All wealth in assets is optimal under interest rate uncertainty*

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This note shows that a long term investor who faces considerable Knightian uncertainty about the future evolution of interest rates optimally puts all his wealth into risky assets.

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1 Introduction

Optimal consumption and portfolio decisions play a fundamental role for individual investors, pension funds, and insurance companies alike. Life-cycle models also form the basic building block for more complex economic models that are used in economic policy and governance discussions. In this note, we consider an otherwise standard life-cycle consumption and portfolio problem for an investor who faces Knightian uncertainty about interest rates.

Interest rates vary considerably over time. For a long term investor, there is thus no riskless asset. While the consumption and portfolio choice problem for investors who face risky interest rates has been amply studied (see below), the role of Knightian or model uncertainty of interest rates has not been tackled so far. Various factors lead to such Knightian uncertainty. Equilibrium interest rates depend on an economy's growth, the time preferences of agents, and the volatility of the market. All these parameters are hard to predict or estimate in the long run and it thus makes sense to ask for robust policies. Moreover, interest rates are also influenced by central bank policies. Some twenty years ago, few investors would have predicted the current zero interest rates or even negative

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interest rate policy that many central banks currently pursue. We can thus conclude that long term investors face considerable model uncertainty about interest rates.

The main new finding that we report in this note is the following. *If interest rate uncertainty is sufficiently high, it is optimal to put all wealth into risky assets.* This is in sharp contrast to other studies involving uncertainty-averse investors. For example, in Dow and Werlang (1992) pioneering study, it is shown that ambiguity-averse investors rather shy away from risky assets when they face Knightian uncertainty about expected returns. In a one period model, the effect of interest rate uncertainty cannot be studied, of course. When there is no or only small interest rate uncertainty, a similar results holds true in our continuous-time model. However, we find it important to stress the relevance of interest rate uncertainty for long term investors.

We take model uncertainty about the short rate and, in fact, the whole term structure, into account[1]. We consider an investor who is willing to work with fixed bounds $\underline{r} < \overline{r}$ for the short rate. Every adapted process (r_t) with values in the interval $[\underline{r}, \overline{r}]$ is considered as a possible trajectory at time 0. At time t, the past values of the short rate $(r_s)_{s \leq t}$ including the current value r_t are known, of course. The investor still faces model uncertainty about the future realizations r_u for $u > t$. In this sense, we assume here that the ambiguity about the future short rates is persistent and no learning occurs[2].

Knightian uncertainty has recently attracted a great deal of attention, both in practice, as the sensitivity of many financial decisions with respect to questionable probabilistic assumptions became clear, and in theory, where an extensive theory of decision making and risk measurement under uncertainty has emerged. Gilboa and Schmeidler (1989) lay the foundation for a new approach to decisions under Knightian uncertainty by weakening the strong independence axiom or sure thing principle used previously by Savage (1954) and Anscombe and Aumann (1963) to justify (subjective) expected utility. The models are closely related to monetary risk measures (Artzner, Delbaen, Eber, and Heath (1999)). Subsequently, the theory has been generalized to variational preferences (Maccheroni, Marinacci, and Rustichini (2006a), Föllmer and Schied (2002)) and dynamic time-consistent models (Epstein and Schneider (2003b), Maccheroni, Marinacci, and Rustichini (2006b), Riedel (2004), Föllmer and Penner (2006)).

Concerning portfolio and consumption choice, the pioneering results of Merton (1969, 1971) are still the basic reference for life-cycle consumption and portfolio choice under uncertainty. As mean return, volatility, and interest rates are known constants in Merton'smodel, the consequences of having stochastic, time-varying dynamics for these parameters have been studied in great detail. Mean–reverting drift (or "predictable returns"), stochastic volatility models and models with stochastic term structures have been studied in detail. These models all work under the expected utility paradigm as they assume a known distribution for the parameters; for example, Barberis (2000) studies mean–reverting returns and estimation errors. Korn and Kraft (2001) studies portfolio problems with stochastic interest rates. Chacko and Viceira (2005), and Kraft (2005) allow for stochastic volatility. We refer the reader to Liu (2007) for a recent general approach with

[1]As there is no money in our model, the interest rate can be considered as the real (not the nominal) interest rate. We can thus also interpret our model as a model of Knightian uncertainty about inflation.

[2]From a conceptual perspective, the short rate is determined by independent and *indistinguishable* experiments in the sense of Epstein and Schneider (2003a); in every (infinitesimal) period, a new ambiguous experiment is carried out which is independent from the past to determine the next short rate. As a consequence, the agent cannot learn from past data.

stochastic interest rates and volatilities. The typical result in this literature identifies additional terms next to the classic optimal portfolio of Merton that are related to the demand for hedging the new diffusive factors. In particular, the portfolio weights vary stochastically with the factor estimates over time.

Robust statistics and robust control as well as the decision–theoretic literature on Knightian uncertainty share a lot of formal and conceptual similarities. The typical "penalty" approach to robust control used by Hansen and Sargent (2011) can be viewed as a special case of variational preferences (Maccheroni, Marinacci, and Rustichini (2006a,b)) where the agent uses entropy as a penalty function. Our approach of a pessimistic multiple prior model is also a special case of variational preferences but with a different penalty function. Using the robust control approach of Anderson, Hansen, and Sargent (2003), Trojani and Vanini (2002), Maenhout (2004), and Luo (2017) study the robust portfolio choice problem with drift ambiguity. Drift ambiguity in continuous time is also discussed in Chen and Epstein (2002), Schied (2005), Quenez (2004), Schied (2008), Miao (2009), Liu (2010, 2011) among others. Föllmer, Schied, and Weber (2009) survey this literature. Drift and volatility uncertainty have been studied in the recent papers Biagini and Pınar (2017), Epstein and Ji (2013), and Neufeld and Nutz (2018). These papers all work with a known interest rate.

Many different dynamic models for the short rate have been developed over time, ranging from Vasicek (1977) over Heath, Jarrow, and Morton (1992) to general affine models (Duffie, Filipović, and Schachermayer (2003)), to name a few outstanding contributions; see also Brigo and Mercurio (2007) for an overview.

The paper is organized as follows. The next section formulates the interest rate ambiguity model within the new framework and states the main theorem. We give the proofs of our results in Section 3.

2 Knightian uncertainty about interest rates and the main theorem

This paper investigates optimal consumption and investment policies under Knightian uncertainty. We extend the Samuelson model of continuous time financial markets to allow for Knightian uncertainty about drift, volatility and interest rates. Investors know neither the future realization of the risky asset's payoff, nor the probability of its occurring. We consider ambiguity–averse agents who neither know the specific parameters nor their probability laws, but are willing to work with certain bounds for the relevant parameters. Investor's preference under Knightian uncertainty is represented essentially by "max-min" expected utility, given a set of probability measures about the financial market. In other words, investors evaluate the outcome of an investment with respect to a set of models and then choose the model that leads to the lowest expected utility.

In continuous time, Knightian uncertainty leads to some subtle issues. Uncertainty about volatility, as well as uncertainty about the short rate, requires the use of singular probability measures, which is a curious, but natural fact in an ambiguous world.

We start with a continuous-time Merton model with Knightian uncertainty. Let us consider asset prices with continuous sample paths. Let Ω be the one-dimensional Wiener space of continuous functions, and $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ be its natural filtration.

In the continuous–time diffusion framework, three parameter processes, drift (or expected return), volatility and interest rate uncertainty, describe all uncertainty. We thus model the Knightian uncertainty by a convex and compact subset $\Theta \subset \mathbb{R}^3$. The investor is not sure about the exact value or distribution of the drift process $\mu = (\mu_t)$ with values in $\mathbb{R}$ nor about the exact value or distribution of the volatility process $\sigma = (\sigma_t)$ with values in $\mathbb{R}$. The investor is also not sure about the exact value or distribution of the interest rate process $r = (r_t)$. The only restriction is that $\theta \equiv (\mu_t, \sigma_t, r_t) \in \Theta$.

Let Y be the canonical process $(\Omega, \mathcal{F})$. For $(\mu, \sigma, r) \in \Theta$, let $\mathbb{P}^{\mu,\sigma}$ be a probability measure on $(\Omega, \mathbb{F})$ such that Y is the unique strong solution of the following stochastic differential equation

$$dY_t^{\mu,\sigma,r} = \begin{pmatrix} dS_t \\ dP_t \end{pmatrix} = \begin{pmatrix} \mu_t S_t dt + \sigma_t S_t dW_t^{\mu,\sigma,r} \\ r_t P_t dt \end{pmatrix},$$

where $W^{\mu,\sigma,r}$ is a $\mathbb{P}^{\mu,\sigma,r}$-Brownian motion with $\theta = (\mu, \sigma, r) \in \Theta$. Let $\mathcal{P}_0$ be the set of all probability measures $\mathbb{P}^{\mu,\sigma,r}$ constructed in this way. The set of priors $\mathcal{P}$ is the closure of $\mathcal{P}_0$ under the topology of weak convergence.

In a financial market, there are one risky asset and one riskless asset. For $\theta = (\mu, \sigma, r) \in \Theta$, under $P^\theta \in \mathcal{P}$, the price of the riskless asset is described by

$$dP_t = P_t r_t dt, \ \ P_0 = 1,$$

and the prices of risky assets evolve as

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t^\theta, \ \ S_0 = 1,$$

where W^θ is a $\mathbb{P}^\theta$-Brownian motion.

Let us give the definitions of consumption and portfolio strategies. We call a pair (π, c) a consumption–portfolio strategy, if (π, c) are $\mathcal{F}-$progressively measurable, and $\int_0^T (\pi_s^2 + c_s^2) ds < \infty, P - a.s.$, for all $P \in \mathcal{P}$. The wealth of the investor with some initial endowment $x_0 > 0$ and portfolio–consumption policy (π, c) is given by

$$dX_t = r_t X_t (1 - \pi_t) dt + X_t \pi_t \mu_t dt - c_t dt + X_t \pi_t \sigma_t dW_t^\theta, \tag{2.1}$$

under P^θ.

The consumption and portfolio strategy (π, c) is *admissible* if for all $P^\theta \in \mathcal{P}$, $X_t \geq 0, P^\theta - a.s., t \in [0, T]$. We denote by Π the set of admissible consumption and portfolio strategies.

We consider an ambiguity–averse agent who maximizes the minimal expected utility over the set of priors. The investor's utility of consuming c and bequesting a terminal wealth X_T is defined by

$$U(c, X) = \inf_{P \in \mathcal{P}} E_P\Big[\int_0^T u(s, c_s) ds + \Phi(T, X_T)\Big],$$

where

$$u(t, c) = \exp(-\delta t) \frac{c^{1-\alpha}}{1-\alpha}, \ \ \Phi(T, x) = \frac{K x^{1-\alpha}}{1-\alpha}, \tag{2.2}$$

for some $\delta > 0, \alpha > 0, K > 0$, and $\alpha \neq 1$.[3]

We are now able to state our main insight concerning Knightian uncertainty about interest rates. More details on the general solution of the problem for other parameter constellations can be found in the next section as well as in our accompanying working paper Lin and Riedel (2014).

Proposition 2.1 *With sufficient ambiguity about interest rates, more precisely, if*

$$\underline{r} \leq \underline{\mu} - \alpha\overline{\sigma}^2 \leq \overline{r}\,,$$

the investor does not participate in the money market and puts all capital into the stock.

The intuition for the above result is as follows. If the investor is convinced that investing in stocks is more profitable than keeping money in the savings account, while accounting for risk aversion, i.e. if $\underline{r} \leq \underline{\mu} - \alpha\overline{\sigma}^2$, then he does not want to stay away from the opportunities that the stock market promises. Given that the investor wants to go long, the worst case parameter for expected returns is the lower bound $\underline{\mu}$. The worst case volatility for an ambiguity-averse investor is always the maximal volatility $\overline{\sigma}$.

As the analysis below will show in more detail, putting all wealth into risky assets hedges the investor from interest rate uncertainty as local expected returns are then independent of the interest rate. For the identified parameters, this kind of hedging is indeed optimal for the agent. Another way of interpreting the results is as follows. With interest rate uncertainty, it is optimal to put all wealth into the risky asset if the set of Merton rations $\frac{\mu-r}{a\overline{\sigma}^2}$ contains 1. In this case, due to ambiguity aversion, the investor does not want to borrow additional money in order to buy even more of the risky asset as interest rates might turn out high, nor does he want to reduce his exposure as interest rates might turn out too low.

3 Proof of the main theorem and extensions

The following verification theorem shows that our portfolio–consumption choice problem is solved when we find a solution to the suitably adjusted Hamilton–Jacobi–Bellman-Isaacs equation tbhat Merton derived. We denote by $Q = [0, T) \times \mathbb{R}$ and $\overline{Q} = Q \cup \partial Q$, where ∂Q is the boundary of Q.

Theorem 3.1 *Suppose $\varphi \in C^{1,2}(Q) \cap C(\overline{Q})$ is a solution of the following* Hamilton–Jacobi–Bellman–Isaacs (HJBI) *equation*

$$\sup_{(\pi,c)\in\mathbb{R}\times\mathbb{R}_+} \Big\{ u(t,c) + \varphi_t(t,x) - c\varphi_x(t,x) + \inf_{r\in[\underline{r},\overline{r}]} \{xr\varphi_x(t,x)(1-\pi)\} \\ + \inf_{(\mu,\sigma)\in[\underline{\mu},\overline{\mu}]\times[\underline{\sigma},\overline{\sigma}]} \{\varphi_x(t,x)x\pi\mu + \frac{1}{2}x^2\varphi_{xx}(t,x)\pi^2\sigma^2\} \Big\} = 0, \tag{3.1}$$

with boundary condition $\varphi(T, x) = \Phi(T, x)$.

[3] The solution for $\alpha = 1$ which corresponds to log-utility can be easily read off our solutions by setting formally $\alpha = 1$ in our formulas. The proof is easily adapted.

For any $(\pi(t,x),c(t,x))\in\mathbb{R}\times\mathbb{R}_+$*, there exists* $(\tilde{\mu},\tilde{\sigma},\tilde{r})$ *such that*

$$\begin{aligned}\inf_{r\in[\underline{r},\overline{r}]}\{xr\varphi_x(t,x)(1-\pi)\}+\inf_{(\mu,\sigma)\in[\underline{\mu},\overline{\mu}]\times[\underline{\sigma},\overline{\sigma}]}\{\varphi_x(t,x)x\pi\mu+\frac{1}{2}x^2\varphi_{xx}(t,x)\pi^2\sigma^2\}\\ =x\tilde{r}\varphi_x(t,x)(1-\pi)+\varphi_x(t,x)x\pi\tilde{\mu}+\frac{1}{2}x^2\varphi_{xx}(t,x)\pi^2(\tilde{\sigma})^2.\end{aligned}\tag{3.2}$$

Let $(\hat{\pi}(t,x),\hat{c}(t,x))\in\mathbb{R}\times\mathbb{R}_+$ *satisfy*

$$\begin{aligned}(\hat{\pi}(t,x),\hat{c}(t,x))\in\mathbb{R}\times\mathbb{R}_+\in\arg\sup_{(\pi,c)\in\mathbb{R}\times\mathbb{R}_+}\Big\{u(t,c)+\varphi_t(t,x)-c\varphi_x(t,x)\\ +\inf_{r\in[\underline{r},\overline{r}]}\{xr\varphi_x(t,x)(1-\pi)\}+\inf_{(\mu,\sigma)\in[\underline{\mu},\overline{\mu}]\times[\underline{\sigma},\overline{\sigma}]}\{\varphi_x(t,x)x\pi\mu+\frac{1}{2}x^2\varphi_{xx}(t,x)\pi^2\sigma^2\}\Big\}.\end{aligned}$$

and (μ^*,σ^*,r^*) *satisfy*

$$\begin{aligned}&\sup_{(\pi,c)\in\mathbb{R}\times\mathbb{R}_+}\Big\{u(t,c)+\varphi_t(t,x)-c\varphi_x(t,x)+\inf_{r\in[\underline{r},\overline{r}]}\{xr\varphi_x(t,x)(1-\pi)\}\\ &\qquad+\inf_{(\mu,\sigma)\in[\underline{\mu},\overline{\mu}]\times[\underline{\sigma},\overline{\sigma}]}\{\varphi_x(t,x)x\pi\mu+\frac{1}{2}x^2\varphi_{xx}(t,x)\pi^2\sigma^2\}\Big\}\\ &=\sup_{(\pi,c)\in\mathbb{R}\times\mathbb{R}_+}\Big\{u(t,c)+\varphi_t(t,x)-c\varphi_x(t,x)+xr^*\varphi_x(t,x)(1-\pi)\\ &\qquad+\varphi_x(t,x)x\pi\mu^*+\frac{1}{2}x^2\varphi_{xx}(t,x)\pi^2(\sigma^*)^2\Big\}=0.\end{aligned}\tag{3.3}$$

Let X^* *be the unique solution of the stochastic differential equation*

$$\begin{cases} dX_t^*=r_t^*X_t^*(1-\hat{\pi}(t,X_t^*))dt+X_t^*\hat{\pi}(t,X_t^*)\mu_t^*dt-\hat{c}(t,X_t^*)dt\\ \qquad\qquad +X_t^*\hat{\pi}(t,X_t^*)\sigma_t^*dW_t^{\mu^*,\sigma^*,r^*},\\ X_0^*=x_0.\end{cases}\tag{3.4}$$

Moreover, let c^* *and* π^* *be two processes defined by* $c_t^*=\hat{c}(t,X_t^*)$ *and* $\pi_t^*=\hat{\pi}(t,X_t^*),t\in[0,T]$*, respectively. If* $(\pi^*,c^*)\in\Pi$*, then* (π^*,c^*) *is an optimal portfolio–consumption policy.*

Proof: As usual, we will verify that for all admissible policies (π,c), the (minimal) expected utility under Knightian uncertainty is bounded by $\varphi(0,x_0)$ and that the candidate optimal policy attains the upper bound.

The idea is as follows: we show that for all admissible policies, the process

$$M_t^{\pi,c}=\int_0^t u(s,c_s)ds+\varphi(t,X_t)$$

is a multiple prior supermartingale in the sense of Riedel (2009), and the upper bound estimate follows. To this end, it is sufficient to find one prior $P\in\mathcal{P}$ such that $M^{\pi,c}$ is a P–supermartingale.

For the candidate optimal policy, we will show that the upper bound $\varphi(0,x_0)$ is attained. To this end, it is sufficient to show that M^{π^*,c^*} is a multiple prior martingale. This is equivalent to the fact that M^{π^*,c^*} is a submartingale for all priors $P\in\mathcal{P}$ and a martingale for the worst case prior P^*.

For any admissible policy (π, c), we let $\tilde{X}$ be the wealth process as follows:

$$\begin{cases} d\tilde{X}_t = \tilde{r}_t \tilde{X}_t (1-\pi_t)dt + \tilde{X}_t \pi_t \tilde{\mu}_t dt - c_t dt + \tilde{X}_t \pi_t \tilde{\sigma}_t dW_t^{\tilde{\mu},\tilde{\sigma},\tilde{r}}, \\ \tilde{X}_0 = x_0. \end{cases}$$

By Itô's lemma, we have

$$\begin{aligned} dM_t^{\pi,c} = & \Big(u(t,c_t) - c_t \varphi_x(t,\tilde{X}_t) + \varphi_t(t,\tilde{X}_t) + \varphi_x(t,\tilde{X}_t)(\tilde{r}\tilde{X}_t + \pi_t \tilde{X}_t(\tilde{\mu}_t - \tilde{r}_t)) \\ & + \frac{1}{2}\pi_t^2 \tilde{X}_t^2 \tilde{\sigma}_t^2 \varphi_{xx}(t,\tilde{X}_t) \Big) dt + \varphi_x(t,\tilde{X}_t)\pi_t \tilde{\sigma}\tilde{X}_t dW_t^{\tilde{\mu},\tilde{\sigma},\tilde{r}} \\ \le & \ \varphi_x(t,\tilde{X}_t)\pi_t \tilde{X}_t dW_t^{\tilde{\mu},\tilde{\sigma},\tilde{r}}. \end{aligned}$$

Therefore, from (3.1) and (3.2) we have $dM_t^{\pi,c} \le \varphi'(\bar{X}_t)\pi_t \bar{X}_t dB_t$, and we conclude that $M^{\pi,c}$ is a $P^{(\tilde{\mu},\tilde{\sigma},\tilde{r})}$–supermartingale, hence a multiple prior supermartingale. If we plug in the candidate optimal policies, the same argument shows that M^{π^*,c^*} is a $P^{(\mu^*,\sigma^*,r^*)}$–supermartingale.

We still need to show that M^{π^*,c^*} is a a martingale for the worst case prior $P^{(\mu^*,\sigma^*,r^*)}$. To this end, note that our value function also solves the HJBI equation (3.3). We thus get

$$\begin{aligned} dM_t^{\pi^*,c^*} = & \Big(u(c_t^*) - c_t^* \varphi_x(t,X_t^*) + \varphi_t(t,X_t^*) + \varphi'(X_t^*)(r^* X_t^* + \pi_t^* X_t^*(\mu^* - r^*)) \\ & + \frac{1}{2}{\pi^*}_t^2 (X_t^*)^2 (\sigma^*)^2 \varphi_{xx}(t,X_t^*) \Big) dt + \varphi_x(t,X_t^*)\pi_t^* X_t^* \sigma^* dW_t^{\mu^*,\sigma^*,r^*} \\ = & \ \varphi_x(t,X_t^*)\pi_t^* X_t^* \sigma^* dW_t^{\mu^*,\sigma^*,r^*}, \end{aligned}$$

and we conclude that M^{π^*,c^*} is a $P^{(\mu^*,\sigma^*,r^*)}$–martingale. The proof is complete. □

Before we give a complete solution to the utility maximization problem, we sketch here the main ideas. As we focus on interest rate uncertainty, let us fix the drift and volatility parameters μ and σ. Let us look at the HJBI equation (3.1). Let us assume (what we shall verify later on) that the value function ϕ is differentiably strictly concave, and let us write $\alpha = -\frac{\phi_{xx}x}{\phi_x}$ for the indirect relative risk aversion coefficient. For the optimal portfolio choice π, we need to study

$$\inf_{r \in [\underline{r}, \overline{r}]} r(1-\pi) + \pi\mu - \frac{1}{2}\alpha\pi^2\sigma^2.$$

If $\pi > 1$, i.e. if we borrow money, then the worst case interest rate is $\overline{r}$. For $\pi < 1$, when we save money, the worst case interest rate is $\underline{r}$. We are thus left with the minimum of two quadratic functions,

$$\min \left\{ \underline{r}(1-\pi) + \pi\mu - \frac{1}{2}\alpha\pi^2\sigma^2, \overline{r}(1-\pi) + \pi\mu - \frac{1}{2}\alpha\pi^2\sigma^2 \right\}.$$

The two quadratic functions intersect at $\pi = 1$ when the dependence on r disappears.

It is optimal to invest all wealth in the risky asset if and only if the left derivative at $\pi = 1$ is nonnegative and the right derivative at $\pi = 1$ is nonpositive. This gives us the two equations

$$\mu - \underline{r} - \alpha\sigma^2 \ge 0\,, \mu - \overline{r} - \alpha\sigma^2 \le 0,$$

or the desired condition

$$\underline{r} \le \mu - \alpha\sigma^2 \le \overline{r}.$$

Theorem 3.2 *1. If* $\underline{r} \leq \underline{\mu} - \alpha\overline{\sigma}^2 \leq \overline{r}$, *then the value function of the utility maximization problem has the form*

$$\varphi(t,x) = f(t)\frac{x^{1-\alpha}}{1-\alpha}$$

for

$$f(t) = \Big[K^{\alpha^{-1}} e^{\beta\alpha^{-1}(T-t)} + \alpha(\beta-\delta)^{-1} e^{-\delta\alpha^{-1}t}\big(e^{(\beta-\delta)\alpha^{-1}(T-t)} - 1\big)\Big]^{\alpha},$$

where $\beta = \big(\underline{\mu} - \frac{1}{2}\alpha\overline{\sigma}^2\big)(1-\alpha)$.

2. The optimal portfolio choice is $\hat{\pi} = 1$ *and the optimal consumption choice is*

$$\hat{c} = \Big[K^{\alpha^{-1}} e^{\beta\alpha^{-1}(T-t)} + \alpha(\beta-\delta)^{-1} e^{-\delta\alpha^{-1}t}\big(e^{(\beta-\delta)\alpha^{-1}(T-t)} - 1\big)\Big]^{-1} x\exp(-\delta\alpha^{-1}t)\,.$$

Proof: Let $\varphi \in C^{1,2}([0,T)) \times \mathbb{R}^+)$ with polynomial growth be a solution of (3.1) and $\varphi_{xx} < 0$. Then, from the first order condition it follows that

$$\hat{c} = v(\varphi_x(t,x)).$$

where v is the inverse of $u_c(t,c)$.

We denote by $a = \dfrac{1}{2}\overline{\sigma}^2 x^2 \varphi_{xx}(t,x) < 0$ and $b = \varphi_x(t,x)x > 0$. Let us consider the following function

$$f(x) = \begin{cases} a\pi^2 + b\pi(\underline{\mu} - \overline{r}) + b\overline{r}, \ \pi > 1, \\ a\pi^2 + b\pi(\underline{\mu} - \underline{r}) + b\underline{r}, \ 0 \leq \pi \leq 1, \\ a\pi^2 + b\pi(\overline{\mu} - \underline{r}) + b\underline{r}, \ \pi \leq 0. \end{cases}$$

If $\underline{\mu} \geq \overline{r}$ and $-\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)}\frac{\underline{\mu}-\overline{r}}{\overline{\sigma}^2} \leq x \leq -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)}\frac{\underline{\mu}-\underline{r}}{\overline{\sigma}^2}$, then

$$\sup_{\pi\geq 1} f(\pi) = f(1) = a + b\underline{\mu}, \qquad \sup_{\pi<0} f(\pi) = f(0) = b\underline{r},$$

and

$$\sup_{0\leq\pi\leq 1} f(\pi) = f(1) = a + b\underline{\mu} > f(0).$$

Therefore, the optimal portfolio choice is $\hat{\pi} = 1$.

We suppose that $\varphi(t,x)$ has the following form

$$\varphi(t,x) = f(t)\frac{x^{1-\alpha}}{1-\alpha},$$

where $f(t)$ is a function and given later. Therefore, substituting the above form of $\varphi(t,x)$ in to (3.1), we obtain the following equation

$$\begin{cases} \alpha\exp(-\delta\alpha^{-1}t)f(t)^{1-\alpha^{-1}} + \beta f(t) + f'(t) = 0, \\ f(T) = K, \end{cases}$$

where $\beta = \left(\underline{\mu} - \frac{1}{2}\alpha\overline{\sigma}^2\right)(1-\alpha)$. The solution of the above equation is given by

$$f(t) = \left[K^{\alpha^{-1}} e^{\beta\alpha^{-1}(T-t)} + \alpha(\beta-\delta)^{-1} e^{-\delta\alpha^{-1}t}\left(e^{(\beta-\delta)\alpha^{-1}(T-t)} - 1\right)\right]^{\alpha}.$$

Therefore, the optimal consumption is

$$\hat{c} = \left[K^{\alpha^{-1}} e^{\beta\alpha^{-1}(T-t)} + \alpha(\beta-\delta)^{-1} e^{-\delta\alpha^{-1}t}\left(e^{(\beta-\delta)\alpha^{-1}(T-t)} - 1\right)\right]^{-1} x \exp(-\delta\alpha^{-1}t).$$

□

In the remainder of this section, we provide the optimal portfolio for the parameter cases that we did not study above.

Theorem 3.3 *(i) For $\overline{\mu} \le \underline{r}$, the optimal portfolio choice is*

$$\hat{\pi} = \frac{\overline{\mu} - \underline{r}}{\alpha\overline{\sigma}^2}.$$

(ii) For $\underline{\mu} < \underline{r} < \overline{\mu}$, the optimal portfolio choice is $\hat{\pi} = 0$.

(iii) For $\underline{\mu} \ge \underline{r}$, the optimal portfolio choice is

$$\hat{\pi} = \begin{cases} \dfrac{\underline{\mu} - \underline{r}}{\alpha\overline{\sigma}^2}, & \text{if } \underline{r} - \alpha\overline{\sigma}^2 \le \underline{\mu} - \alpha\overline{\sigma}^2 < \underline{r}; \\ \dfrac{\underline{\mu} - \overline{r}}{\alpha\overline{\sigma}^2}, & \text{if } \underline{\mu} - \alpha\overline{\sigma}^2 \ge \overline{r}. \end{cases}$$

If the risky asset is known to be dominated by bonds in the sense that the highest expected return $\overline{\mu}$ is below the lowest possible interest rate, the investor short sells the asset and uses the adapted Merton formula for the portfolio with the worst case parameters highest expected return and lowest possible interest rate. We obtain a generalized version of the Dow-Werlang result if Knightian uncertainty about expected returns overweighs the Knightian uncertainty about interest rates (case (ii)). Last not least, if the investor knows that the asset's expected return dominate interest rates, he uses again an adapted Merton portfolio, the worst case interest rate being the lowest one if he saves ($\hat{\pi} \le 1$), and the highest one, if he borrows money.

The proof is provided in the appendix.

4 Conclusion

We study continuous–time consumption and portfolio choice in the presence of Knightian uncertainty about interest rates. For robust parameter sets, the investor puts all his wealth into the asset market when interest rate uncertainty is sufficiently high. Both saving and borrowing are considered to be too uncertain to be worthwhile activities. This insight might have important consequences for policy makers; while central bankers might prefer to remain vague about their future interest rate policies, they should bear in mind that this behavior can have substantial implications for the bond market.

A Appendix: Proof of Theorem 3.3

As a preparation for the proof, we need the following lemma.

Lemma A.1 *Let $\varphi \in C^{1,2}([0,T)) \times \mathbb{R}^+)$ with polynomial growth be a solution of (3.1) and $\varphi_{xx} < 0$, then the optimal consumption is*

$$\hat{c} = v(\varphi_x(t,x)),$$

where v is the inverse of u_c, and

(i) if $\overline{\mu} \le \underline{r}$, then the optimal portfolio choice is

$$\hat{\pi} = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x} \frac{\overline{\mu} - \underline{r}}{\overline{\sigma}^2};$$

(ii) if $\underline{\mu} < \underline{r} < \overline{\mu}$, then the optimal portfolio choice is $\hat{\pi} = 0$;

(iii) if $\underline{r} < \underline{\mu} < \overline{r}$ and $-\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)} \frac{\underline{\mu}-\underline{r}}{\overline{\sigma}^2} < x$, then the optimal portfolio choice is

$$\hat{\pi} = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x} \frac{\underline{\mu} - \underline{r}}{\overline{\sigma}^2}.$$

(iv) if $\underline{\mu} \ge \overline{r}$, and if $-\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)} \frac{\underline{\mu}-\underline{r}}{\overline{\sigma}^2} < x$, then the optimal portfolio choice is

$$\hat{\pi} = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x} \frac{\underline{\mu} - \underline{r}}{\overline{\sigma}^2};$$

and if $-\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)} \frac{\underline{\mu}-\overline{r}}{\overline{\sigma}^2} > x$, then the optimal portfolio choice is

$$\hat{\pi} = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x} \frac{\underline{\mu} - \overline{r}}{\overline{\sigma}^2}.$$

Proof: From the first order condition it follows that $\hat{c} = v(\varphi_x(t,x))$, where v is the inverse of $u_c(t,c)$.

We denote by $a = \frac{1}{2}\overline{\sigma}^2 x^2 \varphi_{xx}(t,x) < 0$ and $b = \varphi_x(t,x)x > 0$. Let us consider the following function

$$f(x) = \begin{cases} a\pi^2 + b\pi(\underline{\mu} - \overline{r}) + b\overline{r}, \ \pi > 1, \\ a\pi^2 + b\pi(\underline{\mu} - \underline{r}) + b\underline{r}, \ 0 \le \pi \le 1, \\ a\pi^2 + b\pi(\overline{\mu} - \underline{r}) + b\underline{r}, \ \pi \le 0. \end{cases}$$

Let us consider $\sup_\pi f(\pi)$ in the following cases.

Case I: If $\overline{\mu} \le \underline{r}$, then $\sup_{\pi > 1} f(\pi) = f(1) = a + b\underline{\mu}$, $\sup_{0 \le \pi \le 1} f(\pi) = f(0) = b\underline{r}$, $\sup_{\pi < 0} f(\pi) = f(\overline{\pi}) = b\underline{r} - \frac{b^2(\overline{\mu} - \underline{r})^2}{4a}$, where $\overline{\pi} = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x} \frac{\overline{\mu}-\underline{r}}{\overline{\sigma}^2}$. Since $a < 0$ and $b > 0$, we have $\sup_\pi f(\pi) = f(\hat{\pi})$, where $\hat{\pi} = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x} \frac{\overline{\mu}-\underline{r}}{\overline{\sigma}^2}$.

Case II: If $\underline{\mu} < \underline{r} < \overline{\mu}$, then it follows that $\sup_{\pi>1} f(\pi) = f(1) = a + b\underline{\mu}$, $\sup_{0\leq\pi\leq1} f(\pi) = f(0) = b\underline{r}$, $\sup_{\pi<0} f(\pi) = f(0) = b\underline{r}$. Since $a < 0$ and $b > 0$, we have $\sup_{\pi} f(\pi) = f(\hat{\pi})$, where $\hat{\pi} = 0$.

Case III: If $\underline{r} < \underline{\mu} < \overline{r}$ and $-\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x}\frac{\underline{\mu}-\underline{r}}{\overline{\sigma}^2} < 1$, then $\sup_{\pi>1} f(\pi) = f(1) = a + b\underline{\mu}$, $\sup_{0\leq\pi\leq1} f(\pi) = f(\bar{\pi}) > f(0), f(\bar{\pi}) > f(1)$, where $\bar{\pi} = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x}\frac{\underline{\mu}-\underline{r}}{\overline{\sigma}^2}$, and $\sup_{\pi<0} f(\pi) = f(0) = b\underline{r}$. Therefore, $\sup_{\pi} f(\pi) = f(\hat{\pi})$, where $\hat{\pi} = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x}\frac{\underline{\mu}-\underline{r}}{\overline{\sigma}^2}$.

Case IV: $\underline{\mu} \geq \overline{r}$. (a) If $-\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)}\frac{\underline{\mu}-\underline{r}}{\overline{\sigma}^2} < x$, then $\sup_{\pi>1} f(\pi) = f(1) = a + b\underline{\mu}$, $\sup_{\pi<0} f(\pi) = f(0) = b\underline{r}$, $\sup_{0\leq\pi\leq1} f(\pi) = f(\bar{\pi}) > f(0), f(\bar{\pi}) > f(1)$, where $\bar{\pi} = \frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x}\frac{\underline{\mu}-\underline{r}}{\overline{\sigma}^2}$, Consequently, $\sup_{\pi} f(\pi) = f(\hat{\pi})$, and the optimal portfolio choice is $\hat{\pi} = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x}\frac{\underline{\mu}-\underline{r}}{\overline{\sigma}^2}$.

(b) If $-\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)}\frac{\underline{\mu}-\overline{r}}{\overline{\sigma}^2} > x$, then $\sup_{\pi\geq1} f(\pi) = f(\bar{\pi}) > f(1)$, where $\bar{\pi} = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x}\frac{\underline{\mu}-\overline{r}}{\overline{\sigma}^2}$, $\sup_{\pi<0} f(\pi) = f(0) = b\underline{r}$, $\sup_{0\leq\pi\leq1} f(\pi) = f(1) = a + b\underline{\mu} > f(0)$. From the above it follows that the optimal portfolio choice is $\hat{\pi} = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x}\frac{\underline{\mu}-\overline{r}}{\overline{\sigma}^2}$. The proof is complete. $\square$

We can now provide the proof of Theorem 3.3. From Lemma A.1 we obtain

(i) if $\overline{\mu} \leq \underline{r}$, then the optimal portfolio choice is $\hat{\pi} = \frac{\overline{\mu}-\underline{r}}{\alpha\overline{\sigma}^2}$;

(ii) if $\underline{\mu} < \underline{r} < \overline{\mu}$, then the optimal portfolio choice is $\hat{\pi} = 0$;

(iii) if $\underline{r} < \underline{\mu} < \overline{r}$ and $\underline{\mu} - \alpha\overline{\sigma}^2 < \underline{r}$, then the optimal portfolio choice is $\hat{\pi} = \frac{\underline{\mu}-\underline{r}}{\alpha\overline{\sigma}^2}$.

We now consider $\underline{\mu} \geq \overline{r}$ in the following cases.

Case I. Suppose $\dfrac{\underline{\mu}-\underline{r}}{\alpha\overline{\sigma}^2} < 1$, from Lemma A.1, then the equation (3.1) has the following form

$$\begin{cases} \alpha e^{-\delta\alpha^{-1}t}\dfrac{\varphi_x^{1-\alpha^{-1}}}{1-\alpha} + \varphi_t + \varphi_x xr - \dfrac{\varphi_x^2(\underline{\mu}-\underline{r})^2}{2\overline{\sigma}^2\varphi_{xx}} = 0, \\ \varphi(T,x) = \dfrac{Kx^{1-\alpha}}{1-\alpha}. \end{cases} \tag{A.1}$$

We suppose that $\varphi(t,x)$ has the following form $\varphi(t,x) = f(t)\dfrac{x^{1-\alpha}}{1-\alpha}$, where $f(t)$ is a function and given later. Therefore, substituting the above form of $\varphi(t,x)$ in to (A.1), we obtain the following equation

$$\begin{cases} \alpha\exp(-\delta\alpha^{-1}t)f(t)^{1-\alpha^{-1}} + \beta f(t) + f'(t) = 0, \\ f(T) = K, \end{cases}$$

where $\beta = \left[r + \frac{(\underline{\mu}-\underline{r})^2}{2\overline{\sigma}^2\alpha}\right](1-\alpha)$. The solution of the above equation is given by

$$f(t) = \left[K^{\alpha^{-1}}e^{\beta\alpha^{-1}(T-t)} + \alpha(\beta-\delta)^{-1}e^{-\delta\alpha^{-1}t}\left(e^{(\beta-\delta)\alpha^{-1}(T-t)} - 1\right)\right]^{\alpha}.$$

Therefore, the optimal portfolio choice is

$$\hat{\pi} = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x}\frac{\underline{\mu} - \underline{r}}{\overline{\sigma}^2} = \frac{\underline{\mu} - \underline{r}}{\alpha\overline{\sigma}^2}.$$

Case II. Suppose $\frac{\underline{\mu} - \overline{r}}{\alpha\overline{\sigma}^2} > 1$, from Lemma A.1, then the equation (3.1) has the following form

$$\begin{cases} \alpha e^{-\delta\alpha^{-1}t}\frac{\varphi_x^{1-\alpha^{-1}}}{1-\alpha} + \varphi_t + \varphi_x x r - \frac{\varphi_x^2(\underline{\mu} - \overline{r})^2}{2\overline{\sigma}^2\varphi_{xx}} = 0, \\ \varphi(T,x) = \frac{Kx^{1-\alpha}}{1-\alpha}. \end{cases} \tag{A.2}$$

Using a similar argument of solving equation (A.1), we obtain that the solution of the above equation is

$$\varphi(t,x) = f(t)\frac{x^{1-\alpha}}{1-\alpha},$$

where $f(t) = \Big[K^{\alpha^{-1}}e^{\beta\alpha^{-1}(T-t)} + \alpha(\beta-\delta)^{-1}e^{-\delta\alpha^{-1}t}(e^{(\beta-\delta)\alpha^{-1}(T-t)} - 1)\Big]^{\alpha}$, and $\beta = \big[r + \frac{(\underline{\mu}-\overline{r})^2}{2\overline{\sigma}^2\alpha}\big](1-\alpha)$. Therefore, the optimal portfolio choice is

$$\hat{\pi} = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x}\frac{\underline{\mu} - \overline{r}}{\overline{\sigma}^2} = \frac{\underline{\mu} - \overline{r}}{\alpha\overline{\sigma}^2}.$$

$\square$

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