$H^{s,p}$ REGULARITY THEORY FOR A CLASS OF NONLOCAL ELLIPTIC EQUATIONS

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Abstract. In this paper, we study the regularity of weak solutions to a class of nonlocal elliptic equations in Bessel potential spaces $H^{s,p}$. Our main results can be seen as an extension of the well-known $W^{1,p}$ regularity theory for local second-order elliptic equations in divergence form to the nonlocal setting.

Contents

1. Introduction

1.1. **Basic setting.** In this work, we study the regularity of weak solutions to nonlocal elliptic equations of the form

$$L_A u + bu = \sum_{i=1}^{m} L_{D_i} g_i + f \text{ in } \Omega \subset \mathbb{R}^n \tag{1}$$

in Bessel potential spaces $H^{s,p}$. Roughly speaking, the purpose of this paper is to prove the implication $u \in H^{s,2} \implies u \in H^{s,p}$ for the whole range of exponents $p \in (2,\infty)$ in the case

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of possibly very irregular data. Here $s \in (0,1)$, $\Omega \subset \mathbb{R}^n$ $(n > 2s)$ is a domain (= open set), $b, f, g_i : \mathbb{R}^n \to \mathbb{R}$ $(i = 1, ..., m,\ m \in \mathbb{N})$ are given functions and

$$L_A u(x) = 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{A(x,y)}{|x-y|^{n+2s}} (u(x) - u(y)) dy, \quad x \in \Omega,$$

is a nonlocal operator. Furthermore, the function $A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is measurable and we assume that there exists a constant $\lambda \geq 1$ such that

$$\lambda^{-1} \leq A(x,y) \leq \lambda \text{ for almost all } x, y \in \mathbb{R}^n. \tag{2}$$

Moreover, we require A to be symmetric, i.e.

$$A(x,y) = A(y,x) \text{ for almost all } x, y \in \mathbb{R}^n. \tag{3}$$

We call such a function A a kernel coefficient. We define $\mathcal{L}_0(\lambda)$ as the class of all such measurable kernel coefficients A that satisfy the conditions (2) and (3). Note that in our main results, we additionally assume that A is translation invariant, cf. section 1.3. Moreover, throughout this work $D_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ $(i = 1, ..., m)$ are assumed to be measurable functions that are symmetric and bounded by some $\Lambda > 0$, i.e.

$$\sum_{i=1}^m |D_i(x,y)| \leq \Lambda \text{ for almost all } x, y \in \mathbb{R}^n. \tag{4}$$

Define the spaces

$$H^s(\Omega|\mathbb{R}^n) = \left\{ u : \mathbb{R}^n \to \mathbb{R} \text{ measurable } \;\Big|\; \int_\Omega u(x)^2 dx + \int_\Omega \int_{\mathbb{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}} dy dx < \infty \right\}$$

and

$$H^s_0(\Omega|\mathbb{R}^n) = \{ u \in H^s(\Omega|\mathbb{R}^n) \mid u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \}.$$

For all measurable functions $u, \varphi : \mathbb{R}^n \to \mathbb{R}$ we define the bilinear form associated to the operator L_A by

$$\mathcal{E}_A(u,\varphi) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{A(x,y)}{|x-y|^{n+2s}} (u(x)-u(y))(\varphi(x)-\varphi(y)) dy dx,$$

provided that the above expression is well-defined and finite, this is e.g. the case if $u \in H^s(\Omega|\mathbb{R}^n)$ and $\varphi \in H^s_0(\Omega|\mathbb{R}^n)$. Analogously we consider the bilinear forms $\mathcal{E}_{D_i}(u,\varphi)$ associated to the operators L_{D_i}.

Definition. *Given $b \in L^\infty(\Omega)$, $f \in L^2(\Omega)$ and $g_i \in H^s(\Omega|\mathbb{R}^n)$, $i = 1, ..., m$, we say that $u \in H^s(\Omega|\mathbb{R}^n)$ is a weak solution to the equation $L_A u + bu = \sum_{i=1}^m L_{D_i} g_i + f$ in Ω, if*

$$\mathcal{E}_A(u,\varphi) + (bu,\varphi)_{L^2(\Omega)} = \sum_{i=1}^m \mathcal{E}_{D_i}(g_i,\varphi) + (f,\varphi)_{L^2(\Omega)} \quad \forall \varphi \in H^s_0(\Omega|\mathbb{R}^n).$$

1.2. **Some previous results.** Studying the regularity of weak solutions to equations of the form (1) has been a very active area of research in recent years. Results concerning Hölder regularity were e.g. obtained in [18], [13], [31], [22], [29] and [28], while results concerning higher differentiability in Sobolev spaces were e.g. obtained in [12] and [5]. Regarding higher integrability, in [3] and [1] it was shown that under the assumptions from section 1.1 there exists some small $\sigma > 0$, such that for any weak solution u of $L_A u = f$ in $\mathbb{R}^n$ the function

$$\nabla^s u(x) = \left(\int_{\mathbb{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}} dy \right)^{\frac{1}{2}} \tag{5}$$

belongs to $L^{2+\sigma}(\mathbb{R}^n)$ whenever $f \in L^2(\mathbb{R}^n)$. In view of a classical characterization of Bessel potential spaces due to Stein (cf. Theorem 3.3), this actually implies that u belongs to the Bessel potential space $H^{s,2+\sigma}(\mathbb{R}^n)$. Similar results were proved in [21], [30] and [1], where it was shown that under the assumptions from section 1.1 u actually not only possesses a higher integrability but also a slightly higher differentiabillity.

1.3. **Main results.** The aim of this work is to prove the $H^{s,p}$ regularity for solutions u to equations of the form (1) not only for some $p > 2$ close enough to 2, but for the full range $p \in (2, \infty)$. In order to accomplish this, we restrict our attention to the following class of translation invariant kernel coefficients.

Definition. *Let $\lambda \geq 1$. We say that a kernel coefficient $A \in \mathcal{L}_0(\lambda)$ belongs to the class $\mathcal{L}_1(\lambda)$, if there exists a measurable function $a : \mathbb{R}^n \to \mathbb{R}$ such that $A(x, y) = a(x - y)$ for all $x, y \in \mathbb{R}^n$, that is, if A is translation invariant.*

Our main result concerning local regularity in Bessel potential spaces $H^{s,p}$ is the following.

Theorem 1.1. (Local $H^{s,p}$ regularity in domains)
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $p \in (2, \infty)$, $s \in (0, 1)$, $b \in L^\infty(\Omega)$, $g_i \in H^s(\Omega|\mathbb{R}^n) \cap H^{s,p}(\mathbb{R}^n)$ and $f \in L^{p_\star}(\Omega)$, where $p_\star = \max\left\{\frac{pn}{n+ps}, 2\right\}$. If A belongs to the class $\mathcal{L}_1(\lambda)$ and if all D_i are symmetric and satisfy (4), then for any weak solution $u \in H^s(\Omega|\mathbb{R}^n)$ of the equation

$$L_A u + bu = \sum_{i=1}^{m} L_{D_i} g_i + f \text{ in } \Omega \tag{6}$$

we have $u \in H^{s,p}_{loc}(\Omega)$ and $u \in W^{s,p}_{loc}(\Omega)$.

Remark. We actually obtain a slightly stronger result (cf. Theorem 7.1) than the one given by Theorem 1.1 in terms of certain function spaces $H^{s,p}(\Omega|\mathbb{R}^n)$ that generalize the space $H^s(\Omega|\mathbb{R}^n)$ to the case when $p \neq 2$, cf. section 3. By a useful alternative characterization of Bessel potential spaces (cf. Theorem 3.3), this space $H^{s,p}(\Omega|\mathbb{R}^n)$ is actually contained in $H^{s,p}(\Omega)$ whenever Ω is regular enough, so that this result then implies Theorem 1.1. Moreover, since Theorem 1.1 is concerned with local regularity, the above result remains true if we generalize the notion of weak solutions to an appropriate notion of local weak solutions, cf. section 7.

Remark. Although in Theorem 1.1 and in our other main results we are primarily concerned with regularity in Bessel potential spaces $H^{s,p}$, due to the classical embedding $H^{s,p} \hookrightarrow W^{s,p}$ for $p \in [2, \infty)$ (cf. Proposition 3.2), we also obtain regularity in Sobolev-Slobodeckij spaces $W^{s,p}$.

If the equation is posed on the whole space $\mathbb{R}^n$, we are actually able to establish the following global regularity result.

Theorem 1.2. ($H^{s,p}$ regularity on the whole space $\mathbb{R}^n$)
Let $p \in (2, \infty)$, $s \in (0, 1)$, $b \in L^\infty(\mathbb{R}^n)$, $g_i \in H^s(\mathbb{R}^n) \cap H^{s,p}(\mathbb{R}^n)$ and $f \in L^2(\mathbb{R}^n) \cap L^{p_\star}(\mathbb{R}^n)$, where $p_\star = \max\left\{\frac{pn}{n+ps}, 2\right\}$. If A belongs to $\mathcal{L}_1(\lambda)$ and if all D_i are symmetric and satisfy (4), then for any weak solution $u \in H^s(\mathbb{R}^n)$ of the equation

$$L_A u + bu = \sum_{i=1}^{m} L_{D_i} g_i + f \text{ in } \mathbb{R}^n \tag{7}$$

we have $u \in H^{s,p}(\mathbb{R}^n)$ and $u \in W^{s,p}(\mathbb{R}^n)$.

In the case when $b = g_i = 0$ $(i = 1, ..., m)$, by writing $A(x, y) = a(x - y)$, Theorem 1.2 can also be deduced by applying [2, Theorem 1] to the symbol

$$M(\xi) = \frac{\int_{\mathbb{R}^n} (\cos(\xi \cdot y) - 1) a^{-1}(y) V(dy)}{\int_{\mathbb{R}^n} (\cos(\xi \cdot y) - 1) V(dy)}, \quad V(dy) = \frac{a(y)}{|y|^{n+2s}} dy.$$

The main achievement of this paper is that we develop the $H^{s,p}$ regularity theory for weak solutions of the equation (1) for a general right-hand side and especially in the setting of arbitrary domains $\Omega \subset \mathbb{R}^n$. This is because although in the case of local elliptic equations local regularity in domains Ω can be deduced from the corresponding result in $\mathbb{R}^n$ by using a cutoff argument, in the nonlocal setting such a cutoff argument requires an additional assumption on the solution in the complement of Ω (cf. [4] or [19]), which is not required in Theorem 1.1. Another advantage of our approach is that it can be generalized to an appropriate class of nonlinear nonlocal equations. We plan to do this in a future work.

We also want to mention that in [15] a somewhat related result was proved. Building on an approach used by Krylov in [20] to obtain $W^{1,p}$ estimates for local second-order elliptic equations, the authors obtained $H^{2s,p}$ a priori estimates for strong solutions of the equation $L_A = f$ in $\mathbb{R}^n$.

In order to prove our main results, we instead apply a variation of another approach commonly used in order to obtain $W^{1,p}$ regularity results for local elliptic equations in divergence form which we briefly describe in section 1.4. This enables us to simultaneously treat the problems of local $H^{s,p}$ regularity in domains Ω and the problem of global $H^{s,p}$ regularity on the whole space $\mathbb{R}^n$.

1.4. **$W^{1,p}$ regularity theory for second-order elliptic equations in divergence form.** Let us briefly review the well-known $W^{1,p}$ regularity theory for local second-order elliptic equations in divergence form treated for example in [7], [8] or [9], where the authors build on an approach first introduced by Caffarelli and Peral in [10]. Consider the equation

$$\operatorname{div}(B\nabla u) = \operatorname{div} g + f \quad \text{in } \Omega, \tag{8}$$

where $\Omega \subset \mathbb{R}^n$ is a domain, the matrix of coefficients $B : \mathbb{R}^n \to \mathbb{R}^{n \times n}$, $B(x) = (B_{ij}(x))_{i,j=1,...,n}$ has measurable entries, is uniformly elliptic and bounded, while $g : \Omega \to \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ are given functions. Furthermore, solutions are understood in an appropriate weak sense, cf. [17] or [7]. Let $2 < p < \infty$. A natural question corresponding to Theorem 1.1 to ask in this context is the following: Under which assumptions on B, f and g does any weak solution $u \in H^1_{loc}(\Omega)$ of (8) in fact belong to the space $u \in W^{1,p}_{loc}(\Omega)$? The minimal assumptions on g and f for this property to hold are $g \in L^p_{loc}(\Omega, \mathbb{R}^n)$ and $f \in L^{\frac{np}{n+p}}_{loc}(\Omega)$, while the coefficients B_{ij} are required to have small enough BMO-seminorms, cf. [7]. The strategy to obtain such local $W^{1,p}$ estimates used e.g. in [7], [8] or [9] is as follows. One approximates the gradient of the weak solution u of (8) in L^2 by the gradient of a weak solution v of a suitable equation $\operatorname{div}(B_0 \nabla u) = 0$, where B_0 has constant coefficients. One then uses the fact that v satisfies a local $C^{0,1}$ estimate along with a real-variable argument based on the Vitali covering lemma, the Hardy-Littlewood maximal function and an alternative characterization of L^p spaces in order to obtain an L^p estimate for ∇u, which in view of interpolation then implies the desired local $W^{1,p}$ estimate.

The main idea of our approach in the nonlocal setting is to apply similar arguments with the gradient ∇u replaced by the nonlocal s-gradient $\nabla^s u$ defined in (5). However, due to the nonlocal nature of the operator ∇^s and the equations we consider, in our setting we have to overcome a number of difficulties that are not present in the local case.

1.5. **Some notation.** For convenience, let us fix some notation which we use throughout the paper. By C and C_i, $i \in \mathbb{N}$, we always denote positive constants, while dependences on parameters of the constants will be shown in parentheses. As usual, by

$$B_r(x_0) := \{x \in \mathbb{R}^n \mid |x - x_0| < r\}$$

we denote the open ball with center $x_0 \in \mathbb{R}^n$ and radius $r > 0$. Moreover, if $E \subset \mathbb{R}^n$ is measurable, then by $|E|$ we denote the n-dimensional Lebesgue-measure of E. If $0 < |E| < \infty$, then for any $u \in L^1(E)$ we define

$$\overline{u}_E := \fint_E u(x)dx := \frac{1}{|E|} \int_E u(x)dx.$$

2. Some tools from real analysis

In this section, we discuss some results from real analysis that will play key roles in our treatment of the $H^{s,p}$ regularity theory for nonlocal elliptic equations.

The following result can be proved by using the well-known Vitali covering lemma, cf. [7, Theorem 2.7].

Lemma 2.1. *Assume that E and F are measurable sets in $\mathbb{R}^n$ that satisfy $E \subset F \subset B_1$. Assume further that there exists some $\varepsilon \in (0,1)$ such that*

$$|E| < \varepsilon |B_1|,$$

and that for all $x \in B_1$ and any $r \in (0,1)$ with $|E \cap B_r(x)| \geq \varepsilon |B_r(x)|$ we have

$$B_r(x) \cap B_1 \subset F.$$

Then we have

$$|E| \leq 10^n \varepsilon |F|.$$

Another tool we use is the Hardy-Littlewood maximal function.

Definition. *Let $f \in L^1_{loc}(\mathbb{R}^n)$. Then the Hardy-Littlewood maximal function $\mathcal{M}f : \mathbb{R}^n \to [0, \infty]$ of f is defined by*

$$\mathcal{M}f(x) := \mathcal{M}(f)(x) := \sup_{\rho > 0} \fint_{B_\rho(x)} |f(y)|dy.$$

Moreover, for any domain $\Omega \subset \mathbb{R}^n$ and any function $f \in L^1(\Omega)$, consider the zero extension of f to $\mathbb{R}^n$

$$f_\Omega(x) := \begin{cases} f(x), & \text{if } x \in \Omega \\ 0 & , \text{if } x \notin \Omega. \end{cases}$$

We then define

$$\mathcal{M}_\Omega f := \mathcal{M} f_\Omega.$$

Rather straightforward but important features of the Hardy-Littlewood maximal function are its scaling and translation invariance, given by the following Lemma which can be proved by using a change of variables.

Lemma 2.2. *Let $f \in L^1_{loc}(\mathbb{R}^n)$, $r > 0$ and $y \in \mathbb{R}^n$. Then for the function $f_{r,y}(x) := f(rx + y)$ and any $x \in \mathbb{R}^n$ we have*

$$\mathcal{M}f_{r,y}(x) = \mathcal{M}f(rx + y).$$

Similarly, for any domain $\Omega \subset \mathbb{R}^n$, any function $f \in L^1(\Omega)$ and any $x \in \Omega$ we have

$$\mathcal{M}_{\Omega'} f_{r,y}(x) = \mathcal{M}_\Omega f(rx+y),$$

where $\Omega' := \{\frac{x-y}{r} \mid x \in \Omega\}$.

We remark that for any $f \in L^1_{loc}(\mathbb{R}^n)$, $\mathcal{M}f$ is Lebesgue-measurable. Intuitively, the Hardy-Littlewood maximal function of a function f in general seems to be much larger than the function f itself. However, the following results show that when measured appropriately, the size of $\mathcal{M}f$ can actually be controlled by the size of f, cf. [33].

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a domain.*

(i) (weak 1-1 estimate) If $f \in L^1(\Omega)$ and $t > 0$, then

$$|\{x \in \Omega \mid \mathcal{M}_\Omega(f)(x) > t\}| \leq \frac{C}{t} \int_\Omega |f| dx,$$

where $C = C(n) > 0$.

(ii) (strong p-p estimates) If $f \in L^p(\Omega)$ for some $p \in (1, \infty]$, then

$$||f||_{L^p(\Omega)} \leq ||\mathcal{M}_\Omega f||_{L^p(\Omega)} \leq C||f||_{L^p(\Omega)},$$

where $C = C(n,p) > 0$.

(iii) If $f \in L^p(\Omega)$ for some $p \in [1, \infty]$, then the function $\mathcal{M}_\Omega f$ is finite almost everywhere.

We conclude this section by giving an alternative characterization of L^p spaces, cf. [11, Lemma 7.3]. It can be proved by using the well-known formula

$$||f||^p_{L^p(\Omega)} = p \int_0^\infty t^{p-1} \left|\{x \in \Omega \mid f(x) > t\}\right| dt.$$

Lemma 2.4. *Let $0 < p < \infty$. Furthermore, suppose that f is a nonnegative and measurable function in a bounded domain $\Omega \subset \mathbb{R}^n$ and let $\tau > 0$, $\beta > 1$ be constants. Then for*

$$S := \sum_{k=1}^\infty \beta^{kp} |\{x \in \Omega \mid f(x) > \tau \beta^k\}|,$$

we have

$$C^{-1} S \leq ||f||^p_{L^p(\Omega)} \leq C(|\Omega| + S)$$

for some constant $C = C(\tau, \beta, p) > 0$. In particular, we have $f \in L^p(\Omega)$ if and only if $S < \infty$.

3. Fractional Sobolev spaces and the s-gradient

We start this section by defining a first type of fractional Sobolev spaces which is probably the most widely used type of such spaces in the literature concerned with elliptic equations.

Definition. *Let $\Omega \subset \mathbb{R}^n$ be a domain. For $p \in [1, \infty)$ and $s \in (0,1)$, we define the Sobolev-Slobodeckij space*

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) \;\Big|\; \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dy dx < \infty \right\}$$

with norm

$$||u||_{W^{s,p}(\Omega)} := \left(\int_\Omega |u(x)|^p dx + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dy dx \right)^{1/p}.$$

Moreover, we define the corresponding local Sobolev-Slobodeckij spaces by

$$W^{s,p}_{loc}(\Omega) := \{u \in L^p_{loc}(\Omega) \mid u \in W^{s,p}(\Omega') \text{ for any domain } \Omega' \subset\subset \Omega\}.$$

Finally, set

$$H^s(\Omega) := W^{s,2}(\Omega).$$

Remark. The space $H^s(\Omega)$ is a Hilbert space with respect to the inner product

$$(u,v)_{H^s(\Omega)} := (u,v)_{L^2(\Omega)} + \int_\Omega \int_\Omega \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} dydx. \tag{9}$$

We use the following fractional Poincaré inequality, cf. [14, Lemma 3.10].

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $s \in (0,1)$. For any $u \in H^s(\Omega)$ we have*

$$\int_\Omega |u(x) - \overline{u}_\Omega|^2 \, dx \leq C \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dydx,$$

where $C = C(s,\Omega) > 0$.

We also use the following type of fractional Sobolev spaces.

Definition. *For $p \in [1,\infty)$ and $s \in \mathbb{R}$, consider the Bessel potential space*

$$H^{s,p}(\mathbb{R}^n) := \left\{ u \in L^p(\mathbb{R}^n) \mid \mathcal{F}^{-1}\left[\left(1+|\xi|^2\right)^{\frac{s}{2}} \mathcal{F}f \right] \in L^p(\mathbb{R}^n) \right\},$$

where $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. We equip $H^{s,p}(\mathbb{R}^n)$ with the norm

$$||u||_{H^{s,p}(\mathbb{R}^n)} := \left|\left| \mathcal{F}^{-1}\left[\left(1+|\xi|^2\right)^{\frac{s}{2}} \mathcal{F}f \right] \right|\right|_{L^p(\mathbb{R}^n)}.$$

Moreover, for any domain $\Omega \subset \mathbb{R}^n$ we define

$$H^{s,p}(\Omega) := \left\{ u\big|_\Omega \mid u \in H^{s,p}(\mathbb{R}^n) \right\}$$

with norm

$$||u||_{H^{s,p}(\Omega)} := \inf \left\{ ||v||_{H^{s,p}(\mathbb{R}^n)} \mid v\big|_\Omega = u \right\}.$$

Furthermore, we define the corresponding local Bessel potential spaces by

$$H^{s,p}_{loc}(\Omega) := \left\{ u \in L^p_{loc}(\Omega) \mid u \in H^{s,p}(\Omega') \text{ for any domain } \Omega' \subset\subset \Omega \right\}.$$

The following result gives some relations between Bessel potential spaces and Sobolev-Slobodeckij spaces.

Proposition 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a domain.*

- *(i) If Ω is a bounded Lipschitz domain or $\Omega = \mathbb{R}^n$, then for all $s \in (0,1)$, $p \in (1,2]$ we have $W^{s,p}(\Omega) \hookrightarrow H^{s,p}(\Omega)$.*
- *(ii) For any $s \in (0,1)$ and any $p \in [2,\infty)$ we have $H^{s,p}(\Omega) \hookrightarrow W^{s,p}(\Omega)$.*

For a proof of Proposition 3.2, we refer to Theorem 5 in chapter V of [33] for the case when $\Omega = \mathbb{R}^n$. For general domains Ω, part (i) then follows by extending an arbitrary function $u \in W^{s,p}(\Omega)$ to a function that belongs to $W^{s,p}(\mathbb{R}^n)$, for which an additional assumption on Ω is required, cf. [23, Theorem 5.4]. Part (ii) follows similarly by extending an arbitrary function $u \in H^{s,p}(\Omega)$ to a function that belongs to $H^{s,p}(\mathbb{R}^n)$, which by definition of $H^{s,p}(\Omega)$ is possible for arbitrary domains.

We now define a function that can be viewed as a nonlocal analogue to the euclidean norm of the gradient of a function in the local context.

Definition. *Let $s \in (0,1)$. For any domain $\Omega \subset \mathbb{R}^n$ and any measurable function $u : \Omega \to \mathbb{R}$, we define the s-gradient $\nabla^s_\Omega u : \Omega \to [0,\infty]$ by*

$$\nabla^s_\Omega u(x) := \left(\int_\Omega \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}} dy \right)^{\frac{1}{2}}.$$

Moreover, for any measurable function $u : \mathbb{R}^n \to \mathbb{R}$ we define $\nabla^s u := \nabla^s_{\mathbb{R}^n} u$.

In view of Proposition 3.2, for any bounded Lipschitz domain Ω we have $u \in H^{s,2}(\Omega)$ if and only if $u \in L^2(\Omega)$ and $\nabla^s_\Omega u \in L^2(\Omega)$. The following result shows that a similar alternative characterization of Bessel potential spaces in terms of the s-gradient is also true for a much wider range of exponents p. This characterization was first given by Stein in [32] in the case when $\Omega = \mathbb{R}^n$. For the case when Ω is an arbitrary Lipschitz domain we refer to [25, Theorem 1.3] or [24, Theorem 2.10], where this characterization is proved in the more general context of Triebel-Lizorkin spaces and so-called uniform domains.

Theorem 3.3. *Let $s \in (0,1)$, $p \in \left(\frac{2n}{n+2s}, \infty\right)$ and assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain or that $\Omega = \mathbb{R}^n$. Then we have $u \in H^{s,p}(\Omega)$ if and only if $u \in L^p(\Omega)$ and $\nabla^s_\Omega u \in L^p(\Omega)$. Moreover, we have*

$$||u||_{H^{s,p}(\Omega)} \simeq ||u||_{L^p(\Omega)} + ||\nabla^s_\Omega u||_{L^p(\Omega)}$$

in the sense of equivalent norms.

We remark that the above result holds in particular for any $p \geq 2$.

Even though we primarily work in some domain $\Omega \subset \mathbb{R}^n$, we obtain most results in this work in terms of the global s-gradient ∇^s instead of the localized s-gradient ∇^s_Ω, which is mostly due to the nonlocal character of the equations we consider. In order to state our main result in domains in an optimal way (cf. Theorem 7.1), we therefore also define the following natural nonstandard function spaces.

Definition. *Let $\Omega \subset \mathbb{R}^n$ be a domain. For $p \in [1,\infty)$ and $s \in (0,1)$, we define the linear space*

$$H^{s,p}(\Omega|\mathbb{R}^n) := \{u : \mathbb{R}^n \to \mathbb{R} \text{ measurable } \mid u \in L^p(\Omega) \text{ and } \nabla^s u \in L^p(\Omega)\}.$$

Moreover, we define the corresponding local spaces by

$$H^{s,p}_{loc}(\Omega|\mathbb{R}^n) = \{u : \mathbb{R}^n \to \mathbb{R} \text{ measurable } \mid u \in H^{s,p}(\Omega'|\mathbb{R}^n) \text{ for any domain } \Omega' \subset\subset \Omega\}.$$

Also, we use the spaces

$$H^{s,p}_0(\Omega|\mathbb{R}^n) := \{u \in H^{s,p}(\Omega|\mathbb{R}^n) \mid u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}.$$

Furthermore, set

$$H^s(\Omega|\mathbb{R}^n) := H^{s,2}(\Omega|\mathbb{R}^n),\ H^s_{loc}(\Omega|\mathbb{R}^n) := H^{s,2}_{loc}(\Omega|\mathbb{R}^n) \text{ and } H^s_0(\Omega|\mathbb{R}^n) := H^{s,2}_0(\Omega|\mathbb{R}^n).$$

Remark. Since for any $u \in H^s_0(\Omega|\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} u(x)^2 dx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}} dydx \leq \int_\Omega u(x)^2 dx + 2\int_\Omega (\nabla^s u(x))^2 dx < \infty,$$

$H^s_0(\Omega|\mathbb{R}^n)$ clearly is a closed subspace of $H^s(\mathbb{R}^n)$ and thus also a Hilbert space with respect to the inner product $(u,v)_{H^s(\mathbb{R}^n)}$ defined in (9).

Remark. In view of Theorem 3.3, for any bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ and all $s \in (0,1)$, $p \in \left(\frac{2n}{n+2s}, \infty\right)$ we have the inclusions

$$H^{s,p}(\mathbb{R}^n) \subset H^{s,p}(\Omega|\mathbb{R}^n) \subset H^{s,p}(\Omega).$$

In the case when $\Omega \subset \mathbb{R}^n$ is an arbitrary domain this implies the inclusions

$$H^{s,p}(\mathbb{R}^n) \subset H^{s,p}_{loc}(\Omega|\mathbb{R}^n) \subset H^{s,p}_{loc}(\Omega).$$

We also use the following embedding theorems of Bessel potential spaces. Parts (i) and (ii) follow from [34, Remark 1.96 (iii)], while the last two parts follow from the corresponding embeddings of Sobolev-Slobodeckij spaces (cf. [23]) and part (ii) of Proposition 3.2.

Theorem 3.4. *Let $1 < p \leq p_1 < \infty$, $s, s_1 \geq 0$ and assume that $\Omega \subset \mathbb{R}^n$ is a domain.*

(i) If $sp < n$, then for any $q \in [p, \frac{np}{n-sp}]$ we have

$$H^{s,p}(\Omega) \hookrightarrow L^q(\Omega).$$

(ii) More generally, if $s - \frac{n}{p} = s_1 - \frac{n}{p_1}$, then

$$H^{s,p}(\Omega) \hookrightarrow H^{s_1,p_1}(\Omega).$$

(iii) If $sp = n$, then for any $q \in [p, \infty)$ we have

$$H^{s,p}(\Omega) \hookrightarrow L^q(\Omega).$$

(iv) If $sp > n$, then we have

$$H^{s,p}(\Omega) \hookrightarrow C^\alpha(\Omega),$$

where $\alpha = s - \frac{n}{p}$.

4. Some preliminary regularity results

For the rest of this paper, we fix real numbers $s \in (0,1)$ and $\lambda \geq 1$.

4.1. L^∞ **estimates.** The following Lemma relates the nonlocal tail of a function that often appears naturally in the literature to the L^2 norm of its s-gradient.

Lemma 4.1. *For all $r, R > 0$ and any $u \in H^s(B_R|\mathbb{R}^n)$ we have*

$$\int_{\mathbb{R}^n \setminus B_r} \frac{u(y)^2}{|y|^{n+2s}} dy \leq C(||\nabla^s u||^2_{L^2(B_R)} + ||u||^2_{L^2(B_R)}), \tag{10}$$

where $C = C(n,s,r,R) > 0$.

Proof. First of all, integration in polar coordinates yields

$$\int_{\mathbb{R}^n \setminus B_r} \frac{dz}{|z|^{n+2s}} = \omega_n \int_r^\infty \frac{\rho^{n-1}}{\rho^{n+2s}} d\rho = \frac{\omega_n}{2sr^{2s}} =: C_1 < \infty, \tag{11}$$

where ω_n denotes the surface area of the $n-1$ dimensional unit sphere $\mathbb{S}^{n-1}$. By the Cauchy-Schwarz inequality, Fubini's theorem and (11) we have

$$\begin{aligned}\int_{\mathbb{R}^n \setminus B_r} \frac{u(y)^2}{|y|^{n+2s}} dy &= \int_{\mathbb{R}^n \setminus B_r} \frac{\left(u(y) - \fint_{B_R} u(x)dx + \fint_{B_R} u(x)dx\right)^2}{|y|^{n+2s}} dy \\ &\leq 2 \left(\int_{\mathbb{R}^n \setminus B_r} \frac{\left(\fint_{B_R} (u(x) - u(y))dx\right)^2}{|y|^{n+2s}} dy + \int_{\mathbb{R}^n \setminus B_r} \frac{\left(\fint_{B_R} u(x)dx\right)^2}{|y|^{n+2s}} dy \right)\end{aligned}$$

$$\leq 2 \left(\int_{\mathbb{R}^n \setminus B_r} \fint_{B_R} \frac{(u(x)-u(y))^2}{|y|^{n+2s}} dxdy + \int_{\mathbb{R}^n \setminus B_r} \frac{\fint_{B_R} u^2(x)dx}{|y|^{n+2s}} dy \right)$$
$$= \frac{2}{|B_R|} \left(\int_{B_R} \int_{\mathbb{R}^n \setminus B_r} \frac{(u(x)-u(y))^2}{|y|^{n+2s}} dydx + C_1 \int_{B_R} u^2(x)dx \right).$$

Moreover, since for any $x \in B_R$ and any $y \in \mathbb{R}^n \setminus B_r$ we have

$$|x-y| \leq |x| + |y| < R + |y| = \left(\frac{R}{|y|} + 1 \right) |y| \leq \left(\frac{R}{r} + 1 \right) |y|,$$

we see that

$$\int_{B_R} \int_{\mathbb{R}^n \setminus B_r} \frac{(u(x)-u(y))^2}{|y|^{n+2s}} dydx \leq C_2 \int_{B_R} \int_{\mathbb{R}^n \setminus B_r} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}} dydx \leq C_2 \int_{B_R} (\nabla^s u)^2(x)dx,$$

where $C_2 = \left(\frac{R}{r}+1\right)^{n+2s}$. By combining the above computations, we see that (10) holds with $C = \frac{2}{|B_R|} \max\{C_2, C_1\}$. ■

We also use the following local L^∞ estimate for weak solutions to homogeneous nonlocal equations, cf. [13, Theorem 1.1]. We remark that although in [13] the below result is stated under the stronger assumption that $u \in H^s(\mathbb{R}^n)$, an inspection of the proof shows that this is not necessary.

Theorem 4.2. *Consider a kernel coefficient $A \in \mathcal{L}_0(\lambda)$. For all $0 < r < R < \infty$ and any weak solution $u \in H^s(B_R|\mathbb{R}^n)$ of the equation*

$$L_A u = 0 \text{ in } B_R$$

we have the estimate

$$||u||_{L^\infty(B_r)} \leq C \left(\int_{\mathbb{R}^n \setminus B_r} \frac{|u(y)|}{|y|^{n+2s}} dy + ||u||_{L^2(B_R)} \right),$$

where $C = C(n,s,r,R,\lambda) > 0$.

By combining the above two results, we obtain the following.

Corollary 4.3. *Consider a kernel coefficient $A \in \mathcal{L}_0(\lambda)$. For all $0 < r < R < \infty$ and any weak solution $u \in H^s(B_R|\mathbb{R}^n)$ of the equation*

$$L_A u = 0 \text{ in } B_R$$

we have the estimate

$$||u||_{L^\infty(B_r)} \leq C(||\nabla^s u||_{L^2(B_R)} + ||u||_{L^2(B_R)}), \tag{12}$$

where $C = C(n,s,r,R,\lambda) > 0$.

Proof. By Theorem 4.2, (11) and Lemma 4.1 we have

$$||u||_{L^\infty(B_r)} \leq C_1 \left(\int_{\mathbb{R}^n \setminus B_r} \frac{|u(y)|}{|y|^{n+2s}} dy + ||u||_{L^2(B_R)} \right)$$
$$\leq C_1 \left(C_2^{\frac{1}{2}} \left(\int_{\mathbb{R}^n \setminus B_r} \frac{u(y)^2}{|y|^{n+2s}} dy \right)^{\frac{1}{2}} + ||u||_{L^2(B_R)} \right)$$
$$\leq C_1 \left(C_2^{\frac{1}{2}} C_3^{\frac{1}{2}} + 1 \right) \left(||\nabla^s u||_{L^2(B_R)} + ||u||_{L^2(B_R)} \right),$$

where C_1 is given by Theorem 4.2, C_2 is given by (11) and C_3 is given by Lemma 4.1. This proves (12) with $C = C_1\left(C_2^{\frac{1}{2}}C_3^{\frac{1}{2}}+1\right)$. ■

Corollary 4.4. *Consider a kernel coefficient $A \in \mathcal{L}_0(\lambda)$. For all $0 < r < R < \infty$ and any weak solution $u \in H^s(B_R|\mathbb{R}^n)$ of the equation*

$$L_A u = 0 \text{ in } B_R$$

we have the estimate

$$||\nabla^s_{\mathbb{R}^n \setminus B_R} u||_{L^\infty(B_r)} \leq C||\nabla^s u||_{L^2(B_R)}, \tag{13}$$

where $C = C(n,s,r,R,\lambda) > 0$.

Proof. For any $x \in B_r$ and any $y \in \mathbb{R}^n \setminus B_R$ we have

$$|y| \leq |x-y| + |x| < |x-y| + R = \left(1 + \frac{R}{|x-y|}\right)|x-y| \leq \left(1 + \frac{R}{R-r}\right)|x-y|.$$

For almost every $x \in B_r$, it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^n \setminus B_R} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}} dy \\
\leq& C_1 \int_{\mathbb{R}^n \setminus B_R} \frac{(u(x)-u(y))^2}{|y|^{n+2s}} dy \\
\leq& 2C_1 \left(\int_{\mathbb{R}^n \setminus B_R} \frac{u(x)^2}{|y|^{n+2s}} dy + \int_{\mathbb{R}^n \setminus B_R} \frac{u(y)^2}{|y|^{n+2s}} dy \right) \\
\leq& 2C_1 \left(C_2 ||u||^2_{L^\infty(B_r)} + C_3 \left(\int_{B_R} \int_{\mathbb{R}^n} \frac{(u(z)-u(y))^2}{|z-y|^{n+2s}} dydz + \int_{B_R} u^2(z) dz \right) \right) \\
\leq& 2C_1(C_2C_4 + C_3) \left(\int_{B_R} \int_{\mathbb{R}^n} \frac{(u(z)-u(y))^2}{|z-y|^{n+2s}} dydz + \int_{B_R} u^2(z) dz \right),
\end{aligned}$$

where $C_1 := \left(1 + \frac{R}{R-r}\right)^{n+2s}$, $C_2 = C_2(n,s,R)$ is given as in (11) in the proof of Lemma 4.1, while $C_3 = C_3(n,s,R)$ is given by Lemma 4.1 and $C_4 = C_4(n,s,\lambda,r,R)$ is given by Corollary 4.3. Set $C_5 := 2C_1(C_2C_4 + C_3)$. Since the function $u - \overline{u}_{B_R} \in H^s(B_R|\mathbb{R}^n)$ also solves the equation

$$L_A(u - \overline{u}_{B_R}) = 0 \text{ weakly in } B_R,$$

the above estimate also applies to the function $u - \overline{u}_{B_R}$, so that together with the fractional Poincaré inequality (Lemma 3.1) for almost every $x \in B_r$ we deduce

$$\begin{aligned}
|\nabla^s_{\mathbb{R}^n \setminus B_R} u(x)|^2 &= \int_{\mathbb{R}^n \setminus B_R} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}} dy = \int_{\mathbb{R}^n \setminus B_R} \frac{((u(x)-\overline{u}_{B_R}) - (u(y)-\overline{u}_{B_R}))^2}{|x-y|^{n+2s}} dy \\
&\leq C_5 \left(\int_{B_R} \int_{\mathbb{R}^n} \frac{((u(z)-\overline{u}_{B_R}) - (u(y)-\overline{u}_{B_R}))^2}{|z-y|^{n+2s}} dydz + \int_{B_R} (u(z)-\overline{u}_{B_R})^2 dz \right) \\
&\leq C_5 \left(\int_{B_R} \int_{\mathbb{R}^n} \frac{(u(z)-u(y))^2}{|z-y|^{n+2s}} dydz + C_6 \int_{B_R} \int_{B_R} \frac{(u(z)-u(y))^2}{|z-y|^{n+2s}} dydz \right) \\
&\leq C_7 ||\nabla^s u||^2_{L^2(B_R)},
\end{aligned}$$

where $C_6 = C_6(n,s,R)$ and $C_7 := C_5(1 + C_6)$, which proves (13) with $C = C_7^{\frac{1}{2}}$. ■

4.2. **Higher Hölder regularity.** In the basic case when $A \in \mathcal{L}_0(\lambda)$, it can be shown that any weak solution to the corresponding homogeneous nonlocal equation is C^α for some $\alpha > 0$, cf. [13, Theorem

1.2]. The following result shows that if A is of class $\mathcal{L}_1(\lambda)$, then weak solutions to the corresponding homogeneous nonlocal equation enjoy better Hölder regularity than in general.

Theorem 4.5. *Consider a kernel coefficient $A \in \mathcal{L}_1(\lambda)$ and assume that $u \in H^s(B_5|\mathbb{R}^n)$ is a weak solution of the equation $L_A u = 0$ in B_5. Then for any $0 < \alpha < \min\{2s, 1\}$ we have*

$$[u]_{C^\alpha(B_3)} \leq C||\nabla^s u||_{L^2(B_5)},$$

where $C = C(n, s, \lambda, \alpha) > 0$ and

$$[u]_{C^\alpha(B_3)} := \sup_{\substack{x,y \in B_3 \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

We will derive Theorem 4.5 from the following analogue of [6, Theorem 5.2], where a corresponding result is proved for weak solutions to the fractional p-Laplace equation.

Theorem 4.6. *Consider a kernel coefficient $A \in \mathcal{L}_1(\lambda)$ and assume that $u \in H^s(B_5|\mathbb{R}^n)$ is a weak solution of the equation $L_A u = 0$ in B_5. Then for any $0 < \alpha < \min\{2s, 1\}$ we have*

$$[u]_{C^\alpha(B_3)} \leq C \left(||u||_{L^\infty(B_4)} + \int_{B_4} \int_{B_4} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dy dx + \int_{\mathbb{R}^n \setminus B_4} \frac{|u(y)|}{|y|^{n+2s}} dy \right),$$

where $C = C(n, s, \lambda, \alpha) > 0$.

Since the proof in [6] is done only in the case when $A(x, y) \equiv 1$ but naturally applies to the setting of arbitrary kernel coefficients $A \in \mathcal{L}_1(\lambda)$, let us briefly explain the modifications that are necessary in order to prove the result in this more general setting. Fix $0 < r < R$, $h \in \mathbb{R}^n \setminus \{0\}$ such that $|h| \leq \frac{R-r}{2}$ and a test function $\varphi \in H^s_0(B_{(R+r)/2}|\mathbb{R}^n)$. Moreover, suppose that $u \in H^s(B_R|\mathbb{R}^n)$ is a weak solution of

$$L_A u = 0 \text{ in } B_R. \tag{14}$$

Since the function $\varphi_{-h}(x) := \varphi(x - h)$ belongs to $H^s_0(B_R|\mathbb{R}^n)$, we can use φ_{-h} as a test function in (14). Setting $u_h(x) := u(x + h)$, along with a change of variables and the assumption that $A \in \mathcal{L}_1(\lambda)$ this yields

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{A(x, y)}{|x - y|^{n+2s}} (u(x) - u(y))(\varphi_{-h}(x) - \varphi_{-h}(y)) dy dx \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{A(x + h, y + h)}{|x - y|^{n+2s}} (u_h(x) - u_h(y))(\varphi(x) - \varphi(y)) dy dx \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{A(x, y)}{|x - y|^{n+2s}} (u_h(x) - u_h(y))(\varphi(x) - \varphi(y)) dy dx. \end{aligned} \tag{15}$$

Moreover, testing (14) with φ yields

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{A(x, y)}{|x - y|^{n+2s}} (u(x) - u(y))(\varphi(x) - \varphi(y)) dy dx = 0. \tag{16}$$

By subtracting (16) from (15) and dividing by $|h| > 0$, we obtain

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{A(x, y)}{|x - y|^{n+2s}} \frac{(u_h(x) - u_h(y)) - (u(x) - u(y))}{|h|} (\varphi(x) - \varphi(y)) dy dx = 0, \tag{17}$$

which corresponds to formula (4.3) in [6, Proposition 4.1]. The further proof of Theorem 4.6 can now be done in almost exactly the same way as in section 4 and 5 of [6] by additionally using the bounds (2) of A when appropriate.

Proof of Theorem 4.5. Since $u_0 := u - \overline{u}_{B_5} \in H^s(B_5|\mathbb{R}^n)$ also solves the equation

$$L_A u_0 = 0 \text{ weakly in } B_5,$$

we have

$$\begin{aligned}
[u]_{C^\alpha(B_3)} &= [u_0]_{C^\alpha(B_3)} \\
&\leq C_1 \left(||u_0||_{L^\infty(B_4)} + \int_{B_4}\int_{B_4} \frac{|u_0(x)-u_0(y)|^2}{|x-y|^{n+2s}} dydx + \int_{\mathbb{R}^n \setminus B_4} \frac{|u_0(y)|}{|y|^{n+2s}} dy \right) \\
&\leq C_1 \left(C_2(||\nabla^s u_0||_{L^2(B_5)} + ||u_0||_{L^2(B_5)}) + ||\nabla^s u_0||_{L^2(B_5)} + C_3 \left(\int_{\mathbb{R}^n \setminus B_4} \frac{|u_0(y)|^2}{|y|^{n+2s}} dy \right)^{\frac{1}{2}} \right) \\
&\leq C_1(C_2 + 1 + C_3 C_4) \left(||\nabla^s u||_{L^2(B_5)} + ||u_0||_{L^2(B_5)} \right) \\
&\leq C_1(C_2 + 1 + C_3 C_4)(1 + C_5) ||\nabla^s u||_{L^2(B_5)},
\end{aligned}$$

where $C_1 = C_1(n,s,\lambda,\alpha)$ is given by Theorem 4.6, $C_2 = C_2(n,s,\lambda)$ is given by Corollary 4.3, $C_3 = C_3(n,s)$ is given by (11), $C_4 = C_4(n,s)$ is given by Lemma 4.1 and $C_5 = C_5(n,s)$ is given by the fractional Poincaré inequality (Lemma 3.1). This proves Theorem 4.5 with $C = C_1(C_2 + 1 + C_3C_4)(1 + C_5)$. ■

Remark. Theorem 4.5 can also be proved by the following alternative approach. In the case when u belongs to $L^\infty(\mathbb{R}^n)$ and is a weak solution of an inhomogeneous equation of the form $L_A = f$ in B_4 with $f \in L^\infty(B_4)$, the additional Hölder regularity from Theorem 4.5 can be proved by essentially the same approach used to prove [28, Theorem 1.1], cf. the lecture notes [26]. Theorem 4.5 can then be deduced by a cutoff argument similar to the one applied in [29, Corollary 2.4].

5. The Dirichlet problem

In what follows, we fix measurable functions $D_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ $(i = 1, ..., m)$ that are symmetric and bounded by some $\Lambda > 0$.

Proposition 5.1. *Consider a kernel coefficient $A \in \mathcal{L}_0(\lambda)$. Let $\Omega \subset \mathbb{R}^n$ be a domain, $g_i, h \in H^s(\Omega|\mathbb{R}^n)$, $f \in L^2(\Omega)$, $b \in L^\infty(\Omega)$ and $l := \operatorname{ess\,inf}_{x\in\Omega} b(x)$. If Ω is bounded, then we assume that $l \geq 0$, otherwise we assume that $l > 0$. Then there exists a unique solution $u \in H^s(\Omega|\mathbb{R}^n)$ of the weak Dirichlet problem*

$$\begin{cases} L_A u + bu = \sum_{i=1}^m L_{D_i} g_i + f & \text{weakly in } \Omega \\ u = h & \text{a.e. in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{18}$$

Moreover, if Ω is bounded and $b \equiv 0$, then u satisfies the estimate

$$||\nabla^s u||_{L^2(\Omega)} \leq C \left(||\nabla^s h||_{L^2(\Omega)} + \sum_{i=1}^m ||\nabla^s g_i||_{L^2(\Omega)} + ||f||_{L^2(\Omega)} \right), \tag{19}$$

where $C = C(n,s,\lambda,\Lambda,|\Omega|)$.

Proof. Consider the symmetric bilinear form

$$\mathcal{E} : H^s_0(\Omega|\mathbb{R}^n) \times H^s_0(\Omega|\mathbb{R}^n) \to \mathbb{R}, \quad \mathcal{E}(w,\varphi) := \mathcal{E}_A(w,\varphi) + (bw,\varphi)_{L^2(\Omega)}.$$

First of all, fix some $w \in H^s_0(\Omega|\mathbb{R}^n)$. We have

$$\mathcal{E}(w,w) \leq \lambda \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(w(x)-w(y))^2}{|x-y|^{n+2s}} dydx + ||b||_{L^\infty(\Omega)} ||w||^2_{L^2(\Omega)} \leq \max\{\lambda, ||b||_{L^\infty(\Omega)}\} ||w||^2_{H^s(\mathbb{R}^n)}.$$

Let us first consider the case when Ω is unbounded, in this case we have $l > 0$ and therefore

$$\mathcal{E}(w,w) \geq \lambda^{-1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(w(x)-w(y))^2}{|x-y|^{n+2s}} dydx + l||w||^2_{L^2(\mathbb{R}^n)} \geq C_1 ||w||^2_{H^s(\mathbb{R}^n)},$$

where $C_1 = \min\{\lambda^{-1}, l\} > 0$. If Ω is bounded, then we have $l \geq 0$. Since we have $w = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$ and $w \in H^s(\mathbb{R}^n)$, in this case Hölder's inequality and the fractional Sobolev inequality (cf. [23, Theorem 6.5]) yield

$$\begin{aligned} \int_{\mathbb{R}^n} w^2 dx = \int_\Omega w^2 dx &\leq |\Omega|^{\frac{2s}{n}} \left(\int_\Omega |w|^{\frac{2n}{n-2s}} dx \right)^{\frac{n-2s}{n}} \\ &\leq C_2 |\Omega|^{\frac{2s}{n}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(w(x)-w(y))^2}{|x-y|^{n+2s}} dydx, \end{aligned} \tag{20}$$

where $C_2 = C_2(n,s) > 0$. We deduce

$$\begin{aligned} \mathcal{E}(w,w) &\geq \lambda^{-1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(w(x)-w(y))^2}{|x-y|^{n+2s}} dydx \\ &\geq \frac{\lambda^{-1}}{2} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(w(x)-w(y))^2}{|x-y|^{n+2s}} dydx + C_2^{-1} |\Omega|^{-\frac{2s}{n}} \int_{\mathbb{R}^n} w^2 dx \right) \geq C_3 ||w||^2_{H^s(\mathbb{R}^n)}, \end{aligned}$$

where $C_3 = \frac{\lambda^{-1}}{2} \min\left\{1, C_2^{-1}|\Omega|^{-\frac{2s}{n}}\right\} > 0$. We obtain that in both cases $\mathcal{E}(\cdot,\cdot)$ is positive definite and hence an inner product in $H^s_0(\Omega|\mathbb{R}^n)$ that is equivalent to the inner product $(\cdot,\cdot)_{H^s(\mathbb{R}^n)}$ defined in section 3. Therefore $H^s_0(\Omega|\mathbb{R}^n)$ with the inner product $\mathcal{E}(\cdot,\cdot)$ is a Hilbert space. Since moreover by Hölder's inequality the expression

$$-\mathcal{E}_A(h,\varphi) - (bh,\varphi)_{L^2(\Omega)} + \sum_{i=1}^m \mathcal{E}_{D_i}(g_i,\varphi) + (f,\varphi)_{L^2(\Omega)}$$

is a bounded linear functional of $\varphi \in H^s_0(\Omega|\mathbb{R}^n)$, by the Riesz representation theorem there exists a unique $w \in H^s_0(\Omega|\mathbb{R}^n)$ such that

$$\begin{aligned} &\mathcal{E}_A(w,\varphi) + (bw,\varphi)_{L^2(\Omega)} \\ = &-\mathcal{E}_A(h,\varphi) - (bh,\varphi)_{L^2(\Omega)} + \sum_{i=1}^m \mathcal{E}_{D_i}(g_i,\varphi) + (f,\varphi)_{L^2(\Omega)} \quad \forall \varphi \in H^s_0(\Omega|\mathbb{R}^n). \end{aligned} \tag{21}$$

But then the function $u := w + h \in H^s(\Omega|\mathbb{R}^n)$ solves the Dirichlet problem (18). Furthermore, if u and v both solve the Dirichlet problem (18), then $u-h$ and $v-h$ both satisfy (21), so that by the uniqueness part of the Riesz representation theorem we deduce $u-h = v-h$ a.e. in $\mathbb{R}^n$ and therefore $u = v$ a.e. in $\mathbb{R}^n$, so that the Dirichlet problem (18) has a unique solution.

Let us now prove that if Ω is bounded and $b \equiv 0$, then the unique solution $u \in H^s_0(\Omega|\mathbb{R}^n)$ of (18) satisfies the estimate (19). In order to accomplish this, note that by (20) for any $w \in H^s_0(\Omega|\mathbb{R}^n)$ we have

$$\begin{aligned} \int_\Omega |f(x)||w(x)|dx &\leq ||f||_{L^2(\Omega)} ||w||_{L^2(\Omega)} \\ &\leq C_2^{\frac{1}{2}} |\Omega|^{\frac{s}{n}} ||f||_{L^2(\Omega)} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(w(x)-w(y))^2}{|x-y|^{n+2s}} dydx \right)^{\frac{1}{2}} \\ &\leq 2C_2^{\frac{1}{2}} |\Omega|^{\frac{s}{n}} ||f||_{L^2(\Omega)} ||\nabla^s w||_{L^2(\Omega)}. \end{aligned}$$

Since $w := u - h \in H_0^s(\Omega|\mathbb{R}^n)$ satisfies (21), using $\varphi = w$ as a test function in (21) along with the Cauchy-Schwarz inequality yields

$$
\begin{aligned}
||\nabla^s w||^2_{L^2(\Omega)} \leq & \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(w(x)-w(y))^2}{|x-y|^{n+2s}} dydx \\
\leq & \lambda \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A(x,y) \frac{(w(x)-w(y))^2}{|x-y|^{n+2s}} dydx \\
= & \lambda \Big(- \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A(x,y) \frac{(h(x)-h(y))(w(x)-w(y))}{|x-y|^{n+2s}} dydx \\
& + \sum_{i=1}^m \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} D_i(x,y) \frac{(g_i(x)-g_i(y))(w(x)-w(y))}{|x-y|^{n+2s}} dydx + \int_\Omega f(x)w(x)dx \Big) \\
\leq & \lambda \Big(\lambda \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|h(x)-h(y)||w(x)-w(y)|}{|x-y|^{n+2s}} dydx \\
& + \Lambda \sum_{i=1}^m \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|g_i(x)-g_i(y)||w(x)-w(y)|}{|x-y|^{n+2s}} dydx + \int_\Omega |f(x)||w(x)|dx \Big) \\
\leq & 2\lambda \max\{\lambda, \Lambda, 2C_2^{\frac{1}{2}}|\Omega|^{\frac{s}{n}}\} \Big(\int_\Omega \int_{\mathbb{R}^n} \frac{|h(x)-h(y)||w(x)-w(y)|}{|x-y|^{n+2s}} dydx \\
& + \sum_{i=1}^m \int_\Omega \int_{\mathbb{R}^n} \frac{|g_i(x)-g_i(y)||w(x)-w(y)|}{|x-y|^{n+2s}} dydx + ||f||_{L^2(\Omega)} ||\nabla^s w||_{L^2(\Omega)} \Big) \\
\leq & C_4 ||\nabla^s w||_{L^2(\Omega)} \left(||\nabla^s h||_{L^2(\Omega)} + \sum_{i=1}^m ||\nabla^s g_i||_{L^2(\Omega)} + ||f||_{L^2(\Omega)} \right),
\end{aligned}
$$

where $C_4 := 2\lambda \max\{\lambda, \Lambda, 2C_2^{\frac{1}{2}}|\Omega|^{\frac{s}{n}}\}$. We obtain

$$
\begin{aligned}
||\nabla^s u||_{L^2(\Omega)} & \leq 2(||\nabla^s w||_{L^2(\Omega)} + ||\nabla^s h||_{L^2(\Omega)}) \\
& \leq 2 \left(C_4 \left(||\nabla^s h||_{L^2(\Omega)} + \sum_{i=1}^m ||\nabla^s g_i||_{L^2(\Omega)} + ||f||_{L^2(\Omega)} \right) + ||\nabla^s h||_{L^2(\Omega)} \right) \\
& \leq C \left(||\nabla^s h||_{L^2(\Omega)} + \sum_{i=1}^m ||\nabla^s g_i||_{L^2(\Omega)} + ||f||_{L^2(\Omega)} \right),
\end{aligned}
$$

where $C = 2(C_4 + 1)$. ■

For a treatment of the nonlocal Dirichlet problem for a much more general class of kernels, we refer to [16].

6. Higher integrabillity of $\nabla^s u$

For the rest of this paper, we assume that the kernel coefficient A belongs to the class $\mathcal{L}_1(\lambda)$.

6.1. **An approximation argument.** A key step in the proof of the higher integrability of $\nabla^s u$ is given by the following approximation lemma.

Lemma 6.1. *Let M be an arbitrary positive real number. For any $\varepsilon > 0$ there exists some $\delta = \delta(\varepsilon, n, s, \lambda, \Lambda, M) > 0$, such that for any weak solution $u \in H^s(B_5|\mathbb{R}^n)$ of the equation*

$$L_A u = \sum_{i=1}^m L_{D_i} g_i + f \ \textit{in}\ B_5$$

under the assumptions that

$$\fint_{B_5} |\nabla^s u|^2 dx \leq M \tag{22}$$

and that

$$\fint_{B_5} \left(f^2 + \sum_{i=1}^m |\nabla^s g_i|^2 \right) dx \leq M\delta^2, \tag{23}$$

there exists a weak solution $v \in H^s(B_5|\mathbb{R}^n)$ of the equation

$$L_A v = 0 \text{ in } B_5 \tag{24}$$

that satisfies

$$||\nabla^s (u-v)||_{L^2(B_5)} \leq \varepsilon. \tag{25}$$

Moreover, v satisfies the estimate

$$||\nabla^s v||_{L^\infty(B_2)} \leq N_0 \tag{26}$$

for some constant $N_0 = N_0(n,s,\lambda,\Lambda,M)$.

Proof. Fix $\varepsilon > 0$ and let $\delta > 0$ to be chosen. Let $v \in H^s(B_5|\mathbb{R}^n)$ be the unique weak solution of the problem

$$\begin{cases} L_A v = 0 & \text{weakly in } B_5 \\ v = u & \text{a.e. in } \mathbb{R}^n \setminus B_5, \end{cases} \tag{27}$$

note that v exists by Proposition 5.1. Observe that we have

$$\begin{cases} L_A(u-v) = \sum_{i=1}^m L_{D_i} g_i + f & \text{weakly in } B_5 \\ u - v = 0 & \text{a.e. in } \mathbb{R}^n \setminus B_5. \end{cases} \tag{28}$$

Thus, by the estimate (19) from Proposition 5.1 and (23), there exists a constant $C_1 = C_1(n,s,\lambda,\Lambda)$ such that

$$\int_{B_5} |\nabla^s (u-v)|^2 dx \leq C_1 \left(\sum_{i=1}^m \int_{B_5} |\nabla^s g_i|^2 dx + \int_{B_5} f^2 dx \right) \leq C_1 |B_5| M\delta^2 \leq \varepsilon^2. \tag{29}$$

where the last inequality follows by choosing δ sufficiently small. This completes the proof of (25). Let us now proof the estimate (26). For almost every $x \in B_2$, by Corollary 4.4 we have

$$\int_{\mathbb{R}^n \setminus B_3} \frac{(v(x)-v(y))^2}{|x-y|^{n+2s}} dy \leq C_2 \int_{B_3} \int_{\mathbb{R}^n} \frac{(v(z)-v(y))^2}{|z-y|^{n+2s}} dydz,$$

where $C_2 = C_2(n,s,\lambda)$. Now choose $\gamma > 0$ small enough such that $\gamma < s$ and $s + \gamma < 1$. In view of the assumption that $A \in \mathcal{L}_1(\lambda)$, by Theorem 4.5 we have

$$[v]_{C^{s+\gamma}(B_3)} \leq C_3 ||\nabla^s v||_{L^2(B_5)}$$

for some constant $C_3 = C_3(n,s,\lambda,\gamma)$. Thus, for almost every $x \in B_2$ we have

$$\begin{aligned} \int_{B_3} \frac{(v(x)-v(y))^2}{|x-y|^{n+2s}} dy &\leq [v]^2_{C^{s+\gamma}(B_3)} \int_{B_3} \frac{dy}{|x-y|^{n-2\gamma}} \\ &= C_4 [v]^2_{C^{s+\gamma}(B_3)} \leq C_4 C_3^2 \int_{B_5} \int_{\mathbb{R}^n} \frac{(v(z)-v(y))^2}{|z-y|^{n+2s}} dydz, \end{aligned}$$

where $C_4 = C_4(n,\gamma) < \infty$. Applying the estimate (19) from Proposition 5.1 to (27) yields

$$\int_{B_5}\int_{\mathbb{R}^n} \frac{(v(z)-v(y))^2}{|z-y|^{n+2s}}dydz \leq C_5 \int_{B_5}\int_{\mathbb{R}^n} \frac{(u(z)-u(y))^2}{|z-y|^{n+2s}}dydz,$$

where $C_5 = C_5(n,s,\lambda,\Lambda)$. By combining the above estimates, along with (22) we conclude that

$$\begin{aligned}(\nabla^s v)^2(x) &= \int_{\mathbb{R}^n\setminus B_3} \frac{(v(x)-v(y))^2}{|x-y|^{n+2s}}dy + \int_{B_3} \frac{(v(x)-v(y))^2}{|x-y|^{n+2s}}dy \\ &\leq C_2 \int_{B_5}\int_{\mathbb{R}^n} \frac{(v(z)-v(y))^2}{|z-y|^{n+2s}}dydz + C_4C_3^2 \int_{B_5}\int_{\mathbb{R}^n} \frac{(v(z)-v(y))^2}{|z-y|^{n+2s}}dydz \\ &\leq C_5(C_2+C_4C_3^2) \int_{B_5}\int_{\mathbb{R}^n} \frac{(u(z)-u(y))^2}{|z-y|^{n+2s}}dydz \\ &\leq C_5(C_2+C_4C_3^2)|B_5|M\end{aligned}$$

for almost every $x \in B_2$, so that (26) holds with $N_0 = (C_5(C_2+C_4C_3^2)|B_5|M)^{\frac{1}{2}}$. ■

6.2. **A real variable argument.** We now combine the above approximation lemma with the techniques from section 2.

Lemma 6.2. *There is a constant $N_1 = N_1(n,s,\lambda,\Lambda) > 1$, such that the following holds. For any $\varepsilon > 0$ there exists some $\delta = \delta(\varepsilon,n,s,\lambda,\Lambda) > 0$, such that for any $z \in \mathbb{R}^n$, any $r \in (0,1]$, any bounded domain $U \subset \mathbb{R}^n$ such that $B_{5r}(z) \subset U$ and any weak solution $u \in H^s(B_{5r}(z)|\mathbb{R}^n)$ of the equation*

$$L_A u = \sum_{i=1}^m L_{D_i} g_i + f \text{ in } B_{5r}(z)$$

with

$$\left\{x \in B_r(z) \mid \mathcal{M}_U(|\nabla^s u|^2)(x) \leq 1\right\} \cap \left\{x \in B_r(z) \mid \mathcal{M}_U\left(|f|^2 + \sum_{i=1}^m |\nabla^s g_i|^2\right)(x) \leq \delta^2\right\} \neq \emptyset,$$

we have

$$\left|\left\{x \in B_r(z) \mid \mathcal{M}_U(|\nabla^s u|^2)(x) > N_1^2\right\}\right| < \varepsilon |B_r|. \tag{30}$$

Proof. Let $\theta > 0$ and $M > 0$ to be chosen and consider the corresponding $\delta = \delta(\theta,n,s,\lambda,\Lambda,M) > 0$ given by Lemma 6.1. Fix $r \in (0,1]$ and $z \in \mathbb{R}^n$. For any $x \in U' := \{\frac{x-z}{r} \mid x \in U\}$, define

$$\widetilde{A}(x,y) := A(rx+z,ry+z) = A(rx,ry), \quad \widetilde{D}_i(x,y) := D_i(rx+z,ry+z),$$
$$\widetilde{u}(x) := r^{-s}u(rx+z), \quad \widetilde{g}_i(x) := r^{-s}g_i(rx+z), \quad \widetilde{f}(x) := r^s f(rx+z)$$

and note that under the above assumptions $\widetilde{A}$ belongs to the class $\mathcal{L}_1(\lambda)$ and that $\widetilde{u} \in H^s(B_5|\mathbb{R}^n)$ satisfies

$$L_{\widetilde{A}}\widetilde{u} = \sum_{i=1}^m L_{\widetilde{D}_i}\widetilde{g}_i + \widetilde{f} \text{ weakly in } B_5.$$

Hence, by Lemma 6.1 there exists a weak solution $\widetilde{v} \in H^s(B_5|\mathbb{R}^n)$ of

$$L_{\widetilde{A}}\widetilde{v} = 0 \text{ in } B_5$$

such that

$$\int_{B_2} |\nabla^s(\widetilde{u}-\widetilde{v})|^2 dx \leq \theta^2, \tag{31}$$

provided that the conditions (22) and (23) are satisfied. By assumption, there exists a point $x \in B_r(z)$ such that

$$\mathcal{M}_U(|\nabla^s u|^2)(x) \leq 1, \quad \mathcal{M}_U\left(|f|^2 + \sum_{i=1}^m |\nabla^s g_i|^2\right)(x) \leq \delta^2.$$

By the scaling and translation invariance of the Hardy-Littlewood maximal function (Lemma 2.2), for the point $x_0 := \frac{x-z}{r} \in B_1$ we thus have

$$\mathcal{M}_{U'}(|\nabla^s \widetilde{u}|^2)(x_0) = \mathcal{M}_U(|\nabla^s u|^2)(x) \leq 1$$

and

$$\mathcal{M}_{U'}\left(|\widetilde{f}|^2 + \sum_{i=1}^m |\nabla^s \widetilde{g}_i|^2\right)(x_0) = \mathcal{M}_U\left(r^{2s}|f|^2 + \sum_{i=1}^m |\nabla^s g_i|^2\right)(x) \leq \delta^2.$$

Therefore, for any $\rho > 0$ we have

$$\fint_{B_\rho(x_0)} |\nabla^s \widetilde{u}|^2 dx \leq 1, \quad \fint_{B_\rho(x_0)} \left(|\widetilde{f}|^2 + \sum_{i=1}^m |\nabla^s \widetilde{g}_i|^2\right) dx \leq \delta^2, \tag{32}$$

where the values of $\nabla^s \widetilde{u}$, $\nabla^s \widetilde{g}_i$ and $\widetilde{f}$ outside of U' are replaced by 0, which we also do for the rest of the proof. Since $B_5 \subset B_6(x_0)$, by (32) we have

$$\fint_{B_5} |\nabla^s \widetilde{u}|^2 dx \leq \frac{|B_6|}{|B_5|} \fint_{B_6(x_0)} |\nabla^s \widetilde{u}|^2 dx \leq \left(\frac{6}{5}\right)^n$$

and

$$\fint_{B_5} \left(|\widetilde{f}|^2 + \sum_{i=1}^m |\nabla^s \widetilde{g}_i|^2\right) dx \leq \frac{|B_6|}{|B_5|} \fint_{B_6(x_0)} \left(|\widetilde{f}|^2 + \sum_{i=1}^m |\nabla^s \widetilde{g}_i|^2\right) dx \leq \left(\frac{6}{5}\right)^n \delta^2,$$

so that we get that $\widetilde{u}$, $\widetilde{g}_i$ and $\widetilde{f}$ satisfy the conditions (22) and (23) with $M = \left(\frac{6}{5}\right)^n$. Therefore, (31) is satisfied by $\widetilde{u}$ and the corresponding approximate solution $\widetilde{v}$. Considering the function $v \in H^s(U|\mathbb{R}^n)$ given by $v(x) := r^s \widetilde{v}\left(\frac{x-z}{r}\right)$ and rescaling back yields

$$\int_{B_{2r}(y)} |\nabla^s (u-v)|^2 dx = r^n \int_{B_2} |\nabla^s (\widetilde{u} - \widetilde{v})|^2 dx \leq \theta^2 r^n. \tag{33}$$

By Lemma 6.1, there exists a constant $N_0 = N_0(n, s, \lambda, \Lambda) > 0$ such that

$$||\nabla^s \widetilde{v}||^2_{L^\infty(B_2)} \leq N_0^2. \tag{34}$$

Next, we define $N_1 := (\max\{4N_0^2, 3^n\})^{1/2} > 1$ and claim that

$$\left\{x \in B_1 \mid \mathcal{M}_{U'}(|\nabla^s \widetilde{u}|^2)(x) > N_1^2\right\} \subset \left\{x \in B_1 \mid \mathcal{M}_{B_2}(|\nabla^s (\widetilde{u} - \widetilde{v})|^2)(x) > N_0^2\right\}. \tag{35}$$

To see this, assume that

$$x_1 \in \left\{x \in B_1 \mid \mathcal{M}_{B_2}(|\nabla^s (\widetilde{u} - \widetilde{v})|^2)(x) \leq N_0^2\right\}. \tag{36}$$

For $\rho < 1$, we have $B_\rho(x_1) \subset B_1(x_1) \subset B_2$, so that together with (36) and (34) we deduce

$$\begin{aligned}
\fint_{B_\rho(x_1)} |\nabla^s \widetilde{u}|^2 dx &\leq 2 \fint_{B_\rho(x_1)} \left(|\nabla^s (\widetilde{u} - \widetilde{v})|^2 + |\nabla^s \widetilde{v}|^2\right) dx \\
&\leq 2 \fint_{B_\rho(x_1)} |\nabla^s (\widetilde{u} - \widetilde{v})|^2 dx + 2\ ||\nabla^s \widetilde{v}||^2_{L^\infty(B_\rho(x_1))} \\
&\leq 2\ \mathcal{M}_{B_2}(|\nabla^s (\widetilde{u} - \widetilde{v})|^2)(x_1) + 2\ ||\nabla^s \widetilde{v}||^2_{L^\infty(B_2)} \leq 4N_0^2.
\end{aligned}$$

On the other hand, for $\rho \geq 1$ we have $B_\rho(x_1) \subset B_{3\rho}(x_0)$, so that (32) implies

$$\fint_{B_\rho(x_1)} |\nabla^s \widetilde{u}|^2 dx \leq \frac{|B_{3\rho}|}{|B_\rho|} \fint_{B_{3\rho}(x_0)} |\nabla^s \widetilde{u}|^2 dx \leq 3^n.$$

Thus, we have

$$x_1 \in \left\{ x \in B_1 \mid \mathcal{M}_{U'}(|\nabla^s \widetilde{u}|^2)(x) \leq N_1^2 \right\},$$

which implies (35). In view of the scaling and translation invariance of the Hardy-Littlewood maximal function (Lemma 2.2), (35) is equivalent to

$$\left\{ x \in B_r(z) \mid \mathcal{M}_U(|\nabla^s u|^2)(x) > N_1^2 \right\} \subset \left\{ x \in B_r(z) \mid \mathcal{M}_{B_{2r}(z)}(|\nabla^s (u-v)|^2)(x) > N_0^2 \right\}. \tag{37}$$

For any $\varepsilon > 0$, using (37), the weak 1-1 estimate from Theorem 2.3 and (33), we conclude that there exists some constant $C = C(n) > 0$ such that

$$\begin{aligned} \left| \left\{ x \in B_r(z) \mid \mathcal{M}_U(|\nabla^s u|^2)(x) > N_1^2 \right\} \right| &\leq \left| \left\{ x \in B_r(z) \mid \mathcal{M}_{B_{2r}(z)}(|\nabla^s (u-v)|^2)(x) > N_0^2 \right\} \right| \\ &\leq \frac{C}{N_0^2} \int_{B_{2r}(z)} |\nabla^s (u-v)|^2 dx \\ &\leq \frac{C}{N_0^2} \theta^2 r^n < \varepsilon |B_r|, \end{aligned}$$

where the last inequality is obtained by choosing θ and thus also δ sufficiently small.
This finishes our proof. ■

Remark. Note that in the above proof, the choice of θ and thus also the choice of a sufficiently small δ does not depend on the radius r, which is due to the fact that $|B_r| = cr^n$ for some constant $c = c(n) > 0$. This is vital in our further proof of the $H^{s,p}$ regularity.

Next, we refine the statement of Lemma 6.2 in order make it applicable for proving the assumptions of Lemma 2.1.

Corollary 6.3. *There is a constant $N_1 = N_1(n, s, \lambda, \Lambda) > 1$, such that the following holds. For any $\varepsilon > 0$ there exists some $\delta = \delta(\varepsilon, n, s, \lambda, \Lambda) > 0$, such that for any $z \in B_1$, any $r \in (0,1)$ and any weak solution $u \in H^s(B_6|\mathbb{R}^n)$ of the equation*

$$L_A u = \sum_{i=1}^m L_{D_i} g_i + f \text{ in } B_6$$

with

$$\left| \left\{ x \in B_r(z) \mid \mathcal{M}_{B_6}(|\nabla^s u|^2)(x) > N_1^2 \right\} \cap B_1 \right| \geq \varepsilon |B_r|, \tag{38}$$

we have

$$\begin{aligned} B_r(z) \cap B_1 \subset & \left\{ x \in B_1 \mid \mathcal{M}_{B_6}(|\nabla^s u|^2)(x) > 1 \right\} \\ & \cup \left\{ x \in B_1 \mid \mathcal{M}_{B_6} \left(|f|^2 + \sum_{i=1}^m |\nabla^s g_i|^2 \right)(x) > \delta^2 \right\}. \end{aligned} \tag{39}$$

Proof. Let $N_1 = N_1(n, s, \lambda, \Lambda) > 1$ be given by Lemma 6.2. Fix $\varepsilon > 0$, $r \in (0,1)$, $z \in \mathbb{R}^n$ and consider the corresponding $\delta = \delta(\varepsilon, n, s, \lambda, \Lambda) > 0$ given by Lemma 6.2. We argue by contradiction. Assume that (38) is satisfied but that (39) is false, so that there exists some $x_0 \in B_r(z) \cap B_1$ such that

$$\begin{aligned}
x_0 \in B_r(z) &\cap \big\{x \in B_1 \mid \mathcal{M}_{B_6}(|\nabla^s u|^2)(x) \le 1\big\} \\
&\cap \left\{x \in B_1 \mid \mathcal{M}_{B_6}\left(|f|^2 + \sum_{i=1}^m |\nabla^s g_i|^2\right)(x) \le \delta^2\right\} \\
&\subset \big\{x \in B_r(z) \mid \mathcal{M}_{B_6}(|\nabla^s u|^2)(x) \le 1\big\} \\
&\cap \left\{x \in B_r(z) \mid \mathcal{M}_{B_6}\left(|f|^2 + \sum_{i=1}^m |\nabla^s g_i|^2\right)(x) \le \delta^2\right\}.
\end{aligned}$$

Since moreover we have $B_{5r}(z) \subset B_6$, Lemma 6.2 with $U = B_6$ yields

$$\begin{aligned}
&\left|\big\{x \in B_r(z) \mid \mathcal{M}_{B_6}(|\nabla^s u|^2)(x) > N_1^2\big\} \cap B_1\right| \\
&\le \left|\big\{x \in B_r(z) \mid \mathcal{M}_{B_6}(|\nabla^s u|^2)(x) > N_1^2\big\}\right| < \varepsilon |B_r|,
\end{aligned}$$

which contradicts (38). ■

Lemma 6.4. *Let $N_1 = N_1(n,s,\lambda,\Lambda) > 1$ be given by Corollary 6.3. Moreover, let $k \in \mathbb{N}$, $\varepsilon \in (0,1)$, set $\varepsilon_1 := 10^n \varepsilon$ and consider the corresponding $\delta = \delta(\varepsilon,n,s,\lambda,\Lambda) > 0$ given by Corollary 6.3. Then for any weak solution $u \in H^s(B_6|\mathbb{R}^n)$ of the equation*

$$L_A u = \sum_{i=1}^m L_{D_i} g_i + f \text{ in } B_6$$

with

$$\left|\big\{x \in B_1 \mid \mathcal{M}_{B_6}(|\nabla^s u|^2)(x) > N_1^2\big\}\right| < \varepsilon |B_1|, \tag{40}$$

we have

$$\begin{aligned}
&\left|\big\{x \in B_1 \mid \mathcal{M}_{B_6}(|\nabla^s u|^2)(x) > N_1^{2k}\big\}\right| \\
&\le \sum_{j=1}^k \varepsilon_1^j \left|\left\{x \in B_1 \mid \mathcal{M}_{B_6}\left(|f|^2 + \sum_{i=1}^m |\nabla^s g_i|^2\right)(x) > \delta^2 N_1^{2(k-j)}\right\}\right| \\
&\quad + \varepsilon_1^k \left|\big\{x \in B_1 \mid \mathcal{M}_{B_6}(|\nabla^s u|^2)(x) > 1\big\}\right|.
\end{aligned}$$

Proof. We proof this Lemma by induction on k. In view of (40) and Corollary 6.3, the case $k=1$ is a direct consequence of Lemma 2.1 applied to the sets

$$E := \big\{x \in B_1 \mid \mathcal{M}_{B_6}(|\nabla^s u|^2)(x) > N_1^2\big\}$$

and

$$F := \big\{x \in B_1 \mid \mathcal{M}_{B_6}(|\nabla^s u|^2)(x) > 1\big\} \cup \left\{x \in B_1 \mid \mathcal{M}_{B_6}\left(|f|^2 + \sum_{i=1}^m |\nabla^s g_i|^2\right)(x) > \delta^2\right\}.$$

Next, assume that the conclusion is valid for some $k \in \mathbb{N}$. Define $\widehat{u} := u/N_1$, $\widehat{g}_i := g_i/N_1$ and $\widehat{f} := f/N_1$. Then $\widehat{u}$ clearly satisfies

$$L_A \widehat{u} = \sum_{i=1}^m L_{D_i} \widehat{g}_i + \widehat{f} \text{ weakly in } B_6.$$

Moreover, since $N_1 > 1$ we have

$$\begin{aligned}
\left|\big\{x \in B_1 \mid \mathcal{M}_{B_6}(|\nabla^s \widehat{u}|^2)(x) > N_1^2\big\}\right| &= \left|\big\{x \in B_1 \mid \mathcal{M}_{B_6}(|\nabla^s u|^2)(x) > N_1^4\big\}\right| \\
&\le \left|\big\{x \in B_1 \mid \mathcal{M}_{B_6}(|\nabla^s u|^2)(x) > N_1^2\big\}\right| < \varepsilon |B_1|.
\end{aligned}$$

Thus, using the induction assumption yields

$$
\begin{aligned}
&\left|\left\{x \in B_1 \mid \mathcal{M}_{B_6}(|\nabla^s u|^2)(x) > N_1^{2(k+1)}\right\}\right| \\
=&\left|\left\{x \in B_1 \mid \mathcal{M}_{B_6}(|\nabla^s \widehat{u}|^2)(x) > N_1^{2k}\right\}\right| \\
\leq&\sum_{j=1}^{k} \varepsilon_1^j \left|\left\{x \in B_1 \mid \mathcal{M}_{B_6}\left(|\widehat{f}|^2 + \sum_{i=1}^m |\nabla^s \widehat{g}_i|^2\right)(x) > \delta^2 N_1^{2(k-j)}\right\}\right| \\
&+ \varepsilon_1^k \left|\left\{x \in B_1 \mid \mathcal{M}_{B_6}(|\nabla^s \widehat{u}|^2)(x) > 1\right\}\right| \\
=&\sum_{j=1}^{k} \varepsilon_1^j \left|\left\{x \in B_1 \mid \mathcal{M}_{B_6}\left(|f|^2 + \sum_{i=1}^m |\nabla^s g_i|^2\right)(x) > \delta^2 N_1^{2(k+1-j)}\right\}\right| \\
&+ \varepsilon_1^k \left|\left\{x \in B_1 \mid \mathcal{M}_{B_6}(|\nabla^s u|^2)(x) > N_1^2\right\}\right|.
\end{aligned}
$$

Moreover, by using the case $k = 1$ we obtain

$$
\begin{aligned}
=&\sum_{j=1}^{k} \varepsilon_1^j \left|\left\{x \in B_1 \mid \mathcal{M}_{B_6}\left(|f|^2 + \sum_{i=1}^m |\nabla^s g_i|^2\right)(x) > \delta^2 N_1^{2(k+1-j)}\right\}\right| \\
&+ \varepsilon_1^k \left|\left\{x \in B_1 \mid \mathcal{M}_{B_6}(|\nabla^s u|^2)(x) > N_1^2\right\}\right| \\
\leq&\sum_{j=1}^{k} \varepsilon_1^j \left|\left\{x \in B_1 \mid \mathcal{M}_{B_6}\left(|f|^2 + \sum_{i=1}^m |\nabla^s g_i|^2\right)(x) > \delta^2 N_1^{2(k+1-j)}\right\}\right| \\
&+ \varepsilon_1^k \Bigg(\varepsilon_1 \left|\left\{x \in B_1 \mid \mathcal{M}_{B_6}\left(|f|^2 + \sum_{i=1}^m |\nabla^s g_i|^2\right)(x) > \delta^2\right\}\right| \\
&+ \varepsilon_1 \left|\left\{x \in B_1 \mid \mathcal{M}_{B_6}(|\nabla^s u|^2)(x) > 1\right\}\right|\Bigg) \\
=&\sum_{j=1}^{k+1} \varepsilon_1^j \Bigg(\left|\left\{x \in B_1 \mid \mathcal{M}_{B_6}\left(|f|^2 + \sum_{i=1}^m |\nabla^s g_i|^2\right)(x) > \delta^2 N_1^{2(k+1-j)}\right\}\right| \\
&+ \varepsilon_1^{k+1} \left|\left\{x \in B_1 \mid \mathcal{M}_{B_6}(|\nabla^s u|^2)(x) > 1\right\}\right|,
\end{aligned}
$$

so that by combining the last two displays we see that the conclusion is valid for $k + 1$, which completes the proof. ∎

We are now set to prove the higher integrability of $\nabla^s u$ in the case of balls. The approach to the proof can be summarized as follows. First of all, we consider an appropriately scaled version of u that satisfies the condition (40) from Lemma 6.4 and also corresponding scaled versions of g_i and f. Then we use Lemma 2.4 in order to derive from Lemma 6.4 the desired L^p estimate in terms of the Hardy-Littlewood maximal functions of the scaled versions of u, g_i and f, which in view of the strong p-p estimates from Theorem 2.3 and rescaling then yields the desired L^p estimate for $\nabla^s u$.

Theorem 6.5. *Let $2 < p < \infty$, $g_i \in H^{s,p}(B_6|\mathbb{R}^n)$ and $f \in L^p(B_6)$. If A belongs to $\mathcal{L}_1(\lambda)$ and if all D_i are symmetric and bounded by $\Lambda > 0$, then for any weak solution $u \in H^s(B_6|\mathbb{R}^n)$ of the equation*

$$
L_A u = \sum_{i=1}^m L_{D_i} g_i + f \text{ in } B_6
$$

we have $\nabla^s u \in L^p(B_1)$. Moreover, there exists a constant $C = C(p, n, s, \lambda, \Lambda) > 0$ such that

$$
||\nabla^s u||_{L^p(B_1)} \leq C \left(||f + \sum_{i=1}^m \nabla^s g_i||_{L^p(B_6)} + ||\nabla^s u||_{L^2(B_6)} \right). \tag{41}
$$

Proof. Fix $p > 2$ and let $N_1 = N_1(n, s, \lambda, \Lambda) > 1$ be given by Lemma 6.4. Moreover, select $\varepsilon \in (0,1)$ such that

$$N_1^p 10^n \varepsilon \leq \frac{1}{2}. \tag{42}$$

Consider also the corresponding $\delta = \delta(\varepsilon, n, s, \lambda, \Lambda) > 0$ given by Corollary 6.3. If $\nabla^s u = 0$ a.e. in B_6, then the assertion is trivially satisfied, so that we can assume $||\nabla^s u||_{L^2(B_6)} > 0$. Next, we define

$$\widehat{u} := \frac{\gamma u}{||\nabla^s u||_{L^2(B_6)}}, \; \widehat{g}_i := \frac{\gamma g_i}{||\nabla^s u||_{L^2(B_6)}} \text{ and } \widehat{f} := \frac{\gamma f}{||\nabla^s u||_{L^2(B_6)}},$$

where $\gamma > 0$ remains to be chosen independently of u, g_i and f, note that we have

$$L_A \widehat{u} = \sum_{i=1}^m L_{D_i} \widehat{g}_i + \widehat{f} \text{ weakly in } B_6.$$

Moreover, we have

$$\int_{B_6} |\nabla^s \widehat{u}|^2 dx = \gamma^2.$$

Combining this observation with the weak 1-1 estimate from Theorem 2.3, it follows that there is a constant $C_1 = C_1(n) > 0$ such that

$$\left|\left\{x \in B_1 \mid \mathcal{M}_{B_6}(|\nabla^s \widehat{u}|^2)(x) > N_1^2\right\}\right| \leq \frac{C_1}{N_1^2} \int_{B_6} |\nabla^s \widehat{u}|^2 dx = \frac{C_1 \gamma^2}{N_1^2} < \varepsilon |B_1|,$$

where the last inequality is obtained by choosing γ small enough. Therefore, all assumptions made in Lemma 6.4 are satisfied by $\widehat{u}$. Furthermore, in view of Lemma 2.4 with $\tau = \delta^2$, $\beta = N_1^2$ and with p replaced by $p/2$, and also taking into account the strong p-p estimates for the Hardy-Littlewood maximal function (cf. Theorem 2.3), we deduce that there exist constants $C_2 = C_2(n, s, \lambda, \Lambda, p) > 0$ and $C_3 = C_3(n, p) > 0$ such that

$$\begin{aligned} &\sum_{k=1}^\infty N_1^{pk} \left|\left\{x \in B_1 \mid \mathcal{M}_{B_6}\left(|\widehat{f}|^2 + \sum_{i=1}^m |\nabla^s \widehat{g}_i|^2\right)(x) > \delta^2 N_1^{2k}\right\}\right| \\ \leq &C_2 ||\mathcal{M}_{B_6}\left(|\widehat{f}|^2 + \sum_{i=1}^m |\nabla^s \widehat{g}_i|^2\right)||_{L^{p/2}(B_6)}^{p/2} \\ \leq &C_2 C_3^p ||\widehat{f} + \sum_{i=1}^m \nabla^s \widehat{g}_i||_{L^p(B_6)}^p. \end{aligned} \tag{43}$$

Setting $\varepsilon_1 := 10^n \varepsilon$, by (42) we see that

$$\sum_{j=1}^\infty (N_1^p \varepsilon_1)^j \leq \sum_{j=1}^\infty \left(\frac{1}{2}\right)^j = 1. \tag{44}$$

Using Lemma 6.4, the Cauchy product, (44), (43), and setting $C_4 := C_2 C_3^p$, we compute

$$\begin{aligned} &\sum_{k=1}^\infty N_1^{pk} \left|\left\{x \in B_1 \mid \mathcal{M}_{B_6}(|\nabla^s \widehat{u}|^2)(x) > N_1^{2k}\right\}\right| \\ \leq &\sum_{k=1}^\infty N_1^{pk} \Bigg(\sum_{j=1}^k \varepsilon_1^j \left|\left\{x \in B_1 \mid \mathcal{M}_{B_6}\left(|\widehat{f}|^2 + \sum_{i=1}^m |\nabla^s \widehat{g}_i|^2\right)(x) > \delta^2 N_1^{2(k-j)}\right\}\right| \\ &+ \varepsilon_1^k \left|\left\{x \in B_1 \mid \mathcal{M}_{B_6}(|\nabla^s \widehat{u}|^2)(x) > 1\right\}\right| \Bigg) \end{aligned}$$

$$
\begin{aligned}
=&\left(\sum_{k=0}^{\infty} N_1^{pk}\left|\left\{x\in B_1 \mid \mathcal{M}_{B_6}\left(|\widehat{f}|^2+\sum_{i=1}^m |\nabla^s \widehat{g}_i|^2\right)(x)>\delta^2 N_1^{2k}\right\}\right|\right)\left(\sum_{j=1}^{\infty}(N_1^p\varepsilon_1)^j\right)\\
&+\left(\sum_{k=1}^{\infty}(N_1^p\varepsilon_1)^k\right)\left|\left\{x\in B_1 \mid \mathcal{M}_{B_6}(|\nabla^s\widehat{u}|^2)(x)>1\right\}\right|\\
\leq&\left(\sum_{k=1}^{\infty} N_1^{pk}\left|\left\{x\in B_1 \mid \mathcal{M}_{B_6}\left(|\widehat{f}|^2+\sum_{i=1}^m |\nabla^s \widehat{g}_i|^2\right)(x)>\delta^2 N_1^{2k}\right\}\right|+2|B_1|\right)\left(\sum_{j=1}^{\infty}(N_1^p\varepsilon_1)^j\right)\\
\leq&\sum_{k=1}^{\infty} N_1^{pk}\left|\left\{x\in B_1 \mid \mathcal{M}_{B_6}\left(|\widehat{f}|^2+\sum_{i=1}^m |\nabla^s \widehat{g}_i|^2\right)(x)>\delta^2 N_1^{2k}\right\}\right|+2|B_1|\\
\leq& C_4||\widehat{f}+\sum_{i=1}^m \nabla^s\widehat{g}_i||^p_{L^p(B_6)}+2|B_1|.
\end{aligned}
$$

Therefore, by Theorem 2.3 and Theorem 2.4 we find that there exists another constant $C_5=C_5(n,s,\lambda,\Lambda,p)>0$ such that

$$
\begin{aligned}
||\nabla^s\widehat{u}||^p_{L^p(B_1)} &\leq ||\mathcal{M}_{B_6}(|\nabla^s\widehat{u}|^2)||^{p/2}_{L^{p/2}(B_1)}\\
&\leq C_5\left(\sum_{k=1}^{\infty} N_1^{pk}\left|\left\{x\in B_1 \mid \mathcal{M}_{B_6}(|\nabla^s\widehat{u}|^2)(x)>N_1^{2k}\right\}\right|+|B_1|\right)\\
&\leq C_5\left(C_4\left(||\widehat{f}||^p_{L^p(B_6)}+\sum_{i=1}^m||\nabla^s\widehat{g}_i||^p_{L^p(B_6)}\right)+3|B_1|\right)\\
&\leq C_6^p\left(||\widehat{f}+\sum_{i=1}^m\nabla^s\widehat{g}_i||^p_{L^p(B_6)}+1\right),
\end{aligned}
$$

where $C_6:=(C_5\max\{C_4,3|B_1|\})^{1/p}$. It follows that

$$
||\nabla^s\widehat{u}||_{L^p(B_1)}\leq C_6\left(||\widehat{f}+\sum_{i=1}^m\nabla^s\widehat{g}_i||^p_{L^p(B_6)}+1\right)^{1/p}\leq C_6\left(||\widehat{f}+\sum_{i=1}^m\nabla^s\widehat{g}_i||_{L^p(B_6)}+1\right),
$$

so that

$$
\begin{aligned}
||\nabla^s u||_{L^p(B_1)} &\leq C_6\left(||f+\sum_{i=1}^m\nabla^s g_i||_{L^p(B_6)}+\frac{||\nabla^s u||_{L^2(B_6)}}{\gamma}\right)\\
&\leq C_6\gamma^{-1}\left(||f+\sum_{i=1}^m\nabla^s g_i||_{L^p(B_6)}+||\nabla^s u||_{L^2(B_6)}\right),
\end{aligned}
$$

which proves (41) with $C:=C_6\gamma^{-1}$. ■

7. Proofs of the main results

In order to state our main result on local regularity in an optimal way, we define the following notion of local weak solutions.

Definition. *Let $\Omega\subset\mathbb{R}^n$ be a domain. Given $b\in L^\infty_{loc}(\Omega)$, $f\in L^2_{loc}(\Omega)$ and $g_i\in H^s_{loc}(\Omega|\mathbb{R}^n)$, we say that $u\in H^s_{loc}(\Omega|\mathbb{R}^n)$ is a local weak solution to the equation $L_Au+bu=\sum_{i=1}^m L_{D_i}g_i+f$ in Ω, if*

$$
\mathcal{E}_A(u,\varphi)+(bu,\varphi)_{L^2(\Omega)}=\sum_{i=1}^m\mathcal{E}_{D_i}(g_i,\varphi)+(f,\varphi)_{L^2(\Omega)}\quad\forall\varphi\in H^s_c(\Omega|\mathbb{R}^n),
$$

where by $H^s_c(\Omega|\mathbb{R}^n)$ we denote the set of all functions that belong to $H^s(\Omega|\mathbb{R}^n)$ and are compactly supported in Ω.

In view of the inclusions

$$H^{s,p}(\mathbb{R}^n) \subset H^{s,p}_{loc}(\Omega|\mathbb{R}^n) \subset H^{s,p}_{loc}(\Omega) \subset W^{s,p}_{loc}(\Omega)$$

for $p \in [2,\infty)$ which we discussed in section 3, Theorem 1.1 follows directly from the following slightly stronger result in terms of the spaces $H^{s,p}_{loc}(\Omega|\mathbb{R}^n)$ defined in section 3.

Theorem 7.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain, $p \in (2,\infty)$, $s \in (0,1)$, $b \in L^\infty_{loc}(\Omega)$, $g_i \in H^{s,p}_{loc}(\Omega|\mathbb{R}^n)$ and $f \in L^{p_\star}_{loc}(\Omega)$, where $p_\star = \max\left\{\frac{pn}{n+ps}, 2\right\}$. If A belongs to $\mathcal{L}_1(\lambda)$ and if all D_i are symmetric and bounded by $\Lambda > 0$, then for any local weak solution $u \in H^s_{loc}(\Omega|\mathbb{R}^n)$ of the equation*

$$L_A u + bu = \sum_{i=1}^m L_{D_i} g_i + f \;\text{ in } \Omega \tag{45}$$

we have $u \in H^{s,p}_{loc}(\Omega|\mathbb{R}^n)$.

Proof. Fix $p \in (2,\infty)$. We first prove the result under the stronger assumption that $f \in L^p_{loc}(\Omega)$. Fix relatively compact bounded open sets $U \subset\subset V \subset\subset \Omega$. Moreover, fix a smooth domain $U_\star$ such that $U \subset\subset U_\star \subset\subset V$. Let $\widetilde{f} := f - bu$, so that u is a local weak solution of

$$L_A u = \sum_{i=1}^m L_{D_i} g_i + \widetilde{f} \text{ in } \Omega. \tag{46}$$

In particular, u is a weak solution of (46) in V. For any $z \in V$, fix some small enough $r_z \in (0,1)$ such that $B_{6r_z}(z) \subset\subset V$. Define

$$A_z(x,y) := A\left(r_z x + z, r_z y + z\right) = A\left(r_z x, r_z y\right), \quad D_{iz}(x,y) := D_i\left(r_z x + z, r_z y + z\right),$$
$$u_z(x) := r_z^{-s} u\left(r_z x + z\right), \quad g_{iz}(x) := r_z^{-s} g\left(r_z x + z\right), \quad \widetilde{f}_z(x) := r_z^s \widetilde{f}\left(r_z x + z\right)$$

and note that for any $z \in V$, A_z belongs to the class $\mathcal{L}_1(\lambda)$ and that u_z satisfies

$$L_{A_z} u_z = \sum_{i=1}^m L_{D_{iz}} g_{iz} + \widetilde{f}_z \text{ weakly in } B_6.$$

Using Theorem 6.5, for any $q \in (2,\infty)$ we obtain the estimate

$$\begin{aligned}
&||\nabla^s u||_{L^q(B_{r_z}(z))} = r_z^{\frac{n}{q}} ||\nabla^s u_z||_{L^q(B_1)} \\
\leq & r_z^{\frac{n}{q}} C_1 \left(||\widetilde{f}_z + \sum_{i=1}^m \nabla^s g_{iz}||_{L^q(B_6)} + ||\nabla^s u_z||_{L^2(B_6)} \right) \\
= & C_1 \left(||r_z^s \widetilde{f} + \sum_{i=1}^m \nabla^s g_i||_{L^q(B_{6r_z}(z))} + r_z^{\frac{n}{q}-\frac{n}{2}} ||\nabla^s u||_{L^2(B_{6r_z}(z))} \right) \\
\leq & C_1 \max\{1, r_z^{\frac{n}{q}-\frac{n}{2}}\} \left(||\widetilde{f}||_{L^q(B_{6r_z}(z))} + ||\sum_{i=1}^m \nabla^s g_i||_{L^q(B_{6r_z}(z))} + ||\nabla^s u||_{L^2(B_{6r_z}(z))} \right),
\end{aligned}$$

where $C_1 = C_1(q,n,s,\lambda,\Lambda) > 0$. Since $\{B_{r_z}(z)\}_{z \in \overline{U}_\star}$ is an open covering of $\overline{U}_\star$ and $\overline{U}_\star$ is compact, there is a finite subcover $\left\{B_{r_{z_i}}(z_i)\right\}_{i=1}^k$ of $\overline{U}_\star$ and hence of $U_\star$. Let $\{\phi_i\}_{i=1}^k$ be a partition of unity subordinate to the covering $\left\{B_{r_{z_i}}(z_i)\right\}_{i=1}^k$ of $\overline{U}_\star$, that is, the ϕ_i are non-negative functions on $\mathbb{R}^n$, we have $\phi_i \in C_0^\infty(B_{r_{x_i}}(x_i))$ for all $i = 1,...,k$, $\sum_{i=1}^k \phi_j \equiv 1$ in an open neighbourhood of $\overline{U}_\star$ and $\sum_{i=1}^k \phi_j \leq 1$ in $\mathbb{R}^n$. Setting $C_2 := C_1 \max\{1, \max_{i=1,...,k} r_{z_i}^{\frac{n}{q}-\frac{n}{2}}\}$ and summing the above estimates over $i = 1, ..., k$, we conclude

$$\begin{aligned}
||\nabla^s u||_{L^q(U_\star)} &= || \sum_{i=1}^k |\nabla^s u|\phi_i||_{L^q(U_\star)} \\
&\leq \sum_{i=1}^k |||\nabla^s u|\phi_i||_{L^q(B_{r_{z_i}}(z_i))} \\
&\leq \sum_{i=1}^k ||\nabla^s u||_{L^q(B_{r_{z_i}}(z_i))} \\
&\leq \sum_{i=1}^k C_2 \left(||\widetilde{f}||_{L^q(B_{6r_z}(z))} + || \sum_{i=1}^m \nabla^s g_i||_{L^q(B_{6r_z}(z))} + ||\nabla^s u||_{L^2(B_{6r_z}(z))} \right) \\
&\leq \sum_{i=1}^k C_2 \left(||\widetilde{f}||_{L^q(V)} + || \sum_{i=1}^m \nabla^s g_i||_{L^q(V)} + ||\nabla^s u||_{L^2(V)} \right) \\
&= C_2 k \left(||\widetilde{f}||_{L^q(V)} + || \sum_{i=1}^m \nabla^s g_i||_{L^q(V)} + ||\nabla^s u||_{L^2(V)} \right),
\end{aligned}$$

which implies that for any $q \in (2, \infty)$ we have

$$||\nabla^s u||_{L^q(U_\star)} \leq C_3 \left(||f||_{L^q(V)} + ||u||_{L^q(V)} + \sum_{i=1}^m ||\nabla^s g_i||_{L^q(V)} + ||\nabla^s u||_{L^2(V)} \right), \tag{47}$$

where $C_3 = C_2 k \max\{1, ||b||_{L^\infty(V)}\}$. In particular, since by assumption and Theorem 3.3 we have $f, \nabla^s g_i \in L^p(V)$, for any $q \in [2, p]$ we have $\nabla^s u \in L^q(U_\star)$ whenever $u \in L^q(V)$. For any $r \in [1, p]$, define

$$r^\star := \begin{cases} \min\{\frac{rn}{n-rs}, p\}, & \text{if } rs < n \\ p, & \text{if } rs \geq n, \end{cases}$$

note that $r^\star \in [1, p]$. By the embedding theorem of Bessel potential spaces (Theorem 3.4), for any $r \geq 1$ we have

$$H^{s,r}(U_\star) \hookrightarrow L^{r^\star}(U_\star).$$

Since $u \in H^s(V)$, we have $u \in L^{2^\star}(V)$ and therefore $\nabla^s u \in L^{2^\star}(U_\star)$. If $p = 2^\star$, we have $u \in L^p(U_\star)$, $\nabla^s u \in L^p(U_\star)$ and therefore $u \in H^{s,p}(U_\star|\mathbb{R}^n)$. If $p > 2^\star$, then we have $u, \nabla^s_{U_\star} u \in L^{2^\star}(U_\star)$, so that Theorem 3.3 yields $u \in H^{s,2^\star}(U_\star)$. We therefore arrive at $u \in L^{2^{\star\star}}(U_\star)$. By replacing $U_\star$ with an arbitrary relatively compact smooth open subset of $U_\star$ which contains U if necessary, we therefore obtain $\nabla^s u \in L^{2^{\star\star}}(U_\star)$. If $2^{\star\star} = p$, then we have $u, \nabla^s_{U_\star} u \in L^p(U_\star)$ and therefore $u \in H^{s,p}(U_\star|\mathbb{R}^n)$. If $2^{\star\star} > p$, then iterating the above procedure also yields $u \in H^{s,p}(U_\star|\mathbb{R}^n)$ and therefore $u \in H^{s,p}(U|\mathbb{R}^n)$ at some point. Since U is an arbitrary relatively compact open subset of Ω, we conclude that $u \in H^{s,p}_{loc}(\Omega|\mathbb{R}^n)$. This finishes the proof when $f \in L^p_{loc}(\Omega)$.

Next, consider the general case when $f \in L^{p_\star}_{loc}(\Omega)$, where $p_\star = \max\left\{\frac{pn}{n+ps}, 2\right\}$. Define the function $f_\Omega : \mathbb{R}^n \to \mathbb{R}$ by

$$f_\Omega(x) := \begin{cases} f(x), & \text{if } x \in \Omega \\ 0, & \text{if } x \in \mathbb{R}^n \setminus \Omega \end{cases}$$

and note that $f_\Omega \in L^{p_\star}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. By Proposition 5.1, there exists a unique weak solution $g \in H^s(\mathbb{R}^n) \subset H^s_{loc}(\Omega|\mathbb{R}^n)$ of the equation

$$(-\Delta)^s g + g = f_\Omega \quad \text{in } \mathbb{R}^n, \tag{48}$$

where
$$(-\Delta)^s g(x)=C_{n,s}\int_{\mathbb{R}^n}\frac{g(x)-g(y)}{|x-y|^{n+2s}}dy$$
is the fractional Laplacian. In view of the classical $H^{2s,p}$ regularity for the fractional Laplacian on the whole space $\mathbb{R}^n$ (cf. for example [19, Lemma 3.5]), we have $g\in H^{2s,p_\star}(\mathbb{R}^n)\hookrightarrow H^{s,p}(\mathbb{R}^n)$ and therefore in particular $g\in H^{s,p}_{loc}(\Omega|\mathbb{R}^n)$. Since furthermore u is a local weak solution of
$$L_Au+bu=\left(\sum_{i=1}^m L_{D_i}g_i+(-\Delta)^s g\right)+g \text{ in } \Omega,$$
by the first part of the proof we obtain that $u\in H^{s,p}_{loc}(\Omega|\mathbb{R}^n)$. This finishes the proof. ■

Proof of Theorem 1.2. Fix $p\in(2,\infty)$ and let $\delta=\delta(p,n,s,\lambda,\Lambda)>0$ be given by Theorem 6.5. We first prove the result under the stronger assumption that $f\in L^2(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)$. For any $k\in\mathbb{Z}^n$, let $E_k:=B_{\sqrt{n}}(k)$ and $F_k:=B_{2\sqrt{n}}(k)$. We then have $\mathbb{R}^n=\bigcup_{k\in\mathbb{Z}^n}E_k$, moreover, there exists some $N\in\mathbb{N}$ depending only on n such that no point in $\mathbb{R}^n$ is contained in more than N of the balls F_k. In other words, we have $\sum_{k\in\mathbb{Z}^d}\chi_{F_k}\leq N$, where χ_{F_k} is the characteristic function of F_k. Since for $\widetilde{f}:=f-bu$ we have
$$L_Au=\sum_{i=1}^m L_{D_i}g_i+\widetilde{f} \text{ weakly in } \mathbb{R}^n,$$
by the same argument as in the proof of Theorem 7.1 for any $k\in\mathbb{Z}^d$ and any $q\in(2,\infty)$ we have
$$||\nabla^s u||_{L^q(E_k)}\leq C\left(||\widetilde{f}||_{L^q(F_k)}+||\sum_{i=1}^m\nabla^s g_i||_{L^q(F_k)}+||\nabla^s u||_{L^2(F_k)}\right)$$
for some constant $C=C(n,s,q,\lambda,\Lambda)>0$. It follows that
$$\begin{aligned}
&\int_{\mathbb{R}^n}|\nabla^s u(x)|^q dx\leq\sum_{k\in\mathbb{Z}^n}\int_{E_k}|\nabla^s u(x)|^q dx\\
\leq& C^q\sum_{k\in\mathbb{Z}^n}\left(\left(\int_{F_k}|\widetilde{f}(x)|^q dx\right)^{\frac{1}{q}}+\left(\int_{F_k}\sum_{i=1}^m|\nabla^s g_i(x)|^q dx\right)^{\frac{1}{q}}+\left(\int_{F_k}|\nabla^s u(x)|^2 dx\right)^{\frac{1}{2}}\right)^q\\
\leq& C_1C^q\left(\left(\sum_{k\in\mathbb{Z}^n}\int_{F_k}|\widetilde{f}(x)|^q+\sum_{i=1}^m|\nabla^s g_i(x)|^q dx\right)+\sum_{k\in\mathbb{Z}^n}\left(\int_{F_k}|\nabla^s u(x)|^2 dx\right)^{\frac{q}{2}}\right)\\
\leq& C_1C^q\left(\left(\sum_{k\in\mathbb{Z}^n}\int_{F_k}|\widetilde{f}(x)|^q+\sum_{i=1}^m|\nabla^s g_i(x)|^q dx\right)+\left(\sum_{k\in\mathbb{Z}^n}\int_{F_k}|\nabla^s u(x)|^2 dx\right)^{\frac{q}{2}}\right)\\
=& C_1C^q\left(\left(\int_{\mathbb{R}^n}\left(|\widetilde{f}(x)|^q+\sum_{i=1}^m|\nabla^s g_i(x)|^q\right)\sum_{k\in\mathbb{Z}^d}\chi_{F_k}(x)dx\right)+\left(\int_{\mathbb{R}^n}|\nabla^s u(x)|^2\sum_{k\in\mathbb{Z}^d}\chi_{F_k}(x)dx\right)^{\frac{q}{2}}\right)\\
\leq& N^{\frac{q}{2}}C_1C^q\left(\left(\int_{\mathbb{R}^n}|\widetilde{f}(x)|^q+\sum_{i=1}^m|\nabla^s g_i(x)|^q dx\right)+\left(\int_{\mathbb{R}^n}|\nabla^s u(x)|^2 dx\right)^{\frac{q}{2}}\right),
\end{aligned}$$
where $C_1=C_1(q)>0$. This implies that for any $q\in(2,\infty)$ we have
$$||\nabla^s u||_{L^q(\mathbb{R}^n)}\leq C_2\left(||f||_{L^q(\mathbb{R}^n)}+||u||_{L^q(\mathbb{R}^n)}+\sum_{i=1}^m||\nabla^s g_i||_{L^q(\mathbb{R}^n)}+||\nabla^s u||_{L^2(\mathbb{R}^n)}\right), \tag{49}$$
where $C_2:=N^{\frac{1}{2}}C_1^{\frac{1}{q}}C\max\{1,||b||_{L^\infty(\mathbb{R}^n)}\}$. In particular, since for any $q\in[2,p]$ we have $L^2(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)\hookrightarrow L^q(\mathbb{R}^n)$ and in view of the assumptions and Theorem 3.3 we have $f,\nabla^s g_i\in L^2(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)$, for any $q\in[2,p]$ it follows that $\nabla^s u\in L^q(\mathbb{R}^n)$ whenever $u\in L^q(\mathbb{R}^n)$. The proof in the

case when $f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ can now be concluded by using essentially the same iteration argument as the one in the proof of Theorem 7.1. The general case when $f \in L^2(\mathbb{R}^n) \cap L^{p_\star}(\mathbb{R}^n)$ then can once again be treated by solving the equation (48) under optimal regularity, as we did in the proof of Theorem 7.1. ■

Remark on boundary regularity. An interesting question is if it is possible to prove a global $H^{s,p}$ regularity result in smooth enough bounded domains Ω corresponding to our local regularity result Theorem 7.1. Our approach is based on a $C^{s+\gamma}$ estimate ($\gamma > 0$) for nonlocal equations with translation invariant kernels, however it is known that already in the case of the fractional Laplacian in a unit ball the optimal regularity up to the boundary is $C^s(\overline{B_1})$, cf. [27, section 7.1]. Therefore, at least with our methods proving such a global $H^{s,p}$ regularity result for the equations we consider in this work seems to be unattainable even in the case when Ω is very regular.

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