathcal{A}\psi(t) + \psi_t(t) + \varepsilon$$

for all $0 < h < h_0$. Since $\big(\psi(t)\big)(x) = \big(u(t)\big)(x)$ we obtain that

$$0 \leq -\big(\mathcal{A}\psi(t)\big)(x) + \big(\psi_t(t)\big)(x) + \varepsilon.$$

Letting $\varepsilon \searrow 0$ yields $\big(\psi_t(t)\big)(x) \geq \big(\mathcal{A}\psi(t)\big)(x)$. $\square$

In order to complete the proof of Theorem 2.5, it remains to show that there exists a set $\mathcal{P}$ of probability measures and a stochastic process $(X_t)_{t\geq 0}$ on a measurable space $(\Omega, \mathcal{F})$ such that the viscosity solution in Proposition 5.11 is of the form

$$u(t,x) = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}(f(x + X_t))$$

for $t \geq 0$ and $x \in G$. This is shown in the following proposition for which the assumption (A2) is not needed.

Proposition 5.12. *There exists a set $\mathcal{P}$ of probability measures on a measurable space $(\Omega,\mathcal{F})$ and an $\mathcal{E}$-Lévy process $(X_t)_{t\geq 0}$ such that*

$$\big(\mathscr{S}(t)f\big)(x)=\mathcal{E}(f(x+X_t))=\sup_{\mathbb{P}\in\mathcal{P}}\mathbb{E}_{\mathbb{P}}(f(x+X_t))$$

for all $f\in \mathrm{BUC}(G)$, $t\geq 0$ and $x\in G$, where $\mathcal{E}(Y)=\sup_{\mathbb{P}\in\mathcal{P}}\mathbb{E}_{\mathbb{P}}(Y)$, $Y\in\mathcal{L}^\infty(\Omega,\mathcal{F})$.

Proof. Let $(\Omega_0,\mathcal{F}_0,\mathbb{Q})$ be a probability space such that there exists an independent family of Lévy processes $(X^\lambda)_{\lambda\in\Lambda}$, where X^λ is a Lévy process with generator A_λ, i.e.

$$\mathbb{E}_{\mathbb{Q}}(f(x+X_t^\lambda))=\big(S_\lambda(t)f\big)(x)$$

for all $x\in G$, $t\geq 0$ and $f\in\mathrm{BUC}(G)$. We call $\phi=(\phi_t)_{t\geq 0}$ Λ-simple if there exist $k\in\mathbb{N}$ and $0=t_0<t_1<\ldots<t_k$ such that

$$\phi_t=\sum_{j=0}^{k-1}\phi^{(j)}1_{(t_j,t_{j+1}]}(t)+\phi^{(k)}1_{(t_k,\infty)}(t), \tag{5.11}$$

where for $j\in\{0,\ldots,k\}$ the mapping

$$\phi^{(j)}\colon G\to\Lambda$$

is measurable with finite range. For such a ϕ we define $X_{t_0}^\phi=X_0^\phi:=0$ and inductively, for $j\in\{0,\ldots,k-1\}$, we then define $\lambda_j:=\phi^{(j)}(X_{t_j}^\phi)$ and

$$X_t^\phi:=X_{t_j}^\phi+X_t^{\lambda_j}-X_{t_j}^{\lambda_j}$$

for all $t_j<t\leq t_{j+1}$. Finally, we define $\lambda_k:=\phi^{(k)}(X_{t_k}^\phi)$ and

$$X_t^\phi:=X_{t_k}^\phi+X_t^{\lambda_k}-X_{t_k}^{\lambda_k}$$

for $t>t_k$. Then, $X^\phi=(X_t^\phi)_{t\geq 0}$ can be interpreted as a stochastic integral w.r.t. the Λ-simple process ϕ. Let $h_1,h_2>0$. Then,

$$\big(J_{h_1}J_{h_2}f\big)(x)=\sup_{\eta\in\mathcal{Q}}\int_G f(x+\,\cdot\,)\,\mathrm{d}\eta$$

for $f\in\mathrm{BUC}(G)$, where $\mathcal{Q}$ is the set of probability measures of the form

$$\eta\colon\mathcal{B}\to[0,1],\quad B\mapsto\left[S_{\lambda_0}(h_1)\bigg(\sum_{k=1}^n 1_{B_k}S_{\lambda_k}(h_2)1_B\bigg)\right](0)$$

with $k\in\mathbb{N}$, $\lambda_0,\lambda_1,\ldots,\lambda_n\in\Lambda$ and $B_1\ldots,B_n\in\mathcal{B}$ is a measurable partition of G. Here, we identify $S_\lambda(h)$ for $\lambda\in\Lambda$ and $h>0$ with the unique translation invariant kernel associated to it. Inductively, we therefore obtain that J_π with $\pi=\{t_0,\ldots,t_k\}$ admits a dual representation in terms of distributions of stochastic integrals w.r.t. Λ-simple processes of the form (5.11). For $t\geq 0$ and $f\in\mathrm{BUC}(G)$, we thus obtain that

$$\big(\mathscr{S}(t)f\big)(0)=\sup_{\phi\ \Lambda\text{-simple}}\mathbb{E}_{\mathbb{Q}}(f(X_t^\phi)).$$

Now, consider $\Omega:=G^{[0,\infty)}$, the product σ-algebra $\mathcal{F}$, the canonical process $(X_t)_{t\geq 0}$ and

$$\mathcal{P}:=\big\{\mathbb{Q}\circ\big(X^\phi\big)^{-1}\,\big|\,\phi\text{ is }\Lambda\text{-simple}\big\}. \tag{5.12}$$

Then, for $\mathcal{E}:=\sup_{\mathbb{P}\in\mathcal{P}}\mathbb{E}_{\mathbb{P}}(\,\cdot\,)$ we have that

$$\big(\mathscr{S}(t)f\big)(x)=\mathcal{E}(f(x+X_t))=\sup_{\mathbb{P}\in\mathcal{P}}\mathbb{E}_{\mathbb{P}}(f(x+X_t))$$

for all $t \geq 0$ and $f \in \mathrm{BUC}(G)$. Then, $(X_t)_{t\geq 0}$ satisfies all properties in the definition of an $\mathcal{E}$-Lévy process. Note that (iii) and (iv) follow immediately from the dual representation of the constructed expectation $\mathcal{E}(\cdot) = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}(\cdot)$ and (v) follows as in the proof of Theorem 2.3. □

The construction of the last proposition allows us to finish the proof of Theorem 2.5.

Proof of Theorem 2.5. By Proposition 5.12, there exists a sublinear expectation space $(\Omega, \mathcal{F}, \mathcal{E})$ and an $\mathcal{E}$-Lévy process $(X_t)_{t\geq 0}$ such that

$$\big(\mathscr{S}(t)f\big)(x) = \mathcal{E}(f(x + X_t))$$

for all $t \geq 0$, $x \in G$ and $f \in \mathrm{BUC}(G)$. By Proposition 5.11,

$$u(t, x) := \big(\mathscr{S}(t)f\big)(x) = \mathcal{E}(f(x + X_t))$$

is a D-viscosity solution to (2.4) - (2.5). Moreover, by Proposition 5.12 there exists a set of probability measures $\mathcal{P}$ on $(\Omega, \mathcal{F})$ such that $\mathcal{E}(Y) = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}(Y)$ for all $Y \in \mathcal{L}^\infty(\Omega, \mathcal{F})$. □

Remark 5.13. a) We note that the set $\mathcal{P}$ is constructed in (5.12), based on Λ-simple functions. This description is in some sense different from the approach by Neufeld and Nutz [19] and Kühn [15], where the set of measures is a priori given in relation to the family of Lévy triplets. The question of characterizing the set $\mathcal{P}$ in the general situation considered in Theorem 2.5 seems to be hard and is outside the scope of the present paper.

b) The construction of the nonlinear semigroup is similar to Nisio's approach [21], where only equidistant partitions of the time interval are used. As we have shown in Lemma 5.4, $\mathscr{S}(t)$ is the limit of J_{π_n} for any increasing sequence $(\pi_n)_{n\in\mathbb{N}}$ with $|\pi_n|_\infty \to 0$. Taking equidistant partitions, we obtain the analogue of Nisio's semigroup. However, in Nisio [21] strongly continuous semigroups on $L^\infty(G)$ were considered, and it is now well known that such semigroups have a bounded generator ([2, Corollary 4.3.19]), which does not cover the interesting cases. Therefore, we work with $\mathrm{BUC}(G)$ as the basic space. We plan to consider other relevant spaces like L^p or the space of bounded continuous functions in the future.

We finish with the proof of Proposition 2.8 which now follows essentially from Lemma 5.3.

Proof of Proposition 2.8. Fix $t > 0$ and $\varepsilon > 0$. Let $\varphi \in \bigcap_{\lambda\in\Lambda}\big(D(A_\lambda) \cap C_0(G)\big)$ with $0 \leq \varphi \leq 1$, $\varphi(0) = 1$ and $\sup_{\lambda\in\Lambda} \|A_\lambda\varphi\|_\infty \leq \frac{\varepsilon}{2t}$. Since $\varphi(0) = 1$ and $1-\varphi \in \bigcap_{\lambda\in\Lambda} D(A_\lambda)$ with $A_\lambda(1 - \varphi) = -A_\lambda\varphi$, it follows from Lemma 5.3 that

$$\begin{aligned}\big|\big(\mathscr{S}(t)(1 - \varphi)\big)(0)\big| &\leq \|\mathscr{S}(t)(1 - \varphi) - (1 - \varphi)\|_\infty \\ &\leq t \sup_{\lambda\in\Lambda} \|A_\lambda\varphi\|_\infty \leq \frac{\varepsilon}{2}.\end{aligned}$$

Note that the estimate in Lemma 5.3 holds for all functions in $\bigcap_{\lambda\in\Lambda} D(A_\lambda)$. Since $\varphi \in C_0(G)$ there exists a compact set $K \subset G$ such that

$$1_{G\setminus K}(y) \leq 1 - \varphi(y) + \frac{\varepsilon}{2} \quad \text{for all } y \in G.$$

Hence,

$$\mathcal{E}(1_{G\setminus K}(X_t)) \leq \mathcal{E}(1 - \varphi(X_t)) + \frac{\varepsilon}{2} = \big|\big(\mathscr{S}(t)(1 - \varphi)\big)(0)\big| + \frac{\varepsilon}{2} \leq \varepsilon.$$

In a second step, let (f_n) be a decreasing sequence in $\mathrm{BUC}(G)$ which converges pointwise to f. Fix $x \in G$ and $\varepsilon > 0$. By the first part, there exists a compact $K \subset G$ such that $\mathcal{E}(1_{G\setminus K}(X_t)) \le \varepsilon/4(1+\|f_1\|_\infty+\|f\|_\infty)$. By Dini's lemma there exists $n_0 \in \mathbb{N}$ such that $f_n(x+y) - f(x+y) \le \varepsilon/2$ for all $y \in K$ and $n \ge n_0$. Hence,

$$\begin{aligned}
&\mathcal{E}\big(f_n(x+X_t)\big) - \mathcal{E}\big(f(x+X_t)\big)\\
\le&\mathcal{E}\big((f_n(x+X_t)-f(x+X_t))1_K(X_t)\big) + \mathcal{E}\big((f_n(x+X_t)-f(x+X_t))1_{G\setminus K}(X_t)\big)\\
\le&\frac{\varepsilon}{2} + 2(\|f_1\|_\infty + \|f\|_\infty)\mathcal{E}(1_{G\setminus K}(X_t)) \le \varepsilon
\end{aligned}$$

for all $n \ge n_0$. This shows that $f \mapsto \big(\mathscr{S}(t)f\big)(x)$ is continuous from above on $\mathrm{BUC}(G)$. □

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