**Abstract.** In this paper, we study the regularity of weak solutions to a class of nonlocal elliptic equations in Bessel potential spaces $H^{s,p}$ under very mild assumptions on the data. Our main results can be seen as an extension of the well-known $W^{1,p}$ regularity theory for local second-order elliptic equations in divergence form with BMO coefficients to the nonlocal setting.

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1. **Introduction**

1.1. **Basic setting.** In this work, we study the regularity of weak solutions to nonlocal elliptic equations of the form

\[ L_A u + bu = \sum_{i=1}^{m} L_{D_i} g_i + f \text{ in } \Omega \subset \mathbb{R}^n \]

in Bessel potential spaces $H^{s,p}$. Roughly speaking, the purpose of this paper is to prove the implication $u \in H^{s,2} \implies u \in H^{s,p}$ for the whole range of exponents $p \in (2, \infty)$ under very
general assumptions on the data. Here \( s \in (0, 1) \), \( \Omega \subset \mathbb{R}^n \) \((n > 2s)\) is a domain (= open set), \( b, f, g_i : \mathbb{R}^n \to \mathbb{R} \) are given functions and

\[
L_A u(x) = 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_{\varepsilon}(x)} \frac{A(x, y)}{|x-y|^{n+2s}} (u(x) - u(y)) dy, \quad x \in \Omega,
\]

is a nonlocal operator. Furthermore, the function \( A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) is jointly measurable and we assume that there exist constants \( 0 < \nu \leq \lambda \) such that

\[
\nu \leq A(x, y) \leq \lambda \quad \text{for almost all} \quad x, y \in \mathbb{R}^n.
\]

Moreover, we require \( A \) to be symmetric, i.e.

\[
A(x, y) = A(y, x) \quad \text{for almost all} \quad x, y \in \mathbb{R}^n.
\]

We call such a function \( A \) a kernel coefficient. We define \( \mathcal{L}_0(\nu, \lambda) \) as the class of all such measurable kernel coefficients \( A \) that satisfy the conditions (2) and (3). Moreover, throughout this work \( D_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) \((i = 1, \ldots, m, \, m \in \mathbb{N})\) are jointly measurable functions that satisfy (3) and are bounded by some \( \Lambda > 0 \), i.e.

\[
\sum_{i=1}^m |D_i(x, y)| \leq \Lambda \quad \text{for almost all} \quad x, y \in \mathbb{R}^n.
\]

Define the spaces

\[
H^s(\Omega|\mathbb{R}^n) = \left\{ u : \mathbb{R}^n \to \mathbb{R} \text{ measurable} \mid \int_{\Omega} u(x)^2 dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x-y|^{n+2s}} dy dx < \infty \right\}
\]

and

\[
H^s_0(\Omega|\mathbb{R}^n) = \{ u \in H^s(\Omega|\mathbb{R}^n) \mid u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \}.
\]

For all measurable functions \( u, \varphi : \mathbb{R}^n \to \mathbb{R} \) we define the bilinear form associated to the operator \( L_A \) by

\[
\mathcal{E}_A(u, \varphi) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{A(x, y)}{|x-y|^{n+2s}} (u(x) - u(y)) (\varphi(x) - \varphi(y)) dy dx,
\]

provided that the above expression is well-defined and finite, this is e.g. the case if \( u \in H^s(\Omega|\mathbb{R}^n) \) and \( \varphi \in H^s_0(\Omega|\mathbb{R}^n) \). Analogously we consider the bilinear forms \( \mathcal{E}_{D_i}(u, \varphi) \) associated to the operators \( L_{D_i} \).

**Definition.** Given \( b \in L^\infty(\Omega) \), \( f \in L^2(\Omega) \) and \( g_i \in H^s(\Omega|\mathbb{R}^n) \), \( i = 1, \ldots, m \), we say that \( u \in H^s(\Omega|\mathbb{R}^n) \) is a weak solution to the equation \( L_A u + bu = \sum_{i=1}^m L_{D_i} g_i + f \) in \( \Omega \), if

\[
\mathcal{E}_A(u, \varphi) + (bu, \varphi)_{L^2(\Omega)} = \sum_{i=1}^m \mathcal{E}_{D_i}(g_i, \varphi) + (f, \varphi)_{L^2(\Omega)} \quad \forall \varphi \in H^s_0(\Omega|\mathbb{R}^n).
\]

1.2. **Some previous results.** Studying the regularity of weak solutions to equations of the form (1) has been a very active area of research in recent years. Results concerning Hölder regularity were e.g. obtained in [15], [14], [27], [22], [26] and [25], while results concerning higher differentiability in Sobolev spaces were e.g. obtained in [11]. Regarding higher integrability, Bass and Ren in [8] showed that under the assumptions from section 1.1 there exists some small \( \sigma > 0 \), such that for any weak solution \( u \) of \( L_A u = f \) in \( \mathbb{R}^n \) the function

\[
\nabla^s u(x) = \left( \int_{\mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x-y|^{n+2s}} dy \right)^{1/2}
\]

belongs to \( L^{2+\sigma}(\mathbb{R}^n) \) whenever \( f \in L^2(\mathbb{R}^n) \). In view of a classical characterization of Bessel potential spaces due to Stein (cf. Theorem 3.3), this actually implies that \( u \) belongs to the Bessel
potential space $H^{s,2+\alpha}(\mathbb{R}^n)$. Similar results were proved in [21] and [1], where it was shown that under the assumptions from section 1.1 $u$ actually not only possesses a higher integrability but also a slightly higher differentiability.

1.3. The small BMO assumption and the main results. The aim of this work is to prove the $H^{s,p}$ regularity for solutions $u$ to equations of the form (1) not only for some $p > 2$ close enough to 2, but for the full range $p \in (2, \infty)$.

The following class of kernel coefficients plays a key role in our theory.

**Definition.** Let $0 < \nu \leq \lambda$. We say that a kernel coefficient $A_0 \in L_0(\nu, \lambda)$ belongs to the class $L_1(\nu, \lambda)$, if there exists a measurable function $a : \mathbb{R}^n \to \mathbb{R}$ such that $A_0(x,y) = a(x-y)$ for all $x,y \in \mathbb{R}^n$, that is, if $A_0$ is translation invariant.

As it is well-known and as we will briefly discuss below, the $W^{1,p}$ regularity for local second-order elliptic equations fails without any additional assumption on the matrix of coefficients. Therefore, also in the nonlocal setting corresponding results cannot be expected to hold without imposing any additional assumption on the kernel coefficient $A \in L_0(\nu, \lambda)$. The following definition contains our additional assumption on $A$. It can be seen as an analogue to the small BMO assumption commonly used in the regularity theory of local second-order elliptic equations.

**Definition.** Let $\delta, R > 0$. We say that $A \in L_0(\nu, \lambda)$ is $(\delta, R)$-BMO, if

$$\sup_{x_0 \in \mathbb{R}^n} \sup_{r \in (0, R]} \int_{B_r(x_0)} \text{ess sup}_{x \in \mathbb{R}^n} |A(x,y) - \int_{B_r(x_0)} A(x-y+z,z)dz| dy \leq \delta.$$  

It is easy to see that if $A(x,y) = a(x-y)$ for some measurable $a : \mathbb{R}^n \to \mathbb{R}$, then the left-hand side of (6) identically vanishes, so that any kernel coefficient of class $L_1(\nu, \lambda)$ is $(\delta, R)$-BMO for any $\delta > 0$ and any $R > 0$. Moreover, if $A \in L_0(\nu, \lambda)$ satisfies the uniform continuity condition

$$\lim_{a \to 0} \sup_{x,y \in \mathbb{R}^n} |A(x+a,y+a) - A(x,y)| = 0,$$

then for any $\delta > 0$ there exists some $R > 0$ such that $A$ is $(\delta, R)$-BMO, as can be seen by a straightforward computation.

Our main result concerning local regularity in Bessel potentials spaces $H^{s,p}$ is the following.

**Theorem 1.1.** (Local $H^{s,p}$ regularity in domains) 

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $p \in (2, \infty)$, $s \in (0,1)$, $p_s = \max \left\{ \frac{pm}{n+ps}, 2 \right\}$, $g_i \in H^{s}(\Omega|\mathbb{R}^n) \cap H^{s,p}(\mathbb{R}^n)$, $f \in L^{p_s}(\Omega)$, $b \in L^{\infty}(\Omega)$ and $R, \nu, \lambda, \Lambda > 0$. Then there exists some small enough $\delta = \delta(p, n, s, \nu, \lambda, \Lambda) > 0$, such that if a kernel coefficient $A \in L_0(\nu, \lambda)$ is $(\delta, R)$-BMO, and if the $D_i$ satisfy (3) and (4), then for any weak solution $u \in H^{s}(\Omega|\mathbb{R}^n)$ of the equation

$$L_{Au} + bu = \sum_{i=1}^{m} L_{D_i}g_i + f \quad \text{in } \Omega,$$

we have $u \in H^{s,p}_{loc}(\Omega)$ and $u \in W^{s,p}_{loc}(\Omega)$.

**Remark.** As we have mentioned above, the small BMO assumption (6) is satisfied for any arbitrarily small $\delta > 0$ whenever $A$ satisfies the uniform continuity condition (7) and in particular whenever $A \in L_1(\nu, \lambda)$. Therefore, for such kernel coefficients $A$ the claim of Theorem 1.1 holds for the full range of exponents $p \in (2, \infty)$, so that we have $u \in H^{s,p}_{loc}(\Omega)$ for any $p \in (2, \infty)$.

**Remark.** We actually obtain a slightly stronger result (cf. Theorem 7.1) than the one given by Theorem 1.1 in terms of certain function spaces $H^{s,p}(\Omega|\mathbb{R}^n)$ that generalize the space $H^{s}(\Omega|\mathbb{R}^n)$.
to the case when \( p \neq 2 \), cf. section 3. By a useful alternative characterization of Bessel potential spaces (cf. Theorem 3.3), this space \( H^{s,p}(\mathbb{R}^n) \) is actually contained in \( H^{s,p}(\Omega) \) whenever \( \Omega \) is regular enough, so that this result then implies Theorem 1.1. Moreover, since Theorem 1.1 is concerned with local regularity, the above result remains true if we generalize the notion of weak solutions to an appropriate notion of local weak solutions, cf. section 7.

**Remark.** Although in Theorem 1.1 and in our other main results we are primarily concerned with regularity in Bessel potential spaces \( H^{s,p} \), due to the classical embedding \( H^{s,p} \hookrightarrow W^{s,p} \) for \( p \in [2, \infty) \) (cf. Theorem 3.2), we also obtain regularity in Sobolev-Slobodeckij spaces \( W^{s,p} \).

If the equation is posed on the whole space \( \mathbb{R}^n \), we are actually able to establish the following global regularity result.

**Theorem 1.2.** (\( H^{s,p} \) regularity on the whole space \( \mathbb{R}^n \))

Let \( p \in (2, \infty) \), \( s \in (0, 1) \), \( \kappa = \max \left\{ \frac{p}{n+sp}, \frac{1}{2} \right\} \), \( g_i \in H^{s}(\mathbb{R}^n) \cap H^{s,p}(\mathbb{R}^n) \), \( f \in L^2(\mathbb{R}^n) \cap L^{p_s}(\mathbb{R}^n) \), \( b \in L^\infty(\mathbb{R}^n) \) and \( R, \nu, \lambda, \Lambda > 0 \). Then there exists some small enough \( \delta = \delta(p, n, s, \nu, \lambda, \Lambda) > 0 \), such that if a kernel coefficient \( A \in \mathcal{L}_0(\nu, \lambda) \) is \((\delta, R)\)-BMO, and if the \( D_i \) satisfy (3) and (4), then for any weak solution \( u \in H^{s}(\mathbb{R}^n) \) of the equation

\[
L_A u + bu = \sum_{i=1}^m L_{D_i} g_i + f \text{ in } \mathbb{R}^n
\]

we have \( u \in H^{s,p}(\mathbb{R}^n) \) and \( u \in W^{s,p}(\mathbb{R}^n) \).

In the case when \( b = g_i = 0 \) \((i = 1, \ldots, m)\) and \( A(x, y) = a(x - y) \) is an arbitrary translation invariant kernel coefficient, Theorem 1.2 can be deduced by applying Theorem 1 in [2] to the symbol

\[
M(\xi) = \frac{\int_{\mathbb{R}^n} (\cos(\xi \cdot y) - 1)a^{-1}(y)V(dy)}{\int_{\mathbb{R}^n} (\cos(\xi \cdot y) - 1)V(dy)}, \quad V(dy) = \frac{a(y)}{|y|^{n+2s}}dy.
\]

We also want to mention that in [13] a related result was proved. Using an approach introduced by Krylov in [20], the authors obtained \( H^{2s,p} \) a priori estimates for strong solutions of the equation (1) in \( \mathbb{R}^n \) in the case of translation invariant but possibly nonsymmetric kernels.

The main achievements of this paper are that we extend the \( H^{s,p} \) regularity theory for weak solutions of the equation (1) to a large class of possibly non-translation invariant kernels and to the setting of arbitrary domains \( \Omega \subset \mathbb{R}^n \). In particular, to the best of our knowledge Theorem 1.1 is already new in the case when \( A \) is translation invariant. This is because although in the case of local elliptic equations local regularity in domains \( \Omega \) can be deduced from the corresponding result in \( \mathbb{R}^n \) by using a cutoff argument, in the nonlocal setting such a cutoff argument requires an additional assumption on the solution in the complement of \( \Omega \) (cf. [4] or [19]), which is not required in Theorem 1.1.

In order to prove our main results, we apply a variation of an approach commonly used in order to obtain \( W^{1,p} \) regularity results for local elliptic equations in divergence form. This enables us to simultaneously treat the problems of local \( H^{s,p} \) regularity in domains \( \Omega \) and the problem of global \( H^{s,p} \) regularity on the whole space \( \mathbb{R}^n \).

### 1.4. \( W^{1,p} \) regularity theory for second-order elliptic equations in divergence form.

Let us briefly review the well-known \( W^{1,p} \) regularity theory for local second-order elliptic equations in divergence form with BMO coefficients treated for example in [5], [6] or [7], where the authors build on an approach first introduced by Caffarelli and Peral in [8]. Consider the equation

\[
\text{div}(Bu) = \text{div} g + f \quad \text{in } \Omega,
\]
where $\Omega \subset \mathbb{R}^n$ is a domain, the matrix of coefficients $B : \mathbb{R}^n \to \mathbb{R}^{n \times n}$, $B(x) = (B_{ij}(x))_{i,j=1,...,n}$ has measurable coefficients, is uniformly elliptic and uniformly bounded, while $g : \Omega \to \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ are given functions. Furthermore, solutions are understood in an appropriate weak sense, cf. [17] or [9]. Let $2 < p < \infty$. A natural question corresponding to Theorem 1.1 to ask in this context is the following: Under which assumptions on $B, f$ and $g$ does any weak solution $u \in H_{loc}^1(\Omega)$ of (10) in fact belong to the space $u \in W_{loc}^{1,p}(\Omega)$? The minimal assumptions on $g$ and $f$ for this property to hold are $g \in L^p_{loc}(\Omega, \mathbb{R}^n)$ and $f \in L_{loc}^\infty(\Omega)$, while $B$ has to satisfy a small BMO assumption which is similar to our assumption (6), cf. [5]. The strategy to obtain such local $W^{1,p}$ estimates used e.g. in [5], [6] or [7] is as follows. One uses the small BMO assumption imposed on $B$ in order to approximate the gradient of the weak solution $u$ of (10) in $L^2$ by the gradient of a weak solution $v$ of a suitable equation $\nabla(vB_0u) = 0$, where $B_0$ has constant coefficients. One then uses the fact that $v$ satisfies a local $C^{0,1}$ estimate along with a real-variable argument based on the Vitali covering lemma, the Hardy-Littlewood maximal function and an alternative characterization of $L^p$ spaces in order to obtain an $L^p$ estimate for $\nabla u$, which in view of interpolation then implies the desired local $W^{1,p}$ estimate.

The main idea of our approach in the nonlocal setting is to apply similar arguments with the gradient $\nabla u$ replaced by the nonlocal $s$-gradient $\nabla^s u$ defined in [5] and with the constant coefficient matrix $B_0$ replaced by some appropriate translation invariant kernel coefficient $A_0$. However, due to the nonlocal nature of the operator $\nabla^s$ and the equations we consider, in our setting we have to overcome a number of difficulties that are not present in the local case.

1.5. Some notation. For convenience, let us fix some notation which we use throughout the paper. By $C$ and $C_i$, $i \in \mathbb{N}$, we always denote positive constants, while dependences on parameters of the constants will be shown in parentheses. As usual, by

$$B_r(x_0) := \{ x \in \mathbb{R}^n \mid |x - x_0| < r \}$$

we denote the open ball with center $x_0 \in \mathbb{R}^n$ and radius $r > 0$. Moreover, if $E \subset \mathbb{R}^n$ is measurable, then by $|E|$ we denote the $n$-dimensional Lebesgue-measure of $E$. If $0 < |E| < \infty$, then for any $u \in L^1(E)$ we define

$$\overline{u}_E := \int_E u(x)dx := \frac{1}{|E|} \int_E u(x)dx.$$
Another tool we use is the Hardy-Littlewood maximal function.

**Definition.** Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then the Hardy-Littlewood maximal function $Mf : \mathbb{R}^n \to [0, \infty]$ of $f$ is defined by

$$Mf(x) := \mathcal{M}(f)(x) := \sup_{\rho > 0} \int_{B_\rho(x)} |f(y)| dy.$$ 

Moreover, for any domain $\Omega \subset \mathbb{R}^n$ and any function $f \in L^1(\Omega)$, consider the zero extension of $f$ to $\mathbb{R}^n$

$$f_\Omega(x) := \begin{cases} f(x), & \text{if } x \in \Omega \\ 0, & \text{if } x \notin \Omega. \end{cases}$$

We then define $M_\Omega f := Mf_\Omega$.

Rather straightforward but important features of the Hardy-Littlewood maximal function are its scaling and translation invariance, given by the following Lemma which can be proved by using a change of variables.

**Lemma 2.2.** Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, $r > 0$ and $y \in \mathbb{R}^n$. Then for the function $f_{r,y}(x) := f(rx + y)$ and any $x \in \mathbb{R}^n$ we have

$$Mf_{r,y}(x) = Mf(rx + y).$$

Similarly, for any domain $\Omega \subset \mathbb{R}^n$, any function $f \in L^1(\Omega)$ and any $x \in \Omega$ we have

$$M_\Omega f_{r,y}(x) = M_\Omega f(rx + y),$$

where $\Omega' := \{ \frac{x-y}{r} \mid x \in \Omega \}$.

We remark that for any $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, $Mf$ is Lebesgue-measurable. Intuitively, the Hardy-Littlewood maximal function of a function $f$ in general seems to be much larger than the function $f$ itself. However, the following results show that when measured appropriately, the size of $Mf$ can actually be controlled by the size of $f$, cf. [30].

**Theorem 2.3.** Let $\Omega \subset \mathbb{R}^n$ be a domain.

(i) (weak 1-1 estimate) If $f \in L^1(\Omega)$ and $t > 0$, then

$$|\{x \in \Omega \mid M_\Omega(f)(x) > t\}| \leq C t \int_\Omega |f| dx,$$

where $C = C(n) > 0$.

(ii) (strong $p$-$p$ estimates) If $f \in L^p(\Omega)$ for some $p \in (1, \infty]$, then

$$\|f\|_{L^p(\Omega)} \leq \|M_\Omega f\|_{L^p(\Omega)} \leq C \|f\|_{L^p(\Omega)},$$

where $C = C(n, s) > 0$.

(iii) If $f \in L^p(\Omega)$ for some $p \in [1, \infty]$, then the function $M_\Omega f$ is finite almost everywhere.

We conclude this section by giving an alternative characterization of $L^p$ spaces, cf. Lemma 7.3 in [9]. It can be proved by using the well-known formula

$$\|f\|^p_{L^p(\Omega)} = p \int_0^\infty t^{p-1} |\{x \in \Omega \mid f(x) > t\}| dt.$$
Lemma 2.4. Let $0 < p < \infty$. Furthermore, suppose that $f$ is a nonnegative and measurable function in a bounded domain $\Omega \subset \mathbb{R}^n$ and let $\tau > 0$, $\beta > 1$ be constants. Then for
\[ S := \sum_{k=1}^{\infty} \beta^k \| \{ x \in \Omega \mid f(x) > \tau \beta^k \} \|, \]
we have
\[ C^{-1} S \leq \| f \|_{L_p(\Omega)}^p \leq C(\| \Omega \| + S) \]
for some constant $C = C(\tau, \beta, p) > 0$. In particular, we have $f \in L^p(\Omega)$ if and only if $S < \infty$.

3. Fractional Sobolev spaces and the $s$-gradient

We start this section by defining a first type of fractional Sobolev spaces which is probably the most widely used type of such spaces in the literature concerned with elliptic equations.

Definition. Let $\Omega \subset \mathbb{R}^n$ be a domain. For $p \in [1, \infty)$ and $s \in (0, 1)$, we define the Sobolev-Slobodeckij space
\[ W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) \mid \int_{\Omega} \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, dy \, dx < \infty \right\} \]
with norm
\[ \|u\|_{W^{s,p}(\Omega)} := \left( \int_{\Omega} u(x)^p \, dx + \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, dy \, dx \right)^{1/p}. \]

Moreover, we define the corresponding local Sobolev-Slobodeckij spaces by
\[ W^{s,p}_{loc}(\Omega) := \{ u \in L^p_{loc}(\Omega) \mid u \in W^{s,p}(\Omega') \text{ for any domain } \Omega' \subset \subset \Omega \}. \]

Finally, set
\[ H^s(\Omega) := W^{s,2}(\Omega). \]

Remark. The space $H^s(\Omega)$ is a Hilbert space with respect to the inner product
\[ (u,v)_{H^s(\Omega)} := (u,v)_{L^2(\Omega)} + \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+sp}} \, dy \, dx. \]

We use the following fractional Poincaré inequality, cf. Lemma 3.10 in [12].

Lemma 3.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $s \in (0, 1)$. For any $u \in H^s(\Omega)$ we have
\[ \int_{\Omega} |u(x) - u_\Omega|^2 \, dx \leq C \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dy \, dx, \]
where $C = C(s, \Omega) > 0$.

We also use the following type of fractional Sobolev spaces.

Definition. For $p \in [1, \infty)$ and $s \in \mathbb{R}$, consider the Bessel potential space
\[ H^{s,p}(\mathbb{R}^n) := \left\{ u \in L^p(\mathbb{R}^n) \mid \mathcal{F}^{-1} \left[ (1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F} u \right] \in L^p(\mathbb{R}^n) \right\}, \]
where $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. We equip $H^{s,p}(\mathbb{R}^n)$ with the norm
\[ \|u\|_{H^{s,p}(\mathbb{R}^n)} := \left\| \mathcal{F}^{-1} \left[ (1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F} u \right] \right\|_{L^p(\mathbb{R}^n)}. \]

Moreover, for any domain $\Omega \subset \mathbb{R}^n$ we define
\[ H^{s,p}(\Omega) := \left\{ u|_\Omega \mid u \in H^{s,p}(\mathbb{R}^n) \right\}. \]
with norm
\[ \|u\|_{H^{s,p}(\Omega)} := \inf \left\{ \|v\|_{H^{s,p}(\mathbb{R}^n)} \mid v|_{\Omega} = u \right\}. \]

Furthermore, we define the corresponding local Bessel potential spaces by
\[ H^{s,p}_{loc}(\Omega) := \{ u \in L^p_{loc}(\Omega) \mid u \in H^{s,p}(\Omega') \text{ for any domain } \Omega' \subset \subset \Omega \}. \]

The following result gives some relations between Bessel potential spaces and Sobolev-Slobodeckij spaces.

**Proposition 3.2.** Let \( \Omega \subset \mathbb{R}^n \) be a domain.

(i) If \( \Omega \) is a bounded Lipschitz domain or \( \Omega = \mathbb{R}^n \), then for all \( s \in (0, 1) \), \( p \in (1, 2] \) we have
\[ W^{s,p}(\Omega) \hookrightarrow H^{s,p}(\Omega). \]

(ii) For any \( s \in (0, 1) \) and any \( p \in [2, \infty) \) we have
\[ H^{s,p}(\Omega) \hookrightarrow W^{s,p}(\Omega). \]

For a proof of Proposition 3.2 we refer to Theorem 5 in chapter V of [30] for the case when \( \Omega = \mathbb{R}^n \). For general domains \( \Omega \), part (i) then follows by extending an arbitrary function \( u \in W^{s,p}(\Omega) \) to a function that belongs to \( W^{s,p}(\mathbb{R}^n) \), for which an additional assumption on \( \Omega \) is required, cf. Theorem 5.4 in [15]. Part (ii) follows similarly by extending an arbitrary function \( u \in H^{s,p}(\Omega) \) to a function that belongs to \( H^{s,p}(\mathbb{R}^n) \), which by definition of \( H^{s,p}(\Omega) \) is possible for arbitrary domains.

We now define a function that can be viewed as a nonlocal analogue to the euclidean norm of the gradient of a function in the local context.

**Definition.** For any domain \( \Omega \subset \mathbb{R}^n \) and any measurable function \( u : \Omega \to \mathbb{R} \), we define the s-gradient \( \nabla^s_{\Omega} u : \Omega \to [0, \infty] \) by
\[ \nabla^s_{\Omega} u(x) := \left( \int_{\Omega} \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} \, dy \right)^{\frac{1}{2}}. \]

Moreover, for any measurable function \( u : \mathbb{R}^n \to \mathbb{R} \) we define \( \nabla^s u := \nabla^s_{\mathbb{R}^n} u \).

In view of Proposition 3.2 for any bounded Lipschitz domain \( \Omega \) we have \( u \in H^{s,2}(\Omega) \) if and only if \( u \in L^2(\Omega) \) and \( \nabla^s_{\Omega} u \in L^2(\Omega) \). The following result shows that a similar alternative characterization of Bessel potential spaces in terms of the s-gradient is also true for a much wider range of exponents \( p \). This characterization was first given by Stein in [29] in the case when \( \Omega = \mathbb{R}^n \). For the case when \( \Omega \) is an arbitrary Lipschitz domain we refer to Theorem 2.10 in [28], where this characterization is proved in the more general context of Triebel-Lizorkin spaces and so-called uniform domains.

**Theorem 3.3.** Let \( s \in (0, 1) \) and \( p \in \left( \frac{2n}{n+2s}, \infty \right) \) and assume that \( \Omega \subset \mathbb{R}^n \) is a bounded Lipschitz domain or that \( \Omega = \mathbb{R}^n \). Then we have \( u \in H^{s,p}(\Omega) \) if and only if \( u \in L^p(\Omega) \) and \( \nabla^s_{\Omega} u \in L^p(\Omega) \). Moreover, we have
\[ \|u\|_{H^{s,p}(\Omega)} \simeq \|u\|_{L^p(\Omega)} + \|\nabla^s_{\Omega} u\|_{L^p(\Omega)} \]
in the sense of equivalent norms.

We remark that the above result holds in particular for any \( p \geq 2 \).

Even though we primarily work in some domain \( \Omega \subset \mathbb{R}^n \), we obtain most results in this work in terms of the global s-gradient \( \nabla^s u \) instead of the localized s-gradient \( \nabla^s_{\Omega} u \), which is mostly due to the nonlocal character of the equations we consider. In order to state our main result in domains in an optimal way (cf. Theorem 7.1), we therefore also define the following natural nonstandard function spaces.
Definition. Let $\Omega \subset \mathbb{R}^n$ be a domain. For $p \in [1, \infty)$ and $s \in (0, 1)$, we define the linear space

$$H^{s,p}(\Omega; \mathbb{R}^n) := \{ u : \mathbb{R}^n \to \mathbb{R} \text{ measurable} \mid u \in L^p(\Omega) \text{ and } \nabla^s u \in L^p(\Omega) \}.$$ 

Moreover, we define the corresponding local spaces by

$$H^{s,p}_{\text{loc}}(\Omega; \mathbb{R}^n) = \{ u : \mathbb{R}^n \to \mathbb{R} \text{ measurable} \mid u \in H^{s,p}(\Omega; \mathbb{R}^n) \text{ for any domain } \Omega' \subset \subset \Omega \}.$$ 

Also, we use the spaces

$$H^0_{s,p}(\Omega; \mathbb{R}^n) := \{ u \in H^{s,p}(\Omega; \mathbb{R}^n) \mid u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \}.$$ 

Furthermore, set

$$H^s(\Omega; \mathbb{R}^n) := H^{s,2}(\Omega; \mathbb{R}^n), \quad H^s_{\text{loc}}(\Omega; \mathbb{R}^n) := H^{s,2}_{\text{loc}}(\Omega; \mathbb{R}^n) \text{ and } H^s_0(\Omega; \mathbb{R}^n) := H^{0,2}_0(\Omega; \mathbb{R}^n).$$

Remark. Since for any $u \in H^0_{s,p}(\Omega; \mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} u(x)^2 \, dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x-y|^{n+2s}} \, dy \, dx \leq \int_{\Omega} u(x)^2 \, dx + 2 \int_{\Omega} (\nabla^s u(x))^2 \, dx < \infty,$$

$H^s_0(\Omega; \mathbb{R}^n)$ clearly is a closed subspace of $H^s(\mathbb{R}^n)$ and thus also a Hilbert space with respect to the inner product $(u, v)_{H^s(\mathbb{R}^n)}$ defined in (11).

Remark. In view of Theorem 3.2, for any bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ and all $s \in (0, 1)$, $p \in \left(\frac{2n}{n+2s}, \infty\right)$ we have the inclusions

$$H^{s,p}(\mathbb{R}^n) \subset H^{s,p}(\Omega; \mathbb{R}^n) \subset H^{s,p}(\Omega).$$

In the case when $\Omega \subset \mathbb{R}^n$ is an arbitrary domain this implies the inclusions

$$H^{s,p}(\mathbb{R}^n) \subset H^{s,p}_{\text{loc}}(\Omega; \mathbb{R}^n) \subset H^{s,p}_{\text{loc}}(\Omega).$$

We also use the following embedding theorems of Bessel potential spaces. Parts (i) and (ii) follow from Remark 1.96 (iii) in [31], while the last two parts follow from the corresponding embeddings of Sobolev-Slobodeckij spaces (cf. [15]) and part (ii) of Theorem 3.2

Theorem 3.4. Let $1 < p \leq p_1 < \infty$, $s, s_1 \geq 0$ and assume that $\Omega \subset \mathbb{R}^n$ is a domain.

(i) If $sp < n$, then for any $q \in [p, \frac{np}{n-sp}]$ we have

$$H^{s,p}(\Omega) \hookrightarrow L^q(\Omega).$$

(ii) More generally, if $s - \frac{n}{p} = s_1 - \frac{n}{p_1}$, then

$$H^{s,p}(\Omega) \hookrightarrow H^{s_1,p_1}(\Omega).$$

(iii) If $sp = n$, then for any $q \in [p, \infty)$ we have

$$H^{s,p}(\Omega) \hookrightarrow C^\alpha(\Omega),$$

where $\alpha = s - \frac{n}{p}.$

(iv) If $sp > n$, then we have

$$H^{s,p}(\Omega) \hookrightarrow C^\alpha(\Omega),$$

where $\alpha = s - \frac{n}{p}.$

4. Some preliminary regularity results

For the rest of this paper, we fix real numbers $s \in (0, 1)$, $0 < \nu \leq \lambda$ and a kernel coefficient $A \in \mathcal{L}_0(\nu, \lambda).$
4.1. $L^\infty$ estimates. The following Lemma relates the nonlocal tail of a function that often appears naturally in the literature to the $L^2$ norm of its $s$-gradient.

**Lemma 4.1.** For all $r, R > 0$ and any $u \in H^s(B_R; \mathbb{R}^n)$ we have

$$
\int_{\mathbb{R}^n \setminus B_r} \frac{u(y)^2}{|y|^{n+2s}} dy \leq C(||\nabla^s u||^2_{L^2(B_r)} + ||u||^2_{L^2(B_r)}),
$$

where $C = C(n, s, r, R) > 0$.

**Proof.** First of all, integration in polar coordinates yields

$$
\int_{\mathbb{R}^n \setminus B_r} \frac{dz}{|z|^{n+2s}} = \omega_n \int_r^{\infty} \rho^{n-1} |\rho s|^{n+2s} d\rho = \frac{\omega_n}{2sr^{2s}} = C_1 < \infty,
$$

where $\omega_n$ denotes the surface area of the $n - 1$ dimensional unit sphere $S^{n-1}$. Moreover, for any $x \in B_R$ and any $y \in \mathbb{R}^n \setminus B_r$ we have

$$
|x - y| \leq |x| + |y| < R + |y| = \left(\frac{R}{|y|} + 1\right) |y| \leq \left(\frac{R}{r} + 1\right) |y|.
$$

Along with the Cauchy-Schwartz inequality, Fubini’s theorem and (13) we obtain

$$
\int_{\mathbb{R}^n \setminus B_r} \frac{u(y)^2}{|y|^{n+2s}} dy = \int_{\mathbb{R}^n \setminus B_r} \left(\frac{u(y) - \mathbf{f}_{B_r}(u(x) - u(y))}{|y|^{n+2s}} dx + \mathbf{f}_{B_r}(u(x) dy \right)^2 dy
$$

$$
\leq 2 \int_{\mathbb{R}^n \setminus B_r} \left(\mathbf{f}_{B_r}(u(x) - u(y))^{2} \frac{u(y)}{|y|^{n+2s}} dy + \mathbf{f}_{B_r}(u(x) dy \right)^2 dy
$$

$$
\leq 2 \left(\int_{\mathbb{R}^n \setminus B_r} \frac{(u(x) - u(y))^2}{|y|^{n+2s}} dx dy + \int_{\mathbb{R}^n \setminus B_r} \frac{u^2(x) dx}{|y|^{n+2s}} dy \right)
$$

$$
\leq 2 \frac{1}{|B_r|} \left(\int_{B_r} \int_{\mathbb{R}^n \setminus B_r} \frac{(u(x) - u(y))^2}{|y|^{n+2s}} dy dx + C_1 \int_{B_r} u^2(x) dx \right)
$$

$$
\leq 2 \frac{1}{|B_r|} \left(C_2 \int_{B_r} \int_{\mathbb{R}^n \setminus B_r} \frac{(u(x) - u(y))^2}{|x-y|^{n+2s}} dy dx + C_1 \int_{B_r} u^2(x) dx \right)
$$

$$
\leq 2 \frac{1}{|B_r|} \max\{C_2, C_1\} \left(\int_{B_r} (\nabla^s u)^2(x) dx + \int_{B_r} u^2(x) dx \right),
$$

where $C_2 = (\frac{R}{r} + 1)^{n+2s}$. This proves (12) with $C = \frac{2}{|B_r|} \max\{C_2, C_1\}$. 

We also use the following local $L^\infty$ estimate for weak solutions to homogeneous nonlocal equations, cf. Theorem 1.1 in [14]. We remark that although in [14] the below result is stated under the stronger assumption that $u \in H^s(\mathbb{R}^n)$, an inspection of the proof shows that this is not necessary.

**Theorem 4.2.** For all $0 < r < R < \infty$ and any weak solution $u \in H^s(B_R; \mathbb{R}^n)$ of the equation

$$
L_A u = 0 \text{ in } B_R
$$

we have the estimate

$$
||u||_{L^\infty(B_r)} \leq C \left(\int_{\mathbb{R}^n \setminus B_r} \frac{|u(y)|}{|y|^{n+2s}} dy + ||u||_{L^2(B_R)}\right),
$$

where $C = C(n, s, r, R, \nu, \lambda) > 0$. 

By combining the above two results, we obtain the following.

**Corollary 4.3.** For all $0 < r < R < \infty$ and any weak solution $u \in H^s(B_R;\mathbb{R}^n)$ of the equation

$$L_A u = 0 \text{ in } B_R$$

we have the estimate

$$||u||_{L^\infty(B_r)} \leq C(||\nabla^s u||_{L^2(B_n)} + ||u||_{L^2(B_n)}),$$

where $C = C(n, s, r, R, \nu, \lambda) > 0$.

**Proof.** By Theorem 4.2, (13) and Lemma 4.1 we have

$$||u||_{L^\infty(B_r)} \leq C_1 \left( \int_{\mathbb{R}^n \setminus B_r} \frac{|u(y)|}{|y|^{n+2s}} dy + ||u||_{L^2(B_n)} \right)$$

$$\leq C_1 \left( \frac{C_2^1}{2} \left( \int_{\mathbb{R}^n \setminus B_r} \frac{|u(y)^2|}{|y|^{n+2s}} dy \right)^{\frac{1}{2}} + ||u||_{L^2(B_n)} \right)$$

$$\leq C_1 \left( C_2^1 C_3^1 + 1 \right) \left( ||\nabla^s u||_{L^2(B_n)} + ||u||_{L^2(B_n)} \right),$$

where $C_1$ is given by Theorem 4.2, $C_2$ is given by (13) and $C_3$ is given by Lemma 4.1. This proves (14) with $C = C_1 \left( C_2^1 C_3^1 + 1 \right)$. \hfill \blacksquare

**Corollary 4.4.** For all $0 < r < R < \infty$ and any weak solution $u \in H^s(B_R;\mathbb{R}^n)$ of the equation

$$L_A u = 0 \text{ in } B_R$$

we have the estimate

$$||\nabla^s \nabla^s u||_{L^\infty(B_r)} \leq C||\nabla^s u||_{L^2(B_n)},$$

where $C = C(n, s, \nu, \lambda, r, R) > 0$.

**Proof.** For any $x \in B_r$ and any $y \in \mathbb{R}^n \setminus B_r$ we have

$$|y| \leq |x - y| + |x| < |x - y| + R = \left( 1 + \frac{R}{|x - y|} \right) |x - y| \leq \left( 1 + \frac{R}{R - r} \right) |x - y|.$$  

For almost every $x \in B_r$, it follows that

$$\int_{\mathbb{R}^n \setminus B_r} \frac{(v(x) - v(y))^2}{|x - y|^{n+2s}} dy$$

$$\leq C_1 \int_{\mathbb{R}^n \setminus B_r} \frac{(v(x) - v(y))^2}{|y|^{n+2s}} dy$$

$$\leq 2C_1 \left( \int_{\mathbb{R}^n \setminus B_r} \frac{v(x)^2}{|y|^{n+2s}} dy + \int_{\mathbb{R}^n \setminus B_r} \frac{|v(y)|^2}{|y|^{n+2s}} dy \right)$$

$$\leq 2C_1 \left( C_2^1 ||v||^2_{L^2(B_n)} + C_3 \left( \int_{B_R} \int_{\mathbb{R}^n} \frac{(v(z) - v(y))^2}{|z - y|^{n+2s}} dy dz + \int_{B_R} v^2(z) dz \right) \right)$$

$$\leq 2C_1 (C_2 C_4 + C_3) \left( \int_{B_R} \int_{\mathbb{R}^n} \frac{(v(z) - v(y))^2}{|z - y|^{n+2s}} dy dz + \int_{B_R} v^2(z) dz \right),$$

where $C_1 := \left( 1 + \frac{R}{R - r} \right)^{n+2s}$, $C_2 = C_2(n, s, R)$ is given as in (13) in the proof of Lemma 4.1 while $C_3 = C_3(n, s, R)$ is given by Lemma 4.4 and $C_4 = C_4(n, s, \nu, \lambda, r, R)$ is given by Corollary 4.3. Set
$C_5 := 2C_1(C_2C_4 + C_3)$. Since the function $v - \overline{v}_{B_R} \in H^s(B_R|\mathbb{R}^n)$ also solves the equation

$$L_{A_0}(v - \overline{v}_{B_R}) = 0 \text{ weakly in } B_R,$$

the above estimate also applies to the function $v - \overline{v}_{B_R}$, so that together with the fractional Poincaré inequality (Lemma 3.1) for almost every $x \in B_r$ we deduce

$$\left|\nabla_{\mathbb{R}^n \setminus B_R} v(x)\right|^2 = \int_{\mathbb{R}^n \setminus B_R} \frac{(v(x) - v(y))^2}{|x - y|^{n+2s}}dy = \int_{\mathbb{R}^n \setminus B_R} \frac{(v(x) - \overline{v}_{B_R}) - (v(y) - \overline{v}_{B_R}))^2}{|x - y|^{n+2s}}dy$$

$$\leq C_5 \left( \int_{B_R} \int_{\mathbb{R}^n} \frac{(v(z) - \overline{v}_{B_R}) - (v(y) - \overline{v}_{B_R})^2}{|z - y|^{n+2s}}dydz + \int_{B_R} (v(z) - \overline{v}_{B_R})^2dz \right)$$

$$\leq C_5 \left( \int_{B_R} \int_{\mathbb{R}^n} \frac{(v(z) - v(y))^2}{|z - y|^{n+2s}}dydz + C_6 \int_{B_R} \int_{B_R} \frac{(v(y) - \overline{v}_{B_R})^2}{|z - y|^{n+2s}}dydz \right)$$

$$\leq C_7 \left|\nabla_{\mathbb{R}^n} w\right|^2_{L^2(B_R)},$$

where $C_6 = C_6(n, s, R)$ and $C_7 := C_5(1 + C_6)$, which proves (15) with $C = C_7^2$.

4.2. $C^{2s}$ regularity in the translation invariant case. In the basic case when $A \in \mathcal{L}_0(\nu, \lambda)$, it can be shown that any weak solution to the corresponding homogeneous nonlocal equation is $C^\alpha$ for some $\alpha > 0$, cf. Theorem 1.2 in [14]. Let us now show that if $A$ is of class $\mathcal{L}_1(\nu, \lambda)$, then weak solutions to the corresponding homogeneous nonlocal equation enjoy better Hölder regularity than in general. To do this, we need the following result, cf. [25] and page 19 in [23].

**Theorem 4.5.** Consider a kernel coefficient $A_0 \in \mathcal{L}_1(\nu, \lambda)$ and assume that $w \in H^s(B_4|\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is a weak solution of the equation $L_{A_0}w = f$ in $B_4$, where $f \in L^\infty(B_4)$.

(i) If $s \neq 1/2$, then

$$\|w\|_{C^{2s}(B_3)} \leq C(\|w\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_4)}),$$

where $C = C(n, s, \nu, \lambda) > 0$.

(ii) If $s = 1/2$, then

$$\|w\|_{C^{2s - \varepsilon}(B_3)} \leq C(\|w\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_4)}) \quad \forall \varepsilon > 0,$$

where $C = C(n, s, \nu, \lambda, \varepsilon) > 0$.

**Corollary 4.6.** Consider a kernel coefficient $A_0 \in \mathcal{L}_1(\nu, \lambda)$ and assume that $v \in H^s(B_7|\mathbb{R}^n)$ is a weak solution of the equation $L_{A_0}v = 0$ in $B_7$.

(i) If $s \neq 1/2$, then

$$\|v\|_{C^{2s}(B_3)} \leq C\|\nabla^s v\|_{L^2(B_3)},$$

where $C = C(n, s, \nu, \lambda) > 0$.

(ii) If $s = 1/2$, then

$$\|v\|_{C^{2s - \varepsilon}(B_3)} \leq C\|\nabla^s v\|_{L^2(B_3)} \quad \forall \varepsilon > 0,$$

where $C = C(n, s, \nu, \lambda, \varepsilon) > 0$.

**Proof.** It suffices to prove part (i), since part (ii) can be shown in the same way. Let $\eta \in C_0^\infty(B_6)$ be a smooth cutoff function of $B_5$ in $B_6$ with the properties

$$0 \leq \eta \leq 1 \text{ in } B_6, \quad \eta = 1 \text{ in } B_5, \quad \eta = 0 \text{ in } \mathbb{R}^n \setminus B_6.$$

For $x \in B_4$, define

$$f(x) := 2 \int_{\mathbb{R}^n \setminus B_5} A_0(x, y) \frac{(v - \overline{v}_{B_5})(1 - \eta))(y)}{|x - y|^{n+2s}}dy.$$
By \((13)\), Lemma \(4.1\) applied to \(v - \tau_{B_7} \in H^s(B_7)\) and the fractional Poincaré inequality (Lemma \(3.1\)) we have
\[
\|f\|_{L^\infty(B_4)} \leq 5^{n+2s}2\lambda \int_{\mathbb{R}^n \setminus B_5} \frac{|(v - \tau_{B_5})(1 - \eta)(y)|}{|y|^{n+2s}} \, dy \\
\leq 5^{n+2s}2\lambda \int_{\mathbb{R}^n \setminus B_5} \frac{|(v - \tau_{B_5})(y)|}{|y|^{n+2s}} \, dy \\
\leq 5^{n+2s}2\lambda C_1 \left( \int_{\mathbb{R}^n \setminus B_5} \frac{|(v - \tau_{B_5})(y)|}{|y|^{n+2s}} \, dy \right)^{\frac{1}{2}} \\
\leq 5^{n+2s}2\lambda C_1 C_2 (||\nabla^s(v - \tau_{B_7})||_{L^2(B_7)}^2 + ||v - \tau_{B_7}||_{L^2(B_7)}^2)^{\frac{1}{2}} \\
\leq 5^{n+2s}2\lambda C_1 C_2 (C_3 + 1)^{\frac{1}{2}} ||\nabla^s v||_{L^2(B_7)} = C_4 ||\nabla^s v||_{L^2(B_7)},
\]
where \(C_1, C_2\) and \(C_3\) depend only on \(n\) and \(s\) and \(C_4 := 5^{n+2s}2\lambda C_1 C_2 (C_3 + 1)^{\frac{1}{2}}\). Now set \(w := (v - \tau_{B_7})\eta \in H^s(\mathbb{R}^n)\). \(w\) is a weak solution of the equation
\[
L_{A_0}w = L_{A_0}((v - \tau_{B_7})(1 - \eta)) \text{ in } B_4.
\]
Since \(f \in L^\infty(B_4) \subset L^2(B_4)\), for any \(\varphi \in H_0^s(B_4)\) we can therefore write
\[
\mathcal{E}_{A_0}(w, \varphi) = \mathcal{E}_{A_0}((v - \tau_{B_7})(1 - \eta), \varphi) = \int_{B_4} A_0((v - \tau_{B_7})(1 - \eta)) \varphi(x) \, dx \\
= 2 \int_{B_4} \left( \int_{\mathbb{R}^n \setminus B_5} A_0(x, y) \frac{((v - \tau_{B_7})(1 - \eta)(y))}{|x - y|^{n+2s}} \, dy \right) \varphi(x) \, dx,
\]
so that we have
\[
L_{A_0}w = f \text{ weakly in } B_4.
\]
By Corollary \(4.3\) and the fractional Poincaré inequality (Lemma \(3.1\)) we have
\[
||w||_{L^\infty(\mathbb{R}^n)} = ||w||_{L^\infty(B_6)} \leq ||v - \tau_{B_7}||_{L^\infty(B_6)} \\
\leq C_5 (||\nabla^s(v - \tau_{B_7})||_{L^2(B_7)} + ||v - \tau_{B_7}||_{L^2(B_7)}) \\
\leq C_5 (1 + C_4^2) ||\nabla^s v||_{L^2(B_7)},
\]
where \(C_5 = C_5(n, s, \nu, \lambda)\). Therefore, by combining Theorem \(4.5\) with the above calculations we obtain the estimate
\[
||v||_{C^2(B_3)} = ||v||_{C^2(B_3)} \leq C_6 (||w||_{L^\infty(\mathbb{R}^n)} + ||f||_{L^\infty(B_4)}) \leq C ||\nabla^s v||_{L^2(B_7)},
\]
where \(C_6 = C_6(s, n, \nu, \lambda)\) and \(C = C_6(C_5(C_3 + 1)^{\frac{1}{2}} + C_4)\). This concludes the proof of part (i). As mentioned, part (ii) can be shown in the same way. \(\blacksquare\)

4.3. **Higher integrability of \(\nabla^s u\) for a small range of exponents.** The aim of this section is to prove a higher integrability result for a small range of exponents which holds for general kernel coefficients \(A \in L_0(\nu, \lambda)\). We do this by using a similar approach as the authors do in \(3\), where a corresponding result is obtained in the case when the equation is posed on \(\mathbb{R}^n\).

**Theorem 4.7.** There exists some small \(\sigma = \sigma(n, s, \nu, \lambda) > 0\) and some constant \(C = C(n, s, \nu, \lambda, \sigma) > 0\), such that for any weak solution \(u \in H^s(B_{12})\) of the equation
\[
L_{A}u = 0 \text{ in } B_{12}
\]
we have
\[ ||\nabla^s u||_{L^{2+s}(B_R)} \leq C ||\nabla^s u||_{L^2(B_{1/2})} \]
and in particular \( \nabla^s u \in L^{2+s}(B_R) \).

One of the main ingredients of the proof is the following Caccioppoli-type inequality.

**Lemma 4.8.** Let \( R > 0 \), \( x_0 \in \mathbb{R}^n \) and \( f \in L^2(B_R(x_0)) \). Then for any weak solution \( u \in H^s(B_R(x_0)[\mathbb{R}^n]) \) of the equation
\[ L_A u = f \text{ in } B_R(x_0) \]
we have
\[ \int_{B_{R/2}(x_0)} (\nabla^s u)^2 dx \leq C \left( \int_{\mathbb{R}^n} u^2(x)\psi(x) dx + \int_{B_R(x_0)} |f(x)u(x)| dx \right) \]
where \( C = C(n, s, \nu, \lambda) > 0 \) and
\[ \psi(x) = \min \left\{ R^{-2s}, \frac{|B_R|}{|x-x_0|^{n+2s}} \right\}. \]

Lemma 4.8 can be proved in exactly the same way as Theorem 3.1 in [3]. This is because although in [3] the equation is assumed to hold on the whole space \( \mathbb{R}^n \), it actually suffices to assume that the equation holds in \( B_R(x_0) \), since the test function used in [3] in fact belongs to \( H^s_0(B_R(x_0)[\mathbb{R}^n]) \) whenever the solution \( u \) belongs to \( H^s(B_R(x_0)[\mathbb{R}^n]) \).

We now proof a version of Theorem 4.7 for globally bounded solutions of possibly inhomogeneous equations.

**Proposition 4.9.** Let \( f \in L^2(B_R) \). Then there exists some small \( \sigma = \sigma(n, s, \nu, \lambda) > 0 \) and some constant \( C = C(n, s, \nu, \lambda, \sigma) > 0 \), such that for any weak solution \( u \in H^s(B_0[\mathbb{R}^n]) \cap L^\infty(\mathbb{R}^n) \) of the equation
\[ L_A u = f \text{ in } B_0 \]
we have
\[ ||\nabla^s u||_{L^{2+s}(B_0)} \leq C(||\nabla^s u||_{L^2(B_0)} + ||u||_{L^\infty(\mathbb{R}^n)} + ||f||_{L^2(B_0)}) \]
and in particular \( \nabla^s u \in L^{2+s}(B_0) \).

**Proof.** Let \( x_0 \in \mathbb{R}^n \) and \( R > 0 \) be such that \( B_R(x_0) \subset B_0 \). Since we also have
\[ L_A (u - \overline{u}_{B_R(x_0)}) = f \text{ weakly in } B_R(x_0), \]
Lemma 4.8 yields
\[
\int_{B_{R/2}(x_0)} (\nabla^s u)^2 dx = \int_{B_{R/2}(x_0)} (\nabla^s (u(x) - \overline{u}_{B_R(x_0)}))^2 dx \\
\leq C_1 \left( \int_{\mathbb{R}^n} (u(x) - \overline{u}_{B_R(x_0)})^2 \psi(x) dx + \int_{B_R(x_0)} |f(x)(u(x) - \overline{u}_{B_R(x_0)})| dx \right) \\
\leq C_1 \left( R^{-2s} \int_{B_R(x_0)} (u(x) - \overline{u}_{B_R(x_0)})^2 dx + |B_R| \int_{\mathbb{R}^n \setminus B_R(x_0)} \frac{(u(x) - \overline{u}_{B_R(x_0)})^2}{|x-x_0|^{n+2s}} dx \right. \\
\left. + \int_{B_R(x_0)} |f(x)(u(x) - \overline{u}_{B_R(x_0)})| dx \right).
\]
where $C_1 = C_1(n,s,\nu,\lambda)$. Fix some $0 < s_1 < s$. By Corollary 4.2 in [3], for $q_1 := \frac{2n}{n + 2s} \in (1,2)$ we have

$$\int_{B_R(x_0)} (u(x) - \Pi_{B_R(x_0)})^2 \, dx \leq C_2 R^{2s - 2s_1} \left( \int_{B_R(x_0)} (\nabla^s u)^{q_1}(x) \, dx \right)^{\frac{2}{q_1}},$$

where $C_2 = C_2(n,s)$. Moreover, we have

$$|B_R| \int_{\mathbb{R}^n \setminus B_R(x_0)} \frac{(u(x) - \Pi_{B_R(x_0)})^2}{|x - x_0|^{n + 2s}} \, dx + \int_{B_R(x_0)} |f(x)(u(x) - \Pi_{B_R(x_0)})| \, dx$$

$$\leq 4C_3 \int_{B_R(x_0)} ||u||_{L^\infty(\mathbb{R}^n)}^2 \, dx + 2 \int_{B_R(x_0)} ||u||_{L^\infty(\mathbb{R}^n)} f(x) \, dx$$

$$\leq C_4 \int_{B_R(x_0)} \left( ||u||_{L^\infty(\mathbb{R}^n)}^2 + ||u||_{L^\infty(\mathbb{R}^n)} |f(x)| \right) \, dx,$$

where $C_3 := \int_{\mathbb{R}^n \setminus B_R(x_0)} \frac{dx}{|x - x_0|^{n + 2s}} < \infty$ and $C_4 := \max\{4C_3, 2\}$. Combining the above estimates now yields

$$\int_{B_R/2} (\nabla^s u)^2(x) \, dx$$

$$\leq C_5 \left( R^{-2s_1} \left( \int_{B_R(x_0)} (\nabla^s u)^{q_1}(x) \, dx \right)^{\frac{2}{q_1}} + \int_{B_R(x_0)} \left( ||u||_{L^\infty(\mathbb{R}^n)}^2 + ||u||_{L^\infty(\mathbb{R}^n)} |f(x)| \right) \, dx \right),$$

where $C_5 := C_1 \max\{C_2, C_4\}$. Setting

$$g(x) := (\nabla^s u)^{q_1}(x), \quad h(x) := \left( ||u||_{L^\infty(\mathbb{R}^n)}^2 + ||u||_{L^\infty(\mathbb{R}^n)} |f(x)| \right)^{\frac{q_1}{2}}$$

and dividing both sides of the above inequality by $|B_R|$ yields

$$\int_{B_R/2} g(x) \, dx \leq 2^n C_5 C_6 \left( \left( \int_{B_R(x_0)} g(x) \, dx \right)^{\frac{2}{q_1}} + \int_{B_R(x_0)} h(x) \, dx \right),$$

where $C_6 = C_6(n)$. Therefore, by the reverse Hölder inequality (cf. Theorem 4.1 in chapter 12 of [10]) there exists some small $\varepsilon > 0$ such that for any $t \in \left[ \frac{2}{q_1}, \frac{4}{q_1} + \varepsilon \right)$ and all $x_0 \in \mathbb{R}^n$, $R > 0$ such that $B_R(x_0) \subset B_0$ we have

$$\left( \int_{B_R/2} g^t(x) \, dx \right)^{\frac{1}{t}} \leq C_7 \left( \left( \int_{B_R(x_0)} g(x) \, dx \right)^{\frac{2}{q_1}} + \left( \int_{B_R(x_0)} h(x) \, dx \right)^{\frac{1}{t}} \right),$$

where $C_7 = C_7(n,s,\nu,\lambda)$. For any such $t$ we thus have

$$\left( \int_{B_R/2} (\nabla^s u)^{q_1 t}(x) \, dx \right)^{\frac{1}{t}}$$

$$\leq C_7 \left( \left( \int_{B_R(x_0)} (\nabla^s u)^2(x) \, dx \right)^{\frac{q_1 t}{2}} + ||u||_{L^\infty(\mathbb{R}^n)}^2 + ||u||_{L^\infty(\mathbb{R}^n)} \left( \int_{B_R(x_0)} f_{\pi}^2(x) \, dx \right)^{\frac{1}{t}} \right).$$

Now choose $t \in \left[ \frac{2}{q_1}, \frac{4}{q_1} + \varepsilon \right)$ such that $q_1 t \leq 4$ and set $p_0 := q_1 t \in [2,4]$. Taking $q_1^{th}$ roots yields

$$||\nabla^s u||_{L^{p_0}(B_R/2(x_0))} \leq C_8 (||\nabla^s u||_{L^2(B_R(x_0))} + ||u||_{L^\infty(\mathbb{R}^n)} + ||u||_{L^\infty(\mathbb{R}^n)})^{\frac{1}{2}} \leq 2 C_8 (||\nabla^s u||_{L^2(B_R(x_0))} + ||u||_{L^\infty(\mathbb{R}^n)} + ||f||_{L^1(B_R(x_0))}),$$

$$||\nabla^s u||_{L^{p_0}(B_R/2(x_0))} \leq C_9 (||\nabla^s u||_{L^2(B_R(x_0))} + ||u||_{L^\infty(\mathbb{R}^n)} + ||u||_{L^\infty(\mathbb{R}^n)})^{\frac{1}{2}} \leq 2 C_9 (||\nabla^s u||_{L^2(B_R(x_0))} + ||u||_{L^\infty(\mathbb{R}^n)} + ||f||_{L^1(B_R(x_0))}),$$

$$||\nabla^s u||_{L^{p_0}(B_R/2(x_0))} \leq C_{10} (||\nabla^s u||_{L^2(B_R(x_0))} + ||u||_{L^\infty(\mathbb{R}^n)} + ||u||_{L^\infty(\mathbb{R}^n)})^{\frac{1}{2}} \leq 2 C_{10} (||\nabla^s u||_{L^2(B_R(x_0))} + ||u||_{L^\infty(\mathbb{R}^n)} + ||f||_{L^1(B_R(x_0))}).$$
where \( C_8 \) and \( C_9 \) depend only on \( n, s, \nu, \lambda \) and \( p_0 \). The proof can now be finished by setting 
\[
\sigma := p_0 - 2 \text{ and by covering } B_8 \text{ with finitely many balls that are subsets of } B_9 \text{ and then using a partition of unity in order to sum the corresponding estimates up, cf. our proof of Theorem 4.7.}
\]

**Proof of Theorem 4.7.** Let \( \eta \in C_0^\infty(B_{11}) \) be a smooth cutoff function of \( B_{10} \) in \( B_{11} \) with the properties
\[
0 \leq \eta \leq 1 \text{ in } B_{11}, \quad \eta = 1 \text{ in } B_{10}, \quad \eta = 0 \text{ in } \mathbb{R}^n \setminus B_{11}.
\]
For \( x \in B_9 \), define 
\[
f(x) := 2 \int_{\mathbb{R}^n \setminus B_9} A(x, y) \frac{((u - \overline{u}_{B_7})(1 - \eta))(y)}{|x - y|^{n+2s}} \, dy.
\]
Moreover, set \( w := (u - \overline{u}_{B_{12}})\eta \in H^s(\mathbb{R}^n) \). By the same arguments as in the proof of Corollary 4.6, we have
\[
L_A w = f \text{ weakly in } B_9
\]
and the estimates
\[
||f||_{L^2(B_9)} \leq |B_9|^\frac{1}{2} ||f||_{L^\infty(B_9)} \leq C_1 |B_9|^\frac{1}{2} ||\nabla^s u||_{L^2(B_{12})},
\]
\[
||w||_{L^\infty(\mathbb{R}^n)} \leq C_2 ||\nabla^s u||_{L^2(B_{12})},
\]
where \( C_1 = C_1(n, s, \lambda) \) and \( C_2 = C_2(n, s, \nu, \lambda) \). Furthermore, by the same argument as in the proof of Lemma 5.3 in [15] and the fractional Poincaré inequality (Lemma 3.1) we have
\[
||\nabla^s w||_{L^2(B_9)} = ||\nabla^s((u - \overline{u}_{B_7})\eta)||_{L^2(B_9)} \leq C_3(||\nabla^s(u - \overline{u}_{B_7})||_{L^2(B_9)} + ||u - \overline{u}_{B_7}||_{L^2(B_9)})
\]
\[
\leq C_3(||\nabla^s u||_{L^2(B_{12})} + ||u - \overline{u}_{B_7}||_{L^2(B_{12})})
\]
\[
\leq C_3(1 + C_4)||\nabla^s u||_{L^2(B_{12})},
\]
where \( C_3 > 0 \) and \( C_4 > 0 \) depend only on \( n \) and \( s \). Therefore, by combining Proposition 4.9 with the above estimates we obtain
\[
||\nabla^s u||_{L^{2+s}(B_{11})} = ||\nabla^s w||_{L^{2+s}(B_{11})} \leq C_5(||\nabla^s u||_{L^2(B_9)} + ||u||_{L^\infty(\mathbb{R}^n)} + ||f||_{L^2(B_9)})
\]
\[
\leq C_6 ||\nabla^s u||_{L^2(B_{11})},
\]
where \( C_5 = C_5(n, s, \nu, \lambda) > 0 \) and \( C_6 := C_5 \left(C_3(1 + C_4) + C_2 + C_1|B_9|^\frac{1}{2}\right) \). Moreover, by Corollary 4.4, we have
\[
||\nabla^s u||_{L^{2+s}(B_7)} \leq |B_7|^\frac{1}{2+s} ||\nabla^s u||_{L^\infty(B_7)} \leq C_7 |B_7|^\frac{1}{2+s} ||\nabla^s u||_{L^2(B_{12})},
\]
where \( C_7 = C_7(n, s, \nu, \lambda) > 0 \). We conclude that
\[
||\nabla^s u||_{L^{2+s}(B_7)} \leq ||\nabla^s u||_{L^2(B_{12})} + ||\nabla^s u||_{L^{2+s}(B_7)} \leq C ||\nabla^s u||_{L^2(B_{12})},
\]
where \( C = C_6 + C_7 |B_7|^\frac{1}{2+s} \). This finishes the proof. ■

5. The Dirichlet problem

In what follows, we also fix jointly measurable functions \( D_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) \( (i = 1, ..., m) \) that are symmetric and bounded by some \( \Lambda > 0 \).

**Proposition 5.1.** Let \( \Omega \subset \mathbb{R}^n \) be a domain, \( g_i, h \in H^s(\Omega; \mathbb{R}^n) \), \( f \in L^2(\Omega) \), \( b \in L^\infty(\Omega) \) and \( l := \text{ess inf}_{x \in \Omega} b(x) \). If \( \Omega \) is bounded, then we assume that \( l \geq 0 \), otherwise we assume that \( l > 0 \).
Then there exists a unique solution \( u \in H^s(\Omega|\mathbb{R}^n) \) of the weak Dirichlet problem

\[
\begin{aligned}
L_A u + bu &= \sum_{i=1}^m L_{D_i} g_i + f \\
u &= 0
\end{aligned}
\]

weakly in \( \Omega \)

Moreover, if \( \Omega \) is bounded and \( b \equiv 0 \), then \( u \) satisfies the estimate

\[
||\nabla^s u||_{L^2(\Omega)} \leq C \left( ||\nabla^s h||_{L^2(\Omega)} + \sum_{i=1}^m ||\nabla^s g_i||_{L^2(\Omega)} + ||f||_{L^2(\Omega)} \right),
\]

where \( C = C(n, s, \nu, \lambda, \Lambda, |\Omega|) \).

**Proof.** Consider the symmetric bilinear form

\[
\mathcal{E} : H^s_0(\Omega|\mathbb{R}^n) \times H^s_0(\Omega|\mathbb{R}^n) \to \mathbb{R}, \quad \mathcal{E}(w, \varphi) := \mathcal{E}_A(w, \varphi) + (bw, \varphi)_{L^2(\Omega)}.
\]

First of all, fix some \( w \in H^s_0(\Omega|\mathbb{R}^n) \). We have

\[
\mathcal{E}(w, w) \leq \lambda \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|w(x) - w(y)|^2}{|x - y|^n+2s} \, dy \, dx + \mu \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|w|^2}{|x - y|^n+2s} \, dy \, dx \leq \max \{ \lambda, \mu \} \|w\|^2_{H^s(\mathbb{R}^n)}.
\]

Let us first consider the case when \( \Omega \) is unbounded, in this case we have \( l > 0 \) and therefore

\[
\mathcal{E}(w, w) \geq \nu \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|w(x) - w(y)|^2}{|x - y|^n+2s} \, dy \, dx + l \|w\|^2_{L^2(\mathbb{R}^n)} \geq C_1 \|w\|^2_{H^s(\mathbb{R}^n)},
\]

where \( C_1 = \min \{ \nu, l \} > 0 \). If \( \Omega \) is bounded, then we have \( l \geq 0 \). Since we have \( w = 0 \) a.e. in \( \mathbb{R}^n \setminus \Omega \) and \( w \in H^s(\mathbb{R}^n) \), in this case Hölder’s inequality and the fractional Sobolev inequality (cf. Theorem 6.5 in [15]) yield

\[
\int_{\mathbb{R}^n} w^2 \, dx = \int_{\Omega} w^2 \, dx \leq |\Omega|^{\frac{n}{2s}} \left( \int_{\Omega} \frac{w^{-2s}}{\nabla w} \, dx \right)^{\frac{1}{2s}} \leq C_2 |\Omega|^{\frac{n}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|w(x) - w(y)|^2}{|x - y|^n+2s} \, dy \, dx,
\]

where \( C_2 = C_2(n, s) > 0 \). We deduce

\[
\mathcal{E}(w, w) \geq \nu \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|w(x) - w(y)|^2}{|x - y|^n+2s} \, dy \, dx \geq \nu \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|w(x) - w(y)|^2}{|x - y|^n+2s} \, dy \, dx + C_2^{-1} |\Omega|^{-\frac{n}{2s}} \int_{\mathbb{R}^n} w^2 \, dx \right) \geq C_3 \|w\|^2_{H^s(\mathbb{R}^n)},
\]

where \( C_3 = \frac{\nu}{2} \min \left\{ 1, C_2^{-1} |\Omega|^{-\frac{n}{2s}} \right\} > 0 \). We obtain that in both cases \( \mathcal{E}(\cdot, \cdot) \) is positive definite and hence an inner product in \( H^s_0(\Omega|\mathbb{R}^n) \) that is equivalent to the inner product \( (\cdot, \cdot)_{H^s(\mathbb{R}^n)} \) defined in section 3. Therefore \( H^s_0(\Omega|\mathbb{R}^n) \) with the inner product \( \mathcal{E}(\cdot, \cdot) \) is a Hilbert space. Since moreover by Hölder’s inequality the expression

\[
-\mathcal{E}_A(h, \varphi) - (bh, \varphi)_{L^2(\Omega)} + \sum_{i=1}^m \mathcal{E}_{D_i}(g_i, \varphi) + (f, \varphi)_{L^2(\Omega)}
\]

is a bounded linear functional of \( \varphi \in H^s_0(\Omega|\mathbb{R}^n) \), by the Riesz representation theorem there exists a unique \( w \in H^s_0(\Omega|\mathbb{R}^n) \) such that

\[
\mathcal{E}_A(w, \varphi) + (bw, \varphi)_{L^2(\Omega)} = -\mathcal{E}_A(h, \varphi) - (bh, \varphi)_{L^2(\Omega)} + \sum_{i=1}^m \mathcal{E}_{D_i}(g_i, \varphi) + (f, \varphi)_{L^2(\Omega)} \quad \forall \varphi \in H^s_0(\Omega|\mathbb{R}^n).
\]
But then the function \( u := w + h \in H^s(\Omega; \mathbb{R}^n) \) solves the Dirichlet problem \([16]\). Furthermore, if \( u \) and \( v \) both solve the Dirichlet problem \([16]\), then \( u - h \) and \( v - h \) both satisfy \([19]\), so that by the uniqueness part of the Riesz representation theorem we deduce \( u - h = v - h \) a.e. in \( \mathbb{R}^n \) and therefore \( u = v \) a.e. in \( \mathbb{R}^n \), so that the Dirichlet problem \([16]\) has a unique solution.

Let us now prove that if \( \Omega \) is bounded and \( b \equiv 0 \), then the unique solution \( u \in H^s_0(\Omega; \mathbb{R}^n) \) of \([16]\) satisfies the estimate \([17]\). In order to accomplish this, note that by \([18]\) for any \( \varphi \in \mathcal{D}(\Omega) \) and \( s \in \mathbb{R} \), and by \([18]\) for any \( w \in H^s_0(\Omega; \mathbb{R}^n) \) we have

\[
\int_{\Omega} |f(x)||w(x)|dx \leq ||f||_{L^2(\Omega)}||w||_{L^2(\Omega)}
\]

\[
\leq C_2^\frac{3}{2} ||\varphi||_{L^2(\Omega)} \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(w(x) - w(y))^2}{|x-y|^{n+2s}} dydx \right)^\frac{1}{2}
\]

\[
\leq 2C_2^\frac{3}{2} ||\varphi||_{L^2(\Omega)} ||\nabla^s w||_{L^2(\Omega)}
\]

Since \( w := u - h \in H^s_0(\Omega; \mathbb{R}^n) \) satisfies \([19]\). using \( \varphi = w \) as a test function in \([19]\) along with the Cauchy-Schwartz inequality yields

\[
||\nabla^s w||_{L^2(\Omega)} \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(w(x) - w(y))^2}{|x-y|^{n+2s}} dydx
\]

\[
\leq \nu^{-1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A(x, y) \frac{(w(x) - w(y))^2}{|x-y|^{n+2s}} dydx
\]

\[
= \nu^{-1} \left( - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A(x, y) \frac{(h(x) - h(y))(w(x) - w(y))}{|x-y|^{n+2s}} dydx + \sum_{i=1}^m \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} D_i(x, y) \frac{(g_i(x) - g_i(y))(w(x) - w(y))}{|x-y|^{n+2s}} dydx \right)
\]

\[
\leq \nu^{-1} \left( \lambda \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|h(x) - h(y)||w(x) - w(y)|}{|x-y|^{n+2s}} dydx
\]

\[
+ \Lambda \sum_{i=1}^m \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|g_i(x) - g_i(y)||w(x) - w(y)|}{|x-y|^{n+2s}} dydx + \int_{\Omega} |f(x)||w(x)|dx \right)
\]

\[
\leq 2\nu^{-1} \text{max}\{\lambda, \Lambda, 2C_2^\frac{3}{2} ||\varphi||^{\frac{1}{2}}\} \left( \int_{\Omega} \int_{\mathbb{R}^n} \frac{|h(x) - h(y)||w(x) - w(y)|}{|x-y|^{n+2s}} dydx
\]

\[
+ \sum_{i=1}^m \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|g_i(x) - g_i(y)||w(x) - w(y)|}{|x-y|^{n+2s}} dydx + ||f||_{L^2(\Omega)} ||\nabla^s w||_{L^2(\Omega)} \right)
\]

\[
\leq C_4 ||\nabla^s w||_{L^2(\Omega)} \left( ||\nabla^s h||_{L^2(\Omega)} + \sum_{i=1}^m ||\nabla^s g_i||_{L^2(\Omega)} + ||f||_{L^2(\Omega)} \right)
\]

where \( C_4 := 2\nu^{-1} \text{max}\{\lambda, \Lambda, 2C_2^\frac{3}{2} ||\varphi||^{\frac{1}{2}}\} \). We obtain

\[
||\nabla^s w||_{L^2(\Omega)} \leq 2(||\nabla^s w||_{L^2(\Omega)} + ||\nabla^s h||_{L^2(\Omega)})
\]

\[
\leq 2 \left( C_4 \left( ||\nabla^s h||_{L^2(\Omega)} + \sum_{i=1}^m ||\nabla^s g_i||_{L^2(\Omega)} + ||f||_{L^2(\Omega)} \right) + ||\nabla^s h||_{L^2(\Omega)} \right)
\]

\[
\leq C \left( ||\nabla^s h||_{L^2(\Omega)} + \sum_{i=1}^m ||\nabla^s g_i||_{L^2(\Omega)} + ||f||_{L^2(\Omega)} \right),
\]

where \( C = 2(C_4 + 1) \).

For a treatment of the nonlocal Dirichlet problem for a much more general class of kernels, we refer to \([16]\).
6. Higher integrability of $\nabla^s u$ for the full range of exponents

The aim of this section is to prove an analogue of Theorem 4.7 for the full range of exponents $p \in (2, \infty)$ in the case when $A$ additionally satisfies the small BMO assumption (6).

6.1. An approximation argument. A key step in the proof of the higher integrability of $\nabla^s u$ is given by the following approximation lemma.

**Key Lemma 6.1.** Let $M$ be an arbitrary positive real number. For any $\varepsilon > 0$ there exists some $\delta = \delta(\varepsilon, n, s, \nu, \lambda, \Lambda, M) > 0$, such that for any weak solution $u \in H^s(B_{12}|\mathbb{R}^n)$ of the equation

$$L_A u = \sum_{i=1}^m L_{D_i} g_i + f \text{ in } B_{12}$$

under the assumptions that

(20) $A$ is $(\delta, \gamma)$-BMO,

that

(21) $\int_{B_{12}} (\nabla^s u)^2 dx \leq M$

and that

(22) $\int_{B_{12}} \left( f^2 + \sum_{i=1}^m (\nabla^s g_i)^2 \right) dx \leq M\delta^2$,

there exist a kernel coefficient $A_0 \in L_1(\nu, \lambda)$ and a weak solution $v \in H^s(B_7|\mathbb{R}^n)$ of the equation

(23) $L_{A_0} v = 0 \text{ in } B_7$

that satisfies

(24) $||\nabla^s (u - v)||_{L^2(B_7)} \leq \varepsilon$.

Moreover, $v$ satisfies the estimate

(25) $||\nabla^s v||_{L^\infty(B_7)} \leq N_0$

for some constant $N_0 = N_0(n, s, \nu, \lambda, \Lambda, M)$.

**Proof.** Fix $\varepsilon > 0$ and let $\delta > 0$ to be chosen. Moreover, we define the kernel coefficient $A_0 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$A_0(x, y) := \begin{cases} \frac{1}{2} \left( f_{B_7} A(x - y + z, z) dz + f_{B_7} A(y - x + z, z) dz \right), & \text{if } x, y \in B_7 \\ f_{B_7} A(x - y + z, z) dz & \text{if } x \in \mathbb{R}^n \setminus B_7, y \in B_7 \\ f_{B_7} A(y - x + z, z) dz & \text{if } x \in B_7, y \in \mathbb{R}^n \setminus B_7 \\ \frac{1}{2} \left( f_{B_7} A(x - y + z, z) dz + f_{B_7} A(y - x + z, z) dz \right), & \text{if } x, y \in \mathbb{R}^n \setminus B_7, \end{cases}$$

note that $A_0 \in L_1(\nu, \lambda)$. Let $u_0 \in H^s(B_{12}|\mathbb{R}^n)$ be the unique weak solution of the problem

(26) $\begin{cases} L_{A_0} u_0 = 0 \text{ weakly in } B_{12} \\ u_0 = u \text{ a.e. in } \mathbb{R}^n \setminus B_{12}, \end{cases}$
note that $u_0$ exists by Proposition 5.1. Observe that we have

\begin{align}
L_A(u - u_0) &= \sum_{i=1}^m L_{D_i} g_i + f \quad \text{weakly in } B_{12} \\
u - u_0 &= 0 \quad \text{a.e. in } \mathbb{R}^n \setminus B_{12}.
\end{align}

Thus, by the estimate \([17]\) from Proposition 5.1 and \((22)\), there exists a constant $C_1 = C_1(n,s,\nu,\lambda,A)$ such that

\begin{equation}
\int_{B_{12}} |\nabla^s (u - u_0)|^2 \, dx \leq C_1 \left( \sum_{i=1}^m \int_{B_{12}} |\nabla^s g_i|^2 \, dx + \int_{B_{12}} f^2 \, dx \right) \leq C_1 |B_{12}| M \delta^2.
\end{equation}

Next, let $v \in H^s(B_7;\mathbb{R}^n)$ be the unique solution of the problem

\begin{equation}
\begin{cases}
L_{A_0} v = 0 & \text{weakly in } B_7 \\
v = u_0 & \text{a.e. in } \mathbb{R}^n \setminus B_7.
\end{cases}
\end{equation}

Observe that $w := u_0 - v \in H^s(B_7;\mathbb{R}^n)$ satisfies

\begin{equation}
\begin{cases}
L_{A_0} w = L_{A_0 - A} u_0 & \text{weakly in } B_7 \\
w = 0 & \text{a.e. in } \mathbb{R}^n \setminus B_7.
\end{cases}
\end{equation}

Using $w \in H^s_0(B_7;\mathbb{R}^n)$ as a test function in the above equation yields

\begin{align*}
\int_{B_7} \int_{\mathbb{R}^n} \frac{(w(x) - w(y))^2}{|x - y|^{n+2s}} \, dy \, dx &\leq \nu^{-1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(w(x) - w(y))^2}{|x - y|^{n+2s}} \, dy \, dx \\
&= \nu^{-1} \left( \int_{B_7} \int_{\mathbb{R}^n} \left( A(x,y) \frac{|w(x) - w(y)|}{|x - y|^{n+2s}} \, dy \right) \right) \\
&\leq \nu^{-1} \left( \int_{B_7} \int_{\mathbb{R}^n} \int_{B_7} A(x - y + z, z) \, dz - A(x,y) \frac{|w(x) - w(y)|}{|x - y|^{n+2s}} \, dy \, dx \\
&+ \int_{\mathbb{R}^n} \int_{B_7} \int_{B_7} A(x - y + z, z) \, dz - A(x,y) \frac{|w(x) - w(y)|}{|x - y|^{n+2s}} \, dy \, dx \right) \\
&= 2 \nu^{-1} \left( \int_{B_7} \int_{\mathbb{R}^n} \int_{B_7} A(x - y + z, z) \, dz - A(x,y) \frac{|w(x) - w(y)|}{|x - y|^{n+2s}} \, dy \, dx \\
&\leq 2 \nu^{-1} \left( \int_{B_7} \int_{\mathbb{R}^n} \int_{B_7} A(x - y + z, z) \, dz - A(x,y) \frac{2}{|x - y|^{n+2s}} \, dy \, dx \right) \right)^rac{1}{2} \\
&= \left( \int_{B_7} \int_{\mathbb{R}^n} \frac{(w(x) - w(y))^2}{|x - y|^{n+2s}} \, dy \, dx \right) ^\frac{1}{2}
\end{align*}

where we also used the symmetry of $A$ and the Cauchy-Schwartz inequality. Thus for $C_2 := 4 \nu^{-2}$ along with Hölder’s inequality, Theorem \([17,20,21]\), and the estimate \([17]\) from Proposition 5.1 applied to \((26)\) we obtain

\begin{align*}
\int_{B_7} |\nabla^s (u_0 - v)|^2 \, dx &= \int_{B_7} |\nabla^s w|^2 \, dx = \int_{B_7} \int_{\mathbb{R}^n} \frac{(w(x) - w(y))^2}{|x - y|^{n+2s}} \, dy \, dx \\
&\leq C_2 \left( \int_{B_7} \int_{\mathbb{R}^n} \frac{|A(x,y) - \int_{B_7} A(x - y + z, z) \, dz|^2}{|x - y|^{n+2s}} \, dy \, dx \right) \\
&\leq C_2 \left( \int_{B_7} \esssup_{x \in \mathbb{R}^n} |A(x,y) - \int_{B_7} A(x - y + z, z) \, dz|^2 \, dy \right) \left( \nabla^s u_0(y) \right)^2 \, dy
\end{align*}
we conclude

\[ C \leq 2 \left( \int_{B_7} \text{ess sup}_{x \in \mathbb{R}^n} |A(x, y) - \int_{B_7} A(x - y + z, z) dz| \right)^{\frac{s+2}{s}} dy \]

\[ \leq C_2 C_3 (2 \lambda)^2 \left( \int_{B_{7r}} \text{ess sup}_{x \in \mathbb{R}^n} |A(x, y) - \int_{B_7} A(x - y + z, z) dz| \right)^{\frac{s}{s+2}} dy \]

\[ \leq C_2 C_3 (2 \lambda)^2 |B_7| \frac{s}{s+2} \delta \frac{s}{s+2} C_1 \left( \int_{B_{12}} (\nabla^s u(y))^2 dy \right) \]

\[ \leq C_1 C_2 C_3 (2 \lambda)^2 |B_7| \frac{s}{s+2} \delta \frac{s}{s+2} |B_{12}| M, \]

where \( \sigma = \sigma(n, s, \nu, \lambda, \sigma) > 0 \) and \( C_3 = C_3(n, s, \nu, \lambda, \sigma) > 0 \) are given by Theorem 4.7. Together with (28) we conclude

\[ \int_{B_7} |\nabla^s (u - v)|^2 dx \leq 2 \left( \int_{B_7} |\nabla^s (u - u_0)|^2 dx + \int_{B_7} |\nabla^s (u_0 - v)|^2 dx \right) \leq 2C_1 C_2 C_3 (2 \lambda)^2 |B_7| \frac{s}{s+2} \delta \frac{s}{s+2} |B_{12}| M + 2C_1 |B_{12}| M \delta^2 \leq \varepsilon^2, \]

where the last inequality follows by choosing \( \delta \) sufficiently small. This completes the proof of (24).

Let us now proof the estimate (25). For almost every \( x \in B_2 \), by Corollary 4.4 we have

\[ \int_{\mathbb{R}^n \setminus B_3} \frac{(v(x) - v(y))^2}{|x - y|^{n+2s}} dy \leq C_4 \int_{B_3} \int_{\mathbb{R}^n} \frac{(v(z) - v(y))^2}{|z - y|^{n+2s}} dy dz, \]

where \( C_4 = C_4(n, s, \nu, \lambda) \). Now choose \( \gamma > 0 \) small enough such that \( 2\gamma < s \) and \( s + 2\gamma < 1 \). Since \( A_0 \) belongs to the class \( L_1(\nu, \lambda) \), by Corollary 4.6 we have

\[ ||v||_{C^{s+\gamma}(B_3)} \leq C_5 ||\nabla^s v||_{L^2(B_3)} \]

for some constant \( C_5 = C_5(n, s, \nu, \lambda, \gamma) \). Thus for almost every \( x \in B_2 \) we have

\[ \int_{B_2} \frac{(v(x) - v(y))^2}{|x - y|^{n+2s}} dy \leq ||v||_{C^{s+\gamma}(B_3)}^2 \int_{B_2} \frac{dy}{|x - y|^{n-2s}} \]

\[ = C_6 ||v||_{C^{s+\gamma}(B_3)}^2 \leq C_6 C_5^2 \int_{B_3} \int_{\mathbb{R}^n} \frac{(v(z) - v(y))^2}{|z - y|^{n+2s}} dy dz, \]

where \( C_6 = C_6(n, \gamma) < \infty \). Applying the estimate (17) from Proposition 5.1 to (29) and (28) yields

\[ \int_{B_7} \int_{\mathbb{R}^n} \frac{(v(z) - v(y))^2}{|z - y|^{n+2s}} dy dz \leq C_7 \int_{B_{12}} \int_{\mathbb{R}^n} \frac{(u_0(z) - u_0(y))^2}{|z - y|^{n+2s}} dy dz \]

\[ \leq C_7 C_8 \int_{B_{12}} \int_{\mathbb{R}^n} \frac{(u(z) - u(y))^2}{|z - y|^{n+2s}} dy dz, \]

where \( C_7 \) and \( C_8 \) depend only on \( n, s, \nu, \lambda \) and \( \Lambda \). By combining the above estimates, along with (21) we deduce

\[ (\nabla^s v)^2(x) = \int_{\mathbb{R}^n \setminus B_3} \frac{(v(x) - v(y))^2}{|x - y|^{n+2s}} dy + \int_{B_3} \frac{(v(x) - v(y))^2}{|x - y|^{n+2s}} dy \]

\[ \leq C_4 \int_{B_7} \int_{\mathbb{R}^n} \frac{(v(z) - v(y))^2}{|z - y|^{n+2s}} dy dz + C_6 C_5^2 \int_{B_3} \int_{\mathbb{R}^n} \frac{(v(z) - v(y))^2}{|z - y|^{n+2s}} dy dz \]

\[ \leq C_7 C_8 (C_4 + C_6 C_5^2) \int_{B_{12}} \int_{\mathbb{R}^n} \frac{(u(z) - u(y))^2}{|z - y|^{n+2s}} dy dz \]

\[ \leq C_7 C_8 (C_4 + C_6 C_5^2) ||B_{12}| M, \]

for almost every \( x \in B_2 \), so that (25) holds with \( N_0 = (C_7 C_8 (C_4 + C_6 C_5^2)) |B_{12}| M \). \( \blacksquare \)
6.2. A real variable argument. We now combine the above approximation lemma with the techniques from section 2.

**Lemma 6.2.** There is a constant $N_1 = N_1(n, s, \nu, \lambda, \Lambda) > 1$, such that the following holds. For any $\varepsilon > 0$ there exists some $\delta = \delta(\varepsilon, n, s, \nu, \lambda, \Lambda) > 0$, such that for any $z \in \mathbb{R}^n$, any $r \in (0, 1]$, any bounded domain $U \subset \mathbb{R}^n$ such that $B_{12r}(z) \subset U$ and any weak solution $u \in H^s(B_{12r}(z))_{\mathbb{R}^n}$ of the equation

$$L_A u = \sum_{i=1}^{m} L_{D_i} g_i + f \text{ in } B_{12r}(z)$$

with $A$ being $(\delta, 7r)$-BMO and

$$\{x \in B_r(z) \mid M_U(|\nabla^s u|^2)(x) \leq 1\} \cap \left\{x \in B_r(z) \mid M_U \left(|f|^2 + \sum_{i=1}^{m} \left|\nabla^s g_i\right|^2\right)(x) \leq \delta^2\right\} \neq \emptyset,$$

we have

$$(31) \quad \left|\left\{x \in B_r(z) \mid M_U(|\nabla^s u|^2)(x) > N_1^2\right\}\right| < \varepsilon |B_r|.$$  

**Proof.** Let $\theta > 0$ and $M > 0$ to be chosen and consider the corresponding $\delta = \delta(\theta, n, s, \nu, \lambda, \Lambda, M) > 0$ given by Key Lemma 6.1. Fix $r \in (0, 1]$ and $z \in \mathbb{R}^n$. For any $x \in U^0 := \{\frac{z-rx}{r^2} \mid x \in U\}$, define

$$\tilde{A}(x, y) := A(rx + z, ry + z), \quad \tilde{D}_i(x, y) := D_i(rx + z, ry + z),$$

$$\tilde{u}(x) := r^{-s} u(rx + z), \quad \tilde{g}_i(x) := r^{-s} g_i(rx + z), \quad \tilde{f}(x) := r^s f(rx + z)$$

and note that under the above assumptions $\tilde{A}$ belongs to the class $L_0(\nu, \lambda)$ and is $(\delta, 7)$-BMO and that $\tilde{u} \in H^s(B_{12r})_{\mathbb{R}^n}$ satisfies

$$L_{\tilde{A}} \tilde{u} = \sum_{i=1}^{m} L_{\tilde{D}_i} \tilde{g}_i + \tilde{f} \text{ weakly in } B_{12r}.$$  

Hence, by Key Lemma 6.1 there exists a kernel coefficient $\tilde{A}_0 \in L_1(\nu, \lambda)$ and a weak solution $\tilde{v} \in H^s(B_{12r})_{\mathbb{R}^n}$ of

$$L_{\tilde{A}_0} \tilde{v} = 0 \text{ in } B_7$$

such that

$$(32) \quad \int_{B_2} |\nabla^s(\tilde{u} - \tilde{v})|^2 dx \leq \theta^2,$$  

provided that the conditions (20), (21) and (22) are satisfied. By assumption, there exists a point $x \in B_r(z)$ such that

$$M_U(|\nabla^s u|^2)(x) \leq 1, \quad M_U \left(|f|^2 + \sum_{i=1}^{m} \left|\nabla^s g_i\right|^2\right)(x) \leq \delta^2.$$  

By the scaling and translation invariance of the Hardy-Littlewood maximal function (Lemma 2.2), for the point $x_0 := \frac{z-rx}{r^2} \in B_1$ we thus have

$$M_U^s(|\nabla^s \tilde{u}|^2)(x_0) = M_U(|\nabla^s u|^2)(x) \leq 1$$

and

$$M_U^s \left(|\tilde{f}|^2 + \sum_{i=1}^{m} \left|\nabla^s \tilde{g}_i\right|^2\right)(x_0) = M_U \left(r^2|f|^2 + \sum_{i=1}^{m} \left|\nabla^s g_i\right|^2\right)(x) \leq \delta^2.$$
Therefore, for any $\rho > 0$ we have

$$\int_{B_\rho(x_0)} |\nabla^s \tilde{u}|^2 \, dx \leq 1,$$

$$\int_{B_\rho(x_0)} \left( |\tilde{f}|^2 + \sum_{i=1}^{m} |\nabla^s \tilde{g}_i|^2 \right) \, dx \leq \delta^2,$$

where the values of $\nabla^s \tilde{u}$, $\nabla^s \tilde{g}_i$ and $\tilde{f}$ outside of $U'$ are replaced by 0, which we also do for the rest of the proof. Since $B_{12} \subset B_{13}(x_0)$, by (33) we have

$$\int_{B_{12}} |\nabla^s \tilde{u}|^2 \, dx \leq \frac{|B_{13}|}{|B_{12}|} \int_{B_{13}(x_0)} |\nabla^s \tilde{u}|^2 \, dx \leq \left( \frac{13}{12} \right)^n$$

and

$$\int_{B_{12}} \left( |\tilde{f}|^2 + \sum_{i=1}^{m} |\nabla^s \tilde{g}_i|^2 \right) \, dx \leq \frac{|B_{13}|}{|B_{12}|} \int_{B_{13}(x_0)} \left( |\tilde{f}|^2 + \sum_{i=1}^{m} |\nabla^s \tilde{g}_i|^2 \right) \, dx \leq \left( \frac{13}{12} \right)^n \delta^2,$$

so that together with fact that $\tilde{A}$ is $(\delta,7)$-BMO, we get that $\tilde{A}$, $\tilde{u}$, $\tilde{g}_i$ and $\tilde{f}$ satisfy the conditions (20), (21) and (22) with $M = \left( \frac{13}{12} \right)^n$. Therefore, (32) is satisfied by $\tilde{u}$ and the corresponding approximate solution $\tilde{v}$. Considering the function $v \in H^s(U[\mathbb{R}^n])$ given by $v(x) := r^s \tilde{v} \left( \frac{x-x_0}{r} \right)$ and rescaling back yields

$$\int_{B_{2r}(y)} |\nabla^s (u-v)|^2 \, dx = r^n \int_{B_2} |\nabla^s (\tilde{u} - \tilde{v})|^2 \, dx \leq \theta^2 r^n.$$

By Key Lemma 6.1 there exists a constant $N_0 = N_0(n,s,\nu,\lambda,\Lambda) > 0$ such that

$$||\nabla^s \tilde{v}||^2_{L^\infty(B_2)} \leq N_0^2.$$

Next, we define $N_1 := (\max\{4N_0^2, 2^n\})^{1/2} > 1$ and claim that

$$\{ x \in B_1 \mid \mathcal{M}_{U'}(|\nabla^s \tilde{u}|^2)(x) > N_1^2 \} \subset \{ x \in B_1 \mid \mathcal{M}_{B_2}(|\nabla^s (\tilde{u} - \tilde{v})|^2)(x) > N_0^2 \}.$$

To see this, assume that

$$x_1 \in \{ x \in B_1 \mid \mathcal{M}_{B_2}(|\nabla^s (\tilde{u} - \tilde{v})|^2)(x) \leq N_0^2 \}. $$

For $\rho < 1$, we have $B_{\rho}(x_1) \subset B_{1}(x_1) \subset B_2$, so that together with (37) and (33) we deduce

$$\int_{B_{\rho}(x_1)} |\nabla^s \tilde{u}|^2 \, dx \leq 2 \int_{B_{\rho}(x_1)} (|\nabla^s (\tilde{u} - \tilde{v})|^2 + |\nabla^s \tilde{v}|^2) \, dx$$

$$\leq 2 \int_{B_{\rho}(x_1)} |\nabla^s (\tilde{u} - \tilde{v})|^2 \, dx + 2 ||\nabla^s \tilde{v}||^2_{L^\infty(B_{\rho}(x_1))}$$

$$\leq 2 \mathcal{M}_{B_2}(|\nabla^s (\tilde{u} - \tilde{v})|^2)(x_1) + 2 ||\nabla^s \tilde{v}||^2_{L^\infty(B_2)} \leq 4N_0^2.$$
For any \( \varepsilon > 0 \), using (38), the weak 1-1 estimate from Theorems 2.3 and 34, we conclude that there exists some constant \( C = C(n) > 0 \) such that

\[
\left\{ x \in B_r(y) \mid \mathcal{M}_{B_{2r}((y)}(|\nabla^s u|^2)(x) > N_1^2 \right\} \leq \left\{ x \in B_r(y) \mid \mathcal{M}_{B_{2r}}(|\nabla^s (u - v)|^2)(x) > N_0^2 \right\}
\]

\[
\leq \frac{C}{N_0^2} \int_{B_{2r}(y)} |\nabla^s (u - v)|^2 \, dx
\]

\[
\leq \frac{C}{N_0^2} \theta^2 r^m < \varepsilon |B_r|,
\]

where the last inequality is obtained by choosing \( \theta \) and thus also \( \delta \) sufficiently small.

This finishes our proof. \( \blacksquare \)

**Remark.** Note that in the above proof, the choice of \( \theta \) and thus also the choice of a sufficiently small \( \delta \) does not depend on the radius \( r \), which is due to the fact that \( |B_r| = cr^n \) for some constant \( c = c(n) > 0 \). This is vital in our further proof of the \( H^{s,p} \) regularity.

Next, we refine the statement of Lemma 6.2 in order make it applicable for proving the assumptions of Lemma 2.1.

**Corollary 6.3.** There is a constant \( N_1 = N_1(n,s,\nu,\lambda,\Lambda) > 1 \), such that the following holds. For any \( \varepsilon > 0 \) there exists some \( \delta = \delta(\varepsilon, n, s, \nu, \lambda, \Lambda) > 0 \), such that for any \( z \in B_1 \), any \( r \in (0,1) \) and any weak solution \( u \in H^s(B_{13}|\mathbb{R}^n) \) of the equation

\[ L_Au = \sum_{i=1}^{m} L_{D_i}g_i + f \text{ in } B_{13} \]

with \( A \) being \((\delta,7)\)-BMO and

\[ \left\{ x \in B_r(z) \mid \mathcal{M}_{B_{13}}(|\nabla^s u|^2)(x) > N_1^2 \right\} \cap B_1 \geq \varepsilon |B_r|, \]

we have

\[ B_r(z) \cap B_1 \subset \left\{ x \in B_1 \mid \mathcal{M}_{B_{13}}(|\nabla^s u|^2)(x) > 1 \right\} \]

\[ \cup \left\{ x \in B_1 \mid \mathcal{M}_{B_{13}} \left( |f|^2 + \sum_{i=1}^{m} |\nabla^s g_i|^2 \right) (x) > \delta^2 \right\}. \]

**Proof.** Let \( N_1 = N_1(\nu,\lambda,\Lambda,n,s) > 1 \) be given by Lemma 6.2. Fix \( \varepsilon > 0 \), \( r \in (0,1) \), \( z \in \mathbb{R}^n \) and consider the corresponding \( \delta = \delta(\varepsilon, n, s, \nu, \lambda, \Lambda) > 0 \) given by Lemma 6.2. We argue by contradiction. Assume that (39) is satisfied but that (40) is false, so that there exists some \( x_0 \in B_r(z) \cap B_1 \) such that

\[ x_0 \in B_r(z) \cap \left\{ x \in B_1 \mid \mathcal{M}_{B_{13}}(|\nabla^s u|^2)(x) \leq 1 \right\} \]

\[ \cap \left\{ x \in B_1 \mid \mathcal{M}_{B_{13}} \left( |f|^2 + \sum_{i=1}^{m} |\nabla^s g_i|^2 \right) (x) \leq \delta^2 \right\} \]

\[ \subset \left\{ x \in B_r(z) \mid \mathcal{M}_{B_{13}}(|\nabla^s u|^2)(x) \leq 1 \right\} \]

\[ \cap \left\{ x \in B_r(z) \mid \mathcal{M}_{B_{13}} \left( |f|^2 + \sum_{i=1}^{m} |\nabla^s g_i|^2 \right) (x) \leq \delta^2 \right\}. \]

Since moreover we have \( B_{12r}(z) \subset B_{13} \) and \( A \) is \((\delta,7r)\)-BMO, Lemma 6.2 with \( U = B_{13} \) yields

\[ \left| \left\{ x \in B_r(z) \mid \mathcal{M}_{B_{13}}(|\nabla^s u|^2)(x) > N_1^2 \right\} \cap B_1 \right| \]

\[ \leq \left| \left\{ x \in B_r(z) \mid \mathcal{M}_{B_{13}}(|\nabla^s u|^2)(x) > N_0^2 \right\} \right| < \varepsilon |B_r|, \]
Corollary

A with which contradicts (39).

Thus, using the induction assumption and the case

\[ L_{Ak} = \sum_{i=1}^{m} L_{Di} g_i + f \text{ in } B_{13} \]

with \( A \) being \((\delta, \tau)\)-BMO and

\[ \left\{ x \in B_1 \mid M_{B_{13}}(|\nabla^s u|^2)(x) > N_1^2 \right\} \]

we have

\[ \left| \left\{ x \in B_1 \mid M_{B_{13}}(|\nabla^s u|^2)(x) > N_1^{2k} \right\} \right| \]

\[ \leq \sum_{j=1}^{k} \frac{c^1}{j^2} \left| \left\{ x \in B_1 \mid M_{B_{13}} \left( |f|^2 + \sum_{i=1}^{m} |\nabla^s g_i|^2 \right)(x) > \delta^2 N_1^{2(k-j)} \right\} \right| \]

\[ + \frac{c^1}{k^2} \left| \left\{ x \in B_1 \mid M_{B_{13}}(|\nabla^s u|^2)(x) > 1 \right\} \right| . \]

Proof. We proof this Lemma by induction on \( k \). In view of (41) and Corollary 6.3, the case \( k = 1 \) is a direct consequence of Lemma 2.1 applied to the sets

\[ E := \left\{ x \in B_1 \mid M_{B_{13}}(|\nabla^s u|^2)(x) > N_1^2 \right\} \]

and

\[ F := \left\{ x \in B_1 \mid M_{B_{13}}(|\nabla^s u|^2)(x) > 1 \right\} \cup \left\{ x \in B_1 \mid M_{B_{13}} \left( |f|^2 + \sum_{i=1}^{m} |\nabla^s g_i|^2 \right)(x) > \delta^2 \right\} . \]

Next, assume that the conclusion is valid for some \( k \in \mathbb{N} \). Define \( \hat{u} := u/N_1 \), \( \hat{g}_i := g_i/N_1 \) and \( \hat{f} := f/N_1 \). Then \( \hat{u} \) clearly satisfies

\[ L_A \hat{u} = \sum_{i=1}^{m} L_{D_i} \hat{g}_i + \hat{f} \text{ weakly in } B_{13} . \]

Moreover, since \( N_1 > 1 \) we have

\[ \left| \left\{ x \in B_1 \mid M_{B_{13}}(|\nabla^s \hat{u}|^2)(x) > N_1^2 \right\} \right| = \left| \left\{ x \in B_1 \mid M_{B_{13}}(|\nabla^s u|^2)(x) > N_1^2 \right\} \right| \]

\[ \leq \left| \left\{ x \in B_1 \mid M_{B_{13}}(|\nabla^s u|^2)(x) > N_1^2 \right\} \right| < \varepsilon |B_1| \]

Thus, using the induction assumption and the case \( k = 1 \) we deduce

\[ \left| \left\{ x \in B_1 \mid M_{B_{13}}(|\nabla^s u|^2)(x) > N_1^{2(k+1)} \right\} \right| \]

\[ = \left| \left\{ x \in B_1 \mid M_{B_{13}}(|\nabla^s u|^2)(x) > N_1^{2k} \right\} \right| \]

\[ \leq \sum_{j=1}^{k} \frac{c^1}{j^2} \left| \left\{ x \in B_1 \mid M_{B_{13}} \left( |\hat{f}|^2 + \sum_{i=1}^{m} |\nabla^s \hat{g}_i|^2 \right)(x) > \delta^2 N_1^{2(k-j)} \right\} \right| \]

\[ + \frac{c^1}{k^2} \left| \left\{ x \in B_1 \mid M_{B_{13}}(|\nabla^s u|^2)(x) > 1 \right\} \right| . \]
\[
\leq \sum_{j=1}^{k} \varepsilon_j^* \left\{ x \in B_1 \mid \mathcal{M}_{B,13} \left( |f|^2 + \sum_{i=1}^{m} |\nabla^s g_i|^2 \right) (x) > \delta^2 N_1^{2(k+1-j)} \right\} \\
+ \varepsilon_j^* \left\{ x \in B_1 \mid \mathcal{M}_{B,13} \left( |f|^2 + \sum_{i=1}^{m} |\nabla^s g_i|^2 \right) (x) > \delta^2 \right\} \\
+ \varepsilon_j^* \left\{ x \in B_1 \mid \mathcal{M}_{B,13}(|\nabla^s u|^2)(x) > 1 \right\} \\
= \sum_{j=1}^{k+1} \varepsilon_j^* \left\{ x \in B_1 \mid \mathcal{M}_{B,13} \left( |f|^2 + \sum_{i=1}^{m} |\nabla^s g_i|^2 \right) (x) > \delta^2 N_1^{2(k+1-j)} \right\} \\
+ \varepsilon_j^* \left\{ x \in B_1 \mid \mathcal{M}_{B,13}(|\nabla^s u|^2)(x) > 1 \right\},
\]
so that the conclusion is valid for \( k + 1 \), which completes the proof. \( \blacksquare \)

We are now set to prove the higher integrability of \( \nabla^s u \) in the case of balls. The approach to
the proof can be summarized as follows. First of all, we consider an appropriately scaled version of
\( u \) that satisfies the condition \( (41) \) from Lemma \( 6.4 \) and also corresponding scaled versions of \( g_i \) and \( f \). Then we use Theorem \( 2.3 \) in order to derive from Lemma \( 6.4 \) the desired \( L^p \) estimate in terms of
the Hardy-Littlewood maximal functions of the scaled versions of \( u, g_i \) and \( f \), which in view of the
strong \( p-p \) estimates from Theorem \( 2.3 \) and rescaling then yields the desired \( L^p \) estimate for \( \nabla^s u \).

**Theorem 6.5.** Let \( 2 < p < \infty \), \( g_i \in H^{s,p}(B_{13} | \mathbb{R}^s \), \( f \in L^p(B_{13}) \) and \( \nu, \lambda, \Lambda > 0 \). Then there exists
some small enough \( \delta = \delta(p, n, s, \nu, \lambda, \Lambda) > 0 \), such that if a kernel coefficient \( A \in \mathcal{L}_{p}(\nu, \lambda) \) is \( (\delta, 7)\)-BMO, and if the \( D_i \) are symmetric and bounded by \( \Lambda \), then for any weak solution \( u \in H^s(B_{13} | \mathbb{R}^s \) of the equation
\[
L_{A} u = \sum_{i=1}^{m} L_{D_i} g_i + f \text{ in } B_{13}
\]
we have \( \nabla^s u \in L^p(B_{13}) \). Moreover, there exists a constant \( C = C(p, n, s, \nu, \lambda, \Lambda) > 0 \) such that
\[
\|\nabla^s u\|_{L^p(B_{13})} \leq C \left( \|f\| + \sum_{i=1}^{m} \|\nabla^s g_i\|_{L^p(B_{13})} + \|\nabla^s u\|_{L^2(B_{13})} \right).
\]

**Proof.** Fix \( p > 2 \) and let \( N_1 = N_1(n, s, \nu, \lambda, \Lambda) > 1 \) be given by Lemma \( 6.4 \). Moreover, select
\( \varepsilon \in (0, 1) \) such that
\[
N_1^{p} 10^n \varepsilon \leq 1 \frac{1}{2}.
\]
Consider also the corresponding \( \delta = \delta(\varepsilon, n, s, \nu, \lambda, \Lambda) > 0 \) given by Corollary \( 6.3 \). If \( \nabla^s u = 0 \) a.e. in
\( B_{13} \), then the assertion is trivially satisfied, so that we can assume \( \|\nabla^s u\|_{L^2(B_{13})} > 0 \). Next, we define
\[
\hat{u} := \frac{\gamma u}{\|\nabla^s u\|_{L^2(B_{13})}}, \quad \hat{g}_i := \frac{\gamma g_i}{\|\nabla^s u\|_{L^2(B_{13})}} \text{ and } \hat{f} := \frac{\gamma f}{\|\nabla^s u\|_{L^2(B_{13})}},
\]
where \( \gamma > 0 \) remains to be chosen independently of \( u, g_i \) and \( f \), note that we have
\[
L_{A} \hat{u} = \sum_{i=1}^{m} L_{D_i} \hat{g}_i + \hat{f} \text{ weakly in } B_{13}.
\]
Moreover, we have
\[
\int_{B_{13}} |\nabla^s \hat{u}|^2 dx = \gamma^2.
\]
Combining this observation with the weak 1-1 estimate from Theorem 2.3, it follows that there is a constant $C_1 = C_1(n) > 0$ such that

$$\| \{ x \in B_1 \mid \mathcal{M}_{B_{13}}(|\nabla^s\hat{u}|^2)(x) > N_1^2 \} \| \leq C_1 \frac{\gamma^2}{N_1^2} \leq \epsilon |B_1|,$$

where the last inequality is obtained by choosing $\gamma$ small enough. Therefore, all assumptions made in Lemma 6.4 are satisfied by $\hat{u}$. Furthermore, in view of Theorem 2.3 and the strong $p$-$p$ estimates for the Hardy-Littlewood maximal function (cf. Theorem 2.3), we deduce that there exist constants $C_2 = C_2(n, s, \nu, \lambda, \Lambda, p) > 0$ and $C_3 = C_3(n, p) > 0$ such that

$$\sum_{k=1}^{\infty} N_1^{p_k} \left\{ x \in B_1 \mid \mathcal{M}_{B_{13}} \left( |\hat{f}|^2 + \sum_{i=1}^{m} |\nabla^s\hat{g}_i|^2 \right)(x) > \delta^2 N_1^{2k} \right\}$$

$$\leq C_2 \mathcal{M}_{B_{13}} \left( |\hat{f}|^2 + \sum_{i=1}^{m} |\nabla^s\hat{g}_i|^2 \right) \|_{L^{p/2}(B_{13})}$$

$$\leq C_2 C_3^p \| \hat{f} + \sum_{i=1}^{m} \nabla^s\hat{g}_i \|_{L^p(B_{13})}.$$

Setting $\epsilon_1 := 10^n \epsilon$, by (43) we see that

$$\sum_{i=1}^{\infty} (N_1^p \epsilon_1)^i \leq \sum_{i=1}^{\infty} \left( \frac{1}{2} \right)^i = 1.$$

Using Lemma 6.4, the Cauchy product, (45), (44), and setting $C_4 := C_2 C_3^p$, we compute

$$\sum_{k=1}^{\infty} N_1^{p_k} \left\{ x \in B_1 \mid \mathcal{M}_{B_{13}}(|\nabla^s\hat{u}|^2)(x) > N_1^{2k} \right\}$$

$$\leq \sum_{k=1}^{\infty} N_1^{p_k} \left( \sum_{i=1}^{k} \epsilon_i \left\{ x \in B_1 \mid \mathcal{M}_{B_{13}}(|\hat{f}|^2)(x) + \sum_{i=1}^{m} |\nabla^s\hat{g}_i|^2(x) > \delta^2 N_1^{2(k-1)} \right\} \right)$$

$$+ \sum_{i=1}^{\infty} (N_1^p \epsilon_1)^i \left\{ x \in B_1 \mid \mathcal{M}_{B_{13}}(|\nabla^s\hat{u}|^2)(x) > 1 \right\}$$

$$= \left( \sum_{k=0}^{\infty} N_1^{p_k} \left\{ x \in B_1 \mid \mathcal{M}_{B_{13}}(|\hat{f}|^2)(x) + \sum_{i=1}^{m} |\nabla^s\hat{g}_i|^2(x) > \delta^2 N_1^{2k} \right\} \right) \left( \sum_{i=1}^{\infty} (N_1^p \epsilon_1)^i \right)$$

$$+ \left( \sum_{k=1}^{\infty} (N_1^p \epsilon_1)^k \right) \left\{ x \in B_1 \mid \mathcal{M}_{B_{13}}(|\nabla^s\hat{u}|^2)(x) > 1 \right\}$$

$$\leq \left( \sum_{k=1}^{\infty} N_1^{p_k} \left\{ x \in B_1 \mid \mathcal{M}_{B_{13}}(|\hat{f}|^2)(x) + \sum_{i=1}^{m} |\nabla^s\hat{g}_i|^2(x) > \delta^2 N_1^{2k} \right\} \right) + 2|B_1| \left( \sum_{i=1}^{\infty} (N_1^p \epsilon_1)^i \right)$$

$$\leq \sum_{k=1}^{\infty} N_1^{p_k} \left\{ x \in B_1 \mid \mathcal{M}_{B_{13}}(|\hat{f}|^2)(x) + \sum_{i=1}^{m} |\nabla^s\hat{g}_i|^2(x) > \delta^2 N_1^{2k} \right\} \right\} + 2|B_1|$$

$$\leq C_4 \| \hat{f} + \sum_{i=1}^{m} \nabla^s\hat{g}_i \|_{L^p(B_{13})} + 2|B_1|.$$

Therefore, by Theorem 2.3 and Theorem 2.4 we find that there exists another constant $C_5 = C_5(n, s, \nu, \lambda, \Lambda, p) > 0$ such that

$$\|\nabla^s\hat{u}\|_{L^p(B_i)} \leq \|\mathcal{M}_{B_{13}}(|\nabla^s\hat{u}|^2)|_{L^{p/2}(B_i)}.$$
are symmetric and bounded by Λ where by u we have

\[ u \in \Omega \]

\[ \text{if } \]

\[ \text{supported in } \Omega \]

\[ u \]

\[ \text{say that } \]

\[ \text{Theorem 7.1.} \]

\[ \text{Let } \]

\[ C \]

\[ U \]

\[ \text{Fix relatively compact bounded open sets } \]

\[ (46) \]

\[ \text{slightly stronger result in terms of the spaces } \]

\[ p \]

\[ \text{notion of local weak solutions.} \]

\[ \text{Proof.} \]

\[ \delta = \text{loc } \]

\[ \in \]

\[ \text{In view of the inclusions } \]

\[ \in \]

\[ \sum_{i=1}^n \{ x \in B_1 \mid M_{B_13}(|\nabla^s \hat{u}|^2)(x) > N_{12}k \} \mid |B_1| \}

\[ \leq C_5 \left( \sum_{k=1}^\infty N_1^{p_k} \left| \left\{ x \in B_1 \mid M_{B_13}(|\nabla^s \hat{u}|^2)(x) > N_{12}k \right\} \right| \right) \]

\[ \leq C_5 \left( C_4 \left( \| \hat{f} \|_{L^p(B_{13})} + \sum_{i=1}^m \| \nabla^s \hat{g}_i \|_{L^p(B_{13})} \right) \right) + 3|B_1| \]

\[ \leq C_6^p \left( \| \hat{f} + \sum_{i=1}^m \nabla^s \hat{g}_i \|_{L^p(B_{13})} + 1 \right) \]

where \( C_6 := (C_5 \max \{C_4, 3|B_1|\})^{1/p} \). It follows that

\[ \| \nabla^s \hat{u} \|_{L^p(B_1)} \leq C_6 \left( \| \hat{f} + \sum_{i=1}^m \nabla^s \hat{g}_i \|_{L^p(B_{13})} + 1 \right)^{1/p} \]

so that

\[ \| \nabla^s u \|_{L^p(B_1)} \leq C_6 \left( \| f + \sum_{i=1}^m \nabla^s g_i \|_{L^p(B_{13})} + \frac{\| \nabla^s u \|_{L^2(B_{13})}}{\gamma} \right) \]

\[ \leq C_6 \gamma^{-1} \left( \| f + \sum_{i=1}^m \nabla^s g_i \|_{L^p(B_{13})} + \| \nabla^s u \|_{L^2(B_{13})} \right) \]

which proves \( \text{[42]} \) with \( C := C_6 \gamma^{-1} \).

\[ \boxed{\text{7. PROOFS OF THE MAIN RESULTS}} \]

In order to state our main result on local regularity in an optimal way, we define the following notion of local weak solutions.

**Definition.** Let \( \Omega \subset \mathbb{R}^n \) be a domain. Given \( b \in L^\infty_{loc}(\Omega) \), \( f \in L^1_{loc}(\Omega) \) and \( g_i \in H^s_{loc}(\Omega;\mathbb{R}^n) \), we say that \( u \in H_s^{\ast}(\Omega;\mathbb{R}^n) \) is a local weak solution to the equation \( L_A u + bu = \sum_{i=1}^m L_{D_i} g_i + f \) in \( \Omega \), if

\[ \mathcal{E}_A(u, \varphi) + (bu, \varphi)_{L^2(\Omega)} = \sum_{i=1}^m \mathcal{E}_{D_i}(g_i, \varphi) + (f, \varphi)_{L^2(\Omega)} \quad \forall \varphi \in H_0^s(\Omega;\mathbb{R}^n), \]

where by \( H^s_\ast(\Omega;\mathbb{R}^n) \) we denote the set of all functions that belong to \( H^s(\Omega;\mathbb{R}^n) \) and are compactly supported in \( \Omega \).

In view of the inclusions

\[ H^{s,p}(\mathbb{R}^n) \subset H_{loc}^{s,p}(\Omega;\mathbb{R}^n) \subset H^{s,p}_{loc}(\Omega) \subset W^{s,p}_{loc}(\Omega) \]

for \( p \in [2, \infty) \) which we discussed in section 3, Theorem \( \boxed{[1]} \) follows directly from the following slightly stronger result in terms of the spaces \( H_{loc}^{s,p}(\Omega;\mathbb{R}^n) \) defined in section 3.

**Theorem 7.1.** Let \( \Omega \subset \mathbb{R}^n \) be a domain, \( p \in (2, \infty) \), \( s \in (0, 1) \), \( p_* = \max \left\{ \frac{pn}{n+ps}, 2 \right\} \), \( g_i \in H^{s,p}_{loc}(\Omega;\mathbb{R}^n) \), \( f \in L^{p_*}_{loc}(\Omega) \), \( b \in L^\infty_{loc}(\Omega) \) and \( R, \nu, \lambda, \Lambda > 0 \). Then there exists some small enough \( \delta = \delta(p, n, s, \nu, \lambda, \Lambda) > 0 \), such that if a kernel coefficient \( A \in L_0(\nu, \lambda) \) is \( (\delta, R)\)-BMO, and if the \( D_i \) are symmetric and bounded by \( \Lambda \), then for any local weak solution \( u \in H^s_{loc}(\Omega;\mathbb{R}^n) \) of the equation

\[ L_A u + bu = \sum_{i=1}^m L_{D_i} g_i + f \text{ in } \Omega \]

we have \( u \in H^{s,p}_{loc}(\Omega;\mathbb{R}^n) \).

**Proof.** Fix \( p \in (2, \infty) \). We first prove the result under the stronger assumption that \( f \in L^p_{loc}(\Omega) \). Fix relatively compact bounded open sets \( U \subset V \subset \Omega \). Moreover, fix a smooth domain \( U \), such
that $U \subset U_* \subset V$. Let $\delta = \delta(p, u, s, \nu, \Lambda) > 0$ be given by Theorem 6.5. Let $\tilde{f} := f - bu$, so that $u$ is a local weak solution of

$$L_A u = \sum_{i=1}^{m} L_{D_i} g_i + \tilde{f} \text{ in } \Omega.$$  

In particular, $u$ is a weak solution of (47) in $V$. For any $z \in V$, fix some small enough $r_z \in (0, 1)$ such that $B_{13r_z}(z) \subset V$ and $13r_z \leq R$, so that in particular $A$ is $(\delta, 13r_z)$-BMO. Define

$$A_z(x, y) := A(r_z x + z, r_z y + z), \quad D_z(x, y) := D_1(r_z x + z, r_z y + z),$$

and note that for any $z \in V$, $A_z$ belongs to the class $\mathcal{L}_0(\nu, \lambda)$ and is $(\delta, 13)$-BMO and that $u_z$ satisfies

$$L_{A_z} u_z = \sum_{i=1}^{m} L_{D_{iz}} g_{iz} + \tilde{f}_z \text{ weakly in } B_{13}.$$ 

Using Theorem 6.5, for any $q \in (2, \infty)$ we obtain the estimate

$$||\nabla^s u||_{L^q(B_{13r_z}(z))} = r_z^\frac{n}{2} ||\nabla^s u_z||_{L^q(B_1)}$$

$$\leq r_z^\frac{n}{2} C_1 \left( ||\tilde{f}_z + \sum_{i=1}^{m} \nabla^s g_{iz}||_{L^q(B_{13r_z})} + ||\nabla^s u_z||_{L^q(B_{13r_z})} \right)$$

$$= C_1 \left( ||\tilde{f}_z + \sum_{i=1}^{m} \nabla^s g_i||_{L^q(B_{13r_z}(z))} + ||\nabla^s u||_{L^q(B_{13r_z}(z))} \right)$$

$$\leq C_1 \max \{1, r_z^\frac{n}{2} \} \left( ||\tilde{f}_z||_{L^q(B_{13r_z}(z))} + ||\sum_{i=1}^{m} \nabla^s g_i||_{L^q(B_{13r_z}(z))} + ||\nabla^s u||_{L^q(B_{13r_z}(z))} \right),$$

where $C_1 = C_1(n, s, q, \nu, \lambda, \Lambda) > 0$. Since $\{B_{r_z}(z)\}_{z \in \overline{U}_*}$ is an open covering of $\overline{U}_*$ and $\overline{U}_*$ is compact, there is a finite subcover $\{B_{r_{z_i}}(x_i)\}_{i=1}^{k}$ of $\overline{U}_*$ and hence of $U_*$. Let $\{\phi_i\}_{i=1}^{k}$ be a partition of unity subordinate to the covering $\{B_{r_{z_i}}(z_i)\}_{i=1}^{k}$ of $\overline{U}_*$, that is, the $\phi_i$ are non-negative functions on $\mathbb{R}^n$, we have $\phi_i \in C_0^\infty(B_{r_{z_i}}(x_i))$ for all $i = 1, \ldots, k$, $\sum_{i=1}^{k} \phi_i \equiv 1$ in an open neighbourhood of $\overline{U}_*$ and $\sum_{i=1}^{k} \phi_i \leq 1$ in $\mathbb{R}^n$. Setting $C_2 := C_1 \max \{1, \max_{i=1, \ldots, k} r_{z_i}^\frac{n}{2} \}$ and summing the above estimates over $i = 1, \ldots, k$, we conclude

$$||\nabla^s u||_{L^q(U_*)} = \left( \sum_{i=1}^{k} ||\nabla^s u|\phi_i||_{L^q(U_*)} \right)$$

$$\leq \sum_{i=1}^{k} \left( ||\tilde{f}_z||_{L^q(B_{13r_z}(z_i))} + ||\sum_{j=1}^{m} \nabla^s g_{iz_j}||_{L^q(B_{13r_z}(z_i))} + ||\nabla^s u||_{L^q(B_{13r_z}(z_i))} \right)$$

$$\leq \sum_{i=1}^{k} C_2 \left( ||\tilde{f}_z||_{L^q(B_{13r_z}(z_i))} + ||\sum_{j=1}^{m} \nabla^s g_{iz_j}||_{L^q(B_{13r_z}(z_i))} + ||\nabla^s u||_{L^q(B_{13r_z}(z_i))} \right)$$

$$\leq \sum_{i=1}^{k} C_2 \left( ||\tilde{f}_z||_{L^q(V)} + ||\sum_{j=1}^{m} \nabla^s g_{iz_j}||_{L^q(V)} + ||\nabla^s u||_{L^q(V)} \right)$$

$$= C_2 k \left( ||\tilde{f}_z||_{L^q(V)} + ||\sum_{j=1}^{m} \nabla^s g_{iz_j}||_{L^q(V)} + ||\nabla^s u||_{L^q(V)} \right).$$
which implies that for any $q \in (2, \infty)$ we have

\begin{equation}
||\nabla^s u||_{L^q(U)} \leq C_3 \left( ||f||_{L^q(V)} + ||u||_{L^q(V)} + \sum_{i=1}^{m} ||\nabla^s g_i||_{L^q(V)} + ||\nabla^s u||_{L^2(V)} \right),
\end{equation}

where $C_3 = C_2 k \max\{1, ||b||_{L^\infty(V)}\}$. In particular, since by assumption and Theorem 3.3 we have $f, \nabla^s g_i \in L^p(V)$, for any $q \in [2, p]$ we have $\nabla^s u \in L^q(U)$ whenever $u \in L^q(V)$. For any $r \in [1, p]$, define

$$r^* := \begin{cases}
\min\left\{ \frac{r}{n-rs}, p \right\}, & \text{if } rs < n \\
p, & \text{if } rs \geq n,
\end{cases}$$

note that $r^* \in [1, p]$. By the embedding theorem of Bessel potential spaces (Theorem 3.4), for any $r \geq 1$ we have

$$H^{s,r}(U^*) \hookrightarrow L^{r^*}(U^*).$$

Since $u \in H^s(V)$, we have $u \in L^{2^*}(V)$ and therefore $\nabla^s u \in L^{2^*}(U^*)$. If $p = 2^*$, we have $u \in L^p(U^*)$, $\nabla^s u \in L^p(U^*)$ and therefore $u \in H^{s,p}(U^* \cap \mathbb{R}^n)$. If $p > 2^*$, then we have $u, \nabla^s u \in L^{2^*}(U^*)$, so that Theorem 3.3 yields $u \in H^{s,2^*}(U^*)$. We therefore arrive at $u \in L^{2^*}(U^*)$. By replacing $U^*$ with an arbitrary relatively compact open subset of $U$, we therefore obtain $\nabla^s u \in L^{2^*}(U)$. If $2^* = p$, then we have $u, \nabla^s u \in L^p(U)$. By replacing $U^*$ and therefore $u \in H^{s,p}(U \cap \mathbb{R}^n)$ and thereby $u \in H^{s,p}(U \cap \mathbb{R}^n)$ at some point. Since $U$ is an arbitrary relatively compact open subset of $\Omega$, we conclude that $u \in H^{s,p}_{loc}(\Omega \cap \mathbb{R}^n)$. This finishes the proof when $f \in L^p_{loc}(\Omega)$.

Next, consider the general case when $f \in L^p_{loc}(\Omega)$, where $p_* = \max\left\{ \frac{m}{n+ps}, 2 \right\}$. Define the function $f_\Omega : \mathbb{R}^n \to \mathbb{R}$ by

$$f_\Omega(x) := \begin{cases}
f(x), & \text{if } x \in \Omega \\
0, & \text{if } x \in \mathbb{R}^n \setminus \Omega
\end{cases}$$

and note that $f_\Omega \in L^{p_*}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. By Proposition 5.1 there exists a unique weak solution $g \in H^s(\mathbb{R}^n) \subset H^s_{loc}(\Omega \cap \mathbb{R}^n)$ of the equation

\begin{equation}
(-\Delta)^s g + g = f_\Omega \quad \text{in } \mathbb{R}^n,
\end{equation}

where

$$(-\Delta)^s g(x) = C_{n,s} \int_{\mathbb{R}^n} \frac{g(x) - g(y)}{|x-y|^{n+2s}} \, dy$$

is the fractional Laplacian. In view of the classical $H^{2\alpha,p}$ regularity for the fractional Laplacian on the whole space $\mathbb{R}^n$ (cf. for example Lemma 3.5 in [19]), we have $g \in H^{2\alpha,p}(\mathbb{R}^n) \hookrightarrow H^{s,p}(\mathbb{R}^n)$ and therefore in particular $g \in H^{s,p}_{loc}(\Omega \cap \mathbb{R}^n)$. Since furthermore $u$ is a local weak solution of

$$L_A u + bu = \left( \sum_{i=1}^{m} L_D g_i + (-\Delta)^s g \right) + g \text{ in } \Omega,$$

by the first part of the proof we obtain that $u \in H^{s,p}_{loc}(\Omega \cap \mathbb{R}^n)$. This finishes the proof.

**Proof of Theorem 1.2** Fix $p \in (2, \infty)$ and let $\delta = \delta(p, n, s, \nu, \lambda, \Lambda) > 0$ be given by Theorem 6.5. We first prove the result under the stronger assumption that $f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. For any $k \in \mathbb{Z}^n$, let $E_k := B_{\sqrt{\nu}}(k)$ and $F_k := B_{2\sqrt{\nu}}(k)$. We then have $\mathbb{R}^n = \bigcup_{k \in \mathbb{Z}^n} E_k$, moreover, there exists some $N \in \mathbb{N}$ depending only on $n$ such that no point in $\mathbb{R}^n$ is contained in more than $N$ of the balls $F_k$. In other words, we have $\sum_{k \in \mathbb{Z}^n} \chi_{F_k} \leq N$, where $\chi_{F_k}$ is the characteristic function of
\( F_k \). Since for \( \tilde{f} := f - bu \) we have

\[
L_Au = \sum_{i=1}^{m} L_{D_i}g_i + \tilde{f} \text{ weakly in } \mathbb{R}^n,
\]

by the same argument as in the proof of Theorem 7.1 for any \( k \in \mathbb{Z}^d \) and any \( q \in (2, \infty) \) we have

\[
||\nabla^s u||_{L^q(E_k)} \leq C \left( ||\tilde{f}||_{L^q(E_k)} + ||\sum_{i=1}^{m} \nabla^s g_i||_{L^q(E_k)} + ||\nabla^s u||_{L^q(E_k)} \right)
\]

for some constant \( C = C(n, s, q, \nu, \lambda, \Lambda) > 0 \). It follows that

\[
\int_{\mathbb{R}^n} |\nabla^s u(x)|^q dx \leq \sum_{k \in \mathbb{Z}^n} \int_{E_k} |\nabla^s u(x)|^q dx
\]

\[
\leq C^q \sum_{k \in \mathbb{Z}^n} \left( \int_{E_k} |\tilde{f}(x)| \right)^q dx + \left( \int_{E_k} \sum_{i=1}^{m} |\nabla^s g_i(x)|^q dx \right) + \left( \int_{E_k} |\nabla^s u(x)|^2 dx \right)^q \]

\[
\leq C_1 C_2 \left( \sum_{k \in \mathbb{Z}^n} \left( \int_{E_k} |\tilde{f}(x)|^q dx \right) + \sum_{k \in \mathbb{Z}^n} \left( \int_{E_k} |\nabla^s g_i(x)|^q dx \right) + \left( \int_{E_k} |\nabla^s u(x)|^2 dx \right)^q \right)
\]

\[
= C_1 C_2 \left( \int_{\mathbb{R}^n} |\tilde{f}(x)|^q dx + \sum_{i=1}^{m} |\nabla^s g_i(x)|^q \sum_{k \in \mathbb{Z}^d} \chi_{E_k}(x) dx \right) + \left( \int_{\mathbb{R}^n} |\nabla^s u(x)|^2 \sum_{k \in \mathbb{Z}^d} \chi_{E_k}(x) dx \right)^q \]

\[
\leq N^2 C_1 C_2 \left( \sum_{i=1}^{m} |\nabla^s g_i(x)|^q dx \right) + \left( \int_{\mathbb{R}^n} |\nabla^s u(x)|^2 dx \right)^q,
\]

where \( C_1 = C_1(q) > 0 \). This implies that for any \( q \in (2, \infty) \) we have

\[
||\nabla^s u||_{L^q(\mathbb{R}^n)} \leq C_2 \left( ||f||_{L^q(\mathbb{R}^n)} + ||u||_{L^q(\mathbb{R}^n)} + \sum_{i=1}^{m} ||\nabla^s g_i||_{L^q(\mathbb{R}^n)} + ||\nabla^s u||_{L^q(\mathbb{R}^n)} \right),
\]

where \( C_2 := N^2 C_1 C \max \{1, ||b||_{L^\infty(\mathbb{R}^n)} \} \). In particular, since for any \( q \in [2, p] \) we have \( L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n) \) and in view of the assumptions and Theorem 3.3 we have \( f, \nabla^s g_i \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \), for any \( q \in [2, p] \) it follows that \( \nabla^s u \in L^q(\mathbb{R}^n) \) whenever \( u \in L^q(\mathbb{R}^n) \). The proof in the case when \( f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \) can now be concluded by using essentially the same iteration argument as the one in the proof of Theorem 7.1. The general case when \( f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \) then can once again be treated by solving the equation (49) under optimal regularity, as we did in the proof of Theorem 7.1.

\[ \square \]

**Remark on boundary regularity.** An interesting question is if it is possible to prove a global \( H^{s,p} \) regularity result in smooth enough bounded domains \( \Omega \) corresponding to our local regularity result Theorem 7.1. Our approach is based on a \( C^{s+\gamma} \) estimate (\( \gamma > 0 \)) for nonlocal equations with translation invariant kernels, however it is known that already in the case of the fractional Laplacian in a unit ball the optimal regularity up to the boundary is \( C^s(\mathbb{B}_1) \), cf. section 7.1 in [24]. Therefore, at least with our methods proving such a global \( H^{s,p} \) regularity result for the equations we consider in this work seems to be unattainable even in the case when \( \Omega \) is very regular.
References


