Term Structure Modeling under Volatility Uncertainty: A Forward Rate Model driven by G-Brownian Motion

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Abstract

We show how to set up a forward rate model in the presence of volatility uncertainty by using the theory of G-Brownian motion. In order to formulate the model, we extend the G-framework to integration with respect to two integrators and prove a version of Fubini's theorem for stochastic integrals. The evolution of the forward rate in the model is described by a diffusion process, which is driven by a G-Brownian motion. Within this framework, we derive a sufficient condition for the absence of arbitrage, known as the drift condition. In contrast to the traditional model, the drift condition consists of two equations and two market prices of risk, respectively, uncertainty. Furthermore, we examine the connection to short rate models and discuss some examples.

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1 Introduction

In the literature of financial mathematics, there is a recent trend to investigate volatility uncertainty in financial markets, commonly called robust finance. In the standard literature, the volatility is often assumed to be deterministic or even to be constant. This is unrealistic and leads to the problem of volatility estimation. Therefore, these models always include some statistical uncertainty. The new approach is to assume that the volatility is uncertain in the sense of Knightian uncertainty, which is introduced in Knight (2012). Such an assumption is advantageous, since it only requires the volatility to lie between certain bounds without any further specification.

The aim of this paper is to construct an arbitrage-free forward rate model in the presence of volatility uncertainty. The behavior of the forward rate is characterized by a G-Brownian motion in order to represent the volatility uncertainty. Since the forward rate is a stochastic process, depending on the current time and the maturity of the related bond, we face some technical issues. Hence, we construct a space of processes, which depend on two time indices and for which we can define a stochastic integral with two integrators. Moreover, we prove a version of Fubini's theorem for such processes. These processes are afterwards used to model

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the dynamics of the forward rate, ensuring that all quantities regarding the related bond market are well-defined.

Similar to the traditional model, we derive a drift condition, which implies that the bond market is arbitrage-free and states that the risk-neutral dynamics of the forward rate are fully determined by the diffusion coefficient of the forward rate. The drift condition is based on a certain Girsanov transformation for G-Brownian motion and aims to make the drift terms of the discounted bonds vanish. In the traditional model, the discounted bond has only one drift term and the process, which is obtained by the Girsanov transformation and cancels out the drift, is called the market price of risk, because it measures the instantaneous excess return of the bond in units of the diffusion term. Since the quadratic variation of the G-Brownian motion is an uncertain process, we get a certain and an uncertain drift term in this model. Thus, we need to use a Girsanov transformation, modifying both drift terms. The two related processes are then called the market prices of risk, respectively, uncertainty.

In the classical framework, without volatility uncertainty, every short rate model corresponds to a certain forward rate model, since there is a connection between the forward rate and the short rate. Therefore, we examine the relation in this model and we try to reproduce some of the classical short rate models in the presence of volatility uncertainty as examples of this forward rate model. It turns out that this procedure works well for exogenous models and has some interesting implications.

Regarding the literature, the concept of volatility uncertainty has already been applied to asset markets, which is carried out in Avellaneda, Levy, and Parás (1995), Lyons (1995), Epstein and Ji (2013), and Vorbrink (2014). However, most of these models argue about the absence of arbitrage in a very intuitive way. The notion of arbitrage is a crucial point in this setting, since volatility uncertainty is usually represented by a nondominated set of measures. Thus, the fundamental theorem of asset pricing from the classical literature is not suitable for this setting. The theorem is extended to multiple, possibly nondominated priors by Bouchard and Nutz (2015) and Burzoni, Riedel, and Soner (2017), for the discrete-time case, and Biagini, Bouchard, Kardaras, and Nutz (2017), for the continuous-time case. Apart from that, there are also models dealing with pricing contracts on the interest rate in the presence of volatility uncertainty. This can be found in Avellaneda and Lewicki (1996), Fadina, Neufeld, and Schmidt (2018), and Hölzermann (2018), where all of these models correspond to short rate models.

Historically speaking, the evolution of short rate models began with Vasicek (1977) by adapting the methods from Black and Scholes (1973). Later on, the model was followed by many other approaches like the one from Cox, Ingersoll Jr, and Ross (1985). Typically, these models are referred to be endogenous, since the current term structure is an output of the model. Another approach was introduced by Ho and Lee (1986) and Hull and White (1990), where the latter extends the previously mentioned models. The novelty was to use the current term structure as an input, which admits a perfect fit to the observed prices on the market. The breakthrough of this approach was achieved by the methodology of Heath, Jarrow, and Morton (1992), which directly models the forward rate starting from an initially observed forward curve. However, this is still a missing step in the literature of robust finance.

From a mathematical point of view, there are various approaches to represent volatility uncertainty. First of all, there are quasi-sure approaches like the one from Denis and Martini (2006). Apart from that, there is the theory of G-Brownian motion and related G-expectation, which is introduced in Peng (2010). The first approach starts from a probabilistic framework, whereas the latter is motivated by a nonlinear partial differential equation. However, it holds a duality between both of them, which is shown in Denis, Hu, and Peng (2011). In fact, we use the calculus from Peng (2010) in this model, since the literature regarding the G-Brownian motion is very rich, which equips us with a lot of mathematical tools. In particular, we use the Itô formula from Li and Peng (2011), the Girsanov transformation from Hu, Ji, Peng, and Song

(2014), and the results about G-stochastic differential equations from Gao (2009) and Li, Lin, and Lin (2016). Additionally, the methodology from Heath, Jarrow, and Morton (1992) bears some issues, since we need to deal with stochastic processes depending on two time indices. That means we need to extend the stochastic integral from Peng (2010) and derive a version of Fubini's theorem, which is very important for this model. Although this can actually be found in Ibragimov (2013), we do this in a simple manner for the sake of completeness.

The paper is organized as follows. In Section 2, we construct a set of beliefs, representing the uncertainty about the volatility and leading to a G-expectation and a G-Brownian motion, which is used to describe the behavior of the forward rate in the succeeding section. Moreover, we state the results about the previously mentioned extension of stochastic processes. The details and the derivations of the results can be found in Section B in the appendix. In the succeeding Section 3, we introduce the forward rate model. We start by defining the forward rate, which determines the money-market account and the bond prices on the market, and derive the drift condition afterwards. In Section 4, we examine the relation to short rate models, which means that we derive the related short rate dynamics for a general forward rate and for the case where the drift condition is satisfied. After that, in Section 5, we are able to discuss some examples, including the Ho-Lee model and the Hull and White model with volatility uncertainty. Section 6 gives a conclusion.

2 Model Framework

This preliminary section is devoted to set up a general framework for the forward rate model. We start with a probabilistic setting, leading to a sublinear expectation and a G-Brownian motion. The G-Brownian motion is used to describe the evolution of the forward rate in Section 3. Before formulating the forward rate model, it is also necessary to deal with processes depending on two time indices, since the forward rate does. Therefore, we construct a space of processes, which are suitable for stochastic integration with respect to two integrators and for which we can prove a version of Fubini's theorem. However, we only state the main results in this section. The formal construction of stochastic integrals and the proofs of the remaining propositions and theorems are collected in Section B in the appendix.

Let $(\Omega, \mathcal{F}, P)$ be a probability space such that $\Omega = C_0(\mathbb{R}_+)$, $\mathcal{F} = \mathcal{B}(\Omega)$, and P is the Wiener measure. Furthermore, let $(B_t)_t$ be the canonical process, which is a Brownian motion under P, and denote by $(\mathcal{F}_t)_t$ the filtration generated by $(B_t)_t$ completed by all P-null sets. Now we define the process $(B_t^\sigma)_t$ by

$$B_t^\sigma := \int_0^t \sigma_s dB_s$$

for all $[\underline{\sigma}, \overline{\sigma}]$-valued, progressively measurable processes $\sigma = (\sigma_t)_t$, where $\overline{\sigma} \geq \underline{\sigma} > 0$, and we define the measure P^σ to be the law of the process $(B_t^\sigma)_t$, that is,

$$P^\sigma := P \circ (B^\sigma)^{-1}.$$

Denote by $\mathcal{P}$ the closure of all such measures under the topology of weak convergence. Then we can define the sublinear expectation

$$\hat{\mathbb{E}}[X] := \sup_{P \in \mathcal{P}} \mathbb{E}_P[X].$$

By the results from Denis, Hu, and Peng (2011), $\hat{\mathbb{E}}$ corresponds to the G-expectation on $L_G^1(\Omega)$ and the process $(B_t)_t$ is a G-Brownian motion under $\hat{\mathbb{E}}$. Henceforth, all statements should be understood to hold quasi-surely, that is, P-almost surely for all $P \in \mathcal{P}$.

From now on, we work on the G-expectation space $(\Omega, L_G^1(\Omega), \hat{\mathbb{E}})$, where we fix a terminal time $\tau < \infty$, that is, we restrict our analysis to $L_G^1(\Omega_\tau) \subset L_G^1(\Omega)$. For further insights regarding the calculus of G-Brownian motion, the reader may refer to Section A in the appendix or to Peng (2010). It should be noted that the following results and the succeeding forward rate model could be generalized to the case of a d-dimensional G-Brownian motion. However, we use a one-dimensional G-Brownian motion to simplify the notation.

Similar to the construction of stochastic integrals in Peng (2010), we start with the space of simple processes and extend it to its completion under a norm. Denote by $\tilde{M}_G^{p,0}(0,T)$, for $p \geq 1$ and $T \leq \tau$, the space of all simple processes, i.e., processes ϕ of the form

$$\phi(t,s) = \sum_{i=0}^{N-1} \varphi_t^i 1_{[s_i,s_{i+1})}(s)$$

for a partition $0 = s_0 < s_1 < ... < s_N = \tau$ and processes $\varphi^i \in M_G^p(0,T)$. Denote by $\tilde{M}_G^p(0,T)$ the completion of $\tilde{M}_G^{p,0}(0,T)$ under the norm $\| \cdot \|_{\tilde{M}_G^p(0,T)}$, given by

$$\| \phi \|_{\tilde{M}_G^p(0,T)}^p = \int_0^\tau \hat{\mathbb{E}}\Big[\int_0^T |\phi(t,s)|^p dt\Big] ds := \sum_{i=0}^{N-1} \hat{\mathbb{E}}\Big[\int_0^T |\varphi_t^i|^p dt\Big] (s_{i+1} - s_i).$$

We clearly have $\tilde{M}_G^p(0,T) \subset \tilde{M}_G^q(0,T)$ for $p \geq q \geq 1$ and it is easy to show that $1_\Gamma \phi \in \tilde{M}_G^p(0,t_2)$ for $\phi \in \tilde{M}_G^p(0,T)$ and $\Gamma = [t_1,t_2] \times [s_1,s_2]$, where $0 \leq t_1 < t_2 \leq T$ and $0 \leq s_1 < s_2 \leq \tau$, which also implies that $\tilde{M}_G^p(0,T') \subset \tilde{M}_G^p(0,T)$ for $T \leq T'$. Apart from that, we can show the latter for more complex sets Γ.

Proposition 2.1. *Let $\phi \in \tilde{M}_G^p(0,T)$ and $\Gamma = \{(t,s) \in [0,T] \times [0,\tau] : t \leq s\}$. Then we have that $1_\Gamma \phi \in \tilde{M}_G^p(0,T)$.*

The space $\tilde{M}_G^p(0,T)$ is used as the set of all admissible integrands in the succeeding model and the results from above allow us to define integrals on all kinds of intervals.

On the space $\tilde{M}_G^p(0,T)$, we can introduce the following integral operator in order to define stochastic integrals.

Theorem 2.1. *We can define the linear and continuous integral operator $I : \tilde{M}_G^p(0,T) \to M_G^p(0,T)$ given by*

$$I(\phi) := \int_0^\tau \phi(\cdot,s) ds.$$

Hence, we are allowed to integrate processes in $\tilde{M}_G^p(0,T)$ with respect to the second variable and, more important, we know that the integral operator maps to $M_G^p(0,T)$, which allows us to integrate a second time with respect to the first variable of the process. This enables us to define the integral

$$\int_0^T \int_0^\tau \phi(t,s) ds dB_t$$

for $\phi \in \tilde{M}_G^2(0,T)$ and the integrals

$$\int_0^T \int_0^\tau \phi(t,s) ds d\langle B \rangle_t \quad \text{and} \quad \int_0^T \int_0^\tau \phi(t,s) ds dt.$$

for $\phi \in \tilde{M}_G^1(0,T)$.

Remark 2.1. *By Proposition 2.1, we can define integrals like*

$$\int_0^T \int_t^T \phi(t,s) ds dB_t$$

for $\phi \in \tilde{M}_G^2(0,T)$ *as well. The same holds true for* $\phi \in \tilde{M}_G^1(0,T)$ *and the other mentioned integrators.*

Moreover, we are able to define a stochastic integral for the reversed order of integration.

Theorem 2.2. *We can define the linear and continuous integral operators* $J : \tilde{M}_G^2(0,T) \to L_G^2(\Omega_T)$, $J' : \tilde{M}_G^1(0,T) \to L_G^1(\Omega_T)$, *and* $J'' : \tilde{M}_G^1(0,T) \to L_G^1(\Omega_T)$ *given by*

$$\begin{aligned} J(\phi) &:= \int_0^\tau \int_0^T \phi(t,s) dB_t ds, \\ J'(\phi) &:= \int_0^\tau \int_0^T \phi(t,s) d\langle B\rangle_t ds, \\ J''(\phi) &:= \int_0^\tau \int_0^T \phi(t,s) dt ds. \end{aligned}$$

Additionally, for all $\phi \in \tilde{M}_G^2(0,T)$, *it holds*

$$\hat{\mathbb{E}}\Big[\int_0^\tau \int_0^T \phi(t,s) dB_t ds\Big] = 0.$$

Remark 2.2. *According to Proposition 2.1, we can also define integrals like*

$$\int_0^T \int_0^s \phi(t,s) dB_t ds$$

for $\phi \in \tilde{M}_G^2(0,T)$, *which works for* $\phi \in \tilde{M}_G^1(0,T)$ *and the other integrators too.*

Another important tool when dealing with double integrals is the version of Fubini's theorem which allows us to interchange the order of integration.

Theorem 2.3. *Let* $\phi \in \tilde{M}_G^2(0,T)$. *Then it holds*

$$\int_0^\tau \int_0^T \phi(t,s) dB_t ds = \int_0^T \int_0^\tau \phi(t,s) ds dB_t.$$

Furthermore, for $\psi \in \tilde{M}_G^1(0,T)$, *it holds*

$$\int_0^\tau \int_0^T \psi(t,s) d\langle B\rangle_t ds = \int_0^T \int_0^\tau \psi(t,s) ds d\langle B\rangle_t$$

and

$$\int_0^\tau \int_0^T \psi(t,s) dt ds = \int_0^T \int_0^\tau \psi(t,s) ds dt.$$

Remark 2.3. *Here we can again apply Proposition 2.1 to obtain*

$$\int_0^T \int_t^T \phi(t,s) ds dB_t = \int_0^T \int_0^s \phi(t,s) dB_t ds$$

for $\phi \in \tilde{M}_G^2(0,T)$, *which holds for* $\phi \in \tilde{M}_G^1(0,T)$ *and the other integrators as well.*

In fact, the processes in $\tilde{M}_G^p(0,T)$ can be related to the classical space $M_G^p(0,T)$ from Peng (2010). This is important in order to achieve a certain kind of regularity of the forward rate in the succeeding model.

Proposition 2.2. *Let $\phi \in \tilde{M}_G^p(0,T)$. Then $\phi(\cdot,s) \in M_G^p(0,T)$ for almost every $s \in [0,\tau]$.*

So we know that such a process is still a regular process in the classical sense if we fix the second variable.

In the end, we give some sufficient conditions for processes to lie in the space $\tilde{M}_G^p(0,T)$. In fact, the first condition is satisfied in all of the examples in Section 5. We denote by $\mathcal{L}^p(S)$ for $S \subset \mathbb{R}^d$ and $d \in \mathbb{N}$ the space of all $\mathcal{B}(S)$-measurable functions $f : S \to \mathbb{R}$ such that $|f|^p$ is integrable with respect to the Lebesgue measure on S.

Remark 2.4. *(i) Let $\phi : [0,T] \times [0,\tau] \to \mathbb{R}$ be a function such that $\phi \in \mathcal{L}^p([0,T] \times [0,\tau])$. Then the norm on $\tilde{M}_G^p(0,T)$ corresponds to the classical integral norm on $\mathcal{L}^p([0,T] \times [0,\tau])$, since ϕ is deterministic. Moreover, ϕ can be approximated by a sequence of simple functions, which shows that ϕ belongs to $\tilde{M}_G^p(0,T)$.*

(ii) Let $\phi(t,s) = \eta_t \psi(s)$ for $\eta \in M_G^p(0,T)$ and $\psi \in \mathcal{L}^p([0,\tau])$. Hence, we know that there is a sequence $\psi_n \in \mathcal{L}^p([0,\tau])$ of simple functions, converging to ψ in $\mathcal{L}^p([0,\tau])$. We know that $\eta\psi_n \in \tilde{M}_G^{p,0}(0,T)$ and it is possible to show that $\eta\psi_n$ converges to $\eta\psi$ in $\tilde{M}_G^p(0,T)$, which yields $\phi \in \tilde{M}_G^p(0,T)$.

3 Forward Rate Model

After building up the general framework, we are now able to formulate the desired forward rate model. We start by modeling the forward rate, determining all bond prices and the money-market account on the market, with the G-Brownian motion and the processes introduced in Section 2. In order to have an arbitrage-free model, we derive a drift condition, ensuring that there are no arbitrage opportunities in the bond market. The drift condition is based on a certain Girsanov transformation, which requires an extension of the sublinear expectation space.

First of all, we introduce the forward rate $f(t,T)$, for $t \leq T \leq \tau$, which is modeled as a diffusion process, that is,

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \beta(s,T)dB_s + \int_0^t \gamma(s,T)d\langle B\rangle_s$$

for some initial integrable forward curve $T \to f(0,T)$ and processes α, β, and γ to be determined. It should be noted that we have an additional drift part in comparison to the classical case without volatility uncertainty. This is due to the fact that the quadratic variation of the G-Brownian motion is an uncertain process, which is not equal to t and whose density with respect to Lebesgue measure is not in $M_G^1(0,T)$ if $\overline{\sigma} > \underline{\sigma}$. The latter can be found in Song (2013). Hence, the forward rate has a certain and uncertain drift term. If $\overline{\sigma} = \underline{\sigma}$, we have $\langle B\rangle_t = t$. Then we could drop the first integral in the dynamics and we would be back in the classical case.

Now let us consider the related bond market, offering zero-coupon bonds for all maturities T within the time horizon $[0,\tau]$. The price at time $t \leq T$ of such a bond is denoted by $P(t,T)$ and is defined by

$$P(t,T) := \exp\Big(-\int_t^T f(t,s)ds\Big).$$

In addition, there is the money-market account $(M_t)_t$, which is given by

$$M_t := \exp\Big(\int_0^t r_s ds\Big),$$

where $(r_t)_t$ is the instantaneous spot interest rate, called short rate. The short rate in this model is represented by the instantaneous forward rate,

$$r_t := f(t,t)$$

In fact, the money-market account is only used for discounting, i.e., we restrict our analysis to the discounted bond prices, defined by

$$\tilde{P}(t,T) := M_t^{-1} P(t,T).$$

In order to have a certain degree of regularity, we henceforth impose the following assumptions.

Assumption 1 (Regularity of the Forward Rate and the Short Rate)**.** *The drift and diffusion terms of the forward rate are supposed to be regular, that is,*

$$\alpha, \beta, \gamma \in \tilde{M}_G^2(0,\tau).$$

This assumption ensures that the forward rate can be integrated with respect to the second argument and that we can apply Theorem 2.3, which is essential in this model. By Proposition 2.2, we also have that $\alpha(\cdot,T), \beta(\cdot,T), \gamma(\cdot,T) \in M_G^2(0,\tau)$ for almost every $T \in [0,\tau]$. Therefore, the forward rate itself is well-defined for almost every maturity as well as the short rate.

Furthermore, we define the processes a, b, and c by

$$a(t,T) := \int_t^T \alpha(t,s)ds,$$
$$b(t,T) := \int_t^T \beta(t,s)ds,$$
$$c(t,T) := \int_t^T \gamma(t,s)ds,$$

for which we have $a(\cdot,T), b(\cdot,T), c(\cdot,T) \in M_G^2(0,\tau)$ for all $T \in [0,\tau]$ by Proposition 2.1 and Theorem 2.1. However, we need to impose some additional regularity.

Assumption 2 (Regularity of the Discounted Bonds)**.** *The integral of the diffusion term is supposed to satisfy some additional regularity with respect to the first argument, that is,*

$$b(\cdot,T)^2 \in M_G^2(0,\tau) \quad \textit{for all } T \in [0,\tau].$$

Assumption 2 implies that the discounted bond price is well-defined, which is shown after Proposition 3.1 below.

As it is the aim to find a no-arbitrage condition for the bond market, we are interested in the dynamics of the discounted bond prices. For this purpose, we need the following lemma.

Lemma 3.1. *The integral of the forward rate with respect to the maturity satisfies the dynamics*

$$\int_t^T f(t,u)du = \int_0^T f(0,u)du + \int_0^t \big(a(u,T) - r_u\big)du + \int_0^t b(u,T)dB_u + \int_0^t c(u,T)d\langle B\rangle_u$$

for $t \leq T \leq \tau$.

Proof. The integral of the forward rate with respect to the maturity is given by

$$\int_t^T f(t,u)du = \int_t^T f(0,u)du + \int_t^T \int_0^t \alpha(s,u)dsdu + \int_t^T \int_0^t \beta(s,u)dB_s du + \int_t^T \int_0^t \gamma(s,u)d\langle B\rangle_s du.$$

If we apply Theorem 2.3 to the double integrals and do some computations, we get

$$\begin{aligned}
&\int_t^T \int_0^t \alpha(s,u)dsdu + \int_t^T \int_0^t \beta(s,u)dB_s du + \int_t^T \int_0^t \gamma(s,u)d\langle B\rangle_s du \\
&= \int_0^t \int_t^T \alpha(s,u)duds + \int_0^t \int_t^T \beta(s,u)dudB_s \int_0^t \int_t^T \gamma(s,u)dud\langle B\rangle_s \\
&= \int_0^t \int_s^T \alpha(s,u)duds + \int_0^t \int_s^T \beta(s,u)dudB_s + \int_0^t \int_s^T \gamma(s,u)dud\langle B\rangle_s \\
&\quad - \int_0^t \int_s^t \alpha(s,u)duds - \int_0^t \int_s^t \beta(s,u)dudB_s - \int_0^t \int_s^t \gamma(s,u)dud\langle B\rangle_s \\
&= \int_0^t a(s,T)ds + \int_0^t b(s,T)dB_s + \int_0^t c(s,T)d\langle B\rangle_s \\
&\quad - \int_0^t \int_0^u \alpha(s,u)dsdu - \int_0^t \int_0^u \beta(s,u)dB_s du - \int_0^t \int_0^u \gamma(s,u)d\langle B\rangle_s du \\
&= \int_0^t a(u,T)du + \int_0^t b(u,T)dB_u + \int_0^t c(u,T)d\langle B\rangle_u \\
&\quad - \int_0^t \Big(\int_0^u \alpha(s,u)ds + \int_0^u \beta(s,u)dB_s + \int_0^u \gamma(s,u)d\langle B\rangle_s\Big)du.
\end{aligned}$$

Hence, it holds

$$\begin{aligned}
\int_t^T f(t,u)du &= \int_0^T f(0,u)du + \int_0^t a(u,T)du + \int_0^t b(u,T)dB_u + \int_0^t c(u,T)d\langle B\rangle_u \\
&\quad - \int_0^t \Big(f(0,u) + \int_0^u \alpha(s,u)ds + \int_0^u \beta(s,u)dB_s + \int_0^u \gamma(s,u)d\langle B\rangle_s\Big)du \\
&= \int_0^T f(0,u)du + \int_0^t \big(a(u,T) - r_u\big)du + \int_0^t b(u,T)dB_u + \int_0^t c(u,T)d\langle B\rangle_u
\end{aligned}$$

by the definition of the short rate. □

Now we can use the extended Itô's formula for G-Brownian motion from Li and Peng (2011) to obtain the dynamics of the discounted bond prices.

Proposition 3.1. *The discounted bond price process* $(\tilde{P}(t,T))_t$ *satisfies the G-stochastic differential equation*

$$\begin{aligned}
\tilde{P}(t,T) =& \tilde{P}(0,T) - \int_0^t a(u,T)\tilde{P}(u,T)du - \int_0^t b(u,T)\tilde{P}(u,T)dB_u \\
&- \int_0^t \big(c(u,T) - \tfrac{1}{2}b(u,T)^2\big)\tilde{P}(u,T)d\langle B\rangle_u
\end{aligned}$$

for $t \leq T \leq \tau$.

Proof. By applying Lemma 3.1, we get

$$\int_t^T f(t,u)du + \int_0^t r_u du = \int_0^T f(0,u)du + \int_0^t a(u,T)du + \int_0^t b(u,T)dB_u + \int_0^t c(u,T)d\langle B\rangle_u.$$

If we then apply Itô's formula to the discounted bond price

$$\tilde{P}(t,T) = \exp\Big(-\int_t^T f(t,u)du - \int_0^t r_u du\Big),$$

we get

$$\begin{aligned}\tilde{P}(t,T) =& \tilde{P}(0,T) - \int_0^t a(u,T)\tilde{P}(u,T)du - \int_0^t b(u,T)\tilde{P}(u,T)dB_u \\ & - \int_0^t \big(c(u,T) - \tfrac{1}{2}b(u,T)^2\big)\tilde{P}(u,T)d\langle B\rangle_u,\end{aligned}$$

for the dynamics of $(\tilde{P}(t,T))_t$. □

As it is mentioned above, we can explain why the discounted bond price is well-defined. We notice that the drift and diffusion terms in the dynamics of the discounted bond are linear in $\tilde{P}(t,T)$. Furthermore, Assumption 1 and Assumption 2 ensure that the coefficients $a(\cdot,T), b(\cdot,T), (c(\cdot,T) - \frac{1}{2}b(\cdot,T)^2) \in M_G^2(0,\tau)$. Hence, by the results from Li, Lin, and Lin (2016), which are based on the method from Gao (2009), we know that the G-stochastic differential equation in Proposition 3.1 has a unique solution in $M_G^2(0,\tau)$. Indeed, the solution is given by the discounted bond price $(\tilde{P}(t,T))_t$.

After examining the traded quantities on the market, we introduce the set of admissible market strategies and the notion of arbitrage. Since we can define the integral of a process with an additional time dependence with respect to two integrators, we are able to allow for continuous trading within the set of all maturities, which is inspired by Björk, Di Masi, Kabanov, and Runggaldier (1997).

Definition 3.1. *An admissible market strategy is a process $\pi \in \tilde{M}_G^2(0,\tau)$ such that $\pi a\tilde{P} \in \tilde{M}_G^1(0,\tau)$, $\pi b\tilde{P} \in \tilde{M}_G^2(0,\tau)$, and $\pi(c - \frac{1}{2}b^2)\tilde{P} \in \tilde{M}_G^1(0,\tau)$. The corresponding portfolio value process $(\tilde{v}_t(\pi))_t$ is given by*

$$\tilde{v}_t(\pi) = \int_0^\tau \int_0^{t\wedge T} \pi(s,T)d\tilde{P}(s,T)dT.$$

The integral in the above representation of the portfolio value is well-defined, since we can replace the integrator $d\tilde{P}(t,T)$ with the dynamics from Proposition 3.1.

Furthermore, we use the following notion of arbitrage, which corresponds to the one from Vorbrink (2014).

Definition 3.2. *An admissible market strategy π is called arbitrage strategy if it holds*

$$\tilde{v}_\tau(\pi) \geq 0 \quad \textit{quasi-surely} \quad \textit{and} \quad P\big(\tilde{v}_\tau(\pi) > 0\big) > 0 \quad \textit{for at least one } P \in \mathcal{P}.$$

The next step in the model is to derive a drift condition ensuring the absence of arbitrage. Similar to the classical model, we use a certain Girsanov transformation to achieve this. Indeed, we use the one from Hu, Ji, Peng, and Song (2014), since it enables us to modify the certain and the uncertain part of the drift of the discounted bond, i.e., the dt- and the $d\langle B\rangle_t$-term. However, we need to extend the sublinear expectation space for this purpose.

In the following, we consider the extended $\tilde{G}$-expectation space $(\tilde{\Omega}_\tau, L^1_{\tilde{G}}(\tilde{\Omega}_\tau), \hat{\mathbb{E}}^{\tilde{G}})$ with the canonical process $(B_t, \tilde{B}_t)_t$, where $\tilde{\Omega}_\tau = C_0([0,\tau], \mathbb{R}^2)$ and the sublinear function $\tilde{G}$ is given by

$$\tilde{G}(A) = \tfrac{1}{2} \sup_{\vartheta \in [\underline{\sigma}^2, \bar{\sigma}^2]} \operatorname{tr}\Big(A \begin{pmatrix} \vartheta & 1 \\ 1 & \vartheta^{-1} \end{pmatrix} \Big)$$

for some symmetric 2×2 matrix A. Then we have

$$\hat{\mathbb{E}}^{\tilde{G}}[X] = \hat{\mathbb{E}}[X]$$

for $X \in L^1_G(\Omega_\tau)$. Furthermore, we can define the sublinear expectation $\tilde{\mathbb{E}}$ by

$$\tilde{\mathbb{E}}[X] := \hat{\mathbb{E}}^{\tilde{G}}[\mathcal{E} X]$$

for $X \in L^\beta_G(\Omega_\tau)$, where $\beta > 1$ and

$$\mathcal{E} = \exp\Big(\int_0^\tau \lambda_t dB_t + \int_0^\tau \kappa_t d\tilde{B}_t - \tfrac{1}{2} \int_0^\tau \lambda_t^2 d\langle B\rangle_t - \int_0^\tau \lambda_t \kappa_t dt - \tfrac{1}{2} \int_0^\tau \kappa_t^2 d\langle \tilde{B}\rangle_t \Big)$$

for bounded processes $\kappa, \lambda \in M^\beta_G(0,\tau)$. Then, by Hu, Ji, Peng, and Song (2014), we know that

$$\bar{B}_t := B_t - \int_0^t \kappa_s ds - \int_0^t \lambda_s d\langle B\rangle_s$$

is a G-Brownian motion under $\tilde{\mathbb{E}}$.

Now we can introduce the drift condition. Similar to the traditional model, the processes κ and λ are used to make the drift terms vanish. Since we have a certain and uncertain drift term in this case, the processes κ and λ can be interpreted as the market price of risk and uncertainty, respectively. Moreover, we show that the dynamics of the forward rate under the "risk-neutral" sublinear expectation $\tilde{\mathbb{E}}$ only depend on its diffusion coefficient.

Theorem 3.1. *Suppose that the processes κ and λ satisfy the drift condition*

$$a(t,T) + b(t,T)\kappa_t = 0,$$
$$c(t,T) - \tfrac{1}{2} b(t,T)^2 + b(t,T)\lambda_t = 0$$

for all $t \leq T$ for all $T \in [0,\tau]$. Then the discounted bond price process $(\tilde{P}(t,T))_t$ is a symmetric G-martingale under $\tilde{\mathbb{E}}$ and the forward rate satisfies

$$f(t,T) = f(0,T) + \int_0^t \beta(s,T) d\bar{B}_s + \int_0^t \Big(\beta(s,T) \int_s^T \beta(s,u)du \Big) d\langle B\rangle_s.$$

Proof. By Proposition 3.1, we have

$$\begin{aligned}
\tilde{P}(t,T) =& \tilde{P}(0,T) - \int_0^t a(u,T)\tilde{P}(u,T)du - \int_0^t b(u,T)\tilde{P}(u,T)dB_u \\
& - \int_0^t \big(c(u,T) - \tfrac{1}{2} b(u,T)^2 \big) \tilde{P}(u,T) d\langle B\rangle_u \\
=& \tilde{P}(0,T) - \int_0^t \big(a(u,T) + b(u,T)\kappa_u \big) \tilde{P}(u,T)du - \int_0^t b(u,T)\tilde{P}(u,T)d\bar{B}_u \\
& - \int_0^t \big(c(u,T) - \tfrac{1}{2} b(u,T)^2 + b(u,T)\lambda_u \big) \tilde{P}(u,T) d\langle B\rangle_u \\
=& \tilde{P}(0,T) - \int_0^t b(u,T)\tilde{P}(u,T)d\bar{B}_u.
\end{aligned}$$

Thus, since $(\bar{B}_t)_t$ is a G-Brownian motion under $\tilde{\mathbb{E}}$, we know that $(\tilde{P}(t,T))_t$ is a symmetric G-martingale. Apart from that, we can differentiate the drift condition with respect to T to get

$$\alpha(t,T) + \beta(t,T)\kappa_t = 0,$$
$$\gamma(t,T) - \beta(t,T)\int_t^T \beta(t,u)du + \beta(t,T)\lambda_t = 0.$$

Hence, the forward rate is given by

$$\begin{aligned} f(t,T) =& f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \beta(s,T)dB_s + \int_0^t \gamma(s,T)d\langle B\rangle_s \\ =& f(0,T) + \int_0^t \big(\alpha(s,T) + \beta(s,T)\kappa_s\big)ds + \int_0^t \beta(s,T)d\bar{B}_s \\ &+ \int_0^t \big(\gamma(s,T) + \beta(s,T)\lambda_s\big)d\langle B\rangle_s \\ =& f(0,T) + \int_0^t \beta(s,T)d\bar{B}_s + \int_0^t \Big(\beta(s,T)\int_s^T \beta(s,u)du\Big)d\langle B\rangle_s, \end{aligned}$$

as it is claimed. □

The drift condition corresponds exactly to the classical one if there is no volatility uncertainty, i.e., if $\overline{\sigma} = \underline{\sigma}$. In that case, we would have $\langle B\rangle_t = t$ and the process α would vanish, which means that the first equation of the drift condition becomes redundant. Then we would be back in the classical case.

Since the drift condition implies that the discounted bond is a symmetric G-martingale under a somehow "equivalent" sublinear expectation, we get the following result.

Corollary 3.1. *If the drift condition from Theorem 3.1 is satisfied, then the market is arbitrage-free, i.e., there is no arbitrage strategy.*

Proof. Suppose there exists an arbitrage strategy π. Hence, we know that

$$\tilde{v}_\tau(\pi) \geq 0 \quad \text{quasi-surely} \quad \text{and} \quad P\big(\tilde{v}_\tau(\pi) > 0\big) > 0 \quad \text{for at least one } P \in \mathcal{P},$$

which implies

$$\hat{\mathbb{E}}^{\tilde{G}}[\tilde{v}_\tau(\pi)] = \hat{\mathbb{E}}[\tilde{v}_\tau(\pi)] > 0.$$

Furthermore, it holds

$$\tilde{\mathbb{E}}[\tilde{v}_\tau(\pi)] = \hat{\mathbb{E}}^{\tilde{G}}[\mathcal{E}\tilde{v}_\tau(\pi)] > 0,$$

since the exponential $\mathcal{E}$ is strictly positive. Now we can use Theorem 3.1 and the property of the double integral which was introduced in Theorem 2.2 to show the contradiction.

$$\begin{aligned} \tilde{\mathbb{E}}[\tilde{v}_\tau(\pi)] =& \tilde{\mathbb{E}}\Big[\int_0^\tau \int_0^T \pi(t,T)d\tilde{P}(t,T)dT\Big] \\ =& \tilde{\mathbb{E}}\Big[\int_0^\tau \int_0^T \pi(t,T)\big(-b(t,T)\tilde{P}(t,T)\big)d\bar{B}_t dT\Big] = 0. \end{aligned}$$

Therefore, there is no arbitrage strategy. □

4 Short Rate Dynamics

Since every forward rate model corresponds to a related short rate model, we want to examine this relation closer. This is especially interesting for examining examples, which is the content of Section 5. There we determine the volatility structure of the forward rate in order to reproduce certain short rate models.

Indeed, if we impose more assumptions on the drift and the diffusion, it is possible to show the relation between the forward rate and the short rate for a general setup, i.e., without specifying α, β, and γ.

Lemma 4.1. *Suppose that $T \to f(0,T)$, $T \to \alpha(t,T)$, $T \to \beta(t,T)$, and $T \to \gamma(t,T)$ are differentiable such that $\partial_T\alpha, \partial_T\gamma \in \tilde{M}_G^1(0,\tau)$ and $\partial_T\beta \in \tilde{M}_G^2(0,\tau)$. Then the short rate process satisfies the dynamics*

$$r_t = r_0 + \int_0^t \big(\psi(u) + \alpha(u,u)\big)du + \int_0^t \beta(u,u)dB_u + \int_0^t \gamma(u,u)d\langle B\rangle_u,$$

where

$$\psi(u) = \partial_u f(0,u) + \int_0^u \partial_u\alpha(s,u)ds + \int_0^u \partial_u\beta(s,u)dB_s + \int_0^u \partial_u\gamma(s,u)d\langle B\rangle_s.$$

Proof. The short rate is given by

$$r_t = f(t,t) = f(0,t) + \int_0^t \alpha(s,t)ds + \int_0^t \beta(s,t)dB_s + \int_0^t \gamma(s,t)d\langle B\rangle_s.$$

Using differentiability, we have

$$f(0,t) = f(0,0) + \int_0^t \partial_u f(0,u)du = r_0 + \int_0^t \partial_u f(0,u)du$$

and

$$\begin{aligned}\int_0^t \alpha(s,t)ds &= \int_0^t \alpha(s,s)ds + \int_0^t \big(\alpha(s,t) - \alpha(s,s)\big)ds \\ &= \int_0^t \alpha(s,s)ds + \int_0^t \int_s^t \partial_u\alpha(s,u)duds \\ &= \int_0^t \alpha(s,s)ds + \int_0^t \int_0^u \partial_u\alpha(s,u)dsdu.\end{aligned}$$

Analogously to the last step, it follows

$$\int_0^t \beta(s,t)dB_s = \int_0^t \beta(s,s)dB_s + \int_0^t \int_0^u \partial_u\beta(s,u)dB_sdu$$

and

$$\int_0^t \gamma(s,t)d\langle B\rangle_s = \int_0^t \gamma(s,s)d\langle B\rangle_s + \int_0^t \int_0^u \partial_u\gamma(s,u)d\langle B\rangle_sdu.$$

Substituting all terms in the first equality of the proof yields the assertion. □

If we now assume that the drift condition holds, we get the corresponding dynamics of the short rate under the risk-neutral sublinear expectation $\hat{\mathbb{E}}$.

Proposition 4.1. *Assume that the requirements of Lemma 4.1 are fulfilled and that the drift condition from Theorem 3.1 is satisfied. Then the short rate process satisfies the dynamics*

$$r_t = r_0 + \int_0^t \psi(u)du + \int_0^t \beta(u,u)d\bar{B}_u,$$

where

$$\psi(u) = \partial_u f(0,u) + \int_0^u \partial_u \beta(s,u)d\bar{B}_s + \int_0^u \big(\partial_u\beta(s,u)b(s,u) + \beta(s,u)^2\big)d\langle B\rangle_s.$$

Proof. If we differentiate the drift condition with respect to T, we get

$$\begin{aligned} \alpha(t,T) + \beta(t,T)\kappa_t = 0,\\ \gamma(t,T) - \beta(t,T)b(t,T) + \beta(t,T)\lambda_t = 0. \end{aligned}$$

By setting $T = t$, we know that it holds

$$\begin{aligned} \alpha(t,t) + \beta(t,t)\kappa_t = 0,\\ \gamma(t,t) + \beta(t,t)\lambda_t = 0. \end{aligned}$$

Furthermore, we can differentiate the drift condition a second time with respect to T to obtain

$$\begin{aligned} \partial_T\alpha(t,T) + \partial_T\beta(t,T)\kappa_t = 0,\\ \partial_T\gamma(t,T) - \partial_T\beta(t,T)b(t,T) - \beta(t,T)^2 + \partial_T\beta(t,T)\lambda_t = 0. \end{aligned}$$

Now we can use Lemma 4.1 and write the dynamics in terms of $(\bar{B}_t)_t$ to apply the conditions from above, i.e.,

$$\begin{aligned} r_t =& r_0 + \int_0^t \big(\psi(u) + \alpha(u,u)\big)du + \int_0^t \beta(u,u)dB_u + \int_0^t \gamma(u,u)d\langle B\rangle_u \\ =& r_0 + \int_0^t \big(\psi(u) + \alpha(u,u) + \beta(u,u)\kappa_u\big)du + \int_0^t \beta(u,u)d\bar{B}_u \\ &+ \int_0^t \big(\gamma(u,u) + \beta(u,u)\lambda_u\big)d\langle B\rangle_u \\ =& r_0 + \int_0^t \psi(u)du + \int_0^t \beta(u,u)d\bar{B}_u, \end{aligned}$$

where

$$\begin{aligned} \psi(u) =& \partial_u f(0,u) + \int_0^u \partial_u\alpha(s,u)ds + \int_0^u \partial_u\beta(s,u)dB_s + \int_0^u \partial_u\gamma(s,u)d\langle B\rangle_s \\ =& \partial_u f(0,u) + \int_0^u \big(\partial_u\alpha(s,u) + \partial_u\beta(s,u)\kappa_s\big)ds + \int_0^u \partial_u\beta(s,u)d\bar{B}_s \\ &+ \int_0^u \big(\partial_u\gamma(s,u) + \partial_u\beta(s,u)\lambda_s\big)d\langle B\rangle_s \\ =& \partial_u f(0,u) + \int_0^u \partial_u\beta(s,u)d\bar{B}_s + \int_0^u \big(\partial_u\beta(s,u)b(s,u) + \beta(s,u)^2\big)d\langle B\rangle_s, \end{aligned}$$

as it is claimed. □

Similar to Theorem 3.1, we see that the dynamics of the short rate only depend on the diffusion coefficient.

5 Examples

The Ho-Lee model and the Hull and White extended Vasicek model are probably the most famous short rate models allowing for a perfect fit to the initial term structure. Therefore, they are often used as examples in the HJM framework. In the following, we want to see if we are also able to reproduce these short rate models in this framework and how the related model differs from the one without volatility uncertainty. Apart from that, the Hull and White model is actually a good example to show why the classical Vasicek model cannot be embedded into this framework.

5.1 Ho-Lee Model

Let us suppose that the diffusion coefficient of the forward rate is a constant $\sigma > 0$, i.e.,

$$\beta(t,T) = \sigma$$

for all $t \leq T \leq \tau$. Then, by Theorem 3.1, we know that the forward rate satisfies

$$f(t,T) = f(0,T) + \sigma \bar{B}_t + \sigma^2 \int_0^t (T-s) d\langle B\rangle_s$$

under the risk-neutral sublinear expectation and Proposition 4.1 tells us that the short rate is given by

$$r_t = r_0 + \int_0^t \big(\partial_u f(0,u) + \sigma^2 \langle B\rangle_u\big) du + \sigma \bar{B}_t.$$

Hence, we can see that this corresponds exactly to the (fitted) Ho-Lee model[1] if there is no volatility uncertainty, i.e., if $\underline{\sigma} = 1 = \overline{\sigma}$.

Furthermore, we can calculate the bond prices

$$P(t,T) = \exp\Big(-\int_t^T f(t,u) du\Big).$$

If we perform some computations, we get

$$\begin{aligned}
\int_t^T f(t,u)du &= \int_t^T \Big(f(0,u) + \sigma \bar{B}_t + \sigma^2 \int_0^t (u-s) d\langle B\rangle_s\Big) du \\
&= \int_t^T f(0,u)du + (T-t)\sigma \bar{B}_t + \sigma^2 \int_t^T \int_0^t (u-t) d\langle B\rangle_s du \\
&\quad + \sigma^2 \int_t^T \int_0^T (t-s) d\langle B\rangle_s du \\
&= \int_t^T f(0,u)du - (T-t) f(0,t) + (T-t) f(0,t) + (T-t)\sigma \bar{B}_t \\
&\quad + \tfrac{\sigma^2}{2} \langle B\rangle_t (T-t)^2 + (T-t)\sigma^2 \int_0^t (t-s) d\langle B\rangle_s \\
&= \int_t^T f(0,u)du - (T-t) f(0,t) + \tfrac{\sigma^2}{2} \langle B\rangle_t (T-t)^2 \\
&\quad + (T-t)\Big(f(0,t) + \sigma \bar{B}_t + \sigma^2 \int_0^t (t-s) d\langle B\rangle_s\Big),
\end{aligned}$$

[1] See Subsection 5.4.4 in Filipovic (2009).

where the term in the large brackets corresponds to the short rate. Thus, we have

$$P(t,T) = \exp\Big(- \int_t^T f(0,u)du + (T-t)f(0,t) - \tfrac{\sigma^2}{2}\langle B\rangle_t (T-t)^2 - (T-t)r_t \Big),$$

which is again the same expression as in the traditional model if $\langle B\rangle_t = t$.

So the main difference is the appearance of the quadratic variation process in the drift of the short rate and in the bond prices. This means that, in contrast to the traditional affine models, the bond price does not only depend on the initial term structure and the current value of the short rate, but also on the current value of the quadratic variation of the short rate.

5.2 Hull and White Model

In order to reproduce the Hull and White extended Vasicek model, we assume that

$$\beta(t,T) = \sigma e^{-\theta(T-t)}$$

for constants $\sigma, \theta > 0$. Hence, we get the forward rate

$$f(t,T) = f(0,T) + \int_0^t \sigma e^{-\theta(T-s)} d\bar{B}_s + \int_0^t \tfrac{\sigma^2}{\theta} e^{-\theta(T-s)}\big(1 - e^{-\theta(T-s)}\big) d\langle B\rangle_s$$

and the short rate

$$r_t = r_0 + \int_0^t \psi(u)du + \sigma \bar{B}_t,$$

where

$$\begin{aligned}
\psi(u) =& \partial_u f(0,u) - \theta \int_0^u \sigma e^{-\theta(u-s)} dB_s \\
&+ \int_0^u \Big(\sigma^2 e^{-\theta(u-s)}\big(1 - e^{-\theta(u-s)}\big) + \sigma^2 e^{-2\theta(u-s)}\Big) d\langle B\rangle_s \\
=& \partial_u f(0,u) + \theta f(0,u) + \sigma^2 \int_0^u e^{-2\theta(u-s)} d\langle B\rangle_s \\
&- \theta\Big(f(0,u) + \int_0^u \sigma e^{-\theta(u-s)} d\bar{B}_s + \int_0^u \tfrac{\sigma^2}{\theta} e^{-\theta(u-s)}\big(1 - e^{-\theta(u-s)}\big) d\langle B\rangle_s \Big) \\
=& \partial_u f(0,u) + \theta f(0,u) + \sigma^2 \int_0^u e^{-2\theta(u-s)} d\langle B\rangle_s - \theta r_u.
\end{aligned}$$

Similar to the previous example, these are the same dynamics than the ones in the classical (fitted) Hull and White model[2] if $\langle B\rangle_t = t$.

To obtain the bond prices

$$P(t,T) = \exp\Big(- \int_t^T f(t,u)du \Big),$$

we need to do some computations on the integral of the forward rate,

$$\begin{aligned}
\int_t^T f(t,u)du = & \int_t^T f(0,u)du + \int_t^T \int_0^t \sigma e^{-\theta(u-s)} d\bar{B}_s du \\
&+ \int_t^T \int_0^t \tfrac{\sigma^2}{\theta} e^{-\theta(u-s)}\big(1 - e^{-\theta(u-s)}\big) d\langle B\rangle_s du,
\end{aligned}$$

[2]See Subsection 3.3.1 in Brigo and Mercurio (2007).

where

$$\int_t^T\int_0^t \sigma e^{-\theta(u-s)}d\bar{B}_s du = \Big(\int_t^T e^{-\theta(u-t)}du\Big)\Big(\int_0^t \sigma e^{-\theta(t-s)}d\bar{B}_s\Big)$$
$$= \tfrac{1}{\theta}\big(1-e^{-\theta(T-t)}\big)\int_0^t \sigma e^{-\theta(t-s)}d\bar{B}_s$$

and

$$\int_t^T\int_0^t \tfrac{\sigma^2}{\theta}e^{-\theta(u-s)}\big(1-e^{-\theta(u-s)}\big)d\langle B\rangle_s du$$
$$= \int_t^T\int_0^t \tfrac{\sigma^2}{\theta}e^{-\theta(u-s)}\big(1-e^{-\theta(t-s)}\big)d\langle B\rangle_s du$$
$$+ \int_t^T\int_0^t \tfrac{\sigma^2}{\theta}e^{-\theta(u-s)}\big(e^{-\theta(t-s)}-e^{-\theta(u-s)}\big)d\langle B\rangle_s du$$
$$= \Big(\int_t^T e^{-\theta(u-t)}du\Big)\Big(\int_0^t \tfrac{\sigma^2}{\theta}e^{-\theta(t-s)}\big(1-e^{-\theta(t-s)}\big)d\langle B\rangle_s\Big)$$
$$+ \Big(\int_t^T \tfrac{\sigma^2}{\theta}e^{-\theta(u-t)}\big(1-e^{-\theta(u-t)}\big)du\Big)\Big(\int_0^t e^{-2\theta(t-s)}d\langle B\rangle_s\Big)$$
$$= \tfrac{1}{\theta}\big(1-e^{-\theta(T-t)}\big)\int_0^t \tfrac{\sigma^2}{\theta}e^{-\theta(t-s)}\big(1-e^{-\theta(t-s)}\big)d\langle B\rangle_s$$
$$+ \tfrac{\sigma^2}{2\theta^2}\big(1-e^{-\theta(T-t)}\big)^2\int_0^t e^{-2\theta(t-s)}d\langle B\rangle_s.$$

Thus, it holds

$$\int_t^T f(t,u)du = \int_t^T f(0,u)du - \tfrac{1}{\theta}\big(1-e^{-\theta(T-t)}\big)f(0,t)$$
$$+ \tfrac{\sigma^2}{2\theta^2}\big(1-e^{-\theta(T-t)}\big)^2\int_0^t e^{-2\theta(t-s)}d\langle B\rangle_s$$
$$+ \tfrac{1}{\theta}\big(1-e^{-\theta(T-t)}\big)r_t$$

and

$$P(t,T) = \exp\Big(-\int_t^T f(0,u)du + \tfrac{1}{\theta}\big(1-e^{-\theta(T-t)}\big)f(0,t)$$
$$- \tfrac{\sigma^2}{2\theta^2}\big(1-e^{-\theta(T-t)}\big)^2\int_0^t e^{-2\theta(t-s)}d\langle B\rangle_s$$
$$- \tfrac{1}{\theta}\big(1-e^{-\theta(T-t)}\big)r_t\Big),$$

which is the expression from the traditional Hull and White model[3] if we have $\langle B\rangle_t = t$.

As in the preceeding example, we notice the appearance of the quadratic variation process in the drift of the short rate and in the bond prices. Indeed, this dependence is very important for the model in order to be arbitrage-free and slightly complicates the application of the model. The reason is that the quadratic variation process is not that easy to specify, since it is not directly observable. However, in the examples from above the quadratic variation of the G-Brownian motion is the same than the quadratic variation of the short rate, which can be estimated.

[3]See Subsection 3.3.2 in Brigo and Mercurio (2007).

5.3 Vasicek Model

After examining the Hull and White model, we can argue why it is not possible to treat the Vasicek model in this framework. In fact, the forward rate in the Vasicek model has the same volatility structure as in the Hull and White model,

$$\beta(t,T) = \sigma e^{-\theta(T-t)},$$

which, by the previous example, leads to the short rate dynamics

$$r_t = r_0 + \int_0^t \Big(\partial_u f(0,u) + \theta f(0,u) + \sigma^2 \int_0^u e^{-2\theta(u-s)} d\langle B\rangle_s - \theta r_u\Big) du + \sigma \bar{B}_t.$$

Hence, it is clear that the Vasicek model cannot be fitted to any initial term structure, since this requires a time dependent mean reversion level, which is a constant μ in that model. However, the bond prices in the Vasicek model have a certain structure such that the related forward rates satisfy

$$\partial_t f(0,t) + \theta f(0,t) + \sigma^2 \int_0^t e^{-2\theta(t-s)} d\langle B\rangle_s = \mu$$

if $\langle B\rangle_t = t$. Unfortunately, this cannot hold if we have volatility uncertainty. The reason is that, if we look at the equivalent representation of the equation from above

$$\sigma^2 \int_0^t e^{-2\theta(t-s)} d\langle B\rangle_s = \mu - \partial_t f(0,t) - \theta f(0,t),$$

we see that the left-hand side is uncertain and thus only observable at time t, whereas the initial forward curve $t \to f(0,t)$ on the right-hand side is already observable at time 0. Therefore, the above equation contradicts to the assumptions of the model.

This shows that it is not possible to reproduce the short rate dynamics of the Vasicek model, since the drift has to include an uncertain process. In general, the examples in this section show that there always has to be an uncertain process in the drift of the short rate, somehow depending on the quadratic variation of the G-Brownian motion, if we want to exclude arbitrage by the drift condition. However, this is usually not part of the assumptions in the classical models without volatility uncertainty. Another way how to deal with a short rate model under volatility uncertainty would be to achieve the certain structure of the drift by modifying the dynamics with a Grisanov transformation, which is actually the strategy in Hölzermann (2018).

6 Conclusion

The paper extends the stochastic integral from Peng (2010) to integrands depending on two time indices, which are integrable with respect to two integrators, in order to set up a forward rate model under volatility uncertainty. Within this framework, we obtain a drift condition, which is sufficient for the absence of arbitrage on the related bond market. In particular, the drift condition implies that the dynamics of the forward rate under the risk-neutral sublinear expectation only depend on the diffusion coefficient as in the classical case.

Furthermore, we investigate the connection to short rate models in order to discuss some examples. In the end, it turns out that exogenous models like the Ho-Lee model or the Hull and White model can be reproduced very well in this model. However, those models have some implications that differ from the classical models. This leads to the deduction that other classical models like the Vasicek model have to be suitably modified in order to fit into this framework.

Appendix

A Sublinear Expectations and G-Brownian Motion

In this section, we give a short introduction to the calculus of G-Brownian motion, including the construction of stochastic integrals. The definition and construction of a G-Brownian motion requires knowledge in the theory of sublinear expectation spaces and distributions on them. The G-normal distribution is essential to describe the behavior of the G-Brownian motion. Further results can be found in Peng (2010) as well as the proofs, which are not given in this section.

A.1 Sublinear Expectation Spaces

Let Ω be a given space and $\mathcal{H}$ is a space of real valued functions defined on Ω. $\mathcal{H}$ is supposed to satisfy

$$\varphi(X_1, ..., X_n) \in \mathcal{H} \quad \text{if } X_1, ..., X_n \in \mathcal{H}$$

for all $\varphi \in C_{l,Lip}(\mathbb{R}^n)$, where $C_{l,Lip}(\mathbb{R}^n)$ is the linear space of functions φ satisfying

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y|$$

for all $x, y \in \mathbb{R}^n$ for some $C > 0$ and $m \in \mathbb{N}$, both depending on φ. $X = (X_1, ..., X_n)$ is called n-dimensional random vector, denoted by $X \in \mathcal{H}^n$.

Definition A.1. *A sublinear expectation $\hat{\mathbb{E}}$ is a functional $\hat{\mathbb{E}} : \mathcal{H} \to \mathbb{R}$ satisfying, for all $X, Y \in \mathcal{H}$,*

(i) Monotonicity:

$$\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y] \quad \textit{if } X \geq Y,$$

(ii) Constant preserving:

$$\hat{\mathbb{E}}[c] = c \quad \textit{for } c \in \mathbb{R},$$

(iii) Sub-additivity:

$$\hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y],$$

(iv) Positive homogeneity:

$$\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X] \quad \textit{for } \lambda \geq 0.$$

The tripel $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sublinear expectation space.

Similar to the classical case, it is possible to show the following useful inequalities, which enable us to take the completion of a sublinear expectation space.

Proposition A.1. *For $X, Y \in \mathcal{H}$ and $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\hat{\mathbb{E}}[|XY|] \leq \hat{\mathbb{E}}[|X|^p]^{\frac{1}{p}} \hat{\mathbb{E}}[|Y|^q]^{\frac{1}{q}}$$

and

$$\hat{\mathbb{E}}[|X + Y|^p]^{\frac{1}{p}} \leq \hat{\mathbb{E}}[|X|^p]^{\frac{1}{p}} + \hat{\mathbb{E}}[|Y|^p]^{\frac{1}{p}}.$$

In particular, for $1 \leq p < p'$, we have

$$\hat{\mathbb{E}}[|X|^p]^{\frac{1}{p}} \leq \hat{\mathbb{E}}[|X|^{p'}]^{\frac{1}{p'}}.$$

For a fixed $p \geq 1$, we can take $\mathcal{H}_0^p = \{X \in \mathcal{H} : \hat{\mathbb{E}}[|X|^p] = 0\}$ as our null-space, which is a linear subspace of $\mathcal{H}$, and introduce the quotient space $\mathcal{H}/\mathcal{H}_0^p$. For every $\{X\} \in \mathcal{H}/\mathcal{H}_0^p$ with a representation $X \in \mathcal{H}$, we can define the sublinear expectation $\hat{\mathbb{E}}[\{X\}] := \hat{\mathbb{E}}[X]$. If we set $||X||_p := \hat{\mathbb{E}}[|X|^p]^{\frac{1}{p}}$, we get that $||\cdot||_p$ is a norm on $\mathcal{H}/\mathcal{H}_0^p$ by Proposition A.1. Hence, we can extend $\mathcal{H}/\mathcal{H}_0^p$ to its completion $\hat{\mathcal{H}}_p$ under this norm.

Furthermore, we introduce the mapping $^+ : \mathcal{H} \to \mathcal{H}$, defined by $X^+ := \max\{X, 0\}$, which can be continuously extended to $\hat{\mathcal{H}}_p$, since it is a contraction mapping by the inequality

$$|X^+ - Y^+| \leq |X - Y|.$$

Thus, we can define a partial order $\geq$ on $\hat{\mathcal{H}}_p$, that is, we write $X \geq Y$ if $X - Y = (X - Y)^+$. Since it holds

$$|\hat{\mathbb{E}}[X] - \hat{\mathbb{E}}[Y]| \leq \hat{\mathbb{E}}[|X - Y|] \leq ||X - Y||_p,$$

for $X, Y \in \mathcal{H}$, the sublinear expectation $\hat{\mathbb{E}}$ can be continuously extended to $\hat{\mathcal{H}}_p$ as well, on which it is still a sublinear expectation.

A.2 G-Normal Distribution

First of all, we introduce the notion of distributions and independence on sublinear expectation spaces.

Definition A.2. *Let X and Y be two random variables defined on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. They are called identically distributed, denoted by $X \overset{d}{=} Y$, if*

$$\hat{\mathbb{E}}[\varphi(X)] = \hat{\mathbb{E}}[\varphi(Y)]$$

for all $\varphi \in C_{l,Lip}(\mathbb{R})$.

Definition A.3. *In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ a random vector $Y \in \mathcal{H}^n$ is said to be independent from another random vector $X \in \mathcal{H}^m$ under $\hat{\mathbb{E}}(\cdot)$ if for each test function $\varphi \in C_{l,Lip}(\mathbb{R}^{m+n})$ it holds*

$$\hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x, Y)]_{x=X}].$$

The previous definitions allow us to define the G-normal distribution, which is a generalization of the centralized normal distribution. It can be used to represent the variance uncertainty of a random variable.

Definition A.4. *A random variable X on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called G-normal distributed if*

$$aX + b\bar{X} \overset{d}{=} \sqrt{a^2 + b^2} X$$

for $a, b \geq 0$, where $\bar{X}$ is an independent copy of X.

In order to characterize the distribution of G-normal distributed random variables, we define the sublinear, monotonic function $G : \mathbb{R} \to \mathbb{R}$ by

$$G(a) := \tfrac{1}{2}\hat{\mathbb{E}}[X^2 a].$$

Since G is also continuous, it can be shown that there exists a bounded, convex, and closed subset $[\underline{\sigma}^2, \overline{\sigma}^2] \subset \mathbb{R}$ such that

$$G(a) = \frac{1}{2} \sup_{\sigma \in [\underline{\sigma}^2, \overline{\sigma}^2]} \{\sigma a\}.$$

In this case, we say that the random variable is $\mathcal{N}(\{0\}, [\underline{\sigma}^2, \overline{\sigma}^2])$-distributed.

Conversely, it can also be shown that for an arbitrary sublinear, monotonic, and continuous function there exists a G-normal distributed random variable, satisfying the same G. In particular, this proves the existence of G-normal distributed random variables.

Proposition A.2. *Let $G : \mathbb{R} \to \mathbb{R}$ be a given sublinear, monotonic function, which is continuous. Then there exists a G-normal distributed random variable X on some sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ satisfying*

$$G(a) = \tfrac{1}{2}\hat{\mathbb{E}}[X^2 a].$$

Similar to the standard normal distribution, the expectation of a G-normal distributed random variable can be described by a partial differential equation, which is in this case called the G-heat equation.

Proposition A.3. *For a G-normal distributed random variable X and a function $\varphi \in C_{l,Lip}(\mathbb{R})$, we define*

$$u(t, x) := \hat{\mathbb{E}}[\varphi(x + \sqrt{t}X)]$$

for $(t, x) \in [0, \infty) \times \mathbb{R}$. Then u is the unique viscosity solution of

$$\partial_t u - G(\partial_x^2 u) = 0$$

with boundary condition $u|_{t=0} = \varphi$.

A.3 G-Brownian Motion

Before introducing the G-Brownian motion, we determine what is meant by a stochastic process.

Definition A.5. *Let $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ be a sublinear expectation space. $(X_t)_t$ is called stochastic process if X_t is a random variable in $\mathcal{H}$ for all $t \geq 0$.*

The definition of a G-Brownian motion is similar to the one of the standard Brownian motion with the difference that the increments are now G-normal distributed.

Definition A.6. *A stochastic process $(B_t)_t$ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a G-Brownian motion if*

(i) $B_0(\omega) = 0$,

(ii) *For each $t, s \geq 0$, the increment $B_{t+s} - B_t$ is $\mathcal{N}(\{0\}, s[\underline{\sigma}^2, \overline{\sigma}^2])$-distributed and is independent from $(B_{t_1}, ..., B_{t_n})$ for each $n \in \mathbb{N}$ and $0 \leq t_1 \leq ... \leq t_n \leq t$.*

In order to construct a G-Brownian motion, we set $\Omega := C_0(\mathbb{R}_+)$ and $\Omega_T := \{\omega_{\cdot \wedge T} : \omega \in \Omega\}$ for a fixed $T \in \mathbb{R}_+$. Moreover, we define the canonical process $B_t(\omega) := \omega_t$ and the spaces

$$L_{ip}(\Omega_T) := \{\varphi(B_{t_1 \wedge T}, ..., B_{t_n \wedge T}) : n \in \mathbb{N}, t_1, ..., t_n \in \mathbb{R}_+, \varphi \in C_{l,Lip}(\mathbb{R}^n)\}$$

and

$$L_{ip}(\Omega) := \bigcup_{n=1}^{\infty} L_{ip}(\Omega_n).$$

Now we construct a sublinear expectation on $(\Omega, L_{ip}(\Omega))$ such that the canonical process $(B_t)_t$ is a G-Brownian motion. Let $\{\xi_i\}_{i=1}^{\infty}$ be a sequence of $\mathcal{N}(\{0\}, \Sigma)$-distributed random variables on a sublinear expectation space $(\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{\mathbb{E}})$ such that ξ_{i+1} is independent from $(\xi_1, ..., \xi_i)$ for all $i \in \mathbb{N}$. Then we can introduce a sublinear expectation $\hat{\mathbb{E}}$ on $(\Omega, L_{ip}(\Omega))$ by the following procedure. For each $X \in L_{ip}(\Omega)$ with

$$X = \varphi(B_{t_1} - B_{t_0}, ..., B_{t_n} - B_{t_{n-1}})$$

for some $\varphi \in C_{l,Lip}(\mathbb{R})$ and $0 = t_0 < t_1 < ... < t_n < \infty$, we set

$$\hat{\mathbb{E}}[\varphi(B_{t_1} - B_{t_0}, ..., B_{t_n} - B_{t_{n-1}})] := \tilde{\mathbb{E}}[\varphi(\sqrt{t_1 - t_0}\xi_1, ..., \sqrt{t_n - t_{n-1}}\xi_n)].$$

Then we get that $\hat{\mathbb{E}}$ is a sublinear expectation on $(\Omega, L_{ip}(\Omega))$ and $(B_t)_t$ is a G-Brownian motion under $\hat{\mathbb{E}}$. In particular, $\hat{\mathbb{E}}$ is called G-expectation.

Furthermore, we denote by $L^p_G(\Omega)$ and $L^p_G(\Omega_T)$, for $p \geq 1$, the completions of $L_{ip}(\Omega)$ and $L_{ip}(\Omega_T)$ under the norm $||X||_p := \hat{\mathbb{E}}[|X|^p]^{\frac{1}{p}}$, respectively. According to Subsection A.1, the G-expectation $\hat{\mathbb{E}}$ can be continuously extended to a sublinear expectation on $(\Omega, L^p_G(\Omega))$ still denoted by $\hat{\mathbb{E}}$.

A.4 Stochastic Integrals

For the purpose of defining integrals with respect to a G-Brownian motion, we start by constructing the space of simple processes. For fixed $p \geq 1$ and $T \in \mathbb{R}_+$, we denote by $M^{p,0}_G(0,T)$ the collection of processes of the form

$$\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) 1_{[t_k, t_{k+1})}(t),$$

where $\xi_k \in L^p_G(\Omega_{t_k})$ and $0 = t_0 < t_1 < ... < t_N = T$. For $\eta \in M^{p,0}_G(0,T)$, the related Bochner integral is defined by

$$\int_0^T \eta_t(\omega) dt := \sum_{k=0}^{N-1} \xi_k(\omega)(t_{k+1} - t_k).$$

We denote by $M^p_G(0,T)$ the completion of $M^{p,0}_G(0,T)$ under the norm

$$||\eta||_{M^p_G(0,T)} := \hat{\mathbb{E}}\Big[\int_0^T |\eta_t|^p dt\Big]^{\frac{1}{p}}.$$

In particular, we can define the stochastic integral $\int_0^T \eta_t dB_t \in L^2_G(\Omega_T)$ for all $\eta \in M^2_G(0,T)$ and we have the following result.

Lemma A.1. *For $\eta \in M^2_G(0,T)$, we have*

$$\hat{\mathbb{E}}\Big[\int_0^T \eta_t dB_t\Big] = 0$$

and

$$\hat{\mathbb{E}}\Big[\Big|\int_0^T \eta_t dB_t\Big|^2\Big] \leq \overline{\sigma}^2 \hat{\mathbb{E}}\Big[\int_0^T |\eta_t|^2 dt\Big].$$

A.5 Quadratic Variation Process

First of all, we denote by π_N a sequence of partitions of $[0,t]$ such that

$$\lim_{N\to\infty} \max_{t_k\in\pi_N}\{|t_{k+1}-t_k|\}=0.$$

Then we define the quadratic variation of the G-Brownian motion by

$$\langle B\rangle_t := \lim_{N\to\infty} \sum_{t_k\in\pi_N} (B_{t_{k+1}}-B_{t_k})^2.$$

In fact, we know that the limit is well-defined, since we have the identity

$$\sum_{t_k\in\pi_N} (B_{t_{k+1}}-B_{t_k})^2 = B_t^2 - \sum_{t_k\in\pi_N} 2B_{t_k}(B_{t_{k+1}}-B_{t_k}),$$

where the sum on the right-hand side converges to $2\int_0^t B_s dB_s$ in $L_G^2(\Omega)$. The quadratic variation process is an increasing process with $\langle B\rangle_0=0$, which, in contrast to the classical case, is not deterministic but uncertain.

In addition, it is possible to define the integral $\int_0^T \eta_t d\langle B\rangle_t \in L_G^1(\Omega_T)$ for all $\eta\in M_G^1(0,T)$ and to show the following inequality.

Lemma A.2. *For $\eta\in M_G^1(0,T)$, we have*

$$\hat{\mathbb{E}}\Big[\Big|\int_0^T \eta_t d\langle B\rangle_t\Big|\Big] \leq \overline{\sigma}^2 \hat{\mathbb{E}}\Big[\int_0^T |\eta_t| dt\Big].$$

B Processes in $\tilde{M}_G^p(0,T)$ and Corresponding Properties

Let $(B_t)_t$ be a one-dimensional G-Brownian motion on the G-expectation space $(\Omega, L_G^1(\Omega), \hat{\mathbb{E}})$ and fix a terminal time τ, that is, we focus on $L_G^1(\Omega_\tau)\subset L_G^1(\Omega)$. Denote by $\tilde{M}_G^{p,0}(0,T)$, for $p\geq 1$ and $T\leq\tau$, the space of all simple processes, i.e., processes ϕ of the form

$$\phi(t,s)=\sum_{i=0}^{N-1}\varphi_t^i 1_{[s_i,s_{i+1})}(s)$$

for a partition $0=s_0<s_1<...<s_N=\tau$ and processes $\varphi^i\in M_G^p(0,T)$. Denote by $\tilde{M}_G^p(0,T)$ the completion of $\tilde{M}_G^{p,0}(0,T)$ under the norm $\|\cdot\|_{\tilde{M}_G^p(0,T)}$, given by

$$\|\phi\|^p_{\tilde{M}_G^p(0,T)} = \int_0^\tau \hat{\mathbb{E}}\Big[\int_0^T |\phi(t,s)|^p dt\Big] ds := \sum_{i=0}^{N-1}\hat{\mathbb{E}}\Big[\int_0^T |\varphi_t^i|^p dt\Big](s_{i+1}-s_i).$$

We clearly have $\tilde{M}_G^p(0,T)\subset\tilde{M}_G^q(0,T)$ for $p\geq q\geq 1$ and it is easy to show that $1_\Gamma\phi\in\tilde{M}_G^p(0,t_2)$ for $\phi\in\tilde{M}_G^p(0,T)$ and $\Gamma=[t_1,t_2]\times[s_1,s_2]$, where $0\leq t_1<t_2\leq T$ and $0\leq s_1<s_2\leq\tau$, which also implies that $\tilde{M}_G^p(0,T')\subset\tilde{M}_G^p(0,T)$ for $T\leq T'$. Apart from that, we can show the latter for more complex sets Γ.

Proposition B.1. *Let $\phi\in\tilde{M}_G^p(0,T)$ and $\Gamma=\{(t,s)\in[0,T]\times[0,\tau]: t\leq s\}$. Then we have that $1_\Gamma\phi\in\tilde{M}_G^p(0,T)$.*

Proof. Let $\phi \in \tilde{M}_G^p(0,T)$. Then there exists a sequence $\phi_n \in \tilde{M}_G^{p,0}(0,T)$ such that

$$\lim_{n\to\infty} \int_0^\tau \hat{\mathbb{E}}\Big[\int_0^T \big|\phi(t,s) - \phi_n(t,s)\big|^p dt\Big] ds = 0.$$

Hence, we get

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^\tau \hat{\mathbb{E}}\Big[\int_0^T \big|1_\Gamma(t,s)\phi(t,s) - 1_\Gamma(t,s)\phi_n(t,s)\big|^p dt\Big] ds \\
&= \lim_{n\to\infty} \int_0^\tau \hat{\mathbb{E}}\Big[\int_0^T 1_\Gamma(t,s)\big|\phi(t,s) - \phi_n(t,s)\big|^p dt\Big] ds \\
&\leq \lim_{n\to\infty} \int_0^\tau \hat{\mathbb{E}}\Big[\int_0^T \big|\phi(t,s) - \phi_n(t,s)\big|^p dt\Big] ds = 0.
\end{aligned}$$

However, it is not obvious that $1_\Gamma\phi_n \in \tilde{M}_G^p(0,T)$, since $1_\Gamma\phi_n$ is not a simple process. Therefore, the rest of the proof is dedicated to show that $1_\Gamma\phi \in \tilde{M}_G^p(0,T)$ if $\phi \in \tilde{M}_G^{p,0}(0,T)$.

Let $\phi \in \tilde{M}_G^{p,0}(0,T)$, that is,

$$\phi(t,s) = \sum_{i=0}^{N-1} \varphi_t^i 1_{[s_i,s_{i+1})}(s)$$

for a partition $0 = s_0 < s_1 < ... < s_N = \tau$ and processes $\varphi^i \in M_G^p(0,T)$. First of all, we consider the case where ϕ is bounded, i.e., φ^i is a bounded process for all i. For each i we construct a sequence of partitions $\{s_{ij}^n\}_{j=0}^n$ such that $s_i = s_{i0}^n < s_{i1}^n < ... < s_{in}^n = s_{i+1}$ and

$$\lim_{n\to\infty} \max_{j=0,1,\dots,n-1} \{|s_{ij+1}^n - s_{ij}^n|\} = 0.$$

Now we define ϕ_n by

$$\phi_n(t,s) := \sum_{i=0}^{N-1}\sum_{j=0}^{n-1} \varphi_t^i 1_{[0,s_{ij}^n]}(t) 1_{[s_{ij}^n,s_{ij+1}^n)}(s).$$

Then we get $\phi_n \in \tilde{M}_G^{p,0}(0,T)$, since $\{s_{ij}^n\}_{i,j}$ is again a partition of $[0,\tau]$ and $\varphi^i 1_{[0,s_{ij}^n]} \in M_G^p(0,T)$ for all i and j. Apart from that, we have

$$\begin{aligned}
1_\Gamma(t,s)\phi(t,s) - \phi_n(t,s) =& 1_{[0,s]}(t) \sum_{i=0}^{N-1} \varphi_t^i 1_{[s_i,s_{i+1})}(s) - \sum_{i=0}^{N-1}\sum_{j=0}^{n-1} \varphi_t^i 1_{[0,s_{ij}^n]}(t) 1_{[s_{ij}^n,s_{ij+1}^n)}(s) \\
=& \sum_{i=0}^{N-1}\sum_{j=0}^{n-1} \varphi_t^i 1_{[0,s]}(t) 1_{[s_{ij}^n,s_{ij+1}^n)}(s) - \sum_{i=0}^{N-1}\sum_{j=0}^{n-1} \varphi_t^i 1_{[0,s_{ij}^n]}(t) 1_{[s_{ij}^n,s_{ij+1}^n)}(s) \\
=& \sum_{i=0}^{N-1}\sum_{j=0}^{n-1} \varphi_t^i 1_{(s_{ij}^n,s]}(t) 1_{[s_{ij}^n,s_{ij+1}^n)}(s).
\end{aligned}$$

By using the boundedness of φ^i, we obtain

$$\begin{aligned}
\big|1_\Gamma(t,s)\phi(t,s) - \phi_n(t,s)\big|^p =& \sum_{i=0}^{N-1}\sum_{j=0}^{n-1} |\varphi_t^i|^p 1_{(s_{ij}^n,s]}(t) 1_{[s_{ij}^n,s_{ij+1}^n)}(s) \\
\leq& C \sum_{i=0}^{N-1}\sum_{j=0}^{n-1} 1_{[s_{ij}^n,s_{ij+1}^n]}(t) 1_{[s_{ij}^n,s_{ij+1}^n)}(s)
\end{aligned}$$

for some constant $C > 0$. After all, this yields

$$\lim_{n\to\infty}\int_0^\tau \hat{\mathbb{E}}\Big[\int_0^T \big|1_\Gamma(t,s)\phi(t,s)-\phi_n(t,s)\big|^p dt\Big]ds$$
$$\le \lim_{n\to\infty} C\sum_{i=0}^{N-1}\sum_{j=0}^{n-1}(s^n_{ij+1}-s^n_{ij})^2$$
$$\le \lim_{n\to\infty} C\sum_{i=0}^{N-1}\max_{j=0,1,\dots,n-1}\{|s^n_{ij+1}-s^n_{ij}|\}(s_{i+1}-s_i)=0,$$

since the maximum converges to 0 for all i. Thus, we get $1_\Gamma\phi\in\tilde{M}^p_G(0,T)$.

Now we consider the case where the processes φ^i are not bounded. For $n\in\mathbb{N}$, we define

$$\varphi^{i,n} := (\varphi^i \vee -n)\wedge n$$

for $i = 0, 1, ..., N-1$ and

$$\phi_n(t,s) := \sum_{i=0}^{N-1}\varphi^{i,n}_t 1_{[s_i,s_{i+1})}(s).$$

Then we know that $1_\Gamma\phi_n\in\tilde{M}^p_G(0,T)$, since ϕ_n is bounded, and

$$\lim_{n\to\infty}\int_0^\tau \hat{\mathbb{E}}\Big[\int_0^T \big|1_\Gamma(t,s)\phi(t,s)-1_\Gamma(t,s)\phi_n(t,s)\big|^p dt\Big]ds$$
$$\le \lim_{n\to\infty}\int_0^\tau \hat{\mathbb{E}}\Big[\int_0^T \big|\phi(t,s)-\phi_n(t,s)\big|^p dt\Big]ds$$
$$= \lim_{n\to\infty}\sum_{i=0}^{N-1}\hat{\mathbb{E}}\Big[\int_0^T |\varphi^i_t-\varphi^{i,n}_t|^p dt\Big](s_{i+1}-s_i)$$
$$= \lim_{n\to\infty}\sum_{i=0}^{N-1}\hat{\mathbb{E}}\Big[\int_0^T \big|\varphi^i_t 1_{\{|\varphi^i|>n\}}(t)\big|^p dt\Big](s_{i+1}-s_i)=0,$$

since each of the summands converges to 0 by the characterization of the space $M^p_G(0,T)$ from Hu, Wang, and Zheng (2016). So we have $1_\Gamma\phi\in\tilde{M}^p_G(0,T)$, which completes the proof. □

B.1 Construction of Stochastic Integrals

On the space of simple processes we can define the integral operator $I:\tilde{M}^{p,0}_G(0,T)\to M^p_G(0,T)$ by

$$I(\phi)=\int_0^\tau \phi(\cdot,s)ds := \sum_{i=0}^{N-1}\varphi^i(s_{i+1}-s_i),$$

which is clearly linear. Additionally, we have the following inequality.

Lemma B.1. *Let $\phi\in\tilde{M}^{p,0}_G(0,T)$. Then it holds*

$$\hat{\mathbb{E}}\Big[\int_0^T \Big|\int_0^\tau \phi(t,s)ds\Big|^p dt\Big] \le \tau^{p-1}\int_0^\tau \hat{\mathbb{E}}\Big[\int_0^T \big|\phi(t,s)\big|^p dt\Big]ds.$$

Proof. By applying Hölder's inequality, we get

$$\hat{\mathbb{E}}\Big[\int_0^T \big|\int_0^\tau \phi(t,s)ds\big|^p dt\Big] \leq \tau^{p-1}\hat{\mathbb{E}}\Big[\int_0^T \int_0^\tau \big|\phi(t,s)\big|^p dsdt\Big]$$
$$\leq \tau^{p-1}\int_0^\tau \hat{\mathbb{E}}\Big[\int_0^T \big|\phi(t,s)\big|^p dt\Big]ds,$$

where the last inequality follows from the sublinearity of $\hat{\mathbb{E}}$. □

The inequality implies that the operator I is continuous. Thus, I can be continuously extended to $I : \tilde{M}_G^p(0,T) \to M_G^p(0,T)$ and we know that Lemma B.1 also holds for $\phi \in \tilde{M}_G^p(0,T)$. In particular, this enables us to define the integral

$$\int_0^T \int_0^\tau \phi(t,s)dsdB_t$$

for $\phi \in \tilde{M}_G^2(0,T)$ and the integrals

$$\int_0^T \int_0^\tau \phi(t,s)dsd\langle B\rangle_t \quad \text{and} \quad \int_0^T \int_0^\tau \phi(t,s)dsdt.$$

for $\phi \in \tilde{M}_G^1(0,T)$.

For simple processes we can also define the linear integral operator $J : \tilde{M}_G^{2,0}(0,T) \to L_G^2(\Omega_T)$ by

$$J(\phi) = \int_0^\tau \int_0^T \phi(t,s)dB_t ds := \sum_{i=0}^{N-1}\Big(\int_0^T \varphi_t^i dB_t\Big)(s_{i+1}-s_i).$$

Similar to the construction of the stochastic integral in Peng (2010), we have the following properties. The isometry is of particular interest, since it allows us to extend the operator.

Lemma B.2. *Let $\phi \in \tilde{M}_G^{2,0}(0,T)$. Then it holds*

$$\hat{\mathbb{E}}\Big[\int_0^\tau \int_0^T \phi(t,s)dB_t ds\Big] = 0.$$

and we have the estimate

$$\hat{\mathbb{E}}\Big[\big|\int_0^\tau \int_0^T \phi(t,s)dB_t ds\big|^2\Big] \leq \overline{\sigma}^2\tau\int_0^\tau \hat{\mathbb{E}}\Big[\int_0^T \big|\phi(t,s)\big|^2 dt\Big]ds.$$

Proof. Let $\phi \in \tilde{M}_G^{2,0}(0,T)$. Then we can interchange the order of integration for simple processes, since the integral with respect to a G-Brownian motion is linear, and we know that $\int_0^\tau \phi(\cdot,s)ds \in M_G^2(0,T)$. Thus, we get

$$\hat{\mathbb{E}}\Big[\int_0^\tau \int_0^T \phi(t,s)dB_t ds\Big] = \hat{\mathbb{E}}\Big[\int_0^T \int_0^\tau \phi(t,s)dsdB_t\Big] = 0.$$

By Hölder's inequality and the sublinearity of $\hat{\mathbb{E}}$, we additionally get

$$\hat{\mathbb{E}}\Big[\big|\int_0^\tau \int_0^T \phi(t,s)dB_t ds\big|^2\Big] \leq \tau\int_0^\tau \hat{\mathbb{E}}\Big[\big|\int_0^T \phi(t,s)dB_t\big|^2\Big]ds$$
$$\leq \overline{\sigma}^2\tau\int_0^\tau \hat{\mathbb{E}}\Big[\int_0^T \big|\phi(t,s)\big|^2 dt\Big]ds,$$

where the last step uses the classical isometry from Peng (2010).[4] □

[4] See Lemma A.1.

Hence, the operator J can be continuously extended to $J : \tilde{M}_G^2(0,T) \to L_G^2(\Omega_T)$ and Lemma B.2 still holds for $\phi \in \tilde{M}_G^2(0,T)$.

In a similar fashion, we define $J' : \tilde{M}_G^{1,0}(0,T) \to L_G^1(\Omega_T)$ and $J'' : \tilde{M}_G^{1,0}(0,T) \to L_G^1(\Omega_T)$ by

$$J'(\phi) = \int_0^\tau \int_0^T \phi(t,s) d\langle B\rangle_t ds := \sum_{i=0}^{N-1} \Big(\int_0^T \varphi_t^i d\langle B\rangle_t \Big)(s_{i+1} - s_i)$$

and

$$J''(\phi) = \int_0^\tau \int_0^T \phi(t,s) dt ds := \sum_{i=0}^{N-1} \Big(\int_0^T \varphi_t^i dt \Big)(s_{i+1} - s_i),$$

for which we can show the following inequalities.

Lemma B.3. *Let $\phi \in \tilde{M}_G^{1,0}(0,T)$. Then it holds*

$$\hat{\mathbb{E}}\Big[\big| \int_0^\tau \int_0^T \phi(t,s) d\langle B\rangle_t ds \big|\Big] \leq \overline{\sigma}^2 \int_0^\tau \hat{\mathbb{E}}\Big[\int_0^T \big|\phi(t,s)\big| dt \Big] ds$$

and

$$\hat{\mathbb{E}}\Big[\big| \int_0^\tau \int_0^T \phi(t,s) dt ds \big|\Big] \leq \int_0^\tau \hat{\mathbb{E}}\Big[\int_0^T \big|\phi(t,s)\big| dt \Big] ds.$$

Proof. Let $\phi \in \tilde{M}_G^{1,0}(0,T)$. As in the previous proof, we use the sublinearity of $\hat{\mathbb{E}}$ and a standard result from Peng (2010)[5] to get

$$\begin{aligned} \hat{\mathbb{E}}\Big[\big| \int_0^\tau \int_0^T \phi(t,s) d\langle B\rangle_t ds \big|\Big] &\leq \int_0^\tau \hat{\mathbb{E}}\Big[\big| \int_0^T \phi(t,s) d\langle B\rangle_t \big|\Big] ds \\ &\leq \overline{\sigma}^2 \int_0^\tau \hat{\mathbb{E}}\Big[\int_0^T \big|\phi(t,s)\big| dt \Big] ds. \end{aligned}$$

The second assertion can be deduced by just using the sublinearity of $\hat{\mathbb{E}}$. □

Therefore, the operators J' and J'' can be continuously extended to $J' : \tilde{M}_G^1(0,T) \to L_G^1(\Omega_T)$ and $J'' : \tilde{M}_G^1(0,T) \to L_G^1(\Omega_T)$ and we still have Lemma B.3 for $\phi \in \tilde{M}_G^1(0,T)$.

B.2 Fubini's Theorem

Apart from defining integrals for processes in $\tilde{M}_G^p(0,T)$, we can also show the following version of Fubini's theorem.

Theorem B.1. *Let $\phi \in \tilde{M}_G^2(0,T)$. Then it holds*

$$\int_0^\tau \int_0^T \phi(t,s) dB_t ds = \int_0^T \int_0^\tau \phi(t,s) ds dB_t.$$

Proof. Let $\phi \in \tilde{M}_G^2(0,T)$. Then there exists a sequence $\phi_n \in \tilde{M}_G^{2,0}(0,T)$ such that

$$\lim_{n\to\infty} \int_0^\tau \hat{\mathbb{E}}\Big[\int_0^T \big|\phi(t,s) - \phi_n(t,s)\big|^2 dt \Big] ds = 0.$$

[5] See Lemma A.2.

It clearly holds

$$\int_0^\tau \int_0^T \phi_n(t,s)dB_t ds = \int_0^T \int_0^\tau \phi_n(t,s)ds dB_t,$$

since ϕ_n is a simple process. Additionally, we can apply Lemma B.2 to get

$$\begin{aligned}
&\lim_{n\to\infty} \hat{\mathbb{E}}\Big[\Big|\int_0^\tau \int_0^T \phi(t,s)dB_t ds - \int_0^\tau \int_0^T \phi_n(t,s)dB_t ds\Big|^2\Big] \\
&= \lim_{n\to\infty} \hat{\mathbb{E}}\Big[\Big|\int_0^\tau \int_0^T \big(\phi(t,s)-\phi_n(t,s)\big)dB_t ds\Big|^2\Big] \\
&\leq \lim_{n\to\infty} \overline{\sigma}^2\tau \int_0^\tau \hat{\mathbb{E}}\Big[\int_0^T \big|\phi(t,s)-\phi_n(t,s)\big|^2 dt\Big]ds = 0
\end{aligned}$$

and, by the isometry property from Peng (2010) and Lemma B.1, it holds

$$\begin{aligned}
&\lim_{n\to\infty} \hat{\mathbb{E}}\Big[\Big|\int_0^T \int_0^\tau \phi(t,s)ds dB_t - \int_0^T \int_0^\tau \phi_n(t,s)ds dB_t\Big|^2\Big] \\
&= \lim_{n\to\infty} \hat{\mathbb{E}}\Big[\Big|\int_0^T \int_0^\tau \big(\phi(t,s)-\phi_n(t,s)\big)ds dB_t\Big|^2\Big] \\
&\leq \lim_{n\to\infty} \overline{\sigma}^2\tau \int_0^\tau \hat{\mathbb{E}}\Big[\int_0^T \big|\phi(t,s)-\phi_n(t,s)\big|^2 dt\Big]ds = 0.
\end{aligned}$$

Hence, we have

$$\int_0^\tau \int_0^T \phi(t,s)dB_t ds = \int_0^T \int_0^\tau \phi(t,s)ds dB_t,$$

as desired. □

Moreover, we can prove the same statement for the other integrals we defined.

Theorem B.2. *Let $\phi \in \tilde{M}_G^1(0,T)$. Then it holds*

$$\int_0^\tau \int_0^T \phi(t,s)d\langle B\rangle_t ds = \int_0^T \int_0^\tau \phi(t,s)ds d\langle B\rangle_t$$

and

$$\int_0^\tau \int_0^T \phi(t,s)dt ds = \int_0^T \int_0^\tau \phi(t,s)ds dt.$$

Proof. Let $\phi \in \tilde{M}_G^1(0,T)$. Then there exists a sequence $\phi_n \in \tilde{M}_G^{1,0}(0,T)$ such that

$$\lim_{n\to\infty} \int_0^\tau \hat{\mathbb{E}}\Big[\int_0^T \big|\phi(t,s)-\phi_n(t,s)\big| dt\Big]ds = 0.$$

For the simple processes ϕ_n it clearly holds

$$\int_0^\tau \int_0^T \phi_n(t,s)d\langle B\rangle_t ds = \int_0^T \int_0^\tau \phi_n(t,s)ds d\langle B\rangle_t$$

and

$$\int_0^\tau \int_0^T \phi_n(t,s)dt ds = \int_0^T \int_0^\tau \phi_n(t,s)ds dt.$$

By applying Lemma B.3, we have

$$\begin{aligned}
&\lim_{n\to\infty}\hat{\mathbb{E}}\Big[\Big|\int_0^\tau\int_0^T \phi(t,s)d\langle B\rangle_t ds-\int_0^\tau\int_0^T \phi_n(t,s)d\langle B\rangle_t ds\Big|\Big]\\
&=\lim_{n\to\infty}\hat{\mathbb{E}}\Big[\Big|\int_0^\tau\int_0^T \big(\phi(t,s)-\phi_n(t,s)\big)d\langle B\rangle_t ds\Big|\Big]\\
&\leq \lim_{n\to\infty}\overline{\sigma}^2\int_0^\tau \hat{\mathbb{E}}\Big[\int_0^T \big|\phi(t,s)-\phi_n(t,s)\big|dt\Big]ds=0
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{n\to\infty}\hat{\mathbb{E}}\Big[\Big|\int_0^\tau\int_0^T \phi(t,s)dtds-\int_0^\tau\int_0^T \phi_n(t,s)dtds\Big|\Big]\\
&=\lim_{n\to\infty}\hat{\mathbb{E}}\Big[\Big|\int_0^\tau\int_0^T \big(\phi(t,s)-\phi_n(t,s)\big)dtds\Big|\Big]\\
&\leq \lim_{n\to\infty}\int_0^\tau \hat{\mathbb{E}}\Big[\int_0^T \big|\phi(t,s)-\phi_n(t,s)\big|dt\Big]ds=0.
\end{aligned}$$

Similar to the proof of Theorem B.1, we can use the inequality from Lemma B.1 to get

$$\begin{aligned}
&\lim_{n\to\infty}\hat{\mathbb{E}}\Big[\Big|\int_0^T\int_0^\tau \phi(t,s)dsd\langle B\rangle_t-\int_0^T\int_0^\tau \phi_n(t,s)dsd\langle B\rangle_t\Big|\Big]\\
&=\lim_{n\to\infty}\hat{\mathbb{E}}\Big[\Big|\int_0^T\int_0^\tau \big(\phi(t,s)-\phi_n(t,s)\big)dsd\langle B\rangle_t\Big|\Big]\\
&\leq \lim_{n\to\infty}\overline{\sigma}^2\int_0^\tau \hat{\mathbb{E}}\Big[\int_0^T \big|\phi(t,s)-\phi_n(t,s)\big|dt\Big]ds=0
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{n\to\infty}\hat{\mathbb{E}}\Big[\Big|\int_0^T\int_0^\tau \phi(t,s)dsdt-\int_0^T\int_0^\tau \phi_n(t,s)dsdt\Big|\Big]\\
&=\lim_{n\to\infty}\hat{\mathbb{E}}\Big[\Big|\int_0^T\int_0^\tau \big(\phi(t,s)-\phi_n(t,s)\big)dsdt\Big|\Big]\\
&\leq \lim_{n\to\infty}\int_0^\tau \hat{\mathbb{E}}\Big[\int_0^T \big|\phi(t,s)-\phi_n(t,s)\big|dt\Big]ds=0,
\end{aligned}$$

which completes the proof. □

B.3 Relation to the Space $M_G^p(0,T)$

It can be shown that processes in $\tilde{M}_G^p(0,T)$ belong to the space $M_G^p(0,T)$ with respect to the first variable in a certain sense. The following lemma is essential to show this relation. We denote by $\mathcal{L}^p(S)$ for $S\subset\mathbb{R}^d$ and $d\in\mathbb{N}$ the space of all $\mathcal{B}(S)$-measurable functions $f:S\to\mathbb{R}$ such that $|f|^p$ is integrable with respect to the Lebesgue measure on S.

Lemma B.4. *Let* $\phi\in\tilde{M}_G^p(0,T)$*. Then* $\hat{\mathbb{E}}[\int_0^T|\phi(t,\cdot)|^p dt]\in\mathcal{L}^1([0,\tau])$*.*

Proof. Let $\phi\in\tilde{M}_G^{p,0}(0,T)$. Then

$$\hat{\mathbb{E}}\Big[\int_0^T\big|\phi(t,s)\big|^p dt\Big]=\sum_{i=0}^{N-1}\hat{\mathbb{E}}\Big[\int_0^T|\varphi_t^i|^p dt\Big]1_{[s_i,s_{i+1})}(s),$$

where $\hat{\mathbb{E}}[\int_0^T |\varphi_t^i|^p dt] < \infty$ for all i. Hence, $\hat{\mathbb{E}}[\int_0^T |\phi(t,\cdot)|^p dt] \in \mathcal{L}^1([0,\tau])$. Now let $\phi \in \tilde{M}_G^p(0,T)$. Thus, there exists a sequence $\phi_n \in \tilde{M}_G^{p,0}(0,T)$ such that

$$\lim_{n\to\infty} \int_0^\tau \hat{\mathbb{E}}\Big[\int_0^T |\phi(t,s) - \phi_n(t,s)|^p dt\Big] ds = 0.$$

Furthermore, we know that $\hat{\mathbb{E}}[\int_0^T |\phi_n(t,\cdot)|^p dt] \in \mathcal{L}^1([0,\tau])$ for all n and

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^\tau \Big|\hat{\mathbb{E}}\Big[\int_0^T |\phi(t,s)|^p dt\Big] - \hat{\mathbb{E}}\Big[\int_0^T |\phi_n(t,s)|^p dt\Big]\Big| ds \\
&\leq \lim_{n\to\infty} \int_0^\tau \hat{\mathbb{E}}\Big[\int_0^T |\phi(t,s) - \phi_n(t,s)|^p dt\Big] ds = 0.
\end{aligned}$$

This means that $\hat{\mathbb{E}}[\int_0^T |\phi(t,\cdot)|^p dt] \in \mathcal{L}^1([0,\tau])$ as well, since the space $\mathcal{L}^1([0,\tau])$ is complete. □

Now, since we have the above integrability, we can use a standard argument to show the succeeding proposition.

Proposition B.2. *Let $\phi \in \tilde{M}_G^p(0,T)$. Then $\phi(\cdot,s) \in M_G^p(0,T)$ for almost every $s \in [0,\tau]$.*

Proof. Let $\phi \in \tilde{M}_G^p(0,T)$. Thus, there exists a sequence $\phi_n \in \tilde{M}_G^{p,0}(0,T)$ such that

$$\lim_{n\to\infty} \int_0^\tau \hat{\mathbb{E}}\Big[\int_0^T |\phi(t,s) - \phi_n(t,s)|^p dt\Big] ds = 0.$$

Then we can find a subsequence n_k such that

$$\int_0^\tau \hat{\mathbb{E}}\Big[\int_0^T |\phi(t,s) - \phi_{n_k}(t,s)|^p dt\Big] ds \leq 2^{-k}.$$

Since $\hat{\mathbb{E}}[\int_0^T |\phi(t,\cdot) - \phi_{n_k}(t,\cdot)|^p dt]$ is measurable by Lemma B.4, we can use the monotone convergence theorem to obtain

$$\begin{aligned}
\int_0^\tau \sum_{k=1}^\infty \hat{\mathbb{E}}\Big[\int_0^T |\phi(t,s) - \phi_{n_k}(t,s)|^p dt\Big] ds &= \sum_{k=1}^\infty \int_0^\tau \hat{\mathbb{E}}\Big[\int_0^T |\phi(t,s) - \phi_{n_k}(t,s)|^p dt\Big] ds \\
&\leq \sum_{k=1}^\infty 2^{-k} = 1 < \infty,
\end{aligned}$$

where $\sum_{k=1}^\infty \hat{\mathbb{E}}[\int_0^T |\phi(t,\cdot) - \phi_{n_k}(t,\cdot)|^p dt]$ is again measurable. In particular, this implies

$$\sum_{k=1}^\infty \hat{\mathbb{E}}\Big[\int_0^T |\phi(t,s) - \phi_{n_k}(t,s)|^p dt\Big] < \infty$$

for almost every s, which means that

$$\lim_{k\to\infty} \hat{\mathbb{E}}\Big[\int_0^T |\phi(t,s) - \phi_{n_k}(t,s)|^p dt\Big] = 0$$

for almost every s. Since $\phi_{n_k}(\cdot,s) \in M_G^p(0,T)$ for a fixed s and $M_G^p(0,T)$ is complete, we get $\phi(\cdot,s) \in M_G^p(0,T)$ for almost every s. □

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