ON THE VANISHING VISCOSITY LIMIT OF THE ISENTROPIC NAVIER–STOKES SYSTEM

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Abstract. We show that any weakly converging sequence of solutions to the isentropic Navier–Stokes system on the full physical space R^d, $d = 2, 3$, in the vanishing viscosity limit either **(i)** converges strongly in the energy norm, or **(ii)** the limit is not a weak solution of the associated Euler system. The same result holds for any sequence of approximate solutions in the spirit of DiPerna and Majda. This is in sharp contrast to the incompressible case, where (oscillatory) approximate solutions may converge weakly to solutions of the Euler system. Our approach leans on identifying a system of differential equations satisfied by the associated turbulent defect measures and showing that it only has a trivial solution.

1. Introduction

In the light of the recent results [8, 9, 10, 11] indicating essential ill–posedness of the isentropic Euler system, the vanishing viscosity limit might be seen as a sound selection criterion to identify the physically relevant solutions of systems describing inviscid fluids, although this can be still arguable in view of the examples collected in the recent survey by Buckmaster and Vicol [4] and Constantin and Vicol [12]. The principal difficulties of this approach, caused in particular by the presence of kinematic boundaries, are well understood in the case of incompressible fluids, see e.g. the survey of E [17]. However, much less is known in the compressible case. Leaving apart the boundary layer issue, Sueur [27] proved unconditional convergence in the barotropic case provided the Euler system admits a smooth solution. A similar result was obtained for the full Navier–Stokes/Euler systems in [18].

However, as many solutions of the Euler system are known to develop discontinuities in finite time, it is of essential interest to understand the inviscid limit provided the target solution is not smooth. Very recently, Basarić [2] identified the vanishing viscosity limit with a measure–valued solution to the Euler system on general, possibly unbounded, spatial domains, which can be seen as a "compressible" counterpart of the pioneering work of DiPerna and Majda [14] in the incompressible case. The incompressible setting was further studied

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in space dimension two and for vortex sheet initial data by DiPerna and Majda [15, 16] and Greengard and Thomann [22]. Their results show that the set, where the approximate solutions do not converge strongly is either empty or its projection on the time axis is of positive measure. In addition, explicit examples of *weakly* converging sequences generating a weak solution of the incompressible Euler system are given.

In the present paper, we show even more striking result for the compressible isentropic case in dimension two and three: Either the solutions of the isentropic Navier–Stokes system converge strongly in the vanishing viscosity limit or the limit is not a weak solution of the isentropic Euler system. It is worth noting that the same argument can be applied to any family of approximate solutions satisfying certain consistency conditions in the spirit of DiPerna and Majda [15]. In particular, the result holds for any sequence of weak solutions of the Euler system satisfying the associated energy inequality. Finally, we recall that the problem is well understood if $d = 1$ or in the associated radially symmetric case, where unconditional strong convergence has been established by Chen and Perepelitsa [6], [7].

Let $d = 2, 3$ be the dimension of the physical space. The compressible Navier–Stokes system governs the evolution of density $\varrho : [0, T] \times R^d \to [0, \infty)$ and velocity $\boldsymbol{u} : [0, T] \times R^d \to R^d$ of a viscous barotropic fluid:

$$\partial_t \varrho + \mathrm{div}_x (\varrho \boldsymbol{u}) = 0, \tag{1.1}$$

$$\partial_t (\varrho \boldsymbol{u}) + \mathrm{div}_x (\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla_x p(\varrho) = \mathrm{div}_x\, \mathbb{S}(\nabla_x \boldsymbol{u}). \tag{1.2}$$

The viscous stress tensor is determined by Newton's rheological law

$$\mathbb{S}(\nabla_x \boldsymbol{u}) = \mu \left(\nabla_x \boldsymbol{u} + \nabla_x^t \boldsymbol{u} - \frac{2}{d} \mathrm{div}_x \boldsymbol{u} \mathbb{I} \right) + \lambda \mathrm{div}_x \boldsymbol{u} \mathbb{I}$$

for certain viscosity constants $\mu > 0$, $\lambda \geq 0$; the pressure is given by the isentropic equation of state,

$$p(\varrho) = a \varrho^\gamma, \ a > 0, \ \gamma > 1.$$

As the physical space is unbounded, we impose the far field conditions

$$\boldsymbol{u} \to \boldsymbol{u}_\infty, \ \varrho \to \varrho_\infty \text{ as } |x| \to \infty \tag{1.3}$$

for some constant fields $\boldsymbol{u}_\infty \in R^d$ and $\varrho_\infty \geq 0$. The problem is formally closed by prescribing the initial state

$$\varrho(0) = \varrho_0, \qquad \varrho \boldsymbol{u}(0) = \boldsymbol{m}_0. \tag{1.4}$$

We are interested in the vanishing viscosity limit of *finite (relative) energy* solutions

$$[\varrho_n, \boldsymbol{m}_n \equiv \varrho_n \boldsymbol{u}_n]$$

to the Navier–Stokes system for $\mu = \mu_n \searrow 0$ and $\lambda = \lambda_n \searrow 0$. Formally, the limit $[\varrho, \boldsymbol{m}]$ should satisfy the isentropic Euler system:

$$\partial_t \varrho + \mathrm{div}_x \boldsymbol{m} = 0, \tag{1.5}$$

$$\partial_t \boldsymbol{m} + \mathrm{div}_x \left(\mathbf{1}_{\varrho>0} \frac{\boldsymbol{m} \otimes \boldsymbol{m}}{\varrho} \right) + \nabla_x p(\varrho) = 0. \tag{1.6}$$

Although formally obvious, the convergence to the Euler system has been rigorously proved only in the case when the latter admits a (unique) strong solution, see Basarić [2], Sueur [27]. More recently, a larger class of unconditional limits has been identified in [19]. However, it is still not evident whether the limit system is (1.5), (1.6), even if the difficulties related to boundary layer are left apart.

Our main goal is to show that the convergence is necessarily strong in the associated energy norm whenever the limit is a *weak solution* of the Euler system (1.5), (1.6). Such a scenario implies/suggests the following:

- If a sequence of solutions of the Navier–Stokes system in the vanishing viscosity limit converges only weakly, then the limit cannot be a weak solution of the Euler system but rather a dissipative solutions in the sense of [3].
- The weak solutions of the isentropic Euler system resulting from the vanishing viscosity limit may inherit certain favorable properties of their "viscous" counterparts in the spirit of the celebrated Kolmogorov K41 hypothesis as suggested by Chen and Glimm [5]. Indeed, if the solutions to Navier–Stokes system satisfy the energy equality, then the rate of dissipation of the limit solution is given by the predicted value

$$\epsilon(t) := \lim_{n\to\infty} \int_{R^d} \mathbb{S}_n(\nabla_x \mathbf{u}_n(t)) : \nabla_x \mathbf{u}_n(t)\,\mathrm{d}x.$$

- Certain "wild solutions" obtained via the method of convex integration, see e.g. Chiodaroli [8], DeLellis and Székelyhidi [13], in particular those with increasing total energy, are definitely not limits of viscous approximations.

In accordance with the seminal paper by DiPerna and Majda [14], the vanishing viscosity limit can be identified with a generalized *measure–valued* solution of the Euler system. The compressible analogue of this result was proved by Basarić [2]. The concept of the measure–valued solutions used in [2], however, follows the philosophy: the more general the better, while preserving a suitable weak (measure–valued)/strong uniqueness principle. Such an approach is typically beneficial for a number of applications (e.g. in numerical analysis). As a matter of fact, a more refined description of the asymptotic limit can be obtained via Alibert–Bouchitté's [1] framework employed by Gwiazda, Świerczewska–Gwiazda, and Wiedemann [23]. Nevertheless, similarly to the work by Chen and Glimm [5], the measure–valued solutions are defined for the density ϱ and the weighted velocity $\sqrt{\varrho}\mathbf{u}$ yielding a rather technical definition of a solution.

On the search for physically relevant solutions to the isentropic Euler system in [3], we employed a different strategy than in the above mentioned works. In particular, we identified a stable notion of *dissipative* solution which permits a construction of a solution semiflow, even under the severe ill–posedness of the system and in particular the existence of infinitely many solutions. The main idea was to study the possible oscillations and concentrations in terms of the *conservative variables*, namely the density ϱ and the momentum $\boldsymbol{m} \equiv \varrho \boldsymbol{u}$. From the physical point of view (further mathematically justified by the construction of the semiflow in [3]), this seems to be the correct approach as these are the variables whose evolution is governed by (1.5), (1.6).

In the present paper, we further simplify the definition of dissipative solution: we do not separate the measures governing oscillations and concentrations in the various nonlinearities but rather introduce one *turbulent defect measure* which takes into account all these defects. The key new ingredient is that the defect in the momentum equation directly controls the defect in the energy. (The converse, i.e. the fact that the defect in the energy controls the defect in the momentum equation, is also true and indispensable but has already appeared in the previous definitions.). Furthermore, the turbulent defect measure $\mathbb{D}(t)$ is for a.e. time given by a (symmetric) positive semidefinite matrix–valued finite Borel measure on R^d in the sense that

$$\mathbb{D}(t) : (\xi \otimes \xi) \text{ is a non–negative finite measure on } R^d \text{ for any } \xi \in R^d,$$

and it can be identified along with a system of differential equations it obeys. In particular, we show below that the problem of convergence towards a weak solution reduces to solving a system of differential equations

$$\mathrm{div}_x \mathbb{D}(t) = 0. \tag{1.7}$$

We prove that it admits only a trivial solution $\mathbb{D}(t) \equiv 0$ when restricted to the class of positively semidefinite matrix–valued finite measures. Since the turbulent defect measure also controls the defect of the energy, we are able to conclude the strong convergence of the approximate sequence of solutions in the energy norm, under the assumption that the weak limit is a weak solution. In addition, we treat the case of a bounded domain under certain additional hypotheses concerning the behavior of the defect near the boundary.

The paper is organized as follows. In Section 2, we recall the concept of weak solution for both the Navier–Stokes and the Euler system and state our main result. Section 3 is devoted to the analysis of the turbulent defect measures. In Section 4 we show how the problem of convergence can be transformed to solving (1.7) and demonstrate that the defect measures vanish. The strong convergence is shown in Section 5 and further applications of the method are discussed in Section 6.

2. Preliminary material and the main results

We start by introducing the concept of weak solution for both the primitive and the target system.

2.1. Weak solutions to the Navier–Stokes system.

Let $\boldsymbol{u}_\infty \in R^d$ and $\varrho_\infty \geq 0$ be given. We define $\boldsymbol{m} \equiv \varrho \boldsymbol{u}$ and $\boldsymbol{m}_\infty \equiv \varrho_\infty \boldsymbol{u}_\infty$. We say that $[\varrho, \boldsymbol{u}]$ is a *weak solution* to the Navier–Stokes system (1.1)–(1.3) if:

- $\varrho \geq 0$, $(\varrho - \varrho_\infty) \in L^\infty(0,T; (L^\gamma + L^2)(R^d))$, $(\boldsymbol{m} - \boldsymbol{m}_\infty) \in L^\infty(0,T; (L^{\frac{2\gamma}{\gamma+1}} + L^2)(R^d; R^d))$, $\nabla_x \mathbf{u} \in L^2((0,T) \times R^d; R^{d\times d})$;

- $$\int_0^T \int_{R^d} \Big[\varrho \partial_t \varphi + \varrho \boldsymbol{u} \cdot \nabla_x \varphi \Big] \, {\rm d}x \, {\rm d}t = 0 \text{ for any } \varphi \in C^1_c((0,T) \times R^d), \tag{2.1}$$
  equivalently, equation (1.1) holds in $\mathcal{D}'((0,T) \times R^d)$;
- $$\int_0^T \int_{R^d} \Big[\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho \boldsymbol{u} \otimes \boldsymbol{u} : \nabla_x \boldsymbol{\varphi} + p(\varrho) {\rm div}_x \boldsymbol{\varphi} \Big] \, {\rm d}x \, {\rm d}t = \int_0^T \int_{R^d} \mathbb{S}(\nabla_x \boldsymbol{u}) : \nabla_x \boldsymbol{\varphi} \, {\rm d}x \, {\rm d}t \tag{2.2}$$
  $$\text{for any } \boldsymbol{\varphi} \in C^1_c((0,T) \times R^d; R^d),$$
  equivalently, equation (1.2) holds in $\mathcal{D}'((0,T) \times R^d; R^d)$;
- the energy inequality
  $$\begin{aligned} &\int_{R^d} \left[\frac{1}{2} \varrho |\boldsymbol{u} - \boldsymbol{u}_\infty|^2 + P(\varrho) - P'(\varrho_\infty)(\varrho - \varrho_\infty) - P(\varrho_\infty) \right] (\tau, \cdot) \, {\rm d}x \\ &\quad + \int_0^\tau \int_{R^d} \mathbb{S}(\nabla_x \boldsymbol{u}) : \nabla_x \boldsymbol{u} \, {\rm d}x \, {\rm d}t \leq E_0, \qquad P(\varrho) \equiv \frac{a}{\gamma - 1} \varrho^\gamma = \frac{1}{\gamma - 1} p(\varrho), \end{aligned} \tag{2.3}$$
  holds for any $\tau \in (0,T)$.

The constant E_0 in (2.3) represents the energy of the initial data,

$$E_0 = \int_{R^d} \left[\frac{1}{2} \frac{|\boldsymbol{m}_0|^2}{\varrho_0} - \boldsymbol{m}_0 \boldsymbol{u}_\infty + \frac{1}{2} \varrho_0 |\boldsymbol{u}_\infty|^2 + P(\varrho_0) - P'(\varrho_\infty)(\varrho_0 - \varrho_\infty) - P(\varrho_\infty) \right] \, {\rm d}x.$$

In our setting, this is the natural quantity to measure since it corresponds to the relative energy of $[\varrho, \mathbf{u}]$ with respect to $[\varrho_\infty, \mathbf{u}_\infty]$. Here and hereafter, the kinetic energy is understood as a lower semicontinuous convex function of the variables $[\varrho, \boldsymbol{m}]$ given by

$$e_{\rm kin}(\varrho, \boldsymbol{m}) \equiv \frac{1}{2} \frac{|\boldsymbol{m}|^2}{\varrho} = \begin{cases} \frac{1}{2} \frac{|\boldsymbol{m}|^2}{\varrho} \text{ for } \varrho > 0, \\ 0 \text{ for } \boldsymbol{m} = 0, \ \varrho \geq 0, \\ \infty \text{ otherwise.} \end{cases}$$

Accordingly,

$$\begin{aligned} &e_{\rm kin}(\varrho, \boldsymbol{m}) - \frac{\partial e_{\rm kin}(\varrho_\infty, \boldsymbol{m}_\infty)}{\partial \varrho} (\varrho - \varrho_\infty) - \frac{\partial e_{\rm kin}(\varrho_\infty, \boldsymbol{m}_\infty)}{\partial \boldsymbol{m}} \cdot (\boldsymbol{m} - \boldsymbol{m}_\infty) - e_{\rm kin}(\varrho_\infty, \boldsymbol{m}_\infty) \\ &\qquad = \frac{1}{2} \frac{|\boldsymbol{m}|^2}{\varrho} - \boldsymbol{m} \cdot \boldsymbol{u}_\infty + \frac{1}{2} \varrho |\boldsymbol{u}_\infty|^2 \left(= \frac{1}{2} \varrho |\boldsymbol{u} - \boldsymbol{u}_\infty|^2 \right) \end{aligned} \tag{2.4}$$

is a convex lower semi–continuous function of $[\varrho, \boldsymbol{m}]$.

Similarly, we define the internal energy as well as the total energy as

$$e_{\rm int}(\varrho) \equiv P(\varrho) \text{ and } e(\varrho, \boldsymbol{m}) \equiv e_{\rm kin}(\varrho, \boldsymbol{m}) + e_{\rm int}(\varrho).$$

Finally, it is convenient to introduce the relative energy

$$e\left(\varrho, \boldsymbol{m} | \varrho_\infty, \boldsymbol{m}_\infty\right) \equiv e(\varrho, \boldsymbol{m}) - \frac{\partial e(\varrho_\infty, \boldsymbol{m}_\infty)}{\partial \varrho} (\varrho - \varrho_\infty) - \frac{\partial e(\varrho_\infty, \boldsymbol{m}_\infty)}{\partial \boldsymbol{m}} \cdot (\boldsymbol{m} - \boldsymbol{m}_\infty) - e(\varrho_\infty, \boldsymbol{m}_\infty).$$

The existence of global–in–time weak solutions for the problem (1.1)–(1.4) was proved by Novotný and Pokorný [25] in the case $\boldsymbol{u}_\infty = 0$, $\varrho_\infty > 0$. The case $\boldsymbol{u}_\infty = 0$, $\varrho_\infty = 0$ has been considered by Lions [24] and in [20]. In both cases, the weak solutions exist for any initial data with finite energy E_0 under the technical condition $\gamma > \frac{d}{2}$. The general situation, including the energy inequality (2.3), is is treated by Novotný and Straškraba [26, Chapter 7, Theorem 7.79], see also Lions [24, Section 5.6]. For future analysis, we retain only the piece of information provided by the integral identities (2.1), (2.2), and the energy inequality (2.3).

2.2. **Weak solutions to the Euler system.** We consider the distributional solutions of the Euler system (1.5), (1.6), namely

$$\int_0^T \int_{R^d} \Big[\varrho \partial_t \varphi + \boldsymbol{m} \cdot \nabla_x \varphi \Big] \,\mathrm{d}x\,\mathrm{d}t = 0 \text{ for any } \varphi \in C^1_c((0,T) \times R^d); \tag{2.5}$$

$$\int_0^T \int_{R^d} \Big[\boldsymbol{m} \cdot \partial_t \boldsymbol{\varphi} + \mathbf{1}_{\varrho > 0} \frac{\boldsymbol{m} \otimes \boldsymbol{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \Big] \,\mathrm{d}x\,\mathrm{d}t = 0 \tag{2.6}$$
$$\text{for any } \boldsymbol{\varphi} \in C^1_c((0,T) \times R^d; R^d).$$

As already pointed out in the introduction, the system (2.5), (2.6) is desperately ill–posed and admits infinitely many solutions for any sufficiently regular initial data.

2.3. **Main result.** Our main result concerns the asymptotic behavior of the solutions of the Navier–Stokes system in the vanishing viscosity limit. We first state it in the case of the full physical space R^d and then discuss possible extensions to bounded domains.

Theorem 2.1. *Let $d = 2,3$ and $\gamma > 1$. Let $\{\mu_n\}_{n=1}^\infty$, $\{\lambda_n\}_{n=1}^\infty$ be sequences of viscosity coefficients, $\mu_n \searrow 0$, $\lambda_n \searrow 0$. Let $[\varrho_n, \boldsymbol{u}_n]$ be a sequence of solutions of the Navier–Stokes system (2.1)–(2.3), with the initial energy E_0 bounded uniformly for $n \to \infty$. Suppose that*

$$\varrho_n \to \varrho \ \text{in} \ \mathcal{D}'((0,T) \times R^d), \quad \boldsymbol{m}_n \equiv \varrho_n \boldsymbol{u}_n \to \boldsymbol{m} \ \text{in} \ \mathcal{D}'((0,T) \times R^d; R^d),$$

where ϱ, $\boldsymbol{m}$ is a solution of the Euler system (2.5), (2.6).

Then

$$e(\varrho_n, \boldsymbol{m}_n | \varrho_\infty, \boldsymbol{m}_\infty) \to e(\varrho, \boldsymbol{m} | \varrho_\infty, \boldsymbol{m}_\infty) \ \text{in} \ L^q(0,T; L^1(R^d)) \tag{2.7}$$

as $n \to \infty$ for any $1 \le q < \infty$. In particular,

$$(\varrho_n - \varrho_\infty) \to (\varrho - \varrho_\infty) \ \text{in} \ L^q(0,T; (L^\gamma + L^2)(R^d)),$$
$$(\boldsymbol{m}_n - \boldsymbol{m}_\infty) \to (\boldsymbol{m} - \boldsymbol{m}_\infty) \ \text{in} \ L^q(0,T; (L^{\frac{2\gamma}{\gamma+1}} + L^2)(R^d; R^d)) \ \text{for any} \ 1 \le q < \infty.$$

Theorem 2.1 can be reformulated in the way that either the sequence of solutions of the isentropic Navier–Stokes system converges strongly in the vanishing viscosity regime or the limit *is not* a weak solution of the Euler system. The fact that the energy converges strongly, cf. (2.4), (2.7), rules out certain "wild" solutions constructed by the method of convex integration as solutions reachable in the vanishing viscosity limit, apparently those with increasing total energy.

The proof of Theorem 2.1, elaborated in the remaining part of this paper, is based on careful analysis of the associated turbulent defect measures. They can be seen as "compressible counterpart" of those introduced by DiPerna and Majda [14]. In contrast with [14], however, they are regular with respect to the time variable in accordance with the energy bounds (2.3) that are uniform in time.

The fact that the pressure $p = p(\varrho)$ is explicitly related to the density by the *isentropic* equation of state plays a crucial role together with the fact that we use the *conservative* state variables ϱ and $\boldsymbol{m}$ as the basis for the Young measure describing possible oscillations. As we show below, the total turbulent defect can then be described by a matrix–valued measure $\mathbb{D}$ satisfying certain system of differential equations on the whole physical space R^d. Finally, we use positive semidefinitness and tightness of the defect to show that $\mathbb{D}$ must vanish yielding the desired result.

The extension of Theorem 2.1 faces serious difficulties under the presence of a physical boundary. It turns out, however, that as soon as we control the defect in a neighborhood of the boundary then it must vanish in the interior of the domain. Of course, the specific value of the far fields ϱ_∞, $\boldsymbol{u}_\infty$ are irrelevant on a bounded domain and we may set $\boldsymbol{u}_\infty = \varrho_\infty = 0$ for simplicity.

Theorem 2.2. *Let $d = 2, 3$, $\gamma > 1$, and let $\Omega \subset R^d$ be a bounded domain. Let $\{\mu_n\}_{n=1}^\infty$, $\{\lambda_n\}_{n=1}^\infty$ be sequences of viscosity coefficients, $\mu_n \searrow 0$, $\lambda_n \searrow 0$. Let $[\varrho_n, \boldsymbol{u}_n]$ be a sequence of solutions of the Navier–Stokes system (2.1)–(2.3) in $(0,T) \times \Omega$[1], with the initial energy E_0 bounded uniformly for $n \to \infty$. Suppose that*

$$\varrho_n \to \varrho \text{ in } \mathcal{D}'((0,T) \times \Omega), \quad \boldsymbol{m}_n \equiv \varrho_n \boldsymbol{u}_n \to \boldsymbol{m} \text{ in } \mathcal{D}'((0,T) \times \Omega; R^d),$$

where ϱ, $\boldsymbol{m}$ is a solution of the Euler system (2.5), (2.6). Suppose that for a.e. $\tau \in (0,T)$

$$\limsup_{n \to \infty} \int_{\{x \in \Omega,\ \mathrm{dist}[x, \partial\Omega] \leq \delta\}} \left[e(\varrho_n, \boldsymbol{m}_n) - e(\varrho, \boldsymbol{m}) \right] (\tau, \cdot) \, \mathrm{d}x$$

is of order $o(\delta)$ as $\delta \searrow 0$.

Then

$$e(\varrho_n, \boldsymbol{m}_n) \to e(\varrho, \boldsymbol{m}) \text{ in } L^q(0,T; L^1(\Omega))$$

as $n \to \infty$ for any $1 \leq q < \infty$. In particular,

$$\varrho_n \to \varrho \text{ in } L^q(0,T; L^\gamma(\Omega)),$$

$$\boldsymbol{m}_n \to \boldsymbol{m} \text{ in } L^q(0,T; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d)) \text{ for any } 1 \leq q < \infty.$$

Corollary 2.3. *Let $d = 2, 3$, $\gamma > 1$, and let $\Omega \subset R^d$ be a bounded domain. Let $\{\mu_n\}_{n=1}^\infty$, $\{\lambda_n\}_{n=1}^\infty$ be sequences of viscosity coefficients, $\mu_n \searrow 0$, $\lambda_n \searrow 0$. Let $[\varrho_n, \boldsymbol{u}_n]$ be a sequence of solutions of the Navier–Stokes system (2.1)–(2.3) in $(0,T) \times \Omega$, with the initial energy E_0 bounded uniformly for $n \to \infty$. Suppose that*

$$\varrho_n \to \varrho \text{ in } \mathcal{D}'((0,T) \times \Omega), \quad \boldsymbol{m}_n \equiv \varrho_n \boldsymbol{u}_n \to \boldsymbol{m} \text{ in } \mathcal{D}'((0,T) \times \Omega; R^d),$$

[1] considering test functions compactly supported in $(0,T) \times \Omega$, i.e. $\varphi \in C_c^1((0,T) \times \Omega)$ and $\boldsymbol{\varphi} \in C_c^1((0,T) \times \Omega)$.

where ϱ, $\boldsymbol{m}$ is a solution of the Euler system (2.5), (2.6). In addition, suppose that there exists an open neighborhood $\mathcal{U}$ of $\partial\Omega$ such that

$$\|e(\varrho_n, \boldsymbol{m}_n) - e(\varrho, \boldsymbol{m})\|_{L^1((0,T)\times\mathcal{U})} \to 0.$$

Then

$$e(\varrho_n, \boldsymbol{m}_n) \to e(\varrho, \boldsymbol{m}) \text{ in } L^q(0,T; L^1(\Omega))$$

as $n \to \infty$ for any $1 \le q < \infty$. In particular,

$$\varrho_n \to \varrho \text{ in } L^q(0,T; L^\gamma(\Omega)),$$
$$\boldsymbol{m}_n \to \boldsymbol{m} \text{ in } L^q(0,T; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d)) \text{ for any } 1 \le q < \infty.$$

In other words, if the limit solution is a weak solution of the Euler system and the convergence is strong in a neighborhood of the boundary, then it must be strong everywhere in the interior of the domain.

The rest of the paper is devoted to the proof of Theorems 2.1, 2.2. First of all, assuming the distributional convergence of $[\varrho_n, \boldsymbol{m}_n]$ we identify the equation satisfied by the limit $[\varrho, \boldsymbol{m}]$. This is inspired by the definition of dissipative solution to the Euler system (1.5), (1.6) introduced in [3]. However, here we take it one step further and introduce only one turbulent defect measure which takes into account all possible oscillations as well as concentrations.

Assuming, in the second step, that the limit $[\varrho, \boldsymbol{m}]$ is a weak solution, that is, (2.5), (2.6) hold, we obtain a differential equation satisfied by the turbulent defect measure. Due to the properties of this measure, namely its positive semidefinitness and boundedness, we are able to conclude that it vanishes. Since this also controls the defect of the energy, we conclude that the energies converge strongly and therefore the strong convergence of $[\varrho_n, \boldsymbol{m}_n]$ to $[\varrho, \boldsymbol{m}]$ follows.

3. Turbulent defect measures

In the following, we pass several times to suitable subsequences in the vanishing viscosity sequence without explicit relabeling. However, it is easy to see that it is enough to show the conclusion of Theorems 2.1, 2.2 for a subsequence once the limit $[\varrho, \boldsymbol{m}]$ has been fixed.

It follows from equations (2.1), (2.2), and the bounds imposed by the energy inequality (2.3) that we may suppose

$$(\varrho_n - \varrho_\infty) \to (\varrho - \varrho_\infty) \text{ in } C_{\rm weak}([0,T]; (L^\gamma + L^2)(R^d)),$$

$$(\boldsymbol{m}_n - \boldsymbol{m}_\infty) \to (\boldsymbol{m} - \boldsymbol{m}_\infty) \text{ in } C_{\rm weak}([0,T]; (L^{\frac{2\gamma}{\gamma+1}} + L^2)(R^d; R^d)). \tag{3.1}$$

Indeed we have only to show uniform boundedness in time of the aforementioned quantities whereas convergence in $C_{\rm weak}$ follows immediately from the equations (2.1), (2.2). As the total energy is $e(\varrho, \boldsymbol{m})$ is a strictly convex function of $[\varrho, \boldsymbol{m}]$ it is easy to check that

(3.2)
$$e(\varrho, \boldsymbol{m} | \varrho_\infty, \boldsymbol{m}_\infty) \gtrsim (\varrho - \varrho_\infty)^2 + (\boldsymbol{m} - \boldsymbol{m}_\infty)^2 \ \text{ for } \frac{1}{2}\varrho_\infty \leq \varrho \leq 2\varrho_\infty, \ \ \frac{1}{2}|\boldsymbol{m}_\infty| \leq |\boldsymbol{m}| \leq 2|\boldsymbol{m}_\infty|,$$
$$\gtrsim 1 + \varrho^\gamma + \frac{|\boldsymbol{m}|^2}{\varrho} \text{ otherwise;}$$

whence the desired bounds follow from the energy inequality (2.3).

3.1. **Internal energy and pressure defect.** Next, recall that the sequence

$$0 \leq P(\varrho_n) - P'(\varrho_\infty)(\varrho_n - \varrho_\infty) - P(\varrho_\infty), \ n = 1, 2, \dots,$$

is bounded in $L^\infty(0, T; L^1(R^d))$ uniformly in n by (2.3). It holds

$$L^\infty(0, T; L^1(R^d)) \subset L^\infty_{w^*}(0, T, \mathcal{M}(R^d)),$$

where the symbol $\mathcal{M}(R^d)$ denotes the set of finite Borel measures on R^d and $L^\infty_{w^*}(0, T; \mathcal{M}(R^d))$ stands for the space of weak-(*)-measurable mappings $\nu : [0, T] \to \mathcal{M}(R^d)$ such that

$$\text{esssup}_{t \in [0,T]} \|\nu\|_{\mathcal{M}(R^d)} < \infty.$$

In addition, $L^\infty_{w^*}(0, T, \mathcal{M}(R^d))$ is the dual of $L^1(0, T, C_0(R^d))$ hence passing to a suitable subsequence as the case may be, there is $\mathcal{P} \in L^\infty_{w^*}(0, T; \mathcal{M}(R^d))$ such that

$$P(\varrho_n) - P'(\varrho_\infty)(\varrho_n - \varrho_\infty) - P(\varrho_\infty) \to \mathcal{P} \text{ weakly-(*) in } L^\infty_{w^*}(0, T; \mathcal{M}(R^d)).$$

As the function P is convex and the approximate internal energies are non–negative, we deduce by weak lower semicontinuity that

$$\mathfrak{R}_e \equiv \mathcal{P} - \big[P(\varrho) - P'(\varrho_\infty)(\varrho - \varrho_\infty) - P(\varrho_\infty)\big] \in L^\infty_{w^*}(0, T; \mathcal{M}^+(R^d)),$$

where $\mathcal{M}^+(R^d)$ denotes the set of non–negative finite Borel measures on R^d. This defines the internal energy defect measure $\mathfrak{R}_e$. It is important to note that

$$\int_0^T \int_{R^d} \psi(t)\varphi(x) \, \mathrm{d}\mathfrak{R}_e(t) \, \mathrm{d}t = \lim_{n \to \infty} \int_0^T \int_{R^d} \psi(t)\varphi(x) \left(P(\varrho_n) - P(\varrho)\right) \, \mathrm{d}x \, \mathrm{d}t \tag{3.3}$$
$$\text{for any } \psi \in L^1(0, T), \ \varphi \in C_c(R^d),$$

which will be used later.

3.2. **Viscosity defect.** We proceed by similar arguments and with the help of the bound (2.3) for the kinetic energies

$$0 \leq \frac{1}{2}\varrho_n |\mathbf{u}_n - \mathbf{u}_\infty|^2 = e_{\text{kin}}(\varrho_n, \boldsymbol{m}_n) - \partial e_{\text{kin}}(\varrho_\infty, \boldsymbol{m}_\infty)(\varrho_n - \varrho_\infty; \boldsymbol{m}_n - \boldsymbol{m}_\infty) - e_{\text{kin}}(\varrho_\infty, \boldsymbol{m}_\infty),$$

$n = 1, 2, \dots$. In particular, writing

$$\mathbb{C}_n \equiv \varrho_n(\boldsymbol{u}_n - \boldsymbol{u}_\infty) \otimes (\boldsymbol{u}_n - \boldsymbol{u}_\infty) = \mathbf{1}_{\varrho_n > 0} \left[\frac{\boldsymbol{m}_n \otimes \boldsymbol{m}_n}{\varrho_n} - \boldsymbol{u}_\infty \otimes \boldsymbol{m}_n - \boldsymbol{m}_n \otimes \boldsymbol{u}_\infty + \varrho_n \boldsymbol{u}_\infty \otimes \boldsymbol{u}_\infty \right]$$

we obtain the existence of $\mathbb{C} \in L^\infty_{w^*}(0,T;\mathcal{M}^+(R^d;R^{d\times d}_{\rm sym}))$, where $\mathcal{M}^+(R^d;R^{d\times d}_{\rm sym})$ is the set of finite symmetric positive semidefinite matrix–valued (signed) Borel measures, such that

$$\mathbb{C}_n \to \mathbb{C} \text{ weakly-(*) in } L^\infty_{w^*}(0,T;\mathcal{M}^+(R^d;R^{d\times d}_{\rm sym})).$$

More specifically, each component $C_{i,j}$ is a finite signed measure on R^d, $C_{i,j} = C_{j,i}$, and

$$\mathbb{C}(t) : (\xi \otimes \xi) \in \mathcal{M}^+(R^d) \text{ for any } \xi \in R^d \text{ and a.a. } t \in (0,T). \tag{3.4}$$

The viscosity defect measure is then defined by

$$\mathfrak{R}_v \equiv \mathbb{C} - \mathbf{1}_{\varrho>0}\left[\frac{\boldsymbol{m}\otimes\boldsymbol{m}}{\varrho} - \boldsymbol{u}_\infty \otimes \boldsymbol{m} - \boldsymbol{m}\otimes\boldsymbol{u}_\infty + \varrho \boldsymbol{u}_\infty \otimes \boldsymbol{u}_\infty\right] \in L^\infty_{w^*}(0,T;\mathcal{M}(R^d;R^{d\times d}_{\rm sym})).$$

Now, a simple but crucial observation is that the $\mathfrak{R}_v$ is positive semidefinite. To see this, we compute

$$\begin{aligned}\mathfrak{R}_v : (\xi\otimes\xi) &= \lim_{n\to\infty} \mathbf{1}_{\varrho_n>0}\frac{\boldsymbol{m}_n\otimes\boldsymbol{m}_n}{\varrho_n} : (\xi\otimes\xi) - \mathbf{1}_{\varrho>0}\frac{\boldsymbol{m}\otimes\boldsymbol{m}}{\varrho} : (\xi\otimes\xi) \\ &= \lim_{n\to\infty} \frac{|\boldsymbol{m}_n\cdot\xi|^2}{\varrho_n} - \frac{|\boldsymbol{m}\cdot\xi|^2}{\varrho} \text{ in } \mathcal{D}'((0,T)\times B)\end{aligned}$$

for any bounded ball $B \subset R^d$; whence the desired conclusion follows from the weak lower semicontinuity of the convex function $[\varrho, \boldsymbol{m}] \mapsto \frac{|\boldsymbol{m}\cdot\xi|^2}{\varrho}$, $\xi \in R^d$. We conclude that

$$\mathfrak{R}_v \in L^\infty_{w^*}(0,T;\mathcal{M}^+(R^d;R^{d\times d}_{\rm sym})).$$

Finally, similarly to (3.3), we note that

$$\begin{aligned}&\int_0^T \int_{R^d} \psi(t)\boldsymbol{\varphi}(x) : {\rm d}\mathfrak{R}_v(t)\,{\rm d}t \\ &\quad = \lim_{n\to\infty}\int_0^T\int_{R^d}\psi(t)\boldsymbol{\varphi}(x) : \left(\mathbf{1}_{\varrho_n>0}\frac{\boldsymbol{m}_n\otimes\boldsymbol{m}_n}{\varrho_n} - \mathbf{1}_{\varrho>0}\frac{\boldsymbol{m}\otimes\boldsymbol{m}}{\varrho}\right)\,{\rm d}x\,{\rm d}t \\ &\text{for any } \psi \in L^1(0,T),\ \boldsymbol{\varphi}\in C_c(R^d;R^{d\times d})).\end{aligned} \tag{3.5}$$

3.3. **Total defect.** We introduce the *total defect measure*

$$\mathbb{D} \equiv \mathfrak{R}_v + (\gamma-1)\mathfrak{R}_e\mathbb{I} \in L^\infty_{w^*}(0,T;\mathcal{M}^+(R^d;R^{d\times d}_{\rm sym})), \tag{3.6}$$

which describes the defect in the momentum equation as seen in (4.1) below. Moreover, we get for the total energy

$$e(\varrho_n,\boldsymbol{m}_n|\varrho_\infty,\boldsymbol{m}_\infty) \to e(\varrho,\boldsymbol{m}|\varrho_\infty,\boldsymbol{m}_\infty) + \frac{1}{2}{\rm trace}[\mathfrak{R}_v] + \mathfrak{R}_e \tag{3.7}$$

weakly-(*) in $L^\infty_{w^*}(0,T;\mathcal{M}^+(R^d;R^{d\times d}_{\rm sym}))$. In other words, we have a precise relation of the defect in the momentum equation and the defect of the energy. Finally, we get from (3.7) that

$$\int_0^T \int_{R^d} \psi(t)\varphi(x) \left(\frac{1}{2} \frac{|\boldsymbol{m}_n|^2}{\varrho_n} - \frac{1}{2} \frac{|\boldsymbol{m}|^2}{\varrho} + P(\varrho_n) - P(\varrho) \right) \,\mathrm{d}x\,\mathrm{d}t$$
$$\to \int_0^T \psi(t)\varphi(x)\,\mathrm{d}\left(\frac{1}{2}\mathrm{trace}[\mathfrak{R}_v(t)] + \mathfrak{R}_e(t) \right) \mathrm{d}t$$

for any $\psi \in L^1(0,T)$ and any $\varphi \in C_c(R^d)$.

3.4. **Bounded domain.** The above construction of the turbulent defect measure $\mathbb{D}$ as well as the proof of its properties can be carried out the same way on a bounded domain $\Omega \subset R^d$, while using the dualities

$$L^1(0,T;C(\overline{\Omega}))^* \cong L^\infty_{w^*}(0,T;\mathcal{M}(\overline{\Omega})) \text{ and } L^1(0,T;C_0(\overline{\Omega};R^{d\times d}))^* \cong L^\infty_{w^*}(0,T;\mathcal{M}(\overline{\Omega};R^{d\times d})),$$

respectively, where $\mathcal{M}(\overline{\Omega})$ is the set of bounded Borel measures on $\overline{\Omega}$ (and similarly for the matrix–valued case).

4. Asymptotic limit

It follows from the energy inequality (2.3) that

$$\int_0^T \int_{R^d} \mathbb{S}(\nabla_x \boldsymbol{u}_n) : \nabla_x \boldsymbol{u}_n \,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^T \int_{R^d} \left[\frac{\mu_n}{2} \left| \nabla_x \boldsymbol{u}_n + \nabla_x^t \boldsymbol{u}_n \right|^2 + \left(\lambda_n - \frac{2}{d}\mu_n \right) |\mathrm{div}_x \boldsymbol{u}_n|^2 \right] \mathrm{d}x\,\mathrm{d}t \le E_0,$$

which permits to pass to the limit on the right hand side of the momentum equation (2.2). Consequently, using (3.3), (3.5) we obtain

$$\int_0^T \int_{R^d} \left[\partial_t \psi \boldsymbol{m}\cdot\boldsymbol{\varphi} + \psi \mathbf{1}_{\varrho>0} \frac{\boldsymbol{m}\otimes\boldsymbol{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + \psi p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \right] \mathrm{d}x\,\mathrm{d}t$$
$$= - \int_0^T \psi \Big[\nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathfrak{R}_v(t) + (\gamma - 1)\mathrm{div}_x \boldsymbol{\varphi}\,\mathrm{d}\mathfrak{R}_e(t) \Big] \mathrm{d}t \tag{4.1}$$
$$\text{for any } \psi \in C^1_c(0,T),\ \boldsymbol{\varphi} \in C^1_c(R^d;R^d).$$

Thus, if the limit is a weak solution of the Euler system, that is (2.5), (2.6) hold, then the left hand side of (4.1) vanishes. Hence, in view of the definition of the total defect measure (3.6), we obtain

$$\int_{R^d} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathbb{D}(t) = 0 \text{ for any } \boldsymbol{\varphi} \in C^1_c(R^d;R^d) \text{ for a.a. } t \in (0,T)$$

which is nothing else than (1.7). Similarly, in the situation of Theorem 2.2, we deduce

$$\int_{R^d} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathbb{D}(t) = 0 \text{ for any } \boldsymbol{\varphi} \in C^1_c(\Omega;R^d) \text{ for a.a. } t \in (0,T)$$

extending (formally) $\mathbb{D}$ to be zero outside $\overline{\Omega}$.

4.1. **Equation** $\mathrm{div}_x \mathbb{D} = 0$ **in** R^d**.** The following result, which can be regarded as a version of Liouville's theorem, is crucial in the proof of Theorem 2.1.

Proposition 4.1. *Let* $\mathbb{D} \in \mathcal{M}^+(R^d; R^{d\times d}_{\rm sym})$ *satisfy*

$$\int_{R^d} \nabla_x \boldsymbol{\varphi} : {\rm d}\mathbb{D} = 0 \textit{ for any } \boldsymbol{\varphi} \in C^1_c(R^d; R^d). \tag{4.2}$$

Then $\mathbb{D} \equiv 0$.

Remark 4.2. The assumption that the matrix $\mathbb{D}$ is positive semidefinite (or alternatively negative semidefinite, as a matter of fact), is absolutely essential. Indeed, DeLellis and Székelyhidi in their proof of the so-called oscillatory lemma in [13] showed the existence of infinitely many smooth fields $\mathbb{D} \in C^\infty_c(R^d; R^{d\times d}_{\rm sym})$ satisfying $\mathrm{div}_x \mathbb{D} = 0$.

Proof of Proposition 4.1. The proof relies on the extension of (4.2) to all functions $\boldsymbol{\varphi} \in C^1(R^d; R^d)$ with $\nabla_x \boldsymbol{\varphi} \in L^\infty(R^d; R^{d\times d})$, which is possible since $\mathbb{D}$ is a finite measure. This then permits to test (4.2) by linear functions $\boldsymbol{\varphi}$ and the conclusion follows from the positive semidefinitness of $\mathbb{D}$.

To this end, let us consider a sequence of cut–off functions

$$\psi_n \in C^\infty_c(R^d),\ 0 \leq \psi \leq 1,\ \psi_n(x) = 1 \text{ for } |x| \leq n,\ \psi_n(x) = 0 \text{ for } |x| \geq 2n,\ |\nabla_x \psi| \lesssim \frac{1}{n}$$

uniformly for $n \to \infty$.

For $\boldsymbol{\varphi} \in C^1(R^d; R^d)$, with $\nabla_x \boldsymbol{\varphi} \in L^\infty(R^d; R^{d\times d})$, we have

$$|\boldsymbol{\varphi}(x)| \lesssim (1+n) \text{ for all } x \in \mathrm{supp}\, \psi_n;$$

whence

$$\begin{aligned} 0 &= \int_{R^d} \nabla_x(\psi_n \boldsymbol{\varphi}) : {\rm d}\mathbb{D} = \int_{R^d} \psi_n \nabla_x \boldsymbol{\varphi} : {\rm d}\mathbb{D} + \int_{R^d} (\nabla_x \psi_n) \otimes \boldsymbol{\varphi} : {\rm d}\mathbb{D} \\ &= \int_{|x|\leq n} \nabla_x \boldsymbol{\varphi} : {\rm d}\mathbb{D} + \int_{n<|x|<2n} \psi_n \nabla_x \boldsymbol{\varphi} : {\rm d}\mathbb{D} + \int_{n<|x|<2n} (\nabla_x \psi_n) \otimes \boldsymbol{\varphi} : {\rm d}\mathbb{D} \end{aligned}$$

Seeing that

$$|\psi_n \nabla_x \boldsymbol{\varphi}(x)| + |(\nabla_x \psi_n) \otimes \boldsymbol{\varphi}| \lesssim 1 \text{ whenever } n \leq |x| \leq 2n$$

we may use the fact that $\mathbb{D}$ is a finite (signed) measure together with Lebesgue's dominated convergence theorem to let $n \to \infty$ and conclude that

$$\int_{R^d} \nabla_x \boldsymbol{\varphi} : {\rm d}\mathbb{D} = 0 \text{ for any } \varphi \in C^1(R^d; R^d),\ \nabla_x \boldsymbol{\varphi} \in L^\infty(R^d; R^{d\times d}). \tag{4.3}$$

Finally, given a vector $\xi \in R^d$, we may use

$$\boldsymbol{\varphi}(x) = \xi(\xi \cdot x)$$

as a test function in (4.3) to obtain

$$\int_{R^d} (\xi \otimes \xi) : \mathrm{d}\mathbb{D} = 0 \text{ for any } \xi \in R^d.$$

As $\mathbb{D}$ is positive semidefinite in the sense of (3.4), i.e. $(\xi \otimes \xi) : \mathbb{D}$ is a non–negative finite measure on R^d, this yields $(\xi \otimes \xi) : \mathbb{D} = 0$ for any $\xi \in R^d$. Thus for any $g \in C_b(R^d)$, $g \geq 0$, and the matrix $\int_{R^d} g \, \mathrm{d}\mathbb{D}$ is positive semidefinite and we may infer

$$\int_{R^d} g \, \mathrm{d}D_{i,j} = 0 \text{ for any } i, j.$$

As g was arbitrary, this yields the desired conclusion $\mathbb{D} \equiv 0$. □

4.2. **Equation $\mathrm{div}_x \mathbb{D} = 0$ in a bounded domain.** A trivial example of a constant–valued matrix shows that Proposition 4.1 does not hold if R^d is replaced by a bounded domain Ω unless some extra restrictions are imposed. In addition to the hypotheses of Proposition 4.1, we shall assume that $\mathbb{D}$ vanishes sufficiently fast near the boundary $\partial\Omega$.

Proposition 4.3. *Let $\Omega \subset R^d$ be a bounded domain. Let $\mathbb{D} \in \mathcal{M}^+(\overline{\Omega}; R^{d\times d}_{\mathrm{sym}})$ satisfying*

$$\int_{R^d} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathbb{D} = 0 \text{ for any } \boldsymbol{\varphi} \in C^1_c(\Omega; R^d), \tag{4.4}$$

and

$$\frac{1}{\delta} \int_{\{x \in \Omega; \mathrm{dist}[x, \partial\Omega] \leq \delta\}} \mathrm{d}(\mathrm{trace})[\mathbb{D}] \to 0 \text{ as } \delta \to 0. \tag{4.5}$$

Then $\mathbb{D} \equiv 0$.

Proof. Similarly to the proof of Proposition 4.1, it is enough to show that (4.4) can be extended to a suitable function $\varphi \in C^1(\overline{\Omega}; R^d)$, whose gradient is constant.

It is a routine matter, cf. e.g. Galdi [21], to construct a sequence of cut–off functions ψ_n enjoying the following properties:

$$\psi_n \in C^1_c(\Omega), \ 0 \leq \psi_n \leq 1, \ \psi_n(x) = 1 \text{ whenever } \mathrm{dist}[x, \partial\Omega] > \frac{1}{n}, \ |\nabla_x \psi_n| \lesssim n.$$

Thus, plugging $\psi_n \boldsymbol{\varphi}$, $\boldsymbol{\varphi} \in C^1(\overline{\Omega}; R^d)$ in (4.4) we get

$$\begin{aligned} 0 = \int_\Omega \nabla_x (\psi_n \boldsymbol{\varphi}) : \mathrm{d}\mathbb{D} &= \int_\Omega \psi_n \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathbb{D} + \int_\Omega (\nabla_x \psi_n) \otimes \boldsymbol{\varphi} : \mathrm{d}\mathbb{D} \\ = \int_{\mathrm{dist}[x,\partial\Omega] > \frac{1}{n}} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathbb{D} &+ \int_{\mathrm{dist}[x,\partial\Omega] \leq \frac{1}{n}} \psi_n \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathbb{D} + \int_{\mathrm{dist}[x,\partial\Omega] \leq \frac{1}{n}} (\nabla_x \psi_n) \otimes \boldsymbol{\varphi} : \mathrm{d}\mathbb{D} \end{aligned}$$

Now, we observe that

$$|\psi_n \nabla_x \boldsymbol{\varphi}(x)| + |(\nabla_x \psi_n) \otimes \boldsymbol{\varphi}(x)| \lesssim n \text{ whenever } \mathrm{dist}[x, \partial\Omega] \leq \frac{1}{n},$$

which due to (4.5) allows to pass to the limit as $n \to \infty$ in the second and the third term on the right hand side. The convergence of the first term follows from the fact that by (4.5) the defect vanishes on the boundary, i.e.

$$\int_{\partial\Omega} \mathrm{d}|\mathbb{D}| = 0,$$

and in the interior of Ω we have pointwise convergence of the corresponding integrand. $\square$

5. Strong convergence

Applying Proposition 4.1 in the situation of Theorem 2.1 and Proposition 4.3 in the case of Theorem 2.2 we obtain that $\mathfrak{R}_v \equiv 0$ and $\mathfrak{R}_e \equiv 0$. In accordance with (3.7), this yields

$$e(\varrho_n, \boldsymbol{m}_n | \varrho_\infty, \boldsymbol{m}_\infty) \to e(\varrho, \boldsymbol{m} | \varrho_\infty, \boldsymbol{m}_\infty) \tag{5.1}$$

weakly-(*) in $L^\infty_{w^*}(0,T;\mathcal{M}^+(R^d))$ provided we have extended $\boldsymbol{m}_n = \boldsymbol{m} = 0$, $\varrho_n = \varrho = 0$ outside Ω in the case of Theorem 2.2. We show that this implies the strong convergence claimed in Theorems 2.1, 2.2.

First, we recall that both kinetic and internal energy are convex functions of the density and the momentum so from (5.1) we obtain

$$\int_0^T \int_{R^d} \left[\frac{|\boldsymbol{m}_n|^2}{\varrho_n} - 2\boldsymbol{m}_n \cdot \boldsymbol{u}_\infty + \varrho_n |\boldsymbol{u}_\infty|^2 \right] \mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_{R^d} \left[\frac{|\boldsymbol{m}|^2}{\varrho} - 2\boldsymbol{m} \cdot \boldsymbol{u}_\infty + \varrho |\boldsymbol{u}_\infty|^2 \right] \mathrm{d}x\,\mathrm{d}t,$$

$$\int_0^T \int_{R^d} P(\varrho_n) - P'(\varrho_\infty)(\varrho_n - \varrho_\infty) - P(\varrho_\infty)\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_{R^d} P(\varrho) - P'(\varrho_\infty)(\varrho - \varrho_\infty) - P(\varrho_\infty)\,\mathrm{d}x\,\mathrm{d}t.$$

Moreover, we may apply convexity again to deduce that

$$\int_B \left[\frac{|\boldsymbol{m}_n|^2}{\varrho_n} - 2\boldsymbol{m}_n \cdot \boldsymbol{u}_\infty + \varrho_n |\boldsymbol{u}_\infty|^2 \right] \mathrm{d}x\,\mathrm{d}t \to \int_B \left[\frac{|\boldsymbol{m}|^2}{\varrho} - 2\boldsymbol{m} \cdot \boldsymbol{u}_\infty + \varrho |\boldsymbol{u}_\infty|^2 \right] \mathrm{d}x\,\mathrm{d}t, \tag{5.2}$$

$$\int_B P(\varrho_n) - P'(\varrho_\infty)(\varrho_n - \varrho_\infty) - P(\varrho_\infty)\,\mathrm{d}x\,\mathrm{d}t \to \int_B P(\varrho) - P'(\varrho_\infty)(\varrho - \varrho_\infty) - P(\varrho_\infty)\,\mathrm{d}x\,\mathrm{d}t,$$

for every Borel set $B \subset [0,T] \times R^d$.

Accordingly, choosing $B = [0,T] \times K$ for a compact set $K \subset R^d$, we obtain the convergence of the norms of ϱ_n in $L^\gamma([0,T] \times K)$, hence the strong convergence

$$\varrho_n \to \varrho \ \text{ in } L^\gamma([0,T] \times K).$$

The strong convergence on the full space $[0,T] \times R^d$ now follows by a tightness argument. Indeed, due to the weak convergence of the measures in (5.1), Prokhorov's theorem yields their tightness. In particular, for a given $\varepsilon > 0$ there exists a compact set $K \subset R^d$ such that

$$\sup_{n=1,2,\dots} \int_0^T \int_{K^c} P(\varrho_n) - P'(\varrho_\infty)(\varrho_n - \varrho_\infty) - P(\varrho_\infty)\,\mathrm{d}x\,\mathrm{d}t < \varepsilon,$$

$$\int_0^T \int_{K^c} P(\varrho) - P'(\varrho_\infty)(\varrho - \varrho_\infty) - P(\varrho_\infty)\,\mathrm{d}x\,\mathrm{d}t < \varepsilon.$$

Finally, we write

$$\begin{aligned}
\|\varrho_n-\varrho\|_{(L^\gamma+L^2)([0,T]\times R^d)} &\le \|\varrho_n-\varrho\|_{(L^\gamma+L^2)([0,T]\times K)}+\|\varrho_n-\varrho_\infty\|_{(L^\gamma+L^2)([0,T]\times K^c)}\\
&\quad+\|\varrho-\varrho_\infty\|_{(L^\gamma+L^2)([0,T]\times K^c)}\\
&\le \|\varrho_n-\varrho\|_{(L^\gamma+L^2)([0,T]\times K)}+\int_0^T\int_{K^c}P(\varrho_n)-P'(\varrho_\infty)(\varrho_n-\varrho_\infty)-P(\varrho_\infty)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_0^T\int_{K^c}P(\varrho)-P'(\varrho_\infty)(\varrho-\varrho_\infty)-P(\varrho_\infty)\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

where the first term converges to zero as $n\to\infty$ whereas the second as well as the third term is small uniformly in n.

Let us now establish the strong convergence of the momenta on $[0,T]\times R^d$. To this end, we recall that by the energy bound (2.3) it holds (up to a subsequence)

$$\boldsymbol{h}_n\equiv\frac{\boldsymbol{m}_n}{\sqrt{\varrho_n}}\to\boldsymbol{h}\text{ weakly in }L^2([0,T]\times B;R^d)$$

for some $\boldsymbol{h}\in L^2([0,T]\times B;R^d)$, and by (3.1)

$$\boldsymbol{m}_n\to\boldsymbol{m}\text{ weakly in }(L^{\frac{2\gamma}{\gamma+1}})([0,T]\times B;R^d)$$

for any bounded ball $B\subset R^d$. We shall show that

$$\boldsymbol{h}=\mathbf{1}_{\varrho>0}\frac{\boldsymbol{m}}{\sqrt{\varrho}}\text{ a.a. in }[0,T]\times B.$$

Combining the weak convergence of $\boldsymbol{h}_n$ with the strong convergence of ϱ_n and the weak convergence of $\boldsymbol{m}_n$ we obtain

$$\sqrt{\varrho_n}\boldsymbol{h}_n=\boldsymbol{m}_n\to\mathbf{m}=\sqrt{\varrho}\boldsymbol{h}\text{ weakly in }L^1([0,T]\times B;R^d);$$

whence it is enough to prove that $\boldsymbol{h}=0$ whenever $\varrho=0$. By weak lower semicontinuity of the L^2-norm together with (5.2), we obtain

$$\int_{\varrho<\delta}\mathbf{1}_B|\boldsymbol{h}|^2\,\mathrm{d}x\,\mathrm{d}t\le\lim_{n\to\infty}\int_{\varrho<\delta}\mathbf{1}_B\frac{|\boldsymbol{m}_n|^2}{\varrho_n}\,\mathrm{d}x\,\mathrm{d}t=\int_{\varrho<\delta}\mathbf{1}_B\frac{|\boldsymbol{m}|^2}{\varrho}\,\mathrm{d}x\,\mathrm{d}t.$$

Now, it is enough to observe that in the limit $\delta\to0$, the left hand side converges to

$$\int_{\varrho=0}\mathbf{1}_B|\boldsymbol{h}|^2\,\mathrm{d}x\,\mathrm{d}t,$$

whereas the right hand side vanishes, since due to the integrability of the kinetic energy $\frac{|\boldsymbol{m}|^2}{\varrho}$ it holds that the set, where $\varrho=0$ and $\boldsymbol{m}\neq0$, is of zero Lebesgue measure. Thus $\boldsymbol{h}=0$ whenever $\varrho=0$.

To summarize, we have shown that

$$\frac{\boldsymbol{m}_n}{\sqrt{\varrho_n}}\to\mathbf{1}_{\varrho>0}\frac{\boldsymbol{m}}{\sqrt{\varrho}}\text{ weakly in }L^2([0,T]\times B;R^d)$$

and hence strongly due to (5.2), which implies the strong convergence

$$\boldsymbol{m}_n = \sqrt{\varrho_n}\frac{\boldsymbol{m}_n}{\sqrt{\varrho_n}} \to \boldsymbol{m} \text{ in } L^{\frac{2\gamma}{\gamma+1}}([0,T]\times B; R^d).$$

Finally, a tightness argument as for the density above implies the strong convergence

$$(\boldsymbol{m} - \boldsymbol{m}_\infty) \to (\boldsymbol{m} - \boldsymbol{m}_\infty) \text{ in } (L^{\frac{2\gamma}{\gamma+1}} + L^2)([0,T]\times R^d; R^d).$$

The strong convergence of the densities and the momenta from the statements of Theorems 2.1, 2.2 now follows immediately from the fact that uniformly in n

$$(\varrho_n - \varrho_\infty) \in L^\infty(0,T;(L^\gamma + L^2)(R^d)), \quad (\boldsymbol{m}_n - \boldsymbol{m}_\infty) \in L^\infty(0,T;(L^{\frac{2\gamma}{\gamma+1}} + L^2)(R^d;R^d)),$$

due to the energy bound (2.3). The convergence (up to a subsequence) of the energies in L^1 is then a consequence of the strong convergence of $\frac{|\boldsymbol{m}_n|}{\sqrt{\varrho_n}}$ and ϱ_n together with (5.1) and Vitali's theorem. This completes the proof of Theorem 2.1, 2.2.

6. Concluding remarks

We conclude the paper by a short discussion on possible implications and/or extensions of the results stated in Theorems 2.1, 2.2.

6.1. **General domains.** It is easy to see that Theorem 2.2 can be extended to Lipschitz domains with compact boundaries, in particular to exterior domains, provided the total energy is bounded as in Theorem 2.1. More general domains like infinite strips can be more delicate as the defect near the boundary may interfere with that at the far field.

6.2. **Singular set.** Consider a bounded domain and suppose, similarly to Theorem 2.2, that the limit functions are still solutions of the Euler system. We are interested in the support of the defect measure $\mathbb{D}$ introduced in (3.6). As an immediate consequence of Theorem 2.2 we get the following:

Suppose that $\mathrm{supp}[\mathbb{D}(t)] \subset K$ *for a.a.* $t \in (0,T)$*, where* $K \subset \Omega$ *is compact. Then* $\mathbb{D} \equiv 0$. Refining this observation, we may infer that $\mathrm{supp}[\mathbb{D}(t)]$ cannot contain isolated compact sets.

6.3. **More general approximate solutions.** Following DiPerna and Majda [15] we may consider more general sequences of approximate solutions than those arising in the vanishing viscosity limit. The only piece of information that should be retained is:

- The approximate sequence $[\varrho_n, \boldsymbol{m}_n]$ should have uniformly bounded energy:

$$\mathrm{esssup}_{t\in(0,T)} \int_{R^d} e(\varrho_n, \boldsymbol{m}_n | \varrho_\infty, \boldsymbol{m}_\infty)\, \mathrm{d}x \le E_0.$$

- The momentum equation satisfied by $[\varrho_n, \boldsymbol{m}_n]$ should be *consistent* with the Euler limit:

$$\partial_t \boldsymbol{m}_n + \mathrm{div}_x \left(\mathbf{1}_{\varrho_n > 0} \frac{\boldsymbol{m}_n \otimes \boldsymbol{m}_n}{\varrho_n} \right) + \nabla_x p(\varrho_n) \to 0 \text{ in } \mathcal{D}'((0,T)\times R^d; R^d) \text{ as } n \to \infty.$$

- Isentropic pressure–density equation of state $p(\varrho) = (\gamma - 1)P(\varrho)$, $\gamma > 1$, is valid.

In particular, the approximate solutions may be solutions of the same isentropic Euler system with possibly different initial data, or numerical solutions produced by a consistent energy dissipative numerical scheme. Note that at least some suitable weak solutions of the Euler system admit the finite speed of propagation (see [28]); whence problems on R^d with so-called compactly supported data, e.g. $\varrho_0 = \varrho_\infty$, $\boldsymbol{m}_0 = 0$ outside a bounded ball, can be in particular viewed as problems on a sufficiently large bounded domain.

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