Game Options under Knightian Uncertainty in Discrete Time

Bodo Rubbenstroth

2019

Abstract

This paper studies two player stopping games in a discrete time multiple prior framework with a finite time horizon. Optimal stopping times as well as recursive formulas for the value processes of the games are derived. These results are used to characterize the set of no-arbitrage prices for a game option. The notion of a no-arbitrage price for a game option is based on the idea to consider the payoff for fixed stopping times as an European option.

Keywords Dynkin games, multiple priors, game options, incomplete Markets.

1 Introduction

Game options are a class of financial contracts which extend the class of American options. Like in the case of an American option the buyer of a game option has the right to exercise the contract at any time. But in addition also the seller of the game option has the right to cancel the contract (i.e. forcing the buyer to exercise it) at any time. Intuitively it is clear that if it is favorable for the buyer to not exercise the contract, it should be favorable for the seller to cancel the contract and vice versa. So the contract should be exercised or cancelled immediately. For this reason game options usually include a certain penalty that the seller has to pay to the buyer in addition to the obligations of the financial contract, if the seller cancels the contract. If this penalty is high enough the seller will not cancel the contract at all and the option can be considered as an American option.

From a mathematical point of view game options correspond to the concept of two player stopping games. In a two player stopping game there are given two processes of which one dominates the other and both players choose a stopping time as their strategy. The game ends whenever a player stops. At that time Player 1 pays Player 2 an amount based on the higher process if Player 1 stopped first and if Player 2 stopped first, the amount Player 1 pays to Player 2 is based on the lower process. In the case of game options Player 1 is the seller and Player 2 is the buyer. The difference between the processes is the penalty that the seller has to pay in order to cancel the contract. Since

The author would like to thank Prof. Dr. Frank Riedel and Dr. Hanwu Li for their support and valuable remarks. Financial support through the German Research Foundation via CRC 1283 "Taming Uncertainty" is gratefully acknowledged.

in a stopping game Player 1 tries to minimize the expected payoff and Player 2 tries to maximize it, there is a minimax and maximin value for the game which are called upper and lower values. Typically, if a stopping game is considered the first step is to show that upper and lower values coincide which implies that the game has a value. After that optimal stopping times for both players are of interest. A stopping time for Player 1 (resp. Player 2) is considered to be optimal, if the expected payoff with this stopping time is less (resp. greater) or equal to the game value, no matter what stopping time Player 2 (resp. Player 1) chooses.

In 1969 Dynkin extended the optimal stopping theory which was initiated in (Snell 1952) by considering a two player stopping game. For this reason stopping games are also referred to as Dynkin games. The results of Dynkin were developed further in (Neveu 1975) and (Ohtsubo 1986). The first connection between Dynkin games and financial options was made in (Kifer 2000) by the introduction of game options, which are also often called Israeli options, in the binomial model and the Black-Scholes model. Kifer derived optimal stopping times for the buyer and the seller and a cheapest superhedge for the seller against a game option. Since then Dynkin games as well as game options have been studied in a wide range of different models and applications. In (Kifer 2013a) there is given an extensive overview about different results for Dynkin games and game options and the corresponding literature.

Knightian uncertainty describes the distinction between the terms "risk" and "uncertainty". In general the term risk is used if the randomness of a situation can be described by a single probability measure and if this is not the case the term uncertainty is used. For example if we consider optimal stopping problems, it is possible that the agent is unsure about the distribution of the payoff process. To account for this situation a common way in the literature is to conisder a multiple prior framework. There the uncertainty is modelled by a set of probability measures which includes all probability measures the agent thinks could describe the distribution of the payoff process. Such a set is called a set of priors. It is then assumed that the agent tries to maximize the worst case expectation which leads to a stopping problem under a nonlinear operator.

An axiomatic foundation for multiple prior expected utility was given in (Gilboa and Schmeidler 1989) and extended to an intertemporal setting in (Epstein and Schneider 2003). In (Riedel 2009) the classical optimal stopping theory as well as the classical martingale theory was extended to multiple prior frameworks.

In this paper we consider at first a Dynkin game which is inspired by the payoff of game options in a multiple prior framework. We consider a discrete time model with a finite time horizon which is similar to the multiple prior model in (Riedel 2009). The approach will be to consider the worst case scenarios of seller and buyer separately which leads to two stopping games under nonlinear expectations. We show that these stopping games have a value and obtain optimal stopping times as well as recursive formulas for the value processes. After that game options are considered in incomplete financial markets. We consider a general discrete time financial market model which is taken from (Föllmer and Schied 2016). In incomplete financial markets we are in a multiple prior framework where the set of priors is given by the set of equivalent martingale measures. After defining the notion of no-arbitrage prices for game options in an incomplete market, superhedging strategies for the seller and for the buyer are considered. The idea to derive the notion of no-arbitrage prices will be to consider the payoff of a game option for fixed stopping times as an European option. With the help of the results obtained for the worst case stopping games we derive that there exists a cheapest superhedge for the seller and a most expensive superhedge for the buyer and that the prices of these superhedges coincide with the bounds of the set of no-arbitrage prices.

The stopping game we consider in a multiple prior framework has to the best of my knowledge not been studied yet. There are some works on Dynkin games under Knightian uncertainty in continuous time frameworks, we mention (Yin 2012), (Koo, Tang, and Yang 2015) and (Bayraktar and Yao 2017) here. Due to the continuous time framework considered in these papers their approaches and arguments differ a lot from the discrete time Dynkin game we consider in this paper. Also in a continuous time framework (Dumitrescu, Quenez, and Sulem 2017) considered game options in imperfect markets and also incorporated the possibility of default in their model. They studied superhedging strategies for the seller and showed that the price of the cheapest superhedge for the seller corresponds to the value of a Dynkin game under nonlinear expectations. In a discrete time framework (Dolinsky 2014) studied game options under volatility uncertainty and obtained a duality theorem for the price of the cheapest superhedge for the seller. Dolinsky considered a financial market model without a probabilistic structure. Hence the model and arguments differ from this paper. The approach of game options in incomplete markets in this paper extends the approach for American options in incomplete markets of (Föllmer and Schied 2016). This extension is also, to the best of my knowledge, new. In (Kühn 2004) and (Kallsen and Kühn 2004) game options were considered in incomplete markets and their approach was to use utility maximization. In (Kallsen and Kühn 2004) there was also made a neutral valuation approach which led to a replacement of the unique equivalent martingale measure in a complete market by a neutral pricing measure. There are further papers which deal with game options in incomplete markets. In (Kifer 2013b) game options are considered in a discrete market model with transaction costs. Kifer derives a cheapest superhedge for the seller and a most expensive superhedge for the buyer and a representation for the corresponding prices which includes the notion of randomized stopping times. In continuous time (Kallsen and Kühn 2005) define the notion of no-arbitrage prices by superhedging and obtain a duality theorem for the cheapest superhedging price for the seller which looks similar to the one we obtain in this paper, but by other arguments. In (Dolinsky and Kifer 2007) there is derived a representation of the price of the cheapest superhedge for the seller in multinomial models. This representation coincides with the more general one of this paper and also the general structure of the proof is quite similar. However the proof in (Dolinsky and Kifer 2007) heavily relies on the finite probability space one has in a multinomial model, by using the fact that the infimum over a finite set of stopping times is always attained. Instead of this argument we will use the results we obtain for the Dynkin game under Knightian uncertainty. Moreover we derive a duality result for the price representation as well as an explicit stopping time for the cheapest superhedge for the seller.

This paper is structured in the following way. The main results are provided in Section 2 and Section 3. In Section 2 a two player stopping game is solved in a multiple prior framework. Section 3 consideres game options in incomplete financial markets. The proofs of the main results are provided in Section 4. Section 5 concludes. The Appendix contains a section about the theory of multiple prior conditional expectations.

2 A Two Player Stopping Game with Multiple Priors

Let $(\Omega, \mathscr{F}, P_0)$ be a probability space equipped with a filtration $(\mathscr{F}_t)_{t=0,\dots,T}$ such that

$$\mathscr{F}_0 = \{\emptyset, \Omega\} \subseteq \mathscr{F}_1 \subseteq \ldots \subseteq \mathscr{F}_T = \mathscr{F},$$

where $T \in \mathbb{N}$ denotes a finite time horizon. We consider a two player stopping game between a buyer and a seller. The potential payoffs are modelled by nonnegative adapted processes $(Y_t)_{t=0,\dots,T}$, $(Z_t)_{t=0,\dots,T}$ with $Y_t \leqslant Z_t$ for all $t = 0,\dots,T$. Both, buyer and seller choose a stopping time out of the set

$$\mathscr{T}_0 := \{\tau : \Omega \to \{0,\dots,T\} : \tau \text{ stopping time}\}.$$

If the buyer chooses the stopping time $\tau \in \mathscr{T}_0$ and the seller chooses the stopping time $\sigma \in \mathscr{T}_0$, the seller has to pay the buyer the following payoff at time $\sigma \wedge \tau$:

$$R(\sigma, \tau) := \mathbb{1}_{\{\sigma < \tau\}} Z_\sigma + \mathbb{1}_{\{\sigma \geqslant \tau\}} Y_\tau.$$

We consider the case of Knightian uncertainty which means that the distribution of the payoff processes is not exactly known to the buyer and the seller. To account for this uncertainty, we consider a multiple prior framework and use two nonempty sets of probability measures on $(\Omega, \mathscr{F})$: $\mathscr{Q}_b$ which denotes the set of priors for the buyer and $\mathscr{Q}_s$ which denotes the set of priors for the seller. We consider the possibility of different sets of priors for buyer and seller to allow the possibility that buyer and seller have different information about the distribution of the payoffs. In addition to that we assume that the buyer has no information about the set of priors of the seller and vice versa. As the buyer tries to maximize the expected payoff and the seller tries to minimize it, we have different worst case scenarios for seller and buyer (even if $\mathscr{Q}_b = \mathscr{Q}_s$). Our approach will be to consider two stopping games, one for the worst case of the buyer and the other for the worst case of the seller.
As the buyer has no information about the set of priors of the seller, the buyer has to assume that in the worst case the seller chooses a stopping time that minimizes the expected worst case payoff of the buyer. Hence, in the worst case of the buyer, we have a two player stopping game with upper value

$$\inf_{\sigma \in \mathscr{T}_0} \sup_{\tau \in \mathscr{T}_0} \inf_{P \in \mathscr{Q}_b} \mathbb{E}^P[R(\sigma, \tau)]$$

and lower value

$$\sup_{\tau \in \mathscr{T}_0} \inf_{\sigma \in \mathscr{T}_0} \inf_{P \in \mathscr{Q}_b} \mathbb{E}^P[R(\sigma, \tau)].$$

With the same argument we consider for the worst case of the seller a two player stopping game with upper value

$$\inf_{\sigma \in \mathscr{T}_0} \sup_{\tau \in \mathscr{T}_0} \sup_{P \in \mathscr{Q}_s} \mathbb{E}^P[R(\sigma, \tau)]$$

and lower value

$$\sup_{\tau \in \mathscr{T}_0} \inf_{\sigma \in \mathscr{T}_0} \sup_{P \in \mathscr{Q}_s} \mathbb{E}^P[R(\sigma, \tau)].$$

Let $\mathscr{Q} \neq \emptyset$ be a set of probability measures on $(\Omega, \mathscr{F})$. We define

$$\mathscr{X}_{\mathscr{Q}} := \{X \in \mathscr{L}^0(\Omega, \mathscr{F}, P_0) : \lim_{c \to \infty} \sup_{P \in \mathscr{Q}} \mathbb{E}^P[|X| \mathbb{1}_{\{|X| \geqslant c\}}] = 0\}$$

as the set of all $\mathscr{Q}$-uniformly integrable random variables. We make the following assumptions on the payoff processes and sets of priors.

Assumption 2.1. (i) $Y_t, Z_t \in \mathscr{X}_{\mathcal{Q}_b}$ and $Y_t, Z_t \in \mathscr{X}_{\mathcal{Q}_s}$ for all $t = 0, \dots, T$.

(ii) $P \sim P_0 \sim Q$ for all $P \in \mathcal{Q}_b$ and for all $Q \in \mathcal{Q}_s$.

(iii) $\mathcal{Q}_b$ and $\mathcal{Q}_s$ are time-consistent [1].

Assumption 2.1(ii) economically means that buyer and seller know which events can occur and which not. Time-consistency of the set $\mathcal{Q}_b$ makes sure that if the buyer considers a measure $P \in \mathcal{Q}_b$ until some stopping time $\tau \in \mathscr{T}$ and shifts to another measure $Q \in \mathcal{Q}_b$ afterwards, there exists a measure $R \in \mathcal{Q}_b$ which describes this behavior [2]. Assumption 2.1(i) makes sure that upper and lower values of both stopping games are well defined.

The following theorems solve the stopping games. We will see that upper and lower values coincide and obtain optimal stopping times for buyer and seller as well as recursive formulas for the value processes of the stopping games. We define for all $t = 1, \dots, T$, $\mathscr{T}_t := \{\tau \in \mathscr{T}_0 : \tau \geq t\}$. We start with the worst case scenario of the buyer.

Theorem 2.2. *Define* $(W_t^{\downarrow})_{t=0,\dots,T}$ *recursively by*

$$W_T^{\downarrow} = Y_T,$$
$$W_t^{\downarrow} = \min(Z_t, \max(Y_t, \operatorname*{ess\,inf}_{P \in \mathcal{Q}_b} \mathbb{E}^P[W_{t+1}^{\downarrow} | \mathscr{F}_t])) \text{ for } t = 0, \dots, T-1$$

and define for all $t = 0, \dots, T$*:*

$$\sigma_t^{\downarrow} := \inf\{s \geq t : W_s^{\downarrow} = Z_s\} \wedge T,$$
$$\tau_t^{\downarrow} := \inf\{s \geq t : W_s^{\downarrow} = Y_s\}.$$

Then $\sigma_t^{\downarrow}, \tau_t^{\downarrow} \in \mathscr{T}_t$*,*

$$\begin{aligned} W_t^{\downarrow} &= \operatorname*{ess\,inf}_{\sigma \in \mathscr{T}_t} \operatorname*{ess\,sup}_{\tau \in \mathscr{T}_t} \operatorname*{ess\,inf}_{P \in \mathcal{Q}_b} \mathbb{E}^P[R(\sigma, \tau) | \mathscr{F}_t] \\ &= \operatorname*{ess\,sup}_{\tau \in \mathscr{T}_t} \operatorname*{ess\,inf}_{\sigma \in \mathscr{T}_t} \operatorname*{ess\,inf}_{P \in \mathcal{Q}_b} \mathbb{E}^P[R(\sigma, \tau) | \mathscr{F}_t], \end{aligned}$$

and

$$\operatorname*{ess\,inf}_{P \in \mathcal{Q}_b} \mathbb{E}^P[R(\sigma_t^{\downarrow}, \tau) | \mathscr{F}_t] \leq W_t^{\downarrow} \leq \operatorname*{ess\,inf}_{P \in \mathcal{Q}_b} \mathbb{E}^P[R(\sigma, \tau_t^{\downarrow}) | \mathscr{F}_t], \ \forall \sigma, \tau \in \mathscr{T}_t. \tag{1}$$

We conclude by solving the stopping game which arises by considering the worst case scenario of the seller.

Theorem 2.3. *Define* $((W_t^{\uparrow})_{t=0,\dots,T})$ *recursively by*

$$W_T^{\uparrow} = Y_T,$$
$$W_t^{\uparrow} = \min(Z_t, \max(Y_t, \operatorname*{ess\,sup}_{P \in \mathcal{Q}_s} \mathbb{E}^P[W_{t+1}^{\uparrow} | \mathscr{F}_t])) \text{ for } t = 0, \dots, T-1$$

[1] The multiple prior framework we consider in this section is taken from (Riedel 2009), except that we drop some weak compactness assumption. In the Appendix the assumptions we make are characterized in detail.

[2] See (Riedel 2009) for a detailed interpretation of the time-consistency property for a set of priors.

and define for all $t = 0, \dots, T$

$$\sigma_t^{\uparrow} := \inf\{s \geqslant t : W_s^{\uparrow} = Z_s\} \wedge T,$$
$$\tau_t^{\uparrow} := \inf\{s \geqslant t : W_s^{\uparrow} = Y_s\}.$$

Then $\sigma_t^{\uparrow}, \tau_t^{\uparrow} \in \mathcal{T}_t$,

$$\begin{aligned} W_t^{\uparrow} &= \operatorname*{ess\,inf}_{\sigma \in \mathcal{T}_t} \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \operatorname*{ess\,sup}_{P \in \mathcal{Q}_s} \mathbb{E}^P[R(\sigma, \tau) | \mathcal{F}_t] \\ &= \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \operatorname*{ess\,inf}_{\sigma \in \mathcal{T}_t} \operatorname*{ess\,sup}_{P \in \mathcal{Q}_s} \mathbb{E}^P[R(\sigma, \tau) | \mathcal{F}_t], \end{aligned}$$

and

$$\operatorname*{ess\,sup}_{P \in \mathcal{Q}_s} \mathbb{E}^P[R(\sigma_t^{\uparrow}, \tau) | \mathcal{F}_t] \leqslant W_t^{\uparrow} \leqslant \operatorname*{ess\,sup}_{P \in \mathcal{Q}_s} \mathbb{E}^P[R(\sigma, \tau_t^{\uparrow}) | \mathcal{F}_t], \ \forall \sigma, \tau \in \mathcal{T}_t.$$

3 Game Options in Incomplete Markets

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a filtration $(\mathcal{F}_t)_{t=0,\dots,T}$ such that

$$\mathcal{F}_0 = \{\emptyset, \Omega\} \subseteq \mathcal{F}_1 \subseteq \dots \subseteq \mathcal{F}_T = \mathcal{F},$$

where $T \in \mathbb{N}$ denotes a finite time horizon. We consider a financial market [3] with $D + 1$ assets: one riskless asset which price is modelled by the process

$$S_t^0 := (1 + r)^t, t = 0, \dots, T,$$

where $r \geqslant 0$ denotes a constant interest rate, and D risky assets which prices are modelled by nonnegative, adapted processes $(S_t^d)_{t=0,\dots,T}$, $d = 1, \dots, D$.

$$\tilde{S}_t^d := (1 + r)^{-t} S_t^d, t = 0, \dots, T,$$

denotes the discounted price process of asset $d \in \{0, \dots, D\}$. In this setup a portfolio is a $\mathbb{R}^{d+1}$-valued predictable process

$$\zeta = (\zeta_t)_{t=1,\dots,T} = (\zeta_t^0, \dots, \zeta_t^D)_{t=1,\dots,T}$$

and it's value is denoted by

$$V_0^{\zeta} := \sum_{d=0}^{D} \zeta_1^d S_0^d,$$
$$V_t^{\zeta} := \sum_{d=0}^{D} \zeta_t^d S_t^d, \qquad t = 1, \dots, T.$$

A portfolio ζ is called self-financing if

$$\sum_{d=0}^{D} \zeta_t^d S_t^d = \sum_{d=0}^{D} \zeta_{t+1}^d S_t^d \qquad \text{for } t = 1, \dots, T - 1.$$

[3]The financial market model is taken from (Föllmer and Schied 2016, Section 5.1).

We denote by $(\tilde{V}_t^{\zeta})_{t=0,\dots,T}$ the discounted portfolio value process, by $\mathscr{P}_e$ the set of equivalent martingale measures and define for $t = 0,\dots,T$:

$$\mathscr{T}_t := \{\tau : \Omega \to \{0,\dots,T\} \text{ stopping time} : \tau \geqslant t\}.$$

We assume that the financial market is arbitrage free, but not necessarily complete which is equivalent to the assumption:

$$\mathscr{P}_e \neq \emptyset.$$

Define

$$\mathscr{X} := \{X \in \mathscr{L}^0(\Omega, \mathscr{F}, P) | \lim_{c\to\infty} \sup_{P^* \in \mathscr{P}_e} \mathbb{E}^{P^*}[|X| \mathbb{1}_{\{|X| \geqslant c\}}] = 0\}.$$

Definition 3.1. A game option or Game Contingent Claim (GCC) (Y, Z) is a contract between a seller and a buyer which consists of nonnegative, adapted payoff processes $Y = (Y_t)_{t=0,\dots,T}, Z = (Z_t)_{t=0,\dots,T}$ such that $Y_t \leqslant Z_t$ and $Y_t, Z_t \in \mathscr{X}$, a choice of a cancellation time $\sigma \in \mathscr{T}_0$ by the seller and a choice of an exercise time $\tau \in \mathscr{T}_0$ by the buyer.
The seller pays the buyer at time $\sigma \wedge \tau$ the following payoff:

$$R(\sigma, \tau) := Z_\sigma \mathbb{1}_{\{\sigma < \tau\}} + Y_\tau \mathbb{1}_{\{\sigma \geqslant \tau\}}.$$

In order to define the notion of a no-arbitrage price (NA price) for a GCC in an incomplete financial market, the idea is to consider for fixed stopping times $\tau, \sigma \in \mathscr{T}_0$ for buyer and seller the payoff $R(\sigma, \tau)$ as an European option. To be able to compare these options for different stopping times, we have to take the interest rate into account. Basically the idea is that the payoff $R(\sigma, \tau)$ is invested in the riskless asset at time $\sigma \wedge \tau$. This leads to the the following European option:

$$C(\sigma, \tau) := (1 + r)^{T-(\sigma\wedge\tau)} R(\sigma, \tau).$$

For this option the set of NA prices is given by

$$\Pi(C(\sigma, \tau)) := \{\mathbb{E}^{P^*}[(1 + r)^{-\sigma\wedge\tau} R(\sigma, \tau)] : P^* \in \mathscr{P}_e\}.$$

(Note that since $Z_t \in \mathscr{X}$ for all $t = 0,\dots,T$, $\mathbb{E}^{P^*}[(1 + r)^{-\sigma\wedge\tau} R(\sigma, \tau)] < \infty$ for all $\sigma, \tau \in \mathscr{T}_0$ and for all $P^* \in \mathscr{P}_e$).
But how do we define the set of NA prices of a GCC? It should be possible for the buyer to choose a stopping time such that no matter what stopping time the seller chooses, there exists a NA price for the corresponding European option which is greater or equal than our candidate for a NA price of the GCC. If this condition is not satisfied our candidate price would be too high in the sense that no matter what stopping time the buyer chooses, there would exist a stopping time for the seller such that our candidate price would be strictly above all NA prices for the corresponding European option. If we also consider that the seller should be able to find a stopping time such that our candidate price is not too low in a similar sense we are led to the following definition. (The definition extends the definition of (Föllmer and Schied 2016, Definition 6.29, p. 376) for American options to game options).

Definition 3.2. Let (Y, Z) be a GCC. Denote by $\Pi(Y, Z)$ the set of NA prices for (Y, Z). We define $\pi \in \mathbb{R}$ to be in $\Pi(Y, Z)$ if the following two conditions are satisfied.

(i) "π is not too high": there exists $\tilde{\tau} \in \mathcal{T}_0$ such that for all $\sigma \in \mathcal{T}_0$ there exists $\pi_\sigma \in \Pi(C(\sigma, \tilde{\tau}))$ with $\pi \leq \pi_\sigma$.

(ii) "π is not too low": there exists $\tilde{\sigma} \in \mathcal{T}_0$ such that for all $\tau \in \mathcal{T}_0$ there exists $\pi_\tau \in \Pi(C(\tilde{\sigma}, \tau))$ with $\pi \geq \pi_\tau$.

We will now characterize the set of NA prices of a GCC (Y, Z). Define for $P^* \in \mathcal{P}_e$

$$W_0^{P^*} := \inf_{\sigma \in \mathcal{T}_0} \sup_{\tau \in \mathcal{T}_0} \mathbb{E}^{P^*}[(1+r)^{-\sigma \wedge \tau} R(\sigma, \tau)].$$

Let us consider the case that P^* would be the unique equivalent martingale measure (hence we would have an complete financial market model). In complete markets the fair price of an option is typically defined as the cost of the cheapest superhedge against this option. By the results of (Kifer 2000) it follows that $W_0^{P^*}$ is the (unique) fair price of a GCC [4]. The following theorem characterizes the set of NA prices of a GCC. We will see that if we consider for an equivalent martingale measure the fair price that we would obtain if this measure would be the unique equivalent martingale measure, this price is also a NA price in the incomplete market framework. Furthermore the set of NA prices is an interval with bounds equal to the infimum and supremum over the fair prices obtained in the complete markets corresponding to the set of equivalent martingale measures.

Theorem 3.3. *Let (Y, Z) be a GCC.*

(i) *Then*

$$W_0^{P^*} \in \Pi(Y, Z).$$

(ii) *For all $\pi \in \Pi(Y, Z)$ the following holds*

$$\inf_{\sigma \in \mathcal{T}_0} \sup_{\tau \in \mathcal{T}_0} \inf_{P^* \in \mathcal{P}_e} \mathbb{E}^{P^*}[(1+r)^{-\sigma \wedge \tau} R(\sigma, \tau)] \leq \pi \leq \sup_{\tau \in \mathcal{T}_0} \inf_{\sigma \in \mathcal{T}_0} \sup_{P^* \in \mathcal{P}_e} \mathbb{E}^{P^*}[(1+r)^{-\sigma \wedge \tau} R(\sigma, \tau)].$$

(iii)

$$\begin{aligned} \inf \Pi(Y, Z) &= \inf_{\sigma \in \mathcal{T}_0} \sup_{\tau \in \mathcal{T}_0} \inf_{P^* \in \mathcal{P}_e} \mathbb{E}^{P^*}[(1+r)^{-\sigma \wedge \tau} R(\sigma, \tau)] \\ &= \inf_{P^* \in \mathcal{P}_e} \inf_{\sigma \in \mathcal{T}_0} \sup_{\tau \in \mathcal{T}_0} \mathbb{E}^{P^*}[(1+r)^{-\sigma \wedge \tau} R(\sigma, \tau)] = \inf_{P^* \in \mathcal{P}_e} W_0^{P^*}. \end{aligned}$$

$$\begin{aligned} \sup \Pi(Y, Z) &= \sup_{\tau \in \mathcal{T}_0} \inf_{\sigma \in \mathcal{T}_0} \sup_{P^* \in \mathcal{P}_e} \mathbb{E}^{P^*}[(1+r)^{-\sigma \wedge \tau} R(\sigma, \tau)] \\ &= \sup_{P^* \in \mathcal{P}_e} \sup_{\tau \in \mathcal{T}_0} \inf_{\sigma \in \mathcal{T}_0} \mathbb{E}^{P^*}[(1+r)^{-\sigma \wedge \tau} R(\sigma, \tau)] = \sup_{P^* \in \mathcal{P}_e} W_0^{P^*}. \end{aligned}$$

(iv) *$\Pi(Y, Z)$ is an interval.*

[4] The results of (Kifer 2000) in the binomial model can be extended to the financial market model we consider by quite similar arguments.

We will now discuss the topic of superhedging for game options in our incomplete market framework. Superhedging for both, seller and buyer, will be considered. We will use the results obtained in Section 2 for our superhedging approach. For this purpose we consider the setup of Section 2 with $\mathcal{Q}_b = \mathcal{P}_e = \mathcal{Q}_s$ and the discounted payoff processes of a GCC (Y, Z): $((1+r)^{-t} Y_t)_{t=0,\ldots,T}$ and $((1+r)^{-t} Z_t)_{t=0,\ldots,T}$. By definition every $P^* \in \mathcal{P}_e$ is equivalent to P and it can be shown that $\mathcal{P}_e$ is time-consistent[5]. The assumption that buyer and seller have no information about the set of priors of each other may not seem appropriate in our incomplete market framework. But since the concept of superhedging requires to be safe in any possible scenario, we will consider worst cases for buyer and seller which correspond to the stopping games considered in Section 2.

We start by considering the question how the seller can superhedge against a GCC. In order to superhedge against a GCC the seller has to find a stopping time and a self-financing portfolio such that her liabilities are covered by the portfolio value independent of the exercise time the buyer chooses.

Definition 3.4. Let (Y, Z) be a GCC. A superhedge for the seller is a pair (σ, ζ) consisting of a stopping time $\sigma \in \mathcal{T}_0$ and a self-financing portfolio ζ such that

$$V^{\zeta}_{\sigma \wedge t} \geq R(\sigma, t) \text{ a.s. } \forall\, t = 0, \ldots, T.$$

We are interested in finding the cheapest superhedge. The cost of this superhedge would be the lowest price for which the seller would be able to sell the GCC without facing any risk. In the following theorem we obtain that there exists a cheapest superhedge for the seller and its cost is equal to the upper bound of the set of NA prices of the considered GCC. The stopping time the seller chooses in this superhedge is

$$\sigma^{\uparrow}_0 := \inf\{s \geq 0 : W^{\uparrow}_s = (1+r)^{-s} Z_s\} \wedge T$$

which is an optimal stopping time for the seller in the two player stopping game for the worst case of the seller (see Theorem 2.3).

Theorem 3.5. *Let (Y, Z) be a GCC. There exists a self-financing portfolio $\tilde{\zeta}$ such that $(\sigma^{\uparrow}_0, \tilde{\zeta})$ is a superhedge for the seller with $V^{\tilde{\zeta}}_0 = \sup \Pi(Y, Z)$. Moreover $V^{\tilde{\zeta}}_0 = c$ where*

$$c := \inf\{V^{\zeta}_0 : \text{ there exists } \sigma \in \mathcal{T}_0 \text{ and } \zeta \text{ self-financing such that } (\sigma, \zeta) \text{ is a superhedge for the seller}\}.$$

We will now turn to the question how the buyer can superhedge against a GCC. In contrast to the seller, the buyer can only superhedge for a given price under the assumption that the GCC would be available for this price. In order to superhedge against a GCC, the buyer collects an initial investment to buy the GCC by lending money from the bank or short selling shares of the risky assets. The buyer then has to find a stopping time such that her debts which are managed in a self-financing portfolio are covered by the payoff of the GCC independent of what cancellation time the seller chooses.

Definition 3.6. Let (Y, Z) be a GCC. A superhedge for the buyer is a triple (θ, τ, ζ) consisting of an initial investment $\theta \in \mathbb{R}_+$, a stopping time $\tau \in \mathcal{T}_0$ and a self-financing portfolio ζ such that

$$V^{\zeta}_0 = -\theta \quad \text{and} \quad V^{\zeta}_{\tau \wedge t} + R(t, \tau) \geq 0 \text{ a.s. } \forall\, t = 0, \ldots, T.$$

[5] See (Föllmer and Schied 2016, Proposition 6.43, p. 386), for example.

We are interested in finding the most expensive superhedge for the buyer. The initial investment the buyer has to make for this superhedge would be the highest price the buyer could pay for the GCC without facing any risk. The next theorem states that there exists a most expensive superhedge for the buyer and the initial investment needed for this superhedge is equal to the lower bound of the set of NA prices of the GCC. The stopping time the buyer chooses in this superhedge is

$$\tau_0^{\downarrow} := \inf\{s \geqslant 0 : W_s^{\downarrow} = (1+r)^{-s} Y_s\}$$

which is an optimal stopping time for the buyer in the two player stopping game for the worst case of the buyer (see Theorem 2.2).

Theorem 3.7. *Let (Y, Z) be a GCC. There exists a self-financing portfolio $\tilde{\zeta}$ with $V_0^{\tilde{\zeta}} = -\inf\Pi(Y,Z)$ such that $(\inf\Pi(Y,Z), \tau_0^{\downarrow}, \tilde{\zeta})$ is a superhedge for the buyer. Moreover*

$$\inf\Pi(Y,Z) = m$$

where

$$m := \sup\{c \geqslant 0 : \text{ there exists } \zeta \text{ self-financing and } \tau \in \mathcal{T}_0 \text{ such that } (c, \zeta, \tau) \text{ is a superhedge for the buyer }\}.$$

4 Proofs of the Main Results

This section contains the proofs of the results of Section 2 and Section 3. We start with the proofs of Theorem 2.2 and Theorem 2.3. The key tool we will use is the following lemma which states that the law of iterated expectations also holds for multiple prior conditional expectations [6].

Lemma 4.1. *Let $(\Omega, \mathcal{F}, P_0)$ be a probability space equipped with a filtration $(\mathcal{F}_t)_{t=0,\dots,T}$ such that*

$$\mathcal{F}_0 = \{\emptyset, \Omega\} \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_T = \mathcal{F},$$

where $T \in \mathbb{N}$ denotes a finite time horizon. Let $\mathcal{Q} \neq \emptyset$ be a time -consistent set of probability measures on $(\Omega, \mathcal{F})$ that are equivalent to P_0. Let $X \in \mathcal{X}_{\mathcal{Q}}$ and $t \in \{0, \dots, T-1\}$. Then

$$\operatorname*{ess\,inf}_{P\in\mathcal{Q}} \mathbb{E}^P[\operatorname*{ess\,inf}_{Q\in\mathcal{Q}} \mathbb{E}^Q[X|\mathcal{F}_{t+1}]|\mathcal{F}_t] = \operatorname*{ess\,inf}_{P\in\mathcal{Q}} \mathbb{E}^P[X|\mathcal{F}_t].$$

Proof. Let $X \in \mathcal{X}_{\mathcal{Q}}$ and $t \in \{0, \dots, T-1\}$. At first we show that the set

$$\mathcal{C} := \{\mathbb{E}^Q[X|\mathcal{F}_{t+1}] : Q \in \mathcal{Q}\}$$

is downward directed. For this purpose let $P, Q \in \mathcal{Q}$ and define

$$A := \{\mathbb{E}^P[X|\mathcal{F}_{t+1}] < \mathbb{E}^Q[X|\mathcal{F}_{t+1}]\} \in \mathcal{F}_{t+1}.$$

Then

$$\mathbb{E}^P[X|\mathcal{F}_{t+1}] \wedge \mathbb{E}^Q[X|\mathcal{F}_{t+1}] = \mathbb{1}_A \mathbb{E}^P[X|\mathcal{F}_{t+1}] + \mathbb{1}_{A^c} \mathbb{E}^Q[X|\mathcal{F}_{t+1}]$$

[6] This result was already developed in (Riedel 2009), but under an additional weak compactness assumption that we dropped in this paper.

and Proposition A.1(ii) implies that there exists $R \in \mathscr{Q}$ such that

$$\mathbb{E}^R[X|\mathscr{F}_{t+1}] = \mathbb{1}_A \mathbb{E}^P[X|\mathscr{F}_{t+1}] + \mathbb{1}_{A^c}\mathbb{E}^Q[X|\mathscr{F}_{t+1}].$$

Hence $\mathscr{C}$ is downward directed and so there exists a nonincreasing sequence $(\mathbb{E}^{P_n}[X|\mathscr{F}_{t+1}])_{n\in\mathbb{N}}$ in $\mathscr{C}$ with

$$\lim_{n\to\infty} \mathbb{E}^{P_n}[X|\mathscr{F}_{t+1}] = \operatorname*{ess\,inf}_{Q\in\mathscr{Q}} \mathbb{E}^Q[X|\mathscr{F}_{t+1}] \quad \text{a.s.}$$

By Proposition A.6 and Proposition A.8 we obtain for each $n \in \mathbb{N}$:

$$|\mathbb{E}^{P_n}[X|\mathscr{F}_{t+1}]| \leq |\mathbb{E}^{P_1}[X|\mathscr{F}_{t+1}]| + |\operatorname*{ess\,inf}_{Q\in\mathscr{Q}} \mathbb{E}^Q[X|\mathscr{F}_{t+1}]| \in \mathscr{L}^1(\Omega,\mathscr{F},P), \ \ \forall P \in \mathscr{Q}.$$

Then by Lebesgue's dominated convergence theorem it holds for all $P \in \mathscr{Q}$

$$\begin{aligned}\mathbb{E}^P[\operatorname*{ess\,inf}_{Q\in\mathscr{Q}} \mathbb{E}^Q[X|\mathscr{F}_{t+1}]|\mathscr{F}_t] &= \mathbb{E}^P[\lim_{n\to\infty} \mathbb{E}^{P_n}[X|\mathscr{F}_{t+1}]|\mathscr{F}_t] \\ &= \lim_{n\to\infty} \mathbb{E}^P[\mathbb{E}^{P_n}[X|\mathscr{F}_{t+1}]|\mathscr{F}_t].\end{aligned}$$

By Proposition A.5 for each $n \in \mathbb{N}$ there exists $R_n \in \mathscr{Q}$ such that

$$\mathbb{E}^P[\mathbb{E}^{P_n}[X|\mathscr{F}_{t+1}]|\mathscr{F}_t] = \mathbb{E}^{R_n}[X|\mathscr{F}_t] \geq \operatorname*{ess\,inf}_{P\in\mathscr{Q}} \mathbb{E}^P[X|\mathscr{F}_t].$$

From this it follows for all $P \in \mathscr{Q}$

$$\begin{aligned}\mathbb{E}^P[\operatorname*{ess\,inf}_{Q\in\mathscr{Q}} \mathbb{E}^Q[X|\mathscr{F}_{t+1}]|\mathscr{F}_t] &= \lim_{n\to\infty} \mathbb{E}^P[\mathbb{E}^{P_n}[X|\mathscr{F}_{t+1}]|\mathscr{F}_t] \\ &\geq \operatorname*{ess\,inf}_{P\in\mathscr{Q}} \mathbb{E}^P[X|\mathscr{F}_t]\end{aligned}$$

and so

$$\operatorname*{ess\,inf}_{P\in\mathscr{Q}} \mathbb{E}^P[\operatorname*{ess\,inf}_{Q\in\mathscr{Q}} \mathbb{E}^Q[X|\mathscr{F}_{t+1}]|\mathscr{F}_t] \geq \operatorname*{ess\,inf}_{P\in\mathscr{Q}} \mathbb{E}^P[X|\mathscr{F}_t].$$

The other inequality just follows from the classical law of iterated expectations:

$$\begin{aligned}&\mathbb{E}^P[\operatorname*{ess\,inf}_{Q\in\mathscr{Q}} \mathbb{E}^Q[X|\mathscr{F}_{t+1}]|\mathscr{F}_t] \leq \mathbb{E}^P[\mathbb{E}^P[X|\mathscr{F}_{t+1}]|\mathscr{F}_t] = \mathbb{E}^P[X|\mathscr{F}_t], \ \forall P \in \mathscr{Q} \\ &\Rightarrow \operatorname*{ess\,inf}_{P\in\mathscr{Q}} \mathbb{E}^P[\operatorname*{ess\,inf}_{Q\in\mathscr{Q}} \mathbb{E}^Q[X|\mathscr{F}_{t+1}]|\mathscr{F}_t] \leq \operatorname*{ess\,inf}_{P\in\mathscr{Q}} \mathbb{E}^P[X|\mathscr{F}_t].\end{aligned}$$

□

Proof of Theorem 2.2. We start by showing some integrability conditions. Since for all $t = 0,\dots,T$, $Y_t, Z_t \in \mathscr{X}_{\mathscr{Q}_b}$ are nonnegative and

$$\begin{aligned}&0 \leq R(\sigma,\tau) \leq Z_{\sigma\wedge\tau} \leq \sum_{t=0}^{T} Z_t \in X_{\mathscr{Q}_b}, \ \forall \sigma,\tau \in \mathscr{T}_0, \\ &0 \leq Y_t \leq W_t^{\downarrow} \leq Z_t \in X_{\mathscr{Q}_b}, \ \forall t = 0,\dots,T,\end{aligned}$$

we obtain $R(\sigma,\tau) \in X_{\mathscr{Q}_b}$ for all $\sigma,\tau \in \mathscr{T}_0$ and $W_t^{\downarrow} \in \mathscr{X}_{\mathscr{Q}_b}$ for all $t = 0,\dots,T$, by Proposition A.3. Furthermore we obtain by considering any $P \in \mathscr{Q}_b$

$$0 \leq \operatorname*{ess\,inf}_{P\in\mathscr{Q}_b} \mathbb{E}^P[R(\sigma,\tau)|\mathscr{F}_t] \leq \mathbb{E}^P[R(\sigma,\tau)|\mathscr{F}_t] \in X_{\mathscr{Q}_b}.$$

Hence $\operatorname{ess\,inf}_{P\in\mathcal{Q}_b}\mathbb{E}^P[R(\sigma,\tau)|\mathcal{F}_t]\in X_{\mathcal{Q}_b}$ for all $\sigma,\tau\in\mathcal{T}_0$ and for all $t=0,\dots,T$.
We show (1) by backward induction (notice that by definition $\sigma_t^{\downarrow},\tau_t^{\downarrow}\in\mathcal{T}_t$, because $W_T^{\downarrow}=Y_T$). By definition $W_T^{\downarrow}=Y_T$ and for all $\sigma,\tau\in\mathcal{T}_T$ we have $\sigma\equiv T\equiv\tau$. Since $R(T,T)=Y_T\in\mathcal{X}_{\mathcal{Q}_b}$ is $\mathcal{F}_T$-measurable, (1) is satisfied by Proposition A.9. Now assume that (1) is satisfied for $t+1$. Let $\sigma\in\mathcal{T}_t$ and consider the following decomposition of $W_t^{\downarrow}$:

$$W_t^{\downarrow}=\mathbb{1}_{\{t=\tau_t^{\downarrow}\}}W_t^{\downarrow}+\mathbb{1}_{\{t<\tau_t^{\downarrow}\}\cap\{t<\sigma\}}W_t^{\downarrow}+\mathbb{1}_{\{t<\tau_t^{\downarrow}\}\cap\{t=\sigma\}}W_t^{\downarrow}.$$

Since $\mathbb{1}_{\{t=\tau_t^{\downarrow}\}}$, $\mathbb{1}_{\{t<\tau_t^{\downarrow}\}\cap\{t<\sigma\}}$ and $\mathbb{1}_{\{t<\tau_t^{\downarrow}\}\cap\{t=\sigma\}}$ are $\mathcal{F}_t$-measurable, by Proposition A.9 we obtain on the set $\{t=\tau_t^{\downarrow}\}$

$$W_t^{\downarrow}=Y_t=\operatorname*{ess\,inf}_{P\in\mathcal{Q}_b}\mathbb{E}^P[Y_t|\mathcal{F}_t]=\operatorname*{ess\,inf}_{P\in\mathcal{Q}_b}\mathbb{E}^P[R(\sigma,t)|\mathcal{F}_t]=\operatorname*{ess\,inf}_{P\in\mathcal{Q}_b}\mathbb{E}^P[R(\sigma,\tau_t^{\downarrow})|\mathcal{F}_t],$$

on the set $\{t<\tau_t^{\downarrow}\}\cap\{t<\sigma\}$

$$\begin{aligned}W_t^{\downarrow}&\leq\max(Y_t,\operatorname*{ess\,inf}_{P\in\mathcal{Q}_b}\mathbb{E}^P[W_{t+1}^{\downarrow}|\mathcal{F}_t])=\operatorname*{ess\,inf}_{P\in\mathcal{Q}_b}\mathbb{E}^P[W_{t+1}^{\downarrow}|\mathcal{F}_t]\\&\leq\operatorname*{ess\,inf}_{P\in\mathcal{Q}_b}\mathbb{E}^P[\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_b}\mathbb{E}^Q[R(\max(\sigma,t+1),\tau_{t+1}^{\downarrow})|\mathcal{F}_{t+1}]|\mathcal{F}_t]\\&=\operatorname*{ess\,inf}_{P\in\mathcal{Q}_b}\mathbb{E}^P[R(\max(\sigma,t+1),\tau_{t+1}^{\downarrow})|\mathcal{F}_t]=\operatorname*{ess\,inf}_{P\in\mathcal{Q}_b}\mathbb{E}^P[R(\sigma,\tau_t^{\downarrow})|\mathcal{F}_t],\end{aligned}$$

and on the set $\{t<\tau_t^{\downarrow}\}\cap\{t=\sigma\}$

$$W_t^{\downarrow}\leq Z_t=\operatorname*{ess\,inf}_{P\in\mathcal{Q}_b}\mathbb{E}^P[Z_t|\mathcal{F}_t]=\operatorname*{ess\,inf}_{P\in\mathcal{Q}_b}\mathbb{E}^P[R(\sigma,\tau_t^{\downarrow})|\mathcal{F}_t].$$

Hence the second inequality in (1) holds true. Let $\tau\in\mathcal{T}_t$ and consider the following decomposition of $W_t^{\downarrow}$:

$$W_t^{\downarrow}=\mathbb{1}_{\{t=\sigma_t^{\downarrow}\}}W_t^{\downarrow}+\mathbb{1}_{\{t<\sigma_t^{\downarrow}\}\cap\{t<\tau\}}W_t^{\downarrow}+\mathbb{1}_{\{t<\sigma_t^{\downarrow}\}\cap\{t=\tau\}}W_t^{\downarrow}.$$

We obtain on the set $\{t=\sigma_t^{\downarrow}\}$

$$W_t^{\downarrow}=Z_t=\operatorname*{ess\,inf}_{P\in\mathcal{Q}_b}\mathbb{E}^P[Z_t|\mathcal{F}_t]\geq\operatorname*{ess\,inf}_{P\in\mathcal{Q}_b}\mathbb{E}^P[R(t,\tau)|\mathcal{F}_t]=\operatorname*{ess\,inf}_{P\in\mathcal{Q}_b}\mathbb{E}^P[R(\sigma_t^{\downarrow},\tau)|\mathcal{F}_t],$$

on the set $\{t<\sigma_t^{\downarrow}\}\cap\{t<\tau\}$

$$\begin{aligned}W_t^{\downarrow}&\geq\min(Z_t,\operatorname*{ess\,inf}_{P\in\mathcal{Q}_b}\mathbb{E}^P[W_{t+1}^{\downarrow}|\mathcal{F}_t])=\operatorname*{ess\,inf}_{P\in\mathcal{Q}_b}\mathbb{E}^P[W_{t+1}^{\downarrow}|\mathcal{F}_t]\\&\geq\operatorname*{ess\,inf}_{P\in\mathcal{Q}_b}\mathbb{E}^P[\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_b}\mathbb{E}^Q[R(\sigma_{t+1}^{\downarrow},\max(\tau,t+1))|\mathcal{F}_{t+1}]|\mathcal{F}_t]\\&=\operatorname*{ess\,inf}_{P\in\mathcal{Q}_b}\mathbb{E}^P[R(\sigma_{t+1}^{\downarrow},\max(\tau,t+1))|\mathcal{F}_t]=\operatorname*{ess\,inf}_{P\in\mathcal{Q}_b}\mathbb{E}^P[R(\sigma_t^{\downarrow},\tau)|\mathcal{F}_t],\end{aligned}$$

and on the set $\{t<\sigma_t^{\downarrow}\}\cap\{t=\tau\}$

$$W_t^{\downarrow}\geq Y_t=\operatorname*{ess\,inf}_{P\in\mathcal{Q}_b}\mathbb{E}^P[Y_t|\mathcal{F}_t]=\operatorname*{ess\,inf}_{P\in\mathcal{Q}_b}\mathbb{E}^P[R(\sigma_t^{\downarrow},\tau)|\mathcal{F}_t].$$

So we have shown that equation (1) holds true. From this inequality we obtain for all $t = 0, \dots, T$:

$$
\begin{aligned}
W_t^{\downarrow} &\leqslant \operatorname*{ess\,inf}_{P \in \mathcal{Q}_b} \mathbb{E}^P[R(\sigma, \tau_t^{\downarrow}) | \mathscr{F}_t], \ \forall \sigma \in \mathscr{F}_t \\
\Rightarrow W_t^{\downarrow} &\leqslant \operatorname*{ess\,inf}_{\sigma \in \mathscr{T}_t} \operatorname*{ess\,inf}_{P \in \mathcal{Q}_b} \mathbb{E}^P[R(\sigma, \tau_t^{\downarrow}) | \mathscr{F}_t] \\
&\leqslant \operatorname*{ess\,sup}_{\tau \in \mathscr{T}_t} \operatorname*{ess\,inf}_{\sigma \in \mathscr{T}_t} \operatorname*{ess\,inf}_{P \in \mathcal{Q}_b} \mathbb{E}^P[R(\sigma, \tau) | \mathscr{F}_t], \\
W_t^{\downarrow} &\geqslant \operatorname*{ess\,inf}_{P \in \mathcal{Q}_b} \mathbb{E}^P[R(\sigma_t^{\downarrow}, \tau) | \mathscr{F}_t], \ \forall \tau \in \mathscr{F}_t \\
\Rightarrow W_t^{\downarrow} &\geqslant \operatorname*{ess\,sup}_{\tau \in \mathscr{T}_t} \operatorname*{ess\,inf}_{P \in \mathcal{Q}_b} \mathbb{E}^P[R(\sigma_t^{\downarrow}, \tau) | \mathscr{F}_t] \\
&\geqslant \operatorname*{ess\,inf}_{\sigma \in \mathscr{T}_t} \operatorname*{ess\,sup}_{\tau \in \mathscr{T}_t} \operatorname*{ess\,inf}_{P \in \mathcal{Q}_b} \mathbb{E}^P[R(\sigma, \tau) | \mathscr{F}_t].
\end{aligned}
$$

Hence

$$
\begin{aligned}
\operatorname*{ess\,inf}_{\sigma \in \mathscr{T}_t} \operatorname*{ess\,sup}_{\tau \in \mathscr{T}_t} \operatorname*{ess\,inf}_{P \in \mathcal{Q}_b} \mathbb{E}^P[R(\sigma, \tau) | \mathscr{F}_t] \leqslant W_t^{\downarrow} & \\
&\leqslant \operatorname*{ess\,sup}_{\tau \in \mathscr{T}_t} \operatorname*{ess\,inf}_{\sigma \in \mathscr{T}_t} \operatorname*{ess\,inf}_{P \in \mathcal{Q}_b} \mathbb{E}^P[R(\sigma, \tau) | \mathscr{F}_t] \\
&\leqslant \operatorname*{ess\,inf}_{\sigma \in \mathscr{T}_t} \operatorname*{ess\,sup}_{\tau \in \mathscr{T}_t} \operatorname*{ess\,inf}_{P \in \mathcal{Q}_b} \mathbb{E}^P[R(\sigma, \tau) | \mathscr{F}_t]
\end{aligned}
$$

and so

$$
\begin{aligned}
W_t^{\downarrow} &= \operatorname*{ess\,inf}_{\sigma \in \mathscr{T}_t} \operatorname*{ess\,sup}_{\tau \in \mathscr{T}_t} \operatorname*{ess\,inf}_{P \in \mathcal{Q}_b} \mathbb{E}^P[R(\sigma, \tau) | \mathscr{F}_t] \\
&= \operatorname*{ess\,sup}_{\tau \in \mathscr{T}_t} \operatorname*{ess\,inf}_{\sigma \in \mathscr{T}_t} \operatorname*{ess\,inf}_{P \in \mathcal{Q}_b} \mathbb{E}^P[R(\sigma, \tau) | \mathscr{F}_t].
\end{aligned}
$$

□

Proof of Theorem 2.3. Let $X \in \mathscr{X}_{\mathcal{Q}_s}$ and $\tau \in \mathscr{T}_0$. Then by definition of the essential infimum

$$\operatorname*{ess\,sup}_{P \in \mathcal{Q}_s} \mathbb{E}^P[X | \mathscr{F}_\tau] = -\operatorname*{ess\,inf}_{P \in \mathcal{Q}_s} \mathbb{E}^P[-X | \mathscr{F}_\tau].$$

Hence by Proposition A.8 we obtain for all $Q \in \mathcal{Q}_s$

$$\operatorname*{ess\,sup}_{P \in \mathcal{Q}_s} \mathbb{E}^P[X | \mathscr{F}_\tau] \in \mathscr{L}^1(\Omega, \mathscr{F}, Q)$$

and we also obtain that Proposition A.9(i),(ii) and (iv) hold true for the nonlinear operator $\operatorname{ess\,sup}_{P \in \mathcal{Q}_s} \mathbb{E}^P[\cdot | \mathscr{F}_\tau]$ as well.
Let $X \in \mathscr{X}_{\mathcal{Q}_s}$ and $t \in \{0, \dots, T-1\}$. Then by Lemma 4.1

$$
\begin{aligned}
\operatorname*{ess\,sup}_{P \in \mathcal{Q}_s} \mathbb{E}^P[\operatorname*{ess\,sup}_{Q \in \mathcal{Q}_s} \mathbb{E}^Q[X | \mathscr{F}_{t+1}] | \mathscr{F}_t] &= -\operatorname*{ess\,inf}_{P \in \mathcal{Q}_s} \mathbb{E}^P[\operatorname*{ess\,inf}_{Q \in \mathcal{Q}_s} \mathbb{E}^Q[-X | \mathscr{F}_{t+1}] | \mathscr{F}_t] \\
&= -\operatorname*{ess\,inf}_{P \in \mathcal{Q}_s} \mathbb{E}^P[-X | \mathscr{F}_t] \\
&= \operatorname*{ess\,sup}_{P \in \mathcal{Q}_s} \mathbb{E}^P[X | \mathscr{F}_t].
\end{aligned}
$$

and so the law of iterated expectations also holds true for the nonlinear operator $\operatorname{ess\,sup}_{P \in \mathcal{Q}_s} \mathbb{E}^P[\cdot | \mathscr{F}_t]$. Hence the proof follows by the same arguments as used in the proof of Theorem 2.2. The only difference is that we do not necessarily obtain $\operatorname{ess\,sup}_{P\in\mathcal{Q}_s}\mathbb{E}^P[R(\sigma,\tau)|\mathcal{F}_t]\in\mathcal{X}_{\mathcal{Q}_s}$ for all $\sigma,\tau\in\mathcal{T}_0$ and $t=0,\dots,T$. But since $\operatorname{ess\,sup}_{P\in\mathcal{Q}_s}\mathbb{E}^P[R(\sigma,\tau)|\mathcal{F}_t]\in\mathcal{L}^1(\Omega,\mathcal{F},Q)$ for all $Q\in\mathcal{Q}_s$ and Proposition A.9(i) also holds under this weaker integrability condition, this does not cause any problems. □

We will now continue with the proofs of the results of Section 3.

Proof of Theorem 3.3.

(i) Let $P^*\in\mathcal{P}_e$. Let us consider the stopping games of Section 2 with the discounted potential payoff processes of the GCC and $\mathcal{Q}_b=\{P^*\}=\mathcal{Q}_s$. Then by Theorem 2.2 (Theorem 2.3) there exist stopping times $\sigma_0^*,\tau_0^*\in\mathcal{T}_0$ such that for all $\sigma,\tau\in\mathcal{T}_0$:

$$\mathbb{E}^{P^*}[(1+r)^{-\sigma_0^*\wedge\tau}R(\sigma_0^*,\tau)]\leqslant W_0^{P^*}\leqslant\mathbb{E}^{P^*}[(1+r)^{-\sigma\wedge\tau_0^*}R(\sigma,\tau_0^*)].$$

Hence τ_0^* satisfies condition 3.2(i) and σ_0^* satisfies condition 3.2(ii) which implies $W_0^{P^*}\in\Pi(Y,Z)$.

(ii) Let $\pi\in\Pi(Y,Z)$ (we know by (i) that $\Pi(Y,Z)$ is not empty). By Definition 3.2(i) there exists $\tilde\tau\in\mathcal{T}_0$ such that for all $\sigma\in\mathcal{T}_0$

$$\pi\leqslant\sup_{P^*\in\mathcal{P}_e}\mathbb{E}^{P^*}[(1+r)^{-\sigma\wedge\tilde\tau}R(\sigma,\tilde\tau)].$$

Hence

$$\begin{aligned}\pi&\leqslant\inf_{\sigma\in\mathcal{T}_0}\sup_{P^*\in\mathcal{P}_e}\mathbb{E}^{P^*}[(1+r)^{-\sigma\wedge\tilde\tau}R(\sigma,\tilde\tau)]\\&\leqslant\sup_{\tau\in\mathcal{T}_0}\inf_{\sigma\in\mathcal{T}_0}\sup_{P^*\in\mathcal{P}_e}\mathbb{E}^{P^*}[(1+r)^{-\sigma\wedge\tau}R(\sigma,\tau)].\end{aligned}$$

By Definition 3.2(ii) there exists $\tilde\sigma\in\mathcal{T}_0$ such that for all $\tau\in\mathcal{T}_0$

$$\pi\geqslant\inf_{P^*\in\mathcal{P}_e}\mathbb{E}^{P^*}[(1+r)^{-\tilde\sigma\wedge\tau}R(\tilde\sigma,\tau)].$$

Hence

$$\begin{aligned}\pi&\geqslant\sup_{\tau\in\mathcal{T}_0}\inf_{P^*\in\mathcal{P}_e}\mathbb{E}^{P^*}[(1+r)^{-\tilde\sigma\wedge\tau}R(\tilde\sigma,\tau)]\\&\geqslant\inf_{\sigma\in\mathcal{T}_0}\sup_{\tau\in\mathcal{T}_0}\inf_{P^*\in\mathcal{P}_e}\mathbb{E}^{P^*}[(1+r)^{-\sigma\wedge\tau}R(\sigma,\tau)].\end{aligned}$$

(iii) Fix $\sigma\in\mathcal{T}_0$ and define for $t=0,\dots,T$,

$$H_t^\sigma:=(1+r)^{-\sigma\wedge t}R(\sigma,t).$$

Then H_t^σ is $\mathcal{F}_t$-measurable and, since $Z_t\in\mathcal{X}$, we have for all $t=0,\dots,T$:

$$\sup_{P^*\in\mathcal{P}_e}\mathbb{E}^{P^*}[H_t^\sigma]<\infty.$$

With the duality result (Föllmer and Schied 2016, Theorem 6.46, p. 388) we obtain for all $\sigma\in\mathcal{T}_0$:

$$\begin{aligned}\sup_{\tau\in\mathcal{T}_0}\inf_{P^*\in\mathcal{P}_e}\mathbb{E}^{P^*}[(1+r)^{-\sigma\wedge\tau}R(\sigma,\tau)]&=\sup_{\tau\in\mathcal{T}_0}\inf_{P^*\in\mathcal{P}_e}\mathbb{E}^{P^*}[H_\tau^\sigma]\\&=\inf_{P^*\in\mathcal{P}_e}\sup_{\tau\in\mathcal{T}_0}\mathbb{E}^{P^*}[H_\tau^\sigma]\\&=\inf_{P^*\in\mathcal{P}_e}\sup_{\tau\in\mathcal{T}_0}\mathbb{E}^{P^*}[(1+r)^{-\sigma\wedge\tau}R(\sigma,\tau)].\end{aligned}$$

Hence

$$\inf_{\sigma\in\mathcal{T}_0}\sup_{\tau\in\mathcal{T}_0}\inf_{P^*\in\mathcal{P}_e}\mathbb{E}^{P^*}[(1+r)^{-\sigma\wedge\tau}R(\sigma,\tau)] = \inf_{\sigma\in\mathcal{T}_0}\inf_{P^*\in\mathcal{P}_e}\sup_{\tau\in\mathcal{T}_0}\mathbb{E}^{P^*}[(1+r)^{-\sigma\wedge\tau}R(\sigma,\tau)]$$
$$= \inf_{P^*\in\mathcal{P}_e}\inf_{\sigma\in\mathcal{T}_0}\sup_{\tau\in\mathcal{T}_0}\mathbb{E}^{P^*}[(1+r)^{-\sigma\wedge\tau}R(\sigma,\tau)]$$
$$= \inf_{P^*\in\mathcal{P}_e} W_0^{P^*}.$$

With similar arguments we also obtain

$$\sup_{\tau\in\mathcal{T}_0}\inf_{\sigma\in\mathcal{T}_0}\sup_{P^*\in\mathcal{P}_e}\mathbb{E}^{P^*}[(1+r)^{-\sigma\wedge\tau}R(\sigma,\tau)] = \sup_{P^*\in\mathcal{P}_e} W_0^{P^*}.$$

By considering (i) and (ii) we conclude

$$\inf\Pi(Y,Z) = \inf_{P^*\in\mathcal{P}_e} W_0^{P^*},$$
$$\sup\Pi(Y,Z) = \sup_{P^*\in\mathcal{P}_e} W_0^{P^*}.$$

(iv) Let $\pi_1,\pi_2\in\Pi(Y,Z)$ and $\lambda\in[0,1]$. Assume w.l.o.g. that $\pi_1\leqslant\pi_2$. Define

$$x := \lambda\pi_1 + (1-\lambda)\pi_2.$$

Since $\pi_1\leqslant x\leqslant\pi_2$ and $\pi_1,\pi_2\in\Pi(Y,Z)$, x is not too high and not too low in the sense of Definition 3.2. Hence $x\in\Pi(Y,Z)$ which implies that $\Pi(Y,Z)\subseteq\mathbb{R}$ is convex and so an interval. □

In the following we will proof the superhedging results of Theorem 3.5 and Theorem 3.7. We will use the following lemma which states that the discounted stopped value process of a self-financing portfoilio in the market model of Chaper 3 is a uniform $\mathcal{P}_e$-martingale. The proof of the lemma follows by similar arguments as used in (Föllmer and Schied 2016, Theorem 5.14, p. 299).

Lemma 4.2. *Let $\sigma\in\mathcal{T}_0$ and $P^*\in\mathcal{P}_e$. Let ζ be a self-financing portfolio such that $\mathbb{E}^*[(\tilde{V}^\zeta_{\sigma\wedge t})^-]<\infty$ for all $t=0,\dots,T$. Then $(\tilde{V}^\zeta_{\sigma\wedge t})_{t=0,\dots,T}$ is a P^*-martingale.*

Proof of Theorem 3.5. Let (σ,ζ) be a superhedge for the seller. (There always exists a superhedge for the seller, consider $\sigma\equiv 0$ and any self-financing portfolio ζ with $V_0^\zeta\geqslant Z_0$ for example). ζ is self-financing and $V^\zeta_{\sigma\wedge t}\geqslant R(\sigma,t)\geqslant 0$ a.s. for all $t=0,\dots,T$, hence $(\tilde{V}^\zeta_{\sigma\wedge t})_{t=0,\dots,T}$ is a P^*-martingale for each $P^*\in\mathcal{P}_e$. Let $P^*\in\mathcal{P}_e$. Then for each $\tau\in\mathcal{T}_0$:

$$V_0^\zeta = \mathbb{E}^{P^*}[\tilde{V}^\zeta_{\sigma\wedge\tau}]\geqslant\mathbb{E}^{P^*}[(1+r)^{-\sigma\wedge\tau}R(\sigma,\tau)]$$
$$\geqslant\inf_{\sigma\in\mathcal{T}_0}\mathbb{E}^{P^*}[(1+r)^{-\sigma\wedge\tau}R(\sigma,\tau)].$$

So by definition for all $\tau\in\mathcal{T}_0$ and for all $P^*\in\mathcal{P}_e$

$$c\geqslant\inf_{\sigma\in\mathcal{T}_0}\mathbb{E}^{P^*}[(1+r)^{-\sigma\wedge\tau}R(\sigma,\tau)]$$

which gives us

$$c\geqslant\sup_{P^*\in\mathcal{P}_e}\sup_{\tau\in\mathcal{T}_0}\inf_{\sigma\in\mathcal{T}_0}\mathbb{E}^{P^*}[(1+r)^{-\sigma\wedge\tau}R(\sigma,\tau)]\overset{3.3}{=}\sup\Pi(Y,Z). \tag{2}$$

Let $\sigma \in \mathcal{T}_0$ and define $(U_t^\sigma)_{t=0,\dots,T}$ by

$$U_t^\sigma := \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \operatorname*{ess\,sup}_{P^*\in\mathcal{P}_e} \mathbb{E}^{P^*}[(1+r)^{-\sigma\wedge\tau}R(\sigma,\tau)|\mathcal{F}_t].$$

$C_t^\sigma := (1+r)^{-\sigma\wedge t}R(\sigma,t)$ is nonnegative, $\mathcal{F}_t$-measurable and $\sup_{P^*\in\mathcal{P}_e}\mathbb{E}^{P^*}[C_t^\sigma] < \infty$ for all $t = 0,\dots,T$, hence $(U_t^\sigma)_{t=0,\dots,T}$ is the smallest uniform $\mathcal{P}_e$-supermartingale that dominates $(C_t^\sigma)_{t=0,\dots,T}$ [7]. By the uniform Doob decomposition [8] there exists a nondecreasing adapted process $(A_t^\sigma)_{t=0,\dots,T}$ with
$A_0^\sigma = 0$ and a D-dimensional predictable process $(\theta_t^\sigma)_{t=1,\dots,T} = (\theta_t^{\sigma,1},\dots,\theta_t^{\sigma,D})_{t=1,\dots,T}$ such that for all $t = 0,\dots,T$:

$$U_t^\sigma = U_0^\sigma + \sum_{s=1}^{t}\sum_{d=1}^{D}\theta_s^{\sigma,d}(\tilde{S}_s^d - \tilde{S}_{s-1}^d) - A_t^\sigma \text{ a.s.}$$

There exists a self-financing portfolio [9] $(\zeta_t^\sigma)_{t=1,\dots,T} = (\zeta_t^{\sigma,0},\theta_t^\sigma)_{t=1,\dots,T}$ with $V_0^{\zeta^\sigma} = U_0^\sigma$. Since $(\zeta_t^\sigma)_{t=1,\dots,T}$ is self-financing this implies for each $t = 0,\dots,T$:

$$\tilde{V}_t^{\zeta^\sigma} = U_0^\sigma + \sum_{s=1}^{t}\sum_{d=1}^{D}\theta_s^{\sigma,d}(\tilde{S}_s^d - \tilde{S}_{s-1}^d) = U_t^\sigma + A_t^\sigma.$$

Since we have for all $t = 0,\dots,T$

$$\begin{aligned}
\tilde{V}_{\sigma\wedge t}^{\zeta^\sigma} &= \mathbb{1}_{\{t\leqslant\sigma\}}\tilde{V}_t^{\zeta^\sigma} + \mathbb{1}_{\{t>\sigma\}}\tilde{V}_\sigma^{\zeta^\sigma} \\
&\geqslant \mathbb{1}_{\{t\leqslant\sigma\}}U_t^\sigma + \mathbb{1}_{\{t>\sigma\}}U_\sigma^\sigma \\
&\geqslant \mathbb{1}_{\{t\leqslant\sigma\}}C_t^\sigma + \mathbb{1}_{\{t>\sigma\}}U_\sigma^\sigma
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{1}_{\{t>\sigma\}}U_\sigma^\sigma &= \mathbb{1}_{\{t>\sigma\}}\sum_{s=0}^{T}\mathbb{1}_{\{\sigma=s\}}U_s^\sigma \\
&= \mathbb{1}_{\{t>\sigma\}}\sum_{s=0}^{T}\mathbb{1}_{\{\sigma=s\}}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_s}\operatorname*{ess\,sup}_{P^*\in\mathcal{P}_e}\mathbb{E}^{P^*}[(1+r)^{-\sigma\wedge\tau}R(\sigma,\tau)|\mathcal{F}_s] \\
&\geqslant \mathbb{1}_{\{t>\sigma\}}\sum_{s=0}^{T}\mathbb{1}_{\{\sigma=s\}}\operatorname*{ess\,sup}_{P^*\in\mathcal{P}_e}\mathbb{E}^{P^*}[(1+r)^{-\sigma\wedge T}R(\sigma,T)|\mathcal{F}_s] \\
&\geqslant \mathbb{1}_{\{t>\sigma\}}\operatorname*{ess\,sup}_{P^*\in\mathcal{P}_e}(\sum_{s=0}^{T}\mathbb{1}_{\{\sigma=s\}}\mathbb{E}^{P^*}[(1+r)^{-\sigma\wedge T}R(\sigma,T)|\mathcal{F}_s]) \\
&= \mathbb{1}_{\{t>\sigma\}}\operatorname*{ess\,sup}_{P^*\in\mathcal{P}_e}(\sum_{s=0}^{T}\mathbb{1}_{\{\sigma=s\}}\mathbb{E}^{P^*}[(1+r)^{-\sigma\wedge T}R(\sigma,T)|\mathcal{F}_\sigma]) \\
&= \mathbb{1}_{\{t>\sigma\}}\operatorname*{ess\,sup}_{P^*\in\mathcal{P}_e}\mathbb{E}^{P^*}[(1+r)^{-\sigma\wedge T}R(\sigma,T)|\mathcal{F}_\sigma] \\
&\geqslant \operatorname*{ess\,sup}_{P^*\in\mathcal{P}_e}\mathbb{E}^{P^*}[\mathbb{1}_{\{t>\sigma\}}(1+r)^{-\sigma\wedge T}R(\sigma,T)|\mathcal{F}_\sigma] \\
&\geqslant \mathbb{1}_{\{t>\sigma\}}(1+r)^{-\sigma}Z_\sigma \\
&= \mathbb{1}_{\{t>\sigma\}}(1+r)^{-\sigma\wedge t}R(\sigma,t) \\
&= \mathbb{1}_{\{t>\sigma\}}C_t^\sigma
\end{aligned}$$

[7] See (Föllmer and Schied 2016, Theorem 7.2, p. 395), for example.
[8] See (Föllmer and Schied 2016, Theorem 7.5, p. 397), for example.
[9] See (Föllmer and Schied 2016, p. 295), for example.

it follows that

$$V_{\sigma\wedge t}^{\zeta^\sigma} \geqslant R(\sigma, t) \text{ a.s. } \forall\, t = 0, \dots, T.$$

So (σ, ζ^σ) is a superhedge for the seller with

$$V_0^{\zeta^\sigma} = U_0^\sigma = \sup_{\tau\in\mathcal{T}_0} \sup_{P^*\in\mathcal{P}_e} \mathbb{E}^{P^*}[(1+r)^{-\sigma\wedge\tau} R(\sigma,\tau)].$$

Hence

$$\begin{aligned} c \leqslant V_0^{\zeta^{\sigma_0^\uparrow}} &= \sup_{\tau\in\mathcal{T}_0} \sup_{P^*\in\mathcal{P}_e} \mathbb{E}^{P^*}[(1+r)^{-\sigma_0^\uparrow\wedge\tau} R(\sigma_0^\uparrow,\tau)] \\ &\overset{2.3}{=} \sup_{\tau\in\mathcal{T}_0} \inf_{\sigma\in\mathcal{T}_0} \sup_{P^*\in\mathcal{P}_e} \mathbb{E}^{P^*}[(1+r)^{-\sigma\wedge\tau} R(\sigma,\tau)] \\ &\overset{3.3}{=} \sup \Pi(Y,Z) \overset{(2)}{\leqslant} c. \end{aligned}$$

□

Proof of Theorem 3.7. Let (θ, τ, ζ) be a superhedge for the buyer. (If we consider $\theta = 0$ and $\zeta_t^d = 0$ for all $d = 0, \dots, D$ and for all $t = 1, \dots, T$ the conditions of Definition 3.6 are satisfied for any $\tau \in \mathcal{T}_0$ which implies that there always exists a superhedge for the buyer). Then $V_{\tau\wedge t}^\zeta + R(t,\tau) \geqslant 0$ a.s. for all $t = 0, \dots, T$, hence we obtain for all $t = 0, \dots, T$ and for all $P^* \in \mathcal{P}^*$

$$\mathbb{E}^{P^*}[(\tilde{V}_{\tau\wedge t}^\zeta)^-] \leqslant \mathbb{E}^{P^*}[(1+r)^{-t\wedge\tau} R(t,\tau)] \leqslant \mathbb{E}^{P^*}[\sum_{t=0}^T Z_t] < \infty.$$

So $(\tilde{V}_{\tau\wedge t}^\zeta)_{t=0,\dots,T}$ is a P^*-martingale for all $P^* \in \mathcal{P}_e$. Hence for all $\sigma \in \mathcal{T}_0$ and $P^* \in \mathcal{P}_e$, we obtain

$$\begin{aligned} \theta = -V_0^\zeta = \mathbb{E}^{P^*}[-\tilde{V}_{\tau\wedge\sigma}^\zeta] &\leqslant \mathbb{E}^{P^*}[(1+r)^{-\sigma\wedge\tau} R(\sigma,\tau)] \\ &\leqslant \sup_{\tau\in\mathcal{T}_0} \mathbb{E}^{P^*}[(1+r)^{-\sigma\wedge\tau} R(\sigma,\tau)]. \end{aligned}$$

Hence, by definition

$$m \leqslant \inf_{P^*\in\mathcal{P}_e} \inf_{\sigma\in\mathcal{T}} \sup_{\tau\in\mathcal{T}_0} \mathbb{E}^{P^*}[(1+r)^{-\sigma\wedge\tau} R(\sigma,\tau)] \overset{3.3}{=} \inf \Pi(Y,Z). \tag{3}$$

Fix $\tau \in \mathcal{T}_0$ and consider $((1+r)^{t-(t\wedge\tau)} R(t,\tau))_{t=0,\dots,T}$ as an American option. There exists a self-financing portfolio ζ^τ such that [10]

$$\tilde{V}_t^{\zeta^\tau} \geqslant (1+r)^{-t\wedge\tau} R(t,\tau) \geqslant 0 \text{ a.s. } \forall\, t = 0, \dots, T.$$

Hence we can interpret $(\tilde{V}_t^{\zeta^\tau} - (1+r)^{-t\wedge\tau} R(t,\tau))_{t=0,\dots,T}$ also as an American option. Define $(U_t^\tau)_{t=0,\dots,T}$ by

$$U_t^\tau := \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_t} \operatorname*{ess\,sup}_{P^*\in\mathcal{P}_e} \mathbb{E}^{P^*}[\tilde{V}_\sigma^{\zeta^\tau} - (1+r)^{-\sigma\wedge\tau} R(\sigma,\tau) | \mathcal{F}_t].$$

[10] See (Föllmer and Schied 2016, Corollary 7.9, p. 401), for example.

Then $(U_t^\tau)_{t=0,\dots,T}$ is a uniform $\mathscr{P}_e$-supermartingale, because for all $t = 0,\dots,T$:

$$\begin{aligned}
&\mathbb{E}^{P^*}[\tilde{V}_t^{\zeta^\tau} - (1+r)^{-t\wedge\tau}R(t,\tau)] \leqslant \mathbb{E}^{P^*}[\tilde{V}_t^{\zeta^\tau}] = \tilde{V}_0^{\zeta^\tau}, \ \ \forall P^* \in \mathscr{P}_e \\
&\Rightarrow \sup_{P^*\in\mathscr{P}_e} \mathbb{E}^{P^*}[\tilde{V}_t^{\zeta^\tau} - (1+r)^{-t\wedge\tau}R(t,\tau)] < \infty.
\end{aligned}$$

Define for all $t = 0,\dots,T$

$$D_t^\tau := \operatorname*{ess\,inf}_{\sigma\in\mathscr{T}_t} \operatorname*{ess\,inf}_{P^*\in\mathscr{P}_e} \mathbb{E}^{P^*}[(1+r)^{-\sigma\wedge\tau}R(\sigma,\tau)|\mathscr{F}_t].$$

Then

$$\begin{aligned}
U_t^\tau &= \operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_t} \operatorname*{ess\,sup}_{P^*\in\mathscr{P}_e} \mathbb{E}^{P^*}[\tilde{V}_\sigma^{\zeta^\tau} - (1+r)^{-\sigma\wedge\tau}R(\sigma,\tau)|\mathscr{F}_t] \\
&= \operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_t} \operatorname*{ess\,sup}_{P^*\in\mathscr{P}_e} (\tilde{V}_t^{\zeta^\tau} + \mathbb{E}^{P^*}[-(1+r)^{-\sigma\wedge\tau}R(\sigma,\tau)|\mathscr{F}_t]) \\
&= \tilde{V}_t^{\zeta^\tau} + \operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_t} \operatorname*{ess\,sup}_{P^*\in\mathscr{P}_e} \mathbb{E}^{P^*}[-(1+r)^{-\sigma\wedge\tau}R(\sigma,\tau)|\mathscr{F}_t] \\
&= \tilde{V}_t^{\zeta^\tau} - D_t^\tau.
\end{aligned}$$

By the uniform Doob decomposition there exists an adapted nondecreasing process $(B_t^\tau)_{t=0,\dots,T}$ with $B_0^\tau = 0$ and a D-dimensional predictable process $(\gamma_t^\tau)_{t=1,\dots,T} = (\gamma_t^{\tau,1},\dots,\gamma_t^{\tau,D})_{t=1,\dots,T}$ such that for all $t = 0,\dots,T$:

$$\tilde{V}_t^{\zeta^\tau} - D_t^\tau = \tilde{V}_0^{\zeta^\tau} - D_0^\tau + \sum_{s=1}^{t}\sum_{d=1}^{D} \gamma_s^{\tau,d}(\tilde{S}_s^d - \tilde{S}_{s-1}^d) - B_t^\tau \text{ a.s.}$$

Hence, since ζ^τ is self-financing:

$$\begin{aligned}
-D_t^\tau + B_t^\tau &= -D_0^\tau + \tilde{V}_0^{\zeta^\tau} - \tilde{V}_t^{\zeta^\tau} + \sum_{s=1}^{t}\sum_{d=1}^{D} \gamma_s^{\tau,d}(\tilde{S}_s^d - \tilde{S}_{s-1}^d) \\
&= -D_0^\tau - \sum_{s=1}^{t}\sum_{d=1}^{D} \zeta_s^{\tau,d}(\tilde{S}_s^d - \tilde{S}_{s-1}^d) + \sum_{s=1}^{t}\sum_{d=1}^{D} \gamma_s^{\tau,d}(\tilde{S}_s^d - \tilde{S}_{s-1}^d) \\
&= -D_0^\tau + \sum_{s=1}^{t}\sum_{d=1}^{D} (\gamma_s^{\tau,d} - \zeta_s^{\tau,d})(\tilde{S}_s^d - \tilde{S}_{s-1}^d).
\end{aligned}$$

There exists a self-financing portfolio

$$(\eta_t^\tau)_{t=0,\dots,T} = (\eta_t^{\tau,0}, \gamma_t^{\tau,1} - \zeta_t^{\tau,1}, \dots, \gamma_t^{\tau,D} - \zeta_t^{\tau,D})_{t=1,\dots,T}$$

with $V_0^{\eta^\tau} = -D_0^\tau$. Hence $\tilde{V}_t^{\eta^\tau} = -D_t^\tau + B_t^\tau$ for all $t = 0,\dots,T$.

Since

$$
\begin{aligned}
D_\tau^\tau &= \sum_{t=0}^{T} \mathbb{1}_{\{t=\tau\}} D_t^\tau \\
&= \sum_{t=0}^{T} \mathbb{1}_{\{t=\tau\}} \operatorname*{ess\,inf}_{\sigma\in\mathcal{T}_t} \operatorname*{ess\,inf}_{P^*\in\mathcal{P}_e} \mathbb{E}^{P^*}[(1+r)^{-\sigma\wedge\tau} R(\sigma,\tau)|\mathcal{F}_t] \\
&\leqslant \sum_{t=0}^{T} \mathbb{1}_{\{t=\tau\}} \operatorname*{ess\,inf}_{P^*\in\mathcal{P}_e} \mathbb{E}^{P^*}[(1+r)^{-T\wedge\tau} R(T,\tau)|\mathcal{F}_t] \\
&= \sum_{t=0}^{T} \mathbb{1}_{\{t=\tau\}} \operatorname*{ess\,inf}_{P^*\in\mathcal{P}_e} \mathbb{E}^{P^*}[(1+r)^{-\tau} Y_\tau|\mathcal{F}_t] \\
&= \sum_{t=0}^{T} \operatorname*{ess\,inf}_{P^*\in\mathcal{P}_e} \mathbb{E}^{P^*}[\mathbb{1}_{\{t=\tau\}}(1+r)^{-t} Y_t|\mathcal{F}_t] \\
&= \sum_{t=0}^{T} \mathbb{1}_{\{t=\tau\}}(1+r)^{-t} Y_t = (1+r)^{-\tau} Y_\tau,
\end{aligned}
$$

we obtain for all $t = 0, \dots, T$:

$$
\begin{aligned}
D_{t\wedge\tau}^\tau = \mathbb{1}_{\{t<\tau\}} D_t^\tau + \mathbb{1}_{\{t\geqslant\tau\}} D_\tau^\tau &\leqslant \mathbb{1}_{\{t<\tau\}} \operatorname*{ess\,inf}_{P^*\in\mathcal{P}_e} \mathbb{E}^{P^*}[(1+r)^{-t\wedge\tau} R(t,\tau)|\mathcal{F}_t] + \mathbb{1}_{\{t\geqslant\tau\}}(1+r)^{-\tau} Y_\tau \\
&= \mathbb{1}_{\{t<\tau\}}(1+r)^{-t\wedge\tau} R(t,\tau) + \mathbb{1}_{\{t\geqslant\tau\}}(1+r)^{-t\wedge\tau} R(t,\tau) \\
&= (1+r)^{-t\wedge\tau} R(t,\tau).
\end{aligned}
$$

Hence for all $t = 0, \dots, T$:

$$
\tilde{V}_{t\wedge\tau}^{\eta^\tau} + (1+r)^{-t\wedge\tau} R(t,\tau) \geqslant \tilde{V}_{t\wedge\tau}^{\eta^\tau} + D_{t\wedge\tau}^\tau = B_{t\wedge\tau}^\tau \geqslant 0 \text{ a.s.}
$$

and so $(D_0^\tau, \tau, \eta^\tau)$ is a superhedge for the buyer. For the stopping time $\tau_0^\downarrow$ we then obtain

$$
\begin{aligned}
m \geqslant D_0^{\tau_0^\downarrow} &= \inf_{\sigma\in\mathcal{T}_0} \inf_{P^*\in\mathcal{P}_e} \mathbb{E}^{P^*}[(1+r)^{-\sigma\wedge\tau_0^\downarrow} R(\sigma,\tau_0^\downarrow)] \\
&\overset{2.2}{=} \inf_{\sigma\in\mathcal{T}_0} \sup_{\tau\in\mathcal{T}_0} \inf_{P^*\in\mathcal{P}_e} \mathbb{E}^{P^*}[(1+r)^{-\sigma\wedge\tau} R(\sigma,\tau)] \\
&\overset{3.3}{=} \inf \Pi(Y,Z) \overset{(3)}{\geqslant} m.
\end{aligned}
$$

□

5 Conclusion

We extended the optimal stopping theory in multiple prior frameworks (for nonnegative payoff processes) by solving two Dynkin games in multiple prior frameworks, one for each worst case scenario of a player. The solution of these stopping games under nonlinear expectations allowed us to derive for a game option a cheapest superhedge for the seller and a most expensive superhedge for the buyer in a general incomplete financial market model. The costs of these superhedges correspond to the bounds of the set of no-arbitrage prices we derived by the approach to consider the payoff of a game option as an European option for fixed stopping times. These results are a consistent extension of the hedging and pricing results for game options in complete markets.

A step to extend the results of this paper further would be to drop the nonnegativity assumption on the payoff processes for the Dynkin game in a multiple prior framework. We only used this assumption for integrability conditions, but it should be possible to derive them without the assumption or show that the arguments of the proof hold under weaker integrability conditions. Furthermore it would be interesting to also drop the assumption that the players have no information about the set of priors of the other player. Then, instead of considering a stopping game for the worst case scenario of each player, one could consider only one stopping game and search for Nash equilibria. Besides a consideration of an infinite time horizon would be interesting. The main difficulty for the infinite time horizon case is that it is no longer economically justifiable to work with equivalent probability measures.

A Appendix

Let $(\Omega, \mathscr{F}, P_0)$ be a probability space equipped with a filtration $(\mathscr{F}_t)_{t=0,\dots,T}$ such that

$$\mathscr{F}_0 = \{\emptyset, \Omega\} \subseteq \mathscr{F}_1 \subseteq \ldots \subseteq \mathscr{F}_T = \mathscr{F},$$

where $T \in \mathbb{N}$ denotes a finite time horizon. Define

$$\mathscr{T} := \{\tau : \Omega \to \{0, \ldots, T\} : \tau \text{ stopping time}\}.$$

We consider a set of probability measures $\mathscr{Q} \neq \emptyset$ on $(\Omega, \mathscr{F})$ which are equivalent to P_0. We call $\mathscr{Q}$ a set of priors and we assume that $\mathscr{Q}$ is time-consistent: let $P, Q \in \mathscr{Q}$ and denote by $(p_t)_{t=0,\dots,T}$ (resp. $(q_t)_{t=0,\dots,T}$) the density process of P (resp. Q) with respect to P_0. Let $\tau \in \mathscr{T}$ and define R by setting for all $t = 0, \ldots, T$:

$$\frac{dR}{dP_0}|_{\mathscr{F}_t} := \begin{cases} p_t, & t \leq \tau \\ \frac{p_\tau q_t}{q_\tau}, & t > \tau. \end{cases}$$

Then $R \in \mathscr{Q}$.
The definition of time-consistency is taken from (Riedel 2009) and two equivalent characterizations which frequently arise in the literature are stated in the following proposition.

Proposition A.1. (Riedel 2009, Lemma 8)
The following properties are equivalent for the set of priors $\mathscr{Q}$.

(i) *$\mathscr{Q}$ is time-conistent.*

(ii) *$\mathscr{Q}$ is stable: let $\tau \in \mathscr{T}$, $A \in \mathscr{F}_\tau$ and $P, Q \in \mathscr{Q}$. Then there exists a unique probability measure $R \in \mathscr{Q}$ such that*

$$\frac{dR}{dP_0}|_{\mathscr{F}_\tau} = \frac{dP}{dP_0}|_{\mathscr{F}_\tau},$$
$$\mathbb{E}^R[X|\mathscr{F}_\tau] = \mathbb{1}_A \mathbb{E}^Q[X|\mathscr{F}_\tau] + \mathbb{1}_{A^c} \mathbb{E}^P[X|\mathscr{F}_\tau], \quad \forall X \in \mathscr{X}.$$

(iii) *$\mathscr{Q}$ is rectangular: let $\tau \in \mathscr{T}$ and $P, Q \in \mathscr{Q}$. Define*

$$R(B) := \mathbb{E}^P[\mathbb{E}^Q[\mathbb{1}_B|\mathscr{F}_\tau]], \quad \forall B \in \mathscr{F}.$$

Then $R \in \mathscr{Q}$.

Define

$$\mathscr{X} := \{X \in \mathscr{L}^0(\Omega, \mathscr{F}, P_0) : \lim_{c\to\infty} \sup_{P\in\mathscr{Q}} \mathbb{E}^P[|X|\mathbb{1}_{\{|X|\geqslant c\}}] = 0\}.$$

A random variable $X \in \mathscr{X}$ is called $\mathscr{Q}$-uniformly integrable [11]. In the following proposition we obtain an equivalent characterization of $\mathscr{Q}$-uniformly integrable random variables. This characterization is an epsilon delta criterion similar to the epsilon delta criterion which holds for uniformly integrable families of random variables. The proof also follows by similar arguments [12].

Proposition A.2. *Let $X \in \mathscr{L}^0(\Omega, \mathscr{F}, P_0)$. Then the following properties are equivalent:*

(i) $X \in \mathscr{X}$.

(ii) $\sup_{P\in\mathscr{Q}} \mathbb{E}^P[|X|] < \infty$ *and for each* $\epsilon > 0$ *there exists* $\delta > 0$ *such that for all* $A \in \mathscr{F}$ *with* $\sup_{P\in\mathscr{Q}} P(A) \leqslant \delta$*:*

$$\mathbb{E}^P[|X|\mathbb{1}_A] < \epsilon, \ \ \forall P \in \mathscr{Q}.$$

Proof. Assume that (i) holds. Since $X \in \mathscr{X}$ we have

$$\lim_{c\to\infty} \sup_{P\in\mathscr{Q}} \mathbb{E}^P[|X|\mathbb{1}_{\{|X|\geqslant c\}}] = 0.$$

So there exists a constant $K > 0$ such that $\sup_{P\in\mathscr{Q}} \mathbb{E}^P[|X|\mathbb{1}_{\{|X|\geqslant K\}}] < 1$. Hence

$$\sup_{P\in\mathscr{Q}} \mathbb{E}^P[|X|] = \sup_{P\in\mathscr{Q}} (\mathbb{E}^P[|X|\mathbb{1}_{\{|X|<K\}}] + \mathbb{E}^P[|X|\mathbb{1}_{\{|X|\geqslant K\}}]) \leqslant K + 1 < \infty.$$

Let $\epsilon > 0$. Since $X \in \mathscr{X}$ there exists a constant $K_\epsilon > 0$ such that

$$\sup_{P\in\mathscr{Q}} \mathbb{E}^P[|X|\mathbb{1}_{\{|X|\geqslant K_\epsilon\}}] < \frac{\epsilon}{2}.$$

Define $\delta := \frac{\epsilon}{2K_\epsilon}$. Then for all $A \in \mathscr{F}$ with $\sup_{P\in\mathscr{Q}} P(A) \leqslant \delta$ and for all $P \in \mathscr{Q}$:

$$\begin{aligned}\mathbb{E}^P[|X|\mathbb{1}_A] &= \mathbb{E}^P[|X|\mathbb{1}_A\mathbb{1}_{\{|X|\geqslant K_\epsilon\}}] + \mathbb{E}^P[|X|\mathbb{1}_A\mathbb{1}_{\{|X|<K_\epsilon\}}] \\ &\leqslant \mathbb{E}^P[|X|\mathbb{1}_{\{|X|\geqslant K_\epsilon\}}] + K_\epsilon \cdot P(A) \\ &< \frac{\epsilon}{2} + K_\epsilon \cdot \delta = \epsilon.\end{aligned}$$

Assume that (ii) holds. Let $\epsilon > 0$ and δ as in (ii). Since by assumption $\sup_{P\in\mathscr{Q}} \mathbb{E}^P[|X|] < \infty$, it follows for large c that $\frac{1}{c}\sup_{p\in\mathscr{Q}} \mathbb{E}^P[|X|] \leqslant \delta$ and so by the Markov inequality

$$\begin{aligned}&P[|X| \geqslant c] \leqslant \frac{1}{c}\mathbb{E}^P[|X|] \leqslant \delta, \ \ \forall P \in \mathscr{Q} \\ &\Rightarrow \sup_{p\in\mathscr{Q}} P[|X| \geqslant c] \leqslant \delta.\end{aligned}$$

By assumption we then obtain for large c:

$$\begin{aligned}&\mathbb{E}^P[|X|\mathbb{1}_{\{|X|\geqslant c\}}] < \epsilon, \ \ \forall P \in \mathscr{Q} \\ &\Rightarrow \sup_{p\in\mathscr{Q}} \mathbb{E}^P[|X|\mathbb{1}_{\{|X|\geqslant c\}}] \leqslant \epsilon.\end{aligned}$$

From this we can conclude that $\lim_{c\to\infty} \sup_{P\in\mathscr{Q}} \mathbb{E}^P[|X|\mathbb{1}_{\{|X|\geqslant c\}}] = 0$ and so $X \in \mathscr{X}$. □

[11] The definition is taken from (Riedel 2009).

[12] See (Röckner 2016, Lemma 1.8.8), for example.

In the next proposition we state some properties of $\mathcal{Q}$-uniformly integrable random variables.

Proposition A.3. (i) *Let $X \in \mathcal{X}$. Then $X \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$ for all $P \in \mathcal{Q}$.*

(ii) *Let $X \in \mathcal{X}$ and let $Z \in \mathcal{L}^0(\Omega, \mathcal{F}, P_0)$ such that $|Z| \leqslant |X|$. Then $Z \in \mathcal{X}$.*

(iii) *Let $X \in \mathcal{L}^0(\Omega, \mathcal{F}, P_0)$ be bounded. Then $X \in \mathcal{X}$.*

(iv) *Let $Y \in \mathcal{L}^0(\Omega, \mathcal{F}, P_0)$ be bounded and $X \in \mathcal{X}$. Then $YX \in \mathcal{X}$.*

(v) *Let $X_n \in \mathcal{X}$, $n = 1, \dots, N$. Then*

$$\sum_{n=1}^{N} |X_n| \in \mathcal{X}.$$

Proof. Since Proposition A.2 implies

$$\sup_{P \in \mathcal{Q}} \mathbb{E}^P[|X|] < \infty.$$

for all $X \in \mathcal{X}$ the properties (i)-(iv) follow directly by the definition of $\mathcal{Q}$-uniformly integrability.
In order to proof property (v) we notice that for all $P \in \mathcal{Q}$, for all $c \in \mathbb{R}$ and every $n \in \{1, \dots, N\}$:

$$\begin{aligned}
&\mathbb{E}^P[|X_n| \mathbb{1}_{\{\sum_{m=1}^{N} |X_m| \geqslant c\}}] \leqslant \mathbb{E}^P[|X_n| \sum_{m=1}^{N} \mathbb{1}_{\{|X_m| \geqslant \frac{c}{N}\}}] = \sum_{m=1}^{N} \mathbb{E}^P[|X_n| \mathbb{1}_{\{|X_m| \geqslant \frac{c}{N}\}}] \\
&= \sum_{m=1}^{N} (\mathbb{E}^P[|X_n| \mathbb{1}_{\{|X_m| \geqslant \frac{c}{N}\}} \mathbb{1}_{\{|X_n| \geqslant |X_m|\}}] + \mathbb{E}^P[|X_n| \mathbb{1}_{\{|X_m| \geqslant \frac{c}{N}\}} \mathbb{1}_{\{|X_n| < |X_m|\}}]) \\
&\leqslant \sum_{m=1}^{N} \mathbb{E}^P[|X_n| \mathbb{1}_{\{|X_n| \geqslant \frac{c}{N}\}}] + \sum_{m=1}^{N} \mathbb{E}^P[|X_m| \mathbb{1}_{\{|X_m| \geqslant \frac{c}{N}\}} \mathbb{1}_{\{|X_n| < |X_m|\}}] \\
&\leqslant N \mathbb{E}^P[|X_n| \mathbb{1}_{\{|X_n| \geqslant \frac{c}{N}\}}] + \sum_{m=1}^{N} \mathbb{E}^P[|X_m| \mathbb{1}_{\{|X_m| \geqslant \frac{c}{N}\}}].
\end{aligned}$$

From this we can conclude

$$\begin{aligned}
0 &\leqslant \lim_{c \to \infty} \sup_{P \in \mathcal{Q}} \mathbb{E}^P[\sum_{n=1}^{N} |X_n| \mathbb{1}_{\{\sum_{m=1}^{N} |X_m| \geqslant c\}}] = \lim_{c \to \infty} \sup_{P \in \mathcal{Q}} (\sum_{n=1}^{N} \mathbb{E}^P[|X_n| \mathbb{1}_{\{\sum_{m=1}^{N} |X_m| \geqslant c\}}]) \\
&\leqslant \lim_{c \to \infty} \sup_{P \in \mathcal{Q}} (N \sum_{n=1}^{N} \mathbb{E}^P[|X_n| \mathbb{1}_{\{|X_n| \geqslant \frac{c}{N}\}}] + N \sum_{m=1}^{N} \mathbb{E}^P[|X_m| \mathbb{1}_{\{|X_m| \geqslant \frac{c}{N}\}}]) \\
&= \lim_{c \to \infty} \sup_{P \in \mathcal{Q}} (2N \sum_{n=1}^{N} \mathbb{E}^P[|X_n| \mathbb{1}_{\{|X_n| \geqslant \frac{c}{N}\}}]) \\
&\leqslant \lim_{c \to \infty} 2N \sum_{n=1}^{N} \sup_{P \in \mathcal{Q}} \mathbb{E}^P[|X_n| \mathbb{1}_{\{|X_n| \geqslant \frac{c}{N}\}}] \\
&= 2N \sum_{n=1}^{N} \lim_{c \to \infty} \sup_{P \in \mathcal{Q}} \mathbb{E}^P[|X_n| \mathbb{1}_{\{|X_n| \geqslant \frac{c}{N}\}}] = 0.
\end{aligned}$$

□

The following formula is known as Bayes' formula and can for example be found in (Karatzas and Shreve 1997, p. 193).

Proposition A.4. *Let Q be a probability measure on $(\Omega,\mathscr{F})$ such that $Q \sim P_0$ and let $\tau,\sigma \in \mathscr{T}$ such that $\sigma \leqslant \tau$. Let $X \in \mathscr{L}^1(\Omega,\mathscr{F},Q)$ be $\mathscr{F}_\tau$-measurable. Then*

$$\mathbb{E}^Q[X|\mathscr{F}_\sigma] = \mathbb{E}^{P_0}[Xq_\tau|\mathscr{F}_\sigma]\frac{1}{q_\sigma}.$$

The next proposition gives us a time-consistent version of the law of iterated expectations.

Proposition A.5. *Let $P,Q \in \mathscr{Q}$ and $\sigma,\tau \in \mathscr{T}$ such that $\sigma \leqslant \tau$. Then there exists $R \in \mathscr{Q}$ such that for all $X \in \mathscr{X}$:*

$$\mathbb{E}^P[\mathbb{E}^Q[X|\mathscr{F}_\tau]|\mathscr{F}_\sigma] = \mathbb{E}^R[X|\mathscr{F}_\sigma].$$

Proof. Let $(p_t)_{t=0,\ldots,T}$ and $(q_t)_{t=0,\ldots,T}$ be the density processes of P and Q with respect to P_0. Then for all $t = 0,\ldots,T$:

$$\frac{dQ}{dP}|_{\mathscr{F}_t} = \frac{q_t}{p_t}.$$

Define R by

$$\frac{dR}{dP_0} := \frac{p_\tau}{q_\tau}q_T.$$

By time-consistency $R \in \mathscr{Q}$. It holds that

$$\frac{dR}{dP} = \frac{\frac{dR}{dP_0}}{\frac{dP}{dP_0}} = \frac{\frac{p_\tau}{q_\tau}q_T}{p_T} = \frac{q_T}{p_T}\frac{p_\tau}{q_\tau}.$$

Hence by the optional sampling theorem

$$\begin{aligned}\frac{dR}{dP}|_{\mathscr{F}_\sigma} = \mathbb{E}^P[\frac{dR}{dP}|\mathscr{F}_\sigma] &= \mathbb{E}^P[\frac{q_T}{p_T}\frac{p_\tau}{q_\tau}|\mathscr{F}_\sigma]\\ &= \mathbb{E}^P[\frac{p_\tau}{q_\tau}\mathbb{E}^P[\frac{q_T}{p_T}|\mathscr{F}_\tau]|\mathscr{F}_\sigma]\\ &= \mathbb{E}^P[\frac{p_\tau}{q_\tau}\frac{q_\tau}{p_\tau}|\mathscr{F}_\sigma] = 1.\end{aligned}$$

We obtain with Bayes' formula and the law of iterated expectations for all $X \in \mathscr{X}$

$$\begin{aligned}\mathbb{E}^R[X|\mathscr{F}_\sigma] &= \mathbb{E}^P[X\frac{q_T}{p_T}\frac{p_\tau}{q_\tau}|\mathscr{F}_\sigma]\\ &= \mathbb{E}^P[\mathbb{E}^P[X\frac{q_T}{p_T}|\mathscr{F}_\tau]\frac{p_\tau}{q_\tau}|\mathscr{F}_\sigma]\\ &= \mathbb{E}^P[\mathbb{E}^Q[X|\mathscr{F}_\tau]|\mathscr{F}_\sigma].\end{aligned}$$

□

We will now show that conditional expectations of $\mathscr{Q}$-uniformly integrable random variables are $\mathscr{Q}$-uniformly integrable.

Proposition A.6. *Let $X \in \mathscr{X}$, $Q \in \mathscr{Q}$ and $t \in \{0,\ldots,T\}$. Then*

$$\mathbb{E}^Q[X|\mathscr{F}_t] \in \mathscr{X}.$$

Proof. Let $\epsilon > 0$. By Proposition A.2 there exists $\delta > 0$ such that for all $A \in \mathscr{F}$ with $\sup_{P\in\mathscr{Q}} P(A) \leqslant \delta$:

$$\mathbb{E}^P[|X|\mathbb{1}_A] < \epsilon, \;\; \forall P \in \mathscr{Q}.$$

We also know by Proposition A.2 that $\sup_{P\in\mathscr{Q}} \mathbb{E}^P[|X|] < \infty$. So for large c:

$$\frac{1}{c}\sup_{p\in\mathscr{Q}} \mathbb{E}^P[|X|] \leqslant \delta.$$

By the Markov inequality and by Proposition A.5 there exist $R_P \in \mathscr{Q}$ such that for large c:

$$P(\mathbb{E}^Q[|X||\mathscr{F}_t] \geqslant c) \leqslant \frac{1}{c}\mathbb{E}^P[\mathbb{E}^Q[|X||\mathscr{F}_t]] = \frac{1}{c}\mathbb{E}^{R_P}[|X|] \leqslant \delta, \;\; \forall P \in \mathscr{Q}$$
$$\Rightarrow \sup_{p\in\mathscr{Q}} P(\mathbb{E}^Q[|X||\mathscr{F}_t] \geqslant c) \leqslant \delta.$$

Then by Proposition A.5 for each $P \in \mathscr{Q}$ there exists $\tilde{R}_P \in \mathscr{Q}$ such that for large c:

$$\begin{aligned}\mathbb{E}^P[|\mathbb{E}^Q[X|\mathscr{F}_t]|\mathbb{1}_{\{|\mathbb{E}^Q[X|\mathscr{F}_t]|\geqslant c\}}] &\leqslant \mathbb{E}^P[\mathbb{E}^Q[|X||\mathscr{F}_t]\mathbb{1}_{\{\mathbb{E}^Q[|X||\mathscr{F}_t]\geqslant c\}}] \\ &= \mathbb{E}^P[\mathbb{E}^Q[|X|\mathbb{1}_{\{\mathbb{E}^Q[|X||\mathscr{F}_t]\geqslant c\}}|\mathscr{F}_t]] \\ &= \mathbb{E}^{\tilde{R}_P}[|X|\mathbb{1}_{\{\mathbb{E}^Q[|X||\mathscr{F}_t]\geqslant c\}}] < \epsilon.\end{aligned}$$

This leads us to

$$\sup_{P\in\mathscr{Q}} \mathbb{E}^P[|\mathbb{E}^Q[X|\mathscr{F}_t]|\mathbb{1}_{\{|\mathbb{E}^Q[X|\mathscr{F}_t]|\geqslant c\}}] \leqslant \epsilon$$

and so

$$\lim_{c\to\infty}\sup_{P\in\mathscr{Q}} \mathbb{E}^P[|\mathbb{E}^Q[X|\mathscr{F}_t]|\mathbb{1}_{\{|\mathbb{E}^Q[X|\mathscr{F}_t]|\geqslant c\}}] = 0.$$

□

Definition A.7. For $X \in \mathscr{X}$ and $t \in \{0,\dots,T\}$ we define the multiple prior conditional expectation by

$$\mathscr{E}_t(X) := \operatorname*{ess\,inf}_{P\in\mathscr{Q}} \mathbb{E}^P[X|\mathscr{F}_t].$$

We also define the multiple prior conditional expectation for stopping times $\tau \in \mathscr{T}$ by

$$\mathscr{E}_\tau(X) := \operatorname*{ess\,inf}_{P\in\mathscr{Q}} \mathbb{E}^P[X|\mathscr{F}_\tau].$$

If we consider a deterministic stopping time $\tau \equiv t \in \{0,\dots,T\}$, then by definition $\mathscr{E}_\tau(X) = \mathscr{E}_t(X)$ for all $X \in \mathscr{X}$. With the help of the following two propositions it can be shown that for $X \in \mathscr{X}$ and $\tau \in \mathscr{T}$, $\mathscr{E}_\tau(X)$ coincides with the process $(\mathscr{E}_t(X))_{t=0,\dots,T}$ stopped at τ. For $X \in \mathscr{X}$ we have by Proposition A.3(i) that $X \in \mathscr{L}^1(\Omega,\mathscr{F},P)$ for all $P \in \mathscr{Q}$. Hence multiple prior conditional expectations are well defined. The next proposition shows that multiple prior conditional expectations are integrable.

Proposition A.8. *Let $X \in \mathscr{X}$ and $\tau \in \mathscr{T}$. Then for all $Q \in \mathscr{Q}$:*

$$\operatorname*{ess\,inf}_{P\in\mathscr{Q}} \mathbb{E}^P[X|\mathscr{F}_\tau] \in \mathscr{L}^1(\Omega,\mathscr{F},Q).$$

In particular

$$|\operatorname*{ess\,inf}_{P\in\mathscr{Q}} \mathbb{E}^P[X|\mathscr{F}_\tau]| < \infty \text{ a.s.}$$

Proof. Let $X \in \mathscr{X}$, $\tau \in \mathscr{T}$ and $Q \in \mathscr{Q}$. Then

$$\mathbb{E}^Q[|\operatorname*{ess\,inf}_{P\in\mathscr{Q}} \mathbb{E}^P[X|\mathscr{F}_\tau]|] \leq \mathbb{E}^Q[\operatorname*{ess\,sup}_{P\in\mathscr{Q}} \mathbb{E}^P[|X||\mathscr{F}_\tau]].$$

So if we are able to show that the second term is strictly less than infinity, the proof is finished. In order to show this we show at first that the set $\mathscr{C} := \{\mathbb{E}^P[|X||\mathscr{F}_\tau] : P \in \mathscr{Q}\}$ is upward directed. For this purpose let $R, \tilde{R} \in \mathscr{Q}$ and define

$$A := \{\mathbb{E}^R[|X||\mathscr{F}_\tau] \geq \mathbb{E}^{\tilde{R}}[|X||\mathscr{F}_\tau]\} \in \mathscr{F}_\tau.$$

Then

$$\mathbb{E}^R[|X||\mathscr{F}_\tau] \vee \mathbb{E}^{\tilde{R}}[|X||\mathscr{F}_\tau] = \mathbb{1}_A \mathbb{E}^R[|X||\mathscr{F}_\tau] + \mathbb{1}_{A^c} \mathbb{E}^{\tilde{R}}[|X||\mathscr{F}_\tau]$$

and Proposition A.1(ii) implies that there exists $\tilde{P}$ such that

$$\mathbb{E}^{\tilde{P}}[|X||\mathscr{F}_\tau] = \mathbb{1}_A \mathbb{E}^R[|X||\mathscr{F}_\tau] + \mathbb{1}_{A^c} \mathbb{E}^{\tilde{R}}[|X||\mathscr{F}_\tau].$$

Hence $\mathscr{C}$ is upward directed and so there exists a nondecreasing sequence $(\mathbb{E}^{P^n}[|X||\mathscr{F}_\tau])_{n\in\mathbb{N}}$ in $\mathscr{C}$ with

$$\lim_{n\to\infty} \mathbb{E}^{P^n}[|X||\mathscr{F}_\tau] = \operatorname*{ess\,sup}_{P\in\mathscr{Q}} \mathbb{E}^P[|X||\mathscr{F}_\tau] \text{ a.s.}$$

By the monotone convergence theorem of Levi and Proposition A.5 we conclude that there exist $Q^n \in \mathscr{Q}$, $n \in \mathbb{N}$, such that

$$\begin{aligned}
\mathbb{E}^Q[|\operatorname*{ess\,inf}_{P\in\mathscr{Q}} \mathbb{E}^P[X|\mathscr{F}_\tau]|] &\leq \mathbb{E}^Q[\operatorname*{ess\,sup}_{P\in\mathscr{Q}} \mathbb{E}^P[|X||\mathscr{F}_\tau]] \\
&= \mathbb{E}^Q[\lim_{n\to\infty} \mathbb{E}^{P^n}[|X||\mathscr{F}_\tau]] \\
&= \lim_{n\to\infty} \mathbb{E}^Q[\mathbb{E}^{P^n}[|X||\mathscr{F}_\tau]] \\
&= \lim_{n\to\infty} \mathbb{E}^{Q^n}[|X|] \leq \sup_{P\in\mathscr{Q}} \mathbb{E}^P[|X|] < \infty.
\end{aligned}$$

□

Although multiple prior conditional expectations are no longer linear, a lot of properties of classical conditional expectations carry over to the multiple prior case. Some properties that we will use frequently are listed in the following proposition.

Proposition A.9. (Riedel 2009)
Let $X, Y \in \mathscr{X}$ and $\tau \in \mathscr{T}$. Then the following properties hold.

(i) *Monotonicity: $X \geq Y$ a.s. implies:*

$$\mathscr{E}_\tau(X) \geq \mathscr{E}_\tau(Y).$$

(ii) *Conditional homogeneity of degree 1: let $Z \geq 0$ be a bounded $\mathscr{F}_\tau$-measurable random variable. Then*

$$\mathscr{E}_\tau(ZX) = Z\mathscr{E}_\tau(X).$$

(iii) *Superadditivity:*

$$\mathscr{E}_\tau(X+Y) \geq \mathscr{E}_\tau(X) + \mathscr{E}_\tau(Y).$$

(iv) *Conditional additivity: If Y is $\mathscr{F}_\tau$-measurable, then*

$$\mathscr{E}_\tau(X+Y) = \mathscr{E}_\tau(X) + Y.$$

References

BAYRAKTAR, E., AND S. YAO (2017): "On the Robust Dynkin Game," *The Annals of Applied Probability*, 27, 1702–1755.

DOLINSKY, Y. (2014): "Hedging of game options under model uncertainty in discrete time," *Electron. Commun. Probab.*, 19, 1–11.

DOLINSKY, Y., AND Y. KIFER (2007): "Hedging with risk for game options in discrete time," *Stochastics*, 79, 169–195.

DUMITRESCU, R., M.-C. QUENEZ, AND A. SULEM (2017): "Game Options in an Imperfect Market with Default," *SIAM Journal on Financial Mathematics*, 8, 532–559.

EPSTEIN, L. G., AND M. SCHNEIDER (2003): "Recursive multiple-priors," *Journal of Economic Theory*, 113, 1–31.

FÖLLMER, H., AND A. SCHIED (2016): *Stochastic Finance: An Introduction in Discrete Time*, Graduate. de Gruyter, Berlin, fourth revised and extended edition edn.

GILBOA, I., AND D. SCHMEIDLER (1989): "Maxmin Expected Utility With Non-Unique Prior," *Journal of Mathematical Economics*, 18, 141–153.

KALLSEN, J., AND C. KÜHN (2004): "Pricing derivatives of American and game type in incomplete markets," *Finance and Stochastics*, 8, 261–284.

——— (2005): "Convertible bonds: financial derivatives of game type," *in Exotic Option Pricing and Advanced Lévy Models, A. Kyprianou, W. Schoutens and P. Wilmott, John Wiley & Sons, Chichester, UK*, pp. 277–291.

KARATZAS, I., AND S. E. SHREVE (1997): *Brownian Motion and Stochastic Calculus*, vol. 113 of *Graduate texts in mathematics*. Springer, New York [u.a.], second edition edn.

KÜHN, C. (2004): "Game contingent claims in complete and incomplete markets," *Journal of Mathematical Economics*, 40, 889–902.

KIFER, Y. (2000): "Game options," *Finance and Stochastics*, 4, 443–463.

——— (2013a): "Dynkin's Games and Israeli Options," *ISRN Probability and Statistics*, 2013.

——— (2013b): "Hedging Of Game Options In Discrete Markets With Transaction Costs," *Stochastics*, 85, 667–681.

KOO, H. K., S. TANG, AND Z. YANG (2015): "A Dynkin Game Under Knightian Uncertainty," *Discrete & Continuous Dynamical Systems-A*, 35, 5467–5498.

NEVEU, J. (1975): *Discrete-parameter martingales*, vol. 10 of *North-Holland Mathematical Library*. North-Holland Publishing Company, Amsterdam.

OHTSUBO, Y. (1986): "Optimal Stopping in Sequential Games with or without a Constraint of Always Terminating," *Mathematics of Operations Research*, 11, 591–607.

RÖCKNER, M. (2016): *Wahrscheinlichkeitstheorie 1 & 2*, Vorlesung im WS 2015/16 und SS 2016. Universität Bielefeld.

RIEDEL, F. (2009): "Optimal Stopping With Multiple Priors," *Econometrica*, 77, 857–908.

SNELL, J. L. (1952): "Applications of Martingale System Theorems," *Transactions of the American Mathematical Society*, 73, 293–312.

YIN, J. (2012): "Reflected backward stochastic differential equations with two barriers and Dynkin games under Knightian uncertainty," *Bull. Sci. math*, 136, 709–729.