ON THE MALLIAVIN DIFFERENTIABILITY AND FLOW PROPERTY OF SOLUTIONS TO LÉVY NOISE DRIVEN SDE'S WITH IRREGULAR COEFFICIENTS.

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Abstract. In this paper, we investigate the strong solutions to SDE's driven by Lévy processes with Hölder drifts. We show that the singular SDE has a unique strong solution for each starting point and the collection of these strong solutions starting from single points forms a C^1-stochastic flow. Moreover, the Malliavin differentiability of the strong solutions is obtained, which extends the main result in [11]. As an application, we also prove a path-by-path uniqueness result for the related random ODE.

Keywords: Stochastic flow, Lévy process, Zvonkin's transform, Malliavin differentiable

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1. Introduction

Suppose $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathbf{P})$ is a filtered probability space satisfying the usual conditions, Z is an $\mathscr{F}_t$-adapted d-dimensional pure jump Lévy process with Lévy measure ν. The main aim of this paper is to study the stochastic homeomorphism flow of

$$\mathrm{d}X_t(x) = b(X_t(x))\mathrm{d}t + \sigma(X_{t-}(x))\mathrm{d}Z_t,\ \ X_0(x) = x \in \mathbb{R}^d, \tag{1.1}$$

under low regularity assumptions on the coefficients $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$.

The classical subject of SDEs driven by non-degenerated noises with singular drifts dates back at least to [37], where Zovnkin showed that if $d = 1, \sigma = 1$ and b is bounded, then (1.1) has a unique strong solution. And later, Veretennikov [29] extended the similar result for $d \geqslant 1$. Using Girsanovs transformation and results from PDEs, Krylov and Röckner [13] obtained the existence and uniqueness of strong solutions to (1.1) when σ is the identity matrix and b satisfies $\|b\|_{L^q_t L^p_x} < \infty$ with $\frac{d}{p} + \frac{2}{q} < 1$. One can see also [31, 34] for more delicate results about the well-posedness as well as the stochastic homeomorphism flows (1.1). It should be mentioned that in [17, 16, 19], the authors gave another approach based on Malliavin calculus to study the strong existence. Their method does not rely on a pathwise uniqueness argument and can be used to get the Malliavin differentiability of obtained solutions. And we also need to mention that in [8], Davie proved a remarkable result, it says that if b is only bounded and measurable, W_t is a Brownian motion, $b^\omega_t(x) := b(x+W_t(\omega))$, then the random ODE $\mathrm{d}\theta_t(\omega)/\mathrm{d}t = b^\omega_t(\theta_t)$ has a unique solution for almost all $\omega \in \Omega$. His proof was simplified by Shaposhnikov in [24] by using the flow property of strong solutions of SDE driven by the Brownian motion.

When the noise Z is a pure jump Lévy process, for one-dimensional case, Tanaka, Tsuchiya and Watanabe [27] proved that if Z is a symmetric α-stable process with $\alpha \in [1, 2)$, $\sigma(x) \equiv 1$ and b is bounded measurable, then pathwise uniqueness holds for SDE

Research of Guohuan is supported by the German Research Foundation (DFG) through the Collaborative Research Centre(CRC) 1283 Taming uncertainty and profiting from randomness and low regularity in analysis, stochastics and their applications.

(1.1). They further show that if $\alpha \in (0,1)$, and even if b is Hölder continuous, the pathwise uniqueness may fail. For multidimensional case, Priola [21] first proved pathwise uniqueness for (1.1) when $\sigma(x) = \mathbb{I}$, Z is a non-degenerate symmetric but possibly non-isotropic α-stable process with $\alpha \in [1,2)$ and $b \in C^\beta(\mathbb{R}^d)$ with $\beta \in (1-\alpha/2, 1)$. This result was extended to drift b in some fractional Sobolev spaces in the subcritical case in Zhang [32] and also extended to more general Lévy type driven noises in Priola [22]. In [5], the authors established strong existence and pathwise uniqueness for SDE (1.1) when $\sigma(x) = \mathbb{I}$, b is Hölder continuous and the semigroup of Z_t satisfies some regularity assumptions. It partially answers an open question posted in [22] on the pathwise well-posedness of SDE (1.1) in the supercritical case. Later, Chen, Zhang and Zhao [7] drop the constraint in [5] and give an affirmative answer to the above problem. In [26], Song and Xie extend this method to study singular SDEs driven by Poisson measures. We must mention that Haadem and Proske in [11] studied the existence and Malliavin differentiability by the similar approach used in [17, 16]. However, they had to assume that Z_t is a truncated rotational symmetric α-stable process($\alpha > 1$), $\sigma = \mathbb{I}$ and $b \in C^\beta$ with $\beta > 2 - \alpha$, which are much stronger than our assumptions below.

For $\alpha \in (0,2)$, denote by $\mathbb{M}_\alpha$ the space of all non-degenerate α-stable measures $\nu^{(\alpha)}$, that is,

$$\nu^{(\alpha)}(A) = \int_0^\infty \left(\int_{\mathbb{S}^{d-1}} \frac{1_A(r\theta)\Sigma(\mathrm{d}\theta)}{r^{1+\alpha}} \right) \mathrm{d}r, \quad A \in \mathscr{B}(\mathbb{R}^d), \tag{1.2}$$

where Σ is a finite measure over the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$ with

$$\inf_{\theta_0 \in \mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} |\theta_0 \cdot \theta| \, \Sigma(\mathrm{d}\theta) > 0.$$

All the assumptions on ν, b, σ will be used in this paper are following:

($\mathbf{H}_1$) There are two measures $\nu_1, \nu_2 \in \mathbb{M}_\alpha$ and $\rho \in (0,1)$ such that

$$\nu_1(A) \leqslant \nu(A) \leqslant \nu_2(A) \quad \text{for } A \subseteq B_\rho, \tag{1.3}$$

($\mathbf{H}_2$) There are positive constants β, Λ such that

$$\beta \in (1 - \tfrac{\alpha}{2}, 1), \ b \in C^\beta; \tag{1.4}$$

$$\sigma \in C_b^1, \quad \Lambda^{-1}|\xi| \leqslant |\sigma(x)\xi| \leqslant \Lambda|\xi|. \tag{1.5}$$

($\mathbf{H}_3$) $\sigma \in C_b^{1+\delta}$ for some $\delta \in (0,1)$. ν has a compact support and

$$\operatorname{supp} \nu \subseteq B_{r_0}, \quad r_0 < \|\nabla\sigma\|_\infty^{-1}. \tag{1.6}$$

Thought out this paper, we assume ν satisfies **($\mathbf{H}_1$)**, which is the Lévy measure of Z. And the characteristic exponent $\psi(\xi)$ of Z_t is given by

$$\psi(\xi) := -\log(\mathbf{E}e^{i\xi\cdot Z_1}) = -\int_{\mathbb{R}^d} (e^{i\xi\cdot z} - 1 - i\xi \cdot z^{(\alpha)})\nu(\mathrm{d}z),$$

where $z^{(\alpha)} = z\mathbf{1}_{\alpha>1} + z\mathbf{1}_{\alpha=1}\mathbf{1}_{B_1}(z)$.

Our main result is

Theorem 1.1. *(1) Suppose ν, b, σ satisfy assumptions* **($\mathbf{H}_1$)** *and* **($\mathbf{H}_2$)**, *then there is a unique strong solution to equation (1.1). Moreover, if the jumping size of Z_t is bounded, then for each $t \geqslant 0$, the strong solution $X_t(x)$ to (1.1) is Malliavin differentiable.*

(2) Suppose ν, b, σ satisfy assumptions **($\mathbf{H}_1$)**, **($\mathbf{H}_2$)** *and* **($\mathbf{H}_3$)**, *then $\{X_t(x)\}_{t\geqslant 0; x\in\mathbb{R}^d}$ forms a C^1-stochastic diffeomorphism flow.*

We also have the following corollary:

Corollary 1.2. *Suppose ν satisfies* $(\mathbf{H}_1)$*, $\sigma = \mathbb{I}$, $b \in C^\beta$ with $\beta \in (1-\frac{\alpha}{2}, 1)$, then there is a full set $\Omega_0 \subseteq \Omega$ i.e. $\mathbf{P}(\Omega_0) = 1$ such that for any $\omega \in \Omega_0$, the following ODE:*

$$\frac{\mathrm{d}\theta_t(\omega)}{\mathrm{d}t} = b_t^\omega(\theta_t(\omega)), \quad \theta_0 = x \tag{1.7}$$

admits a unique solution, where $b_t^\omega(x) = b(x + Z_t(\omega))$.

As mentioned before, by using the similar method in [7], we will show that all the strong solutions from single points are Malliavin differentiable and they form a C^1-stochastic flow. In order to study the strong well-posedness of SDE (1.1), we use the well known Zvonkin's transform, which requires a deep understanding for the following nonlocal PDE (Resolvent equation):

$$\lambda u - \mathscr{L}u - b \cdot \nabla u = f, \tag{1.8}$$

where

$$\mathscr{L}u(x) := \int_{\mathbb{R}^d} (u(x+\sigma(x)z) - u(x) - \nabla u(x) \cdot \sigma(x) z^{(\alpha)}) \nu(\mathrm{d}z),$$

and $z^{(\alpha)} = z\mathbf{1}_{\alpha>1} + z\mathbf{1}_{\alpha=1}\mathbf{1}_{B_1}(z)$. When $\mathscr{L}$ is the usual fractional Laplacian $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$ with $\alpha \in (0,1)$ and $b \in L^\infty([0,T]; C^\beta)$ with $\beta \in ((1-\alpha)\vee 0, 1)$, Silvestre [25] obtained the following a priori interior estimate:

$$\|u\|_{L^\infty([0,1];C^{\alpha+\beta}(B_1))} \leqslant C\Big(\|u\|_{L^\infty([0,2]\times B_2)} + \|f\|_{L^\infty([0,2];C^\beta(B_2))}\Big).$$

See also [9], [35] and [15] for similar estimates for more general operators. Our approach of studying (1.8) is based on the Littlewood-Paley decomposition and some Bernstein's type inequalities. As showed in [7], this approach allows us to handle a large class of Lévy's type operator in a uniform way, in particular, for Lévy's type operators with singular Lévy measures. However, in [7], the authors worked in the space $B^s_{p,\infty}$, this space does not enjoy the localization principle(see Lemma 3.5 below), so the usual freezing coefficients method does not work for general Lévy type operators, so they can not get a global diffeomorphism Φ by using Zovnkin's transform(see Theorem 3.3 of [7] and the proof of Theorem 1.1 therein). In order to overcome this difficulty, in this paper, we replace the working space $B^s_{p,\infty}$ with $B^s_{p,p}$, which is coincide with the classic Sobolev-Slobodeckij space W^s_p when $s \notin \mathbb{N}$. Due to the classic freezing coefficient method, Zovnkin's transform and a localization technique from [36] and [30], we can get a global C^1-diffeomorphism Φ for any non-degenerate $\sigma \in C^1_b$, provided that b satisfies $(\mathbf{H}_2)$. This helps us to prove the stochastic flow property of (1.1).

This paper is organized as follows: In Section 2, we recall some well-known facts from Littlewood-Paley theory. In Section 3, we study the nonlocal advection equation (1.8) when ν is compactly supported and b is Hölder continuous, and obtain some apriori estimates in Sobolev spaces. In Section 4, we prove our main theorem by Zvonkin's transform. In the appendix, we give a simple proof of a Bernstein's type estimate.

Finally, we introduce some conventions used throughout this paper: The letter c or C with or without subscripts stands for an unimportant constant, whose value may change in difference places. We use $A \asymp B$ to denote that A and B are comparable up to a constant, and use $A \lesssim B$ to denote $A \leqslant C \cdot B$ for some constant C.

2. Preliminary

2.1. **Sobolev space and Besov space.** We first give some definitions about fractional Sobolev space.

Definition 2.1. *Let $H_p^s := (\mathbb{I}-\Delta)^{-s/2}(L^p)$ be the usual Bessel potential space with norm*

$$\|f\|_{H_p^s} := \|(\mathbb{I}-\Delta)^{s/2} f\|_p \asymp \|f\|_p + \|(-\Delta)^{s/2} f\|_p.$$

The Sobolev-Slobodeckij semi-norm is defined by

$$[f]_{\theta,p} := \left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|f(x)-f(y)|^p}{|x-y|^{\theta p+d}}\, \mathrm{d}x\, dy \right)^{\frac{1}{p}}.$$

Let $s>0$ be not an integer and set $\theta = s-\lfloor s\rfloor \in (0,1)$. Sobolev-Slobodeckij space W_p^s is defined as

$$W_p^s := \left\{ f\in W_p^{\lfloor s\rfloor} : \sup_{|\alpha|=\lfloor s\rfloor} [\partial^\alpha f]_{\theta,p} < \infty \right\}, \quad \|f\|_{W_p^s} := \|f\|_{W_p^{\lfloor s\rfloor}} + \sup_{|\alpha|=\lfloor s\rfloor} [\partial^\alpha f]_{\theta,p}.$$

Suppose $s>0, \varepsilon>0, p\geqslant 1$, $0<s-\frac{d}{p}\notin\mathbb{N}$, then

$$H_p^{s+\varepsilon} \hookrightarrow W_p^s \hookrightarrow H_p^{s-\varepsilon}; \quad H_p^s \hookrightarrow C^{s-\frac{d}{p}}; \quad W_p^s \hookrightarrow C^{s-\frac{d}{p}}. \tag{2.1}$$

Let $\chi : \mathbb{R}^d \to [0,1]$ be a smooth radial function with

$$\chi(\xi) = 1,\ |\xi|\leqslant 1,\ \chi(\xi)=0,\ |\xi|\geqslant 3/2,$$

and $\chi^z(x) := \chi(x-z)$. We define the following localized fractional Sobolev space:

Definition 2.2. *Let $s\geqslant 0$, $p\in[1,\infty]$, we define*

$$\mathcal{W}_p^s := \left\{ u\in W_{p,loc}^s : \sup_{z\in\mathbb{R}^d} \|u\chi_z\|_{W_p^\gamma} < \infty \right\},$$

and define the norm

$$\|u\|_{\mathcal{W}_p^s} := \sup_{z\in\mathbb{R}^d} \|u\chi_z\|_{W_p^s}.$$

$\mathcal{W}_p^s$ is a Banach space and the enjoys the following property:

Lemma 2.3. *(1) If $\gamma>\frac{d}{p}$, $\gamma-\frac{d}{p}\notin\mathbb{N}$, then $\mathcal{W}_p^\gamma \hookrightarrow C^{\gamma-\frac{d}{p}}$;*

(2) If $\beta>\gamma\geqslant 0$, then $C^\beta \hookrightarrow \mathcal{W}_p^\gamma$.

Proof. The first conclusion is just a consequence of Sobolev embedding theorem, and we only need to prove the second conclusion when $1>\beta>\gamma\geqslant 0$. Obviously, if $u\in C^\beta$, then

$$\|u\chi_z\|_p \lesssim \|u\|_{L^\infty} \lesssim \|u\|_{C^\beta}.$$

Hence, $C^\beta \hookrightarrow \mathcal{W}_p^0$. If $1>\beta>\gamma>0$, by definition,

$$\begin{aligned}
&\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u\chi_z(x)-u\chi_z(y)|^p}{|x-y|^{d+\gamma p}}\mathrm{d}x\mathrm{d}y \\
&\lesssim \iint_{|x-z|\leqslant\frac{5}{2},\,|x-y|\leqslant 1} \frac{|u\chi_z(x)-u\chi_z(y)|^p}{|x-y|^{d+\gamma p}}\mathrm{d}x\mathrm{d}y + \iint_{|x-y|>1} \frac{|u\chi_z(x)-u\chi_z(y)|^p}{|x-y|^{d+\gamma p}}\mathrm{d}x\mathrm{d}y \\
&\lesssim \iint_{|x-z|\leqslant\frac{5}{2},\,|x-y|\leqslant 1} \frac{|\chi_z(x)|^p\cdot|u(x)-u(y)|^p}{|x-y|^{d+\gamma p}}\mathrm{d}x\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&+\iint_{|x-z|\leqslant\frac{5}{2},\,|x-y|\leqslant1}\frac{|u(y)|^p\cdot|\chi_z(x)-\chi_z(y)|^p}{|x-y|^{d+\gamma p}}\mathrm{d}x\mathrm{d}y+\iint_{|x-y|>1}\frac{|u\chi_z(x)|^p+|u\chi_z(y)|^p}{|x-y|^{d+\gamma p}}\mathrm{d}x\mathrm{d}y\\
&\lesssim\iint_{|x-z|\leqslant\frac{5}{2},\,|x-y|\leqslant1}\frac{\|u\|_{C^\beta}^p|x-y|^{\beta p}}{|x-y|^{d+\gamma p}}\mathrm{d}x\mathrm{d}y+\iint_{|x-y|>1}\frac{|u\chi_z(x)|^p}{|x-y|^{d+\gamma p}}\mathrm{d}x\mathrm{d}y\\
&\lesssim\|u\|_{C^\beta}^p\int_{|x-z|\leqslant\frac{5}{2}}\mathrm{d}x\int_{|w|\leqslant1}|w|^{-d+(\beta-\gamma)p}\mathrm{d}w+\|u\|_{L^\infty}^p\int_{|x-z|\leqslant\frac{3}{2}}\mathrm{d}x\int_{|w|>1}|w|^{-d-\gamma p}\mathrm{d}w\\
&\lesssim\|u\|_{C^\beta}^p.
\end{aligned}
$$

This yields, $\|u\chi_z\|_{W_p^\gamma}\leqslant C\|u\|_{C^\beta}$ and the constant C does not depends on z.

□

Next we recall some basic facts from the Littlewood-Paley theory. Let $\mathscr{S}(\mathbb{R}^d)$ be the Schwartz space of all rapidly decreasing functions, and $\mathscr{S}'(\mathbb{R}^d)$ the dual space of $\mathscr{S}(\mathbb{R}^d)$ called Schwartz generalized function (or tempered distribution) space. Given $f\in\mathscr{S}(\mathbb{R}^d)$, let $\mathscr{F}f=\hat{f}$ be the Fourier transform of f defined by

$$\hat{f}(\xi):=(2\pi)^{-d/2}\int_{\mathbb{R}^d}\mathrm{e}^{-\mathrm{i}\xi\cdot x}f(x)\mathrm{d}x.$$

Let $\chi:\mathbb{R}^d\to[0,1]$ be the function defined above. Define

$$\varphi(\xi):=\chi(\xi)-\chi(2\xi).$$

It is easy to see that $\varphi\geqslant0$ and supp $\varphi\subset B_{3/2}\setminus B_{1/2}$ and

$$\chi(2\xi)+\sum_{j=0}^{k}\varphi(2^{-j}\xi)=\chi(2^{-k}\xi)\overset{k\to\infty}{\to}1. \tag{2.2}$$

In particular, if $|j-j'|\geqslant2$, then

$$\mathrm{supp}\varphi(2^{-j}\cdot)\cap\mathrm{supp}\varphi(2^{-j'}\cdot)=\emptyset.$$

From now on we shall fix such χ and φ, and introduce the following definitions.

Definition 2.4. *The dyadic block operator Δ_j is defined by*

$$\Delta_jf:=\begin{cases}\mathscr{F}^{-1}(\chi(2\cdot)\mathscr{F}f), & j=-1,\\ \mathscr{F}^{-1}(\varphi(2^{-j}\cdot)\mathscr{F}f), & j\geqslant0.\end{cases}$$

For $s\in\mathbb{R}$ and $p,q\in[1,\infty]$, the Besov space $B^s_{p,q}$ is defined as the set of all $f\in\mathscr{S}'(\mathbb{R}^d)$ with

$$\|f\|_{B^s_{p,q}}:=\left(\sum_{j\geqslant-1}2^{jsq}\|\Delta_jf\|_p^q\right)^{1/q}<\infty;$$

The following two Lemmas can be found in [28].

Lemma 2.5. *(1) (Bernstein's inequality) For any $1\leqslant p\leqslant q\leqslant\infty$ and $j\geqslant0$, we have*

$$\|\nabla^k\Delta_jf\|_q\leqslant C_p2^{(k+d(\frac{1}{p}-\frac{1}{q}))j}\|\Delta_jf\|_p,\ \ k=0,1,\cdots, \tag{2.3}$$

and

$$\|(-\Delta)^{s/2}\Delta_jf\|_q\leqslant C_p2^{(s+d(\frac{1}{p}-\frac{1}{q}))j}\|\Delta_jf\|_p,\ \ s\in\mathbb{R}. \tag{2.4}$$

(2) For any $s \geqslant 0$,

$$\|f\|_{H_p^s} \asymp \left\| \Big(\sum_{j \geqslant -1} 2^{2sj} |\Delta_j f|^2 \Big)^{1/2} \right\|_p. \tag{2.5}$$

Lemma 2.6. *If $s > 0, s \notin \mathbb{N}$*

$$B^s_{\infty,\infty} \asymp C^s, B^s_{p,p} \asymp W^s_p,$$

where C^s is the usual Hölder space.

Let $h := \mathscr{F}^{-1}\chi$ be the inverse Fourier transform of χ. Define

$$h_{-1}(x) := \mathscr{F}^{-1}\chi(2\cdot)(x) = 2^{-d}h(2^{-1}x) \in \mathscr{S}(\mathbb{R}^d),$$

and for $j \geqslant 0$,

$$h_j(x) := \mathscr{F}^{-1}\varphi(2^{-j}\cdot)(x) = 2^{jd}h(2^j x) - 2^{(j-1)d}h(2^{j-1}x) \in \mathscr{S}(\mathbb{R}^d). \tag{2.6}$$

By definition it is easy to see that

$$\Delta_j f(x) = (h_j * f)(x) = \int_{\mathbb{R}^d} h_j(x-y) f(y) \mathrm{d}y, \quad j \geqslant -1. \tag{2.7}$$

2.2. **Mallivian Derivate for Lévy processes.** In this subsection, we introduce some basic conceptions of Mallivian calculus for Lévy processes. One can find more details in [20]. Suppose $N(\mathrm{d}t, \mathrm{d}x)$ is a Poisson point process with intensity measure $\nu(\mathrm{d}z)$. Let $\{\mathcal{F}_t\}_{0\leqslant t\leqslant T}$ be the filtration generated by N and $\tilde{N}(\mathrm{d}t, \mathrm{d}z) := N(\mathrm{d}t, \mathrm{d}z) - \nu(\mathrm{d}z)\mathrm{d}t$.

For each $n \in \mathbb{N}$, and $f \in L^2(([0,T]\times\mathbb{R}^d)^n; (\lambda\times\nu)^n)$, define

$$\widetilde{f}(t_1, z_1; \cdots; t_n, z_n) := \frac{1}{n!} \sum_{\sigma \in S_n} f(t_{\sigma(1)}, z_{\sigma(1)}; \cdots; t_{\sigma(n)}, z_{\sigma(n)}).$$

We denote the space of square integrable symmetric functions by $\widetilde{L}^2(([0,T]\times\mathbb{R}^d)^n; (\lambda\times\nu)^n)$(abbreviated by $\widetilde{L}^2((\lambda\times\nu)^n)$).

Definition 2.7. *The stochastic Sobolev space $\mathbb{D}^1_2$ consists of all $\mathcal{F}_T$ measurable random variables $F \in L^2(\mathbf{P})$ with chaos expansion*

$$F = \sum_{n=1}^{\infty} I_n(f_n), \;\; f_n \in L^2((\lambda\times\nu)^n)$$

satisfying

$$\sum_{n=0}^{\infty} n n! \|f_n\|^2_{L^2((\lambda\times\nu)^n)} < \infty.$$

Here

$$I_n(f_n) := \int_{([0,T]\times\mathbb{R}^d)^n} f_n(t_1, z_1; \cdots; t_n, z_n) \widetilde{N}^{\otimes n}(\mathrm{d}\mathbf{t}, \mathrm{d}\mathbf{z}); \quad \mathbf{t} = (t_1, \cdots t_n), \mathbf{z} = (z_1, \cdots, z_n).$$

Define

$$D_{t,z}F := \sum_{n=1}^{\infty} n I_{n-1}(\widetilde{f}_n(\cdot; t, z)),$$

then

$$\|DF\|^2_{L^2((\lambda\times\nu\times\mathbf{P}))} = \sum_{n=1}^{\infty} n n! \|f_n\|^2_{L^2((\lambda\times\nu)^n)}.$$

Thus, $F \in \mathbb{D}_2^1$ if and only if $F \in L^2(\mathbf{P})$ and $DF \in L^2((\lambda \times \nu \times \mathbf{P}))$.

The next lemma is consequence of Theorem 12.6 of [20].

Lemma 2.8 (Closability of Mallivian derivate)**.** *If $F_n \in \mathbb{D}_2^1$, $F_n \to F$ in $L^2(\mathbf{P})$ and*

$$\sup_n \|DF_n\|_{L^2(\lambda \times \nu \times \mathbf{P})} \leqslant M < \infty.$$

Then, $F \in \mathbb{D}_2^1$ and

$$\|DF\|_{L^2((\lambda \times \nu \times \mathbf{P}))} \leqslant M.$$

Proof. By our assumption, $\{DF_n\}_{n\in\mathbb{N}}$ is bounded in $L^2(\lambda \times \nu \times \mathbf{P})$, thus Banach-Saks theorem implies, the Cesàro mean sequence of a suitable subsequence of $\{DF_n\}$, say $\{DF_{n_k}\}$, converges strongly to some $G \in L^2(\lambda \times \nu \times \mathbf{P})$, i.e.

$$D\left(\frac{1}{m}\sum_{k=1}^m F_{n_k}\right) = \frac{1}{m}\sum_{k=1}^m DF_{n_k} \to G, \text{ in } L^2(\lambda \times \nu \times \mathbf{P}).$$

On the other hand, $\frac{1}{m}\sum_{k=1}^m F_{n_k} \to F$ in $L^2(\mathbf{P})$, by Theorem 12.6 of [20], we get $F \in \mathbb{D}_2^1$, $DF = G$ and

$$\|DF\|_{L^2((\lambda \times \nu \times \mathbf{P}))} = \lim_{m\to\infty} \frac{1}{m} \|\sum_{k=1}^m DF_{n_k}\|_{L^2((\lambda \times \nu \times \mathbf{P}))}$$

$$\leqslant \liminf_{m\to\infty} \frac{1}{m}\sum_{k=1}^m \|DF_{n_k}\|_{L^2((\lambda \times \nu \times \mathbf{P}))} \leqslant M.$$

□

3. A study of nonlocal parabolic equations

In this section we study the solvability and regularity of nonlocal elliptic equations with Hölder drift. First of all, we introduce the nonlocal operator studied in this work. Let σ be a invertible $d \times d$-matrix and ν a Lévy measure, that is,

$$\int_{\mathbb{R}^d \setminus \{0\}} (|z|^2 \wedge 1)\nu(\mathrm{d}z) < \infty.$$

We define a Lévy-type operator by

$$\mathcal{L}_\sigma f(x) := \int_{\mathbb{R}^d} \Big(f(x+\sigma z) - f(x) - \nabla f(x) \cdot \sigma z^{(\alpha)}\Big)\nu(\mathrm{d}z)$$

with $z^{(\alpha)} = z\mathbf{1}_{\alpha>1} + z\mathbf{1}_{\alpha=1}\mathbf{1}_{B_1}(z)$. By Fourier's transform, we have

$$\widehat{\mathcal{L}_\sigma f}(\xi) = \psi_\sigma(\xi)\hat{f}(\xi),$$

where the symbol $\psi_\sigma(\xi)$ takes the form

$$\psi_\sigma(\xi) = -\int_{\mathbb{R}^d} (\mathrm{e}^{\mathrm{i}\xi\cdot\sigma z} - 1 - \mathrm{i}\sigma z^{(\alpha)} \cdot \xi)\nu(\mathrm{d}z).$$

Now, let $\sigma(x) : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ be a Borel measurable function. Define

$$\mathscr{L} f(x) := \mathcal{L}_{\sigma(x)} f(x).$$

In this section we want to study the solvability of the following resolvent equation with Hölder drift $b(x) : \mathbb{R}^d \to \mathbb{R}^d$,

$$\lambda u - \mathscr{L} u - b \cdot \nabla u = f; \quad \lambda > 0. \tag{3.1}$$

3.1. **Constant coefficient case:** $\sigma(x) = \sigma$**.** In this subsection we consider equation (3.1) with non-degenerated constant coefficient $\sigma(x) = \sigma \in \mathbb{R}^{d\times d}$. First of all, we establish the following Bernstein's type inequality for nonlocal operator $\mathcal{L}_\sigma$, which plays a crucial role in the sequel.

Lemma 3.1. *Suppose ν satisfies $(\mathbf{H}_1)$, σ is a constant matrix and $\Lambda^{-1} \leqslant \|\sigma\| \leqslant \Lambda$, then for any $p \geqslant 2$, there are constants $c = c(\nu, \Lambda, p) > 0$, $j_0 = j_0(\nu, \Lambda, p) \in \mathbb{N}$ such that for any $j = j_0, j_0 + 1, \cdots$,*

$$\int_{\mathbb{R}^d} |\Delta_j f|^{p-2} \Delta_j f \mathcal{L}_\sigma \Delta_j f \mathrm{d}x \leqslant -c 2^{\alpha j} \|\Delta_j f\|_p^p, \tag{3.2}$$

and for $-1 \leqslant j < j_0$,

$$\int_{\mathbb{R}^d} |\Delta_j f|^{p-2} \Delta_j f \mathcal{L}_\sigma \Delta_j f \mathrm{d}x \leqslant 0.$$

Remark 3.2. *Readers can find the proof of above lemma in [7], this kind of estimate was first proved in [4] for $\mathcal{L}_\sigma = \Delta^{\alpha/2}$. We will give a much simpler proof in the appendix for symmetric operator $\mathcal{L}_\sigma$.*

We also need the following easy commutator estimate:

Lemma 3.3. *For any $j \geqslant -1$, $\beta \in (0, 1)$,*

$$\|[\Delta_j, b \cdot \nabla]u\|_p \lesssim 2^{-\beta j} \|b\|_{C^\beta} \|\nabla u\|_p.$$

Proof. By (2.7) we have

$$[\Delta_j, b \cdot \nabla]u(x) = \int_{\mathbb{R}^d} h_j(y)(b(x-y) - b(x)) \cdot \nabla u(x-y) \mathrm{d}y,$$

by Minkowski 's inequality and (2.6), we have

$$\begin{aligned} \|[\Delta_j, b \cdot \nabla]u\|_p &\leqslant \int_{\mathbb{R}^d} h_j(y) \|b(\cdot - y) - b(\cdot)\|_\infty \|\nabla u\|_p \mathrm{d}y \\ &\lesssim \|b\|_{C^\beta} \|\nabla u\|_p \int_{\mathbb{R}^d} |h_j(y)|\, |y|^\beta \mathrm{d}y \\ &= \|b\|_{C^\beta} \|\nabla u\|_p 2^{-j\beta} \int_{\mathbb{R}^d} |2h(2y) - h(y)|\, |y|^\beta \mathrm{d}y \\ &\lesssim 2^{-\beta j} \|b\|_{C^\beta} \|\nabla u\|_p. \end{aligned} \tag{3.3}$$

□

Now we can state our main result of this subsection.

Theorem 3.4. *Let $\alpha \in (0, 2)$, $\beta, \gamma \in (0, 1)$, $(1 - \alpha) < \gamma < \beta$, $\Lambda > 1$ and $p \in [2, \infty)$. Suppose ν satisfy $(\mathbf{H}_1)$, $b \in C^\beta$ and $\Lambda^{-1} \leqslant \|\sigma\| \leqslant \Lambda$, then for any $f \in W_p^\gamma$, there exists a unique solution $u \in W^{\alpha+\gamma}$ to equation (3.1). Moreover, there is a constant $\lambda_0 > 0$, such that, for all $\lambda \geqslant \lambda_0 > 0$,*

$$\lambda \|u\|_{W_p^\gamma} + \|u\|_{W_p^{\alpha+\gamma}} \leqslant C \|f\|_{W_p^\gamma}, \tag{3.4}$$

λ_0, C depend only on $d, p, \alpha, \beta, \gamma, \Lambda, \nu$ and $\|b\|_{C^\beta}$.

Proof. We first assume

$$b \in C_b^\infty, f \in \cap_{s \geqslant 0} W_p^s.$$

Under this assumption, it is well-known that PDE (3.1) has a unique smooth solution u. Our main task is to show the apriori estimates (3.4). Using operator Δ_j act on both sides of (3.1) and noticing that $\Delta_j\mathscr{L} = \Delta_j\mathcal{L}_\sigma = \mathcal{L}_\sigma\Delta_j = \mathscr{L}\Delta_j$, we have

$$\lambda\Delta_j u = \mathscr{L}\Delta_j u + \Delta_j(b\cdot\nabla u) + \Delta_j f.$$

For $p \geqslant 2$, by the chain rule or multiplying both sides by $|\Delta_j u|^{p-2}\Delta_j u$ and then integrating in x, we obtain

$$\begin{aligned}\lambda\int_{\mathbb{R}^d}|\Delta_j u|^p &= \int_{\mathbb{R}^d}|\Delta_j u|^{p-2}\Delta_j u\Big[\mathscr{L}\Delta_j u + \Delta_j(b\cdot\nabla u) + \Delta_j f\Big]\mathrm{d}x\\ &= \int_{\mathbb{R}^d}|\Delta_j u|^{p-2}\Delta_j u\mathscr{L}\Delta_j u\mathrm{d}x + \int_{\mathbb{R}^d}|\Delta_j u|^{p-2}\Delta_j u\,[\Delta_j, b\cdot\nabla]u\mathrm{d}x\\ &\quad+\int_{\mathbb{R}^d}|\Delta_j u|^{p-2}\Delta_j u\,(b\cdot\nabla)\Delta_j u\mathrm{d}x + \int_{\mathbb{R}^d}|\Delta_j u|^{p-2}\Delta_j u\Delta_j f\mathrm{d}x\\ &=: I_j^{(1)} + I_j^{(2)} + I_j^{(3)} + I_j^{(4)}.\end{aligned}$$

For $I_j^{(1)}$, recalling $\mathscr{L} = \mathcal{L}_\sigma$ and by Lemma 2.5, there is a $c > 0$ such that

$$I_j^{(1)} \leqslant 0,\quad j \geqslant -1;\quad I_j^{(1)} \leqslant -c2^{\alpha j}\|\Delta_j u\|_p^p,\quad j = j_0, j_0+1, \cdots.$$

For $I_j^{(2)}$, using Lemma 3.3 and Hölder's inequality, we have for all $j = -1, 0, 1, \cdots$,

$$\begin{aligned}I_j^{(2)} &\leqslant \|[\Delta_j, b\cdot\nabla]u\|_p\|\Delta_j u\|_p^{p-1}\\ &\lesssim 2^{-\beta j}\|b\|_{C^\beta}\|\nabla u\|_p\|\Delta_j u\|_p^{p-1}.\end{aligned}$$

For $I_j^{(3)}$, let us write

$$\begin{aligned}I_j^{(3)} &= \int_{\mathbb{R}^d}((b - S_j b)\cdot\nabla)\Delta_j u\,|\Delta_j u|^{p-2}\Delta_j u\mathrm{d}x\\ &\quad+\int_{\mathbb{R}^d}(S_j b\cdot\nabla)\Delta_j u\,|\Delta_j u|^{p-2}\Delta_j u\mathrm{d}x =: I_j^{(31)} + I_j^{(32)}.\end{aligned}$$

For $I_j^{(31)}$, by Bernstein's inequality (2.3), we have

$$\begin{aligned}I_j^{(31)} &\leqslant \sum_{k\geqslant j}\|(\Delta_k b\cdot\nabla)\Delta_j u\|_p\|\Delta_j u\|_p^{p-1}\\ &\leqslant \sum_{k\geqslant j}\|\Delta_k b\|_\infty\|\nabla\Delta_j u\|_p\|\Delta_j u\|_p^{p-1}\\ &\lesssim 2^j\|\Delta_j u\|_p^p\sum_{k\geqslant j}\|\Delta_k b\|_\infty \leqslant 2^j\|\Delta_j u\|_p^p\|b\|_{C^\beta}\sum_{k\geqslant j}2^{-\beta k}\\ &\lesssim 2^{(1-\beta)j}\|b\|_{C^\beta}\|\Delta_j u\|_p^p.\end{aligned}$$

For $I_j^{(32)}$, by integration by parts formula and (2.3) again, we have

$$\begin{aligned}I_j^{(32)} &= \frac{1}{p}\int_{\mathbb{R}^d}(S_j b\cdot\nabla)|\Delta_j u|^p\mathrm{d}x = -\frac{1}{p}\int_{\mathbb{R}^d}S_j\mathrm{div}b\,|\Delta_j u|^p\mathrm{d}x\\ &\leqslant \frac{1}{p}\|S_j\mathrm{div}b\|_\infty\|\Delta_j u\|_p^p \leqslant \frac{1}{p}\sum_{k\leqslant j}\|\Delta_k\mathrm{div}b\|_\infty\|\Delta_j u\|_p^p\\ &\lesssim \sum_{k\leqslant j}2^k\|\Delta_k b\|_\infty\|\Delta_j u\|_p^p\end{aligned}$$

$$\lesssim 2^{(1-\beta)j}\|b\|_{C^\beta}\|\Delta_j u\|_p^p.$$

Combining the above calculations, we obtain

$$\begin{aligned}\lambda\|\Delta_j u\|_p^p + c2^{\alpha j}\mathbf{1}_{\{j\geqslant j_0\}}\|\Delta_j u\|_p^p \leqslant& C2^{-\beta j}\|b\|_{C^\beta}\|\nabla u\|_p\|\Delta_j u\|_p^{p-1}\\ &+ C2^{(1-\beta)j}\|b\|_{C^\beta}\|\Delta_j u\|_p^p + C\|\Delta_j u\|_p^{p-1}\|\Delta_j f\|_p\end{aligned}$$

By dividing both sides by $\|\Delta_j u\|_p^{p-1}$, we get

$$\begin{aligned}&\lambda\|\Delta_j u\|_p + c2^{\alpha j}\mathbf{1}_{\{j\geqslant j_0\}}\|\Delta_j u\|_p - C2^{(1-\gamma)j}\|b\|_{C^\beta}\|\Delta_j u\|_p\\ \leqslant& C2^{-\beta j}\|b\|_{C^\beta}\|\nabla u\|_p + C\|\Delta_j f\|_p.\end{aligned}$$

Since $1-\beta<\alpha$, for some λ sufficiently large and all $j\geqslant -1$,

$$\lambda\|\Delta_j u\|_p + c2^{\alpha j}\mathbf{1}_{\{j\geqslant j_0\}}\|\Delta_j u\|_p \leqslant C2^{-\beta j}\|b\|_{C^\beta}\|\nabla u\|_p + C\|\Delta_j f\|_p \tag{3.5}$$

Multiplying both sides of (3.5) by $2^{\gamma j}$ and then taking ℓ^p norm over j, we obtain

$$\lambda\|u\|_{W_p^\gamma} + \|u\|_{W_p^{\alpha+\gamma}} \leqslant C_1\Big(\|\nabla u\|_p + \|f\|_{W_p^\gamma}\Big),$$

where C_1 only depends on $d, p, \alpha, \beta, \gamma, \Lambda, \nu$ and $\|b\|_{C^\beta}$. Recalling that $\alpha+\gamma>1$ and using interpolation theorem, we have $\|\nabla u\|_p \leqslant \frac{1}{2C_1}\|u\|_{W^{\alpha+\gamma}} + C'\|u\|_{W_p^\gamma}$. Choosing $\lambda_0 > 2C_1C'$, we complete the proof for (3.4). □

3.2. **Varying coefficient case.** In this subsection we consider the varying coefficient case. We drop the large jump part below, and consider the following operator

$$\mathscr{L}^R f(x) := \mathcal{L}^R_{\sigma(x)} f(x) := \int_{B_R}\Big(f(x+\sigma(x)z) - f(x) - \nabla f(x)\cdot\sigma(x)z^{(\alpha)}\Big)\nu(\mathrm{d}z), \tag{3.6}$$

where R is any real number larger than zero. We need the following lemma(see [28, Theorem 2.4.7]) in order to localize the resolvent equation.

Lemma 3.5 (localization principle)**.** *Let $c>0$, $\zeta_k\in C_c^\infty, k=1,2,\cdots$. Assume for any multi-index α and $x\in\mathbb{R}^d$, $\sup_{x\in\mathbb{R}^d}\sum_k|\partial^\alpha\zeta_k(x)|\leqslant C_\alpha<\infty$. Then, there is a constant C such that*

$$\sum_k\|u\zeta_k\|_{W_p^s}^p \leqslant C\|u\|_{W_p^s}.$$

Moreover, if $\sum_k|\zeta_k(x)|^p\geqslant c>0$, then we have

$$\|u\|_{W_p^s}^p \asymp \sum_k\|u\zeta_k\|_{W_p^s}^p. \tag{3.7}$$

The following lemma is taken from [18, Lemma 3.5].

Lemma 3.6. *Suppose $s\in(0,2)$, $p>\max\{1,\frac{d}{s}\}$, then*

$$\left\|\sup_{y\neq 0}\frac{|f(\cdot+y)-f(\cdot)-\nabla f(\cdot)\cdot y^{(\alpha)}|}{|y|^s}\right\|_p \leqslant C\|f\|_{H_p^s}. \tag{3.8}$$

The main result of this subsection is

Theorem 3.7. *Suppose ν, b, σ satisfy assumption* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$*, $\beta>\gamma>\max\{0,1-\alpha\}$ and $\mathscr{L}^R$ is defined as (3.6), then there is a constant λ_0 such that for any $\lambda\geqslant\lambda_0$ and $f\in C^\beta$ the following equation:*

$$\lambda u - \mathscr{L}^R u - b\cdot\nabla u = f$$

has a unique solution in $C^{\alpha+\gamma}$. Moreover, we have

$$\lambda\|u\|_{C^\gamma}+\|u\|_{C^{\alpha+\gamma}}\leqslant C\|f\|_{C^\beta}, \tag{3.9}$$

here the constants λ_0, C only depend on $d,\alpha,\beta,\gamma,R,\Lambda,\nu$ and $\|b\|_{C^\beta}$.

The above theorem is just a consequence of Lemma 2.3 and following lemma.

Lemma 3.8. *Suppose ν, b, σ satisfies assumption $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$, $\beta>\gamma>\max\{0,1-\alpha\}$, $p_0=\max\left\{\frac{d^2}{\alpha\wedge 1},\frac{d}{\alpha+\gamma-1}\right\}$, $p\in\big(p_0,\infty\big)$. Then, for any $f\in\mathcal{W}^\gamma_p$, the following equation:*

$$\lambda u-\mathscr{L}^R u-b\cdot\nabla u=f$$

has a unique solution in $\mathcal{W}^{\alpha+\gamma}_p$. Moreover, we have

$$\lambda\|u\|_{\mathcal{W}^\gamma_p}+\|u\|_{\mathcal{W}^{\alpha+\gamma}_p}\leqslant C\|f\|_{\mathcal{W}^\gamma_p}, \tag{3.10}$$

where λ_0, C depend only on $d,p,\alpha,\beta,\gamma,R,\Lambda,\nu$ and $\|b\|_{C^\beta}$.

Remark 3.9. *The above theorem can be improved under weaker conditions, but we do not attempt to do that here, since it is enough for our main propose of this paper.*

In order to prove the above lemma, we need a commutator estimate under the following assumption.

$(\mathbf{A}_\varepsilon)$ There are $\varepsilon\in(0,r)$ and $\Lambda\geqslant 1$ such that

$$|\sigma(x)-\sigma(y)|\leqslant\Lambda|x-y|,\sigma(x)=\sigma(0),\quad |x|\geqslant\varepsilon, \tag{3.11}$$

$$\Lambda^{-1}|\xi|^2\leqslant|\sigma(0)\xi|^2\leqslant\Lambda|\xi|^2,\ \xi\in\mathbb{R}^d. \tag{3.12}$$

Lemma 3.10. *Under $(\mathbf{A}_\varepsilon)$, for any $s\in(0,1), p>1$, we have*

$$\big\|[\Delta^{s/2},\mathscr{L}^R]u\big\|_p\leqslant C\varepsilon^{1-s+\frac{d}{p}}\|u\|_{C^1}, \tag{3.13}$$

where $[\Delta^{s/2},\mathscr{L}^R]u:=\Delta^{s/2}\mathscr{L}^Ru-\mathscr{L}^R\Delta^{s/2}u$, and the constant $C>0$ is independent of ε.

One can find the proof of above lemma was proved in [7].

Lemma 3.11. *Under $(\mathbf{A}_\varepsilon)$, for any $p\in(\frac{d^2}{\alpha\wedge 1},\infty)$, $\gamma\in(0,1)$, we have*

$$\big\|(\mathcal{L}^R_{\sigma(\cdot)}-\mathcal{L}^R_{\sigma(0)})f\big\|_{W^\gamma_p}\leqslant c_\varepsilon\|f\|_{W^{\alpha+\gamma}_p},$$

where $c_\varepsilon\to 0$ as $\varepsilon\to 0$.

Proof. We only prove the estimate for $\alpha\in(0,1)$. The case $\alpha\in[1,2)$ is similar. By [6, (2.19)], we have

$$\|\mathcal{L}^R_{\sigma_1}f-\mathcal{L}^R_{\sigma_2}f\|_p\lesssim(\|\sigma_1-\sigma_2\|^\alpha\wedge 1)\|f\|_{H^\alpha_p}. \tag{3.14}$$

Define

$$\mathcal{T}_\sigma f:=\mathcal{L}^R_{\sigma(\cdot)}f-\mathcal{L}^R_{\sigma(0)}f.$$

By (3.11), we have

$$|\mathcal{T}_\sigma f(x)|\leqslant|\mathcal{L}^R_{\sigma(x)}f(x)-\mathcal{L}^R_{\sigma(0)}f(x)|\leqslant\sup_{\|\sigma-\sigma(0)\|\leqslant\Lambda\varepsilon}|\mathcal{L}^R_\sigma f(x)-\mathcal{L}^R_{\sigma(0)}f(x)|,$$

since $p>d^2/\alpha$, by [6, Lemma 2.2] and (3.14), we have

$$\|\mathcal{T}_\sigma f\|_p\leqslant\left\|\sup_{\|\sigma-\sigma(0)\|\leqslant\Lambda\varepsilon}|\mathcal{L}^R_\sigma f(\cdot)-\mathcal{L}^R_{\sigma(0)}f(\cdot)|\right\|_p\lesssim\varepsilon^\alpha\|f\|_{H^\alpha_p}. \tag{3.15}$$

By Minkowski's inequality,

$$\|\Delta_i \mathcal{T}_\sigma f\|_p \leqslant \sum_{j>i} \|\Delta_i \mathcal{T}_\sigma \Delta_j f\|_p + \sum_{j\leqslant i} \|\Delta_i \mathcal{T}_\sigma \Delta_j f\|_p =: \mathcal{J}_i^{(1)} + \mathcal{J}_i^{(2)}.$$

For $\mathcal{J}_i^{(1)}$, by (3.15), we have

$$\begin{aligned}\mathcal{J}_i^{(1)} &\leqslant \sum_{j>i} \|\mathcal{T}_\sigma \Delta_j f\|_p \lesssim \varepsilon^\alpha \sum_{j>i} \|\Delta_j f\|_{H_p^\alpha}\\ &\overset{(2.4)}{\lesssim} \varepsilon^\alpha \sum_{j>i} 2^{\alpha j} \|\Delta_j f\|_p \lesssim \varepsilon^\alpha \|f\|_{W_p^{\alpha+\gamma}} \sum_{j>i} 2^{-\gamma j} c_j,\end{aligned}$$

where $c_j = 2^{(\alpha+\gamma)j}\|\Delta_j f\|_p / \|f\|_{W_p^{\alpha+\gamma}}$. Thus,

$$\begin{aligned}2^{\gamma i}\mathcal{J}_i^{(1)} \lesssim & \varepsilon^\alpha \|f\|_{W_p^{\alpha+\gamma}} \sum_{j\in\mathbb{Z}} \mathbf{1}_{\{(i-j)<0\}} 2^{\gamma(i-j)} \cdot \mathbf{1}_{\{j\geqslant -1\}} c_j\\ =&\varepsilon^\alpha \|f\|_{W_p^{\alpha+\gamma}} (a * b)_i.\end{aligned}$$

where $a_k = \mathbf{1}_{\{k\geqslant -1\}} c_k$ and $b_k = \mathbf{1}_{\{k<0\}} 2^{\gamma k}$ ($\forall k \in \mathbb{Z}$). Thus,

$$\begin{aligned}\|2^{\gamma i}\mathcal{J}_i^{(1)}\|_{\ell^p} &\lesssim \varepsilon^\alpha \|f\|_{W_p^{\alpha+\gamma}} \|a\|_{\ell^p} \|b\|_{\ell^1}\\ &\lesssim \varepsilon^\alpha \|f\|_{W_p^{\alpha+\gamma}} \left(\sum_{j\geqslant -1} c_j^p\right)^{1/p} \lesssim \varepsilon^\alpha \|f\|_{W_p^{\alpha+\gamma}}.\end{aligned} \tag{3.16}$$

For $\mathcal{J}_i^{(2)}$, if $i = -1$, then

$$\mathcal{J}_{-1}^{(2)} = \|\Delta_{-1}\mathcal{T}_\sigma \Delta_{-1} f\|_p \lesssim \varepsilon^\alpha \|\Delta_{-1} f\|_{H_p^\alpha} \lesssim \varepsilon^\alpha \|f\|_{W_p^{\alpha+\gamma}} c_{-1}.$$

If $i \geqslant 0$, choose $s \in (\max\{\gamma, 1-\alpha+d/p\}, 1)$ in Lemma 3.10. By Bernstein's inequality and Lemma 3.10, we have

$$\begin{aligned}\mathcal{J}_i^{(2)} &= \sum_{-1\leqslant j\leqslant i} \|\Delta_i \Delta^{-s/2}\Delta^{s/2}\mathcal{T}_\sigma \Delta_j f\|_p \overset{(2.4)}{\lesssim} 2^{-si} \sum_{-1\leqslant j\leqslant i} \|\Delta^{s/2}\mathcal{T}_\sigma \Delta_j f\|_p\\ &\leqslant 2^{-si} \sum_{-1\leqslant j\leqslant i} \left(\|[\Delta^{s/2}, \mathcal{T}_\sigma]\Delta_j f\|_p + \|\mathcal{T}_\sigma \Delta^{s/2}\Delta_j f\|_p\right)\\ &= 2^{-si} \sum_{-1\leqslant j\leqslant i} \left(\|[\Delta^{s/2}, \mathscr{L}^R]\Delta_j f\|_p + \|\mathcal{T}_\sigma \Delta^{s/2}\Delta_j f\|_p\right)\\ &\overset{(3.13),(3.15)}{\lesssim} 2^{-si} \sum_{-1\leqslant j\leqslant i} \left(\varepsilon^{1-s+d/p}\|\Delta_j f\|_{C^1} + \varepsilon^\alpha \|\Delta^{s/2}\Delta_j f\|_{H_p^\alpha}\right)\\ &\overset{(2.3)}{\lesssim} 2^{-si} \sum_{-1\leqslant j\leqslant i} \left(\varepsilon^{1-s+d/p} 2^{(1+d/p)j}\|\Delta_j f\|_p + \varepsilon^\alpha 2^{(s+\alpha)j}\|\Delta_j f\|_p\right)\\ &\lesssim \varepsilon^{1-s+d/p} 2^{-si} \sum_{-1\leqslant j\leqslant i} 2^{(\alpha+s)j}\|\Delta_j f\|_p\\ &\lesssim \varepsilon^{1-s+d/p}\|f\|_{W_p^{\alpha+\gamma}} 2^{-si} \sum_{-1\leqslant j\leqslant i} 2^{(s-\gamma)j} c_j\end{aligned}$$

Denoting $a_k = \mathbf{1}_{\{k\geqslant -1\}}c_k$, $d_k = \mathbf{1}_{\{k\geqslant 0\}}2^{(\gamma-s)k}$, then

$$2^{\gamma i}\mathcal{J}_i^{(2)} \leqslant \varepsilon^{1-s+d/p}\|f\|_{W_p^{\alpha+\gamma}}\sum_{-1\leqslant j\leqslant i}2^{(\gamma-s)(i-j)}c_j \leqslant \varepsilon^{1-s+d/p}\|f\|_{W_p^{\alpha+\gamma}}(d*a)_i.$$

Thus,

$$\|2^{\gamma i}\mathcal{J}_i^{(2)}\|_{\ell^p} \lesssim \varepsilon^{1-s+d/p}\|f\|_{W_p^{\alpha+\gamma}}\|d\|_{\ell^1}\|a\|_{\ell^p} \lesssim \varepsilon^{1-s+d/p}\|f\|_{W_p^{\alpha+\gamma}}. \tag{3.17}$$

By (3.16) and (3.17), we get

$$\begin{aligned}\|\mathcal{T}_\sigma f\|_{W_p^\gamma} =& \|2^{\gamma i}\|\Delta_i\mathcal{T}_\sigma f\|_p\|_{\ell^p} \leqslant \|2^{\gamma i}\mathcal{J}_i^{(1)}\|_{\ell^p} + \|2^{\gamma i}\mathcal{J}_i^{(2)}\|_{\ell^p}\\ \lesssim& \varepsilon^{1-s+d/p}\|f\|_{W_p^{\alpha+\gamma}},\end{aligned}$$

this yields our desired result. □

Now we are on the position of proving Lemma 3.8.

Proof of Lemma 3.8. Like before, we only give the aprior estimate here. Let $\{\zeta_k\}_{k\in\mathbb{N}}$ be a standard partition of unity, such that, for any k, the support of ζ_k lies in a ball B_k of radius $\varepsilon/8$, where ε will be determined later. Denote by y_k the center of B_k. Also for any k, we take functions $\eta_k, \xi_k \in C^\infty$, such that, $\eta_k = 1$ on $B_{\varepsilon/4}(y_k)$, $\eta_k = 0$ outside $B_{\varepsilon/2}(y_k)$, and $0 \leqslant \eta_k \leqslant 1$; $\xi_k = 1$ on $B_{\varepsilon/2}(y_k)$, $\xi_k = 0$ outside $B_\varepsilon(y_k)$, and $0 \leqslant \xi_k \leqslant 1$. Define $\sigma_k(x) = \xi_k(x)\sigma(x) + (1-\xi_k(x))\sigma(y_k)$,

$$\mathscr{L}_k^R f(x) := \int_{B_R}\Big(f(x+\sigma_k(x)z) - f(x) - \nabla f(x)\sigma_k(x)z^{(\alpha)}\Big)\nu(\mathrm{d}z).$$

$$\mathcal{L}_k^R f(x) := \int_{B_R}\Big(f(x+\sigma(y_k)z) - f(x) - \nabla f(x)\sigma(y_k)z^{(\alpha)}\Big)\nu(\mathrm{d}z).$$

Multiplying ζ_k on both side of (1.8), we get

$$\lambda(u\zeta_k) - \mathcal{L}_k^R(u\zeta_k) - b\cdot\nabla(u\zeta_k) = f\zeta_k + \zeta_k(b\cdot\nabla u) - b\cdot\nabla(u\zeta_k) + \zeta_k(\mathscr{L}^R u) - \mathcal{L}_k^R(\zeta_k u) \tag{3.18}$$

Theorem 3.4 yields,

$$\lambda\|u\zeta_k\|_{W_p^\gamma} + \|u\zeta_k\|_{W_p^{\alpha+\gamma}} \lesssim \left(\|f\zeta_k\|_{W_p^\gamma} + \|ub\cdot\nabla\zeta_k\|_{W_p^\gamma} + \|\zeta_k(\mathscr{L}^R u) - \mathcal{L}_k^R(\zeta_k u)\|_{W_p^\gamma}\right).$$

Hence, using Lemma 3.5 we have,

$$\begin{aligned}\lambda^p\|u\|_{W_p^\gamma}^p + \|u\|_{W_p^{\alpha+\gamma}}^p \lesssim& \sum_k \lambda^p\|u\zeta_k\|_{W_p^\gamma}^p + \|u\zeta_k\|_{W_p^{\alpha+\gamma}}^p\\ \lesssim& \sum_k\left(\|f\zeta_k\|_{W_p^\gamma}^p + \|ub\cdot\nabla\zeta_k\|_{W_p^\gamma}^p + \|\zeta_k(\mathscr{L}^R u) - \mathcal{L}_k^R(\zeta_k u)\|_{W_p^\gamma}^p\right)\end{aligned} \tag{3.19}$$

Again by Lemma 3.5,

$$\sum_k\|f\zeta_k\|_{W_p^\gamma}^p \asymp \|f\|_{W_p^\gamma}^p,\quad \sum_k\|ub\cdot\nabla\zeta_k\|_{W_p^\gamma}^p \lesssim \|ub\|_{W_p^\gamma}^p \lesssim \|u\|_{W_p^\gamma}^p, \tag{3.20}$$

the last inequality above is due to the following fact:

$$\begin{aligned}\|ub\|_{W_p^\gamma}^p \lesssim& \|ub\|_p^p + \iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|ub(x)-ub(y)|^p}{|x-y|^{d+\gamma p}}\mathrm{d}x\mathrm{d}y\\ \leqslant& \|u\|_p^p\|b\|_{L^\infty}^p\mathrm{d}x\mathrm{d}y + \iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|u(x)-u(y)|^p|b(x)|^p}{|x-y|^{d+\gamma p}}\\ &+ \iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|u(y)|^p|b(x)-b(y)|^p}{|x-y|^{d+\gamma p}}\mathrm{d}x\mathrm{d}y\end{aligned}$$

$$
\begin{aligned}
&\lesssim \|u\|_p^p\|b\|_{L^\infty}^p + \|b\|_{L^\infty}^p[u]_{\gamma,p}^p + \int_{\mathbb{R}^d}|u(y)|^p\mathrm{d}y\int_{\mathbb{R}^d}\frac{[b]_\beta^p|x-y|^{\beta p}\wedge\|b\|_{L^\infty}^p}{|x-y|^{d+\gamma p}}\mathrm{d}x\\
&\lesssim \|u\|_{W_p^\gamma}^p\|b\|_{C^\beta}^p.
\end{aligned}
$$

Next we estimate the third term in the last line of (3.19). We only give the proof for $\alpha<1$ here, because the proof for $\alpha\geqslant 1$ is almost the same.

$$
\begin{aligned}
&\zeta_k(x)(\mathscr{L}^R u)(x) - \mathcal{L}_k^R(\zeta_k u)(x)\\
=&\left[\mathscr{L}^R(u\zeta_k)(x) - \mathcal{L}_k^R(u\zeta_k)(x)\right]\eta_k(x) + \left[\mathscr{L}^R(u\zeta_k)(x) - \mathcal{L}_k^R(u\zeta_k)(x)\right](1-\eta_k(x))\\
&-\left\{u(x)\mathscr{L}^R\zeta_k(x) + \int_{B_R}[u(x+\sigma(x)z)-u(x)]\cdot[\zeta_k(x+\sigma(x)z)-\zeta_k(x)]\nu(\mathrm{d}z)\right\}\\
=:&\, I_k^{(1)}(x) + I_k^{(2)}(x) - I_k^{(3)}(x)
\end{aligned}
$$

For $I_k^{(1)}$, notice $\sigma_k(x)=\sigma(x)$ when x belongs to the support of η_k, so we have

$$
I_k^{(1)}(x) = [\mathscr{L}_k^R(u\zeta_k)(x) - \mathcal{L}_k^R(u\zeta_k)(x)]\eta_k(x).
$$

$\mathscr{L}_k^R$ satisfies assumption $(\mathbf{A}_\varepsilon)$, by Lemma 3.11, we have

$$
\|I_k^{(1)}\|_{W_p^\gamma} \leqslant c_\varepsilon\|u\zeta_k\|_{W_p^{\alpha+\gamma}} \quad (c_\varepsilon\to 0 \text{ as } \varepsilon\to 0). \tag{3.21}
$$

For $I_k^{(2)}(x)$, since $1-\eta_k(x)=0$ if $|x-y_k|\leqslant\frac{\varepsilon}{4}$ and $u\zeta_k(x)=0$ if $|x-y_k|>\frac{\varepsilon}{8}$, we have

$$
I_k^{(2)}(x) = \int_{\frac{\varepsilon}{8\Lambda}\leqslant|z|<R}[u\zeta_k(x+\sigma(x)z) - u\zeta_k(x+\sigma(y_k)z)](1-\eta_k(x))\nu(\mathrm{d}z).
$$

Choosing $s\in(d/p, 1\wedge(\alpha+\gamma-1))$, by Minkowski inequality, Lemma 3.6 and interpolation theorem, we have

$$
\begin{aligned}
\|I_k^{(2)}\|_p \leqslant& \int_{\frac{\varepsilon}{8\Lambda}\leqslant|z|<R}\|u\zeta_k(\cdot+\sigma(\cdot)z) - u\zeta_k(\cdot)\|_p\nu(\mathrm{d}z)\\
&+\int_{\frac{\varepsilon}{8\Lambda}\leqslant|z|<R}\|u\zeta_k(\cdot+\sigma(y_k)z) - u\zeta_k(\cdot)\|_p\nu(\mathrm{d}z)\\
\lesssim& \int_{\frac{\varepsilon}{8\Lambda}\leqslant|z|<R}|z|^s\left\|\sup_{y\in\mathbb{R}^d}\frac{|u\zeta_k(\cdot+y)-u\zeta_k(\cdot)|}{|y|^s}\right\|_p\nu(\mathrm{d}z)\\
\overset{(3.8)}{\lesssim}& \|u\zeta_k\|_{H_p^s}\int_{\frac{\varepsilon}{8\Lambda}\leqslant|z|<R}|z|^s\nu(\mathrm{d}z) \lesssim \varepsilon^{s-\alpha}\|u\zeta_k\|_{H_p^s}\\
\leqslant& \varepsilon\|u\zeta_k\|_{W_p^{\alpha+\gamma}} + C_\varepsilon\|u\zeta_k\|_p.
\end{aligned}
$$

And

$$
\begin{aligned}
\nabla I_k^{(2)}(x) =& \int_{\frac{\varepsilon}{8\Lambda}\leqslant|z|<R}\Big\{[\nabla(u\zeta_k)(x+\sigma(x)z)(\mathbb{I}+\nabla\sigma(x)) - \nabla(u\zeta_k)(x+\sigma(y_k)z)]\\
&(1-\eta_k(x)) - [u\zeta_k(x+\sigma(x)z) - u\zeta_k(x+\sigma(y_k)z)]\nabla\eta_k(x)\Big\}\nu(\mathrm{d}z)\\
=& \int_{\frac{\varepsilon}{8\Lambda}\leqslant|z|<R}\Big\{\nabla(u\zeta_k)(x+\sigma(x)z)\cdot\nabla\sigma(x)z\ (1-\eta_k(x))\\
&+[\nabla(u\zeta_k)(x+\sigma(x)z) - \nabla(u\zeta_k)(x+\sigma(y_k)z)](1-\eta_k(x))\\
&-[u\zeta_k(x+\sigma(x)z) - u\zeta_k(x+\sigma(y_k)z)]\nabla\eta_k(x)\Big\}\nu(\mathrm{d}z).
\end{aligned}
$$

Like the estimates for $\|I_k^{(2)}\|_p$, choosing $s\in(d/p,1\wedge(\alpha+\gamma-1))$, then

$$
\begin{aligned}
\|\nabla I_k^{(2)}\|_p \lesssim & \int_{\frac{\varepsilon}{8\Lambda}\leqslant|z|<R}\Bigg\{\Big[\|\nabla(u\zeta_k)(\cdot+\sigma(\cdot)z)-\nabla(u\zeta_k)(\cdot)\|_p+\|\nabla(u\zeta_k)\|_p\Big]\cdot\|\nabla\sigma\|_\infty|z| \\
&+\|\nabla(u\zeta_k)(\cdot+\sigma(\cdot)z)-\nabla(u\zeta_k)(\cdot+\sigma(y_k)z)\|_p \\
&+\|\nabla\eta_k\|_\infty\cdot\|u\zeta_k(\cdot+\sigma(\cdot)z)-u\zeta_k(\cdot+\sigma(y_k)z)\|_p\Bigg\}\nu(\mathrm{d}z) \\
\overset{(3.8)}{\lesssim_\varepsilon} & \int_{\frac{\varepsilon}{8\Lambda}\leqslant|z|<R}|z|^s\left\|\sup_{y\in\mathbb{R}^d}\frac{|\nabla(u\zeta_k)(\cdot+y)-\nabla(u\zeta_k)(\cdot)|}{|y|^s}\right\|_p\nu(\mathrm{d}z)+\int_{\frac{\varepsilon}{8\Lambda}\leqslant|z|<R}|z|\|\nabla(u\zeta_k)\|_p\nu(\mathrm{d}z) \\
\lesssim_\varepsilon & \|u\zeta_k\|_{H_p^{1+s}}\leqslant\varepsilon\|u\zeta_k\|_{W_p^{\alpha+\gamma}}+C_\varepsilon\|u\zeta_k\|_p.
\end{aligned}
$$

So

$$
\|I_k^{(2)}\|_{W_p^\gamma}\leqslant\|I_k^{(2)}\|_{W_p^1}\leqslant\varepsilon\|u\zeta_k\|_{W_p^{\alpha+\gamma}}+C_\varepsilon\|u\zeta_k\|_p. \tag{3.22}
$$

For $I_k^{(3)}(x)$, for any $|z|<R$, it's not hard to see,

$$
\sup_{x\in\mathbb{R}^d}\sum_k|\zeta_k(x+\sigma(x)z)-\zeta_k(x)|^p\lesssim|z|^p,\quad \sup_{x\in\mathbb{R}^d}\sum_k|(\mathscr{L}^R\zeta_k)(x)|^p\lesssim 1.
$$

Hence, for any $s\in((\alpha-1)\vee 0,\alpha\wedge 1)$,

$$
\begin{aligned}
\left(\sum_k\|I_k^{(3)}\|_p^p\right)^{1/p}\leqslant&\left(\sum_k\int_{\mathbb{R}^d}|u(x)|^p|\mathscr{L}^R\zeta_k(x)|^p\mathrm{d}x\right)^{1/p} \\
&+\int_{|z|<R}\left\{\int_{\mathbb{R}^d}|u(x+\sigma(x)z)-u(x)|^p\sum_k|\zeta_k(x+\sigma(x)z)-\zeta_k(x)|^p\mathrm{d}x\right\}^{1/p}\nu(\mathrm{d}z) \\
\lesssim&\|u\|_p+\int_{|z|<R}\|u\|_{H_p^s}|z|^{s+1}\nu(\mathrm{d}z)\leqslant\varepsilon\|u\|_{W_p^{\alpha+\gamma}}+C_\varepsilon\|u\|_p.
\end{aligned}
$$

Similarly, we have

$$
\left(\sum_k\|\nabla I_k^{(3)}\|_p^p\right)^{1/p}\leqslant\varepsilon\|u\|_{W_p^{\alpha+\gamma}}+C_\varepsilon\|u\|_p.
$$

So,

$$
\left(\sum_k\|I_k^{(3)}\|_{W_p^\gamma}^p\right)^{1/p}\leqslant\left(\sum_k\|I_k^{(3)}\|_{W_p^1}^p\right)^{1/p}\leqslant\varepsilon\|u\|_{W_p^{\alpha+\gamma}}+C_\varepsilon\|u\|_p. \tag{3.23}
$$

Now using Lemma 3.5, combining (3.19), (3.20), (3.21), (3.22), (3.23) and choosing ε sufficiently small and λ_0 sufficiently large, we get

$$
\lambda\|u\|_{W_p^\gamma}+\|u\|_{W_p^{\alpha+\gamma}}\leqslant C\|f\|_{W_p^\gamma}. \tag{3.24}
$$

Now nultiplying both sides of (3.1) by χ_z, we have

$$
\lambda(u\chi_z)-\mathscr{L}^R(u\chi_z)-b\cdot\nabla(u\chi_z)=g_z,
$$

where

$$
g_z:=f\chi_z+\chi_z\mathscr{L}^Ru-\mathscr{L}^R(u\chi_z)-ub\cdot\nabla\chi_z.
$$

We omit the subscript z below and just prove the case when $\alpha<1$. By definition,

$$[\chi\mathscr{L}^R u-\mathscr{L}^R(u\chi)](x)=\int_{|z|<R}u(x+\sigma(x)z)(\chi(x+\sigma(x)z)-\chi(x))\nu(\mathrm{d}z),$$

so

$$\begin{aligned}\|\chi\mathscr{L}^R u-\mathscr{L}^R(u\chi)\|_p\leqslant&\|u\|_\infty\left\|\int_{|z|<R}[\chi(\cdot+\sigma(\cdot)z)-\chi(\cdot)]\nu(\mathrm{d}z)\right\|_p\\ \leqslant&C\|u\|_\infty.\end{aligned}$$

Notice that

$$\begin{aligned}\nabla[\chi\mathscr{L}^R u-\mathscr{L}^R(u\chi)](x)=&\int_{B_R}\nabla u(x+\sigma(x)z)(\mathbb{I}+\nabla\sigma(x)z)(\chi(x+\sigma(x)z)-\chi(x))\nu(\mathrm{d}z)\\&+\int_{B_R}u(x+\sigma(x)z)(\nabla\chi(x+\sigma(x)z)(\mathbb{I}+\nabla\sigma(x)z)-\nabla\chi(x))\nu(\mathrm{d}z),\end{aligned}$$

we have

$$\begin{aligned}\|\nabla[\chi\mathscr{L}^R u-\mathscr{L}^R(u\chi)]\|_p\leqslant&C\|\nabla u\|_\infty\left\|\int_{B_R}[\chi(\cdot+\sigma(\cdot)z)-\chi(\cdot)]\nu(\mathrm{d}z)\right\|_p\\&+C\|u\|_\infty\left\|\int_{B_R}[\nabla\chi(\cdot+\sigma(\cdot)z)-\nabla\chi(\cdot)]\nu(\mathrm{d}z)\right\|_p\\&+C\|u\|_\infty\left\|\int_{B_R}\nabla\chi(\cdot+\sigma(\cdot)z)\cdot\nabla\sigma(x)z\nu(\mathrm{d}z)\right\|_p\\ \leqslant&C\|u\|_{C^1}.\end{aligned}$$

Thus,

$$\begin{aligned}\|g_z\|_{W_p^\gamma}\leqslant&\|f\chi\|_{W_p^\gamma}+C\|u\|_{C^1}+C\|b\|_{C^\beta}\|u\nabla\chi\|_{W_p^\gamma}\\ \leqslant&C(\|f\|_{\mathcal{W}_p^\gamma}+\|u\|_{C^1}+\|u\|_{\mathcal{W}_p^\gamma}).\end{aligned}$$

By (3.24), we get

$$\begin{aligned}\|u\|_{\mathcal{W}_p^{\alpha+\gamma}}+\lambda\|u\|_{\mathcal{W}_p^\gamma}=&\sup_{z\in\mathbb{R}^d}(\|u\chi_z\|_{W_p^{\alpha+\gamma}}+\lambda\|u\|_{\mathcal{W}_p^\gamma})\\ \leqslant&C(\|f\|_{\mathcal{W}_p^\gamma}+\|u\|_{C^1}+\|u\|_{\mathcal{W}_p^\gamma}).\end{aligned}$$

By Lemma 2.3 and interpolation, for any $\delta>0$ there is a constant C_δ such that

$$\|u\|_{C^1}\leqslant\delta\|u\|_{\mathcal{W}^{\alpha+\gamma}}+C_\delta\|u\|_{\mathcal{W}^\gamma},$$

so we complete our proof by choosing δ small and λ_0 sufficiently large.

□

4. Proof of Theorem 1.1

Let $N(\mathrm{d}t,\mathrm{d}z)$ be the Poisson random measure associated with Z, that is,

$$N((0,t]\times E)=\sum_{s\leqslant t}1_E(\Delta Z_s),$$

where E is any compact set of $\mathbb{R}^d\backslash\{0\}$ and $\Delta Z_s := Z_s - Z_{s-}$. The intensity measure of N is denoted by $\mathrm{d}t\nu(\mathrm{d}z)$. Let $\widetilde{N}(\mathrm{d}t,\mathrm{d}z) = N(\mathrm{d}t,\mathrm{d}z) - \mathrm{d}t\nu(\mathrm{d}z)$ and

$$N^{(\alpha)}(\mathrm{d}t,\mathrm{d}z) = \begin{cases} N(\mathrm{d}t,\mathrm{d}z), & \alpha < 1 \\ N(\mathrm{d}t,\mathrm{d}z) - \mathrm{d}t\mathbf{1}_{B_1}(z)\nu(\mathrm{d}z), & \alpha = 1 \\ N(\mathrm{d}t,\mathrm{d}z) - \mathrm{d}t\nu(\mathrm{d}z), & \alpha > 1 \end{cases}$$

Recalling that

$$\mathbf{E}\mathrm{e}^{i\xi\cdot Z_1} = \exp\left[\int_{\mathbb{R}^d} (e^{i\xi\cdot z} - 1 - i\xi\cdot z^{(\alpha)})\nu(\mathrm{d}z)\right],$$

by Lévy-Itô's decomposition, we have

$$Z_t = \int_0^t\int_{\mathbb{R}^d} zN^{(\alpha)}(\mathrm{d}s,\mathrm{d}z).$$

Thus, SDE (1.1) can be rewritten as

$$X_t = X_0 + \int_0^t b(X_s)\mathrm{d}s + \int_0^t\int_{\mathbb{R}^d} \sigma(X_{s-})zN^{(\alpha)}(\mathrm{d}s,\mathrm{d}z). \tag{4.1}$$

Now we are on the point to give the proof of our main results.

Proof of Theorem 1.1. (1). For the well-posedness, one can assume ν compactly supported on B_R i.e. $\sup_{t\geqslant 0}|\Delta Z_t| < R$, otherwise, we can take $\tau_0 := 0, \tau_k := \inf\{t > \tau_{k-1} : \Delta Z_t \geqslant R\}$ for any $k \geqslant 1$, and solve the SDE step by step.

Let u be the solution of equation:

$$\lambda u - \mathscr{L}^R u - b\cdot\nabla u = b, \quad \lambda \geqslant \lambda_0.$$

By Theorem 3.7, for any $\mu \in (\alpha/2, \alpha+\beta-1)$, we have $u \in C^{1+\mu}$ with $\|u\|_{C^{1+\mu}} = c(\lambda,\mu)$ and $c(\lambda,\mu) \to 0$ as $\lambda \to \infty$. Choose λ sufficiently large so that $\|\nabla u\|_\infty \leqslant 1/2$, thus $\Phi : x \mapsto x + u(x)$ is a $C^{1+\mu}$-diffeomorphism. By a generalized version of Itô's formula(c.f. [21]), we get

$$\begin{aligned} u(X_t) =& u(X_0) + \int_0^t [\mathscr{L}^R u + b\cdot\nabla u](X_s)\mathrm{d}s \\ &+ \int_0^t\int_{B_R} [u(X_{s-} + \sigma(X_{s-})z) - u(X_{s-})]\widetilde{N}(\mathrm{d}s,\mathrm{d}z). \end{aligned}$$

Define $Y_t := \Phi(X_t)$, then

$$\begin{aligned} Y_t = \Phi(X_t) &= \Phi(X_0) + \int_0^t \lambda u(X_s)\mathrm{d}s \\ &+ \int_0^t\int_{B_R} [\Phi(X_{s-} + \sigma(X_{s-})z) - \Phi(X_{s-})]\widetilde{N}(\mathrm{d}s,\mathrm{d}z) \\ &= Y_0 + \int_0^t a(Y_s)\mathrm{d}s + \int_0^t\int_{B_R} g(Y_{s-},z)\widetilde{N}(\mathrm{d}s,\mathrm{d}z), \end{aligned} \tag{4.2}$$

where

$$a(y) := \lambda u(\Phi^{-1}(y));$$

$$\begin{aligned} g(y,z) :=& \Phi(\Phi^{-1}(y) + \sigma(\Phi^{-1}(y)z) - y \\ =& u(\Phi^{-1}(y) + \sigma(\Phi^{-1}(y))z) - u(\Phi^{-1}(y)) + \sigma(\Phi^{-1}(y))z. \end{aligned}$$

It has been showed in [21] and [5] that we only need to show the well-posedness of (4.2), in order to get the well-posedness of (4.1). Elementary calculation yields,

$$\nabla a(y) = \lambda \nabla u(\Phi^{-1}(y))\nabla\Phi^{-1}(y);$$

$$\begin{aligned}\nabla_y g(y,z) =&[\nabla u(\Phi^{-1}(y)+\sigma(\Phi^{-1}(y))z)-\nabla u(\Phi^{-1}(y))]\nabla\Phi^{-1}(y)\\&+\nabla u(\Phi^{-1}(y)+\sigma(\Phi^{-1}(y))z)\ \nabla\sigma(\Phi^{-1}(y))\ \nabla\Phi^{-1}(y)z\\&+\nabla\sigma(\Phi^{-1}(y))\ \nabla\Phi^{-1}(y)z.\end{aligned}\tag{4.3}$$

Fix $\mu\in(\alpha/2,\alpha+\beta-1)$, noticing that $u\in C^{1+\mu}$ with $\|u\|_{C^{1+\mu}}=c(\lambda,\mu)$, we have,

$$\|a\|_{C^{1+\mu}}<\infty,\quad |g(y,z)|\leqslant C|z|\tag{4.4}$$

and

$$\begin{aligned}\|\nabla_y g(\cdot,z)\|_\infty\leqslant&\|\nabla u\|_{C^\mu}\|\nabla\Phi^{-1}\|_\infty(\|\sigma\|_\infty\cdot|z|)^\mu\\&+\|\nabla u\|_\infty\|\nabla\sigma\|_\infty\|\nabla\Phi^{-1}\|_\infty|z|+\|\nabla\sigma\|_\infty\|\nabla\Phi^{-1}\|_\infty|z|\\\leqslant&c(\lambda,\mu)(1-c(\lambda,\mu))^{-1}\big(\|\sigma\|_\infty^\mu|z|^\mu+\|\nabla\sigma\|_\infty|z|\big)+\|\nabla\sigma\|_\infty|z|.\end{aligned}\tag{4.5}$$

Thanks to the estimates (4.4) and (4.5), the proof for existence and uniqueness of solution to (4.2) becomes quite standard. Indeed, let

$$Y_t^0=Y_0;\quad Y_t^{n+1}=Y_0+\int_0^t a(Y_s^n)\mathrm{d}s+\int_0^t\!\!\int_{B_R} g(Y_{s-}^n,z)\widetilde{N}(\mathrm{d}s,\mathrm{d}z),$$

then by Doob's inequality and (4.4), (4.5), we get

$$\begin{aligned}\mathbf{E}\sup_{0\leqslant s\leqslant t}|Y_s^{n+1}-Y_s^n|^2\leqslant&C\|\nabla a\|_\infty^2\mathbf{E}\int_0^t|Y_s^n-Y_s^{n-1}|^2\mathrm{d}s\\&+C\mathbf{E}\int_0^t\!\!\int_{B_R}\left|g(Y_{s-}^n,z)-g(Y_{s-}^{n-1},z)\right|^2\nu(\mathrm{d}z)\mathrm{d}s\\\leqslant&C\left(\|\nabla a\|_\infty^2+\int_{B_R}\|\nabla_y g(\cdot,z)\|_\infty^2\nu(\mathrm{d}z)\right)\mathbf{E}\int_0^t|Y_s^n-Y_s^{n-1}|^2\mathrm{d}s\\\leqslant&C\left[1+\int_{B_R}(|z|^{2\mu}+|z|^2)\nu(\mathrm{d}z)\right]\mathbf{E}\int_0^t|Y_s^n-Y_s^{n-1}|^2\mathrm{d}s\\\leqslant&C\mathbf{E}\int_0^t|Y_s^n-Y_s^{n-1}|^2\mathrm{d}s\leqslant Ct\mathbf{E}\sup_{0\leqslant s\leqslant t}|Y_s^n-Y_s^{n-1}|^2.\end{aligned}$$

Choosing T sufficiently small such that $CT\leqslant\frac{1}{2}$, we get

$$\lim_{n,m\to\infty}\mathbf{E}\sup_{0\leqslant t\leqslant T}|Y_t^n-Y_t^m|^2=0.\tag{4.6}$$

And notice that all the estimates above do not depends on the initial data Y_0, so we obtain that (4.6) holds for any $T>0$. And the limit point Y of $\{Y^n\}_n$ is a strong solution to (4.2). The uniqueness for (4.2) can be obtained by using Gronwall's inequality and similar estimates above.

Next we show that for each t, X_t is Mallivian differentiable. Notice that $\nabla(\Phi^{-1})(y)=(\nabla\Phi)^{-1}\circ\Phi^{-1}(y)$ and $(\nabla\Phi)^{-1}=(\nabla\Phi)^*(\det\nabla\Phi)^{-1}\in C^\mu$, we get $\Phi^{-1}\in C^{1+\mu}$. Since $X_t=\Phi^{-1}(Y_t)$, by Theorem 12.8 of [20], we only need to show that Y_t is Mallivian differentiable. And by the closability of Mallivian derivate(Lemma 2.2), the desired result can be obtained after we prove the following estimate:

$$\sup_{n\in\mathbb{N};t\in[0,T]} \|D_{r,z}Y_t^n\|_{L^2(\lambda\times\nu\times\mathbf{P})} < \infty. \tag{4.7}$$

Assume Y_t^n above is Mallivian differentiable for each t and $\sup_{t\in[0,T]}\|D_{r,z}Y_t^n\|_{L^2(\lambda\times\nu\times\mathbf{P})} < \infty$. By (4.4), we have

$$a(Y_s^n) \in L^2(\mathbf{P}), \quad a(Y_s^n + D_{r,z}Y_s^n) - a(Y_s^n) \in L^2(\lambda\times\nu\times\mathbf{P}).$$

Thanks to Theorem 12.8 of [20], we obtain $a(Y_s^n) \in \mathbb{D}_2^1$ and

$$D_{r,z}a(Y_s^n) = a(Y_s^n + D_{r,z}Y_s^n) - a(Y_s^n).$$

Similarly, by (4.4) and (4.5), we have

$$\mathbf{E}\int_0^T\int_{\mathbb{R}^d} |g(Y_{s-},\eta)|^2\nu(\mathrm{d}\eta)\mathrm{d}s < \infty,$$

and by Itô's isometry, for any $t \in [0,T]$,

$$\begin{aligned}
&\mathbf{E}\left[\int_0^t\int_{\mathbb{R}^d}\left(\int_0^t\int_{\mathbb{R}^d} D_{r,z}g(Y_{s-}^n,\eta)\widetilde{N}(\mathrm{d}s,\mathrm{d}\eta)\right)^2\nu(\mathrm{d}z)\mathrm{d}r\right]\\
=&\mathbf{E}\left[\int_0^t\int_{\mathbb{R}^d}\left(\int_0^t\int_{\mathbb{R}^d}[g(Y_s^n + D_{r,z}Y_{s-}^n,\eta) - g(Y_{s-}^n,\eta)]\widetilde{N}(\mathrm{d}s,\mathrm{d}\eta)\right)^2\nu(\mathrm{d}z)\mathrm{d}r\right]\\
\overset{(4.5)}{\leqslant}& C\mathbf{E}\left[\int_0^t\int_{B_R}\left[\int_0^t\int_{B_R}\|\nabla_y g(\cdot,\eta)\|_\infty^2|\eta|^{2\mu}|D_{r,z}Y_{s-}^n|^2\nu(\mathrm{d}\eta)\mathrm{d}s\right]\nu(\mathrm{d}z)\mathrm{d}r\right]\\
\leqslant& C\mathbf{E}\int_0^T\int_{B_R}\left[\int_0^T|D_{r,z}Y_{s-}^n|^2\mathrm{d}s\right]\nu(\mathrm{d}z)\mathrm{d}r \leqslant C\sup_{t\in[0,T]}\|D_{r,z}Y_t^n\|_{L^2(\lambda\times\nu\times\mathbf{P})} < \infty.
\end{aligned}$$

Thanks to Theorem 12.15 of [20], we obtain

$$\begin{aligned}
&D_{r,z}\int_0^t\int_{B_R} g(Y_{s-}^n,\eta)\widetilde{N}(\mathrm{d}s,\mathrm{d}\eta)\\
=&\left[\int_0^t\int_{B_R}[g(Y_{s-}^n + D_{r,z}Y_{s-}^n,\eta) - g(Y_{s-}^n,\eta)]\widetilde{N}(\mathrm{d}s,\mathrm{d}\eta) + g(Y_{r-}^n,z)\right]\mathbf{1}_{[0,t]}(r).
\end{aligned}$$

So we obtain $Y_t^{n+1} \in \mathbb{D}_2^1$ and for almost every $r \in [0,t]$,

$$\begin{aligned}
D_{r,z}Y_t^{n+1} =& g(Y_{r-}^n,z) + \int_r^t[a(Y_s^n + D_{r,z}Y_s^n) - a(Y_s^n)]\mathrm{d}s\\
&+\int_r^t\int_{B_R}[g(Y_{s-}^n + D_{r,z}Y_{s-}^n,\eta) - g(Y_{s-}^n,\eta)]\widetilde{N}(\mathrm{d}s,\mathrm{d}\eta)
\end{aligned}$$

For any $r \in [0,T]$, denote

$$f_r^n = \mathbf{E}\left[\int_{B_R}\left[\sup_{r\leqslant t\leqslant T}|D_{r,z}Y_t^n|^2\right]\nu(\mathrm{d}z)\right]$$

Again by Doob's inequality,

$$\begin{aligned}
f_r^{n+1} =& \mathbf{E}\left[\int_{B_R}\left[\sup_{r\leqslant t\leqslant T}|D_{r,z}Y_t^{n+1}|^2\right]\nu(\mathrm{d}z)\right]\\
\leqslant& C\Bigg\{\int_{B_R}\|g(\cdot,z)\|_\infty^2\nu(\mathrm{d}z) + \|\nabla a\|_\infty^2\mathbf{E}\int_r^T\int_{B_R}|D_{r,z}Y_{s-}^n|^2\nu(\mathrm{d}z)\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\quad + \int_{B_R} \mathbf{E}\left[\int_r^T \int_{B_R} \|\nabla_y g(\cdot,\eta)\|_\infty^2 |D_{r,z} Y_{s-}^n|^2 \nu(\mathrm{d}\eta)\mathrm{d}s\right] \nu(\mathrm{d}z)\Bigg\} \\
&\leqslant C\Bigg\{1 + T\mathbf{E}\left[\int_{B_R}\left[\sup_{r\leqslant t\leqslant T} |D_{r,z}Y_{t-}^n|^2\right]\nu(\mathrm{d}z)\right] \\
&\quad + T\mathbf{E}\int_{B_R}\left[\sup_{r\leqslant t\leqslant T}|D_{r,z}Y_{t-}^n|^2\right]\nu(\mathrm{d}z)\int_{B_R}|\eta|^{2\mu}\nu(\mathrm{d}\eta)\Bigg\} \\
&= C + CTf_r^n,
\end{aligned}
$$

here C is independent with n, r, T. By choosing $T \leqslant T_0 := \frac{1}{2C}$, then we have

$$
f_r^n \leqslant C + \frac{f_r^{n-1}}{2} \leqslant \cdots \leqslant 2C + f_r^0 = 2C + \mathbf{E}\int_{B_R} |D_{r,z}Y_0|^2 \nu(\mathrm{d}z).
$$

Thus,

$$
\sup_{n\in\mathbb{N}; r\in[0,T]} \mathbf{E}\left[\int_{B_R}\left[\sup_{t\in[r,T]} |D_{r,z}Y_t^n|^2\right]\nu(\mathrm{d}z)\right] < \infty.
$$

which implies (4.7) for sufficiently small T. For arbitrary $T > T_0$, by the similar argument above, we can see that

$$
\begin{aligned}
&\sup_{n\in\mathbb{N}; r\in[0,T]} \mathbf{E}\left[\int_{B_R}\left[\sup_{t\in[r,T]} |D_{r,z}Y_t^n|^2\right]\nu(\mathrm{d}z)\right] \leqslant \sup_{n\in\mathbb{N}; r\in[0,T]} f_r^n \\
&\leqslant 2C + \sup_{n\in\mathbb{N}; r\in[0,T]} \mathbf{E}\int_{B_R} |D_{r,z}Y_{T-T_0}^n|^2 \nu(\mathrm{d}z) \\
&\leqslant \cdots \leqslant 2C([T/T_0]+1) + \sup_{r\in[0,T]} \mathbf{E}\int_{B_R} |D_{r,z}Y_0|^2\nu(\mathrm{d}z) < \infty.
\end{aligned}
$$

So we complete our proof.

(2) Choosing λ sufficiently large, by $(\mathbf{H}_3)$, (4.5), for any $z \in B_{r_0}$, we have

$$
\begin{aligned}
|\nabla_y g(y,z)| &\leqslant \|\nabla\sigma\|_\infty |z| + c(\lambda,\mu)(1-c(\lambda,\mu))^{-1}\big(\|\sigma\|_\infty^\mu |z|^\mu + \|\nabla\sigma\|_\infty |z|\big) \\
&\leqslant r_0 \|\nabla\sigma\|_\infty^{-1} + C\cdot c(\lambda,\mu) < 1,
\end{aligned}
$$

which implies that for each $z \in \operatorname{supp}\nu \subseteq B_{r_0}$ the map $y \mapsto y + g(y,z)$ is homeomorphic and $\mathbb{I} + \nabla_y g(y,z)$ is invertible. Again by (4.5), for any $z \in B_{r_0}$, $\|\nabla_y g(\cdot,z)\|_\infty \leqslant K(z) \asymp |z|^\mu$. Since $2\mu > \alpha$, by $(\mathbf{H}_1)$,

$$
\int_{B_{r_0}} K(z)^2 \nu(\mathrm{d}z) \leqslant C\int_{B_{r_0}} |z|^{2\mu}\nu_2(\mathrm{d}z) \leqslant Cr_0^{2\mu-\alpha} < \infty.
$$

Notice that $\sigma \in C^{1+\delta}$, by (4.3) and the regularity estimates for u, one can also check that

$$
|\nabla_y g(y,z) - \nabla_y g(y',z)| \leqslant L(z)|y-y'|^{\min\{\delta,\mu\}}
$$

and $L(z) \asymp |z|^\mu$. So we also have

$$
\int_{B_{r_0}} L(z)^2 \nu(\mathrm{d}z) < \infty.
$$

Thanks to [14, Theorem 3.11], $\{Y_t(x)\}_{t\geqslant 0; x\in\mathbb{R}^d}$ defines a C^1-stochastic flow, so does $\{X_t(x)\}_{t\geqslant 0; x\in\mathbb{R}^d}$. $\square$

Remark 4.1. *By Theorem 3.4 and the proof of Theorem 1.1, one can see that if σ is a constant invertible matrix, then the conclusions in Theorem 1.1 still hold if ν satisfies* $(\mathbf{H}_1)$ *and $b \in C^\beta$ with $\beta \in (1-\frac{\alpha}{2}, 1)$.*

Following the argument in [24], we give the proof of Corollary 1.2. The flow property of strong solutions will play a crucial role in this proof.

Proof of Corollary 1.2. Suppose θ solves (1.7). Denote

$$y_t := \theta_t + Z_t, \quad \phi(t) := X_{1-t}(y_t),$$

where $\{X_t(x)\}_{t\geqslant 0; x\in\mathbb{R}^d}$ is the stochastic flow associated with (1.1). Recalling that there is a full set $\Omega_0 \subseteq \Omega$ such that for any $\omega \in \Omega_0$, For any $t \in [0,1]$ and $\delta \in (0,1)$,

$$|X_t(x,\omega) - X_t(y,\omega)| \leqslant K(\omega)|x-y|^\delta,$$

here K is a integrable variable depending on δ. We will show that $\phi(t,\omega)$ are constant functions for any $\omega \in \Omega_0$. By the above inequality, we obtain that for any $\omega \in \Omega_0$,

$$\begin{aligned}|\phi(t,\omega) - \phi(s,\omega)| &= |X_{1-t}(y_t(\omega),\omega) - X_{1-t}(X_{t-s}(y_s(\omega),\omega),\omega)|\\ &\leqslant K(\omega)|y_t(\omega) - X_{t-s}(y_s(\omega),\omega)|^\delta,\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}|y_r(\omega) - X_{r-s}(y_s(\omega),\omega)| =& |[y_r(\omega) - y_s(\omega)] - [X_{r-s}(y_s(\omega),\omega) - y_s(\omega)]|\\ =& \left|\int_s^r b(y_u(\omega))\mathrm{d}u - \int_s^r b(X_{u-s}(y_s(\omega),\omega))\mathrm{d}u\right|\\ \leqslant& 2\|b\|_\infty |r-s|.\end{aligned} \tag{4.9}$$

Hence,

$$\begin{aligned}|y_t(\omega) - X_{t-s}(y_s(\omega),\omega)| =& \left|\int_s^t b(y_r(\omega))\mathrm{d}r - \int_s^t b(X_{r-s}(y_s(\omega),\omega))\mathrm{d}r\right|\\ \leqslant& [b]_\beta \int_s^t |y_r(\omega) - X_{r-s}(y_s(\omega),\omega)|^\beta \mathrm{d}r\\ \overset{(4.9)}{\leqslant}& C\|b\|^2_{C^\beta}\int_s^t |r-s|^\beta \mathrm{d}r \leqslant C|t-s|^{1+\beta}.\end{aligned}$$

Combining (4.8) and above inequalities, we obtain

$$|\phi(t,\omega) - \phi(s,\omega)| \leqslant CK(\omega)|t-s|^{\delta(1+\beta)}, \quad \omega \in \Omega_0.$$

By choosing $\delta > (1+\beta)^{-1}$, we obtain that for all $\omega \in \Omega_0$

$$X_1(x,\omega) = \phi(0,\omega) = \phi(1,\omega) = X_0(y_1(\omega),\omega) = y_1(\omega).$$

□

5. Appendix

The full proof of (3.2) for general non-degenerate α-stable like operator is quite complicated(c.f. [4] and [7]). He we give a simple proof for (3.2) under the assumption that ν is symmetric.

Proof of Lemma 3.1. By the following elementary inequality:

$$p(a-b)(|a|^{p-2}a - |b|^{p-2}b) \geqslant (a|a|^{\frac{p}{2}-1} - b|b|^{\frac{p}{2}-1})^2, \ \forall p \geqslant 2, \ a,b \in \mathbb{R},$$

we have,

$$\begin{aligned}
&\int_{\mathbb{R}^d} f|f|^{p-2}(-\mathcal{L}_\sigma f)\mathrm{d}x\\
=&\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d}(f(x+\sigma z)-f(x))(|f|^{p-2}f(x+\sigma z)-|f|^{p-2}f(x))\nu(\mathrm{d}z)\mathrm{d}x\\
\geqslant&\frac{1}{2p}\int_{\mathbb{R}^d\times\mathbb{R}^d}(f|f|^{\frac{p}{2}-1}(x+\sigma z)-f|f|^{\frac{p}{2}-1}(x))^2\nu(\mathrm{d}z)\mathrm{d}x\\
=&\frac{1}{p}\int_{\mathbb{R}^d}\left|(-\mathcal{L}_\sigma)^{\frac{1}{2}}(f|f|^{\frac{p}{2}-1})\right|^2\mathrm{d}x.
\end{aligned}\tag{5.1}$$

This implies the second inequality in Lemma 3.1.

Noticing that for all $x\in[0,1]$, $1-\cos x\geqslant cx^2$, so by $(\mathbf{H}_1)$, for any $|\xi|>(\Lambda\rho)^{-1}$(ρ is the constant in $(\mathbf{H}_1)$),

$$\begin{aligned}
\psi_\sigma(\xi)=&\int_{\mathbb{R}^d}(1-\cos(\sigma z\cdot\xi))\nu(\mathrm{d}z)\geqslant c\int_{|z|\leqslant(\Lambda|\xi|)^{-1}}|z\cdot\sigma^t\xi|^2\nu(\mathrm{d}z)\\
\gtrsim&|\sigma^t\xi|^2\int_{|z|\leqslant(\Lambda|\xi|)^{-1}}\left|z\cdot\frac{\sigma^t\xi}{|\sigma^t\xi|}\right|^2\nu(\mathrm{d}z)\gtrsim|\xi|^2\inf_{\theta\in\mathbb{S}^{d-1}}\int_{|z|\leqslant(\Lambda|\xi|)^{-1}}|z\cdot\theta|^2\nu_1(\mathrm{d}z)\\
\gtrsim&|\xi|^2\int_0^{(\Lambda|\xi|)^{-1}}r^{1-\alpha}\mathrm{d}r\gtrsim|\xi|^\alpha,
\end{aligned}$$

and it is easy to see that

$$\psi_\sigma(\xi)\gtrsim|\xi|^2,\quad\forall|\xi|\leqslant(\Lambda\rho)^{-1}.$$

By Plancherel formula,

$$\begin{aligned}
\int_{\mathbb{R}^d}\left|(-\mathcal{L}_\sigma)^{\frac{1}{2}}(f|f|^{\frac{p}{2}-1})\right|^2\mathrm{d}x=&\int_{\mathbb{R}^d}\psi_\sigma(\xi)\left|\mathscr{F}(f|f|^{\frac{p}{2}-1})(\xi)\right|^2\mathrm{d}x\\
\geqslant&c\int_{\mathbb{R}^d}|\xi|^\alpha\left|\mathscr{F}(f|f|^{\frac{p}{2}-1})(\xi)\right|^2\mathrm{d}x-C\int_{\mathbb{R}^d}\left|\mathscr{F}(f|f|^{\frac{p}{2}-1})(\xi)\right|^2\mathrm{d}x\\
\geqslant&c\int_{\mathbb{R}^d}\left|(-\Delta)^{\frac{\alpha}{4}}(f|f|^{\frac{p}{2}-1})\right|^2\mathrm{d}x-C\int_{\mathbb{R}^d}|f|^p\mathrm{d}x.
\end{aligned}\tag{5.2}$$

Combing (5.1), (5.2) and using the elementary inequality:

$$\left|a|a|^{\frac{p}{2}-1}-b|b|^{\frac{p}{2}-1}\right|^2\geqslant c_p|a-b|^p,\ \forall a,b\in\mathbb{R},\ p\geqslant2,$$

we obtain

$$\begin{aligned}
\int_{\mathbb{R}^d}f|f|^{p-2}(-\mathcal{L}_\sigma f)\mathrm{d}x\geqslant&c\int_{\mathbb{R}^d}\left|(-\Delta)^{\frac{\alpha}{4}}(f|f|^{\frac{p}{2}-1})\right|^2\mathrm{d}x-C\int_{\mathbb{R}^d}|f|^p\mathrm{d}x\\
=&c\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|f|f|^{\frac{p}{2}-1}(x)-Cf|f|^{\frac{p}{2}-1}(y)|^2}{|x-y|^{d+\alpha}}\mathrm{d}x\mathrm{d}y-\|f\|_p^p\\
\geqslant&c\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|f(x)-f(y)|^p}{|x-y|^{d+\frac{\alpha}{p}\cdot p}}\mathrm{d}x\mathrm{d}y-C\|f\|_p^p=c[f]^p_{\frac{\alpha}{p},p}-C\|f\|_p^p.
\end{aligned}$$

Now using Theorem 2.36 of [2], for any $j\geqslant0$,

$$[\Delta_jf]^p_{\frac{\alpha}{p},p}=\|f\|_{\dot{B}^{\frac{\alpha}{p}}_{p,p}}=\sum_{k=-\infty}^{\infty}2^{\alpha k}\|\Delta_k\Delta_jf\|_p^p\asymp2^{\alpha j}\|\Delta_jf\|_p^p.$$

Thus,

$$-\int_{\mathbb{R}^d} |\Delta_j f|^{p-2}\Delta_j f \mathcal{L}_\sigma \Delta_j f \mathrm{d}x \geqslant (c2^{\alpha j} - C)\|\Delta_j f\|_p^p.$$

Letting $j_0 = 1 + \log_2(C/c)/\alpha$, we get the desired result. $\square$

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REFERENCES

[1] Bahouri, H., Chemin, J.-Y. and Danchin, R.: *Fourier Analysis and Nonlinear Partial Differential Equations.* Grundlehrem der mathematischen Wissenschen, Vol.343, Springer-Verlag, 2011.

[2] Bergh J. Löfström J.: An Introduction: Interpolation Spaces. Springer-Verlag, 1976.

[3] Bass, R.F., and Chen, Z.Q.: Systems of equations driven by stable processes. *Probability Theory and Related Fields* **134.2**, 175-214, 2006.

[4] Chen, Q., Miao, C. and Zhang, Z.: A New Bernstein's Inequality and the 2D Dissipative Quasi-Geostrophic Equation. *Commun. Math. Phys.* **271**, 821-838, 2007.

[5] Chen, Z.-Q., Song, R. and Zhang, X.: Stochastic flows for Lévy processes with Hölder drifts. *Revista Matemtica Iberoamericana*, 2018.

[6] Chen, Z.Q. and Zhang, X.: Uniqueness of stable-like processes. *arXiv preprint arXiv:1604.02681*, 2016.

[7] Chen, Z.Q., Zhang, X. and Zhao, G.: Well-posedness of supercritical SDE driven by Lévy processes with irregular drifts. *arXiv preprint arXiv:1709.04632*, 2017.

[8] Davie, A. M.: Uniqueness of solutions of stochastic differential equations. *International Mathematics Research Notices*, (2007).

[9] de Raynal, P. . C., Menozzi, S. and Priola, E: Schauder estimates for drifted fractional operators in the supercritical case. *arXiv preprint arXiv:1902.02616*, 2019.

[10] Fedrizzi, E. and Flandoli F.: Pathwise uniqueness and continuous dependence for SDEs with non-regular drift. *arXiv preprint arxiv:1004.3485v1*, 2010.

[11] Haadem, S. and Proske, F.: On the construction and Malliavin differentiability of solutions of Lévy noise driven SDE's with singular coefficients. *Journal of Functional Analysis,* **266(8)**, pp.5321-5359, 2014.

[12] Ikeda N., Watanabe, S.: *Stochastic differential equations and diffusion processes*, 2nd ed., North-Holland/Kodanska, Amsterdam/Tokyo, 1989.

[13] Krylov, N.V. and Röckner, M.: Strong solutions of stochastic equations with singular time dependent drift. *Probab. Theory Relat. Fields* **131** (2005), 154-196.

[14] Kunita, H.: Stochastic differntial equations based on Lévy processes and stochastic flows of diffeomorphisms. In *Real and Stochastic Analysis*, 305–373. Birkhäuser, (2004), Boston.

[15] Ling, C., and Zhao, G.: Nonlocal elliptic equation in Hölder space and the martingale problem. *arXiv:1907.00588*, (2019).

[16] Meyer-Brandis, T., Nilssen, T., Proske, F. and Zhang, T.: A variational approach to the construction and Malliavin differentiability of strong solutions of SDEs.*Mathematische Annalen,* **357(2)**, pp.761-799, 2013.

[17] Meyer-Brandis, T. and Proske, F.: Construction of strong solutions of SDE's via Malliavin calculus. *Journal of Functional Analysis,* **258(11)**, pp.3922-3953, 2010.

[18] Mikulevicius, R. and Pragarauskas, H.: On the Cauchy problem for certain integro-differential operators in Sobolev and Hölder spaces. *Lithuanian Math. Journal,* Vol.32, No.2, 1992.

[19] Mohammed, S.E.A., Nilssen, T.K. and Proske, F.N.:. Sobolev differentiable stochastic flows for SDEs with singular coefficients: Applications to the transport equation. *The Annals of Probability,* **43(3)**, pp.1535-1576, 2015.

[20] Di Nunno, G., ksendal, B.K. and Proske, F.: Malliavin calculus for Lévy processes with applications to finance *Berlin: Springer.*, Vol. 2, 2009.

[21] Priola, E.: Pathwise uniqueness for singular SDEs driven by stable processes. *Osaka Journal of Mathematics*, **49** (2012), 421-447.

[22] Priola, E.: Stochastic flow for SDEs with jumps and irregular drift term. *Banach Center Publications* **105(1)**, pp. 193-210, 2015.
[23] Priola E.: Davies type uniqueness for a class of SDEs with jumps. *Annales de l'Institut Henri Poincar, Probabilits et Statistiques. Institut Henri Poincar,* **54(2)**, 694-725, 2018.
[24] Shaposhnikov, A.V.: Some remarks on Davies uniqueness theorem. *Proceedings of the Edinburgh Mathematical Society*, **59(4)**, pp. 1019-1035, 2016.
[25] Silvestre, L.: On the differentiability of the solution to an equation with drift and fractional diffusion. *Indiana Univ. Math. J.* **61** (2012), 557-584.
[26] Song, R. and Xie, L.: Weak and strong well-posedness of critical and supercritical SDEs with singular coefficients. *arXiv preprint arXiv:1806.09033.* (2018).
[27] Tanaka, H., Tsuchiya, M. and Watanabe, S.: Perturbation of drift-type for Lévy processes. *J. Math. Kyoto Univ.* **14** (1974), 73-92.
[28] Triebel, H.: Theory of function spaces II. *Reprinted 2010 by Springer Basel AG*
[29] Veretennikov, A.: On the strong solutions of stochastic differential equations. *Theory Probab. Appl.* **24** (1979), 354-366.
[30] Xia, P., Xie, L., Zhang, X. and Zhao, G.: $L^q(L^p)$-theory of stochastic differential equations. arXiv preprint, arXiv:1908.01255, 2019.
[31] Zhang, X.: Stochastic homeomorphism flows of SDEs with singular drifts and Sobolev diffusion coefficients. *Electron. J. Probab.* **16** (2011), 1096-1116.
[32] Zhang, X.: Stochastic differential equations with Sobolev drifts and driven by α-stable processes. *Ann. Inst. H. Poincare Probab. Statist.* **49**, 1057-1079, 2013.
[33] Zhang, X.: L^p-maximal regularity of nonlocal parabolic equations and applications. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis.* **30**, 573-614(2013).
[34] Zhang X.: Stochastic differential equations with Sobolev diffusion and singular drift. *Annals of Applied Probability.* **26**, No. 5, 2697-2732(2016).
[35] Zhang, X. and Zhao, G.: Dirichlet problem for supercritical nonlocal operators. arXiv preprint arXiv:1809.05712, 2018.
[36] Zhang, X. and Zhao, G.: Stochastic Lagrangian path for Leray solutions of 3D Navier-Stokes equations. arXiv preprint arXiv:1904.04387, 2019.
[37] Zvonkin, A.K.: A transformation of the phase space of a diffusion process that removes the drift. *Mat. Sbornik*, **93 (135)** (1974), 129-149.

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