Diffusions and PDEs on Wasserstein Space *

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Abstract

We propose a new type SDE, whose coefficients depend on the image of solutions, to investigate the diffusion process on the Wasserstein space $\mathscr{P}_2$ over $\mathbb{R}^d$, generated by the following time-dependent differential operator for $f \in C_b^2(\mathscr{P}_2)$:

$$\begin{aligned}\mathscr{A}_t f(\mu) :=\ & \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} \big\langle \sigma(t,x,\mu)\sigma(t,y,\mu)^*, D^2 f(\mu)(x,y)\big\rangle \mu(\mathrm{d}x)\mu(\mathrm{d}y)\\ &+ \int_{\mathbb{R}^d} \Big(\frac{1}{2}\big\langle(\sigma\sigma^*)(t,x,\mu), \nabla\{Df(\mu)\}(x)\big\rangle + \big\langle b(t,x,\mu), Df(\mu)(x)\big\rangle\Big)\mu(\mathrm{d}x),\ \ \mu\in\mathscr{P}_2,\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the inner product on $\mathbb{R}^d$ or $\mathbb{R}^d\otimes\mathbb{R}^d$, ∇ is the gradient operator on $\mathbb{R}^d$, D is the intrinsic (or Lions) derivative on $\mathscr{P}_2$, and

$$b: [0,\infty)\times\mathbb{R}^d\times\mathscr{P}_2\to\mathbb{R}^d,\ \ \sigma: [0,\infty)\times\mathbb{R}^d\times\mathscr{P}_2\to\mathbb{R}^d\otimes\mathbb{R}^m$$

are measurable. We study the exponential convergence of the diffusion process, and use the diffusion process to solve the following PDE

$$(\partial_t+\mathscr{A}_t)U(t,\mu)+(VU)(t,\mu)+F(t,\mu)=0,\ \ (t,\mu)\in[0,T]\times\mathscr{P}_2,$$

where V and F are functions on $[0,T]\times\mathscr{P}_2$. Moreover, the structure of the invariant probability measure is described.

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1 Introduction

Let $\mathscr{P}_2$ be the space of all probability measures μ on $\mathbb{R}^d$ such that

$$\|\mu\|_2 := \Big(\int_{\mathbb{R}^d} |x|^2 \mu(\mathrm{d}x)\Big)^{\frac{1}{2}} < \infty,$$

where $|\cdot|$ is the norm in $\mathbb{R}^d$. We will use $\|\cdot\|$ to denote the operator norm of a matrix or linear operator, and use $\|\cdot\|_{HS}$ to stand for the Hilbert-Schmidt norm. It is well known that $\mathscr{P}_2$ is a Polish space under the Wasserstein distance

$$\mathbb{W}_2(\mu,\nu) := \inf_{\pi\in\mathscr{C}(\mu,\nu)} \Big(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 \pi(\mathrm{d}x,\mathrm{d}y)\Big)^{\frac{1}{2}},$$

where $\mathscr{C}(\mu,\nu)$ is the set of couplings for μ and ν.

Since 1996 when Albeverio, Kondratiev and Röckner [1] introduced the intrinsic derivative on the configuration space over manifolds, diffusion processes on the space of discrete Radon measures have been investigated by using Dirichlet forms, see [10] and references within. This derivative provides a natural Riemannian structure on the Wasserstein space $(\mathscr{P}_2, \mathbb{W}_2)$, see Subsection 1.2 below.

To develop stochastic analysis and applications on this space, we intend to construct diffusion processes generated by second order differentiable operators and solve the associated PDEs on $\mathscr{P}_2$. Below we first recall the intrinsic/Lions derivative on $\mathscr{P}_2$.

According to [1], let $L^2(\mathbb{R}^d\to\mathbb{R}^d;\mu)$ be the tangent space of $\mathscr{P}_2$ at point $\mu\in\mathscr{P}_2$, and define the directional derivative by

$$D_\phi f(\mu) := \lim_{\varepsilon\downarrow 0} \frac{f(\mu\circ(\mathrm{Id}+\varepsilon\phi)^{-1}) - f(\mu)}{\varepsilon}, \quad \phi\in L^2(\mathbb{R}^d\to\mathbb{R}^d;\mu).$$

When $\phi\mapsto D_\phi f(\mu)$ is a bounded linear functional on $L^2(\mathbb{R}^d\to\mathbb{R}^d;\mu)$, or equivalently the map

$$L^2(\mathbb{R}^d\to\mathbb{R}^d;\mu)\ni\phi\mapsto f(\mu\circ(\mathrm{Id}+\phi)^{-1}) \tag{1.1}$$

is Gateaux differentiable at $\phi=0$, there exists a unique element $Df(\mu)\in L^2(\mathbb{R}^d\to\mathbb{R}^d;\mu)$ such that

$$\langle Df(\mu),\phi\rangle_{L^2(\mu)} = D_\phi f(\mu), \quad \phi\in L^2(\mathbb{R}^d\to\mathbb{R}^d;\mu).$$

In this case, we call f intrinsically differentiable at μ with derivative $Df(\mu)$. According to Lions (see [4]), if $Df(\mu)$ exists and

$$\lim_{\mu(|\phi|^2)\to 0} \frac{f(\mu\circ(\mathrm{Id}+\phi)^{-1}) - f(\mu) - D_\phi f(\mu)}{\sqrt{\mu(|\phi|^2)}} = 0, \tag{1.2}$$

i.e. the map in (1.1) is Fréchet differentiable at $\phi=0$, we call f L-differentiable at $\mu\in\mathscr{P}_2$. If f is L-differentiable at any $\mu\in\mathscr{P}_2$, we call it L-differentiable. Note that $Df(\mu)$ is a μ-a.e. defined $\mathbb{R}^d$-valued function. Let $\{Df(\mu)\}_i$ be its i-th component for $1\le i\le d$.

In this paper, we investigate diffusion processes and applications to PDEs on $\mathscr{P}_2$. Let $m \geq 1$, and let

$$b : [0,\infty) \times \mathbb{R}^d \times \mathscr{P}_2 \to \mathbb{R}^d, \quad \sigma : [0,\infty) \times \mathbb{R}^d \times \mathscr{P}_2 \to \mathbb{R}^d \otimes \mathbb{R}^m$$

be measurable such that $|b(t,\cdot,\mu)| + \|\sigma(t,\cdot,\mu)\|_{HS}^2 \in L^1(\mu)$ for any $(t,\mu) \in [0,\infty) \times \mathscr{P}_2$. We consider the following time-dependent second order differential operators on $\mathscr{P}_2$:

$$\begin{aligned} \mathscr{A}_t f(\mu) := &\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} \big\langle \sigma(t,y,\mu)\sigma(t,z,\mu)^*, D^2 f(\mu)(y,z)\big\rangle \mu(\mathrm{d}y)\mu(\mathrm{d}z) \\ &+ \int_{\mathbb{R}^d} \Big(\frac{1}{2}\big\langle (\sigma\sigma^*)(t,y,\mu), \nabla\{Df(\mu)\}(y)\big\rangle + \big\langle b(t,y,\mu), Df(\mu)(y)\big\rangle\Big)\mu(\mathrm{d}y), \end{aligned} \tag{1.3}$$

where $\langle\cdot,\cdot\rangle$ is the inner product on $\mathbb{R}^d$ or $\mathbb{R}^d \otimes \mathbb{R}^d$. We also consider the following extension of $\mathscr{A}_t$ on $\mathbb{R}^d \times \mathscr{P}_2$:

$$\begin{aligned} \tilde{\mathscr{A}_t} f(x,\mu) := &\mathscr{A}_t f(x,\cdot)(\mu) + \frac{1}{2}\big\langle \sigma(t,x,\mu)\sigma(t,x,\mu)^*, \nabla^2 f(x,\mu)\big\rangle + \big\langle b(t,x,\mu), \nabla f(x,\mu)\big\rangle \\ &+ \int_{\mathbb{R}^d} \big\langle (D\nabla f)(x,\mu)(y), \sigma(t,y,\mu)\sigma(t,x,\mu)^*\big\rangle \mu(\mathrm{d}y). \end{aligned} \tag{1.4}$$

To present reasonable pre-domains of $\mathscr{A}_t$ and $\tilde{\mathscr{A}_t}$, we introduce below some classes of L-differentiable functions.

(1) We write $f \in C^1(\mathscr{P}_2)$, if f is L-differentiable and the derivative has a μ-version $Df(\mu)(x)$ which is jointly continuous in $(\mu,x) \in \mathscr{P}_2 \times \mathbb{R}^d$. If moreover $Df(\mu)(x)$ is bounded in $(x,\mu) \in \mathbb{R}^d \times \mathscr{P}_2$, we denote $f \in C_b^1(\mathscr{P}_2)$.

(2) We write $f \in C^{(1,1)}(\mathscr{P}_2)$, if $f \in C_b^1(\mathscr{P}_2)$ and $Df(\mu)(x)$ is differentiable in x such that the $\mathbb{R}^d \otimes \mathbb{R}^d$-valued function

$$\nabla\{Df(\mu)\}(x) := \big(\partial_{x_j}\{Df(\mu)(x)\}_i\big)_{1\leq i,j\leq d}$$

is jointly continuous in $(\mu,x) \in \mathscr{P}_2 \times \mathbb{R}^d$. If moreover $Df(\mu)(x)$ and $\nabla\{Df(\mu)\}(x)$ are bounded in $(x,\mu) \in \mathbb{R}^d \times \mathscr{P}_2$, we denote $f \in C_b^{(1,1)}(\mathscr{P}_2)$.

(3) We write $f \in C^2(\mathscr{P}_2)$, if $f \in C^{(1,1)}(\mathscr{P}_2)$ and $Df(\mu)(x)$ is L-differentiable in μ such that the $\mathbb{R}^d \otimes \mathbb{R}^d$-valued function

$$D^2 f(\mu)(x,y) := \big(\big\{D[Df(\mu)(x)]_i(y)\big\}_j\big)_{1\leq i,j\leq d}$$

is jointly continuous in $(\mu,x,y) \in \mathscr{P}_2 \times \mathbb{R}^d \times \mathbb{R}^d$. If moreover $f \in C_b^{(1,1)}(\mathscr{P}_2)$ and $D^2 f(\mu)(x,y)$ is bounded in $(x,y,\mu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathscr{P}_2$, we denote $f \in C_b^2(\mathscr{P}_2)$.

(4) We write $f \in C^{2,2}(\mathbb{R}^k \times \mathscr{P}_2)$ for some $k \geq 1$, if f is a continuous function on $\mathbb{R}^k \times \mathscr{P}_2$ such that $f(\cdot,\mu) \in C^2(\mathbb{R}^k)$ for $\mu \in \mathscr{P}_2$, $f(x,\cdot) \in C^2(\mathscr{P}_2)$ for $x \in \mathbb{R}^k$,

$$(D\nabla f)(x,\mu)(y) := \big(\big\{D[\partial_{x_i} f(x,\mu)]\big\}_j\big)_{1\leq i,j\leq d} \in \mathbb{R}^d \otimes \mathbb{R}^d$$

exists, and the derivatives

$$\nabla f(x,\mu), \nabla^2 f(x,\mu), Df(x,\mu)(y), (D\nabla f)(x,\mu)(y), \nabla\{Df(x,\mu)(\cdot)\}(y), D^2 f(x,\mu)(y,z)$$

are bounded and jointly continuous in the corresponding arguments.

Example 1.1. For any $p \geq 1$, consider the following class of cylindrical functions

$$\begin{aligned}\mathscr{F}C_b^p(\mathscr{P}_2) := \big\{ f(\mu) := g(\mu(h_1), \cdots, \mu(h_n)) :\\ n \geq 1, g \in C_b^p(\mathbb{R}^n), h_i \in C_b^p(\mathbb{R}^d), 1 \leq i \leq n \big\}.\end{aligned} \tag{1.5}$$

When $p = 2$, such a function is in the class $C_b^2(\mathscr{P}_2)$ with

$$\begin{aligned} Df(\mu)(x) &= \sum_{i=1}^n (\partial_i g)(\mu(h_1), \cdots, \mu(h_n)) \nabla h_i(x), \\ D^2 f(\mu)(x,y) &= \sum_{i,j=1}^n (\partial_i \partial_j g)(\mu(h_1), \cdots, \mu(h_n)) \{\nabla h_i(x)\} \otimes \{\nabla h_j(y)\}, \end{aligned} \tag{1.6}$$

where $\{\nabla h_i(x)\} \otimes \{\nabla h_j(y)\} \in \mathbb{R}^d \otimes \mathbb{R}^d$ is defined as

$$\big(\{\nabla h_i(x)\} \otimes \{\nabla h_j(y)\}\big)_{kl} = \{\partial_k h_i(x)\} \partial_l h_j(y), \quad 1 \leq k, l \leq d, x, y \in \mathbb{R}^d.$$

Moreover, $f \in C^{2,2}(\mathbb{R}^d \times \mathscr{P}_2)$ if $f(x,\mu) = g(x, \mu(h_1), \cdots, \mu(h_n))$ for some $n \geq 1$, $g \in C_b^2(\mathbb{R}^{n+d})$ and $\{h_i\}_{1 \leq i \leq n} \subset C_b^2(\mathbb{R}^d)$.

We will construct the $\mathscr{A}_t$-diffusion process by solving the following SDE on $\mathbb{R}^d$:

$$\mathrm{d}X_{s,t}^{x,\mu} = b(t, X_{s,t}^{x,\mu}, \Lambda_{s,t}^\mu)\mathrm{d}t + \sigma(t, X_{s,t}^{x,\mu}, \Lambda_{s,t}^\mu)\mathrm{d}W_t, \ \Lambda_{s,t}^\mu := \mu \circ (X_{s,t}^{\cdot,\mu})^{-1}, \ t \geq s, X_{s,s}^{x,\mu} = x, \tag{1.7}$$

where W_t is the m-dimensional Brownian motion on a complete filtration probability space $(\Omega, \{\mathscr{F}_t\}_{t \geq 0}, \mathbb{P})$, $(s, x, \mu) \in [0, \infty) \times \mathbb{R}^d \times \mathscr{P}_2$. Since this SDE depends on the image of solutions, we call it image SDE.

In the remainder of this section, we first summarize the main results of the paper, then present a link of the present model to the Brownian motion on $\mathscr{P}_2$ for further study, and finally introduce some previous work for analysis on the Wasserstein space.

1.1 Summary of main results

Existence and uniqueness. Under a monotone condition, Theorem 2.1 ensures the existence, uniqueness and moment estimates of solutions to the image SDE (1.7), and that the unique solution is the diffusion processes generated by $\mathscr{A}_t$ on $\mathscr{P}_2$ and $\tilde{\mathscr{A}_t}$ on $\mathbb{R}^d \times \mathscr{P}_2$ respectively.

Feyman-Kac formula. By using the diffusion process $(X_{s,t}^{x,\mu}, \Lambda_{s,t}^\mu)$, Theorem 3.1 solves the following PDE for U on $[0,T] \times \mathbb{R}^d \times \mathscr{P}_2$:

$$\begin{aligned} \partial_t U(t,x,\mu) + \tilde{\mathscr{A}_t} U(t,x,\cdot)(\mu) + (VU)(t,x,\mu) + F(t,x,\mu) = 0, \\ U(T,x,\mu) = \Phi(x,\mu), \quad (t,x,\mu) \in [0,T] \times \mathbb{R}^d \times \mathscr{P}_2, \end{aligned} \tag{1.8}$$

where $T > 0$ is a fixed time, Φ is a function on $\mathbb{R}^d \times \mathscr{P}_2$, and V, F are functions on $[0,T] \times \mathbb{R}^d \times \mathscr{P}_2$. When Φ, F and V do not depend on $x \in \mathbb{R}^d$, this PDE reduces to

$$\begin{aligned} \partial_t U(t,\mu) + \mathscr{A}_t U(t,\cdot)(\mu) + (VU)(t,\mu) + F(t,\mu) = 0, \\ U(T,\mu) = \Phi(\mu), \quad (t,\mu) \in [0,T] \times \mathscr{P}_2. \end{aligned} \tag{1.9}$$

Exponential ergodicity and structure of invariant probability measures. Let b and σ do not depend on t. Under a dissipativity condition, Theorem 4.1 provides the exponential convergence rate of the diffusion process $(X_t^{x,\mu}, \Lambda_t^\mu) := (X_{0,t}^{x,\mu}, \Lambda_{0,t}^\mu)$ to its unique invariant probability measure $\tilde{\Pi}$. Consequently, the diffusion process Λ_t^μ converges at the same rate to the invariant probability measure $\Pi := \tilde{\Pi}(\mathbb{R}^d \times \cdot)$.

Moreover, let $b_0(x) = b(x, \delta_x), \sigma_0(x) = \sigma(x, \delta_x)$, and let μ_0 be the unique invariant probability measure for the classical SDE

$$\mathrm{d}X_t = b_0(X_t)\mathrm{d}t + \sigma_0(X_t)\mathrm{d}W_t. \tag{1.10}$$

By Theorem 4.2, $\tilde{\Pi}$ and Π have the representations

$$\tilde{\Pi}(\mathrm{d}x, \mathrm{d}\mu) = \mu_0(\mathrm{d}x)\delta_{\delta_x}(\mathrm{d}\mu), \quad \Pi = \int_{\mathbb{R}^d} \delta_{\delta_x}\mu_0(\mathrm{d}x), \tag{1.11}$$

where δ_{δ_x} is the Dirac measure at point $\delta_x \in \mathscr{P}_2$. This structure describes an asymptotic collision property of the diffusion process Λ_t^μ: starting from any probability measure $\mu \in \mathscr{P}_2$, the measure-valued process eventually decays to a Dirac random variable, for which the whole mass focus on a single random point.

1.2 Link to the Brownian motion on $\mathscr{P}_2$

A Riemannian structure has been introduced in [2] on the Wasserstein space $(\mathscr{P}_2, \mathbb{W}_2)$. With the intrinsic/Lions derivative, this space is an infinite-dimensional Riemannian manifold with gradient D and Riemannian metric $\langle \cdot, \cdot \rangle_{L^2(\mu)}$ on the tangent space $L^2(\mathbb{R}^d \to \mathbb{R}^d; \mu)$; that is, $\mathbb{W}_2$ is the Riemannian distance induced by D.

As in the finite-dimensional Riemannian setting, we introduce the square field

$$\Gamma(f,g)(\mu) := \int_{\mathbb{R}^d} \langle Df(\mu)(x), Dg(\mu)(x) \rangle \mu(\mathrm{d}x), \quad f, g \in C_b^1(\mathscr{P}_2),$$

and the Laplace operator

$$\Delta f(\mu) := \int_{\mathbb{R}^d} \mathrm{tr}\big\{D^2 f(\mu)(x,x)\big\}\mu(\mathrm{d}x), \quad f \in C^2(\mathscr{P}_2).$$

Then by the chain rule we have

$$\Gamma(f,g) = \frac{1}{2}\big\{\Delta(fg) - f\Delta g - g\Delta f\big\}, \quad f, g \in C^2(\mathscr{P}_2).$$

This structure can be easily extended to the Wasserstein space $\mathscr{P}_2(M)$ over a Riemannian manifold M. Note that when M is compact we have $\mathscr{P}_2(M) = \mathscr{P}(M)$, the space of all probability measures on M.

To develop stochastic analysis on $\mathscr{P}_2$, it is interesting to construct the Brownian motion, i.e. the diffusion process generated by $\frac{1}{2}\Delta$; or more generally, to construct diffusion processes on $\mathscr{P}_2$ with square field Γ. This is the main motivation of [15] introduced in the next subsection.

Below we explain that when $\sigma\sigma^* = \mathrm{Id}$ and $\mu = \delta_x$ is a Dirac measure at some point $x \in \mathbb{R}^d$, the process $(\Lambda^\mu_{0,t})_{t\ge 0}$ is such a diffusion process. Indeed, it is easy to check that the square field of the $\mathscr{A}_t$-diffusion process is

$$\begin{aligned}&\Gamma_t(f,g)(\mu) := \big\{\mathscr{A}_t(fg)(\mu) - f\mathscr{A}_t g - g\mathscr{A}_t\big\}(\mu)\\ &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \big\langle \sigma(t,x,\mu)^* Df(\mu)(x), \sigma(t,y,\mu)^* Df(\mu)(y)\big\rangle \mu(\mathrm{d}x)\mu(\mathrm{d}y) \quad f,g \in C^2_b(\mathscr{P}_2), \mu \in \mathscr{P}_2.\end{aligned}$$

In particular, when $\sigma\sigma^* = \mathrm{Id}$, we have

$$\Gamma_t(f,g)(\mu) = \Gamma(f,g)(\mu), \quad \mu \in \mathscr{P}^0_2 := \{\delta_x : x \in \mathbb{R}^d\}.$$

Since when $\mu = \delta_x$ for some $x \in \mathbb{R}^d$, $\Lambda^\mu_{s,t} = \delta_{X^{x,\delta_x}_{s,t}}$ is a diffusion process on $\mathscr{P}^0_2$, Theorem 2.1(2) below implies that $(\Lambda^\mu_{s,t})_{t\ge s}$ for $\mu \in \mathscr{P}^0_2$ is a diffusion process with square field Γ. However, this does not hold for $\mu \notin \mathscr{P}^0_2$.

1.3 Some previous work

Let $\mathscr{P}(\mathbb{S}^1)$ be the space of all probability measures on the unit circle $\mathbb{S}^1$. A family of probability measures $\{\mathbb{P}_\beta\}_{\beta>0}$ on $\mathscr{P}(\mathbb{S}^1)$, called "entropic measures" with inverse temperature $\beta > 0$, have been constructed by von Renesse and Sturm [15] such that for each $\beta > 0$, the bilinear form

$$\mathscr{E}(f,g) := \int_{\mathscr{P}_2(\mathbb{S}^1)} \langle Df(\mu), Dg(\mu)\rangle_{L^2(\mu)} \mathbb{P}_\beta(\mathrm{d}\mu)$$

gives a symmetric Dirichlet form on $L^2(\Pi_\beta)$, which refers to a $\mathbb{P}_\beta$-a.e. starting diffusion process on $\mathscr{P}(\mathbb{S}^1)$. See also [16] for a different Dirichlet form on $\mathscr{P}([0,1])$ with square field Γ. The construction of Dirichlet forms in these papers heavily relies on the one-dimensional property. In contrast, our diffusion process $(\Lambda^\mu_{s,t})_{t\ge s}$ is defined for any staring point $\mu \in \mathscr{P}_2$, and by establishing Itô's formula for the image SDE, we are able to investigate the ergodicity and the corresponding PDEs as in the classical case.

Next, the distribution-dependent (also called mean-field or McKean-Vlasov) SDEs have been used in [3, 7, 8, 11] to solve PDEs on $\mathbb{R}^d \times \mathscr{P}_2$ with the following type of differential operator:

$$\begin{aligned}\mathbf{L}_t f(x,\mu) &:= \frac{1}{2}\big\langle(\sigma\sigma^*)(x,\mu), \nabla^2 f(x,\mu)\big\rangle + \big\langle b(x,\mu), \nabla f(x,\mu)\big\rangle\\ &+ \int_{\mathbb{R}^d}\Big[\frac{1}{2}\big\langle(\sigma\sigma^*)(y,\mu), \nabla\{(Df(x,\mu))(\cdot)\}(y)\big\rangle + \big\langle b(y,\mu), (Df(x,\mu))(y)\big\rangle\Big]\mu(\mathrm{d}y).\end{aligned}$$

Since this operator only involves in the first order derivative in μ, the associated diffusion process on $\mathbb{R}^d \times \mathscr{P}_2$ has a deterministic marginal $(\mu_t)_{t\ge 0}$ on $\mathscr{P}_2$, which solves the nonlinear Fokker-Planck equation

$$\partial_t \mu_t = (L_{\mu_t})^* \mu_t,$$

where $L_\mu := \frac{1}{2}\sum_{i,j=1}^d (\sigma\sigma^*)_{ij}(x,\mu)\partial_i\partial_j + \sum_{i=1}^d b_i(x,\mu)\partial_i$. See also [9] for nonlinear Fokker-Planck equations on the path space. In the present work, $\mathscr{A}_t$ contains the second-order L-derivative, so that the associated process $\Lambda^\mu_{s,t}$ is a non-trivial diffusion process on $\mathscr{P}_2$.

Moreover, Otto [13] introduced a different gradient formula for functions of the probability density. More precisely, let $\nu(\mathrm{d}x) = \mathrm{e}^{-V(x)}\mathrm{d}x$ for some $V \in C^1(\mathbb{R}^d)$, and let

$$U_\nu(\mu) = \int_{\mathbb{R}^d} U(\rho)\mathrm{d}\nu$$

for $U \in C^1([0,\infty))$ and $\mu(\mathrm{d}x) = \rho(x)\nu(\mathrm{d}x) \in \mathscr{P}_2$. The gradient of U_ν at $\mu = \rho\nu$ is given by the signed measure

$$D_\mu U_\nu := -\{\Delta p(\rho) - \langle \nabla V, \nabla p(\rho)\rangle\}\nu,$$

where $p(r) := rU'(r) - U(r)$. In particular, when $V = 0$ and $U(r) = \rho\log\rho$, we have $D_\mu U_\nu = -\Delta\mu := -(\Delta\rho)(x)\mathrm{d}x$ for $\mu(\mathrm{d}x) = \rho(x)\mathrm{d}x$. See [18, pages 430, 431] for remarks on further development in this direction.

2 Image SDE and diffusion processes on $\mathscr{P}_2$

We will construct the $\mathscr{A}_t$-diffusion process by solving the image SDE (1.7). In general, we allow the coefficients

$$b : \Omega \times [0,\infty) \times \mathscr{P}_2 \to \mathbb{R}^d, \quad \sigma : \Omega \times [0,\infty) \times \mathscr{P}_2 \to \mathbb{R}^d \otimes \mathbb{R}^m$$

to be random but progressively measurable with respect to the filtration $\mathscr{F}_t$. We first present the definition of solution.

Definition 2.1. Let $(s,\mu) \in [0,\infty)\times\mathscr{P}_2$. A family of adapted processes $\{(X_{s,t}^{x,\mu})_{t\geq s} : x \in \mathbb{R}^d\}$ is called a solution to (1.7), if the following conditions hold $\mathbb{P}$-a.s.:

(*a*) $X_{s,t}^{x,\mu}$ is continuous in $t \in [s,\infty)$ and measurable in $x \in \mathbb{R}^d$;

(*b*) $\Lambda_{s,t}^\mu := \mu \circ (X_{s,t}^{\cdot,\mu})^{-1} \in \mathscr{P}_2$ is continuous in $t \geq s$;

(*c*) $\mathbb{E}\int_s^t (|b(r, X_{s,r}^{x,\mu}, \Lambda_{s,r}^\mu)| + \|\sigma(r, X_{s,r}^{x,\mu}, \Lambda_{s,r}^\mu)\|_{HS}^2)\mathrm{d}r < \infty$ and

$$X_{s,t}^{x,\mu} = x + \int_s^t b(r, X_{s,r}^{x,\mu}, \Lambda_{s,r}^\mu)\mathrm{d}r + \int_s^t \sigma(r, X_{s,r}^{x,\mu}, \Lambda_{s,r}^\mu)\mathrm{d}W_r, \quad t \geq s, x \in \mathbb{R}^d.$$

The image SDE (1.7) is called well-posed, if it has a unique solution for any $(s,\mu) \in [0,\infty)\times\mathscr{P}_2$.

To ensure the well-posedness of (1.7), we make the following assumption on b and σ.

(A) The progressively measurable coefficients $b(t,x,\mu)$ and $\sigma(t,x,\mu)$ are continuous in $(x,\mu) \in \mathbb{R}^d \times \mathscr{P}_2$, there exists $K \in L^q_{loc}([0,\infty) \to [0,\infty))$ for some $q > 1$ such that $\mathbb{P}$-a.s. for any $t \geq 0$,

$$|b(t,x,\mu)|^2 + \|\sigma(t,x,\mu)\|_{HS}^2 \leq K(t)\big(1 + |x|^2 + \|\mu\|_2^2\big), \quad (x,\mu) \in \mathbb{R}^d \times \mathscr{P}_2, \tag{2.1}$$

$$\begin{aligned} &2\langle b(t,x,\mu) - b(t,y,\nu), x - y\rangle^+ + \|\sigma(t,x,\mu) - \sigma(t,y,\nu)\|_{HS}^2 \\ &\leq K(t)\big(|x-y|^2 + \mathbb{W}_2(\mu,\nu)^2\big), \quad (x,\mu),(y,\nu) \in \mathbb{R}^d \times \mathscr{P}_2. \end{aligned} \tag{2.2}$$

Theorem 2.1. *Assume* **(A)**. *Then the image SDE* (1.7) *is well-posed, and the unique solution* $X_{s,t}^{x,\mu}$ *is jointly continuous in* $(t,x) \in [s,\infty) \times \mathbb{R}^d$. *Moreover:*

(1) *For any* $p \geq 1$, *there exists an increasing function* $C_p : [0,\infty) \to [0,\infty)$ *such that*

$$\mathbb{E} \sup_{r\in[s,t]} \big(|X_{s,r}^{x,\mu}|^{2p} + \mu(|X_{s,r}^{\cdot,\mu}|^2)^p\big) \leq C_p(t)(1+|x|^{2p} + \|\mu\|_2^{2p}), \tag{2.3}$$

$$\sup_{r\in[s,t]} \mathbb{E}\big\{|X_{s,r}^{x,\mu} - X_{s,r}^{y,\nu}|^{2p} + \mathbb{W}_2(\Lambda_{s,r}^\mu, \Lambda_{s,r}^\nu)^{2p}\big\} \leq C_p(t)(|x-y|^{2p} + \mathbb{W}_2(\mu,\nu)^{2p}) \tag{2.4}$$

hold for all $0 \leq s \leq t, x, y \in \mathbb{R}^d$ *and* $\mu, \nu \in \mathscr{P}_2$. *Consequently,* $X_{s,t}^{x,\mu}$ *is jointly continuous in* $(t,x) \in [s,\infty) \times \mathbb{R}^d$.

(2) *When* (b,σ) *is deterministic,* $\{(\Lambda_{s,t}^\mu)_{t\geq s} : \mu \in \mathscr{P}_2\}$ *is a diffusion process on* $\mathscr{P}_2$ *generated by* $\mathscr{A}_t$; *i.e. it is a continuous strong Markov process such that for any* $\mu \in \mathscr{P}_2$ *and any* $f \in C_b^2(\mathscr{P}_2)$,

$$f(\Lambda_{s,t}^\mu) - f(\mu) - \int_s^t \mathscr{A}_r f(\Lambda_{s,r}^\mu)\mathrm{d}r, \quad t \geq s$$

is a martingale.

(3) *When* (b,σ) *is deterministic,* $\{(X_{s,t}^{x,\mu}, \Lambda_{s,t}^\mu)_{t\geq s} : \mu \in \mathscr{P}_2\}$ *is a diffusion on* $\mathbb{R}^d \times \mathscr{P}_2$ *generated by* $\tilde{\mathscr{A}_t}$; *i.e. it is a continuous strong Markov process such that for any* $(x,\mu) \in \mathbb{R}^d \times \mathscr{P}_2$ *and any* $f \in C_b^{2,2}(\mathbb{R}^d \times \mathscr{P}_2)$,

$$f(X_{s,t}^{x,\mu}, \Lambda_{s,t}^\mu) - f(x,\mu) - \int_s^t \tilde{\mathscr{A}_r} f(X_{s,t}^{x,\mu}, \Lambda_{s,r})\mathrm{d}r, \quad t \geq s$$

is a martingale.

In the following two subsections, we prove Theorem 2.1(1) and (2)-(3) respectively.

2.1 Proof of Theorem 2.1(1)

Obviously, the uniqueness follows from (2.4). Below we prove (2.3), (2.4), joint continuity and the existence of the solution respectively.

(I) Estimate (2.3)**.** Let $(X_{s,t}^{x,\mu})_{x\in\mathbb{R}^d, t\geq s}$ be a solution of (1.7). We have

$$\|\Lambda_{s,t}^\mu\|_2^2 = \|\mu \circ (X_{s,t}^{\cdot,\mu})^{-1}\|_2^2 = \mu(|X_{s,t}^{\cdot,\mu}|^2), \quad t \geq s. \tag{2.5}$$

So, by (2.1) and Itô's formula, we may find out $\kappa \in L_{loc}^1([0,\infty) \to [0,\infty))$ such that

$$\mathrm{d}|X_{s,t}^{x,\mu}|^2 \leq \kappa(t)\big(1 + |X_{s,t}^{x,\mu}|^2 + \mu(|X_{s,t}^{\cdot,\mu}|^2)\big)\mathrm{d}t + 2\big\langle X_{s,t}^{x,\mu}, \sigma(t, X_{s,t}^{x,\mu}, \Lambda_{s,t}^\mu)\mathrm{d}W_t\big\rangle, \quad t \geq s. \tag{2.6}$$

Let $\gamma_t^x = 2\sigma(t, X_{s,t}^{x,\mu}, \Lambda_{s,t}^\mu)^* X_{s,t}^{x,\mu}$. Since $(\Lambda_{s,t}^\mu)_{t\ge s}$ is an adapted continuous process on $\mathscr{P}_2$ and due to (2.1), $\sigma(t,x,\mu)$ has linear growth in x, there exists an increasing function $c:[0,\infty)\to[0,\infty)$ such that

$$\mu(|\gamma_t^{\cdot}|) \le c(t)\big\{1+\mu(|X_{s,t}^{\cdot,\mu}|^2)\big\} = c(t)\big\{1+\|\Lambda_{s,t}^\mu\|_2^2\big\} < \infty.$$

So, integrating (2.6) with respect to $\mu(\mathrm{d}x)$ leads to

$$\mathrm{d}\mu(|X_{s,t}^{\cdot,\mu}|^2) \le \kappa(t)\big(1+2\mu(|X_{s,t}^{\cdot,\mu}|^2)\big)\mathrm{d}t + \langle \mu(\gamma_t^{\cdot}), \mathrm{d}W_t\rangle, \quad t\ge s. \tag{2.7}$$

Let $h_{s,t} := \mathrm{e}^{2\int_s^t \kappa(r)\mathrm{d}r}$ and

$$\tau_n = \inf\big\{t\ge s : \mu(|X_{s,t}^{\cdot,\mu}|^2) + |X_{s,t}^{x,\mu}|^2 \ge n\big\}, \quad n\ge 1.$$

Then (2.7) implies

$$\mu(|X_{s,t\wedge\tau_n}^{\cdot,\mu}|^2) \le h_{s,t}\|\mu\|_2^2 + \int_s^t h_{r,t}\kappa(r)\mathrm{d}r + \int_s^{t\wedge\tau_n} h_{r,t}\langle\mu(\gamma_r^{\cdot}), \mathrm{d}W_r\rangle, \quad t\ge s, \tag{2.8}$$

so that by (2.6),

$$\begin{aligned}|X_{s,t\wedge\tau_n}^{x,\mu}|^2 &\le |x|^2 + \int_s^{t\wedge\tau_n}\langle\mu(\gamma_r^{\cdot}), \mathrm{d}W_r\rangle \\ &+ \int_s^{t\wedge\tau_n}\kappa(r)\bigg\{1+|X_{s,r}^{x,\mu}|^2 + h_{s,r}\|\mu\|_2^2 + h_{s,r}\int_s^r\kappa(\theta)\mathrm{d}\theta + \int_s^r h_{\theta,r}\langle\gamma_\theta^x, \mathrm{d}W_\theta\rangle\bigg\}\mathrm{d}r\end{aligned} \tag{2.9}$$

holds for $t\ge s$. Moreover, (2.1) implies

$$|\gamma_t^x|^2 = |2\sigma(t, X_{s,t}^{x,\mu}, \Lambda_{s,t}^\mu)^* X_{s,t}^{x,\mu}|^2 \le 4K(t)|X_{s,t}^{x,\mu}|^2\big(1+|X_{s,t}^{x,\mu}|^2 + \mu(|X_{s,t}^{\cdot,\mu}|^2)\big). \tag{2.10}$$

This together with the Schwarz inequality gives

$$|\mu(\gamma_t^{\cdot})|^2 \le 4K(t)\mu(|X_{s,t}^{\cdot,\mu}|^2)\big(1+2\mu(|X_{s,t}^{\cdot,\mu}|^2)\big). \tag{2.11}$$

Then for any $p\ge 1$ and $\varepsilon>0$, there exists a constant $c=c(p,\varepsilon)>0$ such that

$$\bigg(\int_s^{t\wedge\tau_n}|\mu(\gamma_r^{\cdot})|^2\mathrm{d}r\bigg)^{\frac{p}{2}} \le \varepsilon \sup_{r\in[s,t\wedge\tau_n]}\big\{\mu(|X_{s,r}^{\cdot,\mu}|^2)\big\}^p + c\int_s^{t\wedge\tau_n}K(r)\big(1+\big\{\mu(|X_{s,r}^{\cdot,\mu}|^2)\big\}^p\big)\mathrm{d}r.$$

Combining this with (2.8) and using the BDG inequality, we may find an increasing function $C_0:[0,\infty)\to[0,\infty)$ such that

$$\begin{aligned}&\mathbb{E}\Big[\sup_{r\in[s,t\wedge\tau_n]}\big\{\mu(|X_{s,r}^{\cdot,\mu}|^2)\big\}^p\Big] \\ &\le \frac{1}{2}\mathbb{E}\Big[\sup_{r\in[s,t\wedge\tau_n]}\big\{\mu(|X_{s,r}^{\cdot,\mu}|^2)\big\}^p\Big] + \frac{C_0(t)}{2}\bigg(1+\|\mu\|_2^{2p} + \mathbb{E}\int_s^t\big\{\mu(|X_{s,r\wedge\tau_n}^{\cdot,\mu}|^2)\big\}^p\mathrm{d}r\bigg).\end{aligned}$$

By Gronwall's inequality, this implies

$$\mathbb{E}\Big[\sup_{r\in[s,t\wedge\tau_n]}\big\{\mu(|X^{\cdot,\mu}_{s,r}|^2)\big\}^p\Big]\le C_0(t)\mathrm{e}^{\int_s^t C_0(r)\mathrm{d}r}(1+\|\mu\|_2^{2p}). \tag{2.12}$$

Similarly, by (2.9)-(2.12) and the BDG inequality, we conclude that for any $p\ge 1$ there exist increasing functions $C_1,C_2:[0,\infty)\to[0,\infty)$ such that

$$\begin{aligned}\mathbb{E}\Big[\sup_{r\in[s,t\wedge\tau_n]}|X^{x,\mu}_{s,r}|^{2p}\Big]&\le C_1(t)\big(1+|x|^{2p}+\|\mu\|_2^{2p}\big)+C_1(t)\mathbb{E}\bigg(\int_s^{t\wedge\tau_n}\kappa(r)|X^{x,\mu}_{s,t}|^2\mathrm{d}r\bigg)^p\\&\quad+C_1(t)\mathbb{E}\bigg(\int_s^{t\wedge\tau_n}\big\{|\mu(\gamma^{\cdot}_r)|^2+|\gamma^x_r|^2\big\}\mathrm{d}r\bigg)^{\frac{p}{2}}\\&\le\frac12\mathbb{E}\Big[\sup_{r\in[s,t\wedge\tau_n]}|X^{x,\mu}_{s,r}|^{2p}\Big]+C_2(t)\big(1+|x|^{2p}+\|\mu\|_2^{2p}\big)+C_2(t)\mathbb{E}\int_s^t\kappa(r)|X^{x,\mu}_{s,r}|^{2p}\mathrm{d}r,\quad t\ge s.\end{aligned}$$

By Grownwall's lemma, there exists an increasing function $Q:[0,\infty)\to(0,\infty)$ such that

$$\mathbb{E}\Big[\sup_{r\in[s,t\wedge\tau_n]}|X^{x,\mu}_{s,r}|^{2p}\Big]\le Q(t)(1+|x|^{2p}+\|\mu\|_2^{2p}),\quad t\ge s.$$

By letting $n\to\infty$ in this inequality and (2.12), we prove (2.3) for some increasing function $C_p:[0,\infty)\to[0,\infty)$.

(II) Estimate (2.4). Let $\pi\in\mathscr{C}(\mu,\nu)$ such that

$$\mathbb{W}_2(\mu,\nu)^2=\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|^2\pi(\mathrm{d}x,\mathrm{d}y). \tag{2.13}$$

Then $\pi_{s,t}:=\pi\circ(X^{\cdot,\mu}_{s,t},X^{\cdot,\nu}_{s,t})^{-1}\in\mathscr{C}(\Lambda^\mu_{s,t},\Lambda^\nu_{s,t})$, so that

$$\begin{aligned}\mathbb{W}_2(\Lambda^\mu_{s,t},\Lambda^\nu_{s,t})^2&\le\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|^2\pi_{s,t}(\mathrm{d}x,\mathrm{d}y)\\&=\int_{\mathbb{R}^d\times\mathbb{R}^d}|X^{x,\mu}_{s,t}-X^{y,\nu}_{s,t}|^2\pi(\mathrm{d}x,\mathrm{d}y)=:\ell_{s,t},\quad t\ge s.\end{aligned} \tag{2.14}$$

Thus, by (2.2) and Itô's formula, we obtain

$$\begin{aligned}\mathrm{d}|X^{x,\mu}_{s,t}-X^{y,\nu}_{s,t}|^2&\le K(t)\big\{|X^{x,\mu}_{s,t}-X^{y,\nu}_{s,t}|^2+\ell_{s,t}\big\}\mathrm{d}t\\&\quad+2\big\langle X^{x,\mu}_{s,t}-X^{y,\nu}_{s,t},\{\sigma(r,X^{x,\mu}_{s,t},\Lambda^\mu_{s,t})-\sigma(r,X^{y,\nu}_{s,t},\Lambda^\nu_{s,t})\}\mathrm{d}W_t\big\rangle,\quad t\ge s.\end{aligned} \tag{2.15}$$

Integrating both sides with respect to $\pi_{s,t}(\mathrm{d}x,\mathrm{d}y)$, and letting

$$\eta_t=2\int_{\mathbb{R}^d\times\mathbb{R}^d}\{\sigma(t,X^{x,\mu}_{s,t},\Lambda^\mu_{s,t})-\sigma(t,X^{y,\nu}_{s,t},\Lambda^\nu_{s,t})\}^*(X^{x,\mu}_{s,t}-X^{y,\nu}_{s,t})\,\pi(\mathrm{d}x,\mathrm{d}y),$$

we arrive at

$$\mathrm{d}\ell_{s,t}\le 2K(t)\ell_{s,t}\mathrm{d}t+\langle\eta_t,\mathrm{d}W_t\rangle,\quad t\ge s.$$

This together with $\ell_{s,t} = \mathbb{W}_2(\mu,\nu)^2$ implies

$$\ell_{s,t} \le \mathbb{W}_2(\mu,\nu)^2 \mathrm{e}^{2\int_s^t K(r)\mathrm{d}r} + \int_s^t \mathrm{e}^{2\int_r^t K(\theta)\mathrm{d}\theta} \langle \eta_r, \mathrm{d}W_r\rangle, \quad t \ge s. \tag{2.16}$$

Moreover, **(A)** and the Schwarz inequality yield

$$\begin{aligned} |\eta_r|^2 &\le 4K(r)\ell_{s,r} \int_{\mathbb{R}^d\times\mathbb{R}^d} \big\{|X_{s,r}^{x,\mu} - X_{s,r}^{y,\nu}|^2 + \mathbb{W}_2(\Lambda_{s,r}^\mu, \Lambda_{s,r}^\nu)^2\big\}\pi(\mathrm{d}x,\mathrm{d}y) \\ &\le 8K(r)\ell_{s,r}^2, \quad r \ge s. \end{aligned} \tag{2.17}$$

For given $x, y \in \mathbb{R}^d$ and $\mu,\nu \in \mathscr{P}_2$, let

$$\tau_n = \inf\big\{t \ge s : \|\Lambda_{s,t}^\mu\|_2 + \|\Lambda_{s,t}^\nu\|_2 + |X_{s,t}^{x,\mu}| + |X_{s,t}^{y,\nu}| \ge n\big\}.$$

By (2.16), (2.17) and using the Hölder and BDG inequalities, we may find out increasing functions $c_1, c_2 : [0,\infty) \to [0,\infty)$ such that

$$\begin{aligned} \mathbb{E}\Big[\sup_{r\in[s,t]} \ell_{s,r\wedge\tau_n}^p\Big] &\le c_1(t)\mathbb{W}_2(\mu,\nu)^{2p} + c_1(t)\mathbb{E}\bigg(\int_s^{t\wedge\tau_n} |\eta_r|^2\mathrm{d}r\bigg)^{\frac{p}{2}} \\ &\le c_1(t)\mathbb{W}_2(\mu,\nu)^{2p} + c_2(t)\int_s^t \mathbb{E}\ell_{s,r\wedge\tau_n}^p\mathrm{d}r + \frac{1}{2}\mathbb{E}\Big[\sup_{r\in[s,t]} \ell_{s,r\wedge\tau_n}^p\Big], \quad t \ge s. \end{aligned}$$

Then it follows from Gronwall's lemma that

$$\mathbb{E}\Big[\sup_{r\in[s,t]} \ell_{s,r\wedge\tau_n}^p\Big] \le 2c_1(t)\mathrm{e}^{2tc_2(t)}\mathbb{W}_2(\mu,\nu)^{2p}, \quad t \ge s.$$

By letting $n \to \infty$ and using Fatou's lemma, we obtain

$$\mathbb{E}\Big[\sup_{r\in[s,t]} \ell_{s,r}^p\Big] \le 2c(t)\mathrm{e}^{2tc_p(t)}\mathbb{W}_2(\mu,\nu)^{2p}, \quad t \ge s. \tag{2.18}$$

Similarly, by (2.15), (2.18), assumption **(A)** and using the Hölder and BDG inequality, for any $p \ge 1$ we find out increasing functions $K_1, K_2 : [0,\infty) \to [0,\infty)$ such that

$$\begin{aligned} \mathbb{E}\Big[\sup_{r\in[s,t]} |X_{s,r\wedge\tau_n}^{x,\mu} - X_{s,r\wedge\tau_n}^{y,\nu}|^{2p}\Big] &\le |x-y|^{2p} + K_1(t)\mathbb{E}\int_s^{t\wedge\tau_n} K(r)\big\{|X_{s,r}^{x,\mu} - X_{s,r}^{y,\nu}|^{2p} + \ell_{s,r}^p\big\}\mathrm{d}r \\ &\le |x-y|^{2p} + K_2(t)\mathbb{E}\int_s^t K(r)|X_{s,r\wedge\tau_n}^{x,\mu} - X_{s,r\wedge\tau_n}^{y,\nu}|^{2p}\,\mathrm{d}r + K_2(t)\mathbb{W}_2(\mu,\nu)^{2p}, \quad t \ge s. \end{aligned}$$

Therefore, by Grownwall's lemma, there exists an increasing function $C : [0,\infty) \to (0,\infty)$ such that

$$\mathbb{E}\Big[\sup_{r\in[s,t]} |X_{s,r\wedge\tau_n}^{x,\mu} - X_{s,r\wedge\tau_n}^{y,\nu}|^{2p}\Big] \le C(t)\big(|x-y|^{2p} + \mathbb{W}_2(\mu,\nu)^{2p}\big), \quad t \ge s.$$

Letting $n \to \infty$ and using Fatou's lemma, we arrive at

$$\mathbb{E}\Big[\sup_{r\in[s,t]} |X_{s,r}^{x,\mu} - X_{s,r}^{y,\nu}|^{2p}\Big] \le C(t)\big(|x-y|^{2p} + \mathbb{W}_2(\mu,\nu)^{2p}\big), \quad t \ge s.$$

Combining this with (2.14) and (2.18), we prove (2.4) for some increasing function $C_p : [0,\infty) \to [0,\infty)$.

(III) Joint continuity of $X_{s,t}^{x,\mu}$ in (t,x). Let $K \in L^q_{loc}([0,\infty) \to [0,\infty))$ for some $q > 1$. By (2.1), (2.3) and (2.4), for any $n, p \geq 1$, there exist constants $C_1, C_2 > 0$ such that for any $n \geq t \geq r \geq s$, and $|x|, |y| \leq n$,

$$
\begin{aligned}
&\mathbb{E}\big(|X_{s,t}^{x,\mu} - X_{s,r}^{y,\mu}|^{2p}\big) \leq 2^{2p-1}\big(\mathbb{E}|X_{s,t}^{x,\mu} - X_{s,t}^{y,\mu}|^{2p} + \mathbb{E}|X_{s,t}^{y,\mu} - X_{s,r}^{y,\mu}|^{2p}\big)\\
&\leq C_1|x-y|^{2p} + C_1\mathbb{E}\bigg|\int_r^t K(\theta)\sqrt{1+|X_{s,\theta}^{y,\mu}|^2 + \mu(|X_{s,\theta}^{y,\mu}|^2)}\mathrm{d}\theta\bigg|^{2p}\\
&\quad + C_1\mathbb{E}\bigg(\int_r^t K(\theta)\big\{1+|X_{s,\theta}^{y,\mu}|^2 + \mu(|X_{s,\theta}^{y,\mu}|^2)\big\}\mathrm{d}\theta\bigg)^p\\
&\leq C_1|x-y|^{2p} + C_1\bigg(\int_r^t K(\theta)^q\mathrm{d}\theta\bigg)^{\frac{2p}{q}}\mathbb{E}\bigg|\int_r^t \big(1+|X_{s,\theta}^{y,\mu}|^2 + \mu(|X_{s,\theta}^{y,\mu}|^2)\big)^{\frac{q}{2(q-1)}}\mathrm{d}\theta\bigg|^{\frac{2p(q-1)}{q}}\\
&\quad + C_1\bigg(\int_r^t K(\theta)^q\bigg)^{\frac{p}{q}}\mathbb{E}\bigg(\int_r^t \big\{1+|X_{s,\theta}^{y,\mu}|^2 + \mu(|X_{s,\theta}^{y,\mu}|^2)\big\}^{\frac{q}{q-1}}\mathrm{d}\theta\bigg)^{\frac{p(q-1)}{q}}\\
&\leq C_2\Big\{|x-y|^{2p} + (t-r)^{\frac{p(q-1)}{q}}\Big\}.
\end{aligned}
\tag{2.19}
$$

By Kolmogorov's continuity criterion, for large enough $p > 1$ this implies that $X_{s,t}^{x,\mu}$ has a $\mathbb{P}$-version jointly continuous in $(t,x) \in [s,n] \times \{x \in \mathbb{R}^d : |x| \leq n\}$. Since $n \geq 1$ is arbitrary, $X_{s,t}^{x,\mu}$ has a version jointly continuous in $(t,x) \in [s,\infty) \times \mathbb{R}^d$.

(IV) Existence of solution. It suffices to construct a solution up to an arbitrarily fixed time $T > 0$. To this end, we adopt an iteration argument as in [17].

(1) For fixed $(s,\mu) \in [0,T] \times \mathscr{P}_2$, let $\Lambda_{s,t}^{0,\mu} = \mu$ and $X_{s,t}^{0,x,\mu} = x$ for all $x \in \mathbb{R}^d$ and $t \geq s$.

(2) Assume that for some $n \in \mathbb{Z}_+$ we have constructed adapted $(X_{s,t}^{n,x,\mu})_{t\geq s, x\in\mathbb{R}^d}$ which is jointly continuous in $(t,x) \in [s,\infty) \times \mathscr{P}_2$, and satisfies

$$
\mathbb{E}\Big[\sup_{r\in[s,t]}|X_{s,r}^{n,x,\mu}|^2\Big] \leq c(t)(1+|x|^2+\|\mu\|_2^2), \quad t \geq s, x \in \mathbb{R}^d \tag{2.20}
$$

for some increasing $c : [0,\infty) \to [0,\infty)$. Consequently, $\Lambda_{s,t}^{n,\mu} := \mu \circ (X_{s,t}^{n,\cdot,\mu})^{-1} \in \mathscr{P}_2$ is continuous in $t \geq s$. Indeed, by the Fubini theorem, (2.20) implies

$$
\mathbb{E}\Big[\mu\Big(\sup_{r\in[s,t]}|X_{s,r}^{n,\cdot,\mu}|^2\Big)\Big] \leq c(t)(1+2\|\mu\|_2^2) < \infty, \quad t \geq s,
$$

so that $\mathbb{P}$-a.s

$$
\mu\Big(\sup_{r\in[s,t]}|X_{s,r}^{n,\cdot,\mu}|^2\Big) < \infty, \quad t \geq s.
$$

Then by the dominated convergence theorem and the continuity of $X_{s,t}^{n,x,\mu}$ in $t \geq s$, we obtain $\mathbb{P}$-a.s.

$$
\lim_{r\to t}\mathbb{W}_2(\Lambda_{s,r\vee s}^{n,\mu}, \Lambda_{s,t}^{n,\mu})^2 \leq \lim_{r\to t}\mu\big(|X_{s,r\vee s}^{n,\cdot,\mu} - X_{s,t}^{n,\cdot,\mu}|^2\big) = 0, \quad t \geq s.
$$

(3) Let $(X^{n+1,x,\mu}_{s,t})_{t\ge s}$ solve the SDE

$$\mathrm{d}X^{n+1,x,\mu}_{s,t} = b(t, X^{n+1,x,\mu}_{s,t}, \Lambda^{n,\mu}_{s,t})\mathrm{d}t + \sigma(t, X^{n+1,x,\mu}_{s,t}, \Lambda^{n,\mu}_{s,t})\mathrm{d}W_t, \quad t \ge s, X^{n+1,x,\mu}_{s,s} = x.$$

By **(A)** and (2.20), it is easy to see that this SDE is well-posed, and when x varies the inequality (2.20) holds for $X^{n+1,x,\mu}_{s,t}$ replacing $X^{n,x,\mu}_{s,t}$ with possibly a different function $c : [0,\infty) \to [0,\infty)$. Moreover, as in **(III)**, **(A)** and (2.20) also imply the joint continuity of $X^{n+1,x,\mu}_{s,t}$ in $(t,x) \in [s,\infty) \times \mathbb{R}^d$. Consequently, as shown in step (2) that $\Lambda^{n+1,\mu}_{s,t} := \mu \circ (X^{n+1,\cdot,\mu}_{s,t})^{-1} \in \mathscr{P}_2$ is continuous in $t \ge s$.

Therefore, we have construct a sequence $\{(X^{n,x,\mu}_{s,t}, \Lambda^{n,\mu}_{s,t})_{t\ge s, x\in\mathbb{R}^d}\}_{n\ge 0}$, which satisfies (2.20), $X^{n,x,\mu}_{s,t}$ is jointly continuous in $(t,x) \in [s,\infty) \times \mathbb{R}^d$, and $\mathbb{P}$-a.s.

$$X^{n+1,x,\mu}_{s,t} = x + \int_s^t b(r, X^{n+1,x,\mu}_{s,r}, \Lambda^{n,\mu}_{s,r})\mathrm{d}r + \int_s^t \sigma(r, X^{n+1,x,\mu}_{s,r}, \Lambda^{n,\mu}_{s,r})\mathrm{d}W_r, \quad t \ge s, x \in \mathbb{R}^d. \tag{2.21}$$

The following lemma gives a constant $t_0 > 0$ independent of $(s,x,\mu) \in [0,T] \times \mathbb{R}^d \times \mathscr{P}_2$, such that $\{X^{n,x,\mu}_{s,\cdot}\}_{n\ge 1}$ is a Cauchy sequence in $L^2(\Omega \to C([s,s+t_0] \to \mathbb{R}^d); \mathbb{P})$.

Lemma 2.2. *Assume* **(A)**. *For fixed $T > 0$, there exists a constant $t_0 > 0$ such that*

$$\lim_{n,m\to\infty} \sup_{(s,x,\mu)\in[0,T]\times\mathbb{R}^d\times\mathscr{P}_2} \frac{\mathbb{E}\sup_{t\in[s,s+t_0]} |X^{m,x,\mu}_{s,t} - X^{n,x,\mu}_{s,t}|^2}{1 + |x|^2 + \|\mu\|_2^2} = 0.$$

Proof. As in (2.14), we have $\mathbb{W}_2(\Lambda^{n,\mu}_{s,t}, \Lambda^{n-1,\mu}_{s,t})^2 \le \mu(|X^{n,\cdot,\mu}_{s,t} - X^{n-1,\cdot,\mu}_{s,t}|^2)$ for $n \ge 1$. Combining this with By (2.2) and Itô's formula, we obtain

$$\begin{aligned}\mathrm{d}|X^{n+1,x,\mu}_{s,t} - X^{n,x,\mu}_{s,t}|^2 &\le K(t)\Big\{|X^{n+1,x,\mu}_{s,t} - X^{n,x,\mu}_{s,t}|^2 + \mu(|X^{n,\cdot,\mu}_{s,t} - X^{n-1,\cdot,\mu}_{s,t}|^2)\Big\}\mathrm{d}t\\ &\quad + 2\big\langle X^{n+1,x,\mu}_{s,t} - X^{n,x,\mu}_{s,t}, \{\sigma(t, X^{n+1,x,\mu}_{s,t}, \Lambda^{n,\mu}_{s,t}) - \sigma(t, X^{n,x,\mu}_{s,t}, \Lambda^{n-1,\mu}_{s,t})\}\mathrm{d}W_t\big\rangle, \quad t \ge s.\end{aligned}$$

So, by (2.2) and using the BDG inequality, we may find out constants $c_1, c_2 > 0$ such that

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\in[s,s+t_0]} |X^{n+1,x,\mu}_{s,t} - X^{n,x,\mu}_{s,t}|^2\Big]\\ &\le \int_s^t K(r)\mathbb{E}\big[|X^{n+1,x,\mu}_{s,r} - X^{n,x,\mu}_{s,r}|^2 + \mu(|X^{n,\cdot,\mu}_{s,r} - X^{n-1,\cdot,\mu}_{s,t}|^2)\big]\mathrm{d}r\\ &\quad + c_1\mathbb{E}\bigg(\int_s^t K(r)|X^{n+1,x,\mu}_{s,r} - X^{n,x,\mu}_{s,r}|^2\big\{|X^{n+1,x,\mu}_{s,r} - X^{n,x,\mu}_{s,t}|^2 + \mu(|X^{n,\cdot,\mu}_{s,r} - X^{n-1,\cdot,\mu}_{s,t}|^2)\big\}\mathrm{d}r\bigg)^{\frac{1}{2}}\\ &\le \frac{c_2}{2}\int_s^t K(r)\mathbb{E}\big[|X^{n+1,x,\mu}_{s,r} - X^{n,x,\mu}_{s,r}|^2 + \mu(|X^{n,\cdot,\mu}_{s,r} - X^{n-1,\cdot,\mu}_{s,t}|^2)\big]\mathrm{d}r\\ &\quad + \frac{1}{2}\mathbb{E}\Big[\sup_{t\in[s,s+t_0]} |X^{n+1,x,\mu}_{s,t} - X^{n,x,\mu}_{s,t}|^2\Big], \quad t \ge s.\end{aligned}$$

Since (2.20) holds for all n, this and Grownwall's inequality imply

$$\mathbb{E}\sup_{r\in[s,t]} |X^{n+1,x,\mu}_{s,r} - X^{n,x,\mu}_{s,r}|^2 \le c_2\int_s^t \mathrm{e}^{c_2\int_r^t K(\theta)\mathrm{d}\theta}\mathbb{E}\mu(|X^{n,\cdot,\mu}_{s,r} - X^{n-1,\cdot,\mu}_{s,r}|^2)\mathrm{d}r, \quad t \ge s \tag{2.22}$$

for all $(s,x)\in[0,T]\times\mathbb{R}^d$. Taking integral with respect to $\mu(\mathrm{d}x)$ leads to

$$\sup_{r\in[s,t]}\mathbb{E}\mu(|X^{n+1,\cdot,\mu}_{s,r}-X^{n,\cdot,\mu}_{s,r}|^2)\le c_2(t-s)\mathrm{e}^{c_2\int_s^t K(r)\mathrm{d}r}\sup_{r\in[s,t]}\mathbb{E}\mu(|X^{n,\cdot,\mu}_{s,r}-X^{n-1,\cdot,\mu}_{s,r}|^2),\quad t\ge s.$$

Now, taking $t_0>0$ such that

$$\varepsilon:=c_2t_0\mathrm{e}^{c_2\int_0^{T+t_0}K(r)\mathrm{d}r}<1, \tag{2.23}$$

by iterating in n we arrive at

$$\begin{aligned}&\sup_{s\in[0,T],t\in[s,s+t_0]}\mathbb{E}\mu(|X^{n+1,\cdot,\mu}_{s,t}-X^{n,\cdot,\mu}_{s,t}|^2)\le\varepsilon\sup_{s\in[0,T],t\in[s,s+t_0]}\mathbb{E}\mu(|X^{n,\cdot,\mu}_{s,t}-X^{n-1,\cdot,\mu}_{s,t}|^2)\\&\le\cdots\le\varepsilon^n\sup_{s\in[0,T],t\in[s,s+t_0]}\mathbb{E}\mu(|X^{1,\cdot,\mu}_{s,t}-X^{0,\cdot,\mu}_{s,t}|^2)=c(x,\mu)\varepsilon^n<\infty,\end{aligned}$$

where due to (2.20),

$$c(x,\mu):=\sup_{s\in[0,T]}\sup_{t\in[s,s+t_0]}\mathbb{E}\mu(|X^{1,\cdot,\mu}_{s,t}-x|^2)\le c(1+|x|^2+\|\mu\|_2^2)$$

for some constant $c>0$. Substituting this into (2.22) and using (2.23), we get

$$\sup_{s\in[0,T]}\mathbb{E}\sup_{t\in[s,s+t_0]}|X^{n+1,x,\mu}_{s,t}-X^{n,x,\mu}_{s,t}|^2\le c(1+|x|^2+\|\mu\|_2^2)\varepsilon^n,\quad n\ge1.$$

This finishes the proof. □

By Lemma 2.2, there exist a constant $t_0>0$ depending on $T>0$, such that for any $s\in[0,T)$ we have a family of continuous processes

$$\{(X^{x,\mu}_{s,t})_{t\in[s,s+t_0]}:x\in\mathbb{R}^d,\mu\in\mathscr{P}_2\}$$

which are measurable in x and

$$\lim_{n\to\infty}\mathbb{E}\Big[\sup_{r\in[s,s+t_0]}\big(|X^{n,x,\mu}_{s,r}-X^{x,\mu}_{s,t}|^2+\mu(|X^{n,\cdot,\mu}_{s,r}-X^{\cdot,\mu}_{s,t}|^2)\big)\Big]=0.$$

Letting $\Lambda^\mu_{s,t}=\mu\circ(X^{\cdot,\mu}_{s,t})^{-1}$, by this and (2.14) we obtain

$$\lim_{n\to\infty}\mathbb{E}\Big[\sup_{r\in[s,s+t_0]}\mathbb{W}_2(\Lambda^{n,\mu}_{s,t},\Lambda^\mu_{s,t})^2\Big]\le\mathbb{E}\Big[\sup_{r\in[s,s+t_0]}\mu(|X^{n,\cdot,\mu}_{s,r}-X^{\cdot,\mu}_{s,t}|^2)\Big]=0.$$

Thus, the continuity of $\Lambda^{n,\mu}_{s,t}$ in $t\in[s,s+t_0]$ implies that of $\Lambda^\mu_{s,t}$; due to (2.20) we may find out a constant $c_1>0$ such that

$$\mathbb{E}\Big[\sup_{t\in[s,s+t_0]}\big\{\mu(|X^{\cdot,\mu}_{s,t}|^2)+|X^{x,\mu}_{s,t}|^2\big\}\Big]\le c_1\big(1+|x|^2+\|\mu\|_2^2\big),\quad(s,x,\mu)\in[0,T]\times\mathbb{R}^d\times\mathscr{P}_2; \tag{2.24}$$

and finally, by assumption **(A)** we may let $n\to\infty$ in (2.21) to derive

$$X^{x,\mu}_{s,t}=x+\int_s^t b(r,X^{x,\mu}_{s,r},\Lambda^\mu_{s,r})\mathrm{d}r+\int_s^t\sigma(r,X^{x,\mu}_{s,r},\Lambda^\mu_{s,r})\mathrm{d}W_r,\quad t\in[s,s+t_0],x\in\mathbb{R}^d.$$

So, when $T \le s+t_0$ we have solved the SDE up to time T.

In the case that $T > s+t_0$, let $\bar s = s+t_0, \bar x = X_{s,s+t_0}^{x,\mu}$ and $\bar\mu = \Lambda_{s,s+t_0}^\mu$. Since given $\mathscr{F}_{s+t_0}$ the process $(W_t - W_{\bar s})_{t\ge \bar s}$ is an m-dimensional Brownian motion, and $(\bar x, \bar\mu)$ is given as well, as in above we may construct a solution $(X_{\bar s,t}^{\bar x,\bar\mu}, \Lambda_{\bar s,t}^{\bar\mu})_{t\in[\bar s,\bar s+t_0]}$ for (1.7) with $\bar s$ replacing s. Then extending $(X_{s,t}^{x,\mu}, \Lambda_{s,t}^\mu)$ to $t\in[\bar s,\bar s+t_0]$ by letting

$$X_{s,t}^{x,\mu} = X_{\bar s,t}^{\bar x,\bar\mu},\quad \Lambda_{s,t}^\mu = \Lambda_{\bar s,t}^{\bar\mu},\quad t\in[\bar s,\bar s+t_0],$$

we see that $(X_{s,t}^{x,\mu}, \Lambda_{s,t}^\mu)_{t\in[s,s+2t_0]}$ solves (1.7) up to time $\bar s + t_0 = s+2t_0$. Runing this procedure for k times until $s+kt_0 \ge T$, we construct a solution to (1.7) up to time T.

2.2 Proof of Theorem 2.1(2)-(3)

We first establish Itô's formula for the diffusion process $(\Lambda_{s,t}^\mu)_{t\ge s}$. To this end, we need the following chain rule for the L-derivative, which is essentially due to [4, Theorem 6.5] where the reference probability space is Polish, see also [6, Proposition A.2] for general probability space but bounded random variables $\{\xi_s\}_{s\in[0,\varepsilon]}$ (note that D_k therein is compact).

Lemma 2.3. *Let $\{\xi_s\}_{s\in[0,\varepsilon]}$ for some $\varepsilon>0$ be a family of square integrable random variables on $\mathbb{R}^d$ with respect to a probability space $(\Omega,\mathscr{F},\mathbb{P})$, and let $\mathscr{L}_{\xi_s}$ denote the law of ξ_s. If*

$$\xi_0' := \lim_{s\downarrow 0}\frac{\xi_s-\xi_0}{s}$$

exists in $L^2(\Omega\to\mathbb{R}^d;\mathbb{P})$, then for any $f\in C^1(\mathscr{P}_2)$,

$$\lim_{s\downarrow 0}\frac{f(\mathscr{L}_{\xi_s}) - f(\mathscr{L}_{\xi_0})}{s} = \mathbb{E}\langle Df(\mathscr{L}_{\xi_0})(\xi_0), \xi_0'\rangle.$$

Proof. By a standard extension argument, we may and do assume that $(\Omega,\mathscr{F},\mathbb{P})$ is atomless. For instance, we enlarge $(\Omega,\mathscr{F},\mathbb{P})$ by $(\Omega\times[0,1], \mathscr{F}\times\mathscr{B}([0,1]), \mathbb{P}\times \mathrm{d}r)$ and use $\tilde\xi_s$ to replace ξ_s, where $\tilde\xi_s(\omega,r) := \xi_s(\omega)$ for $(\omega,r)\in\Omega\times[0,1]$, so that $\mathscr{L}_{\tilde\xi_s}$ under $\mathbb{P}\times\mathrm{d}r$ coincides with $\mathscr{L}_{\xi_s}$ under $\mathbb{P}$. Then the proof is completely similar to that of [14, Proposition 3.1] for ξ_s replacing $X+sY$. $\square$

Lemma 2.4 (Itô's formula)**.** *Assume* **(A)** *and let $\{\Lambda_{s,t}^\mu = \mu\circ(X_{s,t}^{\cdot,\mu})^{-1}\}_{t\ge s}$ for the solution to* (1.7)*. Then for any $f\in C_b^2(\mathscr{P}_2)$,*

$$\mathrm{d}f(\Lambda_{s,t}^\mu) = (\mathscr{A}_t f)(\Lambda_{s,t}^\mu)\mathrm{d}t + \bigg\langle \int_{\mathbb{R}^d}\big\{\sigma(t,x,\Lambda_{s,t}^\mu)^*(Df)(\Lambda_{s,t}^\mu)(x)\big\}\mu(\mathrm{d}x), \mathrm{d}W_t\bigg\rangle,\quad t\ge s.$$

Proof. For any $t\ge s$ and small $\varepsilon>0$, let

$$\xi_r = (1-r)X_{s,t}^{\cdot,\mu} + rX_{s,t+\varepsilon}^{\cdot,\mu} : \mathbb{R}^d\to\mathbb{R}^d,\quad r\in[0,1].$$

Then $\mu\circ\xi_r^{-1}$ is the law of ξ_r on the probability space $(\mathbb{R}^d,\mathscr{B}(\mathbb{R}^d),\mu)$. By (2.3),

$$\sup_{r\in[0,1]}\mathbb{E}\|\mu\circ\xi_r^{-1}\|_2^2 \le \mathbb{E}\Big[\sup_{r\in[0,1]}\mu(|\xi_r|^2)\Big] < \infty,\quad t\ge s.$$

Moreover, $\xi_r' := \frac{\mathrm{d}}{\mathrm{d}r}\xi_r = X^{\cdot,\mu}_{s,t+\varepsilon} - X^{\cdot,\mu}_{s,t}$ exists in $L^2(\mathbb{R}^d \to \mathbb{R}^d; \mu)$. So, Lemma 2.3 implies

$$
\begin{aligned}
f(\Lambda^\mu_{s,t+\varepsilon}) - f(\Lambda^\mu_{s,t}) &= f(\mu\circ\xi_1^{-1}) - f(\mu\circ\xi_0^{-1}) = \int_0^1 \Big(\frac{\mathrm{d}}{\mathrm{d}r} f(\mu\circ\xi_r^{-1})\Big)\mathrm{d}r \\
&= \int_{\mathbb{R}^d\times[0,1]} \big\langle Df(\mu\circ\xi_r^{-1})(\xi_r^x), X^{x,\mu}_{s,t+\varepsilon} - X^{x,\mu}_{s,t}\big\rangle \mu(\mathrm{d}x)\mathrm{d}r \\
&= \int_{\mathbb{R}^d} I_1(x)\mu(\mathrm{d}x) + \int_{\mathbb{R}^d\times[0,1]} I_2(x,r)\mu(\mathrm{d}x)\mathrm{d}r + \int_{\mathbb{R}^d\times[0,1]} I_3(x,r)\mu(\mathrm{d}x)\mathrm{d}r,
\end{aligned}
\tag{2.25}
$$

where, since $\mu\circ\xi_0^{-1} = \Lambda^\mu_{s,t}$,

$$
\begin{aligned}
&I_1(x) := \langle Df(\Lambda^\mu_{s,t})(X^{x,\mu}_{s,t}), X^{x,\mu}_{s,t+\varepsilon} - X^{x,\mu}_{s,t}\rangle,\\
&I_2(x,r) := \big\langle Df(\mu\circ\xi_r^{-1})(\xi_r^x) - Df(\mu\circ\xi_0^{-1})(\xi_r^x), X^{x,\mu}_{s,t+\varepsilon} - X^{x,\mu}_{s,t}\big\rangle,\\
&I_3(x,r) := \big\langle Df(\Lambda^\mu_{s,t})(\xi_r^x) - Df(\Lambda^\mu_{s,t})(\xi_0^x), X^{x,\mu}_{s,t+\varepsilon} - X^{x,\mu}_{s,t}\big\rangle.
\end{aligned}
$$

Below, we calculate $I_1(x), I_2(x)$ and $I_3(x)$ respectively.

Firstly, by (1.7) and $f \in C_b^2(\mathscr{P}_2)$, we have

$$
\begin{aligned}
I_1(x) &= \int_t^{t+\varepsilon} \big\langle (Df)(\Lambda^\mu_{s,t})(X^{x,\mu}_{s,t}), \mathrm{d}X^{x,\mu}_{s,u}\big\rangle = \int_t^{t+\varepsilon} \big\langle (Df)(\Lambda^\mu_{s,u})(X^{x,\mu}_{s,u}), \mathrm{d}X^{x,\mu}_{s,u}\big\rangle + \mathrm{o}(\varepsilon)\\
&= \int_t^{t+\varepsilon} \big\langle (Df)(\Lambda^\mu_{s,u})(X^{x,\mu}_{s,u}), b(u, X^{x,\mu}_{s,u}, \Lambda^\mu_{s,u})\big\rangle \mathrm{d}u\\
&\quad + \int_t^{t+\varepsilon} \big\langle (Df)(\Lambda^\mu_{s,u})(X^{x,\mu}_{s,u}), \sigma(u, \mathrm{d}X^{x,\mu}_{s,u}, \Lambda^\mu_{s,u})\mathrm{d}W_u\big\rangle + \mathrm{o}(\varepsilon),
\end{aligned}
\tag{2.26}
$$

where and in the following, $\mathrm{o}(\varepsilon)$ means ε-dependent (real, vector or matrix valued) random variables satisfying $\lim_{\varepsilon\to 0}\varepsilon^{-1}|\mathrm{o}(\varepsilon)| = 0$.

Next, (1.7) implies

$$
(X^{x,\mu}_{s,t+\varepsilon} - X^{x,\mu}_{s,t}) \otimes (X^{y,\mu}_{s,t+\varepsilon} - X^{y,\mu}_{s,t}) = \int_t^{t+\varepsilon} \sigma(u, X^{x,\mu}_{s,u}, \Lambda^\mu_{s,u})\sigma(u, X^{y,\mu}_{s,u}, \Lambda^\mu_{s,u})^*\mathrm{d}u + \mathrm{o}(\varepsilon).
\tag{2.27}
$$

Combining this with $f \in C_b^2(\mathscr{P}_2)$, we deduce from Lemma 2.3 and $\xi_\theta' = X^{\cdot,\mu}_{s,t+\varepsilon} - X^{\cdot,\mu}_{s,t}$ that up to an error term $\mathrm{o}(\varepsilon)$,

$$
\begin{aligned}
I_2(x,r) &= \int_0^r \mathrm{d}\theta \int_{\mathbb{R}^d} \big\langle (D^2 f)(\mu\circ\xi_\theta^{-1})(\xi_r^x, \xi_\theta^y), (X^{x,\mu}_{s,t+\varepsilon} - X^{x,\mu}_{s,t}) \otimes (\xi_\theta^y)'\big\rangle \mu(\mathrm{d}y)\\
&= \int_0^r \mathrm{d}\theta \int_t^{t+\varepsilon} \mathrm{d}u \int_{\mathbb{R}^d} \big\langle (D^2) f(\mu\circ\xi_\theta^{-1})(\xi_r^x, \xi_\theta^y), \sigma(u, X^{x,\mu}_{s,u}, \Lambda^\mu_{s,u})\sigma(u, X^{y,\mu}_{s,u}, \Lambda^\mu_{s,u})^*\big\rangle \mu(\mathrm{d}y)\\
&= r\int_t^{t+\varepsilon} \mathrm{d}u \int_{\mathbb{R}^d} \big\langle (D^2 f)(\Lambda^\mu_{s,u})(X^{x,\mu}_{s,u}, X^{y,\mu}_{s,u}), \sigma(u, X^{x,\mu}_{s,u}, \Lambda^\mu_{s,u})\sigma(u, X^{y,\mu}_{s,u}, \Lambda^\mu_{s,u})^*\big\rangle \mu(\mathrm{d}y).
\end{aligned}
\tag{2.28}
$$

Similarly, by using (2.27) with $x = y$, we obtain that up to an error term $\mathrm{o}(\varepsilon)$,

$$
I_3(x,r) = \big\langle (Df)(\mu\circ\xi_0^{-1})(\xi_r^x) - (Df)(\mu\circ\xi_0^{-1})(\xi_0^x), X^{x,\mu}_{s,t+\varepsilon} - X^{x,\mu}_{s,t}\big\rangle
$$

$$= \int_0^r \big\langle \nabla\{(Df)(\Lambda_{s,t}^\mu)\}(\xi_\theta^x), (X_{s,t+\varepsilon}^{x,\mu} - X_{s,t}^{x,\mu}) \otimes (X_{s,t+\varepsilon}^{x,\mu} - X_{s,t}^{x,\mu})\big\rangle \mathrm{d}r$$
$$= r\int_t^{t+\varepsilon} \big\langle \nabla\{(Df)(\Lambda_{s,u}^\mu)\}(X_{s,u}^{x,\mu}), (\sigma\sigma^*)(t, X_{s,u}^{x,\mu}, \Lambda_{s,u}^\mu)\big\rangle \mathrm{d}u.$$

Combining this with (2.25)-(2.28), we arrive at

$$\begin{aligned}
&\mathrm{d}f(\Lambda_{s,t}^\mu) - \int_{\mathbb{R}^d} \big\langle (Df)(\Lambda_{s,t}^\mu)(x), \sigma(t,x,\Lambda_{s,t}^\mu)\mathrm{d}W_t\big\rangle \mu(\mathrm{d}x)\\
&= \bigg(\int_{\mathbb{R}^d} \big\langle (Df)(\Lambda_{s,t}^\mu)(X_{s,t}^{x,\mu}), b(t, X_{s,t}^{x,\mu}, \Lambda_{s,t}^\mu)\big\rangle \mu(\mathrm{d}x)\bigg)\mathrm{d}t\\
&\quad + \bigg(\frac{1}{2}\int_{\mathbb{R}^d} \big\langle \nabla\{(Df)(\Lambda_{s,t}^\mu)\}(X_{s,t}^{x,\mu}), (\sigma\sigma^*)(t, X_{s,t}^{x,\mu}, \Lambda_{s,t}^\mu)\big\rangle \Big\}\mu(\mathrm{d}x)\bigg)\mathrm{d}t\\
&\quad + \bigg(\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} \big\langle (D^2 f)(\Lambda_{s,t}^\mu)(X_{s,t}^{x,\mu}, X_{s,t}^{y,\mu}), \sigma(t, X_{s,t}^{x,\mu}, \Lambda_{s,t}^\mu)\sigma(t, X_{s,t}^{y,\mu}, \Lambda_{s,t}^\mu)^*\big\rangle \mu(\mathrm{d}x)\mu(\mathrm{d}y)\bigg)\mathrm{d}t\\
&= (\mathscr{A}_t f)(\Lambda_{s,t}^\mu)\mathrm{d}t.
\end{aligned}$$

Then the proof is finished. □

Proof of Theorem 2.1(2)-(3). By the uniqueness result in Theorem 2.1, we have the flow property

$$X_{s,t}^{x,\mu} = X_{r,t}^{X_{s,r}^{x,\mu}, \Lambda_{s,r}^\mu}, \quad \Lambda_{s,t}^\mu = \Lambda_{r,t}^{\Lambda_{s,r}^\mu}, \quad 0 \le s \le r \le t, \tag{2.29}$$

which implies that both $(\Lambda_{s,t}^\mu)_{t\ge s}$ and $(X_{s,t}^{x,\mu}, \Lambda_{s,t}^\mu)_{t\ge s}$ are Markov processes.

Next, by (2.4), these two Markov processes are Feller and hence, strong Markovian. Therefore, Theorem 2.1(2) follows from Lemma 2.4.

Finally, for any $f \in C_b^{2,2}(\mathbb{R}^d, \mathscr{P}_2)$, Lemma 2.4 and the classical Itô's formula for the semimartingale $(X_{s,t}^{x,\mu})_{t\ge s}$ imply

$$\begin{aligned}
\mathrm{d}f(X_{s,t}^{x,\mu}, \Lambda_{s,t}^\mu) &= (\tilde{\mathscr{A}}_t f)(X_{s,t}^{x,\mu}, \Lambda_{s,t}^\mu)\mathrm{d}t + \big\langle \nabla f(\cdot, \Lambda_{s,t}^\mu)(X_{s,t}^{x,\mu}), \sigma(t, X_{s,t}^{x,\mu}, \Lambda_{s,t}^\mu)\mathrm{d}W_t\big\rangle\\
&\quad + \int_{\mathbb{R}^d} \big\langle Df(X_{s,t}^{x,\mu}, \cdot)(\Lambda_{s,t}^\mu)(x), \sigma(t,x,\Lambda_{s,t}^\mu)\mathrm{d}W_t\big\rangle \mu(\mathrm{d}x), \quad t \ge s.
\end{aligned}\tag{2.30}$$

This proves Theorem 2.1(3). □

3 Feyman-Kac formula for PDEs on $\mathbb{R}^d \times \mathscr{P}_2$

In this section, we solve the PDEs (1.8) and (1.9) by using $(X_{s,t}^{x,\mu}, \Lambda_{s,t}^\mu)_{0\le s\le t\le T}$. A function on U on $[0,T]\times\mathbb{R}^d\times\mathscr{P}_2$ is called a solution to (1.8), if $U(t,x,\mu)$ is differentiable in t and $U(t,\cdot,\cdot) \in C^{2,2}(\mathbb{R}^d\times\mathscr{P}_2)$ such that (1.8) holds. If moreover $U(t,x,\mu)$ does not depend on x, it is called a solution to (1.9).

We first introduce the following class $C_b^{0,2,2}([0,T]\times\mathbb{R}^d\times\mathscr{P}_2)$.

Definition 3.1. Let f be a real, vector or matrix valued function on $[0,T]\times\mathbb{R}^k\times\mathscr{P}_2$ for some $k\ge 1$. We write $f\in C_b^{0,2,2}([0,T]\times\mathbb{R}^k\times\mathscr{P}_2)$, if f is jointly continuous, $f(t,\cdot,\cdot)\in C_b^{2,2}(\mathbb{R}^k\times\mathscr{P}_2)$ for every $t\in[0,T]$, and all derivatives

$$\begin{aligned}&\nabla f(t,x,\mu),\quad \nabla^2 f(t,x,\mu),\quad Df(t,x,\mu)(y),\\ &D\{\nabla f(t,x,\mu)\}(y),\quad \nabla\{Df(t,x,\mu)(\cdot)\}(y),\quad D^2 f(t,x,\mu)(y,z)\end{aligned}$$

are bounded and jointly continuous in corresponding arguments. If moreover $f(t,x,\mu)$ does not depend on x, we denote $f\in C_b^{0,2}([0,T]\times\mathscr{P}_2)$.

Theorem 3.1. *Assume that $b,\sigma\in C_b^{0,2,2}([0,T]\times\mathbb{R}^d\times\mathscr{P}_2)$ are deterministic.*

(1) *For any $\Phi\in C_b^{2,2}(\mathbb{R}^d\times\mathscr{P}_2)$, $F\in C_b^{0,2,2}([0,T]\times\mathbb{R}^d\times\mathscr{P}_2)$, and bounded $V\in C_b^{0,2,2}([0,T]\times\mathbb{R}^d\times\mathscr{P}_2)$,*

$$U(t,x,\mu):=\mathbb{E}\Big[\Phi(X_{t,T}^{x,\mu},\Lambda_{t,T}^{\mu})\mathrm{e}^{\int_t^T V(r,X_{t,r}^{x,\mu},\Lambda_{t,r}^{\mu})dr}+\int_t^T F(r,X_{t,r}^{x,\mu},\Lambda_{t,r}^{\mu})\mathrm{e}^{\int_t^r V(\theta,X_{t,\theta}^{x,\mu},\Lambda_{t,\theta}^{\mu})\mathrm{d}\theta}\mathrm{d}r\Big]$$

is the unique solution of (1.8) *in the class $C_b^{0,2,2}([0,T]\times\mathbb{R}^d\times\mathscr{P}_2)$ with $\partial_t U\in C([0,T]\times\mathbb{R}^d\times\mathscr{P}_2)$.*

(2) *For any $\Phi\in C_b^{2}(\mathbb{R}^d\times\mathscr{P}_2)$, $F\in C_b^{0,2}([0,T]\times\mathscr{P}_2)$, and bounded $V\in C_b^{0,2}([0,T]\times\mathscr{P}_2)$,*

$$U(t,\mu):=\mathbb{E}\Big[\Phi(\Lambda_{t,T}^{\mu})\mathrm{e}^{\int_t^T V(r,\Lambda_{t,r}^{\mu})\mathrm{d}r}+\int_t^T F(r,\Lambda_{t,r}^{\mu})\mathrm{e}^{\int_t^r V(\theta,\Lambda_{t,\theta}^{\mu})\mathrm{d}\theta}\mathrm{d}r\Big]$$

is the unique solution of (1.9) *in the class $C_b^{0,2}([0,T]\times\mathscr{P}_2)$ with $\partial_t U\in C([0,T]\times\mathscr{P}_2)$*

Proof. Since $\tilde{\mathscr{A}_t}F(x,\mu)=\mathscr{A}_tF(\mu)$ holds for $F\in C_b^2(\mathscr{P}_2)$, (2) follows from (1). So, it suffices to prove Theorem 3.1(1).

If $U\in C_b^{0,2,2}([0,T]\times\mathbb{R}^d\times\mathscr{P}_2)$ is a solution of (1.8), then (2.30) yields

$$\begin{aligned}\mathrm{d}U(t,X_{s,t}^{x,\mu},\Lambda_{s,t}^{\mu})&=(\partial_t U+\tilde{\mathscr{A}_t})U(t,X_{s,t}^{x,\mu},\Lambda_{s,t}^{\mu})\mathrm{d}t+\mathrm{d}M_t\\ &=\mathrm{d}M_t-\big(VU+F\big)(t,X_{s,t}^{x,\mu},\Lambda_{s,t}^{\mu})\mathrm{d}t,\quad t\in[s,T]\end{aligned}$$

for some martingale $(M_t)_{t\in[s,T]}$. Thus, the process

$$\eta_t:=U(t,X_{s,t}^{x,\mu},\Lambda_{s,t}^{\mu})\mathrm{e}^{\int_s^t V(r,X_{s,r}^{x,\mu},\Lambda_{s,r}^{\mu})\mathrm{d}r}+\int_s^t F(r,X_{s,r}^{x,\mu},\Lambda_{s,r}^{\mu})\mathrm{e}^{\int_s^r V(\theta,X_{s,\theta}^{x,\mu},\Lambda_{s,\theta}^{\mu})\mathrm{d}\theta}\mathrm{d}r,\quad t\in[s,T]$$

satisfies

$$\mathrm{d}\eta_t=\mathrm{e}^{\int_s^t V(r,X_{s,r}^{x,\mu},\Lambda_{s,r}^{\mu})\mathrm{d}r}\mathrm{d}M_t,\quad t\in[s,T].$$

So,

$$\begin{aligned}&U(s,x,\mu)=\mathbb{E}\eta_s=\mathbb{E}\eta_T\\ &=\mathbb{E}\Big[\Phi(X_{s,T}^{x,\mu},\Lambda_{s,T}^{\mu})\mathrm{e}^{\int_s^T V(r,X_{s,r}^{x,\mu},\Lambda_{s,r}^{\mu})dr}+\int_s^T F(r,X_{s,r}^{x,\mu},\Lambda_{s,r}^{\mu})\mathrm{e}^{\int_s^r V(\theta,X_{s,\theta}^{x,\mu},\Lambda_{s,\theta}^{\mu})\mathrm{d}\theta}\mathrm{d}r\Big]\end{aligned}$$

as claimed in Theorem 3.1(1).

On the other hand, let U be given in Theorem 3.1(1). For any $t \in [0,T)$ and $\varepsilon \in (0, T-t)$, by (2.29) and the formula of $U(t,x,\mu)$ in Theorem 3.1(1),

$$U(t,x,\mu) - \mathbb{E}\big[U(t+\varepsilon, X^{x,\mu}_{t,t+\varepsilon}, \Lambda^x_{t,t+\varepsilon})\big] = I_1(\varepsilon) + I_2(\varepsilon) + I_3(\varepsilon)$$

holds for

$$\begin{aligned}
I_1(\varepsilon) &:= \mathbb{E}\Big[\Phi(X^{X^{x,\mu}_{t,t+\varepsilon},\Lambda^\mu_{t,t+\varepsilon}}_{t+\varepsilon,T}, \Lambda^{\Lambda^\mu_{t,t+\varepsilon}}_{t+\varepsilon,T})\big(\mathrm{e}^{\int_t^T V(r,X^{x,\mu}_{r,T},\Lambda^\mu_{r,T})dr} - \mathrm{e}^{\int_{t+\varepsilon}^T V(r,X^{x,\mu}_{r,T},\Lambda^\mu_{r,T})dr}\big)\Big],\\
I_2(\varepsilon) &:= \mathbb{E}\bigg[\int_t^{t+\varepsilon} F(r, X^{x,\mu}_{t,r}, \Lambda^\mu_{t,r})\mathrm{e}^{\int_t^r V(\theta,X^{x,\mu}_{t,\theta},\Lambda^\mu_{t,\theta})\mathrm{d}\theta}\mathrm{d}r\bigg],\\
I_3(\varepsilon) &:= \mathbb{E}\bigg[\int_{t+\varepsilon}^T F(r, X^{x,\mu}_{t,r}, \Lambda^\mu_{t,r})\big(\mathrm{e}^{\int_t^r V(\theta,X^{x,\mu}_{t,\theta},\Lambda^\mu_{t,\theta})\mathrm{d}\theta} - \mathrm{e}^{\int_{t+\varepsilon}^r V(\theta,X^{x,\mu}_{t,\theta},\Lambda^\mu_{t,\theta})\mathrm{d}\theta}\big)\mathrm{d}r\bigg].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{U(t,x,\mu) - \mathbb{E}[U(t+\varepsilon, X^{x,\mu}_{t,t+\varepsilon}, \Lambda^x_{t,t+\varepsilon})]}{\varepsilon}\\
&= V(t,x,\mu)\mathbb{E}\big[\Phi(X^{x,\mu}_{t,T}, \Lambda^\mu_{t,T})\mathrm{e}^{\int_t^T V(r,X^{x,\mu}_{t,r},\Lambda^\mu_{t,r})\mathrm{d}r}\big] + F(t,x,\mu)\\
&\quad + V(t,x,\mu)\mathbb{E}\bigg[\int_t^T F(r, X^{x,\mu}_{t,r}, \Lambda^\mu_{t,r})\mathrm{e}^{\int_t^r V(\theta,X^{x,\mu}_{t,\theta},\Lambda^\mu_{t,\theta})\mathrm{d}\theta}\bigg]\\
&= (VU+F)(t,x,\mu).
\end{aligned} \tag{3.1}$$

By Proposition 3.2 below, $U \in C_b^{0,2,2}([0,T]\times\mathbb{R}^d\times\mathscr{P}_2)$ and $\tilde{\mathscr{A}}_t U(t,x,\mu)$ is continuous in (t,x,μ). Then (2.30) implies

$$\mathbb{E}[U(t+\varepsilon, X^{x,\mu}_{t,t+\varepsilon}, \Lambda^x_{t,t+\varepsilon})] = U(t+\varepsilon, x, \mu) + \mathbb{E}\int_t^{t+\varepsilon} \tilde{\mathscr{A}}_r U(r, X^{x,\mu}_{t,r}, \Lambda^\mu_{t,r})\mathrm{d}r.$$

Combining this with (3.1) we arrive at

$$-\partial_t U(t,x,\mu) = \lim_{\varepsilon\to 0}\frac{U(t,x,\mu) - U(t+\varepsilon,x,\mu)}{\varepsilon} = \tilde{\mathscr{A}}_t U(t,x,\mu) + (UV+F)(t,x,\mu).$$

Therefore, U solves (1.8) with continuous $\tilde{\mathscr{A}}_t U$. □

The remainder of this section devotes to the proof of the following result.

Proposition 3.2. *Under conditions of Theorem* 3.1 *and let U be given in Theorem* 3.1(1). *Then $U \in C_b^{0,2,2}([0,T]\times\mathbb{R}^d\times\mathscr{P}_2)$, so that $\tilde{\mathscr{A}}_t U$ is continuous on $[0,T]\times\mathbb{R}^d\times\mathscr{P}_2$.*

We first introduce some notations which will be used in calculations.

(*a*) For $f \in C^2(\mathbb{R}^d)$,

$$(\nabla f(x))v_1 := \langle \nabla f(x), v_1\rangle = \nabla_{v_1} f(x),\ \ (\nabla^2 f(x))(v_1,v_2) := \mathrm{Hess}_f(v_1,v_2),\ \ x, v_1, v_2 \in \mathbb{R}^d.$$

(*b*) For $f \in C^2(\mathscr{P}_2)$,

$$\{Df(\mu)\}\phi := D_\phi f(\mu) = \int_{\mathbb{R}^d} \langle Df(\mu)(x), \phi(x)\rangle \mu(\mathrm{d}x), \quad \phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d; \mu).$$

(*c*) Derivatives of vector or matrix valued functions are given by those of component functions. For instance, for $f = (f_{ij}) \in C^1(\mathbb{R}^d \times \mathscr{P}_2 \to \mathbb{R}^l \otimes \mathbb{R}^k)$,

$$\nabla_v f(x,\mu) := \big(\langle \nabla f_{ij}(x,\mu), v\rangle\big), \quad D_\phi f(x,\mu) := \big(D_\phi f_{ij}(x,\mu)\big),$$

where $x, v \in \mathbb{R}^d, \mu \in \mathscr{P}_2$ and $\phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d; \mu)$.

We will also need the following notion of uniform boundedness and continuity.

Definition 3.2. Let $\mathbb{B}$ be a Banach space, and let E be a topological space. The family

$$\big\{\eta(x) \in L^1(\Omega \to \mathbb{B}; \mathbb{P}) : x \in E\big\}$$

is called $L^{\infty-}(\mathbb{P})$ bounded continuous, if for any $p \geq 1$,

$$\sup_{x \in E} \mathbb{E}\|\eta(x)\|^p < \infty, \quad \lim_{y \to x} \mathbb{E}\|\eta(x) - \eta(y)\|^p = 0, \quad x \in E.$$

Let $\mathscr{L}(\mathbb{B}_1 \to \mathbb{B}_2)$ denote the space of all bounded linear operators from a Banach space $\mathbb{B}_1$ to the other one $\mathbb{B}_2$. When $\mathbb{B}_1$ and $\mathbb{B}_2$ are finite-dimensional Hilbert spaces, we regard $\mathscr{L}(\mathbb{B}_1 \to \mathbb{B}_2)$ as Euclidean space. The following lemma can be easily proved by using Itô's formula, so we omit the proof to save space.

Lemma 3.3. *Let $k, l \geq 1$, and let*

$$B_1 : \Omega \times [0,T] \times \mathbb{R}^l \times \mathscr{P}_2 \to \mathbb{R}^k, \quad \Sigma_1 : \Omega \times [0,T] \times \mathbb{R}^l \times \mathscr{P}_2 \to \mathbb{R}^k \otimes \mathbb{R}^m,$$
$$B_2 : \Omega \times [0,T] \times \mathbb{R}^l \times \mathscr{P}_2 \to \mathbb{R}^k \otimes \mathbb{R}^k, \quad \Sigma_2 : \Omega \times [0,T] \times \mathbb{R}^l \times \mathscr{P}_2 \to \mathscr{L}(\mathbb{R}^k \to \mathbb{R}^k \otimes \mathbb{R}^m)$$

be progressively measurable. If $\{B_2, \Sigma_2\}$ are uniformly bounded and continuous in $(t,x,\mu) \in [0,T] \times \mathbb{R}^l \times \mathscr{P}_2$, and $\{B_1(t,x,\mu), \Sigma_1(t,x,\mu)\}$ are $L^{\infty-}(\mathbb{P})$ bounded continuous, then for any $e \in \mathbb{R}^k$ and $(x,\mu) \in \mathbb{R}^l \times \mathscr{P}_2$, the solution $(\eta_{s,t}^{x,\mu})_{t \in [s,T]}$ for the SDE

$$\mathrm{d}\eta_{s,t}^{x,\mu} = \big\{B_1(t,x,\mu) + B_2(t,x,\mu)\eta_t^{x,\mu}\big\}\mathrm{d}t + \big\{\Sigma_1(t,x,\mu) + \Sigma_2(t,x,\mu)\eta_t^{x,\mu}\big\}\mathrm{d}W_t, \ \eta_{s,s}^{x,\mu} = e, t \in [s,T]$$

is $L^{\infty-}(\mathbb{P})$ bounded continuous.

In the following subsections, we calculate the first and second order derivatives of $(X_{s,t}^{x,\mu}, \Lambda_{s,t}^{\mu})$ in x and μ respectively, which will be used in the proof of Proposition 3.2.

3.1 Formulas for $\nabla X^{x,\mu}_{s,t}$ and $\nabla^2 X^{x,\mu}_{s,t}$

Let $\{e_i\}_{1\le i\le d}$ be the canonical orthonormal basis of $\mathbb{R}^d$. Given $(\Lambda^\mu_{s,t})_{t\ge s}$, the SDE (1.7) becomes the classical one with random coefficients of bounded and continuous first and second order derivatives in x. So, when $\nabla b(t,x,\mu)$ and $\nabla\sigma(t,x,\mu)$ are $L^{\infty-}(\mathbb{P})$ bounded continuous, by taking ∂_{x_i} to $X^{x,\mu}_{s,t}$ in (1.7), we see that for any $1\le i\le d$,

$$v^{i,x,\mu}_{s,t} := \partial_{x_i} X^{x,\mu}_{s,t},\quad t\ge s$$

solves the linear SDE

$$\begin{aligned}&\mathrm{d}v^{i,x,\mu}_{s,t} = \Big[\big\{\nabla b(t,\cdot,\Lambda^\mu_{s,t})(X^{x,\mu}_{s,t})\big\}v^{i,x,\mu}_{s,t}\Big]\mathrm{d}t + \Big[\big\{\nabla\sigma(t,\cdot,\Lambda^\mu_{s,t})(X^{x,\mu}_{s,t})\big\}v^{i,x,\mu}_{s,t}\Big]\mathrm{d}W_t,\\ &t\ge s,\ v^{i,x,\mu}_{s,s}=e_i.\end{aligned}\tag{3.2}$$

If moreover $\nabla^2 b(t,x,\mu)$ and $\nabla^2\sigma(t,x,\mu)$ are $L^{\infty-}(\mathbb{P})$ bounded continuous, then by taking ∂_j to the SDE (3.2), we see that for $1\le j\le d$

$$v^{i,j,x,\mu}_{s,t} := \partial_{x_i}\partial_{x_j} X^{x,\mu}_{s,t},\quad t\ge s$$

solves the SDEs

$$\begin{aligned}\mathrm{d}v^{i,j,x,\mu}_{s,t} &= \Big[\big\{\nabla b(t,\cdot,\Lambda^\mu_{s,t})(X^{x,\mu}_{s,t})\big\}v^{i,j,x,\mu}_{s,t} + \big\{\nabla^2 b(t,\cdot,,\Lambda^\mu_{s,t})(X^{x,\mu}_{s,t})\big\}(v^{i,x,\mu}_{s,t},v^{j,x,\mu}_{s,t})\Big]\mathrm{d}t\\ &+\Big[\big\{\nabla\sigma(t,\cdot,\Lambda^\mu_{s,t})(X^{x,\mu}_{s,t})\big\}v^{i,j,x,\mu}_{s,t} + \big\{\nabla^2\sigma(t,\cdot,,\Lambda^\mu_{s,t})(X^{x,\mu}_{s,t})\big\}(v^{i,x,\mu}_{s,t},v^{j,x,\mu}_{s,t})\Big]\mathrm{d}W_t,\quad v^{i,j,x,\mu}_{s,s}=0.\end{aligned}$$

Combining these with Lemma 3.3, we obtain the following result.

Lemma 3.4. *Assume* **(A)** *and that $\nabla b(t,x,\mu),\nabla^2 b(t,x,\mu),\nabla\sigma(t,x,\mu)$ and $\nabla^2\sigma(t,x,\mu)$ are $L^{\infty-}(\mathbb{P})$ bounded continuous, then so are $\nabla X^{x,\mu}_{s,t}$ and $\nabla^2 X^{x,\mu}_{s,t}$.*

3.2 Formula for $DX^{x,\mu}_{s,t}$

We will establish the SDE for $DX^{x,\mu}_{s,t}(y)$ under the following condition **(C)** on b and σ.

(C) Assume that b and σ are progressively measurable such that the derivatives

$$\nabla b(t,x,\mu),\quad \nabla\sigma(t,x,\mu),\quad Db(t,x,\mu)(y),\quad D\sigma(t,x,\mu)(y)$$

are uniformly bounded and continuous in $(x,\mu,y)\in\mathbb{R}^d\times\mathscr{P}_2\times\mathbb{R}^d$.

Lemma 3.5. *Assume* **(C)**. *Then for any $(x,\mu,y)\in\mathbb{R}^d\times\mathscr{P}_2\times\mathbb{R}^d$, $w^{x,\mu}_{s,t}(y) := (DX^{x,\mu}_{s,t})(y)$ for $t\in[s,T]$ exists and solves the SDE*

$$\begin{aligned}\mathrm{d}w^{x,\mu}_{s,t}(y) = \Big[&\big\{w^{x,\mu}_{s,t}(y)\big\}^*\nabla b(t,\cdot,\Lambda^\mu_{s,t})(X^{x,\mu}_{s,t}) + (\nabla X^{y,\mu}_{s,t})^*\{Db(t,X^{x,\mu}_{s,t},\cdot)(\Lambda^\mu_{s,t})\}(X^{y,\mu}_{s,t})\\ &+\int_{\mathbb{R}^d}\big\{w^{z,\mu}_{s,t}(y)\big\}^*\big\{Db(t,X^{x,\mu}_{s,t},\cdot)(\Lambda^\mu_{s,t})\big\}(X^{z,\mu}_{s,t})\mu(\mathrm{d}z)\Big]\mathrm{d}t\end{aligned}$$

$$
+ \Big[\big\{w_{s,t}^{x,\mu}(y)\big\}^*\big\{\nabla\sigma(t,\cdot,\Lambda_{s,t}^\mu)(X_{s,t}^{x,\mu})\big\} + (\nabla X_{s,t}^{y,\mu})^*\{D\sigma(t,X_{s,t}^{x,\mu},\cdot)(\Lambda_{s,t}^\mu)\}(X_{s,t}^{y,\mu})
$$

$$
+ \int_{\mathbb{R}^d}\big\{w_{s,t}^{z,\mu}(y)\big\}^*\big\{D\sigma(t,X_{s,t}^{x,\mu},\cdot)(\Lambda_{s,t}^\mu)\big\}(X_{s,t}^{z,\mu})\mu(\mathrm{d}z)\Big]\mathrm{d}W_t, \quad w_{s,s}^{x,\mu,y} = 0,
$$

where $\big\{w_{s,t}^{x,\mu}(y)\big\}^*$ *is the transposition of the matrix* $w_{s,t}^{x,\mu}(y)$*. Consequently,* $(DX_{s,t}^{x,\mu})(y)$ *is* $L^{\infty-}(\mathbb{P})$ *bounded continuous.*

To prove the existence of $DX_{s,t}^{x,\mu}$, for fixed $\phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d;\mu)$, let $\mu_\varepsilon = \mu\circ(\mathrm{Id}+\varepsilon\phi)^{-1}$ and consider

$$
\xi_{s,t}^{x,\varepsilon} := \frac{X_{s,t}^{x,\mu_\varepsilon} - X_{s,t}^{x,\mu}}{\varepsilon}, \quad \varepsilon\in(0,1), t\in[s,T].
$$

We first establish the SDE for $D_\phi X_{s,t}^{x,\mu} := \lim_{\varepsilon\downarrow 0}\xi_{s,t}^{x,\varepsilon}$. To this end, we need the following lemma.

Lemma 3.6. *Assume* **(A)** *and let* $\tilde{\xi}_{s,t}^{x,\varepsilon} := \frac{X_{s,t}^{x+\varepsilon\phi(x),\mu_\varepsilon} - X_{s,t}^{x,\mu_\varepsilon}}{\varepsilon}$*. Then for any* $f\in C^{1,1}(\mathbb{R}^d\times\mathscr{P}_2)$ *with*

$$
K_f := \sup_{(x,\mu)\in\mathbb{R}^d\times\mathscr{P}_2}\big(|\nabla f(x,\mu)|^2 + \|Df(x,\mu)\|_{L^2(\mu)}^2\big) < \infty,
$$

the process

$$
\Xi_{s,t}^{x,\varepsilon}(f) := \frac{f(X_{s,t}^{x,\mu_\varepsilon},\Lambda_{s,t}^{\mu_\varepsilon}) - f(X_{s,t}^{x,\mu},\Lambda_{s,t}^\mu)}{\varepsilon} - \nabla_{\xi_{s,t}^{x,\varepsilon}} f(\cdot,\Lambda_{s,t}^\mu)(X_{s,t}^{x,\mu})
$$

$$
- \int_{\mathbb{R}^d}\big\langle \xi_{s,t}^{z,\varepsilon} + \tilde{\xi}_{s,t}^{z,\varepsilon}, \{Df(X_{s,t}^{z,\mu},\cdot)(\Lambda_{s,t}^\mu)\}(X_{s,t}^{z,\mu})\big\rangle\mu(\mathrm{d}z), \quad t\in[s,T]
$$

satisfies

$$
\big|\Xi_{s,t}^{x,\varepsilon}(f)\big|^2 \le 8K_f\big(|\xi_{s,t}^{x,\varepsilon}|^2 + \mu(|\xi_{s,t}^{\cdot,\varepsilon} + \tilde{\xi}_{s,t}^{\cdot,\varepsilon}|^2)\big), \quad t\in[s,T], \tag{3.3}
$$

$$
\lim_{\varepsilon\downarrow 0}\mathbb{E}\big|\Xi_{s,t}^{x,\varepsilon}(f)\big|^2 = 0. \tag{3.4}
$$

Proof. Let $\eta_r^x = X_{s,t}^{x,\mu} + r(X_{s,t}^{x+\varepsilon\phi(x),\mu_\varepsilon} - X_{s,t}^{x,\mu})$, $r\in[0,1]$. Then $\eta_0^x = X_{s,t}^{x,\mu}, \eta_1^x = X_{s,t}^{x+\varepsilon\phi(x),\mu_\varepsilon}$, so that

$$
\mathscr{L}_{\eta_0|\mu} := \mu\circ(X_{s,t}^{\cdot,\mu})^{-1} = \Lambda_{s,t}^\mu, \quad \mathscr{L}_{\eta_1|\mu} := \mu\circ(X_{s,t}^{\cdot+\varepsilon\phi,\mu_\varepsilon})^{-1} = \mu_\varepsilon\circ(X_{s,t}^{\cdot,\mu_\varepsilon})^{-1} = \Lambda_{s,t}^{\mu_\varepsilon}.
$$

Moreover, $\frac{\mathrm{d}}{\mathrm{d}r}\eta_r^x = \xi_{s,t}^{x,\varepsilon} + \tilde{\xi}_{s,t}^{x,\varepsilon}$. Then by Lemma 2.3, we have

$$
\begin{aligned}
\frac{d}{\mathrm{d}r} f(y,\mathscr{L}_{\eta_r|\mu}) &= \Big\langle Df(y,\cdot)(\mathscr{L}_{\eta_r|\mu})(\eta_r), \frac{\mathrm{d}}{\mathrm{d}r}\eta_r\Big\rangle_{L^2(\mu)} \\
&= \varepsilon\int_{\mathbb{R}^d}\big\langle Df(y,\cdot)(\mathscr{L}_{\eta_r|\mu})(\eta_r^z), \xi_{s,t}^{z,\varepsilon} + \tilde{\xi}_{s,t}^{z,\varepsilon}\big\rangle\mu(\mathrm{d}z), \quad r\in[0,1], y\in\mathbb{R}^d.
\end{aligned}
$$

So, letting $\zeta_r^x = (1-r)X_{s,t}^{x,\mu} + rX_{s,t}^{x,\mu_\varepsilon}$, we obtain

$$\frac{f(X_{s,t}^{x,\mu_\varepsilon}, \Lambda_{s,t}^{\mu_\varepsilon}) - f(X_{s,t}^{x,\mu}, \Lambda_{s,t}^{\mu})}{\varepsilon} = \frac{1}{\varepsilon}\int_0^1 \Big\{\frac{\mathrm{d}}{\mathrm{d}r} f(\zeta_r^x, \mathscr{L}_{\eta_r|\mu})\Big\}\mathrm{d}r$$
$$= \int_0^1 \Big\{\big\langle \nabla f(\cdot, \mathscr{L}_{\eta_r|\mu})(\zeta_r^x), \xi_{s,t}^{x,\varepsilon}\big\rangle + \int_{\mathbb{R}^d} \big\langle Df(\zeta_r^x, \cdot)(\mathscr{L}_{\eta_r|\mu})(\eta_r^z), \xi_{s,t}^{z,\varepsilon} + \tilde{\xi}_{s,t}^{z,\varepsilon}\big\rangle \mu(\mathrm{d}z)\Big\}\mathrm{d}r.$$

This together with the definition of $\Xi_{s,t}^{x,\varepsilon}(f)$ gives

$$\begin{aligned}\big|\Xi_{s,t}^{x,\varepsilon}(f)\big|^2 = &\ \bigg|\int_0^1 \Big\{\big\langle \nabla f(\cdot, \mathscr{L}_{\eta_r|\mu})(\zeta_r^x) - \nabla f(\cdot, \Lambda_{s,t}^{\mu})(X_{s,t}^{x,\mu}), \xi_{s,t}^{x,\varepsilon}\big\rangle \\ &+ \int_{\mathbb{R}^d} \big\langle Df(\eta_r^x, \cdot)(\mathscr{L}_{\eta_r|\mu})(\zeta_r^z) - Df(X_{s,t}^{x,\mu}, \cdot)(\Lambda_{s,t}^{\mu})(X_{s,t}^{z,\mu}), \xi_{s,t}^{z,\varepsilon} + \tilde{\xi}_{s,t}^{z,\varepsilon}\big\rangle \mu(\mathrm{d}z)\Big\}\mathrm{d}r\bigg|^2 \\ \le& 8K_f\big(|\xi_{s,t}^{x,\varepsilon}|^2 + \mu(|\xi_{s,t}^{\cdot,\varepsilon} + \tilde{\xi}_{s,t}^{\cdot,\varepsilon}|^2)\big),\end{aligned} \tag{3.5}$$

which implies (3.3). On the other hand, it is easy to see that (2.4) implies

$$\sup_{x\in\mathbb{R}^d, \varepsilon\in(0,1)} \mathbb{E}\Big[\sup_{t\in[s,T]} \big\{|\xi_{s,t}^{x,\varepsilon}|^2 + \mu(|\tilde{\xi}_{s,t}^{\cdot,\varepsilon}|^2)\big\}\Big] \le c\mu(|\phi|^2), \quad \phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d; \mu) \tag{3.6}$$

for some constant $c > 0$. Combining this with the facts that $(\nabla f, Df)$ is bounded continuous, $\lim_{r\to 0}\zeta_r^z = X_{s,t}^{z,\mu}$, and $\lim_{r\to 0}\mathscr{L}_{\eta_r|\mu} = \Lambda_{s,t}^{\mu}$, we may apply the dominated convergence theorem to deduce (3.4) from the first equality in (3.5) with $\varepsilon \downarrow 0$. □

Lemma 3.7. *Assume* **(C)**. *For any $(s,x,\mu) \in [0,T] \times \mathbb{R}^d, \times \mathscr{P}_2$ and $\phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d)$, $w_{s,t}^{x,\mu,\phi} := D_\phi X_{s,t}^{x,\mu}$ for $t \in [s,T]$ exists in $L^2(\Omega \to C([s,T] \to \mathbb{R}^d); \mathbb{P})$, and there exists a constant $C > 0$ such that*

$$\mathbb{E}\Big[\sup_{s\le t\le T} |w_{s,t}^{x,\mu,\phi}|^2\Big] \le C\mu(|\phi|^2), \quad (s,x,\mu) \in [0,T] \times \mathbb{R}^d \times \mathscr{P}_2. \tag{3.7}$$

Moreover, for any $t \in [s,T]$,

$$\begin{aligned} w_{s,t}^{x,\mu,\phi} = &\int_s^t \Big\{\nabla_{w_{s,r}^{x,\mu,\phi}} b(r, \cdot, \Lambda_{s,r}^{\mu})(X_{s,r}^{x,\mu})\Big\}\mathrm{d}r + \int_s^t \Big\{\nabla_{w_{s,t}^{x,\mu,\phi}} \sigma(t, \cdot, \Lambda_{s,t}^{\mu})(X_{s,t}^{x,\mu})\Big\}\mathrm{d}W_r \\ &+ \int_s^t \bigg(\int_{\mathbb{R}^d} \big\langle \{Db(r, X_{s,r}^{x,\mu}, \cdot)(\Lambda_{s,r}^{\mu})\}(X_{s,r}^{z,\mu}), w_{s,r}^{z,\mu,\phi} + \nabla_{\phi(z)} X_{s,t}^{z,\mu}\big\rangle \mu(\mathrm{d}z)\bigg)\mathrm{d}r \\ &+ \int_s^t \bigg(\int_{\mathbb{R}^d} \big\langle \{D\sigma(r, X_{s,r}^{x,\mu}, \cdot)(\Lambda_{s,t}^{\mu})\}(X_{s,r}^{z,\mu}), w_{s,r}^{z,\mu,\phi} + \nabla_{\phi(z)} X_{s,t}^{z,\mu}\big\rangle \mu(\mathrm{d}z)\bigg)\mathrm{d}W_r. \end{aligned} \tag{3.8}$$

Proof. To prove the existence of $w_{s,t}^{x,\mu,\phi} := D_\phi X_{s,t}^{x,\mu}$ in $L^2(\Omega \to C([s,T] \to \mathbb{R}^d); \mathbb{P})$, it suffices to show

$$\lim_{\varepsilon,\delta\downarrow 0} \mathbb{E}\Big[\sup_{t\in[s,T]} |\xi_{s,t}^{x,\varepsilon} - \xi_{s,t}^{x,\delta}|^2\Big] = 0. \tag{3.9}$$

By the definition of $\xi^{x,\varepsilon}_{s,t}$ and letting

$$\Xi^{x,\varepsilon}_{s,t}(b) = \big(\Xi^{x,\varepsilon}_{s,t}(b_i)\big)_{1\le i\le d},\quad \Xi^{x,\varepsilon}_{s,t}(\sigma) = \big(\Xi^{x,\varepsilon}_{s,t}(\sigma_{i,j})\big)_{1\le i\le d, 1\le j\le m},$$

we obtain

$$\begin{aligned}
\xi^{x,\varepsilon}_{s,t} &= \frac{1}{\varepsilon}\int_s^t \big\{b(r, X^{x,\mu_\varepsilon}_{s,r}, \Lambda^{\mu_\varepsilon}_{s,r}) - b(r, X^{x,\mu}_{s,r}, \Lambda^{\mu}_{s,r})\big\}\mathrm{d}r \\
&\qquad + \frac{1}{\varepsilon}\int_s^t \big\{\sigma(r, X^{x,\mu_\varepsilon}_{s,r}, \Lambda^{\mu_\varepsilon}_{s,r}) - \sigma(r, X^{x,\mu}_{s,r}, \Lambda^{\mu}_{s,r})\big\}\mathrm{d}W_r \\
&= \int_s^t \Big\{\Xi^{x,\varepsilon}_{s,r}(b) + \nabla_{\xi^{x,\varepsilon}_{s,r}} b(r,\cdot,\Lambda^\mu_{s,r})(X^{x,\mu}_{s,r})\Big\}\mathrm{d}r \\
&+ \int_s^t \Big\{\int_{\mathbb{R}^d}\big\langle\{Db(r, X^{x,\mu}_{s,r},\cdot)(\Lambda^\mu_{s,t})\}(X^{z,\mu}_{s,t}), \xi^{z,\varepsilon}_{s,r} + \tilde{\xi}^{z,\varepsilon}_{s,r}\big\rangle\mu(\mathrm{d}z)\Big\}\mathrm{d}r \\
&+ \int_s^t \Big\{\Xi^{x,\varepsilon}_{s,r}(\sigma) + \nabla_{\xi^{x,\varepsilon}_{s,r}} \sigma(r,\cdot,\Lambda^\mu_{s,r})(X^{x,\mu}_{s,r})\Big\}\mathrm{d}W_r \\
&+ \int_s^t \Big\{\int_{\mathbb{R}^d}\big\langle\{D\sigma(r, X^{x,\mu}_{s,r},\cdot)(\Lambda^\mu_{s,t})\}(X^{z,\mu}_{s,t}), \xi^{z,\varepsilon}_{s,r} + \tilde{\xi}^{z,\varepsilon}_{s,r}\big\rangle\mu(\mathrm{d}z)\Big\}\mathrm{d}W_r.
\end{aligned} \tag{3.10}$$

Combining this with **(C)** and using the BDG inequality, we may find out a constant $C > 0$ such that for any $t \in [s,T]$,

$$\begin{aligned}
\mathbb{E}\Big[\sup_{r\in[s,t]}|\xi^{x,\varepsilon}_{s,r} - \xi^{x,\delta}_{s,r}|^2\Big] &\le C\mathbb{E}\int_s^t \Big\{|\Xi^{x,\varepsilon}_{s,r}(b) - \Xi^{x,\delta}_{s,r}(b)|^2 + \|\Xi^{x,\varepsilon}_{s,r}(\sigma) - \Xi^{x,\delta}_{s,r}(\sigma)\|^2 \\
&\qquad + |\xi^{x,\varepsilon}_{s,r} - \xi^{x,\delta}_{s,r}|^2 + \mu(|\xi^{\cdot,\varepsilon}_{s,r} - \xi^{\cdot,\delta}_{s,r}|^2 + |\tilde{\xi}^{\cdot,\varepsilon}_{s,r} - \tilde{\xi}^{\cdot,\delta}_{s,r}|^2)\Big\}\mathrm{d}r.
\end{aligned} \tag{3.11}$$

Integrating both sides with respect to $\mu(\mathrm{d}x)$, we obtain

$$\begin{aligned}
\mathbb{E}\mu(|\xi^{\cdot,\varepsilon}_{s,t} - \xi^{\cdot,\delta}_{s,t}|^2) &\le C\mathbb{E}\int_s^t \mu\big(|\Xi^{\cdot,\varepsilon}_{s,r}(b) - \Xi^{\cdot,\delta}_{s,r}(b)|^2 + \|\Xi^{\cdot,\varepsilon}_{s,r}(\sigma) - \Xi^{\cdot,\delta}_{s,r}(\sigma)\|^2 + |\tilde{\xi}^{\cdot,\varepsilon}_{s,r} - \tilde{\xi}^{\cdot,\delta}_{s,r}|^2\big)\mathrm{d}r \\
&\qquad + 2C\int_s^t \mathbb{E}\mu(|\xi^{\cdot,\varepsilon}_{s,r} - \xi^{\cdot,\delta}_{s,r}|^2)\mathrm{d}r,\quad t\in[s,T].
\end{aligned}$$

Then by Grownwall's inequality, (3.4), (3.6), and the existence of

$$\lim_{\varepsilon\downarrow 0}\tilde{\xi}^{\cdot,\varepsilon}_{s,r} = \nabla_\phi X^{\cdot,\mu}_{s,t} \text{ in } L^2(\mathbb{P})$$

as explained in Subsection 4.1, which implies $\lim_{\varepsilon,\delta\downarrow 0}\mathbb{E}|\tilde{\xi}^{\cdot,\varepsilon}_{s,r} - \tilde{\xi}^{\cdot,\delta}_{s,r}|^2 = 0$, we derive

$$\begin{aligned}
&\lim_{\varepsilon,\delta\downarrow 0}\sup_{t\in[s,T]}\mathbb{E}\mu(|\xi^{\cdot,\varepsilon}_{s,t} - \xi^{\cdot,\delta}_{s,t}|^2) \\
&\le C\mathrm{e}^{2CT}\lim_{\varepsilon,\delta\downarrow 0}\mathbb{E}\int_s^T \mu\big(|\Xi^{\cdot,\varepsilon}_{s,r}(b) - \Xi^{\cdot,\delta}_{s,r}(b)|^2 + \|\Xi^{\cdot,\varepsilon}_{s,r}(\sigma) - \Xi^{\cdot,\delta}_{s,r}(\sigma)\|^2 + |\tilde{\xi}^{\cdot,\varepsilon}_{s,r} - \tilde{\xi}^{\cdot,\delta}_{s,r}|^2\big)\mathrm{d}r = 0.
\end{aligned}$$

Substituting this into (3.11) and using Gronwall's inequality again, we arrive at

$$\begin{aligned}&\lim_{\varepsilon,\delta\downarrow 0}\mathbb{E}\Big[\sup_{t\in[s,T]}|\xi^{x,\varepsilon}_{s,t}-\xi^{x,\delta}_{s,t}|^2\Big]\\&\le C\mathrm{e}^{CT}\lim_{\varepsilon,\delta\downarrow 0}\mathbb{E}\int_s^T\Big\{|\Xi^{x,\varepsilon}_{s,r}(b)-\Xi^{x,\delta}_{s,r}(b)|^2+\|\Xi^{x,\varepsilon}_{s,r}(\sigma)-\Xi^{x,\delta}_{s,r}(\sigma)\|^2\\&\qquad\qquad+\mu\big(|\xi^{\cdot,\varepsilon}_{s,r}-\xi^{\cdot,\delta}_{s,r}|^2+|\tilde{\xi}^{\cdot,\varepsilon}_{s,r}-\tilde{\xi}^{\cdot,\delta}_{s,r}|^2\big)\Big\}\mathrm{d}r=0.\end{aligned}$$

Therefore, (3.9) holds, so that

$$w^{x,\mu,\phi}_{s,t}:=D_\phi X^{x,\mu}_{s,t}=\lim_{\varepsilon\downarrow 0}\xi^{x,\varepsilon}_{s,t},\quad t\in[s,T]$$

exists in $L^2(\Omega\to C([s,T]\to\mathbb{R}^d);\mathbb{P})$, and (3.7) follows from (3.6). Moreover, by **(C)** and Lemma 3.6, we may let $\varepsilon\downarrow 0$ in (3.10) to derive the desired equation for $w^{x,\mu,\phi}_{s,t}$. □

Proof of Lemma 3.5. By (3.7), $(DX^{x,\mu}_{s,t})_{t\in[s,T]}$ exists with

$$\langle DX^{x,\mu}_{s,t},\phi\rangle_{L^2(\mu)}=D_\phi X^{x,\mu}_{s,t}=w^{x,\mu,\phi}_{s,t},\quad \phi\in L^2(\mathbb{R}^d\to\mathbb{R}^d;\mu). \tag{3.12}$$

On the other hand, let $w^{x,\mu}_{s,t}(y)$ solve the SDE in Lemma 3.5. Then $\tilde{w}^{x,\mu,\phi}_{s,t}:=\langle w^{x,\mu}_{s,t},\phi\rangle_{L^2(\mu)}$ solves the SDE in Lemma 3.7 for $w^{x,\mu,\phi}_{s,t}$. By the uniqueness, we have $w^{x,\mu,\phi}_{s,t}=\tilde{w}^{x,\mu,\phi}_{s,t}$. Combining this with (3.12), we obtain μ-a.e. $w^{x,\mu}_{s,t}=DX^{x,\mu}_{s,t}$. Then the proof is finished. □

3.3 Some other derivatives

We first present a formula for $Df(\Lambda^\mu_{s,t})$.

Lemma 3.8. *Assume* **(C)**. *For any* $f\in C^1_b(\mathscr{P}_2)$,

$$\begin{aligned}&\{Df(\Lambda^{\cdot}_{s,t})(\mu)\}(y)\\&=\big(\nabla X^{y,\mu}_{s,t}\big)^*\big\{(Df)(\Lambda^\mu_{s,t})\big\}(X^{y,\mu}_{s,t})+\int_{\mathbb{R}^d}\big(DX^{x,\mu}_{s,t}\big)^*(y)\{(Df)(\Lambda^\mu_{s,t})\}(X^{x,\mu}_{s,t})\mu(\mathrm{d}x).\end{aligned}\tag{3.13}$$

Proof. Let $\phi\in L^2(\mathbb{R}^d\to\mathbb{R}^d;\mu)$. Since $\Lambda^\mu_{s,t}=\mu\circ(X^{\cdot,\mu}_{s,t})^{-1}$, for any $\varepsilon>0$ we have

$$\begin{aligned}\int_{\mathbb{R}^d}h(z)\big(\Lambda^{\mu\circ(\mathrm{Id}+\varepsilon\phi)^{-1}}_{s,t}\big)(\mathrm{d}z)&=\int_{\mathbb{R}^d}h\big(X^{x,\mu\circ(\mathrm{Id}+\varepsilon\phi)^{-1}}_{s,t}\big)\big(\mu\circ(\mathrm{Id}+\varepsilon\phi)^{-1}\big)(\mathrm{d}x)\\&=\int_{\mathbb{R}^d}h\big(X^{x+\varepsilon\phi(x),\mu\circ(\mathrm{Id}+\varepsilon\phi)^{-1}}_{s,t}\big)\mu(\mathrm{d}x),\quad h\in\mathscr{B}_b(\mathbb{R}^d).\end{aligned}$$

So, $\Lambda^{\mu\circ(\mathrm{Id}+\varepsilon\phi)^{-1}}_{s,t}$ is the law of

$$x\mapsto X^{x+\varepsilon\phi(x),\mu\circ(\mathrm{Id}+\varepsilon\phi)^{-1}}_{s,t}$$

on the probability space $(\mathbb{R}^d,\mathscr{B}(\mathbb{R}^d),\mu)$. Therefore, by Lemmas 2.3 and 3.5, we obtain

$$\langle Df(\Lambda^{\cdot}_{s,t})(\mu),\phi\rangle_{L^2(\mu)}:=\frac{\mathrm{d}}{\mathrm{d}\varepsilon}f(\Lambda^{\mu\circ(\mathrm{Id}+\varepsilon\phi)^{-1}}_{s,t})\Big|_{\varepsilon=0}$$

$$= \int_{\mathbb{R}^d} \Big\langle \{(Df)(\Lambda^\mu_{s,t})\}(X^{x,\mu}_{s,t}), \frac{\mathrm{d}}{\mathrm{d}\varepsilon} X^{x+\varepsilon\phi(x),\mu\circ(\mathrm{Id}+\varepsilon\phi)^{-1}}_{s,t}\Big|_{\varepsilon=0} \Big\rangle \mu(\mathrm{d}x)$$
$$= \int_{\mathbb{R}^d} \big\langle \{(Df)(\Lambda^\mu_{s,t})\}(X^{x,\mu}_{s,t}), \nabla_{\phi(x)} X^{x,\mu}_{s,t} + D_\phi X^{x,\mu}_{s,t} \big\rangle \mu(\mathrm{d}x)$$
$$= \int_{\mathbb{R}^d} \big\langle (\nabla X^{x,\mu}_{s,t})^* \{(Df)(\Lambda^\mu_{s,t})\}(X^{x,\mu}_{s,t}), \phi(x) \big\rangle \mu(\mathrm{d}x)$$
$$+ \int_{\mathbb{R}^d\times\mathbb{R}^d} \big\langle (DX^{x,\mu}_{s,t})^*(y) \{(Df)(\Lambda^{x,\mu}_{s,t})\}(X^{x,\mu}_{s,t}), \phi(y) \big\rangle \mu(\mathrm{d}x)\mu(\mathrm{d}y)$$
$$= \Big\langle \big(\nabla X^{\cdot,\mu}_{s,t}\big)^* \{(Df)(\Lambda^\mu_{s,t})\}(X^{\cdot,\mu}_{s,t}) + \int_{\mathbb{R}^d} \big(DX^{x,\mu}_{s,t}\big)^*(\cdot) \{(Df)(\Lambda^\mu_{s,t})\}(X^{x,\mu}_{s,t}) \mu(\mathrm{d}x), \phi \Big\rangle_{L^2(\mu)}.$$

Therefore, (3.13) holds. □

Next, when $b, \sigma \in C_b^{0,2,2}([0,T]\times\mathbb{R}^d\times\mathscr{P}_2)$, by making derivatives to the SDE for $w^{x,\mu}_{s,t}(y)$ presented in Lemma 3.5, we derive the following result.

Lemma 3.9. *Assume that $b, \sigma \in C_b^{0,2,2}([0,T]\times\mathbb{R}^d\times\mathscr{P}_2)$. Then all derivatives*

$$\{D\nabla X^{x,\mu}_{s,t}\}(y),\ \nabla\{DX^{\cdot,\mu}_{s,t}(y)\}(x),\ \nabla\{DX^{y,\mu}_{s,t}(\cdot)\}(y),\ D^2 X^{x,\mu}_{s,t}(y,z)$$

are $L^{\infty-}(\mathbb{P})$ bounded continuous.

Proof. (a) We first consider $\{D\nabla X^{x,\mu}_{s,t}\}(y)$. Since $b, \sigma \in C_b^{0,2,2}([0,T]\times\mathbb{R}^d\times\mathscr{P}_2)$, by (3.2) and Lemmas 3.4-3.5, $v^{x,\mu}_{s,t} := \nabla_v X^{x,\mu}_{s,t}$ for $v\in\mathbb{R}^d$ solves the SDE

$$\mathrm{d}v^{x,\mu}_{s,t} = Z_1(t,x,\mu) v^{x,\mu}_{s,t}\mathrm{d}t + \{Z_2(t,x,\mu)v^{x,\mu}_{s,t}\}\mathrm{d}W_t,\quad v^{x,\mu}_{s,s} = v,$$

where

$$Z_1 : [0,T]\times\mathbb{R}^d\times\mathscr{P}_2 \to \mathbb{R}^d,\quad Z_2 : [0,T]\times\mathbb{R}^d\times\mathscr{P}_2 \to \mathbb{R}^d\otimes\mathbb{R}^d$$

are progressively measurable and satisfy

(D) $Z_1(t,x,\mu)$ and $Z_2(t,x,\mu)$ are uniformly bounded and continuous in $(t,x,\mu)\in[0,T]\times\mathbb{R}^d\times\mathscr{P}_2$; $DZ_1(t,x,\mu)(y)$ and $DZ_2(t,x,\mu)(y)$ are $L^{\infty-}(\mathbb{P})$ bounded continuous.

Then for any $\phi\in L^2(\mathbb{R}^d\to\mathbb{R}^d;\mu)$ and $\mu_\varepsilon := \mu\circ(\mathrm{Id}+\varepsilon\phi)^{-1}$ for small $\varepsilon>0$, $\gamma^\varepsilon_{s,t} := \frac{v^{x,\mu_\varepsilon}_{s,t} - v^{x,\mu}_{s,t}}{\varepsilon}$ solves the SDE

$$\mathrm{d}\gamma^\varepsilon_{s,t} = \{Z_1(t,x,\mu)\gamma^\varepsilon_{s,t}\}\mathrm{d}t + \{Z_2(t,x,\mu)\gamma^\varepsilon_{s,t}\}\mathrm{d}W_t$$
$$+ \frac{\{Z_1(t,x,\mu^\varepsilon) - Z_1(t,x,\mu)\}v^{x,\mu_\varepsilon}_{s,t}}{\varepsilon}\mathrm{d}t + \frac{\{Z_2(t,x,\mu_\varepsilon) - Z_2(t,x,\mu)\}v^{x,\mu_\varepsilon}_{s,t}}{\varepsilon}\mathrm{d}W_t,\quad \eta^\varepsilon_{s,s} = 0.$$

By **(D)**, we may repeat the proof of Lemma 3.7 to conclude that $D_\phi v^{x,\mu}_{s,t} := \lim_{\varepsilon\downarrow 0}\eta^\varepsilon_{s,t}$ exists and solves the SDE

$$\mathrm{d}\{D_\phi v^{x,\mu}_{s,t}\} = \big\{Z_1(t,x,\mu)D_\phi v^{x,\mu}_{s,t} + (D_\phi Z_1(t,x,\mu))v^{x,\mu}_{s,t}\big\}\mathrm{d}t$$
$$+ \big\{Z_2(t,x,\mu)D_\phi v^{x,\mu}_{s,t} + (D_\phi Z_2(t,x,\mu))v^{x,\mu}_{s,t}\big\}\mathrm{d}W_t,\quad D_\phi v^{x,\mu}_{s,s} = 0.$$

Hence, $Dv^{x,\mu}_{s,t}(y)$ solves the SDE

$$\begin{aligned}\mathrm{d}\{Dv^{x,\mu}_{s,t}(y)\} &= \big\{Z_1(t,x,\mu)Dv^{x,\mu}_{s,t}(y) + (DZ_1(t,x,\mu)(y))v^{x,\mu}_{s,t}\big\}\mathrm{d}t\\ &\quad + \big\{Z_2(t,x,\mu)Dv^{x,\mu}_{s,t}(y) + (DZ_2(t,x,\mu)(y))v^{x,\mu}_{s,t}\big\}\mathrm{d}W_t, \quad Dv^{x,\mu}_{s,s}(y) = 0.\end{aligned}$$

Therefore, by Lemma 3.4 and **(D)**, Lemma 3.3 yields that $\{D\nabla X^{x,\mu}_{s,t}\}(y)$ is $L^{\infty-}(\mathbb{P})$ bounded continuous.

(b) To calculate $\nabla\{DX^{\cdot,\mu}_{s,t}(y)\}(x), \nabla\{DX^{x,\mu}_{s,t}(\cdot)\}(y)$ and $D^2X^{x,\mu}_{s,t}(y,z) := D\{DX^{x,\mu}_{s,t}(y)\}(z)$, we reformulate the SDE in Lemma 3.5 for $w^{x,\mu}_{s,t}(y) := DX^{x,\mu}_{s,t}(y)$ as

$$\mathrm{d}w^{x,\mu}_{s,t} = \big\{A_1(t,x,\mu)w^{x,\mu}_{s,t} + A_2(t,x,\mu)\big\}\mathrm{d}t + \big\{B_1(t,x,\mu)w^{x,\mu}_{s,t} + B_2(t,x,\mu)\big\}\mathrm{d}W_t, \quad w^{x,\mu}_{s,s} = 0,$$

where, due to Lemmas 3.4-3.5 and (a), $\{A_i, B_i\}_{i=1,2}$ are progressively measurable maps such that

- A_1 and B_1 are uniformly bounded and continuous in $(t,x,\mu) \in [0,T] \times \mathbb{R}^d \times \mathscr{P}_2$;
- $\{A_i, B_i, \nabla A_i, \nabla B_i, DA_i, DB_i\}_{i=1,2}$ are $L^{\infty-}(\mathbb{P})$ bounded continuous in corresponding arguments.

So, as explained in (a), by taking derivatives $\partial_{x_i}, \partial_{y_i}$ and D_ϕ to this SDE respectively and applying Lemma 3.3, we prove that $\partial_{y_i}DX^{x,\mu}_{s,t}(y)$ and $D^2X^{x,\mu}_{s,t}(y,z)$ are $L^{\infty-}(\mathbb{P})$ bounded continuous in related arguments. We omit the details to save space. □

3.4 Proof of Proposition 3.2

Since $b, \sigma \in C^{0,2,2}_b([0,T] \times \mathbb{R}^d \times \mathscr{P}_2)$, assertions in Lemmas 3.4, 3.5, and 3.9 hold. Then it is straightforward to show that U given in Theorem 3.1(1) is in the class $C^{0,2,2}([0,T] \times \mathscr{P}_2)$.

Firstly, for any $1 \le i \le d$, by taking derivative ∂_{x_i} to the formula of U, we obtain

$$\begin{aligned}\partial_{x_i}U(t,x,\mu) &= \mathbb{E}\Big[\big\langle \nabla\Phi(\cdot, \Lambda^\mu_{t,T})(X^{x,\mu}_{t,T}), \partial_{x_i}X^{x,\mu}_{t,T}\big\rangle \mathrm{e}^{\int_t^T V(r,X^{x,\mu}_{t,r},\Lambda^\mu_{t,r})\mathrm{d}r}\Big]\\ &+ \mathbb{E}\bigg[\Phi(X^{x,\mu}_{t,T}, \Lambda^\mu_{t,T})\mathrm{e}^{\int_t^T V(r,X^{x,\mu}_{t,r},\Lambda^\mu_{t,r})\mathrm{d}r} \int_t^T \big\langle \nabla V(r,\cdot,\Lambda^\mu_{t,r})(X^{x,\mu}_{t,r}), \partial_{x_i}X^{x,\mu}_{t,r}\big\rangle \mathrm{d}r\bigg]\\ &+ \mathbb{E}\int_t^T \big\langle \nabla F(r,\cdot,\Lambda^\mu_{t,r})(X^{x,\mu}_{t,r}), \partial_{x_i}X^{x,\mu}_{t,r}\big\rangle \mathrm{e}^{\int_t^r V(\theta,X^{x,\mu}_{t,\theta},\Lambda^\mu_{t,\theta})\mathrm{d}\theta}\,\mathrm{d}r\\ &+ \mathbb{E}\int_t^T \bigg\{F(r,X^{x,\mu}_{t,r},\Lambda^\mu_{t,r})\mathrm{e}^{\int_t^r V(\theta,X^{x,\mu}_{t,\theta},\Lambda^\mu_{t,\theta})\mathrm{d}\theta} \int_t^r \big\langle \nabla V(\theta,\cdot,\Lambda^\mu_{t,\theta})(X^{x,\mu}_{t,\theta}), \partial_{x_i}X^{x,\mu}_{t,\theta}\big\rangle \mathrm{d}\theta\bigg\}\,\mathrm{d}r.\end{aligned}$$

By assumptions on Φ, V, F and Lemmas 3.4, 3.5 and 3.9, this formula implies that $\nabla U(t,x,\mu)$ is bounded and continuous. Moreover, by taking derivatives ∂_{x_j} and D to the formula, we conclude that $\nabla^2U(t,x,\mu)$ and $D\{\nabla X^{x,\mu}_{s,t}\}(y)$ are bounded and continuous as well.

Similarly, we may prove the assertion for $DU(t,x,\mu)(y), \partial_{x_i}\{DU(t,x,\mu)(y)\}, \partial_{y_i}\{DU(t,x,\mu)(y)\}$ and $D^2U(t,x,\mu)(y,z)$. For simplicity, we only consider the case for $V = F = 0$, for the general case the formulation is only more complicated due to derivatives to F and V, but there is no any essential difference for the proof. For $V = F = 0$ the formula for U becomes

$$U(t,x,\mu) = \mathbb{E}\Phi(X^{x,\mu}_{t,T}, \Lambda^\mu_{s,t}).$$

Then by (3.13) and the chain rule we obtain

$$DU(t,x,\mu)(y) = \mathbb{E}\bigg[\big\{\nabla\Phi(\cdot,\Lambda_{s,t}^{\mu})(X_{s,t}^{x,\mu})\big\}(DX_{s,t}^{x,\mu})(y) + (\nabla X_{s,t}^{y,\mu})^*\big\{(D\Phi(t,X_{s,t}^{x,\mu},\cdot)(\Lambda_{s,t}^{\mu})\big\}(X_{s,t}^{y,\mu})$$
$$+ \int_{\mathbb{R}^d}(DX_{s,t}^{z,\mu})^*(y)\big\{D\Phi(X_{s,t}^{x,\mu},\cdot)(\Lambda_{s,t}^{\mu})\big\}(X_{s,t}^{z,\mu})\mu(\mathrm{d}z)\bigg].$$

Since $\Phi \in C_b^{2,2}(\mathbb{R}^d \times \mathscr{P}_2)$, by Lemmas 3.4, 3.5 and 3.9 we deduce from this formula that $DU(t,x,\mu)(y)$ is bounded and continuous. Moreover, by taking derivatives $\partial_{x_i}, \partial_{y_i}, D$ to this formula, we conclude that $\partial_{x_i}\{DU(t,x,\mu)(y)\}, \partial_{y_i}\{DU(t,x,\mu)(y)\}$ and $D^2U(t,x,\mu)(y,z)$ are bounded and continuous as well. In conclusion, $U \in C^{0,2,2}([0,T]\times\mathbb{R}^d\times\mathscr{P}_2)$.

4 Ergodicity and structure of invariant measures

In this part, we assume that $b(t,x,\mu) = b(x,\mu)$ and $\sigma(t,x,\mu) = \sigma(x,\mu)$ are deterministic, and consider the ergodicity of the diffusion processes generated by $\mathscr{A}$ and $\tilde{\mathscr{A}}$.

Recall that a Markov process is called ergodic, if for any initial distribution, when $t \to \infty$ the process converges weakly to the unique invariant probability measure. For square integrable Markov processes, the weak convergence is equivalent to the convergence under the Wasserstein distance. To estimate the Wasserstein distance for solutions to the image SDE (1.7), we take the following hypothesis:

(H) $b(t,x,\mu) = b(x,\mu)$ and $\sigma(t,x,\mu) = \sigma(x,\mu)$ are deterministic, continuous in (x,μ) and do not depend on t. There exist constants $\lambda \in \mathbb{R}$ and $\kappa, \delta, K \geq 0$ such that

$$2\langle b(x,\mu) - b(y,\nu), x-y\rangle + \|\sigma(x,\mu)-\sigma(y,\nu)\|_{HS}^2 \leq \kappa\mathbb{W}(\mu,\nu)^2 - \lambda|x-y|^2,$$
$$\|\sigma(x,\mu)-\sigma(y,\nu)\|_{HS}^2 \leq K\big\{\mathbb{W}(\mu,\nu)^2 + |x-y|^2\big\},$$
$$|b(x,\mu)|^2 + \|\sigma(x,\mu)\|_{HS}^2 \leq \delta(1+|x|^2+\|\mu\|_2^2), \quad x,y\in\mathbb{R}^d, \mu,\nu\in\mathscr{P}_2.$$

By Theorem 2.1, **(H)** implies the well-posedness of (1.7). In the present time-homogenous case, we only consider the solution from time $s=0$, i.e. $(X_t^{x,\mu},\Lambda_t^{\mu}) := (X_{0,t}^{x,\mu},\Lambda_{0,t}^{\mu})$ for $t \geq 0$.

Let $P_t(\mu;\cdot)$ and $\tilde{P}_t(x,\mu;\cdot)$ denote the laws of Λ_t^{μ} and $(X_t^{x,\mu},\Lambda_t^{\mu})$ respectively. Then the associated Markov semigroups P_t and $\tilde{P}_t$ are given by

$$P_t f(\mu) := \mathbb{E}f(\Lambda_t^{\mu}) = \int_{\mathscr{P}_2} f(\nu)P_t(\mu;\mathrm{d}\nu), \quad f\in\mathscr{B}_b(\mathscr{P}_2),$$
$$\tilde{P}_t g(x,\mu) := \mathbb{E}g(X_t^{x,\mu},\Lambda_t^{\mu}) = \int_{\mathbb{R}^d\times\mathscr{P}_2} g(y,\nu)\tilde{P}_t(x,\mu;\mathrm{d}y,\mathrm{d}\nu), \quad g\in\mathscr{B}_b(\mathbb{R}^d\times\mathscr{P}_2).$$

Let $\mathscr{P}_2(\mathscr{P}_2)$ (resp. $\mathscr{P}_2(\mathbb{R}^d\times\mathscr{P}_2)$) be the set of probability measures on $\mathscr{P}_2$ (resp. $\mathbb{R}^d\times\mathscr{P}_2$) with finite second moments, and let $\mathbf{W}_2^{\mathscr{P}_2}$ be the L^2-Warsserstein distance on $\mathscr{P}_2(\mathscr{P}_2)$ induced by $\mathbb{W}_2$, while $\mathbf{W}_2^{\mathbb{R}^d\times\mathscr{P}_2}$ be that on $\mathscr{P}_2(\mathbb{R}^d\times\mathscr{P}_2)$ induced by the metric

$$\rho((x,\mu),(y,\nu)) := \sqrt{|x-y|^2+\mathbb{W}_2(\mu,\nu)^2}.$$

For any $Q \in \mathscr{P}_2(\mathscr{P}_2)$ and $\tilde{Q} \in \mathscr{P}_2(\mathbb{R}^d \times \mathscr{P}_2)$, let

$$QP_t = \int_{\mathscr{P}_2} P_t(\mu;\cdot)Q(\mathrm{d}\mu), \quad \tilde{Q}\tilde{P}_t = \int_{\mathbb{R}^d \times \mathscr{P}_2} \tilde{P}_t(x,\mu;\cdot)\tilde{Q}(\mathrm{d}x,\mathrm{d}\mu).$$

In the following two subsections, we first investigate the exponential ergodicity of the diffusion processes generated by $\mathscr{A}$ and $\tilde{\mathscr{A}}$, then figure out the structure of the invariant probability measures.

4.1 Exponential ergodicity

Theorem 4.1. *Assume* **(H)**. *Then for any* $(x,\mu) \in \mathbb{R}^d \times \mathscr{P}_2$,

$$\mathbb{E}\mathbb{W}_2(\Lambda_t^\mu, \Lambda_t^\nu)^2 \le \mathbb{W}_2(\mu,\nu)^2 \mathrm{e}^{-(\lambda-\kappa)t}, \quad t \ge 0, \tag{4.1}$$

$$\mathbb{E}|X_t^{x,\mu} - X_t^{y,\nu}|^2 \le |x-y|^2 \mathrm{e}^{-\lambda t} + \mathbb{W}_2(\mu,\nu)^2 \mathrm{e}^{-(\lambda-\kappa)t}, \quad t \ge 0. \tag{4.2}$$

Consequently, if $\lambda > \kappa$ *then:*

(1) $\tilde{P}_t$ *has a unique invariant probability measure* $\tilde{\Pi} \in \mathscr{P}_2(\mathbb{R}^d \times \mathscr{P}_2)$ *such that for any* $\tilde{Q} \in \mathscr{P}_2(\mathbb{R}^d \times \mathscr{P}_2)$,

$$\mathbf{W}_2^{\mathbb{R}^d \times \mathscr{P}_2}(\tilde{Q}\tilde{P}_t, \tilde{\Pi})^2 \le 2\mathrm{e}^{-(\lambda-\kappa)t} \mathbf{W}_2^{\mathbb{R}^d \times \mathscr{P}_2}(\tilde{Q}, \tilde{\Pi})^2, \quad t \ge 0; \tag{4.3}$$

(2) $\Pi := \tilde{\Pi}(\mathbb{R}^d \times \cdot)$ *is the unique invariant probability measure of* P_t *such that for any* $Q \in \mathscr{P}_2(\mathscr{P}_2)$,

$$\mathbf{W}_2^{\mathscr{P}_2}(QP_t(\mu;\cdot), \Pi)^2 \le \mathrm{e}^{-(\lambda-\kappa)t} \mathbf{W}_2^{\mathscr{P}_2}(Q,\Pi)^2, \quad t \ge 0. \tag{4.4}$$

Proof. (a) We first prove (4.1) and (4.2). Let $\pi \in \mathscr{C}(\mu,\nu)$ such that

$$\mathbb{W}_2(\mu,\nu)^2 = \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 \pi(\mathrm{d}x,\mathrm{d}y).$$

Then for any $t \ge 0$,

$$\pi_t := \pi \circ (X_t^{\cdot,\mu}, X_t^{\cdot,\nu})^{-1} \in \mathscr{C}(\Lambda_t^\mu, \Lambda_t^\nu),$$

so that

$$\mathbb{W}_2(\Lambda_t^\mu, \Lambda_t^\nu)^2 \le \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 \pi_t(\mathrm{d}x,\mathrm{d}y) = \int_{\mathbb{R}^d \times \mathbb{R}^d} |X_t^{x,\mu} - X_t^{y,\nu}|^2 \pi(\mathrm{d}x,\mathrm{d}y) =: \ell_t. \tag{4.5}$$

Combining this with **(H)** and Itô's formula, we obtain

$$\mathrm{d}|X_t^{x,\mu} - X_t^{y,\nu}|^2 \le \big\{\kappa \ell_t - \lambda |X_t^{x,\mu} - X_t^{y,\nu}|^2\big\}\mathrm{d}t + \mathrm{d}M_t$$

for some martingale M_t, which implies

$$\mathrm{e}^{\lambda t}\mathbb{E}|X_t^{x,\mu} - X_t^{y,\nu}|^2 \le |x-y|^2 + \kappa \int_0^t \mathrm{e}^{\lambda s}\mathbb{E}\ell_s \mathrm{d}s, \quad t \ge 0. \tag{4.6}$$

Integrating with respect to $\pi(\mathrm{d}x, \mathrm{d}y)$ gives

$$\mathrm{e}^{\lambda t}\mathbb{E}\ell_t \le \mathbb{W}_2(\mu,\nu)^2 + \kappa \int_0^t \mathrm{e}^{\ell s}\mathbb{E}\ell_s \mathrm{d}s, \quad t \ge 0,$$

which together with Grownwall's lemma and (4.5) leads to

$$\mathbb{E}\mathbb{W}_2(\Lambda_t^\mu, \Lambda_t^\nu)^2 \le \mathbb{E}\ell_t \le \mathbb{W}_2(\mu,\nu)^2 \mathrm{e}^{-(\lambda-\kappa)t}, \quad t \ge 0.$$

Thus, (4.1) holds. Substituting (4.1) into (4.6) we arrive at

$$\begin{aligned}\mathbb{E}|X_t^{x,\mu} - X_t^{y,\nu}|^2 &\le \mathrm{e}^{-\lambda t}|x-y|^2 + \kappa \mathbb{W}_2(\mu,\nu)^2 \mathrm{e}^{-\lambda t}\int_0^t \mathrm{e}^{\kappa s}\mathrm{d}s \\ &\le \mathrm{e}^{-\lambda t}|x-y|^2 + \mathbb{W}_2(\mu,\nu)^2\mathrm{e}^{-(\lambda-\kappa)t}.\end{aligned}$$

Hence, (4.2) holds.

(b) Existence of invariant probability measures. Consider, for instance $(X_t^{0,\delta_0}, \Lambda_t^{\delta_0})$, where δ_0 is the Dirac measure at $0 \in \mathbb{R}^d$. Let $\tilde{\Pi}_t = \tilde{P}_t(0, \delta_0; \cdot)$ be the law of $(X_t^{0,\delta_0}, \Lambda_t^{\delta_0})$. By the completeness of the Wasserstein space, if

$$\lim_{s,t\to\infty} \mathbf{W}_2^{\mathbb{R}^d\times\mathscr{P}_2}(\tilde{\Pi}_t, \tilde{\Pi}_s)^2 = 0, \tag{4.7}$$

then there exists a probability measure $\tilde{\Pi}$ on $\mathbb{R}^d \times \mathscr{P}_2$ with $\|\tilde{\Pi}\|_2^2 := \tilde{\Pi}(\rho^2) < \infty$ such that $\lim_{t\to\infty} \mathbf{W}_2^{\mathbb{R}^d\times\mathscr{P}_2}(\tilde{\Pi}_t, \tilde{\Pi}) = 0$. Consequently, $\tilde{\Pi}$ is an invariant probability measure for $\tilde{P}_t$. Moreover, since the law of $\Lambda_t^{\delta_0}$ is $\Pi_t(\mathbb{R}^d \times \cdot)$, which converges to $\Pi := \tilde{\Pi}(\mathbb{R}^d \times \cdot)$ weakly as $t \to \infty$, we see that Π is an invariant probability measure of P_t.

To prove (4.7), let $t > s \ge 0$. By the Markov property we have

$$\tilde{\Pi}_t = P_t(0, \delta_0; \cdot) = \int_{\mathbb{R}^d\times\mathscr{P}_2} P_s(x,\mu;\cdot)\tilde{\Pi}_{t-s}(\mathrm{d}x, \mathrm{d}\mu).$$

Combining this with (4.1) and (4.2) we obtain

$$\begin{aligned}&\mathbf{W}_2^{\mathbb{R}^d\times\mathscr{P}_2}\big(\tilde{\Pi}_t, \tilde{\Pi}_s)^2 \le \int_{\mathbb{R}^d\times\mathscr{P}_2} \mathbf{W}_2^{\mathbb{R}^d\times\mathscr{P}_2}\big(\tilde{P}_s(x,\mu;\cdot), \tilde{P}_s(0,\delta_0;\cdot)\big)^2 \tilde{\Pi}_{t-s}(\mathrm{d}y, \mathrm{d}\nu) \\ &\le \int_{\mathbb{R}^d\times\mathscr{P}_2} \big\{\mathbb{E}|X_s^{0,\delta_0} - X_s^{x,\mu}|^2 + \mathbb{W}_2(\Lambda_s^\mu, \Lambda_s^{\delta_0})^2\big\}\tilde{\Pi}_{t-s}(\mathrm{d}x, \mathrm{d}\mu) \\ &\le \int_{\mathbb{R}^d\times\mathscr{P}_2} \big\{|x|^2\mathrm{e}^{-\lambda s} + 2\mathbb{W}_2(\delta_0,\mu)^2\mathrm{e}^{-(\lambda-\kappa)s}\big\}\tilde{\Pi}_{t-s}(\mathrm{d}x, \mathrm{d}\mu) \\ &= \mathrm{e}^{-\lambda s}\mathbb{E}|X_{t-s}^{0,\delta_0}|^2 + 2\mathrm{e}^{-(\lambda-\kappa)s}\mathbb{E}\mathbb{W}_2(\delta_0, \Lambda_{t-s}^{\delta_0})^2 = (\mathrm{e}^{-\lambda s} + 2\mathrm{e}^{-(\lambda-\kappa)s})\mathbb{E}|X_{t-s}^{0,\delta_0}|^2.\end{aligned}$$

So, to prove (4.7) it remains to show that

$$\sup_{t\ge 0} \mathbb{E}|X_t^{0,\delta_0}|^2 < \infty. \tag{4.8}$$

By assumption **(H)** with $\lambda > \kappa$, for any $\lambda > \lambda' > \kappa' > \kappa$ there exists a constant $c > 0$ such that

$$2\langle b(x,\mu), x\rangle + \|\sigma(x,\mu)\|_{HS}^2 \le c + \kappa'\|\mu\|_2^2 - \lambda'|x|^2, \quad (x,\mu) \in \mathbb{R}^d \times \mathscr{P}_2.$$

Combining this with Itô's formula, and noting that $\|\Lambda_t^{\delta_0}\|_2^2 = \delta_0(|X_t^{\cdot,\delta_0}|^2) = |X_t^{0,\delta_0}|^2$, we obtain

$$\mathrm{d}|X_t^{0,\delta_0}|^2 \le \big\{c + (\kappa' - \lambda')|X_t^{0,\delta_0}|^2\big\}\mathrm{d}t + \mathrm{d}M_t$$

for some martingale M_t. This implies

$$\mathbb{E}|X_t^{0,\delta_0}|^2 \le c\int_0^t \mathrm{e}^{-(\lambda'-\kappa')s}\mathrm{d}s, \quad t \ge 0.$$

Since $\lambda' > \kappa'$, we derive (4.8) and hence finish the proof of the existence of invariant probability measures. Moreover, the invariant probability measure $\tilde{\Pi}$ satisfies

$$\int_{\mathbb{R}^d\times\mathscr{P}_2} (|x|^2 + \|\mu\|_2^2)\tilde{\Pi}(\mathrm{d}x,\mathrm{d}\mu) \le \lim_{t\to\infty}\mathbb{E}|X_t^{0,\delta_0}|^2 \le \frac{c}{\lambda'-\kappa'} < \infty.$$

Hence, $\tilde{\Pi} \in \mathscr{P}_2(\mathbb{R}^d \times \mathscr{P}_2)$.

(c) It is easy to see that (4.3) follows from (4.1) and (4.2). Indeed, letting $\Gamma \in \mathscr{C}(\tilde{Q}, \tilde{\Pi})$ such that

$$\mathbf{W}_2^{\mathbb{R}^d\times\mathscr{P}_2}(\tilde{Q},\tilde{\Pi})^2 = \int_{(\mathbb{R}^d\times\mathscr{P}_2)^2} \rho^2\mathrm{d}\Gamma,$$

we deduce from (4.1), (4.2) and $\tilde{\Pi} = \tilde{\Pi}\tilde{P}_t$ that

$$\begin{aligned}
&\mathbf{W}_2^{\mathbb{R}^d\times\mathscr{P}_2}(\tilde{Q}\tilde{P}_t,\tilde{\Pi})^2 = \mathbf{W}_2^{\mathbb{R}^d\times\mathscr{P}_2}(\tilde{Q}\tilde{P}_t,\tilde{\Pi}\tilde{P}_t)^2\\
&\le \int_{(\mathbb{R}^d\times\mathscr{P}_2)^2} \mathbf{W}_2^{\mathbb{R}^d\times\mathscr{P}_2}(\tilde{P}_t(x,\mu;\cdot),\tilde{P}_t(y,\nu;\cdot))^2\Gamma(\mathrm{d}x,\mathrm{d}\mu;\mathrm{d}y,\mathrm{d}\nu)\\
&\le \int_{(\mathbb{R}^d\times\mathscr{P}_2)^2} \mathbb{E}\big\{|X_t^{x,\mu} - X_t^{y,\nu}|^2 + \mathbb{W}_2(\Lambda_t^\mu,\Lambda_t^\nu)^2\big\}\Gamma(\mathrm{d}x,\mathrm{d}\mu;\mathrm{d}y,\mathrm{d}\nu)\\
&\le \int_{(\mathbb{R}^d\times\mathscr{P}_2)^2} \big\{|x-y|^2\mathrm{e}^{-\lambda t} + 2\mathbb{W}_2(\mu,\nu)^2\mathrm{e}^{-(\lambda-\kappa)t}\big\}\Gamma(\mathrm{d}x,\mathrm{d}\mu;\mathrm{d}y,\mathrm{d}\nu)\\
&\le 2\mathrm{e}^{-(\lambda-\kappa)t}\mathbf{W}_2^{\mathbb{R}^d\times\mathscr{P}_2}(\tilde{Q},\tilde{\Pi})^2, \quad t \ge 0.
\end{aligned}$$

In particular, $\tilde{\Pi}$ is the unique invariant probability measure of P_t.

(d) As shown in (b) and (c), (4.1) for $\lambda > \kappa$ implies that P_t has a unique invariant probability measure Π satisfying the estimate (4.4). Noting that $P_t(\mu;\cdot) = \tilde{P}_t(x,\mu;\mathbb{R}^d\times\cdot)$ holds for all $(x,\mu) \in \mathbb{R}^d \times \mathscr{P}_2$, we have $\Pi = \tilde{\Pi}(\mathbb{R}^d \times \cdot)$. □

4.2 Structure of invariant probability measures

Under condition **(H)**, let $b_0(x) = b(x,\delta_x)$ and $\sigma_0(x) = \sigma(x,\delta_x)$. Then the SDE (1.10) is well-posed. Let P_t^0 be the associated Markov semigroup.

Theorem 4.2. *Assume* **(H)**. *If P_t^0 has an invariant probability measure μ_0, then*

$$\tilde{\Pi}_0(\mathrm{d}x, \mathrm{d}\mu) := \mu_0(\mathrm{d}x)\delta_{\delta_x}(\mathrm{d}\mu)$$

is an invariant probability measure of $\tilde{P}_t$. Consequently, $\Pi_0 := \tilde{\Pi}_0(\mathbb{R}^d \times \cdot) = \int_{\mathbb{R}^d} \delta_{\delta_x}\mu_0(\mathrm{d}x)$ is an invariant probability measure of P_t, and when $\lambda > \kappa$, the unique invariant probability measures $\tilde{\Pi}$ and Π in Theorem 4.1 *satisfy* (1.11).

Proof. Recall that $(X_t^{x,\mu}, \Lambda_t^\mu)$ solve the SDE

$$\mathrm{d}X_t^{x,\mu} = b(X_t^{x,\mu}, \Lambda_t^\mu)\mathrm{d}t + \sigma(X_t^{x,\mu}, \Lambda_t^\mu)\mathrm{d}W_t, \quad X_0^{x,\mu} = x,$$

where $\Lambda_t^\mu := \mu \circ (X_t^{\cdot,\mu})^{-1}$. Then, when $\mu = \delta_x$ we have $\Lambda_t^\mu = \delta_{X_t^{x,\delta_x}}$, so that $(X_t^{x,\delta_x})_{t\geq 0}$ solves the SDE (1.10). By the uniqueness of this SDE and that μ_0 is an invariant probability measure of P_t^0, we obtain

$$\int_{\mathbb{R}^d} \big[\mathbb{E}g(X_t^{x,\delta_x})\big]\mu_0(\mathrm{d}x) = \int_{\mathbb{R}^d} P_t^0 g(x)\mu_0(\mathrm{d}x) = \int_{\mathbb{R}^d} g(x)\mu_0(\mathrm{d}x), \quad t \geq 0, g \in \mathscr{B}_b(\mathbb{R}^d).$$

Combining this with $\tilde{P}_t f(x, \delta_x) = \mathbb{E}f(X_t^{x,\delta_x}, \delta_{X_t^{x,\delta_x}})$ for $f \in \mathscr{B}_b(\mathbb{R}^d \times \mathscr{P}_2)$, and taking $g(x) = f(x, \delta_x)$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d\times\mathscr{P}_2} \tilde{P}_t f(x,\mu)\tilde{\Pi}_0(\mathrm{d}x,\mathrm{d}\mu) = \int_{\mathbb{R}^d} \tilde{P}_t f(x,\delta_x)\mu_0(\mathrm{d}x)\\
&= \int_{\mathbb{R}^d} \big[\mathbb{E}f(X_t^{x,\delta_x}, \delta_{X_t^{x,\delta_x}})\big]\mu_0(\mathrm{d}x) = \int_{\mathbb{R}^d} \big[\mathbb{E}g(X_t^{x,\delta_x})\big]\mu_0(\mathrm{d}x)\\
&= \int_{\mathbb{R}^d} g(x)\mu_0(\mathrm{d}x) = \int_{\mathbb{R}^d} f(x,\delta_x)\mu_0(\mathrm{d}x) = \int_{\mathbb{R}^d\times\mathscr{P}_2} f(x,\mu)\tilde{\Pi}_0(\mathrm{d}x,\mathrm{d}\mu).
\end{aligned}$$

Therefore, $\tilde{\Pi}_0$ is an invariant probability measure of $\tilde{P}_t$. In particular, by taking $f(x,\mu) = f(\mu)$, we see that Π_0 is an invariant probability measure of P_t.

Finally, if $\lambda > \kappa$, by Theorem 4.1, Π and $\tilde{\Pi}$ are the unique invariant probability measures of P_t and $\tilde{P}_t$ respectively. So, $\tilde{\Pi} = \tilde{\Pi}_0$ and $\Pi = \Pi_0$; that is, (1.11) holds. □

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