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GENERALIZED SOLUTIONS TO MODELS OF INVISCID FLUIDS

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Abstract. We discuss several approaches to generalized solutions of problems describing the motion of inviscid fluids. We propose a new concept of *dissipative* solution to the compressible Euler system based on a careful analysis of possible oscillations and/or concentrations in the associated generating sequence. Unlike the conventional measure–valued solutions or rather their expected values, the dissipative solutions comply with a natural compatibility condition – they are classical solutions as long as they enjoy a certain degree of smoothness.

1. **Introduction.** We consider a mathematical model of an *inviscid compressible fluid* with the mass density $\varrho = \varrho(t,x)$ moving with the velocity $\mathbf{u} = \mathbf{u}(t,x)$. Thermal effects being neglected, the evolution of the fluid is governed by the *Euler system*:

$$\begin{aligned} \partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) &= 0, \\ \partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) &= 0. \end{aligned} \tag{1.1}$$

The quantity $p = p(\varrho)$ is the pressure. We suppose the internal energy $e = e(\varrho)$ is related to the pressure through the formula

$$(\gamma - 1)\varrho e(\varrho) = p(\varrho), \tag{1.2}$$

where $\gamma > 1$ is the adiabatic constant. The total energy of the fluid is given by

$$E(\varrho, \mathbf{u}) = \varrho \left[\frac{1}{2} |\mathbf{u}|^2 + e(\varrho) \right]. \tag{1.3}$$

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If not otherwise stated, we suppose the fluid occupies a bounded domain $\Omega \subset R^d$, $d = 2, 3$ with impermeable boundary:

$$\mathbf{u} \cdot \mathbf{n}|_{\partial \Omega} = 0. \tag{1.4}$$

The initial state of the system is given:

$$\varrho(0, \cdot) = \varrho_0, \ \varrho \mathbf{u}(0, \cdot) = \mathbf{m}_0. \tag{1.5}$$

The Euler system has been derived from the basic principles of continuum mechanics on condition that all quantities appearing in (1.1) are at least continuously differentiable and the density ϱ is bounded below away from zero. If the initial data belong to this class then the Euler system admits local–in–time smooth solutions, see e.g. Tani [23]. The life span of such solutions, however, is finite for a fairly general class of the initial data, see Smoller [22].

To continue solutions globally in time, the concept of *weak* solutions is introduced, where all derivatives in (1.1) are understood in the sense of distributions. It is also more convenient to reformulate the problem in the *conservative* variables ϱ and $\mathbf{m} = \varrho \mathbf{u}$:

$$\begin{aligned} \partial_t \varrho + \mathrm{div}_x \mathbf{m} &= 0, \\ \partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) &= 0. \end{aligned} \tag{1.6}$$

Weak solutions are not unique unless a suitable admissibility criterion is imposed. In the context of the Euler system, it is customary to require the energy inequality

$$\partial_t E(\varrho, \mathbf{m}) + \mathrm{div}_x \left[\left(E(\varrho, \mathbf{m}) + p(\varrho) \right) \frac{\mathbf{m}}{\varrho} \right] \leq 0, \tag{1.7}$$

where

$$E(\varrho, \mathbf{m}) = \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + \varrho e(\varrho). \tag{1.8}$$

In view of (1.2), we obtain

$$p(\varrho) = a \varrho^\gamma, \ \varrho e(\varrho) \equiv P(\varrho) = \frac{a}{\gamma - 1} \varrho^\gamma.$$

Indeed the internal energy ϱe coincides (modulo a linear function) with the pressure potential $P = P(\varrho)$:

$$\varrho e(\varrho) = P(\varrho), \text{ where the latter satisfies } P'(\varrho)\varrho - P(\varrho) = p(\varrho).$$

In particular, the energy E is a *convex* function of $\mathbf{m}$ and ϱ.

Even if (1.7) is imposed as an extra admissibility constraint, the Euler system remains ill–posed at least for $N = 2, 3$. As a matter of fact, there exist Lipschitz initial data for which (1.6), (1.7) admits infinitely many solutions on a given time interval $(0, T)$, see Chiodaroli et al. [8], [9], and [10]. Still the question of the *existence* of global–in–time weak solutions to (1.6), (1.7) for *general* initial data remains open.

Our goal is to present several concepts of generalized solutions to the Euler system and discuss their basic properties. In particular, we address the question of *compactness* of the solution set and its *stability* with respect to perturbations. Finally, we introduce a new concept of *dissipative* solution to the Euler system.

The paper is organized as follows. In Section 2 we discuss the problem of compactness of the solution set of the compressible Euler system. Section 3 presents a short review of various concepts of the so–called measure–valued solutions. In

Section 4, we introduce a new concept of dissipative solutions. In Section 5 we introduce a generating sequence and show existence of a dissipative solution to the Euler system for a fairly general class of initial data. Various properties of dissipative solutions including weak–strong uniqueness and conditional regularity are discussed in Section 6. The paper is concluded in Section 7 by introducing admissible dissipative solutions that maximize the mechanical energy dissipation.

2. **Oscillatory solutions.** As revealed by the method of convex integration, bounded sets of solutions to the Euler system may not be precompact even with respect to the natural weak topology, cf. e.g. De Lellis and Székelyhidi [11]. Indeed we claim the following result.

Proposition 2.1. *Let $\Omega \subset R^N$, $N = 2, 3$ be a bounded domain. Let $\varrho_0 \in L^\infty(\Omega)$, $\varrho_0 > 0$ be given.*

Then there exists a sequence of weak solutions $[\varrho_n, \mathbf{m}_n]$ to the Euler system (1.6) in $(0,T) \times \Omega$ with $\varrho_n = \varrho_n(x)$ such that

$$\varrho_n \to \varrho_0 \text{ weakly-(*) in } L^\infty(\Omega),\ \mathbf{m}_n \to 0 \text{ weakly-(*) in } L^\infty((0,T)\times\Omega; R^N), \tag{2.1}$$

$$\liminf_{n\to\infty} \int_\Omega |\varrho_n - \varrho_0| \ \mathrm{d}x > 0. \tag{2.2}$$

Remark 2.2. Relation (2.2) means that the convergence claimed in (2.1) is not strong for $\{\varrho_n\}_{n>0}$.

Proof. The proof is based on the method of convex integration. First, consider a division of the domain Ω,

$$\Omega = \cup_{i\in I}\overline{\Omega}_i,\ \Omega_i \cap \Omega_j = \emptyset \text{ for } i \neq j,$$

where I is a finite index set, and Ω_i are domains. Furthermore, we consider a sequence of endpoints T_i. Next, for each Ω_i, fix $\varrho_i > 0$ - a constant density distribution. Similarly to [17], [21], we consider the following problem:

$$\begin{aligned} \mathrm{div}_x \mathbf{m}_i &= 0,\\ \partial_t \mathbf{m}_i + \mathrm{div}_x \left(\frac{\mathbf{m}_i \otimes \mathbf{m}_i}{\varrho_i} - \frac{1}{N}\frac{|\mathbf{m}_i|^2}{\varrho_i}\mathbb{I} \right) &= 0\\ \frac{1}{2}\frac{|\mathbf{m}_i|^2}{\varrho_i} &= \Lambda - p(\varrho_i)\frac{N}{2} \end{aligned} \tag{2.3}$$

in Ω_i, where $\Lambda > 0$ is a certain positive constant to be determined below. The apparently overdetermined problem (2.3) is supplemented by the initial–end state condition

$$\mathbf{m}_i(0,\cdot) = \mathbf{m}_i(T,\cdot) = 0. \tag{2.4}$$

In addition, we impose the "no flux" boundary conditions specified in the weak sense as follows: We suppose that

$$\int_0^T \int_{\Omega_i} \mathbf{m}_i \cdot \nabla_x \varphi = 0 \ \mathrm{d}x\,\mathrm{d}t \tag{2.5}$$

for any $\varphi \in C^1([0,T]\times\overline{\Omega}_i)$ and

$$\int_0^T \int_{\Omega_i} \left[\mathbf{m}_i \cdot \partial_t \boldsymbol{\varphi} + \left(\frac{\mathbf{m}_i \otimes \mathbf{m}_i}{\varrho_i} - \frac{1}{N}\frac{|\mathbf{m}_i|^2}{\varrho_i}\mathbb{I} \right) : \nabla_x \boldsymbol{\varphi} \right] \mathrm{d}x\,\mathrm{d}t = 0 \tag{2.6}$$

for any $\boldsymbol{\varphi} \in C^1([0,T] \times \overline{\Omega}_i; R^N)$. Accordingly, solutions defined on Ω_i can be "pasted" together to produce a weak solution defined on the whole set Ω. Indeed, for $[\varrho_i, \mathbf{m}_i]$ satisfying (2.4), we can set

$$\varrho = \sum_i 1_{\Omega_i} \varrho_i, \ \mathbf{m}(t,\cdot) = \sum_i 1_{\Omega_i} \mathbf{m}_i(t - mT), \ t \in [mT, (m+1)T) \tag{2.7}$$

for $m = 0, 1, \dots$. It is a routine matter to check that $[\varrho, \mathbf{m}]$ defined through (2.7) is a weak solution of the Euler system (1.6) for $t \in (0, \infty)$, satisfying the impermeability condition (1.4). Note that the momentum equation reads

$$\int_0^\infty \int_\Omega \left[\mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} - \Lambda \mathrm{div}_x \boldsymbol{\varphi} \right] \, \mathrm{d}x \, \mathrm{d}t = 0,$$

where

$$\int_0^\infty \int_\Omega \Lambda \mathrm{div}_x \boldsymbol{\varphi} \, \mathrm{d}x = 0 \text{ whenever } \boldsymbol{\varphi} \in C^1_c([0,\infty) \times \overline{\Omega}; R^d), \ \boldsymbol{\varphi} \cdot \mathbf{n}|_\Omega = 0.$$

Note that, in contrast with (2.6) where no boundary conditions are imposed on test functions, we have effectively used the fact $\boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega} = 0$ here.

Now we claim that problem (2.3)–(2.6) admits, in fact, infinitely many solutions as soon as

$$0 < \underline{\varrho} \le \varrho_i \le \overline{\varrho}, \ i = 1, 2, \dots$$

for certain $\Lambda = \Lambda(\underline{\varrho}, \overline{\varrho}) > 0$. Indeed we refer e.g. to Chiodaroli [7] or [21] for the proof.

Finally, we consider an oscillating sequence

$$\varrho_n = \varrho_i^n \in \Omega_i^n, \ \varrho_n \to \varrho_0 \text{ weakly-(*) in } L^\infty(\Omega) \text{ but not strongly in } L^1(\Omega),$$

with the family of times $T_n = \frac{1}{2^n}$, and

$$\mathbf{m}_n \text{ defined on } [0,\infty), \ \mathbf{m}_n \left(\frac{m}{2^n}, \cdot \right) = 0, \ m = 0, 1, \dots .$$

It can be checked that $[\varrho_n, \mathbf{m}_n]$ enjoys the properties claimed in the conclusion of Proposition 2.1. Indeed we have

$$\mathbf{m}_n \to \mathbf{m} \text{ in } C_{\mathrm{weak}}([0,T]; L^2(\Omega; R^N)) \text{ and weakly-(*) in } L^\infty((0,T) \times \Omega; R^N)$$

for any $T > 0$. Moreover, thanks to the pointwise convergence in $L^2(\Omega; R^N) -$ weak at *any* $t \ge 0$, we have

$$\mathbf{m} \left(\frac{m}{2^n}, \cdot \right) = 0 \text{ for any } m = 0, 1, \dots, \ n = 1, 2, \dots; \text{ whence } \mathbf{m} \equiv 0. \qquad \square$$

Apparently, the limit quantity $\varrho = \varrho_0(x)$, $\mathbf{m} \equiv 0$ is a (weak) solution of the Euler system only if $\varrho_0 = \overline{\varrho}$ - a (positive) constant. Otherwise, the weak closure takes us out of the set of weak solutions. This indicates that a possibly larger class of solutions is necessary to characterize the weak closure. These are the measure–valued solutions discussed in the next section.

3. **Measure–valued solutions.** The concept of *measure–valued* solution was introduced to capture the two major stumbling blocks to strong stability of the Euler system: (i) oscillations discussed in the previous section, and (ii) concentrations due to the kinetic energy "blow up". These two phenomena are conveniently captured by the oscillation–concentration defect measure introduced by Alibert and Bouchitté [1].

Gwiazda et al. [19] used the approach of [1] for the compressible Euler system. This technique requires a certain structure of the nonlinearities to define their recession functions. This structure enforces the introduction of new state variables: the density ϱ and the "weighted velocity" $\sqrt{\varrho}\mathbf{u}$. It is interesting to note that a similar choice of variables has been used by Chen and Glimm [6] in a different context. Within this framework, Gwiazda et al. established the existence as well as the weak–strong uniqueness principle.

The approach of [19] was highly simplified in [14] in the context of the compressible Navier–Stokes. The Alibert–Bouchitté defect measures have been replaced by a combination of the standard Young measure acting on the natural variables ϱ and $\mathbf{u}$ and the concentration defect measures balanced by their dissipation counterpart in the energy inequality. This technique has been adapted by Basarič [2] to the compressible Euler system (1.1) posed on a general, possibly unbounded, domain.

Another simplification, using rather the conservative variables $\varrho, \mathbf{m} \equiv \varrho\mathbf{u}$, has been introduced in [4] in order to construct a solution semiflow to the isentropic Euler system. We refer also to [3] for the application to the complete Euler system and to Section 6 and Section 7 below for further discussion of this subject. These developments led to the work [16], where the underlying ideas for the notion of dissipative solution presented in the sequel can be found. This particularly straightforward formulation allowed to establish the following striking dichotomy: a weakly converging sequence of (weak) solutions to the isentropic Navier–Stokes system on R^N, $N = 2, 3$, in the vanishing viscosity limit either **(i)** converges strongly in the energy norm, or **(ii)** the limit is not a weak solution of the associated Euler system, see [16].

4. **Dissipative solutions.** Motivated by the above mentioned results, we propose the concept of *dissipative* solution adapted to the natural *conservative variables*: the density ϱ and the momentum $\mathbf{m}$ in the Euler system (1.6). They satisfy the following system of equations in the sense of distributions:

$$
\begin{gathered}
\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0, \\
\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = -\mathrm{div}_x \left(\mathfrak{R}_v + \mathfrak{R}_p \mathbb{I} \right) \\
\partial_t \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) + \frac{1}{2} \mathrm{trace}[\mathfrak{R}_v] + \frac{1}{\gamma - 1} \mathfrak{R}_p \right] \,\mathrm{d}x \leq 0,
\end{gathered}
\tag{4.1}
$$

where $\mathfrak{R}_v \in L^\infty(0,T; \mathcal{M}^+(\overline{\Omega}; R^{d\times d}_{\rm sym}))$, $\mathfrak{R}_p \in L^\infty(0,T;\mathcal{M}^+(\overline{\Omega}))$ are the turbulent defect measure associated to the convective term and the pressure, respectively. Here, the symbol $\mathcal{M}^+(\overline{\Omega})$ denotes the space of non–negative Borel measures on $\overline{\Omega}$, while $\mathcal{M}^+(\overline{\Omega}; R^{d\times d}_{\rm sym})$ is the space of matrix valued (signed) measures on $\overline{\Omega}$ ranging in positive semi–definite matrices, meaning

$$
\mathfrak{R}_v : (\xi \otimes \xi) \in \mathcal{M}^+(\overline{\Omega}) \text{ for any } \xi \in R^d.
$$

Observe that dissipative solutions are weakly continuous in time, specifically,

$$
\varrho \in C_{\rm weak}([0,T]; L^\gamma(\Omega)),\ \mathbf{m} \in C_{\rm weak}([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d))
$$

so that one can correctly define the initial conditions. The boundary condition (1.4) is satisfied in the weak sense through suitable choice of the test functions in the weak formulation. The exact definition reads as follows:

Definition 4.1. We say that

$$\varrho \in C_{\rm weak}([0,T];L^\gamma(\Omega)),\ \ \varrho \geq 0,\ \ \mathbf{m} \in C_{\rm weak}([0,T];L^{\frac{2\gamma}{\gamma+1}}(\Omega;R^d)),$$

is a *dissipative solution* to the Euler system (1.1)–(1.5) if there exist turbulent defect measures

$$\mathfrak{R}_v \in L^\infty(0,T;\mathcal{M}^+(\overline{\Omega};R^{d\times d}_{\rm sym})),\ \ \mathfrak{R}_p \in L^\infty(0,T;\mathcal{M}^+(\overline{\Omega}))$$

such that the following holds:

$$\left[\int_\Omega \varrho\varphi \ {\rm d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \Big[\varrho\partial_t\varphi + \mathbf{m}\cdot\nabla_x\varphi\Big] \ {\rm d}x\,{\rm d}t \tag{4.2}$$

for any $0<\tau<T$, and any $\varphi \in C^1_c([0,T)\times\overline{\Omega})$;

$$\begin{split}\left[\int_\Omega \mathbf{m}\cdot\boldsymbol{\varphi} \ {\rm d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\mathbf{m}\cdot\partial_t\boldsymbol{\varphi} + \left(\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}:\nabla_x\boldsymbol{\varphi}\right) + p(\varrho){\rm div}_x\boldsymbol{\varphi}\right] \ {\rm d}x\,{\rm d}t \\ + \int_0^\tau \int_{\overline{\Omega}} \nabla_x\boldsymbol{\varphi} : {\rm d}\Big[\mathfrak{R}_v + \mathfrak{R}_p\mathbb{I}\Big] \ {\rm d}t\end{split} \tag{4.3}$$

for any $0<\tau<T$, and any $\varphi \in C^1_c([0,T)\times\overline{\Omega};R^d)$, $\boldsymbol{\varphi}\cdot\mathbf{n}|_{\partial\Omega} = 0$;

$$\begin{split}\left[\psi\left(\int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)\right] \ {\rm d}x + \int_{\overline{\Omega}}\frac{1}{2}{\rm d}\,{\rm trace}[\mathfrak{R}_v] + \int_{\overline{\Omega}}\frac{1}{\gamma-1}{\rm d}\mathfrak{R}_p\right)\right]_{t=\tau_1-}^{t=\tau_2+} \\ \leq \int_{\tau_1}^{\tau_2}\partial_t\psi\left(\int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)\right] \ {\rm d}x + \int_{\overline{\Omega}}\frac{1}{2}{\rm d}\,{\rm trace}[\mathfrak{R}_v] + \int_{\overline{\Omega}}\frac{1}{\gamma-1}{\rm d}\mathfrak{R}_p\right) \ {\rm d}t\end{split} \tag{4.4}$$

for any $0\leq\tau_1\leq\tau_2<T$, and any $\psi \in C^1_c[0,T)$, $\psi\geq 0$.

Remark 4.2. In (4.4), the initial value of the energy is set

$$\int_\Omega \left[\frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0)\right] \ {\rm d}x.$$

Although the system (4.1) is apparently underdetermined due to the presence of the turbulent defect measures, it reduces to (1.6), (1.7), meaning $\mathfrak{R}_v = \mathfrak{R}_p = 0$ as soon as ϱ and $\mathbf{m}$ are continuously differentiable and $\varrho \geq \underline{\varrho} > 0$ is bounded below away from zero. Indeed we can introduce the velocity $\mathbf{u} = \frac{1}{\varrho}\mathbf{m} \in C^1$, whereas the continuity equation is satisfied in the classical sense:

$$\partial_t\varrho + {\rm div}_x(\varrho\mathbf{u}) = 0.$$

Next, as $\mathbf{u}$ can be used as a test function in the momentum equation (4.3), we easily deduce

$$\begin{split}\int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)\right](\tau,\cdot) \ {\rm d}x = \int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0)\right] \ {\rm d}x \\ + \int_0^\tau \int_{\overline{\Omega}} \nabla_x\mathbf{u} : {\rm d}\left[\mathfrak{R}_v + \mathfrak{R}_p\mathbb{I}\right] \ {\rm d}t.\end{split} \tag{4.5}$$

This expression may be subtracted from the energy inequality (4.4) to obtain

$$\int_{\overline{\Omega}} \left[\frac{1}{2}{\rm d}\,{\rm trace}[\mathfrak{R}_v] + \int_{\overline{\Omega}}\frac{1}{\gamma-1}{\rm d}\mathfrak{R}_p\right](\tau) \leq -\int_0^\tau \int_{\overline{\Omega}} \nabla_x\mathbf{u} : {\rm d}\left[\mathfrak{R}_v + \mathfrak{R}_p\mathbb{I}\right] \ {\rm d}t \tag{4.6}$$

Thus a direct application of Gronwall's lemma yields the desired conclusion $\mathfrak{R}_v = \mathfrak{R}_p = 0$. We have shown the following result.

Theorem 4.3. *Let $\Omega \subset R^d$ be a bounded domain of class C^1. Suppose that a dissipative solution ϱ, $\mathbf{m}$ is continuously differentiable in $[0,T) \times \overline{\Omega}$ and $\varrho \geq \underline{\varrho} > 0$.*
Then $\mathfrak{R}_v = \mathfrak{R}_p = 0$ and ϱ, $\mathbf{m}$ is a classical solution of the Euler system.

A short inspection of (4.6) shows that C^1 regularity is not really necessary. In fact, it is enough that the symmetric velocity gradient

$$\mathbb{D}\mathbf{u} \equiv \frac{\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^t}{2}$$

satisfies a one sided Lipschitz condition, specifically,

$$\mathbb{D}\mathbf{u} + d\mathbb{I} \geq 0 \text{ for certain } d \in L^1(0,T). \tag{4.7}$$

Indeed, as $\mathfrak{R}_v + \mathfrak{R}_p\mathbb{I}$ is positively definite, we get

$$\begin{aligned}
-\int_0^\tau \int_{\overline{\Omega}} \nabla_x \mathbf{u} : \mathrm{d}\left[\mathfrak{R}_v + \mathfrak{R}_p\mathbb{I}\right] \mathrm{d}t &= -\frac{1}{2}\int_0^\tau \int_{\overline{\Omega}} \mathbb{D}\mathbf{u} : \mathrm{d}\left[\mathfrak{R}_v + \mathfrak{R}_p\mathbb{I}\right] \mathrm{d}t \\
&= -\frac{1}{2}\int_0^\tau \int_{\overline{\Omega}} (\mathbb{D}\mathbf{u} + d\mathbb{I}) : \mathrm{d}\left[\mathfrak{R}_v + \mathfrak{R}_p\mathbb{I}\right] \mathrm{d}t + \frac{1}{2}\int_0^\tau d \int_{\overline{\Omega}} \mathrm{d}\,\mathrm{trace}\left[\mathfrak{R}_v + \mathfrak{R}_p\mathbb{I}\right] \mathrm{d}t \\
] &\leq \frac{1}{2}\int_0^\tau d \int_{\overline{\Omega}} \mathrm{d}\,\mathrm{trace}\left[\mathfrak{R}_v + \mathfrak{R}_p\mathbb{I}\right] \mathrm{d}t.
\end{aligned}$$

Consequently, the validity of Theorem 4.3 can be extended to the class of dissipative solutions satisfying the energy equality (4.5) together with the one–sided Lipschitz condition (4.7) for the velocity field. Sufficient conditions for validity of the energy equality of the compressible Euler system have been studied in [15] in the case of periodic boundary conditions. It turns out that (4.5) remains valid if ϱ, $\mathbf{m}$, and $\mathbf{u}$ enjoy extra Besov–type regularity, specifically:

$$\begin{aligned}
&\varrho \in L^\infty((0,T)\times\Omega),\ \varrho \geq \underline{\varrho} > 0,\ \mathbf{m} \in L^\infty((0,T)\times\Omega; R^d), \\
&\varrho,\ \mathbf{m},\ \mathbf{u} \equiv \frac{\mathbf{m}}{\varrho} \in B_3^{\alpha,\infty}((0,T)\times\Omega; R^d),\ \alpha > \frac{1}{3}, \\
&\varrho,\ \mathbf{m} \in L^\infty(0,T; B_q^{\beta,\infty}(\Omega; R^d)) \text{ for some } \beta > 0,\ q > 1.
\end{aligned} \tag{4.8}$$

Remark 4.4. The symbol $B_q^{\alpha,\infty}(Q)$ denotes the Besov space endowed with the norm

$$\|v\|_{B_q^{\alpha,\infty}(Q)} = \|v\|_{L^q(Q)} + \sup_{\xi\in Q} \frac{\|v(\cdot+\xi) - v(\cdot)\|_{L^q(Q\cap(Q-\xi))}}{|\xi|^\alpha}.$$

Now, observe that the impermeability condition (1.4), if imposed on the cube

$$\Omega = (-1,1)^d,$$

can be transformed to the periodic boundary conditions working with classes of functions with certain symmetry, see Ebin [12], and [18]. Summing up the previous observations, we obtain the following extension of Theorem 4.3.

Theorem 4.5. *Let*

$$\Omega = (-1,1)^d$$

be the cube. Suppose that $[\varrho, \mathbf{m}]$ is a dissipative solution of the Euler system belonging to the class (4.8). In addition, let the velocity $\mathbf{u}$ satisfy the one–sided Lipschitz condition

$$\int_0^T \int_\Omega \Big(-\xi\cdot\mathbf{u}\,(\xi\cdot\nabla_x)\varphi + d|\xi|^2\varphi \Big)\,\mathrm{d}x\,\mathrm{d}t \geq 0,\ d \in L^1(0,T),$$

for any $\xi \in R^d$, $\varphi \in C^1_c((0,T) \times \Omega)$.

Then $\mathfrak{R}_v = \mathfrak{R}_p = 0$ *and, consequently,* $[\varrho, \mathbf{m}]$ *is a weak solution of the Euler system.*

5. **Construction of dissipative solutions.** Dissipative solutions appear as a limit of various approximation schemes. To simplify presentation, we consider the *periodic* boundary condition, meaning the spatial domain Ω is identified with the flat torus

$$\Omega = \left([-1,1]|_{\{-1,1\}} \right)^d .$$

The approximate solutions typically solve a system of equations:

$$- \int_\Omega \varrho_{0,n} \varphi \ \mathrm{d}x = \int_0^T \int_\Omega \Big[\varrho_n \partial_t \varphi + \mathbf{m}_n \cdot \nabla_x \varphi \Big] \ \mathrm{d}x \, \mathrm{d}t + E_{1,n}[\varphi] \tag{5.1}$$

for any $\varphi \in C^1_c([0,T) \times \Omega)$;

$$\begin{aligned} - \int_\Omega \mathbf{m}_{0,n} \cdot \boldsymbol{\varphi} \ \mathrm{d}x = \int_0^T \int_\Omega \left[\mathbf{m}_n \cdot \partial_t \boldsymbol{\varphi} + \left(\frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} : \nabla_x \boldsymbol{\varphi} \right) + p(\varrho_n) \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x \, \mathrm{d}t \\ + E_{2,n}[\boldsymbol{\varphi}] \end{aligned} \tag{5.2}$$

for any $\boldsymbol{\varphi} \in C^1_c([0,T) \times \Omega; R^N)$;

$$\begin{aligned} - \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_{0,n}|^2}{\varrho_{0,n}} + P(\varrho_{0,n}) \right] \ \mathrm{d}x \leq \int_0^T \partial_t \psi \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_n|^2}{\varrho_n} + P(\varrho_n) \right] \ \mathrm{d}x \\ + E_{3,n}[\psi] \end{aligned}$$

for any $\psi \in C^1_c[0,T)$, $\psi \geq 0$, $\psi(0) = 1$.

The terms $E_{1,n}$, $E_{2,n}$, $E_{3,n}$ represent *consistency errors*. Furthermore, we suppose

$$E_{1,n}[\varphi] \to 0, \ E_{2,n}[\boldsymbol{\varphi}] \to 0, \ E_{3,n}[\psi] \to 0 \text{ as } n \to \infty \text{ for fixed } [\varphi, \boldsymbol{\varphi}, \psi]. \tag{5.3}$$

Moreover, we require that

$$E_{3,n}[\psi] \lesssim c \left(\|\psi\|_{L^\infty(0,T)} \right) \text{ uniformly for } n \to \infty. \tag{5.4}$$

The approximate solutions $[\varrho_n, \mathbf{m}_n]$ can be obtained via a numerical scheme or a suitable physically relevant approximation. We may consider a *viscosity approximation*:

$$\begin{aligned} \partial_t \varrho_n + \mathrm{div}_x \mathbf{m}_n = 0, \ \varrho_n(0, \cdot) = \varrho_{n,0}, \\ \partial_t \mathbf{m}_n + \mathrm{div}_x \left(\frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} \right) + \nabla_x p(\varrho_n) = \frac{1}{n} \mathrm{div}_x \mathbb{S}_n, \ \mathbf{m}_n(0, \cdot) = \mathbf{m}_{0,n}, \end{aligned} \tag{5.5}$$

together with the relevant energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_n|^2}{\varrho_n} + P(\varrho_n) \right] \ \mathrm{d}x + \frac{1}{n} \int_\Omega \mathbb{S}_n : \mathbb{D}\mathbf{u}_n \ \mathrm{d}x \leq 0, \tag{5.6}$$

where the velocity field $\mathbf{u}_n$ satisfies $\varrho_n \mathbf{u}_n = \mathbf{m}_n$. We suppose the viscous stress depends in a *monotone* way on the velocity gradient $\mathbb{D}$, meaning

$$\mathbb{S}_n : \mathbb{D}\mathbf{u}_n = F(\mathbb{D}\mathbf{u}_n) + F^*(\mathbb{S}_n),$$

where F is a convex l.s.c. function on $R^{d \times d}_{\mathrm{sym}}$ and F^* its conjugate. If, for instance, $\mathrm{Dom}[F] = R^{d \times d}_{\mathrm{sym}}$, $F(0) = 0$, $F \geq 0$, the conjugate F^* is non–negative and superlinear. Accordingly, we may set

$$E_{1,n} = E_{3,n} = 0,\ E_{2,n}[\boldsymbol{\varphi}] = \frac{1}{n}\left| \int_0^T \int_\Omega \mathbb{S}_n : \nabla_x \boldsymbol{\varphi} \ \mathrm{d}x\,\mathrm{d}t \right|,$$

whereas the desired estimates follow from the energy balance (5.6).

Our next goal is to perform the limit for $n \to \infty$ in (5.1)–(5.3). The key tool is the energy inequality (5.3) yielding, together with the consistency bound (5.4), the uniform bounds

$$\int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_n|^2}{\varrho_n} + P(\varrho_n) \right] \ \mathrm{d}x \leq c(\text{data}) \text{ uniformly for } t \in (0,T) \text{ and } n \to \infty, \tag{5.7}$$

in particular,

$$\begin{aligned} \varrho_n &\to \varrho \text{ weakly-(*) in } L^\infty(0,T; L^\gamma(\Omega)), \\ \mathbf{m}_n &\to \mathbf{m} \text{ weakly-(*) in } L^\infty(0,T; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d)), \end{aligned} \tag{5.8}$$

for suitable subsequences as the case may be. Note that the function

$$[\varrho, \mathbf{m}] \in [0,\infty) \times R^d \mapsto \frac{|\mathbf{m}|^2}{\varrho} = \left\{ \begin{array}{l} 0 \text{ whenever } \mathbf{m} = 0, \\ \frac{|\mathbf{m}|^2}{\varrho} \text{ for } \varrho > 0, \\ \infty \text{ otherwise} \end{array} \right.$$

is a convex l.s.c. function.

Next, we have, again for a subsequence,

$$p(\varrho_n) \to \overline{p(\varrho)} \text{ weakly-(*) in } L^\infty(0,T; \mathcal{M}^+(\Omega)).$$

Moreover, as p is convex, we have

$$0 \leq p(\varrho) \leq \overline{p(\varrho)},\ \mathfrak{R}_p \equiv \overline{p(\varrho)} - p(\varrho) \in L^\infty(0,T; \mathcal{M}^+(\Omega)).$$

By the same token,

$$\frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} \to \overline{\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho}} \text{ weakly-(*) in } L^\infty(0,T; \mathcal{M}(\Omega; R^{d\times d}_{\rm sym})),$$

and we set

$$\mathcal{R}_v \equiv \overline{\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho}} - \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho}.$$

The crucial observation now is that

$$\begin{aligned} \mathcal{R}_v : (\xi \otimes \xi) &= \lim_{n\to\infty} \left[\frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} : (\xi \otimes \xi) \right] - \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : (\xi \otimes \xi) \\ &= \lim_{n\to\infty} \left[\frac{|\mathbf{m}_n \cdot \xi|^2}{\varrho_n} - \frac{|\mathbf{m} \cdot \xi|^2}{\varrho} \right] \geq 0 \end{aligned}$$

due to convexity. We therefore conclude that

$$\mathfrak{R}_v \in L^\infty(0,T; \mathcal{M}^+(\Omega; R^{d\times d}_{\rm sym})).$$

Finally, it is a routine matter to show that the limit $[\varrho, \mathbf{m}]$ satisfies (4.2)–(4.4), meaning, it is a dissipative solution of the Euler system.

6. **Properties of the solution set.** Dissipative solutions are not uniquely determined by the initial data unless they enjoy certain extra regularity property similar to (4.8). However, we report the following *weak–strong* uniqueness principle proved in [13, Theorem 2.1].

Theorem 6.1. *Let*

$$\Omega = \left([-1,1]|_{\{-1,1\}} \right)^d$$

be the flat torus. Suppose that the Euler system (1.1)–(1.3) admits a weak solution $\widetilde{\varrho}$, $\widetilde{\mathbf{m}} = \widetilde{\varrho}\mathbf{U}$ *belonging to the regularity class:*

$$\begin{aligned} &\widetilde{\varrho} \in B^{\alpha,\infty}_p((\delta,T)\times\Omega)) \cap C([0,T];L^1(\Omega)),\\ &\mathbf{U} \in B^{\alpha,\infty}_p((\delta,T)\times\Omega;R^d)) \cap C([0,T];L^1(\Omega;R^d)), \ \text{for any } \delta > 0, \end{aligned} \tag{6.1}$$

with

$$\alpha > \frac{1}{2},\ p \geq \frac{4\gamma}{\gamma - 1};$$

$$0 < \underline{\varrho} \leq \widetilde{\varrho}(t,x) \leq \overline{\varrho},\ |\mathbf{U}(t,x)| \leq \overline{U} \ \text{for a.a. } (t,x) \in (0,T)\times\Omega;$$

$$\int_\Omega \left[-\xi\cdot\mathbf{U}(\tau,\cdot)(\xi\cdot\nabla_x)\varphi + d(\tau)|\xi|^2\varphi \right] \ \mathrm{d}x \geq 0,\ d \in L^1(0,T),$$

for any $\xi \in R^d$*, and any* $\varphi \in C(\Omega)$, $\varphi \geq 0$*. Let* ϱ, $\mathbf{m}$ *be a dissipative solution starting from the initial data*

$$\varrho(0,\cdot) = \widetilde{\varrho}(0,\cdot),\ \mathbf{m}(0,\cdot) = \widetilde{\mathbf{m}}(0,\cdot).$$

Then $\mathfrak{R}_p = \mathfrak{R}_v = 0$*, and* $\varrho = \widetilde{\varrho}$, $\mathbf{m} = \widetilde{\mathbf{m}}$.

In the remaining part of this section, we examine the properties of the solution set for fixed finite energy initial data:

$$\varrho_0 \in L^\gamma(\Omega),\ \mathbf{m}_0 \in L^{\frac{2\gamma}{\gamma+1}}(\Omega;R^d),\ \int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] \ \mathrm{d}x \leq E_0 < \infty. \tag{6.2}$$

Let

$$\begin{aligned} \mathcal{U}[\varrho_0,\mathbf{m}_0] = \Big\{ [\varrho,\mathbf{m}] \ \Big| \ &\varrho \in C_{\mathrm{weak}}([0,T];L^\gamma(\Omega)),\ \mathbf{m} \in C_{\mathrm{weak}}([0,T];L^{\frac{2\gamma}{\gamma+1}}(\Omega;R^d)) \\ &\text{is a dissipative solution of the Euler system},\ \varrho(0,\cdot) = \varrho_0,\ \mathbf{m}(0,\cdot) = \mathbf{m}_0 \Big\} \end{aligned}$$

be the set of all dissipative solutions in the sense of Definition 4.1 starting from the initial data $[\varrho_0,\mathbf{m}_0]$.

We claim that for any $[\varrho_0,\mathbf{m}_0]$ satisfying (6.2):

- $\mathcal{U}[\varrho_0,\mathbf{m}_0]$ is non–empty;
- $\mathcal{U}[\varrho_0,\mathbf{m}_0]$ is convex;
- $\mathcal{U}[\varrho_0,\mathbf{m}_0]$ is compact with respect to the metric topology on bounded sets in

$$\left[C_{\mathrm{weak}}([0,T];L^\gamma(\Omega)) \cap L^\infty(0,T;L^\gamma(\Omega)) \right] \times \left[C_{\mathrm{weak}}([0,T];L^{\frac{2\gamma}{\gamma+1}}(\Omega)) \cap L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega)) \right].$$

The fact that the solution set is non–empty was proved in Section 5. Compactness can be shown by the same arguments as the proof of existence. Finally, as a convex combination of two (non–negative) measures is a measure, it is easy to check that the set $\mathcal{U}[\varrho_0,\mathbf{m}_0]$ is convex.

7. **Selection criteria, admissible solutions.** As we have observed in the previous section, the set of dissipative solutions $\mathcal{U}[\varrho_0,\mathbf{m}_0]$ emanating from the initial data $[\varrho_0,\mathbf{m}_0]$ is non–empty, convex, and compact with respect to the weak topology on the trajectory space. Unfortunately, there are numerous examples furnished by the method of convex integration showing the set $\mathcal{U}[\varrho_0,\mathbf{m}_0]$ is not a singleton.

Several criteria could be proposed to rule out the irrelevant solutions. We discuss shortly the *maximal dissipation principle* asserting that the physical solution dissipates the (mechanical) energy at maximal rate. Given $[\varrho_1, \mathbf{m}_1]$, $[\varrho_2, \mathbf{m}_2]$ we define a relation

$$[\varrho_1, \mathbf{m}_1] \prec [\varrho_2, \mathbf{m}_2]$$

if

$$\frac{1}{2}\frac{|\mathbf{m}_1|^2}{\varrho_1} + P(\varrho_1) + \frac{1}{2}\mathrm{trace}[\mathfrak{R}_v^1] + \frac{1}{\gamma-1}\mathfrak{R}_p^1 \leq \frac{1}{2}\frac{|\mathbf{m}_2|^2}{\varrho_2} + P(\varrho_2) + \frac{1}{2}\mathrm{trace}[\mathfrak{R}_v^2] + \frac{1}{\gamma-1}\mathfrak{R}_p^2$$

in the sense of measures on $[0,T] \times \overline{\Omega}$.

Definition 7.1. Let the initial data $[\varrho_0, \mathbf{m}_0]$ be given. We say that a dissipative solution $[\varrho, \mathbf{m}]$ is *admissible* if it is minimal with respect to the relation $\prec$. Specifically, if $[\widetilde{\varrho}, \widetilde{\mathbf{m}}]$ is another dissipative solution starting from the same initial data such that

$$[\widetilde{\varrho}, \widetilde{\mathbf{m}}] \prec [\varrho, \mathbf{m}],$$

then

$$\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) + \frac{1}{2}\mathrm{trace}[\mathfrak{R}_v] + \frac{1}{\gamma-1}\mathfrak{R}_p = \frac{1}{2}\frac{|\widetilde{\mathbf{m}}|^2}{\widetilde{\varrho}} + P(\widetilde{\varrho}) + \frac{1}{2}\mathrm{trace}[\widetilde{\mathfrak{R}}_v] + \frac{1}{\gamma-1}\widetilde{\mathfrak{R}}_p$$

in $[0,T] \times \overline{\Omega}$.

It is easy to see that an admissible solution always exist. It is enough to minimize the energy functional

$$\int_0^T \int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)\right] \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\overline{\Omega}} \left[\frac{1}{2}\mathrm{d}\,\mathrm{trace}[\mathfrak{R}_v] + \frac{1}{\gamma-1}\mathrm{d}\,\mathfrak{R}_p\right]$$

over the set of all dissipative solutions $[\varrho, \mathbf{m}]$ with the associated turbulence defects $\mathcal{R}_v$, $\mathcal{R}_p$ starting from the initial data $[\varrho_0, \mathbf{m}_0]$.

Finally, we point out that a suitable choice of a family of *selection criteria* gives rise to a *semiflow selection* and conditional well posedness. The basic idea goes back to the Krylov [20], where a general selection procedure has been proposed in the context of Markov semigroups. Similar approach in the deterministic setting was used by Cardona and Kapitanskii [5]. Subsequently, the method was adapted to the compressible Euler system in [4]. More precisely, the state variables being enhanced by the associated energy E, there is a measurable mapping

$$U : [t, \varrho_0, \mathbf{m}_0, E_0] \mapsto [\varrho(t), \mathbf{m}(t), E(t)],$$

such that $[\varrho, \mathbf{m}, E]$ solves the Euler system (in the sense of dissipative solutions) with the initial data given by $[\varrho_0, \mathbf{m}_0, E_0]$ and the semigroup property

$$U[t_1 + t_2, \varrho_0, \mathbf{m}_0, E_0] = U[t_2, U[t_1, \varrho_0, \mathbf{m}_0, E_0]] \text{ for any } t_1, t_2 \geq 0,$$

holds true. The interested reader may consult [4] for details.

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