STOCHASTIC LAGRANGIAN PATH FOR LERAY SOLUTIONS OF 3D NAVIER-STOKES EQUATIONS

XICHENG ZHANG AND GUOHUAN ZHAO

Abstract. In this paper we show the existence of stochastic Lagrangian particle trajectory for Leray's solution of 3D Navier-Stokes equations. More precisely, for any Leray's solution $\mathbf{u}$ of 3D-NSE and each $(s,x)\in\mathbb{R}_+\times\mathbb{R}^3$, we show the existence of weak solutions to the following SDE, which has a density $\rho_{s,x}(t,y)$ belonging to $\mathbb{H}^{1,p}_q$ provided $p,q\in[1,2)$ with $\frac{3}{p}+\frac{2}{q}>4$:

$$\mathrm{d}X_{s,t}=\mathbf{u}(s,X_{s,t})\mathrm{d}t+\sqrt{2\nu}\mathrm{d}W_t,\ \ X_{s,s}=x,\ \ t\geqslant s,$$

where W is a three dimensional standard Brownian motion, $\nu>0$ is the viscosity constant. Moreover, we also show that for Lebesgue almost all (s,x), the solution $X^n_{s,\cdot}(x)$ of the above SDE associated with the mollifying velocity field $\mathbf{u}_n$ weakly converges to $X_{s,\cdot}(x)$ so that X is a Markov process in almost sure sense.

Keywords: Leray's solution, Navier-Stokes equation, Stochastic differential equation, De-Giorgi's iteration, Krylov's estimate.

AMS 2010 Mathematics Subject Classification: Primary: 60H10, 35Q30; Secondary: 76D05

1. Introduction

Throughout the paper we assume $d\geqslant 2$. Consider the following Navier-Stokes equation:

$$\partial_t\mathbf{u}=\nu\Delta\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p,\ \ \mathrm{div}\,\mathbf{u}\equiv 0,\ \mathbf{u}_0=\varphi,$$

where $\mathbf{u}=(u_1,\cdots,u_d)$ is the velocity field of the fluid, $\nu>0$ is the viscosity constant, and p stands for the pressure. It is well known that for any divergence free vector field $\varphi\in L^2(\mathbb{R}^d)$, there exists a divergence free Leray weak solution to NSEs in the class

$$\|\mathbf{u}\|_{L^\infty([0,T];L^2(\mathbb{R}^d))}+\|\nabla\mathbf{u}\|_{L^2([0,T];L^2(\mathbb{R}^d))}<\infty,\ \ \forall T>0. \tag{1.1}$$

In a recent remarkable paper, Buckmaster and Vicol [4] showed that there are infinitely many weak solutions $\mathbf{u}\in C(\mathbb{R}_+;L^2(\mathbb{T}^3))$ for 3D-NSEs on the torus. However, it is still unknown whether the above Leray solution is unique and smooth, which is in fact a famous open problem for a long time.

In this work we are interested in the following problem: For any Leray solution $\mathbf{u}$, is it possible to construct the stochastic Lagrangian particle trajectory $X_t=X_t(x)$ associated with the velocity field $\mathbf{u}$? More precisely, for each starting point x, is there a unique solution to the following SDE?

$$\mathrm{d}X_t=\mathbf{u}(t,X_t)\mathrm{d}t+\sqrt{2\nu}\mathrm{d}W_t,\ X_0=x, \tag{1.2}$$

Research of X. Zhang is partially supported by NNSFC grant of China (No. 11731009). Research of G. Zhao is supported by the German Research Foundation (DFG) through the Collaborative Research Centre(CRC) 1283 "Taming uncertainty and profiting from randomness and low regularity in analysis, stochastics and their applications".

where W is a d-dimensional standard Brownian motion on some probability space $(\Omega, \mathscr{F}, \mathbf{P})$. If $\mathbf{u}$ is smooth in x, then by Constantin and Iyer's representation [6] (see also [28, 30]), $\mathbf{u}$ can be reconstructed from $X_t(x)$ as follows:

$$\mathbf{u}(t,x) = \mathcal{P}\mathbb{E}(\nabla^{\mathrm{t}} X_t^{-1}(x) \cdot \varphi(X_t^{-1}(x))),$$

where $\mathcal{P}$ is the Leray projection and $X_t^{-1}(x)$ is the inverse of stochastic flow $x \mapsto X_t(x)$, and ∇^{t} stands for the transpose of the Jacobian matrix. By Krylov and Röckner's result [15], under the following assumption

$$\mathbf{u} \in \cap_{T>0} L^q([0,T]; L^p(\mathbb{R}^d)), \quad p, q \geqslant 2, \quad \tfrac{d}{p} + \tfrac{2}{q} < 1,$$

for any starting point $x \in \mathbb{R}^d$, there is a unique strong solution to SDE (1.2). Moreover, the unique solution $X_t(x)$ is weakly differentiable in x and satisfies (see [7, 27, 32]):

$$\sup_{x\in\mathbb{R}^d} \mathbf{E}\left(\sup_{t\in[0,T]} |\nabla X_t(x)|^p\right) < \infty, \quad \forall p \geqslant 1, \ T > 0.$$

On the other hand, one says that a vector field $\mathbf{u} : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies the so called Ladyzhenskaya-Prodi-Serrin's condition (abbreviated as LPS) if

$$\mathbf{u} \in \cap_{T>0} L^q([0,T]; L^p(\mathbb{R}^d)), \quad p, q \in [2, \infty], \quad \tfrac{d}{p} + \tfrac{2}{q} \leqslant 1. \tag{1.3}$$

It is now well known that any Leray solution $\mathbf{u}$ of 3D-NSE must be smooth under the above LPS conditions (see [20, Theorem 13.11 and Notes, p.261]). Unfortunately, it is still not known whether each Leray solution satisfies (1.3). Indeed, by (1.1) and Sobolev's embedding (see Lemma 2.1 below), we only have

$$\mathbf{u} \in \cap_{T>0} L^q([0,T]; L^p(\mathbb{R}^d)), \quad p, q \geqslant 2, \quad \tfrac{d}{p} + \tfrac{2}{q} = \tfrac{d}{2}. \tag{1.4}$$

Notice that the *deterministic* Lagrangian particle trajectories associated with $\mathbf{u}$ have been studied very well (for example, see [20, Chapter 17] and [5]), which depends on further regularity on Leray's solution. Here we want to solve SDE (1.2) under (1.4) for $d = 3$.

For given $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, we consider the following SDE in $\mathbb{R}^d$ starting from x at time s:

$$\mathrm{d}X_{s,t} = b(t, X_{s,t})\mathrm{d}t + \sqrt{2}\mathrm{d}W_t, \ t > s, \ X_{s,s} = x, \tag{1.5}$$

where $b(t,x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is a measurable vector field. The generator associated with the above SDE is given by

$$\mathscr{L}_t^b := \Delta + b(t, \cdot) \cdot \nabla.$$

In this paper, we focus on the weak solution of SDE (1.5) with *lower regularity* b, that is,

$$b \in \cap_{T>0} L^q([0,T]; L^p(\mathbb{R}^d)) =: L^q_{loc}(L^p), \quad p, q \geqslant 2, \quad \tfrac{d}{p} + \tfrac{2}{q} < 2.$$

Roughly speaking, a weak solution to SDE (1.5) is a *semimartingale* $(X_{s,t})_{t\geqslant s}$ so that

$$\int_s^t |b(r, X_{s,r})|\mathrm{d}r < \infty, \quad \forall t \geqslant s, \quad a.s.,$$

and

$$X_{s,t} = x + \int_s^t b(r, X_{s,r})\mathrm{d}r + \sqrt{2}(W_t - W_s), \quad \forall t \geqslant s, \quad a.s. \tag{1.6}$$

When $b \in L^q_{loc}(L^p)$ for some $p, q \in [2, \infty)$ with $\frac{d}{p} + \frac{2}{q} < 1$, as mentioned above, by Girsanov's transformation and L^p-theory of second order parabolic equations, Krylov and Röckner [15] showed that there is a unique strong solution to SDE (1.5), which extended the main results in [25] and [35]. In [19], Rezakhanlou showed the almost Lipschitz regularity of the associated stochastic flow in the spatial variable and showed some applications in Hamiltonian systems perturbed by white noises. The strong well-posedness of SDE (1.5) driven by multiplicative Brownian noise was studied in [27, 32] by Zvonkin's transformation introduced in [35]. Moreover, the flow property and weak differentiability of $X_{s,t}(x)$ in x are also obtained therein. When $b \in H^{-\alpha,p}$ with $\alpha \in (0, \frac{1}{2})$ and $p \in (\frac{d}{1-\alpha}, \frac{d}{\alpha})$ is time-independent, Flandoli, Issoglio and Russo [9] showed the existence and uniqueness of "virtual" solutions (a class of special weak solutions) to SDE (1.5). Later, the well-posedness of martingale solutions and weak solutions (which may not be a semimartingale but a Dirichlet process) was established in [33] for $b \in H^{-\alpha,p}$ with $\alpha \in (0, \frac{1}{2}]$ and $p \in (\frac{d}{1-\alpha}, \infty)$. We also mention that Bass and Chen in [2] studied the weak well-posedness of SDE (1.5) in the class of semimartingales when b belongs to some generalized Kato's class $\mathbf{K}_{d-1}$, namely b is a signed measure with

$$\lim_{\delta \to 0} \sup_{x \in \mathbb{R}^d} \int_{|y-x|<\delta} \frac{|b|(\mathrm{d}y)}{|x-y|^{d-1}} = 0,$$

(see also [34]). In particular, the space L^p with $p > d$ is included in this class.

It should be emphasized that even in the weak sense, all the works mentioned above do not cover the borderline case $b \in L^q_{loc}(L^p)$ with $\frac{d}{p} + \frac{2}{q} = 1$, not to mention the supercritical case $\frac{d}{p} + \frac{2}{q} > 1$. Let us explain the difficulty firstly. In order to get the weak existence of SDE (1.5) with singular drifts, a straightforward way is to use Girsanov's transform as in [15]. However, this approach does not work in the case when $p \leqslant d$. Let us make a detailed analysis for this point. Let $\mathbb{C}$ be the space of all continuous functions from $\mathbb{R}_+$ to $\mathbb{R}^d$, which is endowed with the topology of locally uniform convergence. Let $\mathcal{B}(\mathbb{C})$ be the Borel σ-field generated by all open subsets of $\mathbb{C}$. The set of all probability measures over $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ is denoted by $\mathscr{P}(\mathbb{C})$. Let ω_t be the canonical process over $\mathbb{C}$. For $t \geqslant 0$, define

$$\mathcal{B}_t(\mathbb{C}) := \sigma\{\omega_s : 0 \leqslant s \leqslant t\}.$$

Let $\mathbb{P} \in \mathscr{P}(\mathbb{C})$ be the classical Wiener measure so that $t \mapsto \omega_t$ is a d-dimensional standard Brownian motion. For $b \in L^p(\mathbb{R}^d)$ with $p \leqslant d$, one can check that the Novikov condition

$$\mathbb{E}^{\mathbb{P}} \exp\left(\frac{1}{2}\int_0^T |b|^2(\omega_t)\mathrm{d}t\right) < \infty \tag{1.7}$$

for the exponential supermartingale

$$\mathcal{E}^b_t = \exp\left(\int_0^t b(\omega_s)\mathrm{d}\omega_s - \frac{1}{2}\int_0^t |b|^2(\omega_s)\mathrm{d}s\right)$$

may not hold. Notice that condition (1.7) is somehow equivalent to say that b belongs to some Kato's class (see [1]). In fact, without other conditions, if b only belongs to $L^{d-\varepsilon}_{loc} \setminus L^d_{loc}$, then the weak existence may be failed. For example, consider the following SDE:

$$X_t = -c\int_0^t X_s|X_s|^{-2}\mathrm{d}s + W_t, \quad c \in \mathbb{R}. \tag{1.8}$$

If $c \geqslant d$, Kinzebulatov and Semenov [14, page 3] explained why the above SDE does not allow a solution (see also [3]). Meanwhile, for $c < c_d$, where $c_d \in (0, d)$ is some constant only depending on d, they proved that there exists a weak solution to the above SDE by utilizing the analytic construction of the semigroup $\mathrm{e}^{-t(\Delta + b\cdot\nabla)}$. By direct calculations, for $b(x) := -cx|x|^{-2}$ and $d \geqslant 3$, we have

$$\mathrm{div} b(x) = -c(d-2)|x|^{-2} \notin L^{d/2}_{loc}.$$

Intuitively, if X is a solution of (1.8) and c is sufficiently large, then the centripetal force is too strong so that the occupation time $\int_0^T \mathbf{1}_{\{X_t=0\}}\mathrm{d}t$ of X at origin during $[0,T]$ must be positive for any $T>0$ even though a random perturbation is added (see [3] for more details), and thus there is no semimartingale solution for SDE (1.8). However, our result below shows that if $b \in L^{d/2+\varepsilon}(\mathbb{R}^d)$ for some $\varepsilon > 0$, then equation (1.5) has at least one semimartingale solution, provided that the negative part of $\mathrm{div} b$ satisfies some integrability condition. We emphasize that Kinzebulatov and Semenov's result in [14] can not be applied to the case $b \in L^{d-\varepsilon}_{loc} \setminus L^d_{loc}$. We believe that the divergence condition is necessary for this case. Moreover, the singular time-dependent drift b is not treated in [14]. If it is not possible, it seems hard to directly construct the two-parameter semigroups associated with time-dependent drifts by analytic method.

Before stating our results, we introduce the following notion of martingale solutions.

Definition 1. *For given $(s,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, we call a probability measure $\mathbb{P}_{s,x} \in \mathscr{P}(\mathbb{C})$ a martingale solution of SDE (1.5) with starting point (s,x) if*

(i) $\mathbb{P}_{s,x}(\omega_t = x, t \leqslant s) = 1$, and for each $t > s$,

$$\mathbb{E}^{\mathbb{P}_{s,x}}\left(\int_s^t |b(r,\omega_r)|\mathrm{d}r\right) < \infty.$$

(ii) For all $f \in C^2_c(\mathbb{R}^d)$, M^f_t is a $\mathcal{B}_t$-martingale under $\mathbb{P}_{s,x}$, where

$$M^f_t(\omega) := f(\omega_t) - f(x) - \int_s^t \mathscr{L}^b_r f(\omega_r)\mathrm{d}r, \quad t \geqslant s.$$

The set of all martingale solutions $\mathbb{P}_{s,x}$ with starting point (s,x) and drift b is denoted by $\mathscr{M}^b_{s,x}$.

Remark 1.1. *Let $\mathbb{P}_{s,x} \in \mathscr{M}^b_{s,x}$. By Lévy's characterization for Brownian motion, one sees that*

$$W_t := \frac{\sqrt{2}}{2}\left(\omega_t - \omega_s - \int_s^t b(r,\omega_r)\mathrm{d}r\right), \quad t \geqslant s,$$

is a d-dimensional standard Browian motion under $\mathbb{P}_{s,x}$ (see [23, Theorem 4.2.1]*), so that*

$$\omega_t = x + \int_s^t b(r,\omega_r)\mathrm{d}r + \sqrt{2}W_t, \quad t \geqslant s.$$

In other words, $(\mathbb{C}, \mathscr{B}(\mathbb{C}), \mathbb{P}_{s,x}, \omega_t, W_t)$ is a weak solution of SDE (1.5).

Our main result is

Theorem 1.1. *Suppose that for some $p_i, q_i \in [2,\infty)$ with $\frac{d}{p_i} + \frac{2}{q_i} < 2$, $i = 1,2$,*

$$-\mathrm{div} b \leqslant \Theta_b, \quad \kappa := |||b|||_{\widetilde{\mathbb{L}}^{p_1}_{q_1}} + |||\Theta_b|||_{\widetilde{\mathbb{L}}^{p_2}_{q_2}} < \infty, \tag{1.9}$$

where $-\mathrm{div} b \leqslant \Theta_b$ is defined in the sense of (2.2) below, and $|||\cdot|||_{\widetilde{\mathbb{L}}^p_q}$ is defined by (2.5) and (2.4) below. Then for each $(s,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, there exists at least one martingale solution $\mathbb{P}_{s,x} \in \mathscr{M}^b_{s,x}$, which satisfies the following Krylov's type estimate: for any $\alpha \in [0,1]$ and $p,q \in (1,\infty)$ with $\frac{d}{p} + \frac{2}{q} < 2 - \alpha$, there exist $\theta = \theta(\alpha,p,q) > 0$ and a constant $C = C(\Pi) > 0$ such that for all $s \leqslant t_0 < t_1 < \infty$ with $t_1 - t_0 \leqslant 1$ and $f \in C^\infty_b(\mathbb{R}^{d+1})$,

$$\mathbb{E}^{\mathbb{P}_{s,x}}\left(\int_{t_0}^{t_1} f(t,\omega_t)\mathrm{d}t \Big| \mathcal{B}_{t_0}\right) \leqslant C(t_1 - t_0)^\theta |||f|||_{\widetilde{\mathbb{H}}^{-\alpha,p}_q}, \tag{1.10}$$

where $\Pi := (\kappa, d, p_i, q_i, p, q, \alpha)$ is the parameter set. Moreover, we have the following conclusions:

(i) (Weak uniqueness) For any mollifying approximation b_n of b, there is a Lebesgue-null set $\mathcal{N} \subset \mathbb{R}_+ \times \mathbb{R}^d$ such that for all $(s,x) \in \mathcal{N}^c$,

$$\mathbb{P}^n_{s,x} \text{ weakly converges to } \mathbb{P}_{s,x} \in \mathscr{M}^b_{s,x}, \text{ where } \mathbb{P}^n_{s,x} \in \mathscr{M}^{b_n}_{s,x}. \tag{1.11}$$

(ii) (Almost surely Markov property) For each $(s,x) \in \mathcal{N}^c$, there is a Lebesgue null set $I_{s,x} \subset [s,\infty)$ such that for all $t_0 \in (s,\infty) \setminus I_{s,x}$, any $t_1 > t_0$ and $f \in C_b(\mathbb{R}^d)$,

$$\mathbb{E}^{\mathbb{P}_{s,x}}(f(\omega_{t_1})|\mathcal{B}_{t_0}) = \mathbb{E}^{\mathbb{P}_{t_0,\omega_{t_0}}}(f(\omega_{t_1})), \ \ \mathbb{P}_{s,x} - a.s. \tag{1.12}$$

(iii) (L^p-semigroup) Let $\mathcal{T}_{s,t}f(x) := \mathbb{E}^{\mathbb{P}_{s,x}} f(\omega_t)$. For any $p \geqslant 1$ and $T > 0$, there is a constant $C = C(T,\Pi) > 0$ such that for Lebesgue almost all $0 \leqslant s < t \leqslant T$ and $f \in L^p(\mathbb{R}^d)$,

$$\|\mathcal{T}_{s,t}f\|_p \leqslant C\|f\|_p. \tag{1.13}$$

We now give some remarks about the above results.

Remark 1.2. *By discretization stopping time approximation, Krylov estimate (1.10) is equivalent to say that for any $\delta \in (0,1)$ and stopping time $\tau \in [s,\infty)$,*

$$\mathbb{E}^{\mathbb{P}_{s,x}}\left(\int_\tau^{\tau+\delta} f(t,\omega_t)\mathrm{d}t\Big|\mathcal{B}_\tau\right) \leqslant C\delta^\theta |||f|||_{\widetilde{\mathbb{H}}^{-\alpha,p}_q}, \tag{1.14}$$

where $\mathcal{B}_\tau := \sigma\{\omega_{t\wedge\tau}, t \geqslant 0\}$ is the stopping σ-field. In fact, suppose that (1.10) holds and let τ_n be a sequence of decreasing stopping times with values in $\mathbb{D} := \{k \cdot 2^{-n} : k,n \in \mathbb{N}\}$ and so that $\tau_n \to \tau$ as $n \to \infty$. For any $f \in C^\infty_b(\mathbb{R}^{d+1})$ and $\delta \in (0,1)$, by the dominated convergence theorem and martingale convergence theorem, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}_{s,x}}\left(\int_\tau^{\tau+\delta} f(t,\omega_t)\mathrm{d}t\Big|\mathcal{B}_\tau\right) &= \lim_{n\to\infty} \mathbb{E}^{\mathbb{P}_{s,x}}\left(\int_{\tau_n}^{\tau_n+\delta} f(t,\omega_t)\mathrm{d}t\Big|\mathcal{B}_{\tau_n}\right)\\
&= \lim_{n\to\infty} \mathbb{E}^{\mathbb{P}_{s,x}}\left(\sum_{a\in\mathbb{D}} \mathbf{1}_{\{\tau_n=a\}}\int_a^{a+\delta} f(t,\omega_t)\mathrm{d}t\Big|\mathcal{B}_{\tau_n}\right)\\
&= \lim_{n\to\infty} \sum_{a\in\mathbb{D}} \mathbf{1}_{\{\tau_n=a\}}\mathbb{E}^{\mathbb{P}_{s,x}}\left(\int_a^{a+\delta} f(t,\omega_t)\mathrm{d}t\Big|\mathcal{B}_a\right)\\
&\overset{(1.10)}{\leqslant} C\delta^\theta |||f|||_{\widetilde{\mathbb{H}}^{-\alpha,p}_q} \lim_{n\to\infty}\sum_{a\in\mathbb{D}} \mathbf{1}_{\{\tau_n=a\}} = C\delta^\theta |||f|||_{\widetilde{\mathbb{H}}^{-\alpha,p}_q}.
\end{aligned}$$

Moreover, let $\mu_{s,x}(t,\mathrm{d}y) := \mathbb{P}_{s,x} \circ \omega_t^{-1}$. For any $\alpha \in [0,1]$ and $p,q \in (1,\infty)$ with $\frac{d}{p} + \frac{2}{q} < 2-\alpha$, by (1.10), for any $T > 0$ there is a constant $C > 0$ such that for all $f \in C^\infty_c([0,T]\times\mathbb{R}^d)$,

$$\left|\int_0^T\int_{\mathbb{R}^d} f(t,y)\mu_{s,x}(t,\mathrm{d}y)\mathrm{d}t\right| \leqslant C\|f\|_{\mathbb{H}^{-\alpha,p}_q},$$

which in turn implies that $\mu_{s,x}(t,\mathrm{d}y) = \rho_{s,x}(t,y)\mathrm{d}y$ with $\rho_{s,x} \in \mathbb{H}^{\alpha,p/(p-1)}_{q/(q-1)}$.

Remark 1.3. *If $(\mathrm{div} b)^- \equiv 0$, then $\|\mathcal{T}_{s,t}f\|_1 \leqslant \|f\|_1$ in (1.13). If $\mathrm{div} b \equiv 0$, then for any nonnegative $f \in L^1(\mathbb{R}^d)$, $\|\mathcal{T}_{s,t}f\|_1 = \|f\|_1$. By (1.4), we can apply the above theorem to the Leray solution of 3D-NSEs.*

Remark 1.4. *Let $d \geqslant 3$ and $\alpha < 3$. Define*

$$b(x) := \sum_{z\in\mathbb{Z}^d} \gamma_z \frac{x-z}{|x-z|^\alpha}\phi(|x-z|),$$

where for some $M > 0$, $\gamma_z \in (0,M)$ is a constant and $\phi \in C^\infty_c(\mathbb{R}_+;[0,1])$ with $\phi(r) = 1$ for $r \in [0,1]$ and $\phi(r) = 0$ for $r > 2$. It is easy to see that (1.9) holds.

Remark 1.5. *It should be compared with the results in* [29, 31]. *Therein, under the assumptions*

$$\nabla b \in \mathbb{L}^1_{loc}, \quad (\mathrm{div} b)^-, b/(1+|x|) \in \mathbb{L}^\infty, \tag{1.15}$$

the existence and uniqueness of almost everywhere stochastic flows are obtained in the framework of DiPerna-Lions' theory. By the estimate (1.13)*, we can weaken the assumption on the boundedness of* $(\mathrm{div} b)^-$ *in* [31] *when the noise is nondegenerate. On the other hand, in* [29, 31] *and recent work* [26]*, under* (1.15)*, the existence of a solution is only shown for Lebesgue almost all* $x \in \mathbb{R}^d$*, while, under* (1.9) *we can show the existence of a solution for all starting point* $x \in \mathbb{R}^d$.

To the best of our knowledge, Theorem 1.1 seems to be the first one that considers the well-posedness of SDE (1.5) beyond the LPS condition. However, it should be also pointed out that the weakness of the present paper is that we can not get the pathwise uniqueness for (1.5) when the drift vector field is the Leray solution of 3D-NSE, not to say the flow property of the solutions and the weak differentiability of the associated stochastic flow with respect to the starting point x. We would like to say that these problems are open to us. We hope to study them in the future.

To prove Theorem 1.1, the key point for us is to establish the maximum principle for the following parabolic equation under (1.9):

$$\partial_t u = \Delta u + b \cdot \nabla u + f, \ u(0) = 0. \tag{1.16}$$

More precisely, for any $\alpha \in [0, 1]$ and $q, p \in (1, \infty)$ with $\frac{d}{p} + \frac{2}{q} < 2 - \alpha$,

$$\|u\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leqslant C |\!|\!| f |\!|\!|_{\widetilde{\mathbb{H}}^{-\alpha,p}_q}. \tag{1.17}$$

When $f \equiv 0$, under (1.9) the local maximum principle is proved by Nazarov and Ural'tseva in [16] by using Moser's iteration. It should be mentioned that when b is divergence-free and smooth, still by Moser's iteration, Qian and Xi [18] recently studied the *apriori* Aronson's type estimate for the heat kernel of operator $\mathscr{L}^b_t = \Delta + b \cdot \nabla$, where the bound depends only on the norm $\|b\|_{\mathbb{L}^p_q}$, where $p, q \in (2, \infty)$ satisfies $1 \leqslant \frac{d}{p} + \frac{2}{q} < 2$. We also refer to [12] for the study of elliptic equations with drift $b \equiv 0$ and $f \in L^p(\mathbb{R}^d)$ for $p > \frac{d}{2}$. Here an open question is that whether we can show (1.11)-(1.13) for all $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, which is closely related to find a continuous solution for PDE (1.16) under (1.9).

This paper is organized as follows: In Section 2, we establish the key maximum estimate (1.17) by De Giorgi's method. In fact, we shall show a more general result by allowing b and f being in negative Sobolev spaces, which are not treated in [12, 16]. In Section 3, we prove our main result Theorem 1.1. In Appendix, some properties of certain local Sobolev spaces are given. Throughout this paper we shall use the following conventions:

- We use $A \lesssim B$ to denote $A \leqslant CB$ for some unimportant constant $C > 0$.
- For any $\varepsilon \in (0, 1)$, we use $A \lesssim \varepsilon B + D$ to denote $A \leqslant \varepsilon B + C_\varepsilon D$ for some constant $C_\varepsilon > 0$.
- $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ := [0, \infty)$, $a \vee b := \max(a, b)$, $a \wedge b := \min(a, b)$, $a^+ := a \vee 0$.
- For $r > 0$, we define $B_r := \{x \in \mathbb{R}^d : |x| < r\}$ and $Q_r := (-r^2, r^2) \times B_r$.

2. Maximum principle for parabolic equations by De Giorgi's method

Let $\mathscr{D} := C^\infty_c(\mathbb{R}^{d+1})$ be the space of all smooth functions with compact supports and $\mathscr{D}'$ the dual space of $\mathscr{D}$, which is also called distribution space. The duality between $\mathscr{D}'$ and $\mathscr{D}$ is denoted by $\langle\!\langle\cdot,\cdot\rangle\!\rangle$. In particular, if $f\in\mathscr{D}'$ is locally integrable and $g\in\mathscr{D}$, then

$$\langle\!\langle f,g\rangle\!\rangle=\int_{\mathbb{R}}\langle f(t),g(t)\rangle\mathrm{d}t\ \text{ with }\ \langle f(t),g(t)\rangle:=\int_{\mathbb{R}^d}f(t,x)g(t,x)\mathrm{d}x. \tag{2.1}$$

For two distributions $f,g\in\mathscr{D}'$, one says that $f\leqslant g$ if for any nonnegative $\varphi\in\mathscr{D}$,

$$\langle\!\langle f,\varphi\rangle\!\rangle\leqslant\langle\!\langle g,\varphi\rangle\!\rangle. \tag{2.2}$$

For $\alpha\in\mathbb{R}$ and $p\in(1,\infty)$, let $H^{\alpha,p}$ be the usual Bessel potential space with norm

$$\|f\|_{\alpha,p}:=\|(\mathbb{I}-\Delta)^{\alpha/2}f\|_p=\left(\int_{\mathbb{R}^d}|(\mathbb{I}-\Delta)^{\alpha/2}f(x)|^p\mathrm{d}x\right)^{1/p}.$$

It is well known that for $\alpha\in(0,1)$, an equivalent norm of $H^{\alpha,p}$ is given by (see [22])

$$\|f\|_{\alpha,p}\asymp\|f\|_p+\|\Delta^{\alpha/2}f\|_p,$$

where $\Delta^{\alpha/2}:=-(-\Delta)^{\alpha/2}$ is the usual fractional Laplacian. For $\alpha\in\mathbb{R}$ and $p,q\in(1,\infty)$, let $\mathbb{H}^{\alpha,p}_q:=L^q(\mathbb{R};H^{\alpha,p})$ be the space of spatial-time functions with norm

$$\|f\|_{\mathbb{H}^{\alpha,p}_q}:=\left(\int_{\mathbb{R}}\|f(t,\cdot)\|^q_{\alpha,p}\mathrm{d}t\right)^{1/q}.$$

If $f\in\mathbb{H}^{\alpha,p}_q$, $g\in\mathbb{H}^{-\alpha,p'}_{q'}$ with $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$, as above we denote

$$\langle\!\langle f,g\rangle\!\rangle:=\int_{\mathbb{R}}\langle f(t),g(t)\rangle\mathrm{d}t=\int_{\mathbb{R}}\langle f,g\rangle.$$

For $\alpha=0$ and $p,q\in[1,\infty]$, we also define

$$\mathbb{H}^{0,p}_q:=\mathbb{L}^p_q:=L^q(\mathbb{R};L^p(\mathbb{R}^d)),$$

and the energy space

$$\mathscr{V}:=\Big\{f\in\mathbb{L}^2_\infty\cap\mathbb{H}^{1,2}_2:\|f\|_{\mathscr{V}}:=\|f\|_{\mathbb{L}^2_\infty}+\|\nabla_xf\|_{\mathbb{L}^2_2}<\infty\Big\}.$$

Throughout this paper we fix a cutoff function

$$\chi\in C^\infty_c(\mathbb{R}^{d+1};[0,1])\text{ with }\chi|_{Q_1}=1\text{ and }\chi|_{Q_2^c}=0,$$

and for $r>0$ and $(s,z)\in\mathbb{R}^{d+1}$, define

$$\chi_r(t,x):=\chi(r^{-2}t,r^{-1}x),\ \chi^{s,z}_r(t,x):=\chi_r(t-s,x-z),\ \ (t,x)\in\mathbb{R}^{d+1}. \tag{2.3}$$

Next we introduce the localized Bessel potential spaces for later use.

Definition 2. *Let $\alpha\neq0$ and $p,q\in(1,\infty)$ or $\alpha=0$ and $p,q\in[1,\infty]$. We define*

$$\mathbb{H}^{\alpha,p}_{q,loc}:=\Big\{f\in\mathscr{D}':f\eta\in\mathbb{H}^{\alpha,p}_q,\ \forall\eta\in C^\infty_c(\mathbb{R}^{d+1})\Big\},$$

and the Banach space: for fixed $r>0$,

$$\widetilde{\mathbb{H}}^{\alpha,p}_q:=\Big\{f\in\mathbb{H}^{\alpha,p}_{q,loc}:|\!|\!|f|\!|\!|_{\widetilde{\mathbb{H}}^{\alpha,p}_q}:=\sup_{s,z}\|f\chi^{s,z}_r\|_{\mathbb{H}^{\alpha,p}_q}<\infty\Big\}. \tag{2.4}$$

For simplicity, we shall write

$$\mathbb{L}^\infty_{loc}:=\mathbb{H}^{0,\infty}_{\infty,loc},\ \ \mathscr{V}_{loc}:=\mathbb{H}^{0,2}_{\infty,loc}\cap\mathbb{H}^{1,2}_{2,loc},\ \ \widetilde{\mathbb{L}}^p_q:=\widetilde{\mathbb{H}}^{0,p}_q,\ \ |\!|\!|f|\!|\!|_{\widetilde{\mathbb{L}}^p_q}:=|\!|\!|f|\!|\!|_{\widetilde{\mathbb{H}}^{0,p}_q}. \tag{2.5}$$

Moreover, we also introduce the localized energy space

$$\widetilde{\mathscr{V}}:=\Big\{f\in\widetilde{\mathbb{L}}^2_\infty\cap\widetilde{\mathbb{H}}^{1,2}_2:|\!|\!|f|\!|\!|_{\widetilde{\mathscr{V}}}:=|\!|\!|f|\!|\!|_{\widetilde{\mathbb{L}}^2_\infty}+|\!|\!|\nabla_xf|\!|\!|_{\widetilde{\mathbb{L}}^2_2}<\infty\Big\}. \tag{2.6}$$

Remark 2.1. *By the very definition, one sees that the definition of $\widetilde{\mathbb{H}}^{\alpha,p}_q$ does not depend on the choice of $r > 0$ (see (i) of Proposition 4.1 in the appendix).*

Moreover, we also introduce the following index set that will be used to state the conditions on the coefficients b, f throughout the paper.

Definition 3. *For $d \geqslant 2$, define*

$$\mathscr{I}_d := \left\{(\alpha, p, q) \in [0,1] \times (1,\infty) \times (1,\infty) : \tfrac{d}{p} + \tfrac{2}{q} < 2 - \alpha\right\}.$$

For given $(\alpha, p, q) \in \mathscr{I}_d$, we define $r, s \in [2,\infty)$ by relations

$$\tfrac{1}{(2-\alpha)p} + \tfrac{1}{r} = \tfrac{1}{2}, \quad \tfrac{1}{(2-\alpha)q} + \tfrac{1}{s} = \tfrac{1}{2}, \tag{2.7}$$

which implies that

$$\tfrac{d}{p} + \tfrac{2}{q} < 2 - \alpha \Leftrightarrow \tfrac{d}{r} + \tfrac{2}{s} > \tfrac{d}{2}. \tag{2.8}$$

In what follows we shall also use the following mollifiers: for $\varepsilon \in (0,1)$ and $n \in \mathbb{N}$,

$$\rho_\varepsilon(x) := \varepsilon^{-d}\rho(\varepsilon^{-1}x), \quad \rho_n(x) := \rho_{1/n}(x),$$

where $0 \leqslant \rho \in C_c^\infty(B_1)$ with $\int \rho = 1$.

2.1. **Localization estimates.** In this subsection we prove an important localization lemma for later use, which is a consequence of Hölder's inequality and the following Gagliado-Nirenberge's interpolation inequality (see [11, Corollary 1.5]): for any $u \in \dot{H}^1_p \cap L^q$,

$$\|\Delta^{\alpha/2}u\|_r \leqslant C\|\nabla u\|_p^\theta \|u\|_q^{1-\theta}, \tag{2.9}$$

where $\alpha \in [0,1]$, $\theta \in [\alpha, 1]$ and $p, q, r \in (1,\infty)$ satisfy

$$\frac{1}{r} = \frac{\alpha}{d} + \theta\left(\frac{1}{p} - \frac{1}{d}\right) + \frac{1-\theta}{q}.$$

Notice that if u has support in B_2 and $r\alpha < d$, then we also have

$$\|u\|_{\alpha,r} \lesssim \|u\|_r + \|\Delta^{\alpha/2}u\|_r \leqslant C\|u\|_{rd/(d-r\alpha)} + C\|\nabla u\|_p^\theta \|u\|_q^{1-\theta} \leqslant C\|\nabla u\|_p^\theta \|u\|_q^{1-\theta}, \tag{2.10}$$

where the last inequality is due to (2.9) with $\alpha = 0$. Here $C = C(\alpha, d, p, q, r)$.

First of all, we have the following interpolation estimates.

Lemma 2.1. *Let $r, s \geqslant 2$ with $\frac{d}{r} + \frac{2}{s} \geqslant \frac{d}{2}$. If $\theta := \frac{d}{2} - \frac{d}{r} \in [0,1)$, then for any $\varepsilon \in (0,1)$, there is a constant $C_\varepsilon = C_\varepsilon(d, r, s) > 0$ such that for all $f \in \mathbb{H}^{1,2}_2 \cap \mathbb{L}^2_{2(1-\theta)s/(2-s\theta)}$,*

$$\|f\|_{\mathbb{L}^r_s} \leqslant \varepsilon\|\nabla f\|_{\mathbb{L}^2_2} + C_\varepsilon\|f\|_{\mathbb{L}^2_{2(1-\theta)s/(2-s\theta)}}.$$

In particular, if $\mathrm{supp} f \subset Q_2$, then for some $C = C(d, r, s) > 0$,

$$\|f\|_{\mathbb{L}^r_s} \leqslant C\|f\|_{\mathscr{V}}, \ f \in \mathscr{V}. \tag{2.11}$$

Proof. For $r \in [2,\infty)$ if $d = 2$ or $r \in [2, 2d/(d-2)]$ if $d \geqslant 3$, by (2.9) we have

$$\|f\|_r \leqslant C\|\nabla f\|_2^\theta \|f\|_2^{1-\theta}.$$

Since $s\theta \leqslant 2$, by Hölder's inequality we further have

$$\|f\|_{\mathbb{L}^r_s} \leqslant C\|\nabla f\|^\theta_{\mathbb{L}^2_2}\|f\|^{1-\theta}_{\mathbb{L}^2_{2(1-\theta)s/(2-s\theta)}},$$

which gives the desired embedding by Young's inequality. □

Lemma 2.2. *Let $Q = I \times D \subset \mathbb{R} \times \mathbb{R}^d$ be a bounded domain. For any $p, q, r, s \in [1, \infty]$, there is a constant $C > 0$ only depending on Q, p, q, r, s such that for any $A \subset Q$,*

$$\|\mathbf{1}_A\|_{\mathbb{L}^p_q} \leqslant C\|\mathbf{1}_A\|^{(r/p)\wedge(s/q)}_{\mathbb{L}^r_s}.$$

Proof. Define

$$A_t := \int_D \mathbf{1}_A(t, x)\mathrm{d}x.$$

If $r/p \leqslant s/q$, then by Hölder's inequality,

$$\|\mathbf{1}_A\|_{\mathbb{L}^p_q} = \left(\int_I A_t^{q/p}\mathrm{d}t\right)^{1/q} \leqslant C\left(\int_I A_t^{s/r}\mathrm{d}t\right)^{r/(sp)} = C\|\mathbf{1}_A\|^{r/p}_{\mathbb{L}^r_s}.$$

If $r/p > s/q$, then by Hölder's inequality,

$$A_t^{q/p} \leqslant CA_t^{s/r} \Rightarrow \|\mathbf{1}_A\|_{\mathbb{L}^p_q} = \left(\int_I A_t^{q/p}\mathrm{d}t\right)^{1/q} \leqslant C\left(\int_I A_t^{s/r}\mathrm{d}t\right)^{1/q} = C\|\mathbf{1}_A\|^{s/q}_{\mathbb{L}^r_s}.$$

The proof is complete. □

The following lemma is the key localization result.

Lemma 2.3. *Let $(\alpha, p, q) \in \mathscr{I}_d$ and $r, s \in [2, \infty)$ be defined by (2.7). Let χ_2 be the cutoff function defined by (2.3). For any $\varepsilon \in (0, 1)$, there is a constant $C_\varepsilon = C_\varepsilon(d, \alpha, p, q) > 0$ such that for any $c, b, f \in \mathbb{H}^{-\alpha,p}_{q,loc}$ and $\eta \in C^\infty_c(Q_2; [0, 1])$, $w \in \mathscr{V}_{loc}$,*

$$|\langle\!\langle c, \eta^2 w^2\rangle\!\rangle| \leqslant \varepsilon\|\eta w\|^2_{\mathscr{V}} + C_\varepsilon\|c\chi_2\|^{2/(2-\alpha)}_{\mathbb{H}^{-\alpha,p}_q}\|\eta w\|^2_{\mathbb{L}^r_s}, \tag{2.12}$$

$$|\langle\!\langle b, \nabla\eta^2 w^2\rangle\!\rangle| \leqslant \varepsilon\|\eta w\|^2_{\mathscr{V}} + C_\varepsilon\Big(1 + \|b\chi_2\|^2_{\mathbb{H}^{-\alpha,p}_q}\Big)\Big(1 + \|\nabla^2\eta\|_\infty + \|\nabla\eta\|^2_\infty\Big)\|w\mathbf{1}_{\eta\neq 0}\|^2_{\mathbb{L}^r_s}, \tag{2.13}$$

$$|\langle\!\langle f, \eta^2 w\rangle\!\rangle| \leqslant \varepsilon\|\eta w\|^2_{\mathscr{V}} + C_\varepsilon\|f\chi_2\|^2_{\mathbb{H}^{-\alpha,p}_q}\Big(1 + \|\nabla\eta\|^2_\infty\Big)\|\mathbf{1}_{\eta w\neq 0}\|^2_{\mathbb{L}^r_s}. \tag{2.14}$$

Proof. Since $\alpha \in [0, 1]$, by relation (2.7), one sees that

$$\frac{1}{p'} := 1 - \frac{1}{p} = \frac{\alpha}{d} + \alpha\left(\frac{r+2}{2r} - \frac{1}{d}\right) + \frac{2(1-\alpha)}{r} > \frac{\alpha}{d}.$$

Thus for any $g \in H^{-\alpha,p}$ and $h \in H^{1,2r/(r+2)} \subset H^{\alpha,p'}$ with support in B_2, we have

$$\langle g, h\rangle \leqslant \|g\|_{-\alpha,p}\|h\|_{\alpha,p'} \overset{(2.10)}{\lesssim} \|g\|_{-\alpha,p}\|h\|^{1-\alpha}_{r/2}\|\nabla h\|^\alpha_{2r/(r+2)}. \tag{2.15}$$

By mollifying approximation, below we assume $c, b, f \in C^\infty$ and fix $\eta \in C^\infty_c(Q_2; [0, 1])$.

(i) Since $\chi_2|_{Q_2} \equiv 1$ and $\eta|_{Q_2^c} = 0$, by (2.15) with $g = c\chi_2$ and $h = \eta^2 w^2$, we have

$$\begin{aligned}\langle c, \eta^2 w^2\rangle = \langle c\chi_2, \eta^2 w^2\rangle &\lesssim \|c\chi_2\|_{-\alpha,p}\|\eta^2 w^2\|^{1-\alpha}_{r/2}\|\nabla(\eta^2 w^2)\|^\alpha_{2r/(r+2)}\\ &\lesssim \|c\chi_2\|_{-\alpha,p}\|\eta w\|^{2(1-\alpha)}_r\|(\eta w)\nabla(\eta w)\|^\alpha_{2r/(r+2)}\\ &\lesssim \|c\chi_2\|_{-\alpha,p}\|\eta w\|^{2-\alpha}_r\|\nabla(\eta w)\|^\alpha_2,\end{aligned}$$

where we drop the time variable t and the last step is due to Hölder's inequality. Integrating both sides in the time variable, and due to $\frac{2-\alpha}{s} + \frac{\alpha}{2} + \frac{1}{q} = 1$, by Hölder's inequality we get

$$|\langle\!\langle c, \eta^2 w^2\rangle\!\rangle| \lesssim \|c\chi_2\|_{\mathbb{H}^{-\alpha,p}_q}\|\eta w\|^{2-\alpha}_{\mathbb{L}^r_s}\|\nabla(\eta w)\|^\alpha_{\mathbb{L}^2_2},$$

which gives (2.12) by Young's inequality.

(ii) By (2.15) with $g = b\chi_2$ and $h = \nabla\eta^2 w^2$, we have

$$\langle b, \nabla\eta^2 w^2\rangle = \langle b\chi_2, \nabla\eta^2 w^2\rangle \lesssim \|b\chi_2\|_{-\alpha,p}\|\nabla\eta^2 w^2\|_{r/2}^{1-\alpha}\|\nabla(\nabla\eta^2 w^2)\|_{2r/(r+2)}^{\alpha}.$$

Notice that

$$\|\nabla\eta^2 w^2\|_{r/2} \leqslant 2\|\nabla\eta\|_\infty \|w\mathbf{1}_{\eta\neq 0}\|_r^2,$$

and by Hölder's inequality,

$$\begin{aligned}\|\nabla(\nabla\eta^2 w^2)\|_{2r/(r+2)} &\leqslant \|\nabla^2\eta^2 w^2\|_{2r/(r+2)} + \|\nabla\eta^2\nabla w^2\|_{2r/(r+2)}\\ &\leqslant \|\nabla^2\eta^2\|_{2r/(r-2)}\|w\mathbf{1}_{\eta\neq 0}\|_r^2 + 4\|w\nabla\eta\|_r \cdot \|\eta\nabla w\|_2.\end{aligned}$$

Hence,

$$\begin{aligned}\langle b, \nabla\eta^2 w^2\rangle &\lesssim \|b\chi_2\|_{-\alpha,p}\|\nabla\eta\|_\infty^{1-\alpha}\|w\mathbf{1}_{\eta\neq 0}\|_r^{2(1-\alpha)}\Big(\|\nabla^2\eta^2\|_{2r/(r-2)}^{\alpha}\|w\mathbf{1}_{\eta\neq 0}\|_r^{2\alpha} + \|w\nabla\eta\|_r^{\alpha}\cdot\|\eta\nabla w\|_2^{\alpha}\Big)\\ &\lesssim \|b\chi_2\|_{-\alpha,p}\Big((\|\nabla\eta\|_\infty^{1-\alpha}\|\nabla^2\eta\|_\infty^{\alpha} + \|\nabla\eta\|_\infty^{1+\alpha})\|w\mathbf{1}_{\eta\neq 0}\|_r^2 + \|\nabla\eta\|_\infty\|w\mathbf{1}_{\eta\neq 0}\|_r^{2-\alpha}\|\eta\nabla w\|_2^{\alpha}\Big)\\ &\lesssim \|b\chi_2\|_{-\alpha,p}\Big((1 + \|\nabla^2\eta\|_\infty + \|\nabla\eta\|_\infty^2)\|w\mathbf{1}_{\eta\neq 0}\|_r^2 + \|\nabla\eta\|_\infty\|w\mathbf{1}_{\eta\neq 0}\|_r^{2-\alpha}\|\nabla(\eta w)\|_2^{\alpha}\Big),\end{aligned}$$

and by Hölder's inequality and due to $\frac{1}{q} + \frac{2-\alpha}{s} + \frac{\alpha}{2} = 1$,

$$\begin{aligned}|\langle\!\langle b, \nabla\eta^2 w^2\rangle\!\rangle| \lesssim\ & \|b\chi_2\|_{\mathbb{H}_q^{-\alpha,p}}(1 + \|\nabla^2\eta\|_\infty + \|\nabla\eta\|_\infty^2)\|w\mathbf{1}_{\eta\neq 0}\|_{\mathbb{L}^r_{2q/(q-1)}}^2\\ &+ \|b\chi_2\|_{\mathbb{H}_q^{-\alpha,p}}\|\nabla\eta\|_\infty\|w\mathbf{1}_{\eta\neq 0}\|_{\mathbb{L}^r_s}^{2-\alpha}\|\nabla(\eta w)\|_{\mathbb{L}^2_2}^{\alpha}.\end{aligned}$$

The desired estimate (2.13) follows by Young's inequality and $2q/(q-1) \leqslant s$.

(iii) By (2.15) with $g = f\eta$ and $h = \eta w$, we have

$$\langle f\eta, \eta w\rangle \lesssim \|f\eta\|_{-\alpha,p}\|\eta w\|_{r/2}^{1-\alpha}\|\nabla(\eta w)\|_{2r/(r+2)}^{\alpha}.$$

Since $\nabla(\eta w) = \nabla(\eta w)^+ - \nabla(\eta w)^- = 0$ a.e. on $\{\eta w = 0\}$ (cf. [10, Lemma 7.6]), we have

$$\nabla(\eta w) = \nabla(\eta w)\mathbf{1}_{\eta w\neq 0}, \quad a.e. \tag{2.16}$$

Thus, by Hölder's inequality, we further have

$$\begin{aligned}|\langle f\eta, \eta w\rangle| &\lesssim \|f\eta\|_{-\alpha,p}\|\eta w\|_{r/2}^{1-\alpha}\|\nabla(\eta w)\|_2^{\alpha}\|\mathbf{1}_{\eta w\neq 0}\|_r^{\alpha}\\ &\lesssim \|f\eta\|_{-\alpha,p}\|\eta w\|_r^{1-\alpha}\|\nabla(\eta w)\|_2^{\alpha}\|\mathbf{1}_{\eta w\neq 0}\|_r,\end{aligned}$$

and due to $\frac{1}{q} + \frac{1-\alpha}{s} + \frac{\alpha}{2} + \frac{1}{s} = 1$ and $\frac{d}{r} + \frac{2}{s} > \frac{d}{2}$,

$$|\langle\!\langle f, \eta^2 w\rangle\!\rangle| \lesssim \|f\eta\|_{\mathbb{H}_q^{-\alpha,p}}\|\eta w\|_{\mathbb{L}^r_s}^{1-\alpha}\|\nabla(\eta w)\|_{\mathbb{L}^2_2}^{\alpha}\|\mathbf{1}_{\eta w\neq 0}\|_{\mathbb{L}^r_s} \overset{(2.11)}{\lesssim} \|f\eta\|_{\mathbb{H}_q^{-\alpha,p}}\|\eta w\|_{\mathscr{V}}\|\mathbf{1}_{\eta w\neq 0}\|_{\mathbb{L}^r_s}. \tag{2.17}$$

Notice that by $\eta = \chi_2\eta$ and (4.2) in the appendix,

$$\|f\eta\|_{\mathbb{H}_q^{-\alpha,p}} \lesssim \|f\chi_2\|_{\mathbb{H}_q^{-\alpha,p}}(1 + \|\nabla\eta\|_\infty).$$

Substituting this into (2.17) and by Young's inequality, we obtain (2.14). $\square$

2.2. **Local energy estimate.** Throughout this paper we shall always assume

$$b \in \mathbb{L}^2_{2,loc}, \quad f \in \mathscr{D}',$$

and consider the following parabolic PDE in $\mathbb{R}^{d+1}$:

$$\partial_t u = \Delta u + b \cdot \nabla u + f. \tag{2.18}$$

Definition 4. *A function $u \in \mathscr{V}_{loc} \cap \mathbb{L}^\infty_{loc}$ is called a weak solution (subsolution or supersolution) of PDE (2.18) with coefficients (b, f) if for any nonnegative smooth function $\varphi \in C^\infty_c(\mathbb{R}^{d+1})$,*

$$-\langle\!\langle u, \partial_t \varphi \rangle\!\rangle = (\leqslant \text{ or } \geqslant) - \langle\!\langle \nabla u, \nabla \varphi \rangle\!\rangle + \langle\!\langle b \cdot \nabla u, \varphi \rangle\!\rangle + \langle\!\langle f, \varphi \rangle\!\rangle, \tag{2.19}$$

where $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is the dual pair between $\mathscr{D}'$ and $\mathscr{D}$ (see also (2.1)).

Now we prove the following local energy estimate.

Lemma 2.4 (Energy estimate)**.** *Suppose that for some $(\alpha_i, p_i, q_i) \in \mathscr{I}_d$, $i = 1, 2, 3$,*

$$b \in \mathbb{H}^{-\alpha_1, p_1}_{q_1, loc}, \quad -\mathrm{div} b \leqslant \Theta_b \in \mathbb{H}^{-\alpha_2, p_2}_{q_2, loc}, \quad f \in \mathbb{H}^{-\alpha_3, p_3}_{q_3, loc}.$$

Let $(r_i, s_i) \in [2, \infty)$ be defined by (2.7) in terms of p_i, q_i and $\kappa \geqslant 0$. For any weak subsolution $u \in \mathscr{V}_{loc} \cap \mathbb{L}^\infty_{loc}$ of PDE (2.18), there is a constant $C > 0$ depending only on $d, \alpha_i, p_i, q_i, i = 1, 2, 3$ and quantities

$$\|b\chi_2\|_{\mathbb{H}^{-\alpha_1, p_1}_{q_1}}, \quad \|\Theta_b \chi_2\|_{\mathbb{H}^{-\alpha_2, p_2}_{q_2}},$$

where χ_2 is defined by (2.3), such that for $w := (u - \kappa)^+$ and any $\eta \in C^\infty_c(Q_2; [0, 1])$ and $t \geqslant 0$,

$$\|\eta w \mathcal{I}_t\|_{\mathscr{V}} \leqslant C \Xi_\eta^{1/2} \left(\|w \mathbf{1}_{\eta \neq 0} \mathcal{I}_t\|_{\mathbb{L}^{r_1}_{s_1}} + \|w \eta \mathcal{I}_t\|_{\mathbb{L}^{r_2}_{s_2}} + \|f \chi_2 \mathcal{I}_t\|_{\mathbb{H}^{-\alpha_3, p_3}_{q_3}} \|\mathbf{1}_{w\eta \neq 0}\|_{\mathbb{L}^{r_3}_{s_3}} \right), \tag{2.20}$$

where $\mathcal{I}_t(\cdot) := \mathbf{1}_{(-\infty, t]}(\cdot)$, and

$$\Xi_\eta := 1 + \|\partial_t \eta\|_\infty + \|\nabla \eta\|^2_\infty + \|\nabla^2 \eta\|_\infty. \tag{2.21}$$

Proof. We divide the proof into three steps.

(i) Let $\eta \in C^\infty_c(Q_2; [0, 1])$, $\kappa \geqslant 0$ and $w := (u - \kappa)^+$. In this step we show that for Lebesgue almost all $t \in \mathbb{R}$,

$$\int_{\mathbb{R}^d} (\eta w)^2(t) \leqslant \int_{-\infty}^t \langle \partial_t \eta^2, w^2 \rangle - 2 \int_{-\infty}^t \langle \nabla u, \nabla(\eta^2 w) \rangle + 2 \int_{-\infty}^t \langle b \cdot \nabla u, \eta^2 w \rangle + 2 \int_{-\infty}^t \langle f, \eta^2 w \rangle. \tag{2.22}$$

Since we want to take the test function $\varphi = w\eta^2$ in (2.19), and $\partial_t u$ only makes sense in the distributional sense, we shall first approximate u by its Steklov's mean:

$$S_h u(t, x) := \frac{1}{h} \int_0^h u(t + s, x) \mathrm{d}s = \frac{1}{h} \int_t^{t+h} u(s, x) \mathrm{d}s, \quad h \in (0, 1). \tag{2.23}$$

Let $u_h := S_h u$. By Definition 2.19 and noticing that $\partial_t u_h \in \mathbb{L}^2_{2,loc}$, one sees that for any nonnegative $\varphi \in C^\infty_c(\mathbb{R}^{d+1})$,

$$\langle\!\langle \partial_t u_h, \varphi \rangle\!\rangle = -\langle\!\langle u_h, \partial_t \varphi \rangle\!\rangle \leqslant -\langle\!\langle \nabla u_h, \nabla \varphi \rangle\!\rangle + \langle\!\langle S_h(b \cdot \nabla u), \varphi \rangle\!\rangle + \langle\!\langle f_h, \varphi \rangle\!\rangle. \tag{2.24}$$

By standard smoothing approximation, it is easy to see that (2.24) still holds for any nonnegative $\varphi \in \mathscr{V}_{loc} \cap \mathbb{L}^\infty_{loc}$ with compact support in Q_2. Now fix $t \in \mathbb{R}$ and define

$$\zeta_{t,\varepsilon}(s) = \mathbf{1}_{(-\infty, t]} + (1 - \varepsilon^{-1}(s - t)) \mathbf{1}_{(t, t+\varepsilon]}(s), \ \varepsilon \in (0, 1).$$

Let $w_h := (u_h - k)^+$. Since

$$2\langle \partial_t u_h, w_h \eta^2 \zeta_{t,\varepsilon} \rangle = 2\langle \partial_t w_h, w_h \eta^2 \zeta_{t,\varepsilon} \rangle = \int_{\mathbb{R}^d} \partial_t (w_h^2 \eta^2 \zeta_{t,\varepsilon}) - \int_{\mathbb{R}^d} w_h^2 \eta^2 \zeta'_{t,\varepsilon} - \int_{\mathbb{R}^d} w_h^2 (\partial_t \eta^2 \zeta_{t,\varepsilon}),$$

by (2.24) with $\varphi = w_h\eta^2\zeta_{t,\varepsilon} \in \mathscr{V}_{loc} \cap \mathbb{L}^\infty_{loc}$ and $\int_{\mathbb{R}^{d+1}} \partial_t(w_h^2\eta^2\zeta_{t,\varepsilon}) = 0$, we obtain

$$-\int_{\mathbb{R}^{d+1}} \eta^2 w_h^2 \zeta'_{t,\varepsilon} \leqslant \int_{\mathbb{R}^{d+1}} w_h^2(\partial_t \eta^2 \zeta_{t,\varepsilon}) - 2\langle\!\langle \nabla u_h, \nabla(w_h\eta^2\zeta_{t,\varepsilon})\rangle\!\rangle \\ + 2\langle\!\langle S_h(b\cdot\nabla u), w_h\eta^2\zeta_{t,\varepsilon}\rangle\!\rangle + 2\langle\!\langle f_h, w_h\eta^2\zeta_{t,\varepsilon}\rangle\!\rangle.$$

Noticing that $u, w \in \mathscr{V}_{loc} \cap \mathbb{L}^\infty_{loc}$ and $b \in \mathbb{L}^2_{2,loc}$, by letting $h \downarrow 0$ and the dominated convergence theorem, we obtain

$$-\int_{\mathbb{R}^{d+1}} (\eta w)^2 \zeta'_{t,\varepsilon} \leqslant \int_{\mathbb{R}^{d+1}} w^2(\partial_t \eta^2 \zeta_{t,\varepsilon}) - 2\langle\!\langle \nabla u, \nabla(w\eta^2\zeta_{t,\varepsilon})\rangle\!\rangle \\ + 2\langle\!\langle b\cdot\nabla u, w\eta^2\zeta_{t,\varepsilon}\rangle\!\rangle + 2\langle\!\langle f, w\eta^2\zeta_{t,\varepsilon}\rangle\!\rangle.$$

Since $\lim_{\varepsilon\downarrow 0} \zeta_{t,\varepsilon}(s) = \mathbf{1}_{(-\infty,t]}(s)$ for each $s \in \mathbb{R}$, the right hand side of the above inequality converges to the right hand side of (2.22) as $\varepsilon \downarrow 0$. On the other hand, by the Lebesgue differential theorem, we also have

$$-\int_{\mathbb{R}^{d+1}} (\eta w)^2 \zeta'_{t,\varepsilon} = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \mathrm{d}s \int_{\mathbb{R}^d} (\eta w)^2(s) \overset{\varepsilon\downarrow 0}{\to} \int_{\mathbb{R}^d} (\eta w)^2(t), \quad a.e.$$

Thus, we obtain (2.22).

(ii) In this step we use (2.22) to show that for Lebesgue almost all $t \in \mathbb{R}$,

$$\int_{\mathbb{R}^d} (\eta w)^2(t) + 2\int_{-\infty}^t \int_{\mathbb{R}^d} |\eta \nabla w|^2 \leqslant \int_{-\infty}^t \langle |\partial_t \eta^2 + \Delta\eta^2|, w^2\rangle \\ + \left|\int_{-\infty}^t \langle \Theta_b, \eta^2 w^2\rangle\right| + \left|\int_{-\infty}^t \langle b, \nabla\eta^2 w^2\rangle\right| + 2\left|\int_{-\infty}^t \langle f, \eta^2 w\rangle\right|. \tag{2.25}$$

Noticing that

$$\nabla u \cdot \nabla w = |\nabla w|^2,\ (\nabla u) w = (\nabla w) w = \nabla w^2/2, \tag{2.26}$$

by the integration by parts formula, we have

$$2\langle \nabla u, \nabla(\eta^2 w)\rangle = 2\int_{\mathbb{R}^d} \eta^2 |\nabla w|^2 - \int_{\mathbb{R}^d} w^2 \cdot \Delta\eta^2. \tag{2.27}$$

Moreover, let w_ε be the mollifying approximation of w. By $-\mathrm{div} b \leqslant \Theta_b$, we also have

$$\begin{aligned} 2\langle b\cdot\nabla u, \eta^2 w\rangle &\overset{(2.26)}{=} \langle b\cdot\nabla w^2, \eta^2\rangle = \lim_{\varepsilon\to 0}\langle b\cdot\nabla w_\varepsilon^2, \eta^2\rangle \\ &= \lim_{\varepsilon\to 0}\Big(-\langle \mathrm{div} b, \eta^2 w_\varepsilon^2\rangle - \langle b, \nabla\eta^2 w_\varepsilon^2\rangle\Big) \\ &\leqslant \lim_{\varepsilon\to 0}\Big(\langle \Theta_b, \eta^2 w_\varepsilon^2\rangle - \langle b, \nabla\eta^2 w_\varepsilon^2\rangle\Big) \\ &= \langle \Theta_b, \eta^2 w^2\rangle - \langle b, \nabla\eta^2 w^2\rangle. \end{aligned} \tag{2.28}$$

Substituting (2.27) and (2.28) into (2.22), we obtain (2.25).

(iii) By (2.25) and definition (2.6), we have

$$\|\eta w \mathcal{I}_t\|^2_{\mathscr{V}} \leqslant \int_{-\infty}^t \langle |\partial_t\eta^2 + \Delta\eta^2| + 2|\nabla\eta^2|, w^2\rangle + \sup_{s\leqslant t}\left|\int_{-\infty}^s \langle \Theta_b, \eta^2 w^2\rangle\right| \\ + \sup_{s\leqslant t}\left|\int_{-\infty}^s \langle b, \nabla\eta^2 w^2\rangle\right| + 2\sup_{s\leqslant t}\left|\int_{-\infty}^s \langle f, \eta^2 w\rangle\right| =: \sum_{i=1}^4 I_i.$$

For I_1, noticing that $r_1, s_1 \geqslant 2$, we have

$$I_1 \leqslant \Big(2\|\partial_t\eta\|_\infty + 2\|\nabla\eta\|_\infty^2 + \|\Delta\eta\|_\infty + 4\|\nabla\eta\|_\infty\Big)\|w\mathbf{1}_{\eta\neq 0}\mathcal{I}_t\|^2_{\mathbb{L}^2_2} \leqslant C\Xi_\eta\|w\mathbf{1}_{\eta\neq 0}\mathcal{I}_t\|^2_{\mathbb{L}^{r_1}_{s_1}}.$$

For I_2, I_3 and I_4, by (2.12), (2.13) and (2.14), we have

$$I_2 \leqslant \varepsilon\|\eta w\mathcal{I}_t\|^2_{\mathscr{V}} + C_\varepsilon\|\Theta_b\chi_2\|^{2/(2-\alpha_2)}_{\mathbb{H}^{-\alpha_2,p_2}_{q_2}}\|w\eta\mathcal{I}_t\|^2_{\mathbb{L}^{r_2}_{s_2}},$$

and

$$I_3 \leqslant \varepsilon\|\eta w\mathcal{I}_t\|^2_{\mathscr{V}} + C_\varepsilon\Big(1 + \|b\chi_2\|^2_{\mathbb{H}^{-\alpha_1,p_1}_{q_1}}\Big)\Big(1 + \|\nabla^2\eta\|_\infty + \|\nabla\eta\|^2_\infty\Big)\|w\mathbf{1}_{\eta\neq 0}\mathcal{I}_t\|^2_{\mathbb{L}^{r_1}_{s_1}},$$

$$I_4 \leqslant \varepsilon\|\eta w\mathcal{I}_t\|^2_{\mathscr{V}} + C_\varepsilon\|f\chi_2\mathcal{I}_t\|^2_{\mathbb{H}^{-\alpha_3,p_3}_{q_3}}(1 + \|\nabla\eta\|^2_\infty)\|\mathbf{1}_{\eta w\neq 0}\|^2_{\mathbb{L}^{r_3}_{s_3}}.$$

Combining the above calculations and letting ε be small enough, we obtain

$$\|\eta w\mathcal{I}_t\|^2_{\mathscr{V}} \lesssim \Xi_\eta\|w\mathbf{1}_{\eta\neq 0}\mathcal{I}_t\|^2_{\mathbb{L}^{r_1}_{s_1}} + \|w\eta\mathcal{I}_t\|^2_{\mathbb{L}^{r_2}_{s_2}} + (1 + \|\nabla\eta\|^2_\infty)\|f\chi_2\mathcal{I}_t\|^2_{\mathbb{H}^{-\alpha_3,p_3}_{q_3}}\|\mathbf{1}_{\eta w\neq 0}\|^2_{\mathbb{L}^{r_3}_{s_3}},$$

where Ξ_η is define by (2.21). From this, we derive (2.20). □

Remark 2.2. *If $\alpha_1 = 0$ and $\frac{d}{p_1} + \frac{2}{q_1} = 1$ or $b(t,x) = b(x) \in L^d_{loc}(\mathbb{R}^d)$, then we can remove the assumption on the divergence of b. In fact, in this case, we can give a direct treatment for the term b in (2.22) as follows: For any $\varepsilon > 0$, let*

$$b_\varepsilon(t,x) := b(t,\cdot) * \rho_\varepsilon(x), \ \ \bar{b}_\varepsilon(t,x) := b(t,x) - b_\varepsilon(t,x).$$

Let $\frac{1}{r_1} + \frac{1}{p_1} = \frac{1}{s_1} + \frac{1}{q_1} = \frac{1}{2}$, which satisfy $\frac{d}{r_1} + \frac{2}{s_1} = \frac{d}{2}$ due to $\frac{d}{p_1} + \frac{2}{q_1} = 1$. Since $\chi_2\eta = \eta$, by (2.26), Hölder's inequality and Lemma 2.1, we have

$$\begin{aligned}\int_{-\infty}^t |\langle b\cdot\nabla u, \eta^2 w\rangle| &\leqslant \int_{-\infty}^t |\langle \bar{b}_\varepsilon\cdot\nabla u, \eta^2 w\rangle| + \int_{-\infty}^t |\langle b_\varepsilon\cdot\nabla u, \eta^2 w\rangle| \\ &\leqslant \|\bar{b}_\varepsilon\chi_2\|_{\mathbb{L}^{p_1}_{q_1}}\|\eta\nabla w\mathcal{I}_t\|_{\mathbb{L}^2_2}\|\eta w\mathcal{I}_t\|_{\mathbb{L}^{r_1}_{s_1}} + \|b_\varepsilon\chi_2\|_\infty\|\eta\nabla w\mathcal{I}_t\|_{\mathbb{L}^2_2}\|\eta w\mathcal{I}_t\|_{\mathbb{L}^2_2} \\ &\leqslant c_\varepsilon\|\eta w\mathcal{I}_t\|^2_{\mathscr{V}} + C_\varepsilon(1 + \|\nabla\eta\|_\infty)\|w\mathbf{1}_{\eta\neq 0}\mathcal{I}_t\|^2_{\mathbb{L}^2_2},\end{aligned}$$

where $\lim_{\varepsilon\to 0} c_\varepsilon = 0$ and $\lim_{\varepsilon\to 0} C_\varepsilon = \infty$. Using this estimate to replace the corresponding estimate about b and taking ε small enough, we still have (2.20). Here the reason that for $p = d$ we assume b being time-independent is that in general

$$\lim_{\varepsilon\to 0}\|\bar{b}_\varepsilon\chi_2\|_{\mathbb{L}^d_\infty} \neq 0 \text{ for } b \in L^\infty_{loc}(L^d_{loc}), \text{ but } \lim_{\varepsilon\to 0}\|\bar{b}_\varepsilon\chi_2\|_d = 0 \text{ for } b \in L^d_{loc}.$$

2.3. **Maximum principle.** The following De Giorgi's iteration lemma is well known [12].

Lemma 2.5. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative numbers. Suppose that for some $C_0, \lambda > 1$ and $\varepsilon > 0$,*

$$a_{n+1} \leqslant C_0\lambda^n a_n^{1+\varepsilon}, \ \ n = 1, 2, \cdots.$$

If $a_1 \leqslant C_0^{-1/\varepsilon}\lambda^{-1/\varepsilon^2}$, then

$$\lim_{n\to\infty} a_n = 0.$$

Now we can show the following local maximum principle for PDE (2.18).

Theorem 2.1 (Local maximum estimate)**.** *Suppose that for some $(\alpha_i, p_i, q_i) \in \mathscr{I}_d$, $i = 1, 2, 3$,*

$$b \in \mathbb{H}^{-\alpha_1,p_1}_{q_1;loc}, \ \ -\mathrm{div} b \leqslant \Theta_b \in \mathbb{H}^{-\alpha_2,p_2}_{q_2;loc}, \ \ f \in \mathbb{H}^{-\alpha_3,p_3}_{q_3;loc}.$$

For any weak subsolution $u \in \mathscr{V}_{loc} \cap \mathbb{L}^\infty_{loc}$ of PDE (2.18), there is a constant $C > 0$ depending only on $d, \alpha_i, p_i, q_i, i = 1, 2, 3$ and the quantities

$$\|b\chi_2\|_{\mathbb{H}^{-\alpha_1,p_1}_{q_1}}, \quad \|\Theta_b\chi_2\|_{\mathbb{H}^{-\alpha_2,p_2}_{q_2}},$$

where χ_2 is defined by (2.3), such that

$$\|u^+\mathbf{1}_{Q_1}\|_\infty \leqslant C\left(\|u^+\chi_2\|_{\mathscr{V}} + \|f\chi_2\|_{\mathbb{H}^{-\alpha_3,p_3}_{q_3}}\right). \tag{2.29}$$

Proof. Let $\kappa > 0$, which will be determined below. For $n \in \mathbb{N}$, define

$$t_n := 4 \cdot (4^{-1} + 3 \cdot 4^{-n}), \quad \lambda_n := 1 + 2^{1-n}, \quad \kappa_n := \kappa\left(1 - 2^{1-n}\right)$$

and

$$\Gamma_n := (-t_n, t_n) \times B_{\lambda_n} \downarrow [-1, 1] \times \bar{B}_1 = \bar{Q}_1.$$

Let $\zeta^{\mathrm{t}}_n \in C^\infty_c((-4, 4); [0, 1])$ be a time-cutoff function so that for some $C > 0$ and any $n \in \mathbb{N}$,

$$\zeta^{\mathrm{t}}_n|_{(-t_{n+1},t_{n+1})} = 1, \quad \zeta^{\mathrm{t}}_n|_{(-t_n,t_n)^c} = 0, \quad |\partial_t\zeta^{\mathrm{t}}_n| \leqslant C4^n.$$

Let $\zeta^{\mathrm{x}}_n \in C^\infty_c(B_2; [0, 1])$ be a spatial-cutoff function so that for some $C > 0$ and any $n \in \mathbb{N}$,

$$\zeta^{\mathrm{x}}_n|_{B_{\lambda_{n+1}}} = 1, \quad \zeta^{\mathrm{x}}_n|_{B^c_{\lambda_n}} = 0, \quad |\nabla^j\zeta^{\mathrm{x}}_n| \leqslant C2^{jn}, \quad j = 1, 2.$$

Now let us define

$$\eta_n(t, x) := \zeta^{\mathrm{t}}_n(t) \cdot \zeta^{\mathrm{x}}_n(x).$$

Let Ξ_{η_n} be defined by (2.21). It is easy to see that for some $C > 0$ and all $n \in \mathbb{N}$,

$$\eta_n|_{\Gamma_{n+1}} = 1, \quad \eta_n|_{\Gamma^c_n} = 0, \quad \Xi_{\eta_n} \leqslant C4^n.$$

Let $(r_i, s_i) \in [2, \infty)$ be defined by (2.7) in terms of p_i, q_i, and define

$$w_n := (u - \kappa_n)^+.$$

Notice that

$$w_n|_{w_{n+1}\neq 0} = (u - \kappa_{n+1} + \kappa_{n+1} - \kappa_n)^+|_{w_{n+1}\neq 0} \geqslant \kappa_{n+1} - \kappa_n = \kappa 2^{-n}.$$

For $i = 1, 2, 3$, due to $\eta_n|_{\Gamma^c_n} = 0$, we have

$$\ell^{(i)}_n := \|w_n\mathbf{1}_{\Gamma_n}\|_{\mathbb{L}^{r_i}_{s_i}} \geqslant \|w_n\mathbf{1}_{\eta_n w_{n+1}\neq 0}\|_{\mathbb{L}^{r_i}_{s_i}} \geqslant \kappa 2^{-n}\|\mathbf{1}_{\eta_n w_{n+1}\neq 0}\|_{\mathbb{L}^{r_i}_{s_i}},$$

which means that

$$\|\mathbf{1}_{\eta_n w_{n+1}\neq 0}\|_{\mathbb{L}^{r_i}_{s_i}} \leqslant 2^n\ell^{(i)}_n/\kappa. \tag{2.30}$$

Since $\frac{2}{r_i} + \frac{d}{s_i} > \frac{d}{2}$ by (2.8), we can choose $\gamma_i, \beta_i > r_i, \theta_i, \tau_i > s_i$ so that

$$\frac{1}{\gamma_i} + \frac{1}{\beta_i} = \frac{1}{r_i}, \quad \frac{1}{\theta_i} + \frac{1}{\tau_i} = \frac{1}{s_i}, \quad \frac{d}{\gamma_i} + \frac{2}{\theta_i} \geqslant \frac{d}{2}.$$

Thus, by $\eta_n|_{\Gamma_{n+1}} = 1$, Hölder's inequality, Lemmas 2.1 and 2.2, we have

$$\begin{aligned}\ell^{(i)}_{n+1} &= \|w_{n+1}\mathbf{1}_{\Gamma_{n+1}}\|_{\mathbb{L}^{r_i}_{s_i}} \leqslant \|\eta_n w_{n+1}\|_{\mathbb{L}^{r_i}_{s_i}} \\ &\leqslant \|\eta_n w_{n+1}\|_{\mathbb{L}^{\gamma_i}_{\theta_i}}\|\mathbf{1}_{\eta_n w_{n+1}\neq 0}\|_{\mathbb{L}^{\beta_i}_{\tau_i}} \\ &\leqslant \|\eta_n w_{n+1}\|_{\mathbb{L}^{\gamma_i}_{\theta_i}}\|\mathbf{1}_{\eta_n w_{n+1}\neq 0}\|^{(s_i/\tau_i)\wedge(r_i/\beta_i)}_{\mathbb{L}^{r_i}_{s_i}} \\ &\leqslant C\|\eta_n w_{n+1}\|_{\mathscr{V}} \cdot (2^n\ell^{(i)}_n/\kappa)^{(s_i/\tau_i)\wedge(r_i/\beta_i)}.\end{aligned} \tag{2.31}$$

Notice $\Gamma_1 = Q_2$. By (2.20) with $\eta = \eta_n$ and $w = w_{n+1}$, for $\kappa \geqslant \|f\chi_2\|_{\mathbb{H}^{-\alpha_3,p_3}_{q_3}}$, we obtain

$$\begin{aligned}\|\eta_n w_{n+1}\|_{\mathscr{V}} &\lesssim 2^n\Big(\|w_{n+1}\mathbf{1}_{\Gamma_n}\|_{\mathbb{L}^{r_1}_{s_1}} + \|w_{n+1}\eta_n\|_{\mathbb{L}^{r_2}_{s_2}} + \|f\chi_2\|_{\mathbb{H}^{-\alpha_3,p_3}_{q_3}}\|\mathbf{1}_{\eta_n w_{n+1}\neq 0}\|_{\mathbb{L}^{r_3}_{s_3}}\Big)\\ &\overset{(2.30)}{\lesssim} C2^n(\ell^{(1)}_n + \ell^{(2)}_n + 2^n\ell^{(3)}_n) \lesssim 4^n(\ell^{(1)}_n + \ell^{(2)}_n + \ell^{(3)}_n),\end{aligned}\tag{2.32}$$

where we have used that $w_{n+1} \leqslant w_n$ and $\eta_n \leqslant \mathbf{1}_{\Gamma_n}$. Now we put

$$a_n := \Big(\ell^{(1)}_n + \ell^{(2)}_n + \ell^{(3)}_n\Big)/\kappa.$$

By (2.31) and (2.32), we obtain that for some $C_0, \varepsilon > 0$ and $\lambda > 1$,

$$a_{n+1} \leqslant C4^n a_n \sum_{i=1}^3 (2^n a_n)^{(s_i/\tau_i)\wedge(r_i/\beta_i)} \leqslant C_0\lambda^n a_n^{1+\varepsilon},\ \forall n \in \mathbb{N},$$

provided $\kappa \geqslant \|f\chi_2\|_{\mathbb{H}^{-\alpha_3,p_3}_{q_3}}$. Notice that by $\chi_2|_{\Gamma_1} = 1$ and Lemma 2.1,

$$a_1 \leqslant \frac{1}{\kappa}\sum_{i=1}^3 \|u^+\mathbf{1}_{\Gamma_1}\|_{\mathbb{L}^{r_i}_{s_i}} \leqslant \frac{1}{\kappa}\sum_{i=1}^3 \|u^+\chi_2\|_{\mathbb{L}^{r_i}_{s_i}} \leqslant \frac{C_1}{\kappa}\|u^+\chi_2\|_{\mathscr{V}}.$$

If $\kappa \geqslant (C_1 C_0^{1/\varepsilon}\lambda^{1/\varepsilon^2}\|u^+\chi_2\|_{\mathscr{V}}) \vee \|f\chi_2\|_{\mathbb{H}^{-\alpha_3,p_3}_{q_3}}$ so that $a_1 \leqslant C_0^{-1/\varepsilon}\lambda^{-1/\varepsilon^2}$, then by Fatou's lemma and Lemma 2.5,

$$\|(u-\kappa)^+\mathbf{1}_{Q_1}\|_{\mathbb{L}^{r_1}_{s_1}} \leqslant \liminf_{n\to\infty}\|w_n\mathbf{1}_{\Gamma_n}\|_{\mathbb{L}^{r_1}_{s_1}} = \liminf_{n\to\infty}\ell^{(1)}_n \leqslant \kappa\cdot\limsup_{n\to\infty} a_n = 0,$$

which implies that for Lebesgue almost all $(t,x) \in \mathbb{R}\times\mathbb{R}^d$,

$$(u^+\mathbf{1}_{Q_1})(t,x) \leqslant C_1 C_0^{1/\varepsilon}\lambda^{1/\varepsilon^2}\|u^+\chi_2\|_{\mathscr{V}} \vee \|f\chi_2\|_{\mathbb{H}^{-\alpha_3,p_3}_{q_3}}.$$

The proof is complete. □

Remark 2.3. *If $\alpha_1 = 0$ and $\frac{d}{p_1} + \frac{2}{q_1} = 1$ or $b(t,x) = b(x) \in L^d_{loc}(\mathbb{R}^d)$, then by Remark 2.2, we can drop the condition on the divergence of b.*

Now we aim to prove the following crucial result.

Theorem 2.2. *(Global maximum estimate) Suppose that for some $(\alpha_i, p_i, q_i) \in \mathscr{I}_d$, $i = 1,2,3$,*

$$b \in \widetilde{\mathbb{H}}^{-\alpha_1,p_1}_{q_1},\quad -\mathrm{div} b \leqslant \Theta_b \in \widetilde{\mathbb{H}}^{-\alpha_2,p_2}_{q_2},\quad f \in \widetilde{\mathbb{H}}^{-\alpha_3,p_3}_{q_3}.\tag{2.33}$$

Let $u \in \mathscr{V}_{loc} \cap \mathbb{L}^\infty_{loc}$ be a weak solution of PDE (2.18) with initial value $u(0) = 0$. For any $T > 0$, there exists a constant $C > 0$ depending only on T, d, α_i, p_i, q_i and the quantity

$$\kappa := |\!|\!|b|\!|\!|_{\mathbb{H}^{-\alpha_1,p_1}_{q_1}} + |\!|\!|\Theta_b|\!|\!|_{\mathbb{H}^{-\alpha_2,p_2}_{q_2}}$$

such that

$$\|u\|_{L^\infty([0,T]\times\mathbb{R}^d)} + |\!|\!|u\mathbf{1}_{[0,T]}|\!|\!|_{\mathscr{V}} \leqslant C|\!|\!|f\mathbf{1}_{[0,T]}|\!|\!|_{\widetilde{\mathbb{H}}^{-\alpha_3,p_3}_{q_3}}.\tag{2.34}$$

Proof. Without loss of generality, we assume $T = 1$ and

$$u(t,x) = f(t,x) \equiv 0,\quad \forall t \leqslant 0.$$

Let χ_1 be as in (2.3) and define for $z \in \mathbb{R}^d$,

$$\eta_z(t,x) := \chi_1(t, x-z).$$

By translation and (2.20) with $\eta = \eta_z$ and $w = u^+, u^-$, there is a constant $C > 0$ depending only on T, d, α_i, p_i, q_i, $|\!|\!|b|\!|\!|_{\mathbb{H}^{-\alpha_1,p_1}_{q_1}}$, $|\!|\!|(\mathrm{div} b)^-|\!|\!|_{\widetilde{\mathbb{H}}^{-\alpha_2,p_2}_{q_2}}$ such that for all $t \in [0, 1]$,

$$\|\eta_z u \mathcal{I}_t\|_{\mathscr{V}} \leqslant C\Big(\|u \mathbf{1}_{\eta_z \neq 0} \mathcal{I}_t\|_{\mathbb{L}^{r_1}_{s_1}} + \|\eta_z u \mathcal{I}_t\|_{\mathbb{L}^{r_2}_{s_2}} + \|f \chi^{0,z}_2 \mathcal{I}_t\|_{\mathbb{H}^{-\alpha_3,p_3}_{q_3}}\Big),$$

where $\chi^{0,z}_2$ is the same as in (2.3). Taking supremum in $z \in \mathbb{R}^d$ for both sides, we obtain

$$\sup_z \|\eta_z u \mathcal{I}_t\|_{\mathscr{V}} \leqslant C\Big(\sup_z \|u \mathbf{1}_{\eta_z \neq 0} \mathcal{I}_t\|_{\mathbb{L}^{r_1}_{s_1}} + \sup_z \|\eta_z u \mathcal{I}_t\|_{\mathbb{L}^{r_2}_{s_2}} + \sup_z \|f \chi^{0,z}_2 \mathcal{I}_t\|_{\mathbb{H}^{-\alpha_3,p_3}_{q_3}}\Big). \tag{2.35}$$

Since for each $z \in \mathbb{R}^d$, there are at most N-points $z_1, \cdots, z_N \in \mathbb{R}^d$ such that

$$B_2(z) \subset \cup_{j=1}^N B_1(z_j),$$

where $N = N(d)$ and $B_r(z) := \{x : |x - z| < r\}$, we have for $t \in [0, 1]$,

$$\|u \mathbf{1}_{\eta_z \neq 0} \mathcal{I}_t\|_{\mathbb{L}^{r_1}_{s_1}} \leqslant \|u \mathbf{1}_{B_2(z)} \mathcal{I}_t\|_{\mathbb{L}^{r_1}_{s_1}} \leqslant \sum_{j=1}^N \|u \mathbf{1}_{B_1(z_j)} \mathcal{I}_t\|_{\mathbb{L}^{r_1}_{s_1}} \leqslant N \sup_z \|\eta_z u \mathcal{I}_t\|_{\mathbb{L}^{r_1}_{s_1}}, \tag{2.36}$$

where the last step is due to $\eta_{z_j}|_{[0,t] \times B_1(z_j)} = 1$. Hence, by (2.35), (2.36) and (4.3) in appendix,

$$\sup_z \|\eta_z u \mathcal{I}_t\|_{\mathscr{V}} \leqslant C\Big(\sup_z \|\eta_z u \mathcal{I}_t\|_{\mathbb{L}^{r_1}_{s_1}} + \sup_z \|\eta_z u \mathcal{I}_t\|_{\mathbb{L}^{r_2}_{s_2}} + |\!|\!|f \mathcal{I}_t|\!|\!|_{\widetilde{\mathbb{H}}^{-\alpha_3,p_3}_{q_3}}\Big). \tag{2.37}$$

Let

$$\theta_i := \frac{d}{2} - \frac{d}{r_i}, \quad s_i' := \frac{2(1 - \theta_i)s_i}{2 - s_i\theta_i}.$$

Since $\frac{d}{r_i} + \frac{2}{s_i} > \frac{d}{2}$, by Lemma 2.1, we have for any $\varepsilon \in (0, 1)$,

$$\|\eta_z u \mathcal{I}_t\|_{\mathbb{L}^{r_i}_{s_i}} \leqslant \varepsilon \|\nabla(\eta_z u) \mathcal{I}_t\|_{\mathbb{L}^2_2} + C_\varepsilon \|\eta_z u \mathcal{I}_t\|_{\mathbb{L}^2_{s_i'}}. \tag{2.38}$$

Combining (2.37) and (2.38), we arrive at

$$\begin{aligned}\sup_z \|\eta_z u \mathcal{I}_t\|_{\mathbb{L}^2_\infty} + \sup_z \|\nabla(\eta_z u) \mathcal{I}_t\|_{\mathbb{L}^2_2} &\leqslant 2 \sup_z \|\eta_z u \mathcal{I}_t\|_{\mathscr{V}} \\ &\leqslant \varepsilon \sup_z \|\nabla(\eta_z u) \mathcal{I}_t\|_{\mathbb{L}^2_2} + C_\varepsilon \sup_z \|\eta_z u \mathcal{I}_t\|_{\mathbb{L}^2_{s_1' \vee s_2'}} + C|\!|\!|f \mathcal{I}_t|\!|\!|_{\widetilde{\mathbb{H}}^{-\alpha_3,p_3}_{q_3}}.\end{aligned}$$

By choosing ε small enough, we obtain

$$\sup_z \|\eta_z u \mathcal{I}_t\|_{\mathbb{L}^2_\infty} + \sup_z \|\nabla(\eta_z u) \mathcal{I}_t\|_{\mathbb{L}^2_2} \leqslant C \sup_z \|\eta_z u \mathcal{I}_t\|_{\mathbb{L}^2_{s_1' \vee s_2'}} + C|\!|\!|f \mathcal{I}_t|\!|\!|_{\widetilde{\mathbb{H}}^{-\alpha_3,p_3}_{q_3}}. \tag{2.39}$$

Since $s_1', s_2' < \infty$ and $u(t) \equiv 0$ for $t \leqslant 0$, the above inequality implies that for any $t \in [0, 1]$,

$$\sup_z \|(\eta_z u)(t)\|_2^{s_1' \vee s_2'} \leqslant C \sup_z \int_0^t \|(\eta_z u)(s)\|_2^{s_1' \vee s_2'} \mathrm{d}s + C|\!|\!|f \mathcal{I}_t|\!|\!|^{s_1' \vee s_2'}_{\mathbb{H}^{-\alpha_3,p_3}_{q_3}}.$$

By Gronwall's inequality we obtain

$$\sup_z \sup_{t \in [0,1]} \|(\eta_z u)(t)\|_2 \leqslant C|\!|\!|f \mathbf{1}_{[0,1]}|\!|\!|_{\widetilde{\mathbb{H}}^{-\alpha_3,p_3}_{q_3}},$$

which together with (2.39) yields

$$|\!|\!|u \mathbf{1}_{[0,1]}|\!|\!|_{\mathscr{V}} \lesssim \sup_z \|\eta_z u \mathbf{1}_{[0,1]}\|_{\mathbb{L}^2_\infty} + \sup_z \|\nabla(\eta_z u) \mathbf{1}_{[0,1]}\|_{\mathbb{L}^2_2} \lesssim |\!|\!|f \mathbf{1}_{[0,1]}|\!|\!|_{\widetilde{\mathbb{H}}^{-\alpha_3,p_3}_{q_3}}. \tag{2.40}$$

Finally, by (2.29) and (2.40), we also have

$$\begin{aligned}\|u\|_{L^\infty([0,1] \times \mathbb{R}^d)} &\leqslant \sup_z \|(u^+ + u^-) \mathbf{1}_{[0,1] \times B_1(z)}\|_\infty \\ &\lesssim |\!|\!|u \mathbf{1}_{[0,1]}|\!|\!|_{\mathscr{V}} + |\!|\!|f \mathbf{1}_{[0,1]}|\!|\!|_{\widetilde{\mathbb{H}}^{-\alpha_3,p_3}_{q_3}} \lesssim |\!|\!|f \mathbf{1}_{[0,1]}|\!|\!|_{\widetilde{\mathbb{H}}^{-\alpha_3,p_3}_{q_3}}.\end{aligned}$$

The proof is complete. $\square$

2.4. **Existence-uniqueness and stability.** In this subsection we prove the existence-uniqueness and stability of weak solutions for PDE (2.18) by using the apriori estimate (2.34). For $T > 0$ and a function f in $\mathbb{R}^{d+1}$, we denote

$$f^T := f\mathbf{1}_{[0,T]}, \ \widetilde{\mathscr{V}_T} := \{f : |||f^T|||_{\mathscr{V}} < \infty\}, \ \mathbb{L}^\infty_T := \{f : \|f^T\|_\infty < \infty\}.$$

Theorem 2.3. *(Existence-uniqueness) Under (2.33), there exists a unique weak solution $u \in \cap_{T>0}\widetilde{\mathscr{V}_T} \cap \mathbb{L}^\infty_T$ to PDE (2.18) with initial value $u(0) = 0$.*

Proof. First of all, the uniqueness is a direct consequence of (2.34). We prove the existence by weak convergence method. Let $b_n(t,x) := b(t,\cdot) * \rho_n(x)$ and $f_n(t,x) := f(t,\cdot) * \rho_n(x)$. By (ii) of Proposition 4.1 in Appendix, we have

$$b_n \in L^{q_1}_{loc}(\mathbb{R}_+; C^\infty_b(\mathbb{R}^d)), \quad f_n \in L^{q_3}_{loc}(\mathbb{R}_+; C^\infty_b(\mathbb{R}^d)),$$

and

$$-\mathrm{div} b_n \leqslant \Theta_b * \rho_n, \quad \sup_n \Big(|||b_n|||_{\widetilde{\mathbb{H}}^{-\alpha_1,p_1}_{q_1}} + |||\Theta_b * \rho_n|||_{\widetilde{\mathbb{H}}^{-\alpha_2,p_2}_{q_2}} + |||f_n|||_{\widetilde{\mathbb{H}}^{-\alpha_3,p_3}_{q_3}} \Big) < \infty. \tag{2.41}$$

It is well known that the following PDE has a unique smooth solution $u_n \in C(\mathbb{R}_+; C^\infty_b(\mathbb{R}^d))$ (see [23]):

$$\partial_t u_n = \Delta u_n + b_n \cdot \nabla u_n + f_n = 0, \quad u_n(0) = 0. \tag{2.42}$$

By (2.41) and Theorem 2.2, we have

$$\sup_n \Big(\|u^T_n\|_\infty + |||u^T_n|||_{\mathscr{V}} \Big) < \infty, \quad \forall T > 0. \tag{2.43}$$

Hence, by the fact that every bounded subset of $\widetilde{\mathbb{H}}^{1,2}_2$ is relatively weak compact, there is a subsequence n_k and $\bar{u} \in \cap_{T>0}\widetilde{\mathscr{V}_T} \cap \mathbb{L}^\infty_T$ such that for any $\varphi \in C^\infty_c(\mathbb{R}^{d+1})$ and $g \in \mathbb{H}^{-1,2}_{2;loc}$,

$$\lim_{k\to\infty} \langle\!\langle u_{n_k}, g\varphi \rangle\!\rangle = \langle\!\langle \bar{u}, g\varphi \rangle\!\rangle. \tag{2.44}$$

By taking weak limits for equation (2.42), one finds that $\bar{u}$ is a weak solution of PDE (2.18). Indeed, it suffices to prove that for any $\varphi \in C^\infty_c(\mathbb{R}^{d+1})$,

$$\lim_{k\to\infty} \langle\!\langle b_{n_k} \cdot \nabla u_{n_k}, \varphi \rangle\!\rangle = \langle\!\langle b \cdot \nabla \bar{u}, \varphi \rangle\!\rangle, \quad \lim_{k\to\infty} \langle\!\langle f_{n_k}, \varphi \rangle\!\rangle = \langle\!\langle f, \varphi \rangle\!\rangle. \tag{2.45}$$

Let the support of φ be contained in Q_R for some $R > 0$. Since $b \in \mathbb{L}^2_{2,loc}$, by (2.43) and Hölder's inequality, we have for some $C > 0$ independent of k,

$$\begin{aligned}\langle\!\langle (b_{n_k} - b) \cdot \nabla u_{n_k}, \varphi \rangle\!\rangle = \langle\!\langle \chi_R(b_{n_k} - b) \cdot \nabla u_{n_k}, \varphi \rangle\!\rangle &\leqslant \|\nabla\varphi\|_\infty \|(b_{n_k} - b)\chi_R\|_{\mathbb{L}^2_2} \|\chi_R \nabla u_{n_k}\|_{\mathbb{L}^2_2} \\ &\leqslant C\|(b_{n_k} - b)\chi_R\|_{\mathbb{L}^2_2} \to 0 \ \text{ as } \ k \to \infty,\end{aligned}$$

where χ_R is the cutoff function defined in (2.3). On the other hand, since $\mathrm{div}(b\varphi) \in \mathbb{H}^{-1,2}_2$ has compact support, by (2.44) we also have

$$\lim_{k\to\infty} \langle\!\langle b \cdot \nabla(u_{n_k} - \bar{u}), \varphi \rangle\!\rangle = \lim_{k\to\infty} \langle\!\langle u_{n_k} - \bar{u}, \mathrm{div}(b\varphi) \rangle\!\rangle = 0.$$

Thus we obtain the first limit in (2.45). The second limit in (2.45) is direct. $\square$

Theorem 2.4. *(Stability) Let $(p_i, q_i) \in [2,\infty)$ with $\frac{d}{p_i} + \frac{2}{q_i} < 2$, where $i = 1,2,3$. For any $n \in \mathbb{N} \cup \{\infty\} =: \mathbb{N}_\infty$, let $b_n, f_n \in \mathscr{D}'$ satisfy*

$$-\mathrm{div} b_n \leqslant \Theta_{b_n}, \quad \sup_{n\in\mathbb{N}_\infty} \left(|||b_n|||_{\widetilde{\mathbb{L}}^{p_1}_{q_1}} + |||\Theta_{b_n}|||_{\widetilde{\mathbb{L}}^{p_2}_{q_2}} + |||f_n|||_{\widetilde{\mathbb{L}}^{p_3}_{q_3}} \right) < \infty. \tag{2.46}$$

For $n \in \mathbb{N}_\infty$, let $u_n \in \widetilde{\mathscr{V}} \cap \mathbb{L}^\infty$ be the unique weak solutions of PDE (2.18) associated with coefficients (b_n, f_n) with initial value $u(0) = 0$. Assume that for any $\varphi \in C_c(\mathbb{R}^{d+1})$,

$$\lim_{n\to\infty} \left(\|(b_n - b_\infty)\varphi\|_{\mathbb{L}^{p_1}_{q_1}} + \|(f_n - f_\infty)\varphi\|_{\mathbb{L}^{p_3}_{q_3}} \right) = 0. \tag{2.47}$$

Then it holds that for Lebesgue almost all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$,

$$\lim_{n\to\infty} u_n(t,x) = u_\infty(t,x). \tag{2.48}$$

Proof. Notice that equation

$$\partial_t u_n = \Delta u_n + b_n \cdot \nabla u_n + f_n = \Delta u_n + \mathrm{div}(b_n u_n) - (\mathrm{div} b_n) u_n + f_n$$

holds in the distributional sense (see (2.19)). Letting $r := \frac{2p_1}{p_1+2} \leqslant 2$ and $s := \frac{2q_1}{q_1+2} \leqslant 2$, by Proposition 4.1 in Appendix, we have

$$\begin{aligned}
|||(\partial_t u_n)\mathbf{1}_{[0,T]}|||_{\widetilde{\mathbb{H}}^{-1,r}_s} &\leqslant |||\Delta u^T_n + \mathrm{div}(b^T_n u^T_n) - (\mathrm{div} b^T_n)u^T_n + f^T_n|||_{\widetilde{\mathbb{H}}^{-1,r}_s} \\
&\lesssim |||u^T_n|||_{\widetilde{\mathbb{H}}^{1,r}_s} + |||b^T_n u^T_n|||_{\widetilde{\mathbb{L}}^r_s} + |||(\mathrm{div} b^T_n)u^T_n|||_{\widetilde{\mathbb{H}}^{-1,r}_s} + |||f^T_n|||_{\widetilde{\mathbb{H}}^{-1,r}_s} \\
&\lesssim |||u^T_n|||_{\widetilde{\mathbb{H}}^{1,2}_2} + |||b^T_n|||_{\widetilde{\mathbb{L}}^r_s} \|u^T_n\|_\infty + |||\mathrm{div} b^T_n|||_{\widetilde{\mathbb{H}}^{-1,p_1}_{q_1}} |||u^T_n|||_{\widetilde{\mathbb{H}}^{1,2}_2} + |||f^T_n|||_{\widetilde{\mathbb{L}}^r_s} \\
&\lesssim |||u^T_n|||_{\widetilde{\mathbb{H}}^{1,2}_2} + |||b^T_n|||_{\widetilde{\mathbb{L}}^{p_1}_{q_1}} \left(\|u^T_n\|_\infty + |||u^T_n|||_{\widetilde{\mathbb{H}}^{1,2}_2} \right) + |||f^T_n|||_{\widetilde{\mathbb{L}}^{p_3}_{q_3}}.
\end{aligned}$$

By (2.46) and Theorem 2.2, we get for any $T > 0$,

$$\sup_n \left(\|u_n\|_{\mathbb{L}^\infty_T} + |||u_n|||_{\mathscr{V}_T} + |||(\partial_t u_n)\mathbf{1}_{[0,T]}|||_{\widetilde{\mathbb{H}}^{-1,r}_s} \right) < \infty.$$

Thus by Aubin-Lions' lemma (cf. [21]), there is a subsequence n_k and $\bar{u} \in \cap_{T>0}(\widetilde{\mathscr{V}}_T \cap \mathbb{L}^\infty_T)$ such that (2.44) holds and

$$\lim_{k\to\infty} \|u_{n_k} - \bar{u}\|_{L^2([0,T]\times B_m)} = 0, \quad \forall T > 0, m \in \mathbb{N}.$$

By selecting a subsubsequence n'_k, it holds that for Lebesgue almost all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$,

$$u_{n'_k}(t,x) \to \bar{u}(t,x), \quad k \to \infty. \tag{2.49}$$

As in showing (2.45), one can show that $\bar{u}$ is a weak solution of PDE (2.18). By the uniqueness, $\bar{u} = u_\infty$, and by a contradiction method, the whole sequence converges almost everywhere. □

3. Proof of Theorem 1.1

Below we always assume that for some $p_i, q_i \in [2,\infty)$ with $\frac{d}{p_i} + \frac{2}{q_i} < 2$, $i = 1,2$,

$$-\mathrm{div} b \leqslant \Theta_b, \quad \kappa := |||b|||_{\widetilde{\mathbb{L}}^{p_1}_{q_1}} + |||\Theta_b|||_{\widetilde{\mathbb{L}}^{p_2}_{q_2}} < \infty.$$

Let $b_n(t,x) = b(t,\cdot) * \rho_n(x)$ be the mollifying approximation of $b(t,\cdot)$. By (ii) of Proposition 4.1 in Appendix, we have

$$-\mathrm{div} b_n \leqslant \Theta_b * \rho_n, \quad \sup_n \left(|||b_n|||_{\widetilde{\mathbb{L}}^{p_1}_{q_1}} + |||\Theta_b * \rho_n|||_{\widetilde{\mathbb{L}}^{p_2}_{q_2}} \right) \leqslant C\kappa, \tag{3.1}$$

and

$$b_n \in L^{q_1}_{loc}(\mathbb{R}_+; C^\infty_b(\mathbb{R}^d)).$$

For $(s,x)\in\mathbb{R}_+\times\mathbb{R}^d$, consider the following SDE:

$$\mathrm{d}X^n_{s,t}=b_n(t,X^n_{s,t})\mathrm{d}t+\sqrt{2}\mathrm{d}W_t,\ \ X^n_{s,s}=x,\ \ t\geqslant s, \tag{3.2}$$

where W is a d-dimensional standard Brownian motion on some complete filtered probability space $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geqslant 0},\mathbf{P})$. It is well known that there is a unique strong solution $X^n_{s,t}(x)$ to the above SDE (cf. [13]).

Now we are in the position to prove our main result.

3.1. **Existence of martingale solutions.** First of all, we prove the following crucial estimate of Krylov's type.

Lemma 3.1. *For any $(\alpha,p,q)\in\mathscr{I}_d$, there are constants $\theta=\theta(\alpha,p,q)>0$ and $C>0$ depending on $\kappa,d,\ \alpha,p,q,\ p_i,q_i$ such that for any $f\in C^\infty_b(\mathbb{R}^{d+1})$ and $0\leqslant s\leqslant t_0<t_1<\infty$ with $t_0-t_1\leqslant 1$,*

$$\sup_n\sup_{x\in\mathbb{R}^d}\mathbf{E}\left(\int_{t_0}^{t_1}f(t,X^n_{s,t}(x))\mathrm{d}t\Big|\mathscr{F}_{t_0}\right)\leqslant C(t_1-t_0)^\theta|||f|||_{\widetilde{\mathbb{H}}^{-\alpha,p}_q}. \tag{3.3}$$

In particular, we have the following Khasminskii's estimate: for any $\lambda\in\mathbb{R}$,

$$\sup_n\sup_{x\in\mathbb{R}^d}\mathbf{E}\exp\left\{\lambda\int_s^{s+1}|f(t,X^n_{s,t}(x))|\mathrm{d}t\right\}\leqslant C=C(\lambda,\kappa,|||f|||_{\widetilde{\mathbb{H}}^{-\alpha,p}_q}). \tag{3.4}$$

Proof. Fix $0\leqslant s\leqslant t_0<t_1<\infty$ with $t_0-t_1\leqslant 1$ and $f\in C^\infty_b(\mathbb{R}^{d+1})$. Let u_n be the unique smooth solution of the following backward PDE:

$$\partial_t u_n+\Delta u_n+b_n\cdot\nabla u_n+f=0,\ u_n(t_1,\cdot)=0. \tag{3.5}$$

By (3.1) and Theorem 2.2, for any $(\alpha',p',q')\in\mathscr{I}_d$, there is a constant $C>0$ depending only on $\kappa,d,\alpha',p',q',p_i,q_i$ such that for all $t_0\in[0,t_1]$,

$$\|u_n(t_0)\|_\infty\leqslant C|||f\mathbf{1}_{[t_0,t_1]}|||_{\widetilde{\mathbb{H}}^{-\alpha',p'}_{q'}}. \tag{3.6}$$

By Itô's formula we have

$$u_n(t_1,X^n_{s,t_1})=u_n(t_0,X^n_{s,t_0})+\int_{t_0}^{t_1}(\partial_t u_n+\Delta u_n+b_n\cdot\nabla u_n)(t,X^n_{s,t})\mathrm{d}t+\sqrt{2}\int_{t_0}^{t_1}\nabla u_n(t,X^n_{s,t})\mathrm{d}W_t.$$

By (3.5) and taking conditional expectation with respect to $\mathscr{F}_{t_0}$, we obtain

$$\mathbf{E}\left(\int_{t_0}^{t_1}f(t,X^n_{s,t})\mathrm{d}t\Big|\mathscr{F}_{t_0}\right)=\mathbf{E}\Big(u_n(t_0,X^n_{s,t_0})|\mathscr{F}_{t_0}\Big)\leqslant\|u_n(t_0)\|_\infty. \tag{3.7}$$

Since $\frac{d}{p}+\frac{2}{q}<2-\alpha$, we can choose $q'<q$ so that $\frac{d}{p}+\frac{2}{q'}<2-\alpha$. Thus by (3.6) and Hölder's inequality, there is constant $C=C(\kappa,d,\alpha,p,q,p_i,q_i)>0$ such that

$$\mathbf{E}\left(\int_{t_0}^{t_1}f(t,X^n_{s,t})\mathrm{d}t\Big|\mathscr{F}_{t_0}\right)\leqslant C|||f\mathbf{1}_{[t_0,t_1]}|||_{\widetilde{\mathbb{H}}^{-\alpha,p}_{q'}}\leqslant C(t_1-t_0)^{1-\frac{q'}{q}}|||f|||_{\widetilde{\mathbb{H}}^{-\alpha,p}_q}.$$

Thus we obtain (3.3). As for (3.4), it is a direct consequence of (3.3) and [17, Lemma 1.1] (or see [27]). □

Lemma 3.2. *For any $T>0$, there is a constant $C>0$ such that for any $f\in L^1(\mathbb{R}^d)$ and $n\in\mathbb{N}$,*

$$\|\mathcal{T}^n_{s,t}f\|_1\leqslant C\|f\|_1,\ \ \forall 0\leqslant s<t\leqslant s+T,$$

where $\mathcal{T}^n_{s,t}f(x):=\mathbf{E}f(X^n_{s,t}(x))$. Moreover, if $(\mathrm{div}b)^-\equiv 0$, then the above C can be 1.

Proof. Let $Y^n_{s,t} := Y^n_{s,t}(x)$ be the inverse flow of $x \mapsto X^n_{s,t}(x)$. Notice that $s \mapsto Y^n_{s,t}$ solves the following backward SDE:

$$Y^n_{s,t} = x - \int_s^t b_n(r, Y^n_{r,t})\mathrm{d}r + \sqrt{2}(W_s - W_t),\ 0 \leqslant s \leqslant t.$$

Letting $J^n_{s,t} := J^n_{s,t}(x) := \nabla Y^n_{s,t}(x)$ be the Jacobian matrix, we have

$$\partial_s J^n_{s,t} = \nabla b_n(s, Y^n_{s,t})J^n_{s,t} \Rightarrow \partial_s \det(J^n_{s,t}) = \mathrm{div} b_n(s, Y^n_{s,t}) \det(J^n_{s,t}).$$

Hence,

$$\det(J^n_{s,t}) = \exp\left\{-\int_s^t \mathrm{div} b_n(r, Y^n_{r,t})\mathrm{d}r\right\} \leqslant \exp\left\{\int_s^t (\Theta_b * \rho_n)(r, Y^n_{r,t})\mathrm{d}r\right\}.$$

Fix $t > 0$. For any $s \in [0, t]$, let $Z^n_{s,t} := Y^n_{t-s,t}$, $\tilde{b}_n(s, y) := b_n(t - s, y)$ and $\tilde{W}_s := W_{t-s} - W_t$. One sees that $\{\tilde{W}_s\}_{s\in[0,t]}$ is a standard Brownian motion on the interval $[0, t]$ and

$$Z^n_{s,t} = x + \int_0^s \tilde{b}_n(r, Z^n_{r,t})\,\mathrm{d}r + \sqrt{2}\tilde{W}_s.$$

Thus, by (3.1) and Khasminskii's estimate (3.4) with $(\alpha, p, q) = (0, p_2, q_2)$, we have

$$\sup_n \sup_{x\in\mathbb{R}^d} \mathbf{E} \det(J^n_{s,t}(x)) \leqslant \sup_n \sup_{x\in\mathbb{R}^d} \mathbf{E} \exp\left\{\int_0^{t-s} |\Theta_b * \rho_n|(t - r, Z^n_{r,t}(x))\mathrm{d}r\right\} < \infty.$$

Now by the change of variables, for any nonnagative $f \in L^1(\mathbb{R}^d)$, we have

$$\|\mathcal{T}^n_{s,t} f\|_1 = \mathbf{E}\left(\int_{\mathbb{R}^d} f(X^n_{s,t}(x))\mathrm{d}x\right) = \mathbf{E}\left(\int_{\mathbb{R}^d} f(x) \det(J^n_{s,t}(x))\mathrm{d}x\right) \leqslant C\|f\|_1.$$

Moreover, if $(\mathrm{div} b)^- \equiv 0$, then $\det(J^n_{s,t}) \leqslant 1$ and the above $C \equiv 1$. □

Lemma 3.3. *For each $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, let $\mathbb{P}^n_{s,x}$ be the law of $X^n_{s,\cdot}(x)$ in $\mathbb{C}$. Then $(\mathbb{P}^n_{s,x})_{n\in\mathbb{N}}$ is tight.*

Proof. Fix $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ and $T > s$. Let $\tau \geqslant s$ be any stopping time less than T. Notice that

$$X^n_{s,\tau+\delta}(x) - X^n_{s,\tau}(x) = \int_\tau^{\tau+\delta} b_n(t, X^n_{s,t}(x))\mathrm{d}t + \sqrt{2}(W_{\tau+\delta} - W_\tau),\quad \delta > 0.$$

By (3.3) with $\alpha = 0$ and Remark 1.2, we have

$$\begin{aligned}\mathbf{E}|X^n_{s,\tau+\delta}(x) - X^n_{s,\tau}(x)| &\leqslant \mathbf{E}\int_\tau^{\tau+\delta} |b_n|(t, X^n_{s,t}(x))\mathrm{d}t + \sqrt{2}\mathbf{E}|W_{\tau+\delta} - W_\tau|\\ &\leqslant C\delta^\theta |||b_n|||_{\tilde{\mathbb{L}}^p_q} + \sqrt{2}\delta^{1/2} \leqslant C\delta^\theta |||b|||_{\tilde{\mathbb{L}}^p_q} + \sqrt{2}\delta^{1/2},\end{aligned}$$

where C is independent of n and x. Thus by [34, Lemma 2.7], we obtain

$$\sup_n \sup_{(s,x)\in[0,T]\times\mathbb{R}^d} \mathbf{E}\left(\sup_{t\in[s,T]} |X^n_{s,t+\delta}(x) - X^n_{s,t}(x)|^{1/2}\right) \leqslant C\left(\delta^{\theta/2}|||b|||^{1/2}_{\tilde{\mathbb{L}}^p_q} + \delta^{1/4}\right).$$

From this, by Chebyshev's inequality, we derive that for any $T, \varepsilon > 0$,

$$\lim_{\delta\to0} \sup_n \sup_{(s,x)\in[0,T]\times\mathbb{R}^d} \mathbf{P}\left(\sup_{t\in[s,T]} |X^n_{s,t+\delta}(x) - X^n_{s,t}(x)| > \varepsilon\right) = 0.$$

Hence, by [23, Theorem 1.3.2], the law of $X^n_{s,\cdot}(x)$ is tight in $\mathbb{C}$. □

Now we can show the existence of martingale solutions.

Lemma 3.4. *Any accumulation point $\mathbb{P}_{s,x}$ of $(\mathbb{P}^n_{s,x})_{n\in\mathbb{N}}$ belongs to $\mathscr{M}^b_{s,x}$. Moreover, for any $(\alpha,p,q)\in\mathscr{I}_d$, there are $\theta=\theta(\alpha,p,q)>0$ and constant $C>0$ such that for any $f\in C^\infty_b(\mathbb{R}^{d+1})$ and $0\leqslant s\leqslant t_0<t_1<\infty$ with $t_1-t_0\leqslant 1$,*

$$\sup_{x\in\mathbb{R}^d}\mathbb{E}^{\mathbb{P}_{s,x}}\left(\int_{t_0}^{t_1} f(t,\omega_t)\mathrm{d}t\Big|\mathcal{B}_{t_0}\right)\leqslant C(t_1-t_0)^\theta|||f|||_{\widetilde{\mathbb{H}}^{-\alpha,p}_q}. \tag{3.8}$$

Proof. Let $(\alpha,p,q)\in\mathscr{I}_d$. By (3.3), there are $\theta=\theta(\alpha,p,q)>0$ and constant $C>0$ such that for any $f\in C^\infty_b(\mathbb{R}^{d+1})$, $0\leqslant s\leqslant t_0<t_1<\infty$ with $t_1-t_0\leqslant 1$, and $G\in C_b(\mathbb{C})$ being $\mathcal{B}_{t_0}$-measurable,

$$\sup_n\sup_{x\in\mathbb{R}^d}\mathbb{E}^{\mathbb{P}^n_{s,x}}\left(\int_{t_0}^{t_1} f(t,\omega_t)\mathrm{d}t\cdot G_{t_0}\right)\leqslant C(t_1-t_0)^\theta|||f|||_{\widetilde{\mathbb{H}}^{-\alpha,p}_q}\mathbb{E}(G_{t_0}). \tag{3.9}$$

Let $\mathbb{P}_{s,x}$ be any accumulation point of $(\mathbb{P}^n_{s,x})_{n\in\mathbb{N}}$, that is, for some subsequence n_k,

$$\mathbb{P}^{n_k}_{s,x}\text{ weakly converges to }\mathbb{P}_{s,x}\text{ as }k\to\infty.$$

By taking weak limits for (3.9) and a standard monotone class method, we obtain (3.8). In order to prove $\mathbb{P}_{s,x}\in\mathscr{M}^b_{s,x}$, it suffices to prove that for any $t_1>t_0\geqslant s$ and $f\in C^2_c(\mathbb{R}^d)$,

$$\mathbb{E}^{\mathbb{P}_{s,x}}(M^f_{t_1}|\mathcal{B}_{t_0})=M^f_{t_0},\quad \mathbb{P}_{s,x}-a.s.,$$

where

$$M^f_t:=f(\omega_t)-f(\omega_s)-\int_s^t(\Delta+b\cdot\nabla)f(r,\omega_r)\mathrm{d}r.$$

By the standard monotone class method, it is enough to show that for any $G\in C_b(\mathbb{C})$ being $\mathcal{B}_{t_0}$-measurable,

$$\mathbb{E}^{\mathbb{P}_{s,x}}\Big(M^f_{t_1}\cdot G_{t_0}\Big)=\mathbb{E}^{\mathbb{P}_{s,x}}\Big(M^f_{t_0}\cdot G_{t_0}\Big).$$

Note that for each $n\in\mathbb{N}$,

$$\mathbb{P}^n_{s,x}\in\mathscr{M}^{b_n}_{s,x}\Rightarrow\mathbb{E}^{\mathbb{P}^n_{s,x}}\Big(M^{n,f}_{t_1}\cdot G_{t_0}\Big)=\mathbb{E}^{\mathbb{P}^n_{s,x}}\Big(M^{n,f}_{t_0}\cdot G_{t_0}\Big),$$

where

$$M^{n,f}_t:=f(\omega_t)-f(\omega_s)-\int_s^t(\Delta+b_n\cdot\nabla)f(r,\omega_r)\mathrm{d}r.$$

We want to take weak limits, where the key point is to show

$$\lim_{k\to\infty}\mathbb{E}^{\mathbb{P}^{n_k}_{s,x}}\left(\int_s^{t_1}(b_{n_k}\cdot\nabla f)(r,\omega_r)\mathrm{d}r\cdot G_{t_0}(\omega)\right)=\mathbb{E}^{\mathbb{P}_{s,x}}\left(\int_s^{t_1}(b\cdot\nabla f)(r,\omega_r)\mathrm{d}r\cdot G_{t_0}(\omega)\right). \tag{3.10}$$

Assume that $\mathrm{supp}(f)\subset Q_R$. By (3.3) with $\alpha=0$ and (4.5) in Appendix, we have

$$\begin{aligned}&\sup_{n\geqslant m}\mathbb{E}^{\mathbb{P}^n_{s,x}}\left|\int_s^{t_1}((b_m-b_n)\cdot\nabla f)(r,\omega_r)\mathrm{d}r\cdot G_{t_0}(\omega)\right|\\&\leqslant\|G_{t_0}\|_\infty\|\nabla f\|_\infty\sup_{n\geqslant m}\mathbb{E}^{\mathbb{P}^n_{s,x}}\left(\int_s^{t_1}|(b_m-b_n)\chi_R|(r,\omega_r)\mathrm{d}r\right)\\&\lesssim\|G_{t_0}\|_\infty\|\nabla f\|_\infty\sup_{n\geqslant m}|||(b_m-b_n)\chi_R|||_{\widetilde{\mathbb{L}}^{p_1}_{q_1}}\to 0,\ m\to\infty,\end{aligned} \tag{3.11}$$

where $\chi_R|_{Q_R}\equiv 1$ is defined in (2.3). Similarly, by (3.8),

$$\mathbb{E}^{\mathbb{P}_{s,x}}\left|\int_s^{t_1}((b_m-b)\cdot\nabla f)(r,\omega_r)\mathrm{d}r\cdot G_{t_0}(\omega)\right|\lesssim|||(b_m-b)\chi_R|||_{\widetilde{\mathbb{L}}^{p_1}_{q_1}}\to 0,\ m\to\infty. \tag{3.12}$$

Moreover, for fixed $m\in\mathbb{N}$, since

$$\omega\mapsto\int_s^{t_1}(b_m\cdot\nabla f)(r,\omega_r)\mathrm{d}r\cdot G_{t_0}(\omega)\in C_b(\mathbb{C}),$$

we also have

$$\lim_{k\to\infty} \mathbb{E}^{\mathbb{P}^{n_k}_{s,x}}\left(\int_s^{t_1} (b_m\cdot\nabla f)(r,\omega_r)\mathrm{d}r\cdot G_{t_0}(\omega)\right) = \mathbb{E}^{\mathbb{P}_{s,x}}\left(\int_s^{t_1} (b_m\cdot\nabla f)(r,\omega_r)\mathrm{d}r\cdot G_{t_0}(\omega)\right),$$

which together with (3.11) and (3.12) yields (3.10). The proof is complete. □

3.2. **Weak convergence of $\mathbb{P}^n_{s,x}$.** In this subsection we show that for Lebesgue almost all (s,x), the accumulation point of $(\mathbb{P}^n_{s,x})_{n\in\mathbb{N}}$ is unique, which in turn implies that

$$\mathbb{P}^n_{s,x} \text{ weakly converges to } \mathbb{P}_{s,x}\in\mathscr{M}^b_{s,x} \text{ as } n\to\infty.$$

For fixed $T>0$ and $f\in\mathbb{L}^\infty_T = L^\infty([0,T]\times\mathbb{R}^d)$, by Theorem 2.3, there is a unique weak solution $u=u_{T,f}\in\widetilde{\mathscr{V}}_T\cap\mathbb{L}^\infty_T$ to the following backward PDE:

$$\partial_t u+\Delta u+b\cdot\nabla u+f=0,\ u(t,\cdot)|_{t\geqslant T}=0. \tag{3.13}$$

Let $\mathbb{Q}\subset\mathbb{R}$ be the set of all rational numbers and $\mathscr{G}_0$ a countable dense subset of $C_c^\infty(\mathbb{R}^d)$. For $m\in\mathbb{N}$, we recursively define a countable set $\mathscr{G}_m$ as follows:

$$\mathscr{G}_m := \Big\{g = fu_{T,h}\in\mathbb{L}^\infty_T : T\in\mathbb{Q},\ f\in\mathscr{G}_0,\ h\in\mathscr{G}_{m-1}\Big\}.$$

Clearly,

$$\mathscr{A} := \cup_{m=0}^\infty \mathscr{G}_m\subset\mathbb{L}^\infty \text{ is a countable set.}$$

Lemma 3.5. *For $T>0$, $f\in\mathbb{L}^\infty_T$ and $n\in\mathbb{N}$, if we define*

$$u^n_{T,f}(s,x) = \mathbb{E}^{\mathbb{P}^n_{s,x}}\left(\int_s^T f(t,\omega_t)\mathrm{d}t\right), \tag{3.14}$$

then $u^n_{T,f}\in\widetilde{\mathscr{V}}_T\cap\mathbb{L}^\infty_T$ uniquely solves PDE (3.13) with $b=b_n$. Moreover, there is a Lebesgue null set $\mathcal{N}\subset\mathbb{R}_+\times\mathbb{R}^d$ such that for all $(s,x)\in\mathcal{N}^c$, $f\in\mathscr{A}$ and $s\leqslant T\in\mathbb{Q}$,

$$\lim_{n\to\infty} u^n_{T,f}(s,x) = u_{T,f}(s,x). \tag{3.15}$$

Proof. For $m\in\mathbb{N}$, let $f_m(t,x) := f(t,\cdot)*\rho_m(x)$ and

$$u^{n,m}_{T,f}(s,x) = \mathbb{E}^{\mathbb{P}^n_{s,x}}\left(\int_s^T f_m(t,\omega_t)\mathrm{d}t\right),\ s\in[0,T],\ x\in\mathbb{R}^d.$$

It is well known that $u^{n,m}_{T,f}$ solves PDE (3.13) with $b=b_n$ and $f=f_m$ (cf. [23]). By Theorem 2.4, for Lebesgue almost all (s,x), we have

$$u^{n,m}_{T,f}(s,x)\to u^{n,\infty}_{T,f}(s,x),\ \ m\to\infty, \tag{3.16}$$

where $u^{n,\infty}_{T,f}\in\widetilde{\mathscr{V}}_T\cap\mathbb{L}^\infty_T$ is the unique weak solution of of PDE (3.13) with $b=b_n$. On the other hand, by Krylov's estimate (3.3), for each $s\leqslant T$ and $x\in\mathbb{R}^d$, we have

$$\lim_{m\to\infty}\mathbb{E}^{\mathbb{P}^n_{s,x}}\left(\int_s^T f_m(t,\omega_t)\mathrm{d}t\right) = \mathbb{E}^{\mathbb{P}^n_{s,x}}\left(\int_s^T f(t,\omega_t)\mathrm{d}t\right) = u^n_{T,f},$$

which together with (3.16) gives $u^{n,\infty}_{T,f} = u^n_{T,f}$ a.e. Moreover, for fixed $T\in\mathbb{Q}$ and $f\in\mathscr{A}$, by Theorem 2.4 again, there is a Lebesgue null set $\mathcal{N}_{T,f}\subset\mathbb{R}_+\times\mathbb{R}^d$ such that (3.15) holds for all $(s,x)\in\mathcal{N}^c_{T,f}$. Finally, we just need to take

$$\mathcal{N} := \cup_{T\in\mathbb{Q}}\cup_{f\in\mathscr{A}}\mathcal{N}_{T,f}.$$

The proof is complete. □

Lemma 3.6. *Let $\mathcal{N}$ be as in Lemma 3.5. For fixed $(s,x)\in\mathcal{N}^c$ and any two accumulation points $\mathbb{P}^{(1)}_{s,x}$ and $\mathbb{P}^{(2)}_{s,x}$ of $(\mathbb{P}^n_{s,x})_{n\in\mathbb{N}}$, we have*

$$\mathbb{P}^{(1)}_{s,x}=\mathbb{P}^{(2)}_{s,x}. \tag{3.17}$$

Proof. Fix $(s,x)\in\mathcal{N}^c$. For $s\leqslant T\in\mathbb{Q}$ and $f\in\mathscr{G}_0$, by (3.15) and taking weak limits for (3.14) along different subsequences for $\mathbb{P}^{(i)}_{s,x}$, $i=1,2$, one finds that

$$u_{T,f}(s,x)=\mathbb{E}^{\mathbb{P}^{(i)}_{s,x}}\left(\int_s^T f(\omega_t)\mathrm{d}t\right),\quad i=1,2,$$

which implies that for all $s\leqslant T\in\mathbb{Q}$ and $f\in\mathscr{G}_0$,

$$\mathbb{E}^{\mathbb{P}^{(1)}_{s,x}}\left(\int_s^T f(\omega_t)\mathrm{d}t\right)=\mathbb{E}^{\mathbb{P}^{(2)}_{s,x}}\left(\int_s^T f(\omega_t)\mathrm{d}t\right).$$

In particular, for all $T\geqslant s$ and $f\in\mathscr{G}_0$,

$$\int_s^T \mathbb{E}^{\mathbb{P}^{(1)}_{s,x}}f(\omega_t)\mathrm{d}t=\int_s^T \mathbb{E}^{\mathbb{P}^{(2)}_{s,x}}f(\omega_t)\mathrm{d}t\Rightarrow \mathbb{E}^{\mathbb{P}^{(1)}_{s,x}}f(\omega_T)=\mathbb{E}^{\mathbb{P}^{(2)}_{s,x}}f(\omega_T).$$

Claim: Let $(s,x)\in\mathcal{N}^c$ and $T>s$. For any sequence $g_m\in\mathbb{L}^\infty_T$ with $\sup_m\|g_m\|_{\mathbb{L}^\infty_T}<\infty$ and being such that $g_m(t,x)\to g(t,x)$ for Lebesgue almost all (t,x), it holds that

$$\lim_{m\to\infty}\sup_n\mathbb{E}^{\mathbb{P}^n_{s,x}}\left(\int_s^T|g_m-g|(t,\omega_t)\mathrm{d}t\right)=0. \tag{3.18}$$

Proof of Claim: For $R>0$, define

$$\tau_R:=\inf\{t\geqslant s:|\omega_t|\geqslant R\}.$$

By (3.3) with $(\alpha,p,q)=(0,d,4)$ and the dominated convergence theorem, we have

$$\lim_{m\to\infty}\sup_n\mathbb{E}^{\mathbb{P}^n_{s,x}}\left(\int_s^{T\wedge\tau_R}|g_m-g|(t,\omega_t)\mathrm{d}t\right)\leqslant C\lim_{m\to\infty}\left\|(g_m-g)\mathbf{1}_{[s,T]\times B_R}\right\|_{\mathbb{L}^d_4}=0. \tag{3.19}$$

On the other hand, by SDE (3.2) and (3.3) again, we also have

$$\mathbf{E}\left(\sup_{t\in[s,T]}|X^n_{s,t}|\right)\lesssim|x|+1+\mathbf{E}\left(\int_0^T|b_n(t,X^n_{s,t})|\mathrm{d}t\right)\leqslant C,$$

where C is independent of n. Hence,

$$\lim_{R\to\infty}\sup_n\mathbb{P}^n_{s,x}(\tau_R<T)=\lim_{R\to\infty}\sup_n\mathbf{P}\left(\sup_{t\in[s,T]}|X^n_{s,t}(x)|>R\right)\leqslant\lim_{R\to\infty}\sup_n\mathbf{E}\left(\sup_{t\in[s,T]}|X^n_{s,t}|\right)/R=0,$$

which together with (3.19) yields the claim.

Next let $s\leqslant T_1<T_2$ be two rational numbers and $f_1,f_2\in\mathscr{G}_0$. Let $(\mathbb{P}^{n_k}_{s,x})_{k\in\mathbb{N}}$ be a subsequence so that $(\mathbb{P}^{n_k}_{s,x})_{k\in\mathbb{N}}$ weakly converges to $\mathbb{P}^{(1)}_{s,x}$. By the Markov property, we have for $f_1,f_2\in\mathscr{G}_0$

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}^{(1)}_{s,x}}\left(\int_s^{T_1}f_1(\omega_{t_1})\left(\int_{t_1}^{T_2}f_2(\omega_{t_2})\mathrm{d}t_2\right)\mathrm{d}t_1\right)\\&=\lim_{k\to\infty}\mathbb{E}^{\mathbb{P}^{n_k}_{s,x}}\left(\int_s^{T_1}f_1(\omega_{t_1})\left(\int_{t_1}^{T_2}f_2(\omega_{t_2})\mathrm{d}t_2\right)\mathrm{d}t_1\right)\\&=\lim_{k\to\infty}\mathbb{E}^{\mathbb{P}^{n_k}_{s,x}}\left(\int_s^{T_1}f_1(\omega_{t_1})\mathbb{E}^{\mathbb{P}^{n_k}_{t_1,\omega_{t_1}}}\left(\int_{t_1}^{T_2}f_2(\omega_s)\mathrm{d}s\right)\mathrm{d}t_1\right)\end{aligned}$$

$$\overset{(3.14)}{=} \lim_{k\to\infty} \mathbb{E}^{\mathbb{P}^{n_k}_{s,x}} \left(\int_s^{T_1} f_1(\omega_{t_1}) u^{n_k}_{T_2,f_2}(t_1,\omega_{t_1}) \mathrm{d}t_1 \right)$$

$$= \lim_{k\to\infty} \mathbb{E}^{\mathbb{P}^{n_k}_{s,x}} \left(\int_s^{T_1} f_1(\omega_{t_1}) u_{T_2,f_2}(t_1,\omega_{t_1}) \mathrm{d}t_1 \right),$$

where the last step is due to (3.15) and the above Claim. Notice that

$$g(s,x) := f_1(x) u_{T_2,f_2}(s,x) \in \mathscr{A}.$$

Hence,

$$\lim_{k\to\infty} \mathbb{E}^{\mathbb{P}^{n_k}_{s,x}} \left(\int_s^{T_1} f_1(\omega_{t_1}) u_{T_2,f_2}(t_1,\omega_{t_1}) \mathrm{d}t_1 \right) \overset{(3.14)}{=} \lim_{k\to\infty} u^{n_k}_{T_1,g}(s,x) \overset{(3.15)}{=} u_{T_1,g}(s,x).$$

Since the right hand side does not depend on the choice of the subsequence n_k, we finally obtain that for any rational numbers $s \leqslant T_1 < T_2$ and $f_1, f_2 \in \mathscr{G}_0$,

$$\mathbb{E}^{\mathbb{P}^{(1)}_{s,x}} \left(\int_s^{T_1} f_1(\omega_{t_1}) \left(\int_{t_1}^{T_2} f_2(\omega_{t_2}) \mathrm{d}t_2 \right) \mathrm{d}t_1 \right) = \mathbb{E}^{\mathbb{P}^{(2)}_{s,x}} \left(\int_s^{T_1} f_1(\omega_{t_1}) \left(\int_{t_1}^{T_2} f_2(\omega_{t_2}) \mathrm{d}t_2 \right) \mathrm{d}t_1 \right).$$

From this, as above we derive that for all $f_1, f_2 \in \mathscr{G}_0$ and $T_2 > T_1 \geqslant s$,

$$\mathbb{E}^{\mathbb{P}^{(1)}_{s,x}} \left(f_1(\omega_{T_1}) f_2(\omega_{T_2}) \right) = \mathbb{E}^{\mathbb{P}^{(2)}_{s,x}} \left(f_1(\omega_{T_1}) f_2(\omega_{T_2}) \right).$$

Similarly, we can prove that for any $T_m > \cdots > T_1 \geqslant s$ and $f_1, \cdots, f_m \in \mathscr{G}_0$,

$$\mathbb{E}^{\mathbb{P}^{(1)}_{s,x}} \left(f_1(\omega_{T_1}) \cdots f_m(\omega_{T_m}) \right) = \mathbb{E}^{\mathbb{P}^{(2)}_{s,x}} \left(f_1(\omega_{T_1}) \cdots f_m(\omega_{T_m}) \right).$$

Thus we obtain (3.17). □

3.3. **Almost surely Markov property.** Let $\mathcal{N}$ be as in Lemma 3.5. We fix $(s,x) \in \mathcal{N}^c$ so that

$$\mathbb{P}^n_{s,x} \text{ weakly converges to } \mathbb{P}_{s,x} \text{ as } n \to \infty. \tag{3.20}$$

Recalling that $\mathscr{G}_0$ is a countable dense subset of $C_c^\infty(\mathbb{R}^d)$, to show (1.12), it suffices to prove the following claim:

Claim 1: For fixed $t_1 \in (s,\infty) \cap \mathbb{Q}$ and $f \in \mathscr{G}_0$, there is a Lebesgue-null set $I^{t_1,f}_{s,x} \subset (s,t_1)$ so that for all $t_0 \in (s,t_1) \setminus I^{t_1,f}_{s,x}$,

$$\mathbb{E}^{\mathbb{P}_{s,x}} \left(f(\omega_{t_1}) | \mathcal{B}_{t_0} \right) = \mathbb{E}^{\mathbb{P}_{t_0,\omega_{t_0}}} \left(f(\omega_{t_1}) \right), \ \ \mathbb{P}_{s,x} - a.s. \tag{3.21}$$

Indeed, if this is proven, then we can take

$$I_{s,x} := \cup_{s<t_1\in\mathbb{Q}} \cup_{f\in\mathscr{G}_0} I^{t_1,f}_{s,x}.$$

Thus for any $t_0 \in (s,\infty) \setminus I_{s,x}$, and all $t_0 < t_1 \in \mathbb{Q}$ and $f \in \mathscr{G}_0$,

$$\mathbb{E}^{\mathbb{P}_{s,x}} \left(f(\omega_{t_1}) | \mathcal{B}_{t_0} \right) = \mathbb{E}^{\mathbb{P}_{t_0,\omega_{t_0}}} \left(f(\omega_{t_1}) \right), \ \ \mathbb{P}_{s,x} - a.s.$$

By a standard approximation argument, the above equality also holds for all $t_1 > t_0$ and $f \in C_c(\mathbb{R}^d)$.

Furthermore, to prove Claim 1, it suffices to prove the following claim:

Claim 2: Let $t_1 \in (s,\infty) \cap \mathbb{Q}$ and $f \in \mathscr{G}_0$. For fixed $m \in \mathbb{N}$, $s_1, \cdots, s_m \in (s,t_1) \cap \mathbb{Q}$ and $g_1, \ldots, g_m \in \mathscr{G}_0$, there exists a null set $I := I^{s_1,\cdots,s_m}_{g_1,\cdots,g_m} \subset [s_m,t_1]$ so that for all $t_0 \in [s_m,t_1] \setminus I$,

$$\mathbb{E}^{\mathbb{P}_{s,x}} \left(g_1(\omega_{s_1}) \cdots g_m(\omega_{s_m}) f(\omega_{t_1}) \right) = \mathbb{E}^{\mathbb{P}_{s,x}} \left(g_1(\omega_{s_1}) \cdots g_m(\omega_{s_m}) \mathbb{E}^{\mathbb{P}_{t_0,\omega_{t_0}}} (f(\omega_{t_1})) \right). \tag{3.22}$$

Indeed, if this is proven, then we can define

$$I^{t_1,f}_{s,x} := \cup_{m\in\mathbb{N}} \cup_{s_1,\cdots,s_m\in(s,t_1)\cap\mathbb{Q}} \cup_{g_1,\cdots,g_m\in\mathscr{G}_0} I^{s_1,\cdots,s_m}_{g_1,\cdots,g_m} \subset (s,t_1).$$

Thus for any $t_0 \in (s,t_1)\setminus I^{t_1,f}_{s,x}$, (3.22) holds for all m and $s_1,\cdots,s_m \in (s,t_0]\cap\mathbb{Q}$ with $s_1 < s_2 < \cdots < s_m$, $g_1,\cdots,g_m \in \mathscr{G}_0$. By a standard monotone class argument, we obtain (3.21) for $t_0 \in (s,t_1)\setminus I^{t_1,f}_{s,x}$ from (3.22).

Proof of Claim 2: For simplicity of notations, we shall write

$$G_{s_m}(\omega) := g_1(\omega_{s_1})\cdots g_m(\omega_{s_m}).$$

By the Lebesgue differential theorem, we only need to prove that for any $t_0 \in [s_m, t_1]$,

$$\mathbb{E}^{\mathbb{P}_{s,x}}\left(G_{s_m}(\omega)f(\omega_{t_1})\right) = \frac{1}{t_1-t_0}\int_{t_0}^{t_1} \mathbb{E}^{\mathbb{P}_{s,x}}\left(G_{s_m}(\omega)\mathbb{E}^{\mathbb{P}_{r,\omega_r}}(f(\omega_{t_1}))\right)\mathrm{d}r. \tag{3.23}$$

Clearly, by the Markov property of $(\mathbb{P}^n_{s,x})_{(s,x)\in\mathbb{R}_+\times\mathbb{R}^d}$, we have

$$\mathbb{E}^{\mathbb{P}^n_{s,x}}\left(G_{s_m}(\omega)f(\omega_{t_1})\right) = \frac{1}{t_1-t_0}\int_{t_0}^{t_1} \mathbb{E}^{\mathbb{P}^n_{s,x}}\left(G_{s_m}(\omega)\mathbb{E}^{\mathbb{P}^n_{r,\omega_r}}(f(\omega_{t_1}))\right)\mathrm{d}r. \tag{3.24}$$

By (3.20) we have

$$\lim_{n\to\infty}\mathbb{E}^{\mathbb{P}^n_{s,x}}\left(G_{s_m}(\omega)f(\omega_{t_1})\right) = \mathbb{E}^{\mathbb{P}_{s,x}}\left(G_{s_m}(\omega)f(\omega_{t_1})\right). \tag{3.25}$$

Define

$$H_n(r,y) := \mathbb{E}^{\mathbb{P}^n_{r,y}} f(\omega_{t_1}) = \mathbf{E} f(X^n_{r,t_1}(y)).$$

Since by (3.20), $H_n(r,y) \to H(r,y)$ for Lebesgue almost all r, y, by (3.18), we have

$$\lim_{n\to\infty}\sup_k \mathbb{E}^{\mathbb{P}^k_{s,x}}\left(\int_{t_0}^{t_1} |H_n(r,\omega_r) - H(r,\omega_r)|\mathrm{d}r\right) = 0.$$

On the other hand, for fixed n and r, since $y \mapsto H_n(r,y)$ is continuous, we also have

$$\lim_{k\to\infty}\int_{t_0}^{t_1} \mathbb{E}^{\mathbb{P}^k_{s,x}}\left(G_{s_m}(\omega)H_n(r,\omega_r)\right)\mathrm{d}r = \int_{t_0}^{t_1} \mathbb{E}^{\mathbb{P}_{s,x}}\left(G_{s_m}(\omega)H_n(r,\omega_r)\right)\mathrm{d}r.$$

Therefore,

$$\lim_{n\to\infty}\int_{t_0}^{t_1} \mathbb{E}^{\mathbb{P}^n_{s,x}}\left(G_{s_m}(\omega)\mathbb{E}^{\mathbb{P}^n_{r,\omega_r}}(f(\omega_{t_1}))\right)\mathrm{d}r = \int_{t_0}^{t_1} \mathbb{E}^{\mathbb{P}_{s,x}}\left(G_{s_m}(\omega)\mathbb{E}^{\mathbb{P}_{r,\omega_r}}(f(\omega_{t_1}))\right)\mathrm{d}r,$$

which together with (3.24) and (3.25) gives (3.23). The proof is complete.

Proof of Theorem 1.1. By Lemma 3.4, we have the existence of $\mathbb{P}_{s,x} \in \mathscr{M}^b_{s,x}$, which satisfies the Krylov estimate (1.10). By Lemma 3.6, we have (i). By Subsection 3.3 we have (ii). By Lemma 3.2 and (i), we have (iii). □

4. Appendix: Properties of space $\widetilde{\mathbb{H}}_q^{\alpha,p}$

In this appendix we prove some important properties about the space $\widetilde{\mathbb{H}}_q^{\alpha,p}$. We need the following lemma, which can be found in [24, p.205] and [33, Lemma 2.2].

Lemma 4.1. *(i) For any $\alpha \in \mathbb{R}$ and $p \in (1,\infty)$, there is a $C = C(d,\alpha,p) > 0$ such that*

$$\|fg\|_{\alpha,p} \leqslant C\|f\|_{\alpha,p}\|g\|_{|\alpha|+1,\infty}. \tag{4.1}$$

(ii) Let $p \in (1,\infty)$ and $\alpha \in (0,1]$ be fixed. For any $p_1 \in [p,\infty)$ and $p_2 \in [\frac{p_1}{p_1-1},\infty)$ with $\frac{1}{p} \leqslant \frac{1}{p_1} + \frac{1}{p_2} < \frac{1}{p} + \frac{\alpha}{d}$, there is a constant $C > 0$ such that for all $f \in H^{-\alpha,p_1}$ and $g \in H^{\alpha,p_2}$,

$$\|fg\|_{-\alpha,p} \leqslant C\|f\|_{-\alpha,p_1}\|g\|_{\alpha,p_2}. \tag{4.2}$$

The following proposition tells us that the localized norm $|||\cdot|||_{\widetilde{\mathbb{H}}_q^{\alpha,p}}$ enjoys the almost same properties as the global norm $\|\cdot\|_{\mathbb{H}_q^{\alpha,p}}$.

Proposition 4.1. *Let $p,q \in (1,\infty)$ and $\alpha \in \mathbb{R}$.*

(i) For $r \neq r' > 0$, there is a constant $C = C(d,\alpha,r,r') \geqslant 1$ such that for all $f \in \widetilde{\mathbb{H}}_q^{\alpha,p}$,

$$C^{-1}\sup_{s,z}\|f\chi_{r'}^{s,z}\|_{\mathbb{H}_q^{\alpha,p}} \leqslant \sup_{s,z}\|f\chi_r^{s,z}\|_{\mathbb{H}_q^{\alpha,p}} \leqslant C\sup_{s,z}\|f\chi_{r'}^{s,z}\|_{\mathbb{H}_q^{\alpha,p}}. \tag{4.3}$$

In other words, the definition of $\widetilde{\mathbb{H}}_q^{\alpha,p}$ does not depend on the choice of r.

(ii) Let $(\rho_n)_{n\in\mathbb{N}}$ be a family of mollifiers in $\mathbb{R}^d$ and $f_n(t,x) := f(t,\cdot)*\rho_n(x)$. For any $f \in \widetilde{\mathbb{H}}_q^{\alpha,p}$, it holds that $f_n \in L^q_{loc}(\mathbb{R};C_b^\infty(\mathbb{R}^d))$ and for some $C = C(d,\alpha,p,q) > 0$,

$$|||f_n|||_{\widetilde{\mathbb{H}}_q^{\alpha,p}} \leqslant C|||f|||_{\widetilde{\mathbb{H}}_q^{\alpha,p}},\ \forall n \in \mathbb{N}, \tag{4.4}$$

and for any $\varphi \in C_c^\infty(\mathbb{R}^{d+1})$,

$$\lim_{n\to\infty}\|(f_n - f)\varphi\|_{\mathbb{H}_q^{\alpha,p}} = 0. \tag{4.5}$$

(iii) For any $k \in \mathbb{N}$, there is a constant $C = C(d,k,\alpha,p,q) \geqslant 1$ such that for all $f \in \widetilde{\mathbb{H}}_q^{\alpha+k,p}$,

$$C^{-1}|||f|||_{\widetilde{\mathbb{H}}_q^{\alpha+k,p}} \leqslant |||f|||_{\widetilde{\mathbb{H}}_q^{\alpha,p}} + |||\nabla^k f|||_{\widetilde{\mathbb{H}}_q^{\alpha,p}} \leqslant C|||f|||_{\widetilde{\mathbb{H}}_q^{\alpha+k,p}}.$$

(iv) Let $p \in (1,\infty)$ and $\alpha \in (0,1]$, $q \in [1,\infty]$. For any $p_1 \in [p,\infty)$ and $p_2 \in [\frac{p_1}{p_1-1},\infty)$ with $\frac{1}{p} \leqslant \frac{1}{p_1} + \frac{1}{p_2} < \frac{1}{p} + \frac{\alpha}{d}$, and $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$, there is a constant $C > 0$ such that

$$|||fg|||_{\widetilde{\mathbb{H}}_q^{-\alpha,p}} \leqslant C|||f|||_{\widetilde{\mathbb{H}}_{q_1}^{-\alpha,p_1}}|||g|||_{\widetilde{\mathbb{H}}_{q_2}^{\alpha,p_2}}.$$

(v) $\mathbb{L}_q^p + \mathbb{L}_\infty^\infty \subsetneq \widetilde{\mathbb{L}}_q^p$.

Proof. (i) Let $r > r'$. We first prove the right hand side inequality in (4.3). Fix $(s,z) \in \mathbb{R}^{d+1}$. Notice that the support of $\chi_r^{s,z}$ is contained in $Q_{2r}^{s,z}$. Clearly, $Q_{2r}^{s,z}$ can be covered by finitely many $Q_{r'}^{s_i,z_i}, i = 1,\cdots,N$, where $N = N(d,r,r')$ does not depend on s,z. Let $(\varphi_i)_{i=1}^N$ be the partition of unity associated with $\{Q_{r'}^{s_i,z_i}, i = 1,\cdots,N\}$ so that

$$(\varphi_1 + \cdots + \varphi_N)|_{Q_{2r}^{s,z}} = 1,\ \mathrm{supp}(\varphi_i) \subset Q_{r'}^{s_i,z_i}.$$

Thus, due to $\chi_{r'}^{s_i,z_i}|_{Q_{r'}^{s_i,z_i}} = 1$, by (4.1) we have

$$\begin{aligned}\|f\chi_r^{s,z}\|_{\mathbb{H}_q^{\alpha,p}} &\leqslant \sum_{i=1}^N \|f\chi_r^{s,z}\varphi_i\|_{\mathbb{H}_q^{\alpha,p}} = \sum_{i=1}^N \|f\chi_{r'}^{s_i,z_i}\varphi_i\|_{\mathbb{H}_q^{\alpha,p}}\\ &\leqslant \sum_{i=1}^N \|f\chi_{r'}^{s_i,z_i}\|_{\mathbb{H}_q^{\alpha,p}}\|\varphi_i\|_{\mathbb{H}_\infty^{|\alpha|+1,\infty}} \leqslant C\sup_{i=1,\cdots,N}\|f\chi_{r'}^{s_i,z_i}\|_{\mathbb{H}_q^{\alpha,p}},\end{aligned}$$

where $C = C(N, \alpha, d, r, r') > 0$, which yields the right hand side inequality in (4.3). On the other hand, since $\chi^{s,z}_{r'} = \chi^{s,z}_{2r}\chi^{s,z}_{r'}$, by what we have proved, we have

$$\|f\chi^{s,z}_{r'}\|_{\mathbb{H}^{\alpha,p}_q} = \|f\chi^{s,z}_{2r}\chi^{s,z}_{r'}\|_{\mathbb{H}^{\alpha,p}_q} \leqslant C\|f\chi^{s,z}_{2r}\|_{\mathbb{H}^{\alpha,p}_q}\|\chi^{s,z}_{r'}\|_{\mathbb{H}^{|\alpha|+1,\infty}_\infty} \leqslant C\|f\chi^{s,z}_{r}\|_{\mathbb{H}^{\alpha,p}_q},$$

where C does not depend on s, z, which gives the left hand side inequality.

(ii) By the definition of convolutions, it is easy to see that

$$(\chi^{s,z}_1 f_n)(t,x) = \chi^{s,z}_1(t,x)\cdot(f\chi^{s,z}_2)(t,\cdot) * \rho_n(x).$$

Hence,

$$\|\chi^{s,z}_1 f_n\|_{\mathbb{H}^{\alpha,p}_q} \lesssim \|\chi^{s,z}_1\|_{\mathbb{H}^{|\alpha|+1,\infty}_\infty}\|(f\chi^{s,z}_2)_n\|_{\mathbb{H}^{\alpha,p}_q} \lesssim \|\chi_1\|_{\mathbb{H}^{|\alpha|+1,\infty}_\infty}\|f\chi^{s,z}_2\|_{\mathbb{H}^{\alpha,p}_q},$$

which gives (4.4). As for (4.5), it follows by a finitely covering technique.

(iii) We only prove it for $k = 1$. By definition and $\chi^{s,z}_2\nabla\chi^{s,z}_1 = \nabla\chi^{s,z}_1$ we have

$$\begin{aligned}\|(\nabla f)\chi^{s,z}_1\|_{\mathbb{H}^{\alpha,p}_q} &\leqslant \|\nabla(f\chi^{s,z}_1)\|_{\mathbb{H}^{\alpha,p}_q} + \|f\nabla\chi^{s,z}_1\|_{\mathbb{H}^{\alpha,p}_q}\\ &\lesssim \|f\chi^{s,z}_1\|_{\mathbb{H}^{\alpha+1,p}_q} + \|f\chi^{s,z}_2\|_{\mathbb{H}^{\alpha,p}_q}\|\nabla\chi^{s,z}_1\|_{\mathbb{H}^{|\alpha|+1,\infty}_\infty},\end{aligned}$$

which in turn gives the right hand side estimate by (i). The left hand side inequality is similar.

(iv) By (4.2) and $\chi^{s,z}_2\chi^{s,z}_1 = \chi^{s,z}_1$, we have

$$\|(fg)\chi^{s,z}_1\|_{\mathbb{H}^{-\alpha,p}_q} = \|(f\chi^{s,z}_2)(g\chi^{s,z}_1)\|_{\mathbb{H}^{-\alpha,p}_q} \leqslant \|f\chi^{s,z}_2\|_{\mathbb{H}^{-\alpha,p_1}_{q_1}}\|g\chi^{s,z}_1\|_{\mathbb{H}^{\alpha,p_2}_{q_2}}.$$

The desired estimate follows by (i).

(v) Let $\mathbb{Z}^d$ be the set of all lattice points. Define

$$f(t,x) := \mathbf{1}_{[0,1]}(t)\sum_{z\in\mathbb{Z}^d}|x-z|^{-d/p}\mathbf{1}_{|x-z|\leqslant 1}.$$

It is easy to see that $f \in \widetilde{\mathbb{L}}^p_q$, but $f \notin \mathbb{L}^p_q + \mathbb{L}^\infty_\infty$. □

Acknowledgement

The authors would like to thanks two referees for their quite useful suggestions.

References

[1] Aizenman M. and Simon B.: Brownian motion and Harnack inequaity for Schrödinger operators. *Comm. Pure Appl. Math.* 35 (1982), 209-273.

[2] Bass R.F. and Chen Z.Q.: Brownian motion with singular drift. *The Annals of Probability*. **31**(2) (2003), 791-817.

[3] Beck L., Flandoli F., Gubinelli M. and Maurelli M.: Stochastic ODEs and stochastic linear PDEs with critical drift: regularity, duality and uniqueness. arXiv:1401.1530, 2014.

[4] Buckmaster T. and Vicol V.: Nonuniqueness of weak solutions to the Navier-Stokes equation. to appear in *Annals of Math.*, arXiv1709.10033v4.

[5] Chemin J.Y. and Lerner N.: Flot de champs de vecteurs non lipschitziens et quations de Navier-Stokes. *Journal of Differential Equations,* **121**(2) (1995), 314-328.

[6] Constantin P. and Iyer G.: A stochastic Lagrangian representation of the three- dimensional incompressible Navier-Stokes equations. *Comm. Pure Appl. Math.* 61 (2008) 330-345.

[7] Fedrizzi E., Flandoli F.: Noise prevents singularities in linear transport equations. *J. Funct. Anal.*, **264**, 1329-1354 (2013).

[8] Flandoli F., Gubinelli M. and Priola E.: Well-posedness of the transport equation by stochastic perturbation. *Inven. Math.* **180** (2010), 1-53.

[9] Flandoli F., Issoglio E. and Russo F.: Multidimensional stochastic differential equations with distributional drift. *Transactions of the American Mathematical Society*. **369**(3), (2017), 1665-1688.

[10] Gilbarg D. and Trudinger N.S.: *Elliptic partial differential equations of second order*. Springer, 2015.

[11] Hajaiej H., Molinet L., Ozawa T. and Wang B.: Sufficient and necessary conditions for the fractional Gagliardo-Nirenberg inequalities and applications to Navier-Stokes and generalized Boson equations. *arXiv preprint arXiv:1004.4287*, 2010.
[12] Han Q. and Lin F.: *Elliptic partial differential equations* Vol. 1. American Mathematical Soc, 2011.
[13] Ikeda N. and Watanabe S.: *Stochastic differential equations and diffusion processes.* North-Holland Mathematical Library, Vol.24. North-Holland Publishing Co., Amsterdam; Kodansha, Ltd., Tokyo, 1989.
[14] Kinzebulatov D. and Semenov Y. A.: Brownian motion with general drift. arXiv:1710.06729v1 (2017).
[15] Krylov N.V. and Röckner M.: Strong solutions of stochastic equations with singular time dependent drift. *Probab. Theory Relat. Fields* **131** (2005), 154-196.
[16] Nazarov A. and Ural'tseva N.N.: The Harnack inequality and related properties for solutions of elliptic and parabolic equations with divergence-free lower-order coefficients. *St. Petersburg Mathematical Journal,* 23(1), (2012), 93-115.
[17] Portenko N. I.: *Generalized diffusion processes*. Nauka, Moscow, 1982 In Russian; English translation: Amer. Math. Soc. Provdence, Rhode Island, 1990.
[18] Qian Z. and Xi G.: Parabolic equations with divergence-free drift in space $L^l_tL^q_x$. *arXiv preprint* arXiv:1704.02173, 2017.
[19] Rezakhanlou F: Regular flows for diffusions with rough drifts. *arXiv preprint* arXiv:1405.5856, 2014.
[20] Robinson J.C. Rodrigo J.L. and Sadowski W.: *The Three-Dimensional Navier-Stokes Equations: Classical Theory*. Cambridge University Press, 2016.
[21] Simon J.: Compact sets in the space $L^p([0,T];B)$. *Annali di Matematica Pura ed Applicata*. 146: 65-96(1986). https://en.wikipedia.org/wiki/Aubin%E2%80%93Lions_lemma
[22] Stein E.M.: *Singular integrals and differentiability properties of functions*. Princeton Mathe- matical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.
[23] Stroock D. W., Varadhan S. R. S.: *Multidimensional diffusion processes*, Grundlehren der Mathematischen Wissenschaften, 233, Springer-Verlag, Berlin-New York, 1979,
[24] Triebel H.: Theory of function spaces II. *Reprinted 2010 by Springer Basel AG*.
[25] Veretennikov A.: On the strong solutions of stochastic differential equations. *Theory Probab. Appl.* **24** (1979), 354-366.
[26] Xi G.: Parabolic equations and diffusion processes with divergence-free vector fields. *Doctoral dissertation, University of Oxford*, 2018.
[27] Zhang X.: Stochastic homeomorphism flows of SDEs with singular drifts and Sobolev diffusion coefficients. *Electron. J. Probab.* **16** (2011), 1096-1116.
[28] Zhang X. A stochastic representation for backward incompressible Navier-Stokes equations. *Probab. Theory Related Fields*, 148(2010) 305-332.
[29] Zhang X.: Stochastic flows of SDEs with irregular coefficients and stochastic transport equations. *Bull. Sci. Math. France*, Vol. **134**, 340–378(2010).
[30] Zhang X. Stochastic Lagrangian particle approach to fractal Navier-Stokes equations. *Comm. Math. Phys.*, 311(2012) 133-155.
[31] Zhang X.: Well-posedness and large deviation for degenerate SDEs with Sobolev coefficients. *Rev. Mat. Iberoam.*, Vol. **29**, no. 1, 25-52(2013).
[32] Zhang X.: Stochastic differential equations with Sobolev diffusion and singular drift. *Annals of Applied Probability*. **26**, No. 5, 2697-2732(2016).
[33] Zhang X. and Zhao G.: Heat kernel and ergodicity of SDEs with distributional drifts. arXiv:1710.10537, 2017.
[34] Zhang X. and Zhao G.: Singular Brownian Diffusion Processes. *Communications in Mathematics and Statistics,* pp.1-49, 2018.
[35] Zvonkin A.K.: A transformation of the phase space of a diffusion process that removes the drift. *Mat. Sbornik*, **93 (135)** (1974), 129-149.

Xicheng Zhang: School of Mathematics and Statistics, Wuhan University, Wuhan, Hubei 430072, P.R.China, Email: XichengZhang@gmail.com

Guohuan Zhao: Fakultät für Mathematik, Universität Bielefeld, 33615, Bielefeld, Germany, Email: zhaoguohuan@gmail.com