On Kantorovich multimarginal optimal transportation problems with density constraints

Anna N. Doledenok

Abstract

New existence and uniqueness results are obtained for the Kantorovich multimarginal optimal transportation problem with density constraints.

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1 Introduction

The Kantorovich problem of optimal transportation was posed in 1942 in [10]. This problem can be formulated as a linear optimization problem on a convex domain: find an optimal measure with given projections, provided that the optimality is measured by a cost function.

In [11], an interesting modification of this problem was posed and solved. This modification includes pointwise constraints on the densities of admissible measures. In the set of all measures with given projections whose densities are bounded by a certain given function, it is required to find an optimal measure. It was shown in [11] that the local nondegeneracy of the cost function implies that an optimal plan is extreme and, as a consequence, unique. In other words, an optimal plan is an extreme point of a convex set. Later, in [12], the same authors proposed a simplified proof of the uniqueness of the optimal plan based on a specific characterization of extreme points of a convex set and appropriate perturbations. Duality issues were considered by the same authors in [13] and [14].

The cited papers deal with the case of measures on the finite-dimensional space $\mathbb{R}^n$ or on the torus with Lebesgue measure. In the case of two marginals the optimal transportation problem with a density constraint in infinite-dimensional spaces was studied by the author n [6]. In the present paper, we study the case of many marginals.

The present paper is organized as follows. The Kantorovich problem with a density constraint and many marginals is posed in Section 2i. The existence of a solution to this problem is proved in Section 3. In Section 2 we obtain a new theorem which is the main tool in our proofs. In this section we also characterize the set of extreme points of the set of admissible densities with fixed projections and a density constraint. Section 5 is devoted to conditions on the cost function sufficient for the uniqueness of the solution. In Section 6 we prove that an optimal plan is an extreme point of the set of admissible densities. The uniqueness of a solution of the problem in question is an almost immediate corollary of this fact. For more information on this problem, see [5]– [17]. Also note that a somewhat different modification of the Kantorovich problem with additional linear constraints is considered in [18].

2 Statement of the problem

Suppose we are given a natural number n and n probability spaces X_i, $i \leqslant n$, equipped with σ-algebras $\mathcal{A}_i$ and probability measures μ_i. Let $X = \prod_{i=1}^n X_i$ be the product space equipped with the product measure $\mu = \bigotimes_{i=1}^n \mu_i$ and let $\chi \in L^1(X, \mu)$ be a nonnegative integrable function. Let $\Gamma_\chi(\mu)$ be the class of densities h with respect to μ which determine probability measures on X whose marginals on X_i are μ_i for all $i \leqslant n$, and satisfy the condition $h \leqslant \chi$. It is obvious that the class $\Gamma_\chi(\mu)$ is a convex set.

Given a cost function $c \in L^\infty(\mu)$ we consider the linear functional on $\Gamma_\chi(\mu)$ defined by

$$I_c(h) = \int_X ch\, d\mu. \tag{1}$$

We pose the following optimal transportation problem with a density constraint: minimize the functional (1) on the convex set of densities h in the class $\Gamma_\chi(\mu)$. A function for which the minimum is attained is called an optimal plan.

3 Existence of solutions

The first main result of the paper is as follows.

Theorem 3.1. *The functional I_c attains its minimum on $\Gamma_\chi(\mu)$.*

Proof. We shall endow $L^1(\mu)$, with a topology in which the class $\Gamma_\chi(\mu)$ is compact and the functional I_c is continuous. The existence of a solution of the optimization problem will then follow from the general fact that a continuous function attains its minimum on a compact set.

For this purpose, we endow $L^1(\mu)$ with the weak topology. The class $\Gamma_\chi(\mu)$ is uniformly integrable, because all functions in $\Gamma_\chi(\mu)$ are bounded by the same integrable function. Therefore, $\Gamma_\chi(\mu)$ has compact closure in the weak topology of $L^1(\mu)$ (see [4, Theorem 4.7.18]). To prove the compactness of the class $\Gamma_\chi(\mu)$, it suffices to show that it is closed. By the convexity of $\Gamma_\chi(\mu)$, it suffices to verify that it is closed with respect to the norm.

Let $\{h_k\}$ be a sequence of densities in $\Gamma_\chi(\mu)$ which converges in norm to a function $h_\infty \in L^1(\mu)$. We must verify that $h_\infty \in \Gamma_\chi(\mu)$, i.e., the measure with density h_∞ has projections $\mu_1, \ldots, \mu_n$ and h_∞ and is majorized by the function χ almost everywhere with respect to the measure μ. The latter is obvious. The former follows from the fact that, for all functions $\varphi \in L^\infty(\mu)$, we have

$$\lim_{k\to\infty} \int_X h_k \varphi\, d\mu = \int_X h_\infty \varphi\, d\mu. \tag{2}$$

Substituting $\varphi = \psi \cdot 1_{\widehat{X_i}}$ into (2), where $\psi \in L^\infty(\mu_i)$ and

$$\widehat{X_i} = X_1 \times \ldots \times X_{i-1} \times X_{i+1} \times \ldots \times X_n,$$

we see that the projection of the measure with density h_∞ on X_i is μ_i. This implies that $\Gamma_\chi(\mu)$ is closed with respect to the norm and, hence, in the weak topology.

The functional I_c is continuous in the weak topology by definition. Thus, we have proved the existence of an optimal plan. $\square$

4 Characterization of extreme points of $\Gamma_\chi(\mu)$

In this section we show that each function $h \in \Gamma_\chi(\mu)$ that is an extreme point of the convex class $\Gamma_\chi(\mu)$ can be represented as $1_W\chi$ for some Lebesgue measurable set $W \subset X$. Recall that a function $h \in \Gamma_\chi(\mu)$ is called an *extreme point* of the convex set $\Gamma_\chi(\mu)$ if h is not the midpoint of a line segment in $\Gamma_\chi(\mu)$.

Let us now use the notation $\mathbf{e_j} = (0, \ldots, 1, \ldots)$ for the infinite-dimensional element whose jth component is equal to 1 and whose other components are zero. Let T denote the space $T = ([0;1]^\infty)^n$ equipped with the Borel σ-algebra and Lebesgue measure.

Lemma 4.1. *Suppose we are given a Lebesgue measurable set $U \subset T$ of positive measure. Then, for each sufficiently small $\delta > 0$ and each set of variables $(t^1_{i_1}, \ldots, t^n_{i_n})$, there exist rational numbers (we call them rational shifts) $\varphi_1, \ldots, \varphi_n \in (-\delta;\delta)\backslash\{0\}$ and a Lebesgue measurable set $V \subset U$ of positive measure such that for every point $(v^1, \ldots, v^n)$ in the set V all 2^n points of the form*

$$(v^1 + \epsilon^1\varphi_1\mathbf{e_{i_1}}, \ldots, v^n + \epsilon^n\varphi_n\mathbf{e_{i_n}}) \in U$$

for each $\forall\, j \leqslant n$ $\epsilon^j \in \{0;1\}$.

Proof. We fix a set of variables $(t^1_{i_1}, \ldots, t^n_{i_n})$. Let $a = \{a_n\}_{n=1}^\infty$ be a sequence of real positive numbers each of which does not exceed $1/3$. Consider the compact set

$$K_a = [a_1; 1-a_1] \times [a_2; 1-a_2] \times \ldots \times [a_n; 1-a_n] \times \ldots \in [0;1]^\infty.$$

Choosing an appropriate sequence a, we can assume that the Lebesgue measure $\prod_{k=1}^\infty(1-2a_k)$ of this compact set is arbitrarily close to 1. Consider the compact set $K = (K_a)^n \in T$. Note that the Lebesgue measure of the set K can be also made arbitrarily close to 1. We fix a sequence a for which the Lebesgue measure of $E = K \cap U$ is positive.

The shift of the set E with respect to the variable $t^1_{i_1}$ by φ is denoted by

$$E(\varphi) = \{(t_1, \ldots, t_n) \in T \colon (t_1 - \varphi\mathbf{e_{i_1}}, \ldots, t_n) \in E\}.$$

Let $\delta = \min(a_{i_1}, \ldots, a_{i_n})/2$. We show that there exists a shift $\varphi_1 \in (-\delta;\delta)\backslash\{0\}$ such that $\lambda(E \cap E(\varphi_1)) > 0$. There are countably many rational shifts in the interval $(-\delta;\delta)\backslash\{0\}$; therefore, we can number them by positive integers, i.e., let q_i denote the ith number ($i \in \mathbb{N}$). If there exist two numbers q_i, q_j such that $q_i > q_j$ and $\lambda(E(q_i) \cap E(q_j)) > 0$, then $\lambda(E \cap E(q_j - q_i)) > 0$. Suppose for each pair of shifts q_i, q_j such that $q_i > q_j$ we have

$$\lambda(E(q_i) \cap E(q_j)) = 0, \;\; \lambda(E \cap E(q_i)) = 0, \;\; \lambda(E \cap E(q_j)) = 0.$$

Since $\lambda(E)$ is positive, there is a natural number M such that $M \cdot \lambda(E) > 1$. Consider M different rational shifts $q_1, q_2, \ldots, q_M \in (-\delta;\delta)\backslash\{0\}$. On the one hand, we have

$$F = \bigcup_{i=1}^M E(\varphi_i) \subset T,$$

hence $\lambda(F) \leq 1$. On the other hand, $\lambda(F) = M \cdot \lambda(E) > 1$. This contradiction proves the existence of a shift φ_1 such that $\lambda(E \cap E(\varphi_1)) > 0$.

Now let us consider the set $E \cap E(\varphi_1)$ of positive measure λ. An analogous reasoning implies the existence of a shift $\varphi_2 \in (-\delta;\delta)\backslash\{0\}$ such that the set $E \cap E(\varphi_1)$ and its shift

$$E(\varphi_1, \varphi_2) = \{(t_1, \ldots, t_n) \in T \colon (t_1, t_2 - \varphi_2\mathbf{e_{i_2}}, \ldots, t_n) \in E \cap E(\varphi_1)\}$$

with respect to the variable $t^2_{i_2}$ by φ_2 have positive Lebesgue measure. We repeat this process $n-2$ times with the remaining variables and obtain a set $W \in T$ of positive measure.

For any natural $k \leqslant n$ we divide the i_kth interval $[0;1]$ in the kth space $[0;1]^\infty$ in the space $T = ([0;1]^\infty)^n$ into N_k equal disjoint half-intervals of length less than $\varphi_1/5$, namely,

$$[0;1] = \Big[0; \frac{1}{N_j}\Big) \cup \bigcup_{k=1}^{N_j-1} \Big[\frac{k}{N_j}; \frac{k+1}{N_j}\Big) \cup 1.$$

Note that

$$W = \bigsqcup_{k_1=1}^{N_1} \dots \bigsqcup_{k_n=1}^{N_n} W_{k_1 \dots k_n} \bigsqcup W_0,$$

where $W_{k_1 \dots k_n}$ contains all points of W whose component $x^j_{i_j}$ belongs to the k_jth half-interval in jth space and W_0 is the remaining set of zero Lebesgue measure. It is clear that, for some numbers $K_1, \dots, K_n$ such that $K_1 \leq N_1, \dots, K_n \leq N_n$, the Lebesgue measure $\lambda(W_{K_1 \dots K_n})$ is positive. The set $W_{K_1 \dots K_n}$ is the desired one. All 2^n shifts of the set $W_{K_1 \dots K_n}$ are disjoint subsets of U of positive Lebesgue measure. $\square$

A similar assertion holds under more general assumptions. Before discussing this assertion, we recall the definition of a point isomorphism.

Definition 4.1. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two measurable spaces with nonnegative measures. A point isomorphism J of these spaces is a one-to-one mapping of X onto Y such that $J(\mathcal{A}) = \mathcal{B}$ and $\mu \circ J^{-1} = \nu$.

Definition 4.2. Spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are said to be isomorphic mod0 if, for some sets $F \in \mathcal{A}_\mu$, $F' \in \mathcal{B}_\nu$ with $\mu(F) = \nu(F') = 0$, there exists a point isomorphism J of the spaces $X \backslash F$ and $Y \backslash F'$ endowed with the restrictions of the measures μ and ν and the σ-algebras $\mathcal{A}_\mu$ and $\mathcal{B}_\nu$.

Lemma 4.2. *Suppose we are given n Souslin spaces $(X_1, \mu_1), \dots, (X_n, \mu_n)$ with nonatomic Borel probability measures and a μ-measurable set $U \subset X$ of positive measure. Then there exists a nonzero function $\xi \in L^1(\mu)$ supported on U such that*

$$\int_{X_1 \times \dots \times X_{i-1} \times X_{i+1} \times \dots \times X_n} \xi \, d\mu_1 \otimes \ldots \otimes \mu_{i-1} \otimes \mu_{i+1} \otimes \ldots \otimes \mu_n = 0$$

μ_i-almost everywhere for any $i \leqslant n$.

Proof. According to [4, Theorem 9.2.2], the space (X_i, μ_i) is isomorphic mod0 to the space $([0;1], \lambda)$ for any $i \leqslant n$, where λ is Lebesgue measure. In other words, for any $i \leqslant n$ there exist sets $X'_i \subset X_i$ and $I_i \subset [0;1]$ with $\mu_i(X'_i) = 1$ and $\lambda(I_i) = 1$ and a point isomorphism J_i between the spaces X'_i and I_i endowed with the restrictions of the measures μ_i and λ. For notational simplicity we shall consider the spaces $I = \prod_{i=1}^n I_i$, $X' = \prod_{i=1}^n X'_i$, $I' = \prod_{i=1}^n I'_i$ and the isomorphism $J = \prod_{i=1}^n J_i$.

We have the following chain of relations:

$$\mu(U) = \mu(U \cap X') = \lambda \otimes \ldots \otimes \lambda(J(U \cap X')).$$

The set $V = J(U \cap X')$ of positive Lebesgue measure lies in the space (I, λ). Following the line of reasoning in Lemma 4.1, for the Lebesgue measurable set $V \subset I$ of positive measure and a sufficiently small $\delta > 0$, we construct sets $V_1, V_2, \dots, V_{2^n} \subset V$ of positive measure obtained by shifts of the set V in the space I.

Note that there is a correspondence between the sets $V_1, V_2, \dots, V_{2^n} \subset V$ and the vertices of the n-dimensional hypercube. Therefore, these sets can be enumerated by binary sequences of length n so that any two sets that differ by shifts with respect to the variable from the jth space correspond to binary sequences which differ only in the jth coordinate.

For every $j \leqslant n$, let $s(j)$ be the sum of all elements of the binary sequence corresponding to the set V_j. Consider the sets

$$V_{odd} = \bigcup_{j \leqslant n,\, s(j) \text{ is odd}} V_j,$$

$$V_{even} = \bigcup_{j \leqslant n,\, s(j) \text{ is even}} V_j.$$

The function $\zeta \in L^1(I, \lambda \otimes \ldots \otimes \lambda)$ defined by

$$\zeta(t_1, \dots, t_n) \colon = \begin{cases} +1 \text{ if } (t_1, \dots, t_n) \in V_{odd}, \\ -1 \text{ if } (t_1, \dots, t_n) \in V_{even}, \\ \ \ 0 \text{ otherwise}, \end{cases}$$

satisfies the condition

$$\int_{I_1 \times \dots \times I_{i-1} \times I_{i+1} \times \dots \times I_n} \xi \, d(\lambda \otimes \ldots \otimes \lambda) = 0$$

for every $i \leqslant n$.

Consider the sets $U_1, U_2, \dots, U_{2^n} \subset X'$ of positive measure μ that are the preimages of $V_1, V_2, \dots, V_{2^n}$ under the mapping $J \colon X' \to I$ and the function $\xi \in L^1(X', \mu)$ defined by

$$\xi(x_1, \dots, x_n) = \zeta(J_1(x_1), \dots, J_n(x_n)).$$

In other words,

$$\xi(x_1, \dots, x_n) \colon = \begin{cases} +1 \text{ if } (x_1, \dots, x_n) \in U_{odd}, \\ -1 \text{ if } (x_1, \dots, x_n) \in U_{even}, \\ \ \ 0 \text{ otherwise}, \end{cases}$$

where the sets U_{odd}, U_{even} are the preimages of V_{odd}, V_{even}.

It is easy to see that the function ξ is well defined and nonzero on a set of positive measure μ, the support of ξ is contained in U and the chain of relations

$$\begin{aligned}
&\int_{[0;1]^{n-1}} \zeta(t_1, \dots, t_n) \, \lambda(dt_2, \dots, dt_n) \\
&= \int_{[0;1]^{n-1}} \zeta(t_1, \dots, t_n) \, \mu_2 \circ J_2^{-1}(dt_2) \ldots \mu_n \circ J_n^{-1}(dt_n) \\
&= \int_{X_2' \times \ldots \times X_n'} \zeta(J_1(x_1), \dots, J_n(x_n)) \, d\mu_2 \ldots d\mu_n = \int_{X_2' \times \ldots \times X_n'} \xi(x_1, \dots, x_n) \, d\mu_2 \ldots d\mu_n \\
&= \int_{X_2 \times \ldots \times X_n} \xi(x_1, \dots, x_n) \, d\mu_2 \ldots d\mu_n = 0
\end{aligned}$$

holds μ_1-almost everywhere. Similar statements hold for other marginals. Thus, ξ is the desired function. $\square$

Lemma 4.2 provides a characterization of extreme points of $\Gamma_\chi(\mu)$.

Theorem 4.1. *Suppose we are given n Suslin spaces with nonatomic Borel probability measures $(X_1, \mu_1), \ldots, (X_n, \mu_n)$. A density $h \in \Gamma_\chi(\mu)$ is an extreme point of the set $\Gamma_\chi(\mu)$ if and only if $h = 1_W \chi$ for a μ-measurable set $W \subset X$.*

Proof. The condition $h \in \Gamma_\chi(\mu)$ implies the inequalities $0 \leqslant h \leqslant \chi$. If h is an extreme point, then these inequalities cannot be simultaneously strict on a set $U \subset X$ of positive measure. Indeed, suppose that such a set U exists. Then, for some $\varepsilon > 0$, the set

$$U_\varepsilon = \{x \in X : \varepsilon < h(x) < \chi(x) - \varepsilon\}$$

has positive measure as well. According to Lemma 4.2, for the set U_ε there exists a function $\xi \in L^1(\mu)$ supported on U_ε such that

$$\int_{X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n} \xi \, d\mu_1 \otimes \ldots \otimes \mu_{i-1} \otimes \mu_{i+1} \otimes \ldots \otimes \mu_n = 0$$

μ_i-almost everywhere for any $i \leqslant n$.

The inequalities $\varepsilon < h < \chi - \varepsilon$ hold on the support of the function ξ. Therefore, both functions $h_\pm = h \pm \varepsilon\xi$ belong to $\Gamma_\chi(\mu)$. They do not coincide with h and are distinct. Since the function $h = (h_+ + h_-)/2$ is a convex combination of h_+ and h_-, it follows that h is not an extreme point of $\Gamma_\chi(\mu)$. This contradiction shows that the extreme point h of the set $\Gamma_\chi(\mu)$ has the form $1_W \chi$.

Conversely, each function $h = 1_W \chi$, where $W \subset X$ is a Lebesgue measurable set, is an extreme point of $\Gamma_\chi(\mu)$. Indeed, suppose that $h = 1_W \chi$ can be represented as a convex combination $h = (h_1 + h_2)/2$, where $h_1, h_2 \in \Gamma_\chi(\mu)$. Since both functions h_1, h_2 are nonnegative, they must vanish at all points at which h vanishes. In other words, $h_{1,2} = 0$ outside the set W. Since $h_{1,2} \leq \chi$, these functions must coincide with χ at those points at which h does. This means that $h_{1,2} = \chi$ on W. Thus, $h_1 = h_2 = h$; hence the function $h = 1_W \chi$ is an extreme point of $\Gamma_\chi(\mu)$. □

5 Conditions on the cost function

The uniqueness of the solution requires much stronger conditions on the involved objects. Hereinafter we assume that for every $i \leqslant n$ the space X_i is $[0;1]^\infty$ equipped with the Borel σ-algebra $\mathcal{A}_i$ and a nonatomic Borel probability measure μ_i.

We also assume that the function $c \in L^\infty(\mu)$ is such that the mixed partial derivative of order n

$$\partial_{i_1,\ldots,i_n} c = \frac{\partial^n c}{\partial x^1_{i_1} \ldots \partial x^n_{i_n}}$$

exists for each set of variables $(x^1_{i_1}, \ldots, x^n_{i_n})$, where $1 \leqslant i_k \leqslant \infty$ for any $k \leqslant n$.

Next, we assume that for some fixed number $N \in \mathbb{N} \bigcup \infty$ there exist at most countably many disjoint open sets $\{G_k\}_{k=1}^N$, $G_k \subset X$, such that the following conditions are satisfied:

(C1) each set G_k has positive Lebesgue measure;

(C2) the union of all sets in $\{G_k\}_{k=1}^N$ has full Lebesgue measure;

(C3) for every $k \leq N$ there exists a set of variables $(x^1_{k_1}, \ldots, x^n_{k_n})$ such that the function $\partial_{k_1,\ldots,k_n} c$ is either strictly positive or strictly negative on G_k.

6 Uniqueness of a solution

Now we show that an optimal plan is an extreme point of the set $\Gamma_\chi(\mu)$.

Theorem 6.1. *Suppose we are given a cost function c on the the space X satisfying conditions* (C1)–(C3)*, a nonnegative constraint-function $\chi \in L^1(\mu)$, and the set $\Gamma_\chi(\mu)$ is nonempty. Then any function h that is an optimal plan is an extreme point of the set $\Gamma_\chi(\mu)$.*

Proof. Suppose that $h \in \Gamma_\chi(\mu)$ is not an extreme point of the set $\Gamma_\chi(\mu)$. We prove the theorem by constructing a perturbation of the function h which decreases the value of the functional I_c. By Theorem 4.1 we have $h \neq 1_W\chi$. Hence

$$U = \{x \in X \colon 0 < h(x) < \chi\}$$

is a set of positive measure. Therefore, for sufficiently small $\varepsilon > 0$ we have

$$U_\varepsilon = \{x \in U \colon \varepsilon < h(x) < \chi - \varepsilon\}$$

is a set of positive measure as well.

By condition *(C2)* on the cost function c,

$$\mu(X \backslash (\bigsqcup_{k=1}^{N} G_k)) = 0.$$

Therefore, the intersection of U_ε with one of the sets G_k is a set of positive measure. We denote this set by G. Condition *(C3)* associates this set with a set of variables $(x^1_{k_1}, \ldots, x^n_{k_n})$ such that the mixed derivative of order n of the cost function $\partial_{k_1,\ldots,k_n} c$ is either positive or negative everywhere on G. Let σ denote the sign of $\partial_{k_1,\ldots,k_n} c$ on G. We also need the sets

$$G^\delta = \{x \in G \colon (x^1 + \alpha_1 \mathbf{e_{k_1}}, \ldots, x^n + \alpha_n \mathbf{e_{k_n}}) \in G \ \ \forall\, 0 < |\alpha_1|, \ldots, |\alpha_n| < \delta\}.$$

For sufficiently small δ the sets G^δ are nonempty, because G is open. These sets form a nested sequence, i.e., $G^{\delta_2} \subset G^{\delta_1}$ if $\delta_1 < \delta_2$. Since G is open, it follows that each point in G belongs G^δ for a sufficiently small δ, hence $\bigcup_{\delta \to 0} G^\delta = G$. Therefore, for a sufficiently small δ_0, the intersection $G^{\delta_0} \cap U_\varepsilon$ has positive measure.

By using Lemma 4.2, for $G^{\delta_0} \cap U_\varepsilon$, the set of variables $(x^1_{k_1}, \ldots, x^n_{k_n})$, and the number $\delta_0/2$, we construct a set of positive measure $V \subset G^{\delta_0} \cap U_\varepsilon$ and shifts $\varphi_1, \ldots, \varphi_n \in (-\delta; \delta) \backslash \{0\}$ such that all 2^n shifts of the set V with respect to the variables $x^1_{k_1}, \ldots, x^n_{k_n}$ are pairwise disjoint subsets of $G^{\delta_0} \cap U_\varepsilon$. Moreover, all 2^n shifts of the set V are contained in G. By using Lemma 4.2, we construct a perturbation ξ with values 1 и -1 such that for every $i \leqslant n$ the equality

$$\int_{X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n} \xi \, d\mu_1 \otimes \ldots \otimes \mu_{i-1} \otimes \mu_{i+1} \otimes \ldots \otimes \mu_n = 0$$

holds μ_i-almost everywhere. Let Σ denote the sign of $\prod_{i \leqslant n} \varphi_i$. The function ξ has support in U_ε, where $\varepsilon < h < \chi - \varepsilon$. Therefore, the function

$$h_\varepsilon = h + (-1)^{n+1} \sigma \varepsilon \Sigma \xi$$

belongs to $\Gamma_\chi(\mu)$ and does not coincide with h. The value of the functional I_c on the function ξ is equal to

$$
\begin{aligned}
I_c(\xi) &= \int_X c\xi \, d\mu \\
&= \int_V \sum_{(i_1,\dots,i_n)\in[0;1]^n} (-1)^{i_1+\dots+i_n} c(x_1 + i_1\varphi_1 \mathbf{e}_{\mathbf{k_1}}, \dots, x_n + i_n\varphi_n \mathbf{e}_{\mathbf{k_n}})\, d\mu \\
&= \varphi_1 \int_V \int_0^1 \sum_{(i_2,\dots,i_n)\in[0;1]^{n-1}} (-1)^{i_2+\dots+i_n+1} \times \\
&\times \frac{\partial c}{\partial x_{k_1}}(x_1 + s_1\varphi_1 \mathbf{e}_{\mathbf{k_1}}, \dots, x_n + i_n\varphi_n \mathbf{e}_{\mathbf{k_n}})\, ds_1\, d\mu \\
&= \prod_{i\leqslant n} \varphi_i \int_V \int_0^1 \dots \int_0^1 (-1)^n \times \\
&\times \frac{\partial^n c}{\partial x_{k_1} \dots \partial x_{k_n}}(x_1 + s_1\varphi_1 \mathbf{e}_{\mathbf{k_1}}, \dots, x_n + s_n\varphi_n \mathbf{e}_{\mathbf{k_n}}) ds_1 \dots ds_n\, d\mu.
\end{aligned}
$$

Since $V \subset G^{\delta_0}$, it follows that the derivative of the cost function has the sign σ everywhere on the domain of integration. Therefore,

$$\operatorname{sgn}(I_c(\xi)) = (-1)^n \sigma \Sigma.$$

Let us observe that the change of the value of the functional generated by the perturbation $(-1)^{n+1}\sigma\varepsilon\Sigma\xi$ equals

$$I_c(h_\varepsilon) - I_c(h) = (-1)^{n+1}\sigma\varepsilon\Sigma I_c(\xi),$$

hence $I_c(h_\varepsilon) < I_c(h)$, which contradicts the optimality of h. Thus, the optimal plan h is an extreme point of the set $\Gamma_\chi(\mu)$. □

Corollary 6.1.1. *Under the conditions of Theorem 6.1, the optimal plan h is unique.*

Proof. If both h_0 and h_1 minimize I_c on $\Gamma_\chi(\mu)$, then, since the functional I_c is linear and the set $\Gamma_\chi(\mu)$ is convex, it follows that the function $h_{1/2} = (h_0+h_1)/2$ minimizes I_c as well. By Theorem 6.1, each of the three functions h_0, h_1 and $h_{1/2}$ is an extreme point of the set $\Gamma_\chi(\mu)$. Hence, by Theorem 4.1, we have $h_0 = 1_{W_0}\chi$, $h_1 = 1_{W_1}\chi$, and $h_{1/2} = 1_{W_{1/2}}\chi$ for some Lebesgue measurable sets $W_0, W_1, W_{1/2} \subset X$. Therefore, $h_0 = h_1 = h_{1/2}$, which completes the proof. □

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Department of Mechanics and Mathematics
Lomonosov Moscow State University
Moscow, Russia