$L^q(L^p)$-THEORY OF STOCHASTIC DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we show the weak differentiability of the unique strong solution with respect to the starting point x as well as Bismut-Elworthy-Li's derivative formula for the following stochastic differential equation in $\mathbb{R}^d$:

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t, \quad X_0 = x \in \mathbb{R}^d,$$

where σ is bounded, uniformly continuous and nondegenerate, $\nabla\sigma \in \widetilde{\mathbb{L}}^{p_1}_{q_1}$ and $b \in \widetilde{\mathbb{L}}^{p_2}_{q_2}$ for some $p_i, q_i \in [2,\infty)$ with $\frac{d}{p_i} + \frac{2}{q_i} < 1$, $i = 1, 2$, where $\widetilde{\mathbb{L}}^{p_i}_{q_i}, i = 1, 2$ are some localized spaces. Moreover, in the endpoint case $b \in \widetilde{\mathbb{L}}^{d;\mathrm{uni}}_{\infty}$, we also show the weak well-posedness.

Keywords: Krylov's estimate, $L^q(L^p)$-estimates, Zvonkin's transformation, duality.

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1. Introduction and main results

Consider the following stochastic differential equation (SDE) in $\mathbb{R}^d$ ($d \geqslant 2$):

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sqrt{2}\mathrm{d}W_t, \quad X_0 = x, \tag{1.1}$$

where $(W_t)_{t\geqslant 0}$ is a d-dimensional standard Brownian motion on some filtered probability space $(\Omega, \mathscr{F}, \mathbf{P}; (\mathscr{F}_t)_{t\geqslant 0})$, and b is a time-dependent measurable vector field. When b is bounded measurable, Veretennikov [16] proved the strong existence and uniqueness of solutions for SDE (1.1). For $T > 0$ and $p, q \in (1,\infty)$, let $\mathbb{L}^p_q(T) := L^q([0,T]; L^p)$. When $b \in \mathbb{L}^p_q := \cap_{T>0}\mathbb{L}^p_q(T)$ for some $p, q \in [2,\infty)$ with $\frac{d}{p} + \frac{2}{q} < 1$, by Girsanov's transformation and some $\mathbb{L}^p_q$-estimate for the associated Kolmogorov equation, Krylov and Röckner [9] showed the strong well-posedness for SDE (1.1) in the class of X that satisfies $\int_0^T |b(t, X_t)|^2\mathrm{d}t < \infty$ a.s. From then on, there are increasing interests of studying the strong and weak well-posedness for SDE (1.1) with singular or even distributional drifts, see [19, 23] and references therein.

After [9], there are also a lot of works devoted to studying the properties of the solution $X_t(x, \omega)$ for SDE (1.1) with singular coefficients. Among all, we mention that when b is bounded measurable, Menoukeu etal [10] showed the weak differentiability of $X_t(x, \omega)$ in x and the Malliavin differentiability of $X_t(x, \omega)$ with respect to the sample point ω. When $b \in \mathbb{L}^p_q$ for some $p, q \in [2,\infty)$ with $\frac{d}{p} + \frac{2}{q} < 1$ and in the multiplicative noise case, the above regularities in x and ω were also shown in

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[22] by Zvonkin's transformation. However, Zvonkin's transformation used in [22] can not be applied to the bounded drift b because the following PDE does not allow an $\mathbb{H}^{2,\infty}$-solution for $b \in L^\infty$ in general:

$$\partial_t u = \Delta u + b \cdot u + b, \quad u(0) = 0.$$

It should be noticed that the weak differentiability of strong solutions in spatial variables enables us to study the well-posedness of the associated stochastic transport equation since it is closely related to SDE (1.1) through the stochastic inverse flow induced by the strong solution, see [3, 11] and references therein. One of the aim of this paper is to provide a unified treatment for the main results in [10] and [22] and extends them to the case of *local* integrable coefficients.

On the other hand, in the critical case $\frac{d}{p} + \frac{2}{q} = 1$ with $p, q \in [2, \infty)$, Beck etal [1] claimed the existence and uniqueness of strong solutions to SDE (1.1) for *almost all* starting point x. Recently, when b belongs to some Lorentz space $L^{q,1}(L^p) \subset L^{q,q}(L^p) = \mathbb{L}^p_q$ for some $p, q \in [2, \infty)$ with $\frac{d}{p} + \frac{2}{q} = 1$, still by Zvonkin's transformation, Nam [12] showed the existence and uniqueness of strong solutions for SDE (1.1). When $b \in L^d(\mathbb{R}^d)$ is *time-independent*, Kinzebulatov and Semenov [4] showed the existence of weak solutions for each starting point $x \in \mathbb{R}^d$, but the uniqueness is left open. Moreover, in the supercritical case $b \in \mathbb{L}^p_q$ for some $p, q \in [2, \infty)$ with $\frac{d}{p} + \frac{2}{q} < 2$, under an extra integrability assumption on $(\mathrm{div} b)^-$, in a recent work [24], the last two authors of the present paper showed the existence of weak solutions. Another goal of this paper is to show the existence and uniqueness of weak solutions for SDE (1.1) with multiplicative noise in the endpoint case $b \in \widetilde{\mathbb{L}}^{d;\mathrm{uni}}_\infty$, which is not covered by all of the above results.

In this paper, we shall consider the following SDE driven by multiplicative Brownian noises:

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t, \quad X_0 = x, \tag{1.2}$$

where $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ and $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ are Borel measurable functions. The generator of this SDE is given by

$$\mathscr{L}^{\sigma,b}_t f(x) := \tfrac{1}{2}(\sigma^{ik}\sigma^{jk})(t, x)\partial_i\partial_j f(x) + b^i(t, x)\partial_i f(x). \tag{1.3}$$

Here and below, we use Einstein's convention that the repeated indices in a product will be summed automatically. Throughout this paper, we assume that

($\mathbf{H}^\sigma$) $\lim_{|x-y|\to 0} \sup_t \|\sigma(t, x) - \sigma(t, y)\|_{HS} = 0$, and for some $c_0 \geqslant 1$ and for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$,

$$c_0^{-1}|\xi|^2 \leqslant |\sigma(t, x)\xi|^2 \leqslant c_0|\xi|^2, \quad \forall \xi \in \mathbb{R}^d,$$

where $\|\cdot\|_{HS}$ stands for the Hilbert-Schmidt norm of a matrix.

Our first main result in this paper is:

Theorem 1.1. *Assume* ($\mathbf{H}^\sigma$) *and* $\nabla\sigma \in \widetilde{\mathbb{L}}^{p_1}_{q_1}, b \in \widetilde{\mathbb{L}}^{p_2}_{q_2}$ *for some* $p_i, q_i \in [2, \infty)$ *with* $\frac{d}{p_i} + \frac{2}{q_i} < 1$, $i = 1, 2$, *where* $\widetilde{\mathbb{L}}^p_q$ *is defined by* (2.2) *below. Then for each* $x \in \mathbb{R}^d$, *there is a unique strong solution* $X_t(x)$ *for SDE* (1.2). *Moreover,* $X_t(x)$ *enjoys the following properties:*

(i) *(Krylov's estimate) For any $p,q\in(1,\infty)$ with $\frac{d}{p}+\frac{2}{q}<2$ and $T>0$, there is a constant $C>0$ such that for all $x\in\mathbb{R}^d$ and $0\leqslant t_0<t_1\leqslant T$, $f\in\widetilde{\mathbb{L}}^p_q(t_0,t_1)$,*
$$\mathbf{E}\left(\int_{t_0}^{t_1} f(s,X_s(x))\mathrm{d}s\Big|\mathscr{F}_{t_0}\right)\leqslant C\|f\|_{\widetilde{\mathbb{L}}^p_q(t_0,t_1)},$$
where $\|\cdot\|_{\widetilde{\mathbb{L}}^p_q(t_0,t_1)}$ is defined by (2.2) below.

(ii) *(Weak differentiability) For each $t\geqslant 0$, the mapping $x\mapsto X_t(x)$ is almost surely weak differentiable and for any $T>0$ and $p\geqslant 1$,*
$$\sup_{x\in\mathbb{R}^d}\mathbf{E}\left(\sup_{t\in[0,T]}|\nabla X_t(x)|^p\right)<\infty. \tag{1.4}$$

(iii) *(Derivative formula) For any $t>0$ and $\varphi\in C^1_b(\mathbb{R}^d)$, it holds that for Lebesgue-almost all $x\in\mathbb{R}^d$,*
$$\nabla\mathbf{E}\varphi(X_t(x))=\frac{1}{t}\mathbf{E}\left(\varphi(X_t(x))\int_0^t\sigma^{-1}(s,X_s(x))\nabla X_s(x)\mathrm{d}W_s\right). \tag{1.5}$$

Remark 1.2. *As we mentioned before, when $\nabla\sigma,b\in\mathbb{L}^p_q$ for some $p,q\in(2,\infty)$ with $\frac{d}{p}+\frac{2}{q}<1$, the above theorem has been obtained in [22]. Notice that $b\in\mathbb{L}^\infty$ is not covered by [22]. The novelty of our result here is that we are considering some localized $\widetilde{\mathbb{L}}^p_q$-spaces so that we still have the global properties (1.4) and (1.5). In particular, we extend the main results in [10,11,22] to more general cases, and our proofs are much simpler than [10].*

Let $\mathbb{C}$ be the space of all continuous functions from $\mathbb{R}_+$ to $\mathbb{R}^d$ endowed with the usual Borel σ-field $\mathcal{B}(\mathbb{C})$, and ω_t the canonical process over $\mathbb{C}$. For $t\geqslant 0$, let $\mathcal{B}_t:=\mathcal{B}_t(\mathbb{C})$ be the natural filtration generated by $\{\omega_s:s\leqslant t\}$. All the probability measures over $(\mathbb{C},\mathcal{B}(\mathbb{C}))$ is denoted by $\mathscr{P}(\mathbb{C})$. We introduce the following notion of martingale solutions.

Definition 1.3. *Given $(s,x)\in\mathbb{R}_+\times\mathbb{R}^d$, we call a probability measure $\mathbb{P}_{s,x}\in\mathscr{P}(\mathbb{C})$ a martingale solution of SDE (1.2) with starting point (s,x) if $\mathbb{P}_{s,x}(\omega_t=x,t\leqslant s)=1$, and for all $f\in C^2_b(\mathbb{R}^d)$, M^f_t is a $\mathcal{B}_t$-martingale under $\mathbb{P}_{s,x}$, where*
$$M^f_t(\omega):=f(\omega_t)-f(x)-\int_s^t\mathscr{L}^{\sigma,b}_r f(\omega_r)\mathrm{d}r,\quad t\geqslant s,$$
and $\mathscr{L}^{\sigma,b}_r$ is defined by (1.3). All the martingale solution $\mathbb{P}_{s,x}$ of SDE (1.2) with starting point (s,x) and coefficients (σ,b) is denoted by $\mathscr{M}^{\sigma,b}_{s,x}$.

Our second main result is the following weak well-posedness of SDE (1.2) in the endpoint case $b\in\widetilde{\mathbb{L}}^{d;\mathrm{uni}}_\infty$ (see (2.3) below for the definition of $\widetilde{\mathbb{L}}^{d;\mathrm{uni}}_\infty$).

Theorem 1.4. *Assume $(\mathbf{H}^\sigma)$ holds and $b\in\widetilde{\mathbb{L}}^{d;\mathrm{uni}}_\infty$. Then for each $(s,x)\in\mathbb{R}_+\times\mathbb{R}^d$, there is a unique martingale solution $\mathbb{P}_{s,x}\in\mathscr{M}^{\sigma,b}_{s,x}$ for SDE (1.2) which satisfies that for any $p,q\in(1,\infty)$ with $\frac{d}{p}+\frac{2}{q}<2$ and $T>0$, there is a constant $C>0$ such that for all $x\in\mathbb{R}^d$ and $s\leqslant t_0<t_1\leqslant T$, $f\in\widetilde{\mathbb{L}}^p_q(t_0,t_1)$,*
$$\mathbb{E}^{\mathbb{P}_{s,x}}\left(\int_{t_0}^{t_1}f(r,\omega_r)\mathrm{d}r\Big|\mathcal{B}_{t_0}\right)\leqslant C\|f\|_{\widetilde{\mathbb{L}}^p_q(t_0,t_1)}. \tag{1.6}$$

The proof of our main results relies on the $\mathbb{L}^p_q$-maximal regularity estimate for the following second order parabolic PDE in $\mathbb{R}_+ \times \mathbb{R}^d$:

$$\partial_t u = a^{ij}\partial_i\partial_j u + f, \quad u(0) = 0, \tag{1.7}$$

where $a(t,x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ is a symmetric matrix-valued Borel function and satisfies

($\mathbf{H}^a$) $\lim_{|x-y|\to 0}\sup_{t\in\mathbb{R}_+}\|a(t,x) - a(t,y)\|_{HS} = 0$ and for some $c_0 \geqslant 1$ and for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$,

$$c_0^{-1}|\xi|^2 \leqslant a^{ij}(t,x)\xi_i\xi_j \leqslant c_0|\xi|^2, \quad \forall \xi \in \mathbb{R}^d. \tag{1.8}$$

More precisely, for any $p,q \in (1,\infty)$, we want to establish the following estimate:

$$\|\partial_t u\|_{\mathbb{L}^p_q(T)} + \|\nabla^2 u\|_{\mathbb{L}^p_q(T)} \leqslant C\|f\|_{\mathbb{L}^p_q(T)}. \tag{1.9}$$

Such type of estimate has been used in [19] to study the strong well-posedness of SDEs with Sobolev diffusion coefficients. Notice that when $p = q$, it is a standard procedure to prove (1.9) by freezing coefficient argument (cf. [22]). While for $p \neq q$, it is non-trivial. When a^{ij} is independent of x, (1.9) was first proved by Krylov in [8]. In the spatial dependent case, Kim [5] showed (1.9) only for $p \leqslant q$. Here we shall drop this restriction by a duality method. In particular, we need to treat the adjoint equation of (1.7) in Sobolev spaces with negative differentiability index, see Theorem 3.3 below, which is of independent interest. Moreover, we also show the estimate (1.9) in localized space $\widetilde{\mathbb{L}}^p_q(T)$.

This paper is organized as follows: In Section 2, we collect some preliminary tools. Section 3 is devoted to the study of $\mathbb{L}^p_q$-maximal regularity estimate for second order parabolic equations. In Section 4, we prove our main theorems. Throughout this paper we shall use the following conventions:

- The letter C denotes a constant, whose value may change in different places.
- We use $A \lesssim B$ and $A \asymp B$ to denote $A \leqslant CB$ and $C^{-1}B \leqslant A \leqslant CB$ for some unimportant constant $C > 0$, respectively.
- For any $\varepsilon \in (0,1)$, we use $A \lesssim \varepsilon B + D$ to denote $A \leqslant \varepsilon B + C_\varepsilon D$ for some constant $C_\varepsilon > 0$.
- $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ := [0,\infty)$, $a \vee b := \max(a,b)$, $a \wedge b := \min(a,b)$, $a^+ := a \vee 0$.
- $\nabla_x := \partial_x := (\partial_{x_1}, \cdots, \partial_{x_d})$, $\partial_i := \partial_{x_i} := \partial/\partial x_i$.

2. Preliminaries

First of all, we introduce some spaces and notations for later use. For $(\alpha,p) \in \mathbb{R} \times (1,\infty)$, let $H^{\alpha,p} := (\mathbb{I} - \Delta)^{-\alpha/2}\big(L^p(\mathbb{R}^d)\big)$ be the usual Bessel potential space with norm

$$\|f\|_{\alpha,p} := \|(\mathbb{I} - \Delta)^{\alpha/2} f\|_p,$$

where $\|\cdot\|_p$ is the usual L^p-norm in $\mathbb{R}^d$, and $(\mathbb{I}-\Delta)^{\alpha/2} f$ is defined through Fourier's transform

$$(\mathbb{I} - \Delta)^{\alpha/2} f := \mathcal{F}^{-1}\big((1 + |\cdot|^2)^{\alpha/2}\mathcal{F}f\big).$$

Notice that for $n \in \mathbb{N}$ and $p \in (1,\infty)$, an equivalent norm in $H^{n,p}$ is given by

$$\|f\|_{n,p} = \|f\|_p + \|\nabla^n f\|_p.$$

Let $\chi \in C_c^\infty(\mathbb{R}^d)$ be a smooth function with $\chi(x) = 1$ for $|x| \leqslant 1$ and $\chi(x) = 0$ for $|x| > 2$. For $r > 0$ and $z \in \mathbb{R}^d$, define

$$\chi_r(x) := \chi(x/r), \quad \chi_r^z(x) := \chi_r(x - z). \tag{2.1}$$

Fix $r > 0$. We introduce the following localized $H^{\alpha,p}$-space:

$$\widetilde{H}^{\alpha,p} := \Big\{ f \in H^{\alpha,p}_{loc}(\mathbb{R}^d), |||f|||_{\alpha,p} := \sup_z \|\chi_r^z f\|_{\alpha,p} < \infty \Big\}.$$

For $T > 0$, $p, q \in (1, \infty)$ and $\alpha \in \mathbb{R}$, we also define space-time function space

$$\mathbb{L}_q^p(T) := L^q\big([0,T]; L^p\big), \quad \mathbb{H}_q^{\alpha,p}(T) := L^q\big([0,T]; H^{\alpha,p}\big),$$

and the localized space $\widetilde{\mathbb{H}}_q^{\alpha,p}(T)$ with norm

$$|||f|||_{\widetilde{\mathbb{H}}_q^{\alpha,p}(T)} := \sup_{z \in \mathbb{R}^d} \|\chi_r^z f\|_{\mathbb{H}_q^{\alpha,p}(T)} < \infty. \tag{2.2}$$

For $q = \infty$ and $p \in [1, \infty)$, we define $\widetilde{\mathbb{L}}_\infty^{p;\mathrm{uni}}(T)$ being all the functions $f \in \widetilde{\mathbb{L}}_\infty^p(T)$ with

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} |||f(t,\cdot) * \rho_\varepsilon - f(t,\cdot)|||_p =: \lim_{\varepsilon \to 0} \kappa_T^f(\varepsilon) = 0, \tag{2.3}$$

where $(\rho_\varepsilon)_{\varepsilon \in (0,1)}$ is a family of mollifiers in $\mathbb{R}^d$. For simplicity we shall write

$$H^{\infty,p} := \cap_{\alpha>0} H^{\alpha,p}, \quad \widetilde{\mathbb{H}}_q^{\alpha,p} := \cap_{T>0} \widetilde{\mathbb{H}}_q^{\alpha,p}(T), \quad \widetilde{\mathbb{L}}_q^p := \cap_{T>0} \widetilde{\mathbb{L}}_q^p(T).$$

It is not hard to show that the definitions of $\widetilde{H}^{\alpha,p}$ and $\widetilde{\mathbb{H}}_q^{\alpha,p}(T)$ do not depend on the choice of r and χ. In fact, we can prove that for any $r, r' > 0$ (cf. [24]),

$$\sup_{z \in \mathbb{R}^d} \|\chi_r^z f\|_{\mathbb{H}_q^{\alpha,p}(T)} \asymp \sup_{z \in \mathbb{R}^d} \|\chi_{r'}^z f\|_{\mathbb{H}_q^{\alpha,p}(T)}. \tag{2.4}$$

Notice that

$$L^q([0,T]; \widetilde{H}^{\alpha,p}) \subset \widetilde{\mathbb{H}}_q^{\alpha,p}(T).$$

Now we list some easy properties about space $\widetilde{\mathbb{H}}_q^{\alpha,p}(T)$ for later use.

- The following Sobolev embedding holds: For any $\alpha > 0$, $p, q \in [1, \infty)$ and $p' \in [p, \frac{pd}{d - p\alpha} \mathbf{1}_{p\alpha < d} + \infty \cdot \mathbf{1}_{p\alpha > d}]$, there is a constant $C > 0$ such that
$$|||f|||_{\widetilde{\mathbb{L}}_q^{p'}(T)} \leqslant C |||f|||_{\widetilde{\mathbb{H}}_q^{\alpha,p}(T)}. \tag{2.5}$$
- For any $f \in \widetilde{\mathbb{H}}_q^{\alpha,p}$, it holds that for any $T, R > 0$ (cf. [24, Proposition 4.1]),
$$\sup_\varepsilon |||f_\varepsilon|||_{\widetilde{\mathbb{H}}_q^{\alpha,p}(T)} \leqslant C |||f|||_{\widetilde{\mathbb{H}}_q^{\alpha,p}(T)}, \ \lim_{\varepsilon \to 0} |||(f_\varepsilon - f)\chi_R|||_{\widetilde{\mathbb{H}}_q^{\alpha,p}(T)} = 0, \tag{2.6}$$
where $f_\varepsilon := f * \rho_\varepsilon$ is the usual mollifying approximation of f.
- Let $p, q \in [2, \infty)$ satisfy $\frac{d}{p} + \frac{2}{q} < 2$. If $u \in \widetilde{\mathbb{H}}_q^{2,p}(T)$ and $\partial_t u \in \widetilde{\mathbb{L}}_q^p(T)$, then $u \in C([0,T] \times \mathbb{R}^d)$ (cf. [9, Lemma 10.2]).

For $R \in (0, \infty)$, we define the local Hardy-Littlewood maximal function by

$$\mathcal{M}_R f(x) := \sup_{r \in (0,R)} \frac{1}{|B_r|} \int_{B_r} f(x+y) \mathrm{d}y,$$

where $B_r := \{x \in \mathbb{R}^d : |x| < r\}$ is the ball in $\mathbb{R}^d$. We have the following results (cf. [14] or [21]).

Lemma 2.1. *(i) For any $R>0$, there exists a constant $C=C(d,R)>0$ such that for any $f\in L^\infty(\mathbb{R}^d)$ with $\nabla f\in L^1_{loc}(\mathbb{R}^d)$ and Lebesgue-almost all $x,y\in\mathbb{R}^d$,*

$$|f(x)-f(y)|\leqslant C|x-y|(\mathcal{M}_R|\nabla f|(x)+\mathcal{M}_R|\nabla f|(y)+\|f\|_\infty). \tag{2.7}$$

(ii) For any $p>1$, $q\geqslant 1$ and $R>0$, there is a constant $C=C(R,d,p)>0$ such that for all $f\in\widetilde{\mathbb{L}}^p_q(T)$,

$$\|\mathcal{M}_R f\|_{\widetilde{\mathbb{L}}^p_q(T)}\leqslant C\|f\|_{\widetilde{\mathbb{L}}^p_q(T)}. \tag{2.8}$$

Proof. (i) If $|x-y|\leqslant R$, then by [21, Lemma 5.4] we have

$$|f(x)-f(y)|\leqslant C|x-y|(\mathcal{M}_R|\nabla f|(x)+\mathcal{M}_R|\nabla f|(y)).$$

If $|x-y|>R$, then

$$|f(x)-f(y)|\leqslant 2|x-y|\,\|f\|_\infty/R.$$

Thus (2.7) is true.

(ii) Noticing that for $|y|\leqslant R$, $\chi_R(x)=\chi_R(x)\chi_{3R}(x+y)$, by definition we have

$$\begin{aligned}\|\chi^z_R\mathcal{M}_R f_s\|_p^p&=\int_{\mathbb{R}^d}\left|\chi_R(x)\sup_{r\in(0,R)}\frac{1}{|B_r|}\int_{B_r}f_s(x+z+y)\mathrm{d}y\right|^p\mathrm{d}x\\&\leqslant\int_{\mathbb{R}^d}\left(\sup_{r\in(0,R)}\frac{1}{|B_r|}\int_{B_r}\chi_{3R}(x+y)|f_s|(x+z+y)\mathrm{d}y\right)^p\mathrm{d}x\\&\leqslant C\|\chi_{3R}\cdot f_s(\cdot+z)\|_p^p=C\|\chi^z_{3R}f_s\|_p^p,\end{aligned}$$

which in turn gives (2.8) by (2.4). □

The following freezing lemma is taken from [23, Lemma 4.1].

Lemma 2.2. *Let ϕ be a nonzero smooth function with compact support. Define $\phi_z(x):=\phi(x-z)$. For any $\alpha\in\mathbb{R}$ and $p\in(1,\infty)$, there exists a constant $C\geqslant 1$ depending only on α,p,ϕ such that for all $f\in H^{\alpha,p}$,*

$$C^{-1}\|f\|_{\alpha,p}\leqslant\left(\int_{\mathbb{R}^d}\|\phi_z f\|^p_{\alpha,p}\mathrm{d}z\right)^{1/p}\leqslant C\|f\|_{\alpha,p}. \tag{2.9}$$

The following lemma was proven in [8] (see also [5, Lemma 2.5]).

Lemma 2.3. *For $k=1,\cdots,n$, let $a_k:\mathbb{R}\to\mathbb{R}^d\otimes\mathbb{R}^d$ be a measurable function and satisfy that for some $c_0\geqslant 1$,*

$$c_0^{-1}|\xi|^2\leqslant a^{ij}_k(t)\xi_i\xi_j\leqslant c_0|\xi|^2,\quad\forall(t,\xi)\in\mathbb{R}\times\mathbb{R}^d,$$

For fixed $\alpha\in\mathbb{R}$, $p\in(1,\infty)$ and $\lambda\geqslant 0$, let $u_k\in\mathbb{H}^{\alpha,p}_p$ solve the following PDE in the distributional sense:

$$\partial_t u_k=a^{ij}_k\partial_{ij}u_k-\lambda u_k+f_k,\quad u(0)=0.$$

Then for any $T\geqslant 0$, there is a constant $N=N(d,\alpha,p,n,c_0)>0$ independent of T,λ such that

$$\int_0^T\prod_{k=1}^n\|\nabla^2u_k(t)\|^p_{\alpha,p}\mathrm{d}t\leqslant N\sum_{k=1}^n\int_0^T\|f_k\|^p_{\alpha,p}\prod_{\ell\neq k}\|\nabla^2u_\ell(t)\|^p_{\alpha,p}\mathrm{d}t.$$

3. $\widetilde{\mathbb{L}}_q^p$-MAXIMAL REGULARITY ESTIMATE FOR PARABOLIC EQUATIONS

Consider the following second order parabolic PDE in $\mathbb{R}_+ \times \mathbb{R}^d$:

$$\partial_t u = a^{ij}\partial_i\partial_j u + b^i\partial_i u - \lambda u + f, \quad u(0) = 0, \tag{3.1}$$

where $\lambda \geqslant 0$, $a(t,x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ and $b(t,x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ are Borel measurable functions. The main aim of this section is to establish the following $\widetilde{\mathbb{L}}_q^p$-maximal regularity estimate for the above equation.

Theorem 3.1. *Let $p, q \in (1,\infty)$. Assume $(\mathbf{H}^a)$ and one of the following conditions holds:*

(i) (Subcritical case) $\frac{d}{p} + \frac{2}{q} < 1$ and for any $T > 0$, $\|b\|_{\widetilde{\mathbb{L}}_q^p(T)} \leqslant \kappa_T^b < \infty$;

(ii) (Critical case) $p \in (1,d)$ and $b \in \widetilde{\mathbb{L}}_\infty^{d;\mathrm{uni}}$.

Then for any $f \in \widetilde{\mathbb{L}}_q^p$ and $\lambda \geqslant 1$, there exists a unique strong solution $u \in \widetilde{\mathbb{H}}_q^{2,p}$ to PDE (3.1), that is, for all $t \geqslant 0$ and Lebesgue almost all $x \in \mathbb{R}^d$,

$$u(t,x) = \int_0^t (a^{ij}\partial_i\partial_j)u(s,x)\mathrm{d}s + \int_0^t (b^i\partial_i u)(s,x)\mathrm{d}s - \lambda\int_0^t u(s,x)\mathrm{d}s + \int_0^t f(s,x)\mathrm{d}s.$$

Moreover, for any $T > 0$ and $\alpha \in [0, 2-\frac{2}{q})$, there is a constant $C > 0$ only depending on α, p, q, d, c_0, T and the continuity modulus of a, as well as κ_T^b in case (i), and $\kappa_T^b(\varepsilon)$ in case (ii), where $\kappa_T^b(\varepsilon)$ is defined by (2.3), such that for any $\lambda \geqslant 1$,

$$\lambda^{1-\frac{\alpha}{2}-\frac{1}{q}}\|u\|_{\widetilde{\mathbb{H}}_\infty^{\alpha,p}(T)} + \|\partial_t u\|_{\widetilde{\mathbb{L}}_q^p(T)} + \|u\|_{\widetilde{\mathbb{H}}_q^{2,p}(T)} \leqslant C\|f\|_{\widetilde{\mathbb{L}}_q^p(T)}. \tag{3.2}$$

Remark 3.2. *In critical case (ii), if $b(t,x) = b(x) \in L^d(\mathbb{R}^d)$ is time-independent, then $b \in \widetilde{\mathbb{L}}_\infty^{d;\mathrm{uni}}$.*

3.1. Smooth a and f.

In this subsection, we study PDE (3.1) with $b \equiv 0$ and a smooth enough, that is, a satisfies $(\mathbf{H}^a)$ and for all $m \in \mathbb{N}$,

$$\|\nabla^m a^{ij}\|_\infty < \infty,$$

where ∇^m stands for the m-order gradient. Given $s < t$, $\lambda \geqslant 0$ and $\varphi, \psi \in C_b^\infty(\mathbb{R}^d)$, consider the following forward heat equation

$$\partial_t u = a^{ij}\partial_{ij}u - \lambda u, \ u(s) = \varphi, \tag{3.3}$$

and backward (adjoint) heat equation

$$\partial_s w = \lambda w - \partial_{ij}(a^{ij}w), \ w(t) = \psi. \tag{3.4}$$

Let $u(t)$ and $w(s)$ be the unique solutions of (3.3) and (3.4) respectively. We shall simply write

$$\mathcal{T}_{s,t}\varphi := u(t), \quad \mathcal{T}_{s,t}^*\psi := w(s).$$

In other words, we have

$$\partial_t\mathcal{T}_{s,t}\varphi = a^{ij}\partial_{ij}\mathcal{T}_{s,t}\varphi - \lambda\mathcal{T}_{s,t}\varphi, \quad \partial_s\mathcal{T}_{s,t}^*\psi = \lambda\mathcal{T}_{s,t}^*\psi - \partial_{ij}(a^{ij}\mathcal{T}_{s,t}^*\psi).$$

Le $p \geqslant 1$. By the chain rule and above equations, it is easy to see that for any $\varphi, \psi \in H^{\infty,p} \subset C_b^\infty(\mathbb{R}^d)$,

$$\langle\mathcal{T}_{s,t}\varphi, \psi\rangle - \langle\varphi, \mathcal{T}_{s,t}^*\psi\rangle = \int_s^t \mathrm{d}_r\langle\mathcal{T}_{s,r}\varphi, \mathcal{T}_{r,t}^*\psi\rangle = 0,$$

where $\langle f, g\rangle := \int_{\mathbb{R}^d} f(x)g(x)\mathrm{d}x$, which means that

$$\langle \mathcal{T}_{s,t}\varphi, \psi\rangle = \langle \varphi, \mathcal{T}^*_{s,t}\psi\rangle. \tag{3.5}$$

Fix $T > 0$ and $p, q \geqslant 1$. For $f \in \mathbb{L}^q_T(H^{\infty,p}) := L^q([0,T]; H^{\infty,p})$, define

$$u(t,x) := \int_0^t \mathcal{T}_{s,t} f(s,x)\mathrm{d}s, \quad w(s,x) := \int_s^T \mathcal{T}^*_{s,t} f(t,x)\mathrm{d}t. \tag{3.6}$$

It is well known that u solves the following forward equation

$$\partial_t u = a^{ij}\partial_{ij}u - \lambda u + f, \quad u(t)|_{t\leqslant 0} = 0, \tag{3.7}$$

and w solves the following backward equation

$$\partial_s w = \lambda w - \partial_{ij}(a^{ij}w) - f, \quad w(s)|_{s\geqslant T} = 0. \tag{3.8}$$

We first prove the following a priori estimates by duality.

Theorem 3.3. *Under* $(\mathbf{H}^a)$*, for any* $p, q \in (1,\infty)$ *and* $T > 0$*, there is a constant* $C > 0$ *only depending on* T, d, p, q, c_0 *and the continuity modulus of* a *such that for any* $f \in \mathbb{L}^q_T(H^{\infty,p})$ *and* $\lambda \geqslant 0$,

$$\|\nabla^2 u_\lambda\|_{\mathbb{L}^p_q(T)} \leqslant C\|f\|_{\mathbb{L}^p_q(T)}, \tag{3.9}$$

$$\|\nabla^2 w_\lambda\|_{\mathbb{H}^{-2,p}_q(T)} \leqslant C\|f\|_{\mathbb{H}^{-2,p}_q(T)}, \tag{3.10}$$

where u_λ *and* w_λ *are solutions of* (3.7) *and* (3.8)*, respectively. Moreover, for any* $\alpha \in [0, 2-\frac{2}{q})$*, we also have*

$$\|u_\lambda\|_{\mathbb{H}^{\alpha,p}_\infty(T)} \leqslant C(1\vee\lambda)^{\frac{\alpha}{2}-1+\frac{1}{q}}\|f\|_{\mathbb{L}^p_q(T)}, \tag{3.11}$$

$$\|w_\lambda\|_{\mathbb{H}^{\alpha-2,p}_\infty(T)} \leqslant C(1\vee\lambda)^{\frac{\alpha}{2}-1+\frac{1}{q}}\|f\|_{\mathbb{H}^{-2,p}_q(T)}. \tag{3.12}$$

Proof. For simplicity of notations, we drop the subscript λ and divide the proof into five steps.

(i) We first claim that it suffices to prove (3.9) and (3.10) for $p \leqslant q$. Indeed, suppose that $q < p$ and let

$$r := \tfrac{p}{p-1} < \theta := \tfrac{q}{q-1}.$$

By duality (3.5) and Hölder's inequality, we have

$$\begin{aligned}
\|\nabla^2 u\|_{\mathbb{L}^p_q(T)} &\overset{(3.6)}{=} \sup_{g\in L^\infty_T(C^\infty_c), \|g\|_{\mathbb{L}^r_\theta(T)}\leqslant 1} \int_0^T\int_{\mathbb{R}^d}\left(\int_0^t \mathcal{T}_{s,t}f(s,x)\mathrm{d}s\right)\nabla^2 g(t,x)\mathrm{d}x\mathrm{d}t\\
&= \sup_{g\in L^\infty_T(C^\infty_c), \|g\|_{\mathbb{L}^r_\theta(T)}\leqslant 1} \int_0^T\int_0^t\left(\int_{\mathbb{R}^d} \mathcal{T}_{s,t}f(s,x)\nabla^2 g(t,x)\mathrm{d}x\right)\mathrm{d}s\mathrm{d}t\\
&\overset{(3.5)}{=} \sup_{g\in L^\infty_T(C^\infty_c), \|g\|_{\mathbb{L}^r_\theta(T)}\leqslant 1} \int_0^T\int_0^t\left(\int_{\mathbb{R}^d} f(s,x)\mathcal{T}^*_{s,t}\nabla^2 g(t,x)\mathrm{d}x\right)\mathrm{d}s\mathrm{d}t\\
&= \sup_{g\in L^\infty_T(C^\infty_c), \|g\|_{\mathbb{L}^r_\theta(T)}\leqslant 1} \int_0^T\int_{\mathbb{R}^d} f(s,x)\left(\int_s^T \mathcal{T}^*_{s,t}\nabla^2 g(t,x)\mathrm{d}t\right)\mathrm{d}x\mathrm{d}s\\
&\leqslant C \sup_{g\in L^\infty_T(C^\infty_c), \|g\|_{\mathbb{L}^r_\theta(T)}\leqslant 1} \|f\|_{\mathbb{L}^p_q(T)}\|\nabla^2 g\|_{\mathbb{H}^{-2,r}_\theta(T)} \leqslant C\|f\|_{\mathbb{L}^p_q(T)},
\end{aligned}$$

where the first inequality is due to (3.10) for $p = r < \theta = q$.

(ii) We only prove (3.10) and (3.12) for $p \leqslant q$ since (3.9) and (3.11) are similar. By Marcinkiewicz's interpolation theorem (see [14]), it suffices to prove that for any $p > 1$ and $n \in \mathbb{N}$,

$$\left\|\nabla^2 w\right\|_{\mathbb{H}^{-2,p}_{np}(T)} \leqslant C\|f\|_{\mathbb{H}^{-2,p}_{np}(T)}. \tag{3.13}$$

Below we fix $p > 1$ and $n \in \mathbb{N}$, and use the freezing coefficient argument to prove (3.13). Let ζ be a nonnegative smooth function with support in the ball B_δ and $\int_{\mathbb{R}^d} \zeta^p \mathrm{d}x = 1$, where $\delta > 0$ is a small constant and will be determined below. For $z \in \mathbb{R}^d$, define

$$\zeta_z(x) := \zeta(x - z), \quad a_z(s) := a(s, z)$$

and

$$w_z(s, x) := w(s, x)\zeta_z(x), \quad f_z(s, x) := f(s, x)\zeta_z(x).$$

It is easy to see that

$$\partial_s w_z + \partial_{ij}(a^{ij}_z w_z) - \lambda w_z + g_z = 0, \quad w_z(T) = 0, \tag{3.14}$$

where

$$g_z := f_z + \partial_{ij}(a^{ij} w)\zeta_z - \partial_{ij}(a^{ij}_z w\zeta_z).$$

Moreover, by Fubini's theorem and $\int_{\mathbb{R}^d} \zeta^p = 1$, we have

$$\int_{\mathbb{R}^d} \|w_z(s)\|_p^p \mathrm{d}z = \int_{\mathbb{R}^d} \|w(s)\zeta_z\|_p^p \mathrm{d}z = \|w(s)\|_p^p. \tag{3.15}$$

Below we drop the time variable for simplicity. Noticing that

$$g_z = f\zeta_z - 2\partial_j(a^{ij} w)\partial_i\zeta_z - a^{ij} w\partial_{ij}\zeta_z + \partial_{ij}((a^{ij} - a^{ij}_z)w\zeta_z),$$

and by Lemma 2.2 with $\phi_z = \zeta_z, \partial_i\zeta_z, \partial_{ij}\zeta_z$ respectively, we have

$$\begin{aligned}\left(\int_{\mathbb{R}^d} \|g_z\|^p_{-2,p}\mathrm{d}z\right)^{1/p} &\leqslant C\|f\|_{-2,p} + C_\delta \sum_{i,j} \|\partial_j(a^{ij} w)\|_{-2,p} \\ &\quad + C_\delta \sum_{i,j} \|a^{ij} w\|_{-2,p} + \omega_a(\delta)\|w\|_p,\end{aligned} \tag{3.16}$$

where

$$\omega_a(\delta) := \sup_{t \geqslant 0} \sup_{|x-y| \leqslant \delta} |a(t, x) - a(t, y)|.$$

Let $a_n(t, x) := a(t, \cdot) * \rho_n(x)$ be the mollifying approximation of a. For every $\varepsilon > 0$, we can take n large enough such that

$$\begin{aligned}&\sum_{i,j} \|\partial_j(a^{ij} w)\|_{-2,p} + \sum_{i,j} \|a^{ij} w\|_{-2,p} \\ &\quad\lesssim \|aw\|_{-1,p} \leqslant \|a_n w\|_{-1,p} + \|(a_n - a)w\|_{-1,p} \\ &\quad\lesssim \|a_n\|_{2,\infty}\|w\|_{-1,p} + \|(a_n - a)w\|_p \\ &\quad\leqslant C_n\|w\|_{-1,p} + \omega_a(\tfrac{1}{n})\|w\|_p \\ &\quad\lesssim \|w\|_{-2,p} + \varepsilon\|w\|_p,\end{aligned}$$

where the last step is due to the interpolation and Young's inequalities. Hence, by (3.16), for any $\varepsilon\in(0,1)$ and $\delta>0$ being small enough,

$$\left(\int_{\mathbb{R}^d}\|g_z\|^p_{-2,p}\mathrm{d}z\right)^{1/p}\lesssim\|f\|_{-2,p}+\|w\|_{-2,p}+\varepsilon\|w\|_p. \tag{3.17}$$

(iii) For any $s\in[0,T]$, notice that by Lemma 2.2 again,

$$\begin{aligned}
\|\nabla^2 w\|^{np}_{\mathbb{H}^{-2,p}_{np}(s,T)}&\lesssim\int_s^T\left(\int_{\mathbb{R}^d}\|\nabla^2 w(t)\zeta_z\|^p_{-2,p}\mathrm{d}z\right)^n\mathrm{d}t\\
&\lesssim\int_s^T\left(\int_{\mathbb{R}^d}\|\nabla^2(w(t)\zeta_z)\|^p_{-2,p}\mathrm{d}z\right)^n\mathrm{d}t\\
&\quad+\int_s^T\left(\int_{\mathbb{R}^d}\|\nabla w(t)\cdot\nabla\zeta_z\|^p_{-2,p}\mathrm{d}z\right)^n\mathrm{d}t\\
&\quad+\int_s^T\left(\int_{\mathbb{R}^d}\|w(t)\cdot\nabla^2\zeta_z\|^p_{-2,p}\mathrm{d}z\right)^n\mathrm{d}t\\
&\lesssim\int_s^T\left(\int_{\mathbb{R}^d}\|\nabla^2 w_z(t)\|^p_{-2,p}\mathrm{d}z\right)^n\mathrm{d}t\\
&\quad+\int_s^T\|\nabla w(t)\|^{np}_{-2,p}\mathrm{d}t+\int_s^T\|w(t)\|^{np}_{-2,p}\mathrm{d}t\\
&\lesssim\int_s^T\!\!\int_{\mathbb{R}^{nd}}\prod_{k=1}^n\|\nabla^2 w_{z_k}(t)\|^p_{-2,p}\mathrm{d}z_1\cdots\mathrm{d}z_n\mathrm{d}t\\
&\quad+\int_s^T\|w(t)\|^{np}_{-1,p}\mathrm{d}t.
\end{aligned}\tag{3.18}$$

Given $z_1,\cdots,z_n\in\mathbb{R}^d$ and by Lemma 2.3, we have

$$\int_s^T\prod_{k=1}^n\|\nabla^2 w_{z_k}(t)\|^p_{-2,p}\mathrm{d}t\leqslant N\sum_{k=1}^n\int_s^T\|g_{z_k}(t)\|^p_{-2,p}\prod_{\ell\neq k}\|\nabla^2 w_{z_\ell}(t)\|^p_{-2,p}\mathrm{d}t,$$

which together with (3.18) and (3.17) yields that for any $\varepsilon\in(0,1)$,

$$\begin{aligned}
\|\nabla^2 w\|^{np}_{\mathbb{H}^{-2,p}_{np}(s,T)}&\lesssim\sum_{k=1}^n\int_s^T\!\!\int_{\mathbb{R}^{nd}}\|g_{z_k}(t)\|^p_{-2,p}\prod_{\ell\neq k}\|w_{z_\ell}(t)\|^p_p\mathrm{d}z_1\cdots\mathrm{d}z_n\mathrm{d}t+\|w\|^{np}_{\mathbb{H}^{-1,p}_{np}(s,T)}\\
&=n\int_s^T\!\left(\int_{\mathbb{R}^d}\|g_z(t)\|^p_{-2,p}\mathrm{d}z\right)\!\left(\int_{\mathbb{R}^d}\|w_z(t)\|^p_p\mathrm{d}z\right)^{n-1}\mathrm{d}t+\|w\|^{np}_{\mathbb{H}^{-1,p}_{np}(s,T)}\\
&\overset{(3.15)}{=}n\int_s^T\left(\int_{\mathbb{R}^d}\|g_z(t)\|^p_{-2,p}\mathrm{d}z\right)\|w(t)\|_p^{(n-1)p}\mathrm{d}t+\|w\|^{np}_{\mathbb{H}^{-1,p}_{np}(s,T)}\\
&\overset{(3.17)}{\lesssim}\|f\|^{np}_{\mathbb{H}^{-2,p}_{np}(s,T)}+\|w\|^{np}_{\mathbb{H}^{-2,p}_{np}(s,T)}+\varepsilon\|\nabla^2 w\|^{np}_{\mathbb{H}^{-2,p}_{np}(s,T)},
\end{aligned}$$

where the last step is due to Hölder's inequality and interpolation's inequality. Taking $\varepsilon=1/2$, we get for any $s\in[0,T]$,

$$\|\nabla^2 w\|^{np}_{\mathbb{H}^{-2,p}_{np}(s,T)}\lesssim\|f\|^{np}_{\mathbb{H}^{-2,p}_{np}(s,T)}+\|w\|^{np}_{\mathbb{H}^{-2,p}_{np}(s,T)}. \tag{3.19}$$

(iv) Let $A^z_{s,t} := \int_s^t a_z(r)\mathrm{d}r$ and

$$P^z_{s,t}f(x) := \frac{1}{(2\pi)^{d/2}\det(A^z_{s,t})^{1/2}}\int_{\mathbb{R}^d} \mathrm{e}^{-\langle (A^z_{s,t})^{-1}y,y\rangle/2} f(x-y)\mathrm{d}y.$$

Notice that the solution of equation (3.14) is explicitly given by

$$w_z(s,x) = \int_s^T \mathrm{e}^{\lambda(s-t)} P^z_{s,t} g_z(t,x)\mathrm{d}t.$$

By (1.8) and a standard interpolation technique, one sees that for any $\alpha\in[0,2)$, there is a constant $C = C(\alpha,d,p,c_0) > 0$ such that for all $z\in\mathbb{R}^d$,

$$\|w_z(s)\|_{\alpha-2,p} \leqslant C\int_s^T \frac{\mathrm{e}^{\lambda(s-t)}}{(t-s)^{\alpha/2}}\|g_z(t)\|_{-2,p}\mathrm{d}t.$$

Thus, for any $\alpha\in[0,2)$, by (2.9) and Minkowski's inequality we have

$$\begin{aligned}\|w(s)\|_{\alpha-2,p} &\lesssim \left(\int_{\mathbb{R}^d}\|w_z(s)\|^p_{\alpha-2,p}\mathrm{d}z\right)^{\frac1p} \leqslant \int_s^T \frac{\mathrm{e}^{\lambda(s-t)}}{(t-s)^{\alpha/2}}\left(\int_{\mathbb{R}^d}\|g_z(t)\|^p_{-2,p}\mathrm{d}z\right)^{\frac1p}\mathrm{d}t\\ &\overset{(3.17)}{\leqslant} \int_s^T \frac{\mathrm{e}^{\lambda(s-t)}}{(t-s)^{\alpha/2}}\Big(\|f(t)\|_{-2,p}+\|w(t)\|_{-2,p}+\|\nabla^2 w(t)\|_{-2,p}\Big)\mathrm{d}t. \quad (3.20)\end{aligned}$$

Now by (3.20) with $\alpha=0$ and (3.19) with $n=1$, we have

$$\begin{aligned}\|w(s)\|^p_{-2,p} &\lesssim \int_s^T \Big(\|f(t)\|^p_{-2,p}+\|w(t)\|^p_{-2,p}+\|\nabla^2 w\|^p_{-2,p}\Big)\mathrm{d}t\\ &\lesssim \int_s^T \Big(\|f(t)\|^p_{-2,p}+\|w(t)\|^p_{-2,p}\Big)\mathrm{d}t.\end{aligned}$$

which by Gronwall's inequality yields

$$\|w\|^p_{\mathbb{H}^{-2,p}_\infty(T)} = \sup_{s\in[0,T]}\|w(s)\|^p_{-2,p} \lesssim \|f\|^p_{\mathbb{H}^{-2,p}_p(T)} \lesssim \|f\|^p_{\mathbb{H}^{-2,p}_{np}(T)}.$$

Substituting this into (3.19) with $s=0$ and noting $\|w\|_{\mathbb{H}^{-2,p}_{np}(T)} \lesssim \|w\|_{\mathbb{H}^{-2,p}_\infty(T)}$, we obtain (3.13).

(v) Finally, letting $q' = \frac{q}{q-1}$, for any $\alpha\in[0,2-\frac2q)$, by (3.20) and Hölder's inequality, we have

$$\begin{aligned}\|w(s)\|^q_{\alpha-2,p} &\lesssim \left(\int_s^T \frac{\mathrm{e}^{q'\lambda(s-t)}}{(t-s)^{\frac{q'\alpha}{2}}}\mathrm{d}t\right)^{\frac{q}{q'}}\int_s^T\Big(\|f(t)\|_{-2,p}+\|w(t)\|_{-2,p}+\|w(t)\|_p\Big)^q\mathrm{d}t\\ &\lesssim (1\vee\lambda)^{(\frac{\alpha}{2}-1+\frac1q)q}\int_s^T\Big(\|f(t)\|^q_{-2,p}+\|w(t)\|^q_{-2,p}+\|\nabla^2 w(t)\|^q_{-2,p}\Big)\mathrm{d}t\\ &\overset{(3.9)}{\lesssim} (1\vee\lambda)^{(\frac{\alpha}{2}-1+\frac1q)q}\left(\|f\|^q_{\mathbb{H}^{-2,p}_q(T)}+\int_s^T\|w(t)\|^q_{-2,p}\mathrm{d}t\right), \quad (3.21)\end{aligned}$$

which yields by choosing $\alpha=0$ and Gronwall's inequality that

$$\|w\|^q_{\mathbb{H}^{-2,p}_\infty(T)} = \sup_{s\in[0,T]}\|w(s)\|^q_{-2,p} \lesssim \|f\|^q_{\mathbb{H}^{-2,p}_q(T)}.$$

The proof is complete by substituting this into (3.21). $\square$

3.2. **Proof of Theorem 3.1.** By standard continuity method (cf. [7]), it suffices to establish the a priori estimate (3.2). We divide the proof into three steps.

(i) (Case $b \equiv 0$) Fix $T > 0$ and $p, q \in (1, \infty)$. Let $u \in \mathbb{H}^{2,p}_q(T)$ and $f \in \mathbb{L}^p_q(T)$ satisfy (3.1). Let ρ_n be a family of mollifiers in $\mathbb{R}^d$. Define

$$u_n(t,x) := u(t,\cdot) * \rho_n(x), \quad a_n(t,x) := a(t,\cdot) * \rho_n(x), \quad f_n(t,x) := f(t,\cdot) * \rho_n(x).$$

It is easy to see that u_n satisfies

$$\partial_t u_n = a^{ij}_n \partial_{ij} u_n - \lambda u_n + g_n, \quad u_n(0) = 0,$$

where

$$g_n := f_n + (a^{ij}\partial_{ij}u) * \rho_n - a^{ij}_n \partial_{ij} u_n.$$

Since a_n satisfies $(\mathbf{H}^a)$ uniformly in n and $g_n \in \mathbb{L}^q_T(H^{\infty,p})$, for any $\alpha \in [0, 2 - \frac{2}{q})$, by (3.7), (3.9) and (3.11), there is a $C > 0$ such that for each $n \in \mathbb{N}$ and $\lambda \geqslant 1$,

$$\begin{aligned}&\lambda^{1-\frac{\alpha}{2}-\frac{1}{q}} \|u_n\|_{\mathbb{H}^{\alpha,p}_\infty(T)} + \|\partial_t u_n\|_{\mathbb{L}^p_q(T)} + \|\nabla^2 u_n\|_{\mathbb{L}^p_q(T)}\\ &\qquad \leqslant C\Big(\|f_n\|_{\mathbb{L}^p_q(T)} + \|(a^{ij}\partial_{ij}u) * \rho_n - a^{ij}_n \partial_{ij} u_n\|_{\mathbb{L}^p_q(T)}\Big).\end{aligned}$$

Letting $n \to \infty$ and by the property of convolutions, we obtain

$$\lambda^{1-\frac{\alpha}{2}-\frac{1}{q}} \|u\|_{\mathbb{H}^{\alpha,p}_\infty(T)} + \|\partial_t u\|_{\mathbb{L}^p_q(T)} + \|\nabla^2 u\|_{\mathbb{L}^p_q(T)} \leqslant C\|f\|_{\mathbb{L}^p_q(T)}. \tag{3.22}$$

Next, let χ^z_r be defined by (2.1). Multiplying both sides of (3.1) by χ^z_r, we have

$$\partial_t(u\chi^z_r) = a^{ij}\partial_{ij}(u\chi^z_r) - \lambda u\chi^z_r + g^z_r,$$

where

$$g^z_r := f\chi^z_r + \chi^z_r a^{ij}\partial_{ij}u - a^{ij}\partial_{ij}(u\chi^z_r).$$

For any $\alpha \in [0, 2 - \frac{2}{q})$, by (3.22) we have

$$\lambda^{1-\frac{\alpha}{2}-\frac{1}{q}} \|u\chi^z_r\|_{\mathbb{H}^{\alpha,p}_\infty(T)} + \|\partial_t u\chi^z_r\|_{\mathbb{L}^p_q(T)} + \|\nabla^2(u\chi^z_r)\|_{\mathbb{L}^p_q(T)} \lesssim \|g^z_r\|_{\mathbb{L}^p_q(T)}.$$

Noticing that

$$a^{ij}\partial_{ij}(u\chi^z_r) - \chi^z_r a^{ij}\partial_{ij}u = a^{ij}u\partial_{ij}\chi^z_r + 2a^{ij}\partial_i u\partial_j\chi^z_r,$$

we have

$$\|g^z_r\|_{\mathbb{L}^p_q(T)} \lesssim \|f\chi^z_r\|_{\mathbb{L}^p_q(T)} + \|u\chi^z_{2r}\|_{\mathbb{L}^p_q(T)} + \|\nabla u \cdot \chi^z_{2r}\|_{\mathbb{L}^p_q(T)}.$$

Hence, for any $\alpha \in [0, 2 - \frac{2}{q})$ and $\varepsilon \in (0, 1)$, by taking supremum in $z \in \mathbb{R}^d$ and using (2.4), we obtain that for all $\lambda \geqslant 1$,

$$\begin{aligned}&\lambda^{1-\frac{\alpha}{2}-\frac{1}{q}} |||u|||_{\widetilde{\mathbb{H}}^{\alpha,p}_\infty(T)} + |||\partial_t u|||_{\widetilde{\mathbb{L}}^p_q(T)} + |||u|||_{\widetilde{\mathbb{H}}^{2,p}_q(T)}\\ &\quad \lesssim |||f|||_{\widetilde{\mathbb{L}}^p_q(T)} + |||u|||_{\widetilde{\mathbb{L}}^p_q(T)} + |||u|||_{\widetilde{\mathbb{H}}^{1,p}_q(T)} \lesssim |||f|||_{\widetilde{\mathbb{L}}^p_q(T)} + |||u|||_{\widetilde{\mathbb{L}}^p_q(T)} + \varepsilon|||u|||_{\widetilde{\mathbb{H}}^{2,p}_q(T)},\end{aligned}$$

which implies by taking $\varepsilon = 1/2$ that

$$\lambda^{1-\frac{\alpha}{2}-\frac{1}{q}} |||u|||_{\widetilde{\mathbb{H}}^{\alpha,p}_\infty(T)} + |||\partial_t u|||_{\widetilde{\mathbb{L}}^p_q(T)} + |||u|||_{\widetilde{\mathbb{H}}^{2,p}_q(T)} \lesssim |||f|||_{\widetilde{\mathbb{L}}^p_q(T)} + |||u|||_{\widetilde{\mathbb{L}}^p_q(T)}.$$

In particular, for $\alpha = 0$, we have

$$|||u(T)|||_p \lesssim |||f|||_{\widetilde{\mathbb{L}}^p_q(T)} + \left(\int_0^T |||u(s)|||^q_p \mathrm{d}s\right)^{1/q}.$$

By Gronwall's inequality again, we obtain

$$\|u\|_{\widetilde{\mathbb{L}}^p_\infty(T)} \leqslant C\|f\|_{\widetilde{\mathbb{L}}^p_q(T)},$$

and so, for any $\alpha \in [0, 2 - \frac{2}{q})$,

$$\lambda^{1-\frac{\alpha}{2}-\frac{1}{q}}\|u\|_{\widetilde{\mathbb{H}}^{\alpha,p}_\infty(T)} + \|\partial_t u\|_{\widetilde{\mathbb{L}}^p_q(T)} + \|u\|_{\widetilde{\mathbb{H}}^{2,p}_q(T)} \lesssim \|f\|_{\widetilde{\mathbb{L}}^p_q(T)}. \tag{3.23}$$

(ii) ($b \neq 0$: subcritical case) Let $q_1 \in (\frac{2p}{p-d}, q]$ and $\lambda \geqslant 1$. For any $\alpha \in [0, 2 - \frac{2}{q_1})$, by (3.23), we have

$$\begin{aligned}\lambda^{1-\frac{\alpha}{2}-\frac{1}{q_1}}\|u\|_{\widetilde{\mathbb{H}}^{\alpha,p}_\infty(T)} &+ \|\partial_t u\|_{\widetilde{\mathbb{L}}^p_q(T)} + \|u\|_{\widetilde{\mathbb{H}}^{2,p}_{q_1}(T)} \\ &\lesssim \|f + b^i\partial_i u\|_{\widetilde{\mathbb{L}}^p_{q_1}(T)} \leqslant \|f\|_{\widetilde{\mathbb{L}}^p_{q_1}(T)} + \|b^i\partial_i u\|_{\widetilde{\mathbb{L}}^p_{q_1}(T)}. \end{aligned} \tag{3.24}$$

Let $\frac{1}{q_2} + \frac{1}{q} = \frac{1}{q_1}$. For any $\theta \in (\frac{d}{p}, 1 - \frac{2}{q_1})$, by Hölder's inequality and Sobolev's embedding (2.5), we have

$$\|b^i\partial_i u\|_{\widetilde{\mathbb{L}}^p_{q_1}(T)} \leqslant \|b\|_{\widetilde{\mathbb{L}}^p_q(T)}\|u\|_{\widetilde{\mathbb{H}}^{1,\infty}_{q_2}(T)} \lesssim \|u\|_{\widetilde{\mathbb{H}}^{1+\theta,p}_{q_2}(T)}. \tag{3.25}$$

Substituting this into (3.24) with $\alpha = 1 + \theta$, we get

$$\lambda^{\frac{1}{2}-\frac{\theta}{2}-\frac{1}{q_1}}\|u\|_{\widetilde{\mathbb{H}}^{1+\theta,p}_\infty(T)} \leqslant C\|f\|_{\widetilde{\mathbb{L}}^p_{q_1}(T)} + \|u\|_{\widetilde{\mathbb{H}}^{1+\theta,p}_{q_2}(T)}.$$

In particular, if $q_1 < q$, then $q_2 < \infty$ and by Gronwall's inequality again, we obtain

$$\|u\|_{\widetilde{\mathbb{H}}^{1+\theta,p}_\infty(T)} \leqslant C\|f\|_{\widetilde{\mathbb{L}}^p_{q_1}(T)} \leqslant C\|f\|_{\widetilde{\mathbb{L}}^p_q(T)}. \tag{3.26}$$

The desired estimate now follows by (3.24), (3.25) with $q_1 = q$ and (3.26).

(iii) ($b \neq 0$: critical case) Let $b_n(t,x) := b(t,\cdot) * \rho_{1/n}(x)$. Since $b \in \widetilde{\mathbb{L}}^{d;\mathrm{uni}}_\infty$, by definition (2.3) we have

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|b_n(t) - b(t)\|_d = 0.$$

Let $p < d$ and $q \in (1, \infty)$. For any $\varepsilon \in (0,1)$, by Sobolev's embedding (2.5) and letting n be large enough so that $\sup_{t\in[0,T]}\|b_n(t) - b(t)\|_d \leqslant \varepsilon$, we have

$$\begin{aligned}\|b^i\partial_i u\|_{\widetilde{\mathbb{L}}^p_q(T)} &\leqslant \|(b^i_n - b^i)\partial_i u\|_{\widetilde{\mathbb{L}}^p_q(T)} + \|b^i_n\partial_i u\|_{\widetilde{\mathbb{L}}^p_q(T)} \\ &\leqslant \sup_{t\in[0,T]}\|b_n(t) - b(t)\|_d\|\nabla u\|_{\widetilde{\mathbb{L}}^{pd/(d-p)}_q(T)} + \|b_n\|_\infty\|u\|_{\widetilde{\mathbb{H}}^{1,p}_q(T)} \\ &\leqslant \varepsilon\|u\|_{\widetilde{\mathbb{H}}^{2,p}_q(T)} + C\|b_n\|_\infty\|u\|^{1/2}_{\widetilde{\mathbb{L}}^p_q(T)}\|u\|^{1/2}_{\widetilde{\mathbb{H}}^{2,p}_q(T)} \\ &\leqslant 2\varepsilon\|u\|_{\widetilde{\mathbb{H}}^{2,p}_q(T)} + C\|b_n\|^2_\infty\|u\|_{\widetilde{\mathbb{L}}^p_q(T)}.\end{aligned}$$

Hence, for any $\alpha \in [0, 2 - \frac{2}{q})$, by (3.24) with $q_1 = q$, we have

$$\lambda^{1-\frac{\alpha}{2}-\frac{1}{q}}\|u\|_{\widetilde{\mathbb{H}}^{\alpha,p}_\infty(T)} + \|\partial_t u\|_{\widetilde{\mathbb{L}}^p_q(T)} + \|u\|_{\widetilde{\mathbb{H}}^{2,p}_q(T)} \lesssim \|f\|_{\widetilde{\mathbb{L}}^p_q(T)} + \varepsilon\|u\|_{\widetilde{\mathbb{H}}^{2,p}_q(T)} + \|u\|_{\widetilde{\mathbb{L}}^p_q(T)},$$

which implies by taking $\varepsilon = 1/2$,

$$\lambda^{1-\frac{\alpha}{2}-\frac{1}{q}}\|u\|_{\widetilde{\mathbb{H}}^{\alpha,p}_\infty(T)} + \|u\|_{\widetilde{\mathbb{H}}^{2,p}_q(T)} \lesssim \|f\|_{\widetilde{\mathbb{L}}^p_q(T)} + \|u\|_{\widetilde{\mathbb{L}}^p_q(T)}.$$

As above, by Gronwall's inequality, we obtain the desired estimate.

4. Subcritical case: Proof of Theorem 1.1

In this section we assume $(\mathbf{H}^\sigma)$ holds and for some $p_i, q_i \in [2,\infty)$ with $\frac{d}{p_i}+\frac{2}{q_i}<1$, $i=1,2$,

$$\nabla\sigma \in \widetilde{\mathbb{L}}^{p_1}_{q_1}, \quad b \in \widetilde{\mathbb{L}}^{p_2}_{q_2}.$$

It is easy to see that $(\mathbf{H}^a)$ holds for

$$a^{ij} := \sigma^{ik}\sigma^{jk}/2.$$

We prepare the following crucial lemma for latter use.

Lemma 4.1. *Let $X_t(x)$ be a solution of SDE (1.2) and $p,q\in(1,\infty)$ with $\frac{d}{p}+\frac{2}{q}<2$.*

(i) (Krylov's estimate) For any $T>0$, there is a constant $C>0$ such that for any $f\in\widetilde{\mathbb{L}}^p_q(T)$ and $x\in\mathbb{R}^d$, $0\leqslant t_0<t_1\leqslant T$,

$$\mathbf{E}\left(\int_{t_0}^{t_1} f(s,X_s(x))\mathrm{d}s\Big|\mathscr{F}_{t_0}\right) \leqslant C|\!|\!|f|\!|\!|_{\widetilde{\mathbb{L}}^p_q(t_0,t_1)}. \tag{4.1}$$

(ii) (Khasminskii's estimate) For any $\gamma\in\mathbb{R}$ and $f\in\widetilde{\mathbb{L}}^p_q(T)$, we have

$$\mathbf{E}\exp\left(\gamma\int_0^T |f(s,X_s)|\mathrm{d}s\right)<\infty. \tag{4.2}$$

(iii) (Generalized Itô's formula) Let $p',q'\in[2,\infty)$ with $\frac{d}{p'}+\frac{2}{q'}<1$. For any $u\in\widetilde{\mathbb{H}}^{2,p'}_{q'}(T)$ with $\partial_t u\in\widetilde{\mathbb{L}}^{p'}_{q'}(T)$, we have

$$\begin{aligned} u(t,X_t) = u(0,x) &+ \int_0^t (\partial_s u + a^{ij}\partial_i\partial_j u + b^i\partial_i u)(s,X_s)\mathrm{d}s \\ &+ \int_0^t (\sigma^{ij}\partial_i u)(s,X_s)\mathrm{d}W^j_s. \end{aligned} \tag{4.3}$$

Proof. (i) By (3.2) and using completely the same argument as in [19, Theorem 5.7], we can prove the Krylov estimate (4.1).

(ii) Since $\frac{d}{p}+\frac{2}{q}<2$, we can choose $q'<q$ so that $\frac{d}{p}+\frac{2}{q'}<2$. Thus by (4.1) and Hölder's inequality we have

$$\mathbf{E}\left(\int_{t_0}^{t_1} f(s,X_s(x))\mathrm{d}s\Big|\mathscr{F}_{t_0}\right) \leqslant C|\!|\!|f|\!|\!|_{\widetilde{\mathbb{L}}^p_{q'}(t_0,t_1)} \leqslant C(t_1-t_0)^{1-\frac{q'}{q}}|\!|\!|f|\!|\!|_{\widetilde{\mathbb{L}}^p_q(T)},$$

which implies (4.2) by [19, Lemma 3.5].

(iii) Let $u_n=(u*\rho_n)(t,x)$ be the mollifying approximation. By Itô's formula we have

$$\begin{aligned} u_n(t,X_t) = u_n(0,X_0) &+ \int_0^t (\partial_s u_n + a^{ij}\partial_{ij}u_n + b^i\partial_i u_n)(s,X_s)\mathrm{d}s \\ &+ \int_0^t (\sigma^{ij}\partial_i u_n)(s,X_s)\mathrm{d}W^j_s. \end{aligned} \tag{4.4}$$

For $R>0$, define a stopping time

$$\tau_R := \inf\{t\geqslant 0: |X_t|\geqslant R\}.$$

Let χ_R be defined by (2.1). By Itô's isometric formula, we have

$$\begin{aligned}
&\mathbf{E}\left|\int_0^{t\wedge\tau_R}(\sigma^{ij}\partial_i(u_n-u))(s,X_s)\mathrm{d}W^j_s\right|^2\\
&\quad\leqslant \|\sigma\|_\infty^2\mathbf{E}\left(\int_0^{t\wedge\tau_R}|\nabla(u_n-u)|^2(s,X_s)\mathrm{d}s\right)\\
&\quad\lesssim \mathbf{E}\left(\int_0^t\chi_R^2(X_s)\cdot|\nabla(u_n-u)|^2(s,X_s)\mathrm{d}s\right)\\
&\quad\overset{(4.1)}{\lesssim} \|\chi_R^2|\nabla(u_n-u)|^2\|_{\widetilde{\mathbb{L}}^{p'/2}_{q'/2}(T)}=\|\chi_R\nabla(u_n-u)\|^2_{\widetilde{\mathbb{L}}^{p'}_{q'}(T)},
\end{aligned}$$

which converges to zero by (2.6) as $n\to\infty$. Similarly, let $\frac{1}{p}:=\frac{1}{p_2}+\frac{1}{p'}$, $\frac{1}{q}:=\frac{1}{q_2}+\frac{1}{q'}$. Since $\frac{d}{p}+\frac{2}{q}<2$, by (4.1) and Hölder's inequality we have

$$\begin{aligned}
&\mathbf{E}\left(\int_0^{t\wedge\tau_R}|b^i\partial_i(u_n-u)|(s,X_s)\mathrm{d}s\right)\leqslant\mathbf{E}\left(\int_0^t\chi_R(X_s)\cdot|b^i\partial_i(u_n-u)|(s,X_s)\mathrm{d}s\right)\\
&\quad\lesssim\|\chi_Rb^i\partial_i(u_n-u)\|_{\widetilde{\mathbb{L}}^p_q(T)}\leqslant\|b\|_{\widetilde{\mathbb{L}}^{p_2}_{q_2}(T)}\|\chi_{2R}\nabla(u_n-u)\|_{\widetilde{\mathbb{L}}^{p'}_{q'}(T)}\overset{n\to\infty}{\to}0,
\end{aligned}$$

and

$$\lim_{n\to\infty}\mathbf{E}\left(\int_0^{t\wedge\tau_R}|(\partial_s+a^{ij}\partial_i\partial_j)(u_n-u)|(s,X_s)\mathrm{d}s\right)=0.$$

By taking limits $n\to\infty$ for both sides of (4.4), we get on $\{t\leqslant\tau_R\}$,

$$u(t,X_t)=u(0,X_0)+\int_0^t(\partial_su+a^{ij}\partial_i\partial_ju+b^i\partial_iu)(s,X_s)\mathrm{d}s+\int_0^t(\sigma^{ij}\partial_iu)(s,X_s)\mathrm{d}W^j_s.$$

Finally, letting $R\to\infty$, we obtain the desired formula. $\square$

Below, we fix a $T>0$. Consider the following backward PDE:

$$\partial_tu+a^{ij}\partial_i\partial_ju-\lambda u+b^i\partial_iu+b=0,\quad u(T)=0.$$

By Theorem 3.1, there is a unique solution $u\in\widetilde{\mathbb{H}}^{2,p_2}_{q_2}(T)$ such that for any $\alpha\in[0,2-\frac{2}{q_2})$ and $\lambda\geqslant1$,

$$\lambda^{1-\frac{\alpha}{2}-\frac{1}{q_2}}\|u\|_{\widetilde{\mathbb{H}}^{\alpha,p_2}_\infty(T)}+\|\partial_tu\|_{\widetilde{\mathbb{L}}^{p_2}_{q_2}(T)}+\|u\|_{\widetilde{\mathbb{H}}^{2,p_2}_{q_2}(T)}\leqslant C\|b\|_{\widetilde{\mathbb{L}}^{p_2}_{q_2}(T)}.$$

In particular, since $\frac{d}{p_2}+\frac{2}{q_2}<1$, by (2.5) one can choose λ large enough so that

$$\|u\|_\infty+\|\nabla u\|_\infty\leqslant\tfrac{1}{2}.\tag{4.5}$$

Define

$$\Phi(t,x):=x+u(t,x).$$

By (4.5), one sees that $x\mapsto\Phi(t,x)$ is a C^1-diffeomorphism and

$$\|\nabla\Phi\|_\infty,\quad\|\nabla\Phi^{-1}\|_\infty\leqslant2.$$

Moreover, we also have

$$\partial_t\Phi+a^{ij}\partial_i\partial_j\Phi+b^i\partial_i\Phi=\lambda u.$$

Define

$$\widetilde{\sigma}(t,y):=(\sigma^{ij}\partial_i\Phi)(t,\Phi^{-1}(t,y))$$

and

$$\widetilde{b}(t,y) := \lambda u(t, \Phi^{-1}(t,y)).$$

By the generalized Itô formula (4.3), we have the following Zvonkin's transformation (see [19, Theorem 3.10]).

Lemma 4.2. *X_t solves SDE (1.2) if and only if $Y_t = \Phi(t, X_t)$ solves the following SDE:*

$$Y_t = y + \int_0^t \widetilde{b}(s, Y_s)\mathrm{d}s + \int_0^t \widetilde{\sigma}(s, Y_s)\mathrm{d}W_s \quad \textit{with} \quad y := \Phi(0,x). \tag{4.6}$$

Now we can use the above lemma to prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 4.2, it suffices to show the conclusions for SDE (4.6). Since the coefficients of SDE (4.6) are bounded and continuous, the existence of a solution Y_t is well known. By Yamada-Watanabe's theorem, we only need to prove the pathwise uniqueness for (4.6) and show (i)-(iii) for Y.

(i) is proven in Lemma 4.1.

(ii) For $i = 1, 2$, let $Y_t^{(i)}$ be two solutions of SDE (4.6) with starting point y_i, that is,

$$Y_t^{(i)} = y_i + \int_0^t \widetilde{b}(s, Y_s^{(i)})\mathrm{d}s + \int_0^t \widetilde{\sigma}(s, Y_s^{(i)})\mathrm{d}W_s.$$

For $p \geqslant 1$, by Itô's formula we have

$$|Y_t^{(1)} - Y_t^{(2)}|^{2p} = |y_1 - y_2|^{2p} + \int_0^t |Y_s^{(1)} - Y_s^{(2)}|^{2p}\mathrm{d}A_s + M_t, \tag{4.7}$$

where M_t is a continuous local martingale given by

$$M_t := \int_0^t 2p|Z_s|^{2p-2}\big[\widetilde{\sigma}(s, Y_s^{(1)}) - \widetilde{\sigma}(s, Y_s^{(2)})\big]^*(Y_s^{(1)} - Y_s^{(2)})\mathrm{d}W_s,$$

where the asterisk stands for the transpose of a matrix, and A_t is defined by

$$\begin{aligned} A_t := &\int_0^t \frac{2p\langle Y_s^{(1)} - Y_s^{(2)}, \widetilde{b}(s, Y_s^{(1)}) - \widetilde{b}(s, Y_s^{(2)})\rangle + p\|\widetilde{\sigma}(s, Y_s^{(1)}) - \widetilde{\sigma}(s, Y_s^{(2)})\|^2}{|Y_s^{(1)} - Y_s^{(2)}|^2}\mathrm{d}s \\ &+ \int_0^t \frac{2p(p-1)|[\widetilde{\sigma}(s, Y_s^{(1)}) - \widetilde{\sigma}(s, Y_s^{(2)})]^*(Y_s^{(1)} - Y_s^{(2)})|^2}{|Y_s^{(1)} - Y_s^{(2)}|^4}\mathrm{d}s. \end{aligned}$$

Notice that by Lemma 2.1,

$$\begin{aligned} |\widetilde{\sigma}(s,x) - \widetilde{\sigma}(s,y)| &\leqslant C|x-y|\Big(\mathcal{M}_1|\nabla\widetilde{\sigma}(s,\cdot)|(x) + \mathcal{M}_1|\nabla\widetilde{\sigma}(s,\cdot)|(y) + \|\widetilde{\sigma}\|_\infty\Big), \\ |\widetilde{b}(s,x) - \widetilde{b}(s,y)| &\leqslant C|x-y|\Big(\mathcal{M}_1|\nabla\widetilde{b}(s,\cdot)|(x) + \mathcal{M}_1|\nabla\widetilde{b}(s,\cdot)|(y) + \|\widetilde{b}\|_\infty\Big). \end{aligned}$$

Thus, by the definitions of $\widetilde{b}$ and $\widetilde{\sigma}$ we have

$$
\begin{aligned}
|A_t| \lesssim & \int_0^t \Big(\mathcal{M}_1|\nabla\widetilde{b}|(s,Y_s^{(1)}) + \mathcal{M}_1|\nabla\widetilde{b}|(s,Y_s^{(2)}) + \|\widetilde{b}\|_\infty\Big)\,\mathrm{d}s \\
& + \int_0^t \Big(\mathcal{M}_1|\nabla\widetilde{\sigma}|^2(s,Y_s^{(1)}) + \mathcal{M}_1|\nabla\widetilde{\sigma}|^2(s,Y_s^{(2)}) + \|\widetilde{\sigma}\|_\infty^2\Big)\,\mathrm{d}s \\
& + \int_0^t \Big(\mathcal{M}_1|\nabla\widetilde{\sigma}|(s,Y_s^{(1)}) + \mathcal{M}_1|\nabla\widetilde{\sigma}|(s,Y_s^{(2)}) + \|\widetilde{\sigma}\|_\infty\Big)\,\mathrm{d}s \\
\lesssim & \, t\Big(\|\nabla\widetilde{b}\|_\infty + \|\widetilde{b}\|_\infty + \|\widetilde{\sigma}\|_\infty^2 + \|\widetilde{\sigma}\|_\infty + 1\Big) \\
& + \int_0^t \Big(\mathcal{M}_1|\nabla\sigma|^2(s,Y_s^{(1)}) + \mathcal{M}_1|\nabla\sigma|^2(s,Y_s^{(2)})\Big)\,\mathrm{d}s \\
& + \int_0^t \Big(\mathcal{M}_1|\nabla^2 u|^2(s,Y_s^{(1)}) + \mathcal{M}_1|\nabla^2 u|^2(s,Y_s^{(2)})\Big)\,\mathrm{d}s,
\end{aligned}
$$

where we have used that $|\nabla\widetilde{\sigma}|(s,x) \lesssim |\nabla\sigma|(s,x) + |\nabla^2 u|(s,x)$.

On the other hand, by (2.8) we have

$$\|\mathcal{M}_1|\nabla\sigma|^2\|_{\widetilde{\mathbb{L}}^{p_1/2}_{q_1/2}(T)} \leqslant C\|\,|\nabla\sigma|^2\|_{\widetilde{\mathbb{L}}^{p_1/2}_{q_1/2}(T)} = C\|\nabla\sigma\|^2_{\widetilde{\mathbb{L}}^{p_1}_{q_1}(T)} < \infty,$$

and

$$\|\mathcal{M}_1|\nabla^2 u|^2\|_{\widetilde{\mathbb{L}}^{p_2/2}_{q_2/2}(T)} \leqslant C\|\,|\nabla^2 u|^2\|_{\widetilde{\mathbb{L}}^{p_2/2}_{q_2/2}(T)} = C\|\nabla^2 u\|^2_{\widetilde{\mathbb{L}}^{p_2}_{q_2}(T)} < \infty.$$

Thus, by Khasminskii's estimate (4.2),

$$\mathbf{E}\mathrm{e}^{\gamma A_T} < \infty, \quad \forall\gamma\in\mathbb{R}.$$

Hence, by (4.7) and stochastic Gronwall's inequality (cf. [13] or [19, Lemma 3.7]),

$$\mathbf{E}\left(\sup_{t\in[0,T]}|Y_t^{(1)} - Y_t^{(2)}|^p\right) \leqslant C|y_1 - y_2|^p, \tag{4.8}$$

which in turn implies by [18, Theorem 1.1] that

$$\sup_{y\in\mathbb{R}^d}\mathbf{E}\left(\sup_{t\in[0,T]}|\nabla Y_t(y)|^p\right) < \infty.$$

Thus, by Lemma 4.2 we obtain (1.4). Moreover, by (4.8) we also have the pathwise uniqueness.

(iii) Let $\widetilde{\sigma}_n(t,y) := \widetilde{\sigma}(t,\cdot) * \rho_n(y)$ be the usual mollifying approximation. Let Y_t^n be the unique strong solution of the following approximation SDE:

$$\mathrm{d}Y_t^n = \widetilde{b}(t,Y_t^n)\mathrm{d}t + \widetilde{\sigma}_n(t,Y_t^n)\mathrm{d}W_t, \quad Y_0^n = y.$$

By the classical Bismut-Elworthy-Li's formula (for example, see [17]), we have for any $h\in\mathbb{R}^d$ and every bounded continuous function φ,

$$\nabla_h\mathbf{E}\varphi\big(Y_t^n(y)\big) = \frac{1}{t}\mathbf{E}\left[\varphi\big(Y_t^n(y)\big)\int_0^t \big[\widetilde{\sigma}_n\big(s,Y_s^n(y)\big)\big]^{-1}\nabla_h Y_s^n(y)\mathrm{d}W_s\right], \tag{4.9}$$

where $\nabla_h Y_t^n(y) := \lim_{\varepsilon\to 0}[Y_t^n(y+\varepsilon h) - Y_t^n(y)]/\varepsilon$. On the other hand, by $(\mathbf{H}^\sigma)$ and the property of convolutions, it is easy to see that

$$\lim_{|x-y|\to 0}\sup_n\sup_t \|\widetilde{\sigma}_n(t,x) - \widetilde{\sigma}_n(t,y)\|_{HS} = 0,$$

and for n_0 large enough,

$$(2c_0)^{-1}|\xi|^2 \leqslant |\widetilde{\sigma}_n(t,x)\xi|^2 \leqslant 2c_0|\xi|^2, \quad \xi\in\mathbb{R}^d.$$

Hence, Y_t^n satisfies the Krylov estimate (4.1) with the constant C independent of n. As a result of [19, Theorem 3.9], we have

$$\lim_{n\to\infty}\mathbf{E}\left(\sup_{t\in[0,T]}|Y_t^n(y) - Y_t(y)|\right) = 0.$$

Moreover, as in the proof of [22, (5.22)], we have

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^d}\mathbf{E}\left(\sup_{t\in[0,T]}|\nabla Y_t^n(y) - \nabla Y_t(y)|\right) = 0.$$

Now taking limits $n\to\infty$ for both sides of (4.9) yields that for every $\varphi\in C_b^1(\mathbb{R}^d)$,

$$\nabla_h\mathbf{E}\varphi\big(Y_t(y)\big) = \frac{1}{t}\mathbf{E}\left[\varphi\big(Y_t(y)\big)\int_0^t\big[\widetilde{\sigma}\big(s,Y_s(y)\big)\big]^{-1}\nabla_h Y_s(y)\mathrm{d}W_s\right].$$

Finally, using $\varphi\circ\Phi_t^{-1}(y)$ in place of φ in the above formula, we obtain (1.5). $\square$

5. Critical case: Proof of Theorem 1.4

In this section we assume that $(\mathbf{H}^\sigma)$ holds and $b\in\widetilde{\mathbb{L}}_\infty^{d;\mathrm{uni}}$. Let

$$b_n(t,x) := b(t,\cdot)*\rho_n(x), \quad \sigma_n(t,x) := \sigma(t,\cdot)*\rho_n(x).$$

By (2.3) and (2.6), it is easy to see that

$$\sup_n \kappa_T^{b_n}(\varepsilon) \leqslant C\kappa_T^b(\varepsilon). \tag{5.1}$$

Without loss of generality we assume $s=0$ and consider the following approximation SDE:

$$\mathrm{d}X_t^n = b_n(t,X_t^n)\mathrm{d}t + \sigma_n(t,X_t^n)\mathrm{d}W_t, \quad X_0^n = x.$$

We first prove the following crucial lemma about Krylov's estimate.

Lemma 5.1. *Let $p\in(1,d)$ and $q\in(1,\infty)$ with $\frac{d}{p}+\frac{2}{q}<2$. For any $T>0$, there are constants $\theta=\theta(p,q)>0$ and $C>0$ such that for any $f\in C_c^\infty(\mathbb{R}^{d+1})$, stopping time $\tau\leqslant T/2$ and $\delta\in(0,T/2)$,*

$$\sup_n\sup_{x\in\mathbb{R}^d}\mathbf{E}\left(\int_\tau^{\tau+\delta} f(s,X_s^n(x))\mathrm{d}s\Big|\mathscr{F}_{t_0}\right) \leqslant C\delta^\theta\|f\|_{\widetilde{\mathbb{L}}_q^p(T)}. \tag{5.2}$$

Proof. By discretizing stopping time approximation (see [24, Remark 1.2]), it suffices to prove that for any $0\leqslant t_0<t_1\leqslant T$ and $f\in C_c^\infty(\mathbb{R}^{d+1})$.

$$\sup_n\sup_{x\in\mathbb{R}^d}\mathbf{E}\left(\int_{t_0}^{t_1} f(s,X_s^n(x))\mathrm{d}s\Big|\mathscr{F}_{t_0}\right) \leqslant C(t_1-t_0)^\theta\|f\|_{\widetilde{\mathbb{L}}_q^p(T)}. \tag{5.3}$$

Let u_n be the smooth solution of the following backward PDE:

$$\partial_t u_n + \tfrac{1}{2}\sigma_n^{ik}\sigma_n^{jk}\partial_i\partial_j u_n + b_n^i\partial_i u_n + f = 0,\ u_n(t_1,\cdot) = 0.$$

Then, by Itô's formula we have

$$u_n(t_1, X^n_{t_1}) = u_n(t_0, X^n_{t_0}) - \int_{t_0}^{t_1} f(s, X^n_s)\mathrm{d}s + \int_{t_0}^{t_1} \sigma_n^{ij}\partial_i u_n(s, X^n_s)\mathrm{d}W^j_s.$$

Taking conditional expectation with respect to $\mathscr{F}_{t_0}$, we obtain

$$\mathbf{E}\left(\int_{t_0}^{t_1} f(s, X^n_s)\mathrm{d}s\Big|\mathscr{F}_{t_0}\right) = u_n(t_0, X^n_{t_0}) \leqslant \|u_n(t_0)\|_\infty.$$

Since $\frac{d}{p} + \frac{2}{q} < 2$, we can choose $q' < q$ so that $\frac{d}{p} + \frac{2}{q'} < 2$. Thus by (5.1), (3.2), (2.5) and Hölder's inequality, there is constant $C > 0$ such that

$$\mathbf{E}\left(\int_{t_0}^{t_1} f(s, X^n_s)\mathrm{d}s\Big|\mathscr{F}_{t_0}\right) \leqslant C|\!|\!|f|\!|\!|_{\widetilde{\mathbb{L}}^p_{q'}(t_0,t_1)} \leqslant C(t_1-t_0)^{1-\frac{q'}{q}}|\!|\!|f|\!|\!|_{\widetilde{\mathbb{L}}^p_q(T)},$$

which in turn gives (5.3). The proof is complete. □

By the above lemma, we can show the following tightness result for X^n.

Lemma 5.2. *For each $x \in \mathbb{R}^d$, let $\mathbb{P}^n_x$ be the law of $X^n_\cdot(x)$ in $\mathbb{C}$. Then $(\mathbb{P}^n_x)_{n\in\mathbb{N}}$ is tight.*

Proof. Let $T > 0$ and $\tau \leqslant T$ be any bounded stopping time. Notice that for every $\delta > 0$,

$$X^n_{\tau+\delta} - X^n_\tau = \int_\tau^{\tau+\delta} b_n(s, X^n_s)\mathrm{d}s + \int_\tau^{\tau+\delta}\sigma_n(s, X_s)\mathrm{d}W_s.$$

Let $p \in (1, d)$ and $q \in (1, \infty)$ with $\frac{d}{p} + \frac{2}{q} < 2$. By (5.2) and Burkhölder's inequality, there exists a $\theta > 0$ such that for any $\delta \in (0, T)$,

$$\begin{aligned}\mathbf{E}|X^n_{\tau+\delta} - X^n_\tau| &\leqslant \mathbf{E}\left(\int_\tau^{\tau+\delta}|b_n(s, X^n_s)|\mathrm{d}s\right) + C\mathbf{E}\left(\int_\tau^{\tau+\delta}|\sigma_n(s, X_s)|^2\mathrm{d}s\right)^{1/2}\\ &\leqslant C\delta^\theta|\!|\!|b_n|\!|\!|_{\widetilde{\mathbb{L}}^p_q(2T)} + C\delta^{1/2} \overset{(2.6)}{\leqslant} C\delta^\theta|\!|\!|b|\!|\!|_{\widetilde{\mathbb{L}}^d_\infty(2T)} + C\delta^{1/2},\end{aligned}$$

where $C > 0$ is independent of n. Thus by [23, Lemma 2.7], we obtain

$$\sup_n \mathbf{E}\left(\sup_{s\in[0,T]}|X^n_{s+\delta} - X^n_s|^{1/2}\right) \leqslant C\left(\delta^{\theta/2}|\!|\!|b|\!|\!|^{1/2}_{\widetilde{\mathbb{L}}^d_\infty(2T)} + \delta^{1/4}\right).$$

By Chebyshev's inequality, we derive that for any $\varepsilon > 0$,

$$\lim_{\delta\to 0}\sup_n \mathbf{P}\left(\sup_{s\in[0,T]}|X^n_{s+\delta} - X^n_s| > \varepsilon\right) = 0,$$

which implies the tightness of $X^n_\cdot$ by [15, Theorem 1.3.2]. □

Now we can give the proof of Theorem 1.4.

Proof of Theorem 1.4. Since $(\mathbb{P}^n_x)_{n\in\mathbb{N}}\subset\mathscr{P}(\mathbb{C})$ is tight, let $\mathbb{P}_x$ be any accumulation point of $(\mathbb{P}^n_x)_{n\in\mathbb{N}}$. By Krylov's estimate (5.2), it is by now easy to show that $\mathbb{P}_x$ is a martingale solution of SDE (1.2), see for example, [23]. Moreover, (1.6) holds. We shall only prove the uniqueness of martingale solutions. Let $\mathbb{P}^{(i)}_x\in\mathscr{M}^{\sigma,b}_{0,x}, i=1,2$ be any two martingale solutions of SDE (1.2) so that for any $T>0$, there is a constant $C>0$ such that for all $x\in\mathbb{R}^d$ and $0\leqslant t_0<t_1\leqslant T$, $f\in\widetilde{\mathbb{L}}^p_q(t_0,t_1)$,

$$\mathbb{E}^{\mathbb{P}^{(i)}_x}\left(\int_{t_0}^{t_1}f(s,\omega_s)\mathrm{d}s\Big|\mathcal{B}_{t_0}\right)\leqslant C|\!|\!|f|\!|\!|_{\widetilde{\mathbb{L}}^p_q(t_0,t_1)}. \tag{5.4}$$

Let $p\in(1,d)$ and $q\in(1,\infty)$ satisfy $\frac{d}{p}+\frac{2}{q}<2$. For $T>0$ and $f\in C^\infty_c([0,T]\times\mathbb{R}^d)$, by Theorem 3.1, there is a unique solution $u\in\widetilde{\mathbb{H}}^{2,p}_q(T)$ to the following backward equation:

$$\partial_t u+\mathscr{L}^{\sigma,b}_t u+f=0,\quad u(T)=0.$$

Let $u_n(t,x):=u(t,\cdot)*\rho_n(x)$ be the mollifying approximation of u. Then we have

$$\partial_t u_n+\mathscr{L}^{\sigma,b}_t u_n+g_n=0,\quad u_n(T)=0,$$

where

$$g_n=f_n+(\mathscr{L}^{\sigma,b}_t u)*\rho_n-\mathscr{L}^{\sigma,b}_t(u*\rho_n).$$

For $R>0$, define

$$\tau_R:=\inf\{t\geqslant 0:|\omega_t|\geqslant R\}.$$

By Itô's formula, we have

$$\mathbb{E}^{\mathbb{P}^{(i)}_x}u_n(T\wedge\tau_R,\omega_{T\wedge\tau_R})=u_n(0,x)-\mathbb{E}^{\mathbb{P}^{(i)}_x}\left(\int_0^{T\wedge\tau_R}g_n(s,\omega_s)\mathrm{d}s\right),\quad i=1,2. \tag{5.5}$$

Since

$$|\!|\!|\mathscr{L}^{\sigma,b}u|\!|\!|_{\widetilde{\mathbb{L}}^p_q(T)}\leqslant\|\sigma\|_\infty|\!|\!|\nabla^2 u|\!|\!|_{\widetilde{\mathbb{L}}^p_q(T)}+|\!|\!|b|\!|\!|_{\widetilde{\mathbb{L}}^d_\infty(T)}\cdot|\!|\!|\nabla u|\!|\!|_{\widetilde{\mathbb{L}}^{pd/(d-p)}_q(T)}\overset{(2.5)}{\lesssim}|\!|\!|u|\!|\!|_{\widetilde{\mathbb{H}}^{2,p}_q(T)},$$

by Krylov's estimate (5.4) and (2.6), we have

$$\begin{aligned}&\lim_{n\to\infty}\mathbb{E}^{\mathbb{P}^{(i)}_x}\left(\int_0^{T\wedge\tau_R}\Big((\mathscr{L}^{\sigma,b}_t u)*\rho_n-\mathscr{L}^{\sigma,b}_t(u*\rho_n)\Big)(s,\omega_s)\mathrm{d}s\right)\\&\leqslant C\lim_{n\to\infty}|\!|\!|\chi_R((\mathscr{L}^{\sigma,b}u)*\rho_n-\mathscr{L}^{\sigma,b}(u*\rho_n))|\!|\!|_{\widetilde{\mathbb{L}}^q_p(T)}=0,\end{aligned}$$

where the cutoff function χ_R is defined by (2.1). Letting $n\to\infty$ for both sides of (5.5) and by the dominated convergence theorem, we obtain

$$\mathbb{E}^{\mathbb{P}^{(i)}_x}u(T\wedge\tau_R,\omega_{T\wedge\tau_R})=u(0,x)-\mathbb{E}^{\mathbb{P}^{(i)}_x}\left(\int_0^{T\wedge\tau_R}f(s,\omega_s)\mathrm{d}s\right),\quad i=1,2,$$

which, by letting $R\to\infty$ and noting $u(T)=0$, yields

$$u(0,x)=\mathbb{E}^{\mathbb{P}^{(i)}_x}\left(\int_0^T f(s,\omega_s)\mathrm{d}s\right),\quad i=1,2.$$

This in particular implies the uniqueness of martingale solutions (see [15]). $\square$

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