DEFAULT AMBIGUITY

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ABSTRACT. This paper discusses ambiguity in the context of single-name credit risk. We focus on uncertainty on the default intensity but also discuss uncertainty on the recovery in a fractional recovery of the market value. This approach is a first step towards integrating uncertainty in credit risky term structure models and can profit from its simplicity. We derive drift conditions in a Heath-Jarrow-Morton forward rate setting in the case of ambiguous default intensity in combination with zero recovery, and in the case of ambiguous fractional recovery of the market value.
Keywords: Model ambiguity, default time, credit risk, no-arbitrage, reduced-form HJM models, recovery process.

1. INTRODUCTION

Recently, an increasing amount of literature focuses on uncertainty as it relates to financial markets. The problem is that, the probability distribution of randomness in these markets is unknown. Typically, the unknown distribution is either estimated by statistical methods or calibrated to given market data by means of a model for the financial market. For example, in credit risk, the default probability is not observed, hence, have to be estimated from observable data. These methods introduce a large *model risk*.

Already, [21] pointed towards a formulation of risk which is able to treat such challenges in a systematic way. He was followed by [14], who called random variables with known probability distribution *certain*, and those where the probability distribution is not known as *uncertain*. In this paper, we address these problems by constructing a model such that the parameters are characterised by uncertainty. Then, a single probability measure in a classical model, is replaced by a family of probability measures, that is, a full class of models.

Following the modern literature in the area, we will call the feature that the probability distribution is not entirely fixed, *ambiguity*. This area has recently renewed the attention of researchers in mathematical finance to fundamental subjects such as arbitrage conditions, pricing mechanisms, and super-hedging, see for example, [4, 8, 27], just to mention a few.

Roughly speaking, ambiguity focuses on a set of probability measures whose role is to determine events that are relevant and those that are negligible. In this paper, we introduce the concept of ambiguity to *defaultable term structure models*. The starting point for term structure models are typically bond prices of the form

$$P(t,T) = e^{-\int_t^T f(t,u)du}, \qquad 0 \le t \le T, \tag{1}$$

Financial support by Carl-Zeiss-Stiftung and German Research Foundation (DFG) via CRC 1283 is gratefully acknowledged. We also thank the Freiburg Institute of Advanced Studies (FRIAS) for its hospitality and financial support.

where $(f(t,T))_{0\leq t\leq T}$ is the instantaneous forward rate and T is the maturity time. This follows the seminal approach proposed in [18]. The presence of credit risk[1] in the model introduces an additional factor known as the default time. In this setting, bond prices are assumed to be absolutely continuous with respect to the maturity of the bond. This assumption is typically justified by the argument that, in practice, only a finite number of bonds are liquidly traded and the full term structure is obtained by interpolation, thus is smooth. There are two classical approaches to model market default risk: the *structural approach* [23] and the *reduced-form approach* (see for example, [2, 12, 22] for some of the first works in this direction).

In structural models of credit risk, the underlying state is the value of a firm's assets which is observable. Default happens at maturity time of the issued bond if the firm value is not sufficient to cover the liabilities. Hence, default is not a surprise. One exception is the structural model of [28], in which the value of the firm's assets is allowed to jump. In fact, the value of the firm's assets is not observable. A credit event usually occurs in correspondence of a missed payment by a corporate entity and, in many cases, the payment dates or coupon dates are publicly known in advance. For example, the missed coupon payments by Argentina on a notional of \$29 billion (on July $30, 2014$), and by Greece on a notional of €1.5 billion (on June $30, 2015$).

Reduced-form (HJM-type) models for defaultable term structure generally assume the existence of a default intensity which implies that default occurs with probability zero at a predictable time. Consequently, reduced-form models typically postulate that default time is totally inaccessible and prior to default, bond prices are absolutely continuous with respect to the maturity. That is, under the assumption of zero *recovery*[2], credit risky bond prices $P(t,T)$ is given by

$$P(t,T) = \mathbb{1}_{\{\tau>t\}} e^{-\int_t^T f(t,u)du} \tag{2}$$

with τ denoting the *random default time*. This approach has been studied in numerous works and up to a great level of generality, see [13, Chapter 3], for an overview of relevant literature. The random default time τ is assumed to have an intensity process λ. For example, with a constant intensity λ, default has a Poisson arrival at intensity λ . More generally, for $\tau > t$, λ_t may be viewed as the conditional rate of arrival of default at time t, given information up to that time. In a situation where the owner of a defaultable claim recovers part of its initial investment upon default, the associated survival process $\mathbb{1}_{\{\tau>t\}}$ in (2), is replaced by a semimartingale.

Under ambiguity, we suggest that there is some a prior information at hand which gives a upper and lower bounds on the *intensity*. The implicit assumption that the probability distribution of default is known is quite restrictive. Thus, we analyse our problem in a multiple priors model which describe uncertainty about the "true probability distribution". By means of the Girsanov theorem, we construct the set of priors from the reference measure. The assumption is that all priors are equivalent.

In view of our framework, it is only important to acknowledge that a rating class provides an estimate of the one-year default probability in terms of a confidence

[1]The risk that an agent fails to fulfil contractual obligations. Example of an instrument bearing credit risk is a corporate bond.

[2]The amount that the owner of a defaulted claim receives upon default.

interval. Also estimates for 3-, and 5-year default probabilities can be obtained from the rating migration matrix. Thus, leading to a certain amount of model risk.

The aim of this paper is to incorporate uncertainty into the context of single-name credit risk. We focus on uncertainty on the default intensity, and also discuss uncertainty on the recovery.

The main results are as follows: we obtain a necessary and sufficient condition for a reference probability measure to be a local martingale measure for credit risky bond markets under default uncertainty, thereby ensuring the absence of arbitrage in a sense to be precisely specified below. Furthermore, we consider the case where we have partial information on the amount that the owner of a defaulted claim receives upon default.

This paper is set up as follows: the next section introduces homogeneous ambiguity, and its example. Section 3 introduces the fundamental theorem of asset pricing (FTAP) under homogeneous ambiguity. In section 4, we derive the no-arbitrage conditions for defaultable term structure models with zero-recovery, and fractional recovery of market value, in our framework.

2. Intensity-based models

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Intensity-based models are the most used model class for modeling credit risk, see [5, Chapter 8] for an overview of relevant literature. The default intensity, however, is difficult to estimate and therefore naturally carries a lot of uncertainty. This has led to the emergence of rating agencies which since the early 20th century estimate bond's credit worthiness[3].

Modeling of credit risk has up to now incorporated uncertainty on the default intensity in a systematic way. On the other side, a number of Bayesian approaches exist, utilizing filtering technologies, see for example [11, 17], among many others.

Here, we introduce an alternative treatment of the lack of precise knowledge of the default intensity based on the concept of *ambiguity* following the seminal ideas from Frank Knight in [21].

Uncertainty in our setting will be captured through a family of probability measures $\mathscr{P}$ replacing the single probability measure $\mathbb{P}$ in classical approaches. Intuitively, each $\mathbb{P}$ represents a model and the family $\mathscr{P}$ collects models which we consider equally likely.

In this spirit, working with a single $\mathbb{P}$, or with a set $\mathscr{P} = \{\mathbb{P}\}$ which contains only one element, is in a one-to-one correspondence to assuming that the parameters of the underlying processes are exactly known. In financial markets, this is certainly not the case and ambiguity helps to incorporate this uncertainty into the used models.

We consider throughout a fixed finite time horizon $T^* > 0$. In light of our discussion above, let $(\Omega, \mathscr{F})$ be a measurable space and $\mathscr{P}$ a set of probability measures on the measurable space $(\Omega, \mathscr{F})$. In particular there is no fixed and known measure $\mathbb{P}$ (except in the special case where $\mathscr{P}$ contains only one element which we treat en passant).

Intensity based default models correspond to the case where the ambiguity is *homogeneous*, i.e. there is a measure $\mathbb{P}'$ such that $\mathbb{P} \sim \mathbb{P}'$ for all $\mathbb{P} \in \mathscr{P}$. Here,

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[3]For a historical account, see [26]: John Moody founded the first rating agency in 1909, in the United States.

$\mathbb{P} \sim \mathbb{P}'$ means that $\mathbb{P}$ and $\mathbb{P}'$ are *equivalent*, that is, they have the same nullsets. The reference measure $\mathbb{P}'$ has only the role of fixing events of measure zero for all probability measures under consideration. Intuitively, this means there is no ambiguity on these events of measure zero. In the following, We write E' for the expectation with respect to the reference measure $\mathbb{P}'$.

Remark 1. As a consequence of the equivalence of all probability measures in $\mathscr{P}$, all equalities and inequalities will hold almost-surely with respect to any probability measure $\mathbb{P} \in \mathscr{P}$, or, respectively, to $\mathbb{P}'$.

Ambiguity in intensity-based models. In this section, we introduce ambiguity in intensity-based models. Our goal is not the most general approach in this setting: we rather focus on simpler, but still practically highly relevant cases. For a more general treatment we refer to [3]. The main mathematical tool we use here is enlargement of filtrations and we refer to [1] for further details and a guide to the literature.

Assume that under $\mathbb{P}'$ we have a d-dimensional Brownian motion W with canonical and augmented[4] filtration $\mathbb{F} = (\mathscr{F}_t)_{0 \le t \le T^*}$ and a standard exponential random variable τ, independent of $\mathscr{F}_{T^*}$, that is, $\mathbb{P}'(t < \tau | \mathscr{F}_t) = \exp(-t)$, $0 \le t \le T^*$. The Brownian motion W has the role of modelling market movements and general information, excluding default information. We therefore call $\mathbb{F}$ the *market filtration* in the following. The filtration $\mathbb{G} = (\mathscr{G}_t)_{0 \le t \le T^*}$ includes default information and is obtained by a progressive enlargement of $\mathbb{F}$ with τ, i.e.,

$$\mathscr{G}_t = \bigcap_{\epsilon > 0} \sigma(\mathbb{1}_{\{t \ge \tau\}}, W_s : 0 \le s \le t + \epsilon), \qquad 0 \le t \le T^*.$$

To finalize our setup, we assume that $\mathscr{F} = \mathscr{G}_{T^*}$.

Note that up to know, everything has been specified under the reference measure $\mathbb{P}'$ and nothing was said about the concrete models we are interested in (except about the nullsets). These models will now be introduced using the Girsanov theorem, i.e., by changing from $\mathbb{P}'$ to the measures we are interested in.

Consequently, the next step is to construct measures $\mathbb{P}^\lambda$ with appropriate processes λ – under $\mathbb{P}^\lambda$, the default time τ will have the intensity λ. More precisely, assume that λ is some positive process which is predictable with respect to the market filtration, $\mathbb{F}$. Define the density process Z^λ by

$$Z_t^\lambda := \begin{cases} \exp\Big(\int_0^t (1 - \lambda_s) ds\Big), & t < \tau \\ \lambda_\tau \exp\Big(\int_0^\tau (1 - \lambda_s) ds\Big) & t \ge \tau. \end{cases} \tag{3}$$

Note that Z^λ is a $\mathbb{G}$-local martingale and corresponds to a Girsanov-type change of measure (see Theorem VI.2.2 in [7]). If $E'[Z^\lambda_{T^*}] = 1$ we obtain an equivalent measure $\mathbb{P}^\lambda \sim \mathbb{P}'$ via

$$\mathbb{P}^\lambda(A) := E'[\mathbb{1}_A Z^\lambda_{T^*}] \quad \forall A \in \mathscr{F}. \tag{4}$$

Under the measure $\mathbb{P}^\lambda$, τ has intensity λ: more precisely, this means that the process

$$M_t^\lambda := \mathbb{1}_{\{t \le \tau\}} - \int_0^{t \wedge \tau} \lambda_s ds, \qquad 0 \le t \le T^* \tag{5}$$

is a $\mathbb{P}^\lambda$-martingale.

[4] Augmentation can be done in a standard fashion with respect to $\mathbb{P}'$.

Now we introduce a precise definition of ambiguity on the default intensity which is very much in spirit of the G-Bronwnian motion: we consider an interval $[\underline{\lambda},\overline{\lambda}] \subset (0,\infty)$ where $\underline{\lambda}$ and $\overline{\lambda}$ denote lower (upper) bounds in the default intensity. Intuitively, we include all possible intensities lying in these bounds in our family of models $\mathscr{P}$. More precisely, we define the set of density generators H by

$$H := \{\lambda : \lambda \text{ is } \mathbb{F}\text{-predictable and } \mathbb{P}'(\underline{\lambda} \le \lambda_t \le \overline{\lambda}, \quad t \in [0,T^*]) = 1\}.$$

Ambiguity on the default intensity is now covered by considering the concrete family of probability measures

$$\mathscr{P} := \{\mathbb{P}^\lambda : \lambda \in H\}. \tag{6}$$

In the following we will always consider this $\mathscr{P}$. First, we observe that this set is convex.

Lemma 2.1. *$\mathscr{P}$ is a convex set.*

Proof. Consider $\mathbb{P}^{\lambda'}, \mathbb{P}^{\lambda''} \in \mathscr{P}$ and $\alpha \in (0,1)$. Then,

$$\alpha\mathbb{P}^{\lambda'}(A) + (1-\alpha)\mathbb{P}^{\lambda''}(A) = E'\big[\mathbb{1}_A(\alpha Z_{T^*}^{\lambda'} + (1-\alpha)Z_{T^*}^{\lambda''})\big].$$

Now consider the (well-defined) intensity λ, given by

$$\int_0^t \lambda_s ds := t - \log\Big[\alpha e^{\int_0^t (1-\lambda'_s)ds} + (1-\alpha)e^{\int_0^t(1-\lambda''_s)ds}\Big],$$

$0 \le t \le T^*$. Then,

$$\alpha Z_{T^*}^{\lambda'} + (1-\alpha)Z_{T^*}^{\lambda''} = Z_{T^*}^{\lambda}$$

such that by (4), $\mathbb{P}^\lambda \sim \mathbb{P}'$ refers to an equivalent change of measure. Finally, we have to check that $\lambda \in H$, which means that $\lambda_{\in}[\underline{\lambda},\overline{\lambda}]$, $0 \le t \le T^*$: note that

$$\begin{aligned} t - \log&\Big[\alpha e^{\int_0^t (1-\lambda'_s)ds} + (1-\alpha)e^{\int_0^t(1-\lambda''_s)ds}\Big] \\ &\le t - \log\Big[\alpha e^{\int_0^t (1-\overline{\lambda})ds} + (1-\alpha)e^{\int_0^t(1-\overline{\lambda})ds}\Big] \\ &\le t - t(1-\overline{\lambda}) = \overline{\lambda}t, \end{aligned}$$

and $\lambda_t \le \overline{\lambda}$ follows. Similarly, $\underline{\lambda} \le \lambda_t$ and the claim follows since t was arbitrary. □

Remark 2. Intuitively, the requirement $\underline{\lambda} > 0$ states that there is always a positive risk of experiencing a default, which is economically reasonable. Technically it has the appealing consequence that all considered measures in $\mathscr{P}$ are equivalent.

It turns out that the set of possible densities will play an important role in connection with measure changes. In this regard, we define *admissible measure changes* with respect to $\mathscr{P}$ by

$$\mathscr{A} := \{\lambda^* : \lambda^* \text{ is positive, } \mathbb{F}\text{-predictable and } E'[Z_{T^*}^{\lambda^*}] = 1\}.$$

The associated Radon-Nikodym derivatives $Z_{T^*}^{\lambda^*}$ for $\lambda^* \in \mathscr{A}$ are the possible Radon-Nikodym derivatives for equivalent measure changes.

Remark 3. It is of course possible to consider an ambiguity setting more general than the specific one in (6). One possibility is to consider only a subset of $\mathscr{P}$. Another possibility is to allow the bounds $\underline{\lambda}$ and $\overline{\lambda}$ to depend on time, or even on the state of the process – this latter case is important for considering affine processes under uncertainty and we refer to [15] for further details. In section 6 we consider indeed such a more general setting.

3. Absence of arbitrage under ambiguity

Absence of arbitrage and the respective generalizations, *no free lunch* (NFL), and *no free lunch with vanishing risk* (NFLVR), are well established concepts when the underlying probability measure $\mathbb{P}$ is known and fixed. Here, we give a small set of sufficient conditions for absence of arbitrage extended to the setting with ambiguity. In this regard, consider, a fixed set $\mathscr{P}$ of probability measures on the measurable space $(\Omega, \mathscr{F})$. In addition, let $\mathbb{G} = (\mathscr{G}_t)_{0 \le t \le T^*}$ be a right-continuous filtration.

Discounted price processes of the traded assets are given by a finite dimensional $\mathbb{G}$-semimartingale $X = (X_t)_{0 \le t \le T^*}$. The semimartingale property holds equivalently in any of the filtration $\mathbb{G}_+$ or the augmentation of $\mathbb{G}_+$, see [24, Proposition 2.2]. It is well known that then X is a semimartingale for all $\mathbb{P} \in \mathscr{P}$.

A self-financing trading strategy is a predictable and X-integrable process Φ and the associated discounted gains process is given by the stochastic integral of Φ with respect to X,

$$(\Phi \cdot X)_t = \int_0^t \Phi_u dX_u, \qquad 0 \le t \le T^*.$$

Intuitively, an *arbitrage* is an admissible self-financing trading strategy which starts from zero initial wealth, has non-negative pay-off under all possible future scenarios, hence for all $P \in \mathscr{P}$, where there is at least one $P \in \mathscr{P}$ such that the pay-off is positive. This is formalized in the following definition, compare for example [27]. As usual a trading strategy is called a-admissible, if $(\Phi \cdot X)_t \ge -a$ for all $0 \le t \le T^*$.

Definition 3.1. A self-financing trading strategy Φ is called *$\mathscr{P}$-arbitrage* if it is a-admissible for some $a > 0$ and

(i) for every $\mathbb{P} \in \mathscr{P}$ we have that $(\Phi \cdot X)_{T^*} \ge 0$, $\mathbb{P}$-almost surely, and
(ii) for at least one $\mathbb{P} \in \mathscr{P}$ it holds that $\mathbb{P}((\Phi \cdot X)_{T^*} > 0) > 0$.

Since all probability measures $\mathbb{P} \in \mathscr{P}$ are considered as possible, a $\mathscr{P}$-arbitrage is a riskless trading strategy for all possible models (i.e., for all $\mathbb{P} \in \mathscr{P}$) while it is a profitable strategy for at least one scenario (i.e., for at least one $\mathbb{P} \in \mathscr{P}$).

The main tool for classifying arbitrage free markets will be local martingale measures, even in the setting with ambiguity. In this regard, we call a probability measure $\mathbb{Q}$ a *local martingale measure* if X is a $\mathbb{Q}$-local martingale.

It is well-known that *no arbitrage* or, more precisely, *no free lunch with vanishing risk* (NFLVR) in a market where discounted price processes are locally bounded semimartingales is equivalent to the existence of an equivalent local martingale measure (ELMM), see [9, 10]. The technically difficult part of this result is to show that a precise criterion of absence of arbitrage implies the existence of an ELMM. In the following we will not aim at such a deep result under ambiguity, but utilize the easy direction, namely that existence of an ELMM implies the absence of arbitrage as formulated below.

From the classical *fundamental theorem of asset pricing* (FTAP), the following result follows easily.

Theorem 3.1. *If, for every $\mathbb{P} \in \mathscr{P}$ there exists an equivalent local martingale measure $\mathbb{Q}(\mathbb{P})$, then there is no arbitrage in the sense of Definition 3.1.*

Proof. Indeed, assume on the contrary that there is an arbitrage Φ with respect to some measure $\mathbb{P} \in \mathscr{P}$ which we fix for the remainder of the proof. If there exists an ELMM $\mathbb{Q}(\mathbb{P})$ then Φ would be an arbitrage strategy together with an ELMM, a contradiction to the classical FTAP. $\square$

This (sufficient) condition directly corresponds to the existing results in the literature (see, for example, [4]) where arbitrages of the first kind are studied under the additional assumption of continuity for the traded assets.

4. Ambiguity on the default intensity

Our aim is to discuss dynamic term structure models under default risk with ambiguity on the default intensity. The relevance of this issue has, for example, already been reported in [25]. Here, we take this as motivation to propose a precise framework taking ambiguity on the default intensity into account.

4.1. **Dynamic defaultable term structures.** We specialize the considerations of absence of arbitrage in section 3 to defaultable bond markets. Recall that we have a filtration $\mathbb{G}$ at hand and that τ is the $\mathbb{G}$-stopping time at which the company defaults. We define the *default indicator process* H by

$$H_t = \mathbb{1}_{\{t \geq \tau\}}, \qquad 0 \leq t \leq T^*.$$

The associated *survival process* is $1 - H$. A credit risky bond with a maturity time $T \leq T^*$ is a contingent claim promising to pay one unit of currency at T. We denote the price of such a bond at time $t \leq T$ by $P(t,T)$. If no default occurs prior to T, $P(T,T) = 1$. We will first consider *zero recovery*, i.e., assume that the bond loses its total value at default. Then $P(t,T) = 0$ on $\{t \geq \tau\}$.

Besides zero recovery, we only make the weak assumption that bond-prices prior to default are positive and absolutely continuous with respect to maturity T. This follows the well-established approach by [18]. More formally, we assume that

$$P(t,T) = \mathbb{1}_{\{\tau > t\}} \exp\left(- \int_t^T f(t,u)du \right) \qquad 0 \leq t \leq T. \tag{7}$$

The initial forward curve $T \mapsto f(0,T)$ is then assumed to be sufficiently integrable and the *forward rate processes* $f(\cdot,T)$ are assumed to follow Itô processes satisfying

$$f(t,T) = f(0,T) + \int_0^t a(s,T)ds + \int_0^t b(s,T)dW_s, \tag{8}$$

for $0 \leq t \leq T$. Recall that W was chosen to be a Brownian motion. We denote by $\mathscr{O}$ the optional σ-algebra and by $\mathscr{B}$ the Borel σ-algebra.

Assumption 1. We require the following technical assumptions:

(i) the initial forward curve is measurable, and integrable on $[0,T^*]$:

$$\int_0^{T^*} |f(0,u)|du < \infty,$$

(ii) the drift parameter $a(\omega,s,t)$ is $\mathbb{R}$-valued $\mathscr{O} \otimes \mathscr{B}$-measurable and integrable on $[0,T^*]$:

$$\int_0^{T^*} \int_0^{T^*} |a(s,t)|dsdt < \infty,$$

(iii) the volatility parameter $b(\omega,s,t)$ is $\mathbb{R}^d$-valued, $\mathscr{O} \otimes \mathscr{B}$-measurable, and

$$\sup_{s,t \leq T^*} \|b(s,t)\| < \infty.$$

(iv) With probability one it holds that

$$0 < f(t,t) - r_t, \qquad 0 \le t \le T^*.$$

Set for $0 \le t \le T \le T^*$,

$$\overline{a}(t,T) = \int_t^T a(t,u)du,$$
$$\overline{b}(t,T) = \int_t^T b(t,u)du.$$

Lemma 4.1. *Under Assumption 1 it holds that,*

$$\int_t^T f(t,u)du = \int_0^T f(0,u)du + \int_0^t \overline{a}(\cdot,u)du + \int_0^t \overline{b}(\cdot,u)dW_u - \int_0^t f(u,u)du$$

for $0 \le t \le T \le T^*$, almost surely.

This follows as in [18]: for the case W is a Brownian motion, this is Lemma 6.1 in [16]. This result could also be generalized where W is replaced by a semimartingale with absolutely continuous characteristics, see Proposition 5.2 in [6]. Note that the strong condition (iii) of uniform boundedness of b in Assumption 1 is needed for the application of the stochastic Fubini theorem.

4.2. **Absence of arbitrage without ambiguity on the default intensity.** The first step towards the study of term structure models with default ambiguity is the study of absence of arbitrage in (classical) intensity-based dynamic term structure models. Consider $\lambda = (\lambda_t)_{0 \le t \le T^*} \in \mathscr{A}$ and the probability measure $\mathbb{P}^\lambda$. Then, the dual predictable projection H^p of H is given by $H^p_t = \int_0^{t \wedge \tau} \lambda_s ds$ (under $\mathbb{P}^\lambda$). Moreover, the Doob-Meyer decomposition yields that

$$M^\lambda := H - \int_0^{\cdot \wedge \tau} \lambda_s ds$$

is $\mathbb{P}^\lambda$-martingale, compare equation (5).

For discounting, we use the bank account. Its value is given by a stochastic process starting with 1 which is then upcounted by the short rate r, i.e., the value process of the bank account is $\gamma(t) = \exp(\int_0^t r_s ds)$ with an $\mathbb{G}$-predictable process r.

In the bond market context considered here, a measure $\mathbb{Q}$ is called local martingale measure if, for any maturity $T \in (0, T^*]$, the discounted bond price process for the bond with maturity T is a $\mathbb{Q}$-local martingale. Then, we obtain the following result.

Proposition 4.2. *Assume that Assumption 1 holds. Consider a measure $\mathbb{Q}$ on $(\Omega, \mathscr{F})$, such that M^λ is a $\mathbb{Q}$-martingale, that W is a $\mathbb{Q}$-Brownian motion and that $\mathbb{Q}(\int_0^{T^*} |r_s| ds < \infty) = 1$. Then $\mathbb{Q}$ is a local martingale measure if and only if*

(i) $f(t,t) = r_t + \lambda_t$,
(ii) the drift condition

$$\bar{a}(t,T) = \frac{1}{2} \left\| \bar{b}(t,T) \right\|^2,$$

holds $dt \otimes dQ$-almost surely for $0 \le t \le T \le T^*$ on $\{\tau > t\}$.

Proof. We set $E = 1 - H$ and $F(t,T) = \exp\left(-\int_t^T f(t,u)du\right)$. Then (7) can be written as $P(t,T) = E(t)F(t,T)$. Integrating by part yields

$$dP(t,T) = F(t-,T)dE(t) + E(t-)dF(t,T) + d[E, F(\cdot,T)]_t.$$

For $\{t < \tau\}$,

$$\begin{aligned} dP(t,T) = {} & P(t-,T)\left(-\lambda_t dt + \left(f(t,t) + \frac{1}{2}\left\|\bar{b}(t,T)\right\|^2 - \bar{a}(t,T)\right)dt\right) \\ & - P(t-,T)\left(dM^\lambda + \bar{b}(t,T)dW_t\right). \end{aligned} \tag{9}$$

The discounted bond price process is a local martingale if and only if the predictable part in the semimartingale decomposition vanishes, i.e.,

$$f(t,t) - r_t - \lambda_t - \bar{a}(t,T) + \frac{1}{2}\left\|\bar{b}(t,T)\right\|^2 = 0. \tag{10}$$

Letting $T = t$ we obtain (i) and (ii) and the result follows. $\square$

4.3. **Absence of arbitrage with ambiguity on the default intensity.** Next, we derive the no-arbitrage conditions for the forward rate in term of the intensity and the short rate, and also the conditions for the drift and volatility parameters, under ambiguity on the default intensity. In this regard, we require a bit more structure: we assume that the setting detailed in section 2 holds, in particular, we consider the family of probability measures $\mathscr{P}$ constructed in equation (6). Recall that for all $\mathbb{P}^\lambda \in \mathscr{P}$, W is a Brownian motion and that $\mathbb{P}^\lambda \sim \mathbb{P}'$. We may, for the moment, safely assume that the market filtration $\mathbb{F}$ satisfies the usual conditions under $\mathbb{P}'$.

For a generic real-valued, $\mathbb{F}$-progressive process $\theta = (\theta_t)_{t\geq 0}$, let the process $z^\theta = (z_t^\theta)_{0\leq t\leq T^*}$ be given as the unique strong solution of

$$dz_t^\theta = \theta_t z_t^\theta dW_t, \quad z_0^\theta = 1. \tag{11}$$

Then, z^θ is a continuous local martingale. If $E'[z_{T^*}^\theta] = 1$, we can define a probability measure $\tilde{\mathbb{P}}^\theta$ by letting

$$\tilde{\mathbb{P}}^\theta(A) := E'[\mathbb{1}_A z_{T^*}^\theta], \quad \forall A \in \mathscr{F}, \tag{12}$$

just as in equation (4). Under $\tilde{\mathbb{P}}^\theta$ the process $\tilde{W} = W - \int_0^\cdot \theta_s ds$ is a Brownian motion, i.e., W itself became a Brownian motion with drift θ, see Theorem 5.1 in Chapter 3 of [20].

Moreover, set $\tilde{\lambda}_t := (f(t,t) - r_t)\cdot\lambda_t^{-1},\ t \in [0,T^*]$. Note that under Assumption 1, $\tilde{\lambda}$ is positive (which is necessary for an equivalent change of measure). The associated density is abbreviated by

$$Z_{T^*}^* = Z_{T^*}^{\tilde{\lambda}}.$$

Theorem 4.3. *Consider $\mathbb{P}^\lambda \in \mathscr{P}$. Under Assumption 1, there exists an ELMM to $\mathbb{P}^\lambda$, if there exists an $\mathbb{F}$-progressive process θ^* such that*

(i) $E^{\mathbb{P}^\lambda}[Z_{T^*}^* z_{T^*}^{\theta^*}] = 1$,

(ii) the drift condition

$$\bar{a}(t,T) = \frac{1}{2}\parallel \bar{b}(t,T)\parallel^2 - \bar{b}(t,T)\theta_t^*, \quad 0 \leq t \leq T \leq T^*$$

holds $dt \otimes d\mathbb{P}^\lambda$-almost surely on $\{t < \tau\}$.

Intuitively, the theorem states that for the probability measure $\mathbb{P}^\lambda$, we find an ELMM if we are able to perform an equivalent change of measure (condition (i)) in such a way that under the new measure the drift condition holds for the Brownian motion with drift θ^* (condition (ii)).

Proof. We start from some $\mathbb{P}^\lambda \in \mathscr{P}$ and fix this measure in the following. This means that, under $\mathbb{P}^\lambda$, W is a Brownian motion and τ has intensity λ. In the search for an ELMM we are looking for an equivalent measure $\mathbb{P}^*$ which satisfies the conditions of Proposition 4.2.

In this regard, note the following: by its definition, (11), together with condition (i), z^{θ^*} is a density process for a change of measure via the Girsanov theorem for Itô processes, see Theorem 5.1 in Chapter 3 of [20]. Moreover, by (3) together with (i), $Z^* = Z^{\tilde\lambda}$ is the density for the change in intensity from λ under $\mathbb{P}^\lambda$ to the intensity given by $\lambda^*_t := (f(t,t) - r_t)_{0 \le t \le T^*}$, see [7, Theorem VI.2.T2]. We set

$$d\mathbb{P}^* := Z^*_{T^*} z^{\theta^*}_{T^*} d\mathbb{P}^\lambda.$$

According to Theorem 3.40 in Chapter III of [19], this refers to a Girsanov-type (and equivalent) change of measure. Moreover, $W^*_t = W_t - \int_0^t \theta^*_s ds,\ 0 \le t \le T^*$, is a $\mathbb{P}^*$-Brownian motion and M^{λ^*} is a $\mathbb{P}^*$-martingale.

We now show that $\mathbb{P}^*$ is also a local martingale measure. Recall from (9) that, under $\mathbb{P}^\lambda$,

$$\frac{dP(t,T)}{P(t-,T)} = \left(f(t,t) + \frac{1}{2} \left\| \bar b(t,T) \right\|^2 - \bar a(t,T) \right) dt - dH_t - \bar b(t,T) dW_t,$$

for $\{t < \tau\}$; here $H_t = \mathbb{1}_{\{t \ge \tau\}}$ is the default indicator. We introduce the martingales W^* and M^{λ^*} into this representation: note that

$$\begin{aligned} \frac{dP(t,T)}{P(t-,T)} &= \left(-\lambda^*(t) + f(t,t) + \frac{1}{2} \left\| \bar b(t,T) \right\|^2 - \bar a(t,T) - \bar b(t,T)\theta^*_t \right) dt \\ &\quad - dM^{\lambda^*}_t - \bar b(t,T) dW^*_t, \end{aligned}$$

again on $\{t < \tau\}$. Since, by assumption, $-\lambda^*_t + f(t,t) = r_t$, together with the drift condition (ii), we obtain for the discounted bond price process $\tilde P(t,T) = P(t,T)/\gamma_t$,

$$d\tilde P(t,T) = \tilde P(t-,T) \cdot \big(- dM^{\lambda^*}_t - \bar b(t,T) dW^*_t \big),$$

which is a $\mathbb{P}^*$-local martingale and the proof is finished. □

5. Examples

The setting proposed in the previous setting can, in dimension one, be directly linked to a special case of the non-linear affine processes introduced in [15]. Indeed, note that for a progressive process λ, the integral

$$X_t := x + \int_0^t \lambda_s ds, \qquad 0 \le t \le T^*$$

is a special semimartingale. Moreover, there are affine bounds on drift and volatility (the bound of the volatility is simply zero) since

$$\underline{\lambda} \le \lambda_t \le \overline{\lambda},$$

such that X is a non-linear affine process.

The major advantage of this setting is that numerical methods via non-linear PDE come into reach. More precisely, Theorem 4.1 in [15] shows that whenever ψ is Lipschitz, the non-linear expectation

$$\mathcal{E}[\psi(X_{T^*})] := \sup_{\mathbb{P}\in\mathscr{P}} E^{\mathbb{P}}[\psi(X_{T^*})] \tag{13}$$

can be expressed as viscosity solution of the fully non-linear PDE

$$\begin{cases} -\partial_t v(t,x) - G\big(x, \partial_x v(t,x)\big) = 0 & \text{on } [0,T^*] \times [\underline{\lambda}, \overline{\lambda}], \\ \qquad\qquad\qquad v(T,x) = \psi(x) & x \in [\underline{\lambda}, \overline{\lambda}], \end{cases} \tag{14}$$

where G is defined by

$$G(x,p) := \sup_{\lambda\in[\underline{\lambda},\overline{\lambda}]} \{\lambda p\} \tag{15}$$

and $v(0,x) = \mathcal{E}[\psi(X_{T^*})]$ (the dependency on x arises through $X_0 = x$).

Clearly, when p is either strictly positive (hence $\partial_x v(t,x)$) or negative, then the supremum in (15) is immediate and the PDE (14) can be solved using standard methods. This means that the solution to the non-linear expectation is obtained simply by the upper bound $\overline{\lambda}$ (or the lower bound $\underline{\lambda}$, respectively). Such a condition holds if ψ is monotone. The more general case has to be solved using numerical methods and we provide a simple example now.

Example 5.1. Consider a butterfly on X_{T^*}, i.e., the derivative with the payoff

$$\psi(x) = (x - K_1)^+ - 2(x - K_2)^+ + (x - K_3)^+,$$

where we choose $K_1 = -0.2$, $K_2 = 0.3$, and $K_3 = 0.8$. Moreover, let $\underline{\lambda} = 0.1$ and $\overline{\lambda} = 0.5$. Then the upper and lower price bounds for the butterfly are shown in Figure 1 (the upper prices are given by the nonlinear expectation in equation (13), while the lower prices are obtained by replacing the supremum in (13) by an infimum).

6. Ambiguity on the recovery

A detailed study of bond markets beyond zero recovery is often neglected, the high degree of uncertainty about the recovery mechanism being a prime reason for this. This motivates us to take some time for developing a deeper understanding of a suitable recovery model under ambiguity.

We start from the observation that intensity-based models always need certain recovery assumptions, as for example, zero recovery, fractional recovery of treasury, and fractional recovery of par value, see [5, Chapter 8]. We have so far considered the case where the credit risky bond becomes worthless at default (zero recovery). In the following, we will consider *fractional recovery of market value* where the credit risky bond looses a fraction of its market value upon default. Other recovery models can be treated in a similar fashion.

6.1. **Fractional recovery without ambiguity.** Fractional recovery of market value (RMV) is specified through a market point process $(T_n, R_n)_{n\geq 1}$ where the stopping times (T_n) denote the default times and $R_n \in (0,1]$ denotes the associated fractional recovery. Let

$$R_t = \prod_{T_n \leq t} R_n, \qquad 0 \leq t \leq T^*, \tag{16}$$

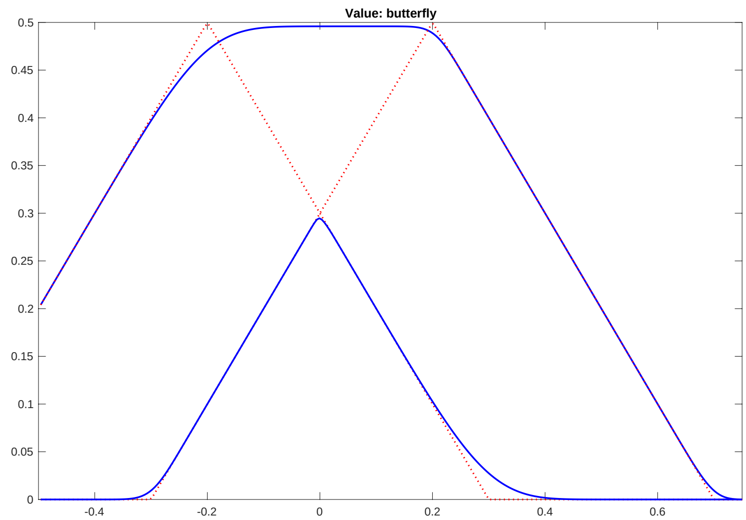

FIGURE 1. This figure shows the solution of the nonlinear PDE in equation (14) with boundary condition $\psi(y) = (y - K_1 + x)^+ - 2(y - K_2 + x)^+ + (y - K_3 + x)^+$, $K_1 = -0.2$, $K_2 = 0.3$, $K_3 = 0.8$, and $x \in [-0.5, 0.7]$ is depicted on the x-axis of the plot. The dashed lines show the solution for the lower bound (upper bound, respectively), i.e., for the constants $\underline{\lambda} = 0.1$ and $\overline{\lambda} = 0.5$. The upper and lower solid lines show the upper and lower price bounds.

denote the *recovery process*. Then, R is non-increasing, positive with $R_0 = 1$. The recovery process replaces the default indicator in (7). More precisely, we assume that the family of defaultable bond prices under RMV satisfy

$$P_R(t,T) = R_t \exp\left(-\int_t^T f(t,u)du\right), \qquad 0 \le t \le T \le T^*. \tag{17}$$

Remark 4. If a default occurs at $t = T_n$, the bond looses a random fraction $q_t = 1 - R_{T_n} \in [0,1)$ of its pre-default value. Thus, the value $(1 - q_t)P(t-, T)$ is immediately available to the bond owner at default. It is still subject to default risk because of the possible future defaults occurring at $T_{n+1}, T_{n+2}, \dots$.

First, we state a generalization of Proposition 4.2 to this setting. To this end, we require more structure and continue in the setting of the section 2. Assume that the market point process $(T_n, R_n)_{n\ge1}$ is independent from W and standard in the following sense: the random times (T_n) are the jumping times from a Poisson process with intensity one, and the recovery values (R_n) are independent from (T_n) and W, and uniformly distributed in $[\underline{r}, \bar{r}] \subset (0,1]$.

The filtration $\mathbb{G} = (\mathscr{G}_t)_{0\le t\le T^*}$ is obtained by a progressive enlargement of $\mathbb{F}$ with default and recovery information (given by R), i.e.,

$$\mathscr{G}_t = \bigcap_{\epsilon>0} \sigma(R_s, W_s : 0 \le s \le t + \epsilon), \qquad 0 \le t \le T^*.$$

We assume again $\mathscr{F} = \mathscr{G}_{T^*}$. As next step, we introduce measure changes for the marked point process. Let

$$\Phi_t = \sum_{T_n \le t} R_n, \qquad 0 \le t \le T^*.$$

Then, Φ is a special semimartingale w.r.t. $\mathbb{G}$. Let

$$\mu^\Phi(dt, dx) = \sum_{n \ge 1} \delta_{(T_n, R_n)}(dt, dx)$$

denote the associated jump measure and let $\nu^\Phi(dt, dx)$ denote its compensator, see Chapter II.1 in [19] or Chapter VIII.1 in [7]. Note that $\nu^\Phi(dt, dx) = \mathbb{1}_{\{x \in [\underline{r}, \overline{r}]\}}(\overline{r} - \underline{r})^{-1} dx dt$.

We introduce the densities

$$L^{\mu,h} = \left(\prod_{T_n \le T^*} \mu_{T_n} h(T_n, R_n) \right) \cdot e^{\int_0^{T^*} \int_{\underline{r}}^{\overline{r}} (1 - \mu_t h(t,x))(\overline{r} - \underline{r})^{-1} dx\, dt}, \tag{18}$$

where the predictable process μ is positive and, for any $x \in [\underline{r}, \overline{r}]$, the $\mathbb{G}$-predictable process $(h(t,x))_{0 \le t \le T^*}$ is also positive. If $E[L_{T^*}] = 1$, we can define the equivalent measure $\mathbb{P}^{\mu,h}$ by

$$d\mathbb{P}^{\mu,h} = L_{T^*} d\mathbb{P}'. \tag{19}$$

By $\mathscr{A}^*$ we denote all pairs (μ, h) which satisfy the above properties. Then, the compensator of the jump measure μ^Φ under $P^{\mu,h}$ is

$$\mu_t h(t,x) \nu^\Phi(dt, dx) = \mu_t h(t,x) \mathbb{1}_{\{x \in [\underline{r}, \overline{r}]\}} (\overline{r} - \underline{r})^{-1} dx dt =: K_t^{\mu,h}(dx) dt, \tag{20}$$

see T10 in Section VIII.3 of [7]. Next, we compute the compensator of R. We obtain from (16) that

$$R_t - \int_0^t \int R_{s-}(x-1) K_s^{\mu,h}(dx) ds, \qquad 0 \le t \le T^* \tag{21}$$

is a $\mathbb{P}^{\mu,h}$-martingale. For a $\mathbb{G}$-progressive process g, we denote

$$M^g = R + \int_0^{\cdot} R_{s-} g_s ds.$$

Proposition 6.1. *Assume that Assumption 1 holds and let g be a positive and $\mathbb{G}$-predictable process. Consider a measure $\mathbb{Q}$ on $(\Omega, \mathscr{F})$, such that M^g is a $\mathbb{Q}$-martingale, W is a $\mathbb{Q}$-Brownian motion, and $\mathbb{Q}(\int_0^{T^*} |r_s| ds < \infty) = 1$. Then $\mathbb{Q}$ is a local martingale measure if and only if*

(i) $f(t,t) = r_t + g_t$,

(ii) the drift condition

$$\bar{a}(t,T) = \frac{1}{2} \left\| \bar{b}(t,T) \right\|^2, \quad 0 \le t \le T \le T^*,$$

holds $dt \otimes d\mathbb{Q}$-almost surely.

Proof. We generalize the proof of Proposition 4.2 to the case of RMV. To this end, let $F(t,T) = \exp\left(- \int_t^T f(t,u) du \right)$. Then (17) reads $P_R(t,T) = R(t) F(t,T)$. Integrating by part yields

$$dP_R(t,T) = F(t-,T) dR(t) + R(t-) dF(t,T) + d[R, F(\cdot, T)]_t.$$

Note that, by assumption,

$$M_t^g = R_t + \int_0^t R_{s-} g_s ds, \qquad 0 \le t \le T^*,$$

is a Q-martingale and that $[R, F(., T)] = 0$ since R is of finite variation and $F(., T)$ is continuous. Hence, by Lemma 4.1,

$$dP_R(t,T) = P_R(t-,T)\left(-g_t + f(t,t) + \frac{1}{2}\left\|\bar{b}(t,T)\right\|^2 - \bar{a}(t,T)\right)dt \tag{22}$$

$$+ P_R(t-,T)\left(r_t dt + dM^g - \bar{b}(t,T)dW_t\right),$$

and we obtain the result as in the proof of Proposition 4.2. □

Example 6.1. A classical example is when the defaults (T_n) arrive at rate $\lambda > 0$, and the recovery values (R_n) are i.i.d. Then, $\int (x-1)\nu^\Phi(dt, dx) = \lambda E[R_1 - 1]dt$. We obtain that the instantaneous forward rate of the defaultable bond $f(t,t)$ equals $r_t + \lambda E[1 - R_1]$. In the case of zero recovery, we recover $f(t,t) = r_t + \lambda$, and, in the case of full recovery (the case without default risk), $f(t,t) = r_t$.

6.2. **Fractional recovery with ambiguity.** We introduce ambiguity in this setting by changing from the standardized measure $\mathbb{P}'$ to various appropriate measures via the Girsanov theorem. We also generalize the setting for ambiguity from the quite specific $\mathscr{P}$ to a general set of probability measures $\mathscr{P}^*$ here, see remark 3. The reason for this is also economic: while bounding the intensity from above and below seems to be quite plausible, an upper / lower bound on the recovery (i.e., on (R_n)) sounds too strong for some applications.

Recall that $\mathscr{A}^*$ was the set of all candidates (μ, h) which induce the measure changes via (19). Ambiguity is introduced by the set $\mathscr{P}^*$ of probability measures satisfying

$$\emptyset \neq \mathscr{P}^* \subset \{\mathbb{P}^{\mu,h} : (\mu, h) \in \mathscr{A}^*\}. \tag{23}$$

If $\mathscr{P}^*$ contains only one probability measure, we are in the classical setting, otherwise there is ambiguity in the market. Measure changes from $\mathbb{P}^{\mu,h}$ to a new measure are done via the density L^{μ^*,h^*} (see (18)) where, as above, $\mu^*,\ h^*(., x),\ x \in [\underline{r}, \overline{r}]$ are positive and progressive. Recall the definition of the density z^θ in (11).

Theorem 6.2. *Let $g_t^* := f(t,t) - r_t,\ t \in [0, T^*]$, and assume that Assumption 1 holds. Then there exists an ELMM for $\mathbb{P}^{\mu,h} \in \mathscr{P}^*$ if*

(i) there exists an $\mathbb{F}$-progressive θ^* such that $E'[z_{T^*}^{\theta^*}] = 1$,

(ii) there exist μ^* and $h^*(t,x)$, such that $E[L^{\mu^*,h^*}] = 1$ and

$$g_t^* = \int (x-1)\mu_t^* h^*(t,x) K_t^{\mu,h}(dx), \quad 0 \le t \le T^*,$$

$dt \otimes dP'$-almost surely, and

(iii) the drift condition

$$\bar{a}(t,T) = \frac{1}{2} \parallel \bar{b}(t,T) \parallel^2 - \bar{b}(t,T)\theta_t^*, \quad 0 \le t \le T \le T^*,$$

holds $dt \otimes dP'$-almost surely.

Absence of arbitrage in this general ambiguity setting can now be classified, thanks to Theorem 3.1 as follows: if an ELMM exists for each $\mathbb{P}^{\mu,h} \in \mathscr{P}^*$, then the market is free of arbitrage in the sense of Definition 3.1.

Proof. Fix $\mathbb{P}^{\mu,h} \in \mathscr{P}^*$. We can define an equivalent measure $\mathbb{P}^* \sim \mathbb{P}^{\mu,h}$ by

$$d\mathbb{P}^* := L^{\mu^*,h^*} z^{\theta^*}_{T^*} d\mathbb{P}^{\mu,h},$$

with μ^* and h^* as in (ii). According to Theorem 3.40 in Chapter III of [19], this refers to a Girsanov-type (and equivalent) change of measure. Moreover, $W^* = W - \int_0^\cdot \theta^*_s ds$ is a $\mathbb{P}^*$-Brownian motion. Next, note that the compensator of the jump measure μ^Φ under $\mathbb{P}^*$ computes, according to T10 in Section VIII.3 in [7], to

$$\nu^*(dt,dx) := \mu^*_t h^*(t,x) K^{\mu,h}_t(dx)dt$$

with $K^{\mu,h}_t$ from Equation (20). This implies that

$$M^{g^*}_t = R_{s-}(x-1)\nu^*(ds,dx) = R_t - \int_0^t R_{s-} g^*_s ds \tag{24}$$

is a $\mathbb{P}^*$-martingale.

Now, we show that $\mathbb{P}^*$ is indeed a martingale measure: from (22) we obtain that

$$\frac{dP_R(t,T)}{P_R(t-,T)} = \Big(f(t,t) + \frac{1}{2}\left\|\bar{b}(t,T)\right\|^2 - \bar{a}(t,T)\Big)dt + dR_t - \bar{b}(t,T)dW_t.$$

It follows that

$$\begin{aligned}\frac{dP_R(t,T)}{P_R(t-,T)} &= \Big(f(t,t) + \frac{1}{2}\left\|\bar{b}(t,T)\right\|^2 - \bar{a}(t,T) - \bar{b}(t,T)\theta^*_t\Big)dt + dM^g_t - \bar{b}(t,T)dW^*_t \\ &= r_t dt + dM^g_t - \bar{b}(t,T)dW^*_t,\end{aligned}$$

by the definition of g^* and the drift condition (iii). Hence, discounted bond prices are $\mathbb{P}^*$-local martingales and the proof is finished. □

Remark 5. We can view zero recovery in the above setting by assuming that $\mathbb{P}'(R_1 = 0) = 1$ and letting $\tau = T_1$. Note that this case is excluded in RMV setting, since, under this assumption, at the first default all prices drop to zero and further defaults can not occur.

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