Convergence rate for Galerkin approximation of the stochastic Allen-Cahn equations on 2D torus

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Abstract

In this paper we discuss the convergence rate for Galerkin approximation of the stochastic Allen-Cahn equations driven by space-time white noise on $\mathbb{T}^2$. First we prove that the convergence rate for stochastic 2D heat equation is of order $\alpha - \delta$ in Besov space $\mathcal{C}^{-\alpha}$ for $\alpha \in (0, 1)$ and $\delta > 0$ arbitrarily small. Then we obtain the convergence rate for Galerkin approximation of the stochastic Allen-Cahn equations of order $\alpha - \delta$ in $\mathcal{C}^{-\alpha}$ for $\alpha \in (0, 2/9)$ and $\delta > 0$ arbitrarily small.

Keywords Stochastic Allen-Cahn equations, convergence rate, Galerkin projection, Besov space, white noise.

Mathematics Subject Classification 60H15, 82C28

1 Introduction

In this paper we study the convergence rate for Galerkin approximation of the stochastic Allen-Cahn equations (see (1.1)) on 2D torus driven by space-time white noise. Such stochastic partial differential equations (SPDEs) contain super-linearly growing nonlinearities in their coefficients and in general they can not be solved explicitly. It is a quite active area to design and analyze approximation

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algorithms, which solve SPDEs with superlinearly growing nonlinearities approximatively. Galerkin approximation has been applied to study solutions to SPDEs (see [26]). It could also be regarded as one of the basic finite elements methods in spatial approximation (see e.g. [5, 6, 15, 17, 19, 21, 22, 39]). There are also a lot of work on other spatial approximation methods including Fourier method, piecewise linear approximation, finite elements methods (see e.g. [2, 15, 20, 41]), and finite differences approximation (see e.g. [6, 8, 10–12, 15]).

Approximations to SPDEs driven by trace-class Wiener process have been studied a lot in the literatures (see e.g. [2, 7, 10, 17, 19, 22, 24, 26, 39]). For SPDEs driven by space-time white noise, we refer to [5, 8, 11, 12, 20, 21, 23] and the references therein for the convergence of spatial approximations and refer to [3, 5, 8, 11, 12, 21, 23] and the references therein for the convergence of temporal approximations. Furthermore, in the references [3, 5, 8, 12, 20] the convergence rates of spatial and temporal approximations were also obtained. To be more specific, in [12] pointwise estimates were considered for the stochastic quasi-linear parabolic PDEs with locally bounded coefficients on one dimensional space driven by multiplicative space-time white noise and the convergence rate of order $1/2$ was obtained. In [8] the rate of convergence for the stochastic heat equation was further improved to 1 for additive noise by estimating space averages of the solution rather than pointwise estimates. S. Becker, B. Gess, A. Jentzen and P. E. Kloeden in [5] obtained the convergence of full-discrete approximations with rates of order $1/2-\epsilon, \epsilon>0$ in space and $1/4-\epsilon, \epsilon>0$ in time for stochastic Allen-Cahn equations driven by space-time white noise on one dimensional space. Using the rough path theory (see [13, 14, 25]), M. Hairer and K. Matetski in [20] showed convergence of spatial approximations with rate of order $1/2-\epsilon$ for Burgers type SPDEs driven by space-time white noise on one dimensional space.

All the references we conclude above are about SPDEs driven by space-time white noise on one dimensional case. For the higher dimensional cases, Yan in [39] obtained the convergence rates of spatial and temporal approximations for linear SPDEs driven by space-time white noise on d ($d=1,2,3$) dimensional space. In [27, 37, 38, 40] the authors studied the stochastic Allen-Cahn equations (the dynamical Φ^4_d, $d>1$ model) and obtained convergence with no description of the convergence rates. To the best of our knowledge, there exists no result in the literature, which establishes the convergence rates for numerical approximation of SPDEs with superlinearly growing nonlinearities driven by space-time white noise on high-dimensional space. In this paper we study the convergence rates for stochastic 2D Allen-Cahn equations driven by space-time white noise.

For spatial dimension $d\geq 2$, stochastic Allen-Cahn equations (1.1) driven by space-time white noise are ill-posed in the classical sense and the main difficulty in this case is that the noise ξ and hence the solution X are so singular that the non-linear terms are not well-defined in the classical sense. In two spatial dimensions, this problem was previously treated in [1, 29]. These kinds of singular SPDEs have received a lot of attention recently (see e.g. [16, 18]) and the renormalization is required. In this paper we define the approximating equations of the stochastic Allen-Cahn equations via Galerkin projection. As explained above, we need to consider the convergence of Galerkin approximation using renormalization (see (1.9)) and we obtain the convergence rate for this approximation.

1.1 Statement and main results

Consider the equation

$$\begin{cases} \partial_t X = \Delta X + :F(X): + \xi, & \text{on } (0,\infty)\times\mathbb{T}^2, \\ X(0) = X_0 \ \ \text{on } \mathbb{T}^2. \end{cases} \tag{1.1}$$

Here $\mathbb{T}^2$ is a torus of size 1 in $\mathbb{R}^2$, ξ is space-time white noise on $\mathbb{R}^+\times\mathbb{T}^2$ (see Definition A.1), Δ is the Laplacian with periodic boundary conditions on $L^2(\mathbb{T}^2)$. $:F(v): := \sum_{j=0}^3 a_j v^{:j:}$, $a_3<0$, where $v^{:0:}=1$, $v^{:1:}=v$ and $v^{:2:}, v^{:3:}$, the Wick powers of v, are defined by approximation in Section 3.1. The initial value $X_0 \in \mathcal{C}^{-\alpha}$, $\alpha\in(0,1)$, which is defined in Section 2. Following [27, 29], we say that X solves the equation (1.1) if

$$X = Y + \bar{Z}, \tag{1.2}$$

where $\bar{Z}$ satisfies the following stochastic heat equation

$$\begin{cases} \partial_t \bar{Z} = A\bar{Z} + \xi, & \text{on } (0,\infty)\times\mathbb{T}^2, \\ \bar{Z}(0) = X_0 \ \ \text{on } \mathbb{T}^2, \end{cases} \tag{1.3}$$

with $A = \Delta - \mathrm{I}$ and Y solves

$$\begin{cases} \partial_t Y = \Delta Y + \Psi(Y, \underline{\bar{Z}}), & \text{on } (0,\infty)\times\mathbb{T}^2, \\ Y(0) = 0, \ \ \text{on } \mathbb{T}^2, \end{cases} \tag{1.4}$$

with $\underline{z} := (z, z^{:2:}, z^{:3:})$ and

$$\Psi(y, \underline{z}) := \sum_{j=0}^3 a_j \sum_{k=0}^j \binom{j}{k} y^k z^{:j-k:} + z. \tag{1.5}$$

We interpret (1.4) in the mild sense, i.e. Y solves (1.4) if for every $t\geq 0$,

$$Y_t = \int_0^t e^{(t-s)\Delta}\Psi(Y_s, \underline{\bar{Z}}_s)ds. \tag{1.6}$$

Next, we consider the Galerkin approximations of (1.3): for $N\geq 1$,

$$\begin{cases} \partial_t \bar{Z}^N = P_N A \bar{Z}^N + \xi^N, & \text{on } (0,\infty)\times\mathbb{T}^2, \\ \bar{Z}^N(0) = P_N X_0, \ \ \text{on } \mathbb{T}^2, \end{cases} \tag{1.7}$$

with P_N, the projection operators on $L^2(\mathbb{T}^2)$, given in (2.2), $\xi^N := P_N\xi$. By [37, (2.3)] and [29, p.4] there exist constants $\mathfrak{R}^N$ given in (3.2), which diverge logarithmically as N goes to ∞, such that

$$(\bar{Z}^N)^{:2:} := (\bar{Z}^N)^2 - \mathfrak{R}^N, \quad (\bar{Z}^N)^{:3:} := (\bar{Z}^N)^3 - 3\mathfrak{R}^N\bar{Z}^N \tag{1.8}$$

converge in $L^p(\Omega; C([0,T]; \mathcal{C}^{-\alpha}))$ with $\alpha \in (0,1)$, $p > 1$ to non-trivial limits denoted by $\bar{Z}^{:2:}$ and $\bar{Z}^{:3:}$, respectively. We refer to Section 3 for details. Then we consider the following Galerkin approximation of (1.1)

$$\begin{cases} \partial_t X^N = P_N\Delta X^N + P_N F^N(X^N) + \xi^N, & \text{on } (0,\infty)\times\mathbb{T}^2, \\ X^N(0,\cdot) = P_N X_0, & \text{on } \mathbb{T}^2, \end{cases} \tag{1.9}$$

with

$$F^N(v) = a_3v^3 + a_2v^2 + a_1^N v + a_0^N,$$
$$a_1^N := a_1 - 3a_3\mathfrak{R}^N, \ a_0^N := a_0 - a_2\mathfrak{R}^N.$$

Then X^N solves (1.9) if

$$X^N = \bar{Z}^N + Y^N \tag{1.10}$$

with $\bar{Z}^N$ solving (1.7) and Y^N solving

$$\begin{cases} \partial_t Y^N = P_N\Delta Y^N + P_N\Psi(Y^N, \underline{\bar{Z}}^N), & \text{on } (0,\infty)\times\mathbb{T}^2, \\ Y^N(0) = 0, & \text{on } \mathbb{T}^2, \end{cases} \tag{1.11}$$

in the mild sense that for every $t \geq 0$,

$$Y_t^N = \int_0^t P_N e^{(t-s)\Delta}\Psi(Y_s^N, \underline{\bar{Z}}_s^N)ds, \tag{1.12}$$

with $\underline{\bar{Z}}^N = (\bar{Z}^N, (\bar{Z}^N)^{:2:}, (\bar{Z}^N)^{:3:})$ given in (1.8).

We mainly discuss the convergence rates for the stochatic Allen-Cahn equations (1.1). The main results obtained is as below. See Theorem 4.4 for more details.

Theorem 1.1. *Let $\alpha \in (0, 2/9), \gamma' > 3\alpha/2$ and $X_0 \in \mathcal{C}^{-\alpha}$. Let X,X^N denote the solutions to equations* (1.1) *and* (1.9) *on* $[0,T]$ *with initial values X_0 and P_NX_0, respectively. Then for any $\delta > 0$*

$$\lim_{N\to\infty} \mathbb{P}\Big(\sup_{t\in[0,T]} t^{\gamma'}\|X_t - X_t^N\|_{-\alpha} \gtrsim N^{\delta-\alpha}\Big) = 0.$$

1.2 Structure

This paper is organized as follows: In Section 2 we collect results related to Besov spaces. In Section 3 we show the convergence rates for linear equation (1.3). In Section 4 we prove the main results, i.e., the convergence rates for stochastic Allen-Cahn equations (1.1).

2 Besov spaces and preliminaries

We first recall Besov spaces from [27, 40]. For general theory we refer to [4, 34, 36]. Throughout the paper, we use the notation $a \lesssim b$ if there exists a constant $c > 0$ independent of the relevant quantities such that $a \leq cb$, we also use the notation $a \gtrsim b$ if $b \lesssim a$, and use the notation $a \backsimeq b$ if $a \lesssim b$ and $b \lesssim a$. For $p \in [1, \infty]$, let $L^p(\mathbb{T}^d)$ denote the usual p integrable space on $\mathbb{T}^d$ with its norm denoted by $\| \cdot \|_{L^p}$. The space of Schwartz functions on $\mathbb{T}^d$ is denoted by $\mathcal{S}(\mathbb{T}^d)$ and its dual, the space of tempered distributions is denoted by $\mathcal{S}'(\mathbb{T}^d)$. The space of real valued infinitely differentiable functions is denoted by $C^\infty(\mathbb{T}^d)$. For any function f on $\mathbb{T}^d$, let $\mathrm{supp}(f)$ denote its support.

Consider the orthonormal basis $\{e_m\}_{m\in\mathbb{Z}^d}$ of trigonometric functions on $\mathbb{T}^d$

$$e_m(x) := e^{\iota 2\pi m\cdot x}, x \in \mathbb{T}^d, \tag{2.1}$$

we write $L^2(\mathbb{T}^d)$ with its inner product $\langle \cdot, \cdot \rangle$ given by

$$\langle f, g \rangle = \int_{\mathbb{T}^d} f(x)\bar{g}(x)\mathrm{d}x, \quad f, g \in L^2(\mathbb{T}^d).$$

Then for any $f \in L^2(\mathbb{T}^d)$, we denote by $\mathcal{F}f$ or $\hat{f}$ its Fourier transform

$$\hat{f}(m) = \langle f, e_m \rangle = \int_{\mathbb{T}^d} e^{-\iota 2\pi m\cdot x} f(x)\mathrm{d}x, \quad m \in \mathbb{Z}^d.$$

For $N \geq 1$, we define P_N the projection operators from $L^2(\mathbb{T}^d)$ onto the s-pace spanned by $\{e_m, \ |m| \leq N\}$ with $|m| := \sqrt{m_1^2 + m_2^2 + \cdots + m_d^2}$, $m = (m_1, \ldots, m_d) \in \mathbb{Z}^d$, i.e.

$$P_N f = \sum_{m:|m|\leq N} \langle f, e_m \rangle \ e_m, \ f \in L^2(\mathbb{T}^d). \tag{2.2}$$

For $\zeta \in \mathbb{R}^d$ and $r > 0$ we denote by $B(\zeta, r)$ the ball of radius r centered at ζ and let the annulus $\mathcal{A} := B(0, \frac{8}{3}) \setminus B(0, \frac{3}{4})$. According to [4, Proposition 2.10], there exist nonnegative radial functions $\chi, \theta \in \mathcal{D}(\mathbb{R}^d)$, the space of real valued infinitely differentiable functions of compact support on $\mathbb{R}^d$, satisfying

i. $\mathrm{supp}(\chi) \subset B(0, \frac{4}{3})$, $\mathrm{supp}(\theta) \subset \mathcal{A}$;

ii. $\chi(z) + \sum_{j\geq 0} \theta(z/2^j) = 1$ for all $z \in \mathbb{R}^d$;

iii. $\mathrm{supp}(\chi) \cap \mathrm{supp}(\theta(\cdot/2^j)) = \emptyset$ for $j \geq 1$ and $\mathrm{supp}(\theta(\cdot/2^i)) \cap \mathrm{supp}(\theta(\cdot/2^j)) = \emptyset$ for $|i - j| > 1$.

(χ, θ) is called a dyadic partition of unity. The above decomposition can be applied to distributions on the torus (see [32, 33]). Let

$$\chi_{-1} := \chi, \quad \chi_j := \theta(\cdot/2^j), \ j \geq 0,$$

we have $\mathrm{supp}(\chi_j) \subset \mathcal{A}_{2^j} := 2^j\mathcal{A}$ for every $j \geq 0$ and $\mathcal{A}_{2^{-1}} \subset B(0, \frac{4}{3})$. For $f \in C^\infty(\mathbb{T}^d)$, the j-Littlewood-Paley block is defined as

$$\Delta_j f(x) = \sum_{m\in\mathbb{Z}^d} \chi_j(m)\hat{f}(m)e^{\iota 2\pi m\cdot x},\ j \geq -1. \tag{2.3}$$

It is noticeable that (2.3) is equivalent to the equality

$$\Delta_j f = \eta_j * f, \quad j \geq -1, \tag{2.4}$$

where

$$\eta_j * f(\cdot) = \int_{\mathbb{T}^d} \eta_j(\cdot - x)f(x)dx, \quad \eta_j(x) := \sum_{m\in\mathbb{Z}^d} \chi_j(m)e^{\iota 2\pi m\cdot x}.$$

For $\alpha \in \mathbb{R},\ p, q \in [1,\infty]$, we define the Besov space on $\mathbb{T}^d$ denoted by $\mathcal{B}^\alpha_{p,q}(\mathbb{T}^d)$ as the completion of $C^\infty(\mathbb{T}^d)$ with respect to the norm ([4, Proposition 2.7])

$$\|u\|_{\mathcal{B}^\alpha_{p,q}(\mathbb{T}^d)} := (\sum_{j\geq-1} 2^{j\alpha q}\|\Delta_j u\|^q_{L^p})^{1/q}, \tag{2.5}$$

with the usual interpretation as l^∞ norm in case $q = \infty$. Note that for $p, q \in [1,\infty)$

$$\mathcal{B}^\alpha_{p,q}(\mathbb{T}^d) = \{u \in \mathcal{S}'(\mathbb{T}^d) :\ \|u\|^q_{\mathcal{B}^\alpha_{p,q}(\mathbb{T}^d)} < \infty\},$$
$$\mathcal{B}^\alpha_{\infty,\infty}(\mathbb{T}^d) \subsetneq \{u \in \mathcal{S}'(\mathbb{T}^d) :\ \|u\|^q_{\mathcal{B}^\alpha_{\infty,\infty}(\mathbb{T}^d)} < \infty\}.$$

Here we choose Besov spaces as completions of smooth functions on the torus, which ensures that the Besov spaces are separable and has a lot of advantages for our analysis below. In the following we give estimates on the torus for later use.

We recall the following Besov embedding theorems on the torus (cf. [34, Theorem 4.6.1], [16, Lemma A.2], [27, Proposition 3.11,Remark 3.3]).

Lemma 2.1. (Besov embedding) (i) *Let $\alpha \leq \beta \in \mathbb{R}$, $p, q \in [1,\infty]$. Then $\mathcal{B}^\beta_{p,q}(\mathbb{T}^d)$ is continuously embedded in $\mathcal{B}^\alpha_{p,q}(\mathbb{T}^d)$.*

(ii) *Let $1 \leq p_1 \leq p_2 \leq \infty$, $1 \leq q_1 \leq q_2 \leq \infty$, and let $\alpha \in \mathbb{R}$. Then $\mathcal{B}^\alpha_{p_1,q_1}(\mathbb{T}^d)$ is continuously embedded in $\mathcal{B}^{\alpha-d(1/p_1-1/p_2)}_{p_2,q_2}(\mathbb{T}^d)$.*

We describe the following Schauder estimates, i.e. the smoothing effect of the heat flow, as measured in Besov spaces (cf. [27, Propositions 3.11,3.12], [16, Lemmas A.7,A.8]).

Lemma 2.2. (Schauder estimates) (i) *Let $f \in \mathcal{B}^\alpha_{p,q}(\mathbb{T}^d)$ for some $\alpha \in \mathbb{R},\ p, q \in [1,\infty]$. Then for every $\delta > 0$, uniformly over $t > 0$*

$$\|e^{tA}f\|_{\mathcal{B}^{\alpha+\delta}_{p,q}(\mathbb{T}^d)} \lesssim t^{-\frac{\delta}{2}}\|f\|_{\mathcal{B}^\alpha_{p,q}(\mathbb{T}^d)}. \tag{2.6}$$

(ii) *Let $\alpha \leq \beta \in \mathbb{R}$ be such that $\beta - \alpha \leq 2$, $f \in \mathcal{B}^\beta_{p,q}(\mathbb{T}^d)$ and $p, q \in [1,\infty]$. Then uniformly over $t > 0$*

$$\|(\mathrm{I} - e^{tA})f\|_{\mathcal{B}^\alpha_{p,q}(\mathbb{T}^d)} \lesssim t^{\frac{\beta-\alpha}{2}}\|f\|_{\mathcal{B}^\beta_{p,q}(\mathbb{T}^d)}. \tag{2.7}$$

The following multiplicative inequalities play a central role later and we treat separately for cases of positive and negative regularity (cf. [27, Corollaries 3.19,3.21], [16, Lemma 2.1]).

Lemma 2.3. (Multiplicative inequalities) (i) *Let $\alpha > 0$ and $p, p_1, p_2 \in [1, \infty]$ be such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then*

$$\|fg\|_{\mathcal{B}^\alpha_{p,q}(\mathbb{T}^d)} \lesssim \|f\|_{\mathcal{B}^\alpha_{p_1,q}(\mathbb{T}^d)} \|g\|_{\mathcal{B}^\alpha_{p_2,q}(\mathbb{T}^d)}. \tag{2.8}$$

(ii) *Let $\beta > 0 > \alpha$ be such that $\beta + \alpha > 0$ and let $p, p_1, p_2 \in [1, \infty]$ be such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then*

$$\|fg\|_{\mathcal{B}^\alpha_{p,q}(\mathbb{T}^d)} \lesssim \|f\|_{\mathcal{B}^\alpha_{p_1,q}(\mathbb{T}^d)} \|g\|_{\mathcal{B}^\beta_{p_2,q}(\mathbb{T}^d)}. \tag{2.9}$$

Throughout the paper we consider the equations on $\mathbb{T}^2$. For notations' simplicity, for any $\alpha \in \mathbb{R}$ and $p, q \in [1, \infty)$, let

$$\mathcal{B}^\alpha_{p,q} := \mathcal{B}^\alpha_{p,q}(\mathbb{T}^2), \quad \mathcal{B}^\alpha_p := \mathcal{B}^\alpha_{p,\infty}(\mathbb{T}^2), \quad \mathcal{C}^\alpha := \mathcal{B}^\alpha_{\infty,\infty}(\mathbb{T}^2) \tag{2.10}$$

and hence we denote their norms by $\|\cdot\|_{\mathcal{B}^\alpha_{p,q}}$, $\|\cdot\|_{\mathcal{B}^\alpha_p}$ and $\|\cdot\|_\alpha$, respectively.

Lemma 2.4. [37, Proposition A.11] *Let $P_N, N \geq 1$ be defined in (2.2). Then for every $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$ and $\lambda > 0$*

$$\|P_N f - f\|_{\mathcal{B}^\alpha_{p,q}} \lesssim \frac{(\log N)^2}{N^\lambda} \|f\|_{\mathcal{B}^{\alpha+\lambda}_{p,q}}, \tag{2.11}$$

$$\|P_N f\|_{\mathcal{B}^\alpha_{p,q}} \lesssim \|f\|_{\mathcal{B}^{\alpha+\lambda}_{p,q}}. \tag{2.12}$$

3 Stochastic heat equation

In this section we prove the convergence rates for stochastic heat equation (1.3).

3.1 Wick powers

Now we follow the idea from [27] to define the Wick powers of the solutions to stochastic heat equations (1.3) in the paths space. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and ξ is space-time white noise on $\mathbb{R} \times \mathbb{T}^2$. Set

$$\tilde{\mathcal{F}}_t := \sigma\big(\{\xi(\phi) : \ \phi|_{(t,\infty)\times\mathbb{T}^2} \equiv 0, \ \phi \in L^2((-\infty,\infty)\times\mathbb{T}^2)\}\big)$$

for $t > -\infty$ and denote by $(\mathcal{F}_t)_{t>-\infty}$ the usual augmentation (see [30, Chapter 1.4]) of the filtration $(\tilde{\mathcal{F}}_t)_{t>-\infty}$. For $n = 1, 2, 3$, consider the multiple stochastic integral given by

$$Z^{:n:}_{-\infty,t}(\phi) := \int_{\{(-\infty,t]\times\mathbb{T}^2\}^n} \langle \phi, \Pi_{i=1}^n H(t-s_i, x_i - \cdot)\rangle \xi(\otimes_{i=1}^n \mathrm{d}s_i, \otimes_{i=1}^n \mathrm{d}x_i) \tag{3.1}$$

for every $t > -\infty$ and $\phi \in C^\infty(\mathbb{T}^2)$. $Z^{:1:}_{-\infty,\cdot}$ is also denoted as $Z_{-\infty,\cdot}$. Here $H(r,\cdot)$, $r \neq 0$, stands for the periodic heat kernel associated to the generator $A = \Delta - \mathrm{I}$ on $\mathbb{T}^2$ given by

$$H(r,x) := \sum_{m\in\mathbb{Z}^2} e^{-rI_m} e_m(x),\ x \in \mathbb{T}^2,\ r \in \mathbb{R} \setminus \{0\}, \tag{3.2}$$

with $I_m := 1 + 4\pi^2|m|^2$ and $e_m = e^{\iota 2\pi m\cdot}$ for $m \in \mathbb{Z}^2$. Let $S(t) := e^{-tA}$ denote the semigroup associated to A in $L^2(\mathbb{T}^2)$. $Z^{:n:}_{-\infty,\cdot}$ is called the n-th Wick power of $Z_{-\infty,\cdot}$. In particular, using Duhamel's principle (c.f. [9, Section 2.3]), we have that

$$Z_t := Z_{-\infty,t} - S(t)Z_{-\infty,0},\ t \geq 0,$$

solves the stochastic heat equation with zero initial condition, i.e.

$$\begin{cases} \partial_t Z = AZ + \xi, \quad \text{on } (0,\infty) \times \mathbb{T}^2, \\ Z(0) = 0 \quad \text{on } \mathbb{T}^2. \end{cases} \tag{3.3}$$

Following the technique in [38, Section 2.1], we set for $n = 1, 2, 3$

$$Z^{:n:}_t := \sum_{k=0}^{n} \binom{n}{k} (-1)^k \Big(S(t)Z_{-\infty,0}\Big)^k Z^{:n-k:}_{-\infty,t}, \tag{3.4}$$

by letting $z^{:1:} = z$ and $z^{:0:} = 1$.

3.2 Finite dimensional approximations

Let $Z^{:n:}_{-\infty,t}, Z^{:n:}_t, t \geq 0$ be defined in the above section. For $N \geq 1$ and let P_N be given in (2.2), we define finite dimensional approximations of $Z_{-\infty,t}$ as its Galerkin projection $Z^N_{-\infty,t} := P_N Z_{-\infty,t}$ and define finite dimensional approximations of $Z^{:n:}_{-\infty,t}$, $n = 2, 3$ by renormalization as

$$(Z^N_{-\infty,t})^{:2:} := (Z^N_{-\infty,t})^2 - \mathfrak{R}^N,\ (Z^N_{-\infty,t})^{:3:} := (Z^N_{-\infty,t})^3 - 3\mathfrak{R}^N Z^N_{-\infty,t}, \tag{3.5}$$

with the renormalization constants

$$\mathfrak{R}^N := \|1_{[0,\infty)} H_N\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2)},$$

where $H_N := P_N H$ and H is given in (3.2). Comparing with (3.4), we define finite dimensional approximations of $Z^{:n:}_t$ as for $N \geq 1$, $t \geq 0$ and $n = 1, 2, 3$

$$(Z^N_t)^{:n:} := \sum_{k=0}^{n} \binom{n}{k} (-1)^k \Big(S(t)Z^N_{-\infty,0}\Big)^k (Z^N_{-\infty,t})^{:n-k:} \tag{3.6}$$

by similarly letting $z^{:1:} = z$ and $z^{:0:} = 1$. Then Z^N_t solve approximating equations (1.7) with initial value zero.

3.3 Main results of stochastic heat equation and its Galerkin approximations

We know that Z_t and $Z_t^N, N \geq 1$ are the solutions to stochastic heat equation (1.3) and its Galerkin approximations (1.7) with initial value zero, respectively. In this subsection, we first prove the convergence rate for stochastic heat equation with initial value zero. Furthermore, we discuss the convergence rate for linear equation with initial value X_0.

To begin with, we recall the following Kolmogorov-type result.

Lemma 3.1. [27, Lemma 5.2] *Let $(t, \phi) \to Z(t, \phi)$ be a map from $(0, \infty) \times L^2(\mathbb{T}^2) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ which is linear and continuous in ϕ. Assume that for some $p > 1$, $\alpha \in \mathbb{R}$ and $\nu > \frac{1}{p}$ and all $T > 0$, there exists a function $K_T \in L^\infty(\mathbb{T}^2)$ such that for all $\kappa \geq -1$, $x \in \mathbb{T}^2$ and $s, t \in [0, T]$*

$$\begin{aligned} \mathbb{E}|Z(t, \eta_\kappa(\cdot - x))|^p &\leq K_T(x)^p 2^{-\kappa\alpha p}, \\ \mathbb{E}|Z(t, \eta_\kappa(\cdot - x)) - Z(s, \eta_\kappa(\cdot - x))|^p &\leq K_T(x)^p 2^{-\kappa(\alpha-\nu)p} |t - s|^{\nu p} \end{aligned} \tag{3.7}$$

with $\eta_\kappa, \kappa \geq -1$ defined in (2.4). Then there exists a random distribution $\tilde{Z}$ which is $C([0, \infty); \mathcal{B}^{\alpha'}_{p,p})$-valued for any $\alpha' < \alpha - \nu$ and satisfies that for all $t > 0$ and $\phi \in \mathcal{S}(\mathbb{T}^2)$

$$Z(t, \phi) = \langle \tilde{Z}(t), \phi \rangle \quad \textit{almost surely.}$$

Furthermore, for every $T > 0$, there exists a constant $C(T, \alpha, \alpha', p) > 0$ such that

$$\mathbb{E} \sup_{0 \leq t \leq T} \|\tilde{Z}(t, \cdot)\|^p_{\mathcal{B}^{\alpha'}_{p,p}} \leq C(T, \alpha, \alpha', p) \|K_T\|^p_{L^p}. \tag{3.8}$$

We use Lemma 3.1 to obtain the regularity properties of $Z_t^{:n:}, (Z_t^N)^{:n:}$.

We remark that properties for stochastic heat equation have been widely discussed in the literature (cf. [38, Propositions 2.2,2.3], [37, Proposition 7.4], [31, Lemma 3.4]). Below we obtain uniform bounds for the approximating equations.

Lemma 3.2. *Let $p > 1$. Then for each $\alpha \in (0, 1)$, $n = 1, 2, 3$ and $N \geq 1$, the processes $Z^{:n:}_{-\infty,\cdot}$ and $(Z^N_{-\infty,\cdot})^{:n:}$ defined in (3.1) and (3.5) belong to $C([0, T]; \mathcal{C}^{-\alpha})$ $\mathbb{P}$-a.s. Moreover, we have*

$$\mathbb{E} \sup_{0 \leq t \leq T} \|Z^{:n:}_{-\infty,t}\|^p_{-\alpha} < \infty, \tag{3.9}$$

$$\sup_{N \geq 1} \mathbb{E} \sup_{0 \leq t \leq T} \|(Z^N_{-\infty,t})^{:n:}\|^p_{-\alpha} < \infty. \tag{3.10}$$

Proof. (3.9) has been obtained in [38, Theorem 2.1] and we only prove (3.10). By (2.4) we set $\phi(\cdot) = \eta_j(x - \cdot)$ in (3.1) and obtain that

$$\Delta_j Z^N_{-\infty,t}(x) = Z^N_{-\infty,t}(\eta_j(x - \cdot)), \quad x \in \mathbb{T}^2, t \in [0, T], \quad j \geq -1.$$

Using the results in the proofs of [38, Theorem 2.1, Proposition 2.3], which are based on (A.1) and the semigroup property of $H(t,x)$ defined in (3.2), we obtain that for $x_1, x_2 \in \mathbb{T}^2$ and $s,t \in [0,T]$

$$\begin{aligned}&\mathbb{E}\Delta_j(Z^N_{-\infty,s})^{:n:}(x_1)\Delta_j(Z^N_{-\infty,t})^{:n:}(x_2) \backsimeq \\ &\qquad n! \sum_{\substack{m_1\in\mathcal{A}_{2^j},\\ m_1\in\mathbb{Z}^2}} \sum_{\substack{|m_i|\le N,\\ m_i\in\mathbb{Z}^2, i=2,\dots,n}} \prod_{i=1}^n \frac{e^{-I_{m_i-m_{i-1}}|s-t|}}{2I_{m_i-m_{i-1}}} e_{m_1}(x_1-x_2),\end{aligned} \tag{3.11}$$

with the convention that $m_0 = 0$ for $I_m = 1+4\pi^2|m|^2$, $e_m = e^{\iota 2\pi m\cdot}$, $m\in\mathbb{Z}^2$. Let $K^\gamma(m) = \frac{1}{(1+|m|^2)^{1-\gamma}}$ for $\gamma\in[0,1)$. If we choose $x_1 = x_2 = x$ and $s=t$, then we get an estimate of the form

$$\mathbb{E}\,|\Delta_j(Z^N_{-\infty,t})^{:n:}(x)\,|^2 \lesssim \sum_{m\in\mathcal{A}_{2^j}, m\in\mathbb{Z}^2} K^0 \star^n_{\le N} K^0(m),$$

with $K^\gamma \star^n_{\le N} K^\gamma, \gamma\in[0,1)$ defined in (A.4), while for $s\neq t$ and every $\gamma\in(0,1)$

$$\mathbb{E}\,|\Delta_j(Z^N_{-\infty,s})^{:n:}(x) - \Delta_j(Z^N_{-\infty,t})^{:n:}(x)\,|^2 \lesssim |s-t|^{n\gamma} \sum_{m\in\mathcal{A}_{2^j}, m\in\mathbb{Z}^2} K^\gamma \star^n_{\le N} K^\gamma(m).$$

Then using the estimates in Lemma A.3 we have for any $\lambda>0$ and $\gamma\in(0,\frac{1}{n})$

$$\begin{aligned}\mathbb{E}\big|\Delta_j(Z^N_{-\infty,t})^{:n:}(x)\big|^2 &\lesssim \sum_{m\in\mathcal{A}_{2^j}, m\in\mathbb{Z}^2} \frac{1}{(1+|m|^2)^{1-\lambda}},\\ \mathbb{E}\,\big|\Delta_j(Z^N_{-\infty,s})^{:n:}(x) - \Delta_j(Z^N_{-\infty,t})^{:n:}(x)\big|^2 &\lesssim \sum_{m\in\mathcal{A}_{2^j}, m\in\mathbb{Z}^2} \frac{|s-t|^{n\gamma}}{(1+|m|^2)^{1-n\gamma}}\end{aligned}$$

uniformly for $x\in\mathbb{T}^2$, $s,t\in[0,T]$, $j\ge -1$, $N\ge 1$. Considering that $|m|\lesssim 2^j$ for $m\in\mathcal{A}_{2^j}$, we further have estimates for the above γ and $\lambda > \frac{n\gamma}{2}$

$$\begin{aligned}\mathbb{E}\big|\Delta_j(Z^N_{-\infty,t})^{:n:}(x)\big|^2 &\lesssim 2^{2j\lambda},\\ \mathbb{E}\,\big|\Delta_j(Z^N_{-\infty,s})^{:n:}(x) - \Delta_j(Z^N_{-\infty,t})^{:n:}(x)\big|^2 &\lesssim |s-t|^{n\gamma}2^{2jn\gamma} \lesssim |s-t|^{n\gamma}2^{2j(\lambda+\frac{n\gamma}{2})}.\end{aligned}$$

Let $p\ge 2$, then by (A.2) and the above estimates we have that for $0<\gamma<\frac{1}{n}$ and $\lambda > \frac{n\gamma}{2}$,

$$\mathbb{E}\,\big|\Delta_j(Z^N_{-\infty,t})^{:n:}(x)\big|^p \lesssim 2^{j\lambda p}, \tag{3.12}$$

$$\mathbb{E}\,\big|\Delta_j(Z^N_{-\infty,s})^{:n:}(x) - \Delta_j(Z^N_{-\infty,t})^{:n:}(x)\big|^p \lesssim |s-t|^{\frac{n\gamma p}{2}} 2^{j(\lambda+\frac{n\gamma}{2})p}, \tag{3.13}$$

where the constants we omit are independent of N. Hence (3.7) holds by Lemma 3.1 and choosing $\nu = \frac{n\gamma}{2}$ and $\alpha = -\lambda$ for $\lambda > \frac{n\gamma}{2}$, and the embedding $\mathcal{B}^{-\alpha+\frac{2}{p}}_{p,p} \hookrightarrow \mathcal{C}^{-\alpha}$ for $\alpha > \frac{2}{p}$. Then we have modifications still denoted by $Z^{:n:}_{-\infty,\cdot}, (Z^N_{-\infty,\cdot})^{:n:}$ in $C([0,T];\mathcal{C}^{-\alpha})$ for any $\alpha > n\gamma + \frac{2}{p}$. Moreover, we conclude that $Z^{:n:}_{-\infty,\cdot}, (Z^N_{-\infty,\cdot})^{:n:} \in C([0,T];\mathcal{C}^{-\alpha})$, $\mathbb{P}$-a.s. for any $\alpha>0$ since by the arbitrariness of γ, p we can choose γ small enough and p sufficiently large. By (3.8), (3.12), (3.13) and Cauchy-Schwarz inequality, (3.10) holds for all $p>1$.

□

Lemma 3.3. *Let $\alpha \in (0,1)$, $n = 1,2,3$ and $p > 1$. Then for every $\alpha' > 0$*

$$\sup_{N\geq 1} \mathbb{E} \sup_{0\leq t\leq T} t^{(n-1)\alpha' p} \|(Z_t^N)^{:n:}\|_{-\alpha}^p < \infty, \tag{3.14}$$

$$\mathbb{E} \sup_{0\leq t\leq T} t^{(n-1)\alpha' p} \|Z_t^{:n:}\|_{-\alpha}^p < \infty. \tag{3.15}$$

Proof. Let $0 < \bar{\alpha} < \alpha \wedge \alpha'$ be fixed and $\epsilon > 0$ small enough. By Lemmas 3.2, 2.2 and 2.3 we have $(Z_{-\infty,t}^N)^{:n:} \in \mathcal{C}^{-\bar{\alpha}}$, $S(t)Z_{-\infty,t}^N \in \mathcal{C}^{\bar{\alpha}+\epsilon}$ for every $t \in [0,T]$ and

$$t^{\bar{\alpha}+\frac{\epsilon}{2}} \|S(t)Z_{-\infty,0}^N\|_{\bar{\alpha}+\epsilon} \lesssim \|Z_{-\infty,0}^N\|_{-\bar{\alpha}},$$

$$t^{(\bar{\alpha}+\frac{\epsilon}{2})(n-1)} \|(S(t)Z_{-\infty,0}^N)^n\|_{-\bar{\alpha}} \lesssim \|Z_{-\infty,0}^N\|_{-\bar{\alpha}}^n.$$

Then using Lemmas 2.3 and 2.1 in (3.6) we have

$$\|(Z_t^N)^{:n:}\|_{-\alpha} \lesssim \sum_{k=0}^{n-1} \|S(t)Z_{-\infty,0}^N\|_{\bar{\alpha}+\epsilon}^k \|(Z_{-\infty,t}^N)^{:n-k:}\|_{-\bar{\alpha}} + \|(S(t)Z_{-\infty,0}^N)^n\|_{-\bar{\alpha}},$$

and (3.14) follows by Lemma 3.2 and Cauchy-Schwarz's inequality.

(3.15) can be similarly obtained and we omit the proof. See also [38, Proposition 2.2] for the details. □

Using the moment bounds obtained in Lemma 3.3, we continue to prove the convergence rate for linear equation (3.3) with initial value zero in Lemma 3.4. Similar convergence results can, e.g., be found in [38, Propositions 2.2,2.3], [37, Proposition 7.4]. The main difference is that we obtain the convergence rates.

Lemma 3.4. *Let $\alpha \in (0,1)$, $n = 1,2,3$ and $p > 1$. Then*

$$\mathbb{E} \sup_{0\leq t\leq T} \|Z_{-\infty,t}^{:n:} - (Z_{-\infty,t}^N)^{:n:}\|_{-\alpha}^p \lesssim (1+N^2)^{-\frac{p\alpha^-}{2}}, \tag{3.16}$$

and for every $\beta > \alpha$

$$\mathbb{E} \sup_{0\leq t\leq T} t^{\frac{(n-1)(\alpha+\beta)p}{2}} \|Z_t^{:n:} - (Z_t^N)^{:n:}\|_{-\alpha}^p \lesssim (1+N^2)^{-\frac{p\alpha^-}{2}}, \tag{3.17}$$

where α^- denotes $\alpha - \delta$ for every $\delta > 0$.

Proof. Let $N \geq 1$, similar as in (3.11) we have that for $x_1, x_2 \in \mathbb{T}^2$ and $s,t \in [0,T]$

$$\begin{aligned} &\mathbb{E}\Delta_j Z_{-\infty,s}^{:n:}(x_1)\Delta_j Z_{-\infty,t}^{:n:}(x_2) \backsimeq \\ &\qquad n! \sum_{m_1\in\mathcal{A}_{2^j}, m_1\in\mathbb{Z}^2} \sum_{m_i\in\mathbb{Z}^2, i=2,\dots,n} \prod_{i=1}^n \frac{e^{-I_{m_i-m_{i-1}}|s-t|}}{2I_{m_i-m_{i-1}}} e_{m_1}(x_1-x_2), \end{aligned} \tag{3.18}$$

$$\begin{aligned} &\mathbb{E}\Delta_j Z_{-\infty,s}^{:n:}(x_1)\Delta_j (Z_{-\infty,t}^N)^{:n:}(x_2) \backsimeq \\ &\qquad n! \sum_{\substack{m_1\in\mathcal{A}_{2^j},\\ m_1\in\mathbb{Z}^2}} \sum_{\substack{|m_i|\leq N,\\ m_i\in\mathbb{Z}^2, i=2,\dots,n}} \prod_{i=1}^n \frac{e^{-I_{m_i-m_{i-1}}|s-t|}}{2I_{m_i-m_{i-1}}} e_{m_1}(x_1-x_2), \end{aligned} \tag{3.19}$$

with $m_0 := 0$, $I_m = 1 + 4\pi^2|m|^2$, $e_m = e^{\iota 2\pi m\cdot}$ for $m \in \mathbb{Z}^2$. Then by letting $x_1 = x_2 = x$ and $s = t$ in the above two estimates and in (3.11) and by equality $(a-b)^2 = a^2 + b^2 - 2ab$ we have

$$\mathbb{E}\big|\Delta_j Z^{:n:}_{-\infty,t}(x) - \Delta_j (Z^N_{-\infty,t})^{:n:}(x)\big|^2 \backsimeq \frac{n!}{2} \sum_{m_1\in\mathcal{A}_{2^j}, m_1\in\mathbb{Z}^2} \Big\{ \sum_{m_i\in\mathbb{Z}^2, i=2,\dots,n} \prod_{i=1}^n \frac{1}{1+|m_i - m_{i-1}|^2}$$

$$- \sum_{|m_i|\leq N, m_i\in\mathbb{Z}^2, i=2,\dots,n} \prod_{i=1}^n \frac{1}{1+|m_i-m_{i-1}|^2}\Big\} \lesssim \sum_{m\in\mathcal{A}_{2^j}, m\in\mathbb{Z}^2} K^0 \star^n_{>N} K^0(m),$$

with $K^0(m) := \frac{1}{1+|m|^2}$ and $K^0 \star^n_{>N} K^0$ defined in (A.5), and by Lemma A.3 we have for any positive λ satisfying $\lambda < 1 - \epsilon$ that

$$\sum_{m\in\mathcal{A}_{2^j}, m\in\mathbb{Z}^2} K^0 \star^n_{>N} K^0(m) \lesssim (1+N^2)^{-\lambda} \sum_{m\in\mathcal{A}_{2^j}, m\in\mathbb{Z}^2} (1+|m|^2)^{\epsilon+\lambda-1} \lesssim (1+N^2)^{-\lambda} 2^{2j(\epsilon+\lambda)}.$$

Then by (A.2), we have for every $p \geq 2$, $\lambda, \epsilon > 0$ such that $\lambda + \epsilon < 1$,

$$\mathbb{E}\left|\Delta_j Z^{:n:}_{-\infty,t}(x) - \Delta_j (Z^N_{-\infty,t})^{:n:}(x)\right|^p \lesssim 2^{j(\lambda+\epsilon)p}(1+N^2)^{-\frac{\lambda p}{2}},$$

uniformly for $x \in \mathbb{T}^2$, $t \in [0,T]$, $j \geq -1$, $N \geq 1$. Then for any $\alpha \in (0,1)$, we choose p sufficiently large, ϵ sufficiently small and $\lambda > 0$ such that $\lambda + \epsilon < \alpha - 2/p$ and thus we obtain

$$\mathbb{E} \sup_{0\leq t\leq T} \|Z^{:n:}_{-\infty,t} - (Z^N_{-\infty,t})^{:n:}\|^p_{\mathcal{B}^{-\alpha+\frac{2}{p}}_{p,p}} \lesssim (1+N^2)^{-\frac{\lambda p}{2}}.$$

Finally, using the embedding $\mathcal{B}^{-\alpha+2/p}_{p,p} \hookrightarrow \mathcal{C}^{-\alpha}$ for any $\alpha > \frac{2}{p}$ and letting λ close to α and ϵ close to 0, we have

$$\mathbb{E} \sup_{0\leq t\leq T} \|Z^{:n:}_{-\infty,t} - (Z^N_{-\infty,t})^{:n:}\|^p_{-\alpha} \lesssim (1+N^2)^{-\frac{p\alpha^-}{2}}$$

for p large enough. Then by the Cauchy-Schwarz inequality, (3.16) holds for all $p > 1$.

We continue to prove (3.17). For every $\beta > \alpha$, proceeding as in Lemma 3.3, we obtain $S(t)Z^N_{-\infty,0} \in \mathcal{C}^\beta$ and

$$t^{\frac{\alpha+\beta}{2}} \|S(t)Z^N_{-\infty,0}\|_\beta \lesssim \|Z^N_{-\infty,0}\|_{-\alpha},$$

$$t^{\frac{n(\alpha+\beta)}{2}} \|(S(t)Z_{-\infty,0})^n - (S(t)Z^N_{-\infty,0})^n\|_\beta \lesssim \|Z_{-\infty,0} - Z^N_{-\infty,0}\|_{-\alpha}(\|Z_{-\infty,0}\|^{n-1}_{-\alpha} + \|Z^N_{-\infty,0}\|^{n-1}_{-\alpha}),$$

$$t^{\frac{(n-1)(\alpha+\beta)}{2}} \|(S(t)Z_{-\infty,0})^n - (S(t)Z^N_{-\infty,0})^n\|_{-\alpha} \lesssim \|Z_{-\infty,0} - Z^N_{-\infty,0}\|_{-\alpha}(\|Z_{-\infty,0}\|^{n-1}_{-\alpha} + \|Z^N_{-\infty,0}\|^{n-1}_{-\alpha}).$$

Then using Lemma 2.3 in (3.6) and (3.4) we have

$$\begin{aligned}\|Z^{:n:}_t - (Z^N_t)^{:n:}\|_{-\alpha} \lesssim \sum_{k=0}^{n-1} \big[&\|(S(t)Z^N_{-\infty,0})\|^k_\beta \cdot \|Z^{:n-k:}_{-\infty,t} - (Z^N_{-\infty,t})^{:n-k:}\|_{-\alpha} \\ &+ \|(S(t)Z_{-\infty,0})^k - (S(t)Z^N_{-\infty,0})^k\|_\beta \cdot \|Z^{:n-k:}_{-\infty,t}\|_{-\alpha}\big] \\ &+ \|(S(t)Z_{-\infty,0})^n - (S(t)Z^N_{-\infty,0})^n\|_{-\alpha},\end{aligned}$$

and (3.17) follows by the Cauchy-Schwarz inequality, (3.10) and (3.16).

□

Now following the techniques in [27, 31], we combine the initial value part with the Wick powers. Let $X_0 \in \mathcal{C}^{-\alpha}, \alpha \in (0,1)$, we set $V_t := S(t)X_0$, $V_t^N := P_N V_t$ and

$$\bar{Z}_t := Z_t + V_t, \;\; \bar{Z}_t^N := Z_t^N + V_t^N,$$

$$\bar{Z}_t^{:n:} := \sum_{k=0}^{n} \binom{n}{k} V_t^{n-k} Z_t^{:k:}, \;\; (\bar{Z}_t^N)^{:n:} := \sum_{k=0}^{n} \binom{n}{k} (V_t^N)^{n-k} (Z_t^N)^{:k:} \tag{3.20}$$

for $n = 1, 2, 3$ with $\bar{Z}_t^{:1:} = \bar{Z}_t$. Then we know that $\bar{Z}_t$, $\bar{Z}_t^N$ are the solutions to (1.3) and (1.7) with initial values X_0 and $P_N X_0$, respectively. By Lemma 2.2 we have $V \in C([0,T]; \mathcal{C}^{-\alpha})$ and $V \in C([0,T]; \mathcal{C}^{\beta})$ for every $\beta > -\alpha$ with the norm $\sup_{t\in[0,T]} t^{\frac{\alpha+\beta}{2}} \| \cdot \|_\beta$. Moreover, together with (2.6) we have

$$\sup_{0\le t\le T} t^{\frac{\alpha+\beta}{2}} \|V_t\|_\beta \lesssim \|X_0\|_{-\alpha}, \;\; \sup_{0\le t\le T} t^{\frac{\alpha+\beta+\kappa}{2}} \|V_t^N\|_\beta \lesssim \|X_0\|_{-\alpha}, \tag{3.21}$$

for $\beta \geq -\alpha$ and $\kappa > 0$. Then we extend the results in Lemmas 3.2-3.4 to the solution to stochastic heat equation (1.3) with initial value X_0.

Theorem 3.5. *Let $\alpha \in (0,1)$, $X_0 \in \mathcal{C}^{-\alpha}$, $n = 1, 2, 3$, $p > 1$. Let $\bar{Z}^{:n:}$ and $(\bar{Z}^N)^{:n:}$ be defined in (3.20). Then $(\bar{Z}^N)^{:n:}$ converges to $\bar{Z}^{:n:}$ in $L^p(C([0,T]; \mathcal{C}^{-\alpha}))$ such that for every $\beta > \alpha$ and $\kappa > 0$*

$$\sup_{N\ge1} \mathbb{E} \sup_{0\le t\le T} t^{\frac{(\alpha+\beta)(n-1)+n\kappa}{2} p} \|(\bar{Z}_t^N)^{:n:}\|_{-\alpha}^p < \infty, \tag{3.22}$$

$$\mathbb{E} \sup_{0\le t\le T} t^{\frac{(\alpha+\beta)(n-1)}{2} p} \|\bar{Z}_t^{:n:}\|_{-\alpha}^p < \infty, \tag{3.23}$$

$$\mathbb{E} \sup_{0\le t\le T} t^{\frac{(\alpha+\beta)(n-1)+n\kappa}{2} p} \|\bar{Z}_t^{:n:} - (\bar{Z}_t^N)^{:n:}\|_{-\alpha}^p \lesssim \frac{(\log N)^{2p}}{N^{\kappa p}} + (1+N^2)^{-\frac{p\alpha^-}{2}}, \tag{3.24}$$

where α^- denotes $\alpha - \delta$ for every $\delta > 0$.

Proof. Using Lemma 2.2 in (3.20), we have for $\beta > \alpha$,

$$\|(\bar{Z}_t^N)^{:n:}\|_{-\alpha} \lesssim \sum_{k=0}^{n-1} \|V_t^N\|_\beta^k \|(Z_t^N)^{:n-k:}\|_{-\alpha} + \|V_t^N\|_{-\alpha} \|V_t^N\|_\beta^{n-1},$$

which together with (3.21) and (3.14), implies (3.22) easily. Similarly we obtain (3.23).

Combining Lemma 2.3 with (3.21), we have for every $\beta > \alpha$, $\kappa > 0$

$$\begin{aligned} t^{\frac{(\alpha+\beta+\kappa)n}{2}} \|(V_t^N)^n - V_t^n\|_\beta &\lesssim \frac{(\log N)^2}{N^\kappa} \|X_0\|_{-\alpha}^n, \\ t^{\frac{(\alpha+\beta)(n-1)+n\kappa}{2}} \|(V_t^N)^n - V_t^n\|_{-\alpha} &\lesssim \frac{(\log N)^2}{N^\kappa} \|X_0\|_{-\alpha}^n. \end{aligned} \tag{3.25}$$

Then finally in (3.20) we get

$$\begin{aligned}\|(\bar{Z}_t^N)^{:n:} - \bar{Z}_t^{:n:}\|_{-\alpha} \lesssim \sum_{k=0}^{n-1} \big[&\|V_t^N\|_\beta^k \|(Z_t^N)^{:n-k:} - Z_t^{:n-k:}\|_{-\alpha} \\ &+ \|(V_t^N)^k - V_t^k\|_\beta \|Z_t^{:n-k:}\|_{-\alpha}\big] + \|(V_t^N)^n - V_t^n\|_{-\alpha}.\end{aligned}$$

Now (3.24) follows by (3.15), (3.17) and (3.25).

□

4 Main results for stochastic Allen-Cahn equations

Now we fix a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,\infty)}, \mathbb{P})$. Suppose that ξ is space-time white noise on $\mathbb{R}^+ \times \mathbb{T}^2$. Let $T > 0$ and $X_0 \in \mathcal{C}^{-\alpha}$ with $\alpha \in (0, 1)$ and let $\bar{Z}, \bar{Z}^N$ be given in Section 3. Existence and uniqueness of the solutions to (1.4) have been obtained in [31, Theorem 3.10] and [27, Theorem 6.2]. By [26, Theorem 5.1] we obtain existence and uniqueness of the solutions to (1.11). The results are concluded in the theorem below.

Theorem 4.1. *Let $\alpha, \beta > 0$ with $\alpha < \beta < \frac{2}{3} - \alpha$ and $p > 1$. Then there exist unique mild solutions Y and Y^N on $[0, T]$ to equations* (1.4) *and* (1.11)*, respectively. Here, $Y, Y^N \in C([0, T]; \mathcal{C}^\beta)$, $Y^N \in C([0, T]; \mathcal{C}^\infty)$ such that for any $\frac{\alpha+\beta}{2} < \gamma' < \frac{1}{3}$*

$$\mathbb{E} \sup_{0\le t\le T} t^{\gamma' p} \|Y_t^N\|_\beta^p < \infty, \quad \mathbb{E} \sup_{0\le t\le T} t^{\gamma' p} \|Y_t\|_\beta^p < \infty. \tag{4.1}$$

Remark 4.2. We notice that if (4.1) holds for $\gamma' < 1/3$, it also holds for $\gamma' \ge 1/3$ since $t^a \lesssim t^b$ for uniform $t \le T$ with $a \ge b$.

We conclude that under the setting of Theorem 4.1,

$$X = Y + \bar{Z}, \quad X^N = \bar{Z}^N + Y^N \tag{4.2}$$

are solutions to (1.1) and (1.9) on $[0, T]$ with initial values X_0 and $P_N X_0$, respectively. Since Theorem 4.1 holds for any $\alpha, \beta, \gamma' > 0$ with $\alpha < \beta$, $\frac{\alpha+\beta}{2} < \gamma' < \frac{1}{3}$, then we let β close to α and together with Theorem 3.5, we have for every $\alpha \in (0, 1/3)$, $\gamma' > \alpha$ and $p > 1$

$$\mathbb{E} \sup_{0\le t\le T} t^{\gamma' p} \|X_t^N\|_{-\alpha}^p < \infty, \quad \mathbb{E} \sup_{0\le t\le T} t^{\gamma' p} \|X_t\|_{-\alpha}^p < \infty. \tag{4.3}$$

4.1 Pathwise error estimates for stochastic Allen-Cahn equations

In the following we fix $\alpha,\beta,p,\gamma',\kappa$ satisfying the following condition:

$\alpha,\beta>0$ with $\alpha<\beta<\frac{2}{3}-\alpha$ and $p>1$, $\gamma'>0$ with $\frac{2\alpha+\beta}{2}<\gamma'<\frac{1}{3}$, and $\kappa>0$ sufficiently small.

Stopping times

Following the notations in the above section, for some fixed $M>0$ sufficiently large, we define stopping times

$$\tau^M:=\inf\{t\leq T: t^{\gamma'}\|Y_t\|_\beta\geq M\}, \tag{4.4}$$

and to approximate Y with Y^N, we define

$$\begin{aligned}\sigma_N: &=\inf\{t\leq T: t^{\gamma'}\|Y_t^N-Y_t\|_\beta>1\},\\ \nu_N^{M,\epsilon}: &=\inf\{t\leq T: \sup_{n=1,2,3} t^{\frac{(\alpha+\beta)(n-1)}{2}}\|\bar{Z}_t^{:n:}\|_\alpha\geq M,\\ &\qquad\qquad \sup_{n=1,2,3} t^{\frac{(\alpha+\beta)(n-1)+n\kappa}{2}}\|\bar{Z}_t^{:n:}-(\bar{Z}_t^N)^{:n:}\|_\alpha>\epsilon\},\end{aligned} \tag{4.5}$$

with $\epsilon>0$ arbitrarily small. For notation's simplicity, we set

$$\|\cdot\|_{\mathcal{M}_\sigma}:=\sup_{t\leq\sigma}t^{\gamma'}\|\cdot\|_\beta$$

for any stopping time σ and $\|v\|_{\bar{\mathcal{L}}}:=\sup_{n=1,2,3}\sup_{t\leq T}t^{\frac{(\alpha+\beta)(n-1)+n\kappa}{2}}\|v_t^{:n:}\|_{-\alpha}$. Then by (1.6) and (1.12)

$$Y_t-Y_t^N=\int_0^t(\mathrm{I}-P_N)e^{(t-s)A}\Psi(Y_s,\underline{\bar{Z}}_s)ds+\int_0^t P_Ne^{(t-s)A}\Big\{\Psi(Y_s,\underline{\bar{Z}}_s)-\Psi(Y_s^N,\underline{\bar{Z}}_s^N)\Big\}ds.$$

In the following we use the decomposition

$$\Psi(u,\underline{z})=F(u)+\widetilde{\Psi}(u,\underline{z}), \tag{4.6}$$

with $u\in\mathcal{C}^\beta,\underline{z}=(z,z^{:2:},z^{:3:})$ that

$$F(u):=\sum_{i=0}^3 a_iu^i,\ \ \widetilde{\Psi}(u,\underline{z}):=\sum_{i=1}^3 a_i\bar{z}^{:i:}+3a_3(u^2\bar{z}+u\bar{z}^{:2:})+2a_2u\bar{z}+\bar{z}.$$

Under the assumption that $0<\alpha<\beta,\alpha+\beta<2$ and applying Lemma 2.3 and Young's inequality, we easily have that for any $u\in\mathcal{C}^\beta,\underline{z}=(z,z^{:2:},z^{:3:})$ with $z^{:n:}\in\mathcal{C}^{-\alpha}$ $F(u)\in\mathcal{C}^\beta,\widetilde{\Psi}(u,\underline{z})\in\mathcal{C}^{-\alpha}$ and

$$\begin{aligned}\|F(u)\|_\beta&\lesssim 1+\|u\|_\beta^3,\\ \|\widetilde{\Psi}(u,\underline{z})\|_{-\alpha}&\lesssim 1+\|u\|_\beta^2\|z\|_{-\alpha}+\|u\|_\beta\|z^{:2:}\|_{-\alpha}+\|z^{:3:}\|_{-\alpha},\end{aligned} \tag{4.7}$$

and that for $v \in \mathcal{C}^\beta, \underline{w} = (w, w^{:2:}, w^{:3:})$ with $w^{:n:} \in \mathcal{C}^{-\alpha}$ we obtain

$$\begin{aligned}
\|F(u) - F(v)\|_\beta &\lesssim \|u - v\|_\beta (1 + \|u\|_\beta^2 + \|v\|_\beta^2), \\
\|\widetilde{\Psi}(u, \underline{z}) - \widetilde{\Psi}(v, \underline{w})\|_{-\alpha} &\lesssim \big\{\|u\|_\beta \|z\|_{-\alpha} + \|v\|_\beta \|z\|_{-\alpha} + \|z^{:2:}\|_{-\alpha}\big\} \|u - v\|_\beta \\
&\quad + \|v\|_\beta^2 \|z - w\|_{-\alpha} + \|v\|_\beta \|z^{:2:} - w^{:2:}\|_{-\alpha} + \|z^{:3:} - w^{:3:}\|_{-\alpha}.
\end{aligned} \tag{4.8}$$

Then by Lemmas 2.4, 2.2 and the inequality $s^{-a} \lesssim s^{-b}$ with $0 \leq a \leq b$ and the assumption $\alpha < \beta$, $\frac{2\alpha+\beta}{2} < \gamma' < \frac{1}{3}$, κ small enough, we deduce that for $t \in [0, \tau^M \wedge \sigma_N \wedge \nu_N^{M,\epsilon}]$

$$\begin{aligned}
\|Y_t - Y_t^N\|_\beta &\lesssim_M \frac{(\log N)^2}{N^\kappa} \int_0^t \Big\{(t-s)^{-\frac{\alpha+\beta+\kappa}{2}} \|\widetilde{\Psi}(Y_s, \underline{\bar{Z}}_s)\|_{-\alpha} + (t-s)^{-\frac{\kappa}{2}} \|F(Y_s)\|_\beta \Big\} ds \\
&\quad + \int_0^t \Big\{(t-s)^{-\frac{\alpha+\beta+\kappa}{2}} \|\widetilde{\Psi}(Y_s, \underline{\bar{Z}}_s) - \widetilde{\Psi}(Y_s^N, \underline{\bar{Z}}_s^N)\|_{-\alpha} + (t-s)^{-\frac{\kappa}{2}} \|F(Y_s) - F(Y_s^N)\|_\beta \Big\} ds \\
&\lesssim_M \frac{(\log N)^2}{N^\kappa} \int_0^t \big\{(t-s)^{-\frac{\alpha+\beta+\kappa}{2}} s^{-2\gamma'} + (t-s)^{-\frac{\kappa}{2}} s^{-3\gamma'}\big\} ds \\
&\quad + \|\bar{Z} - \bar{Z}^N\|_{\bar{\mathcal{L}}} \int_0^t (t-s)^{-\frac{\alpha+\beta+\kappa}{2}} s^{-2\gamma' - \frac{\kappa}{2}} ds \\
&\quad + \int_0^t \big\{(t-s)^{-\frac{\alpha+\beta+\kappa}{2}} s^{-\gamma'} + (t-s)^{-\frac{\kappa}{2}} s^{-2\gamma'}\big\} \|Y_s - Y_s^N\|_\beta ds,
\end{aligned}$$

which, together with

$$\int_0^t (t-s)^{-a} s^{-b} \leq t^{1-a-b}, \tag{4.9}$$

for $a, b > 0$ satisfying $a + b < 1$, implies that

$$\begin{aligned}
\|Y_t - Y_t^N\|_\beta &\lesssim \frac{(\log N)^2}{N^\kappa} t^{1-\frac{\kappa}{2}-3\gamma'} + \|\bar{Z} - \bar{Z}^N\|_{\bar{\mathcal{L}}} t^{1-\frac{\alpha+\beta+2\kappa}{2}-2\gamma'} \\
&\quad + \int_0^t \big\{(t-s)^{-\frac{\alpha+\beta+\kappa}{2}} s^{-\gamma'} + (t-s)^{-\frac{\kappa}{2}} s^{-2\gamma'}\big\} \|Y_s - Y_s^N\|_\beta ds.
\end{aligned}$$

Multiplying by $t^{\gamma'}$ and using the Gronwall's inequality we have for $t \in [0, \tau^M \wedge \sigma_N \wedge \nu_N^{M,\epsilon}]$

$$t^{\gamma'} \|Y_t - Y_t^N\|_\beta \lesssim_M \frac{(\log N)^2}{N^\kappa} t^{1-\frac{\kappa}{2}-2\gamma'} + \|\bar{Z} - \bar{Z}^N\|_{\bar{\mathcal{L}}} t^{1-\frac{\alpha+\beta+2\kappa}{2}-\gamma'},$$

where the constants we omit are independent of N. Then we find $C = C(M, T) > 0$, which is independent of N, such that

$$\|Y - Y^N\|_{\mathcal{M}_{\sigma_N \wedge \tau^M \wedge \nu_N^{M,\epsilon}}} \leq C\big(\frac{(\log N)^2}{N^\kappa} + \|\bar{Z} - \bar{Z}^N\|_{\bar{\mathcal{L}}}\big). \tag{4.10}$$

Let ϵ and $N_0 = N_0(M)$ be such that $\|\bar{Z} - \bar{Z}^N\|_{\bar{\mathcal{L}}} < \epsilon$ and $\epsilon + \frac{(\log N)^2}{N^\kappa} < \frac{1}{C}$ hold for any $N > N_0$. Such ϵ and N_0 can be ensured by (3.24). Then $\|Y - Y^N\|_{\mathcal{M}_{\sigma_N \wedge \tau^M \wedge \nu_N^{M,\epsilon}}} < 1$

for every $N > N_0$, and, recalling the definition of σ_N^M, we have $\sigma_N \wedge \tau^M \wedge \nu_N^{M,\epsilon} = \tau^M \wedge \nu_N^{M,\epsilon}$. As a consequence, for every $N > N_0$ we have

$$\sup_{0\le t\le \tau^M\wedge \nu_N^{M,\epsilon}} t^{\gamma'}\|Y_t - Y_t^N\|_\beta < 1. \tag{4.11}$$

Besides, by the definition of $\tau^M, \nu_N^{M,\epsilon}$, we also have for every $N \ge N_0$

$$\begin{aligned}\sup_{0\le t\le \tau^M\wedge \nu_N^{M,\epsilon}} \sup_{n=1,2,3} \Big\{ & t^{\gamma'}\|Y_t\|_\beta, t^{\gamma'}\|Y_t^N\|_\beta, t^{\frac{(\alpha+\beta)(n-1)}{2}}\|\bar{Z}_t^{:n:}\|_{-\alpha}, \\ & t^{\frac{(\alpha+\beta)(n-1)+n\kappa}{2}}\|(\bar{Z}_t^N)^{:n:}\|_{-\alpha}\Big\} \le M+1.\end{aligned} \tag{4.12}$$

By Theorems 3.5 and 4.1 we can deduce that for the above fixed M, ϵ

$$\lim_{N\to\infty} P(\nu_N^{M,\epsilon} = T) = 1, \tag{4.13}$$

and

$$\lim_{M\to\infty} P(\tau^M = T) = 1. \tag{4.14}$$

In the following we consider pathwise error estimates for Y^N, Z^N, Y and Z before the stopping times $\tau^M \wedge \nu_N^{M,\epsilon}$ with $\epsilon, M > 0$ fixed as above for $N > N_0$. All the constants may depend on M.

Theorem 4.3. *Assume the setting in Section 4.1. Then for $N \ge N_0 = N_0(M)$*

$$\mathbb{E}\Big(\sup_{0\le t\le \tau\wedge\nu_N^\epsilon} t^{\gamma' p}\|P_N X_t - X_t^N\|_\beta^p\Big) \lesssim_M (1+N^2)^{-\frac{p\alpha^-}{2}}, \tag{4.15}$$

where α^- denotes $\alpha - \delta$ for every $\delta > 0$.

Proof. Since

$$P_N X_t - X_t^N = P_N Y_t - Y_t^N,$$

and according to (1.6) and (1.12) with the decomposition in (4.6), we have

$$P_N Y_t - Y_t^N = \int_0^t P_N e^{(t-s)A}\Big\{F(Y_s) - F(Y_s^N) + \widetilde{\Psi}(Y_s, \underline{\bar{Z}}_s) - \widetilde{\Psi}(Y_s^N, \underline{\bar{Z}}_s^N)\Big\} ds.$$

By Lemmas 2.4, 2.2 with $\beta > \alpha$ and the inequality $s^{-a} \lesssim s^{-b}$ with $0 \le a \le b$ we deduce for $t \le \tau^M \wedge \nu_N^{M,\epsilon}$,

$$\begin{aligned}&\|P_N Y_t - Y_t^N\|_\beta \\ \lesssim &\int_0^t \Big\{(t-s)^{-\frac{\alpha+\beta+\kappa}{2}}\|\widetilde{\Psi}(Y_s, \underline{\bar{Z}}_s) - \widetilde{\Psi}(Y_s^N, \underline{\bar{Z}}_s^N)\|_{-\alpha} + (t-s)^{-\frac{\kappa}{2}}\|F(Y_s) - F(Y_s^N)\|_\beta\Big\} ds \\ \lesssim &\int_0^t \Big\{\big((t-s)^{-\frac{\alpha+\beta+\kappa}{2}}s^{-\gamma'} + (t-s)^{-\frac{\kappa}{2}}s^{-2\gamma'}\big)\|Y_s - Y_s^N\|_\beta + (t-s)^{-\frac{\alpha+\beta+\kappa}{2}} \\ &\quad\cdot \big(s^{-2\gamma'}\|\bar{Z}_s - \bar{Z}_s^N\|_{-\alpha} + s^{-\gamma'}\|\bar{Z}_s^{:2:} - (\bar{Z}_s^N)^{:2:}\|_{-\alpha} + \|\bar{Z}_s^{:3:} - (\bar{Z}_s^N)^{:3:}\|_{-\alpha}\big)\Big\} ds,\end{aligned} \tag{4.16}$$

where the second inequality follows by the estimates in (4.8). We note that in the second inequality of (4.16), by (2.11) we have

$$\|Y_s - Y_s^N\|_\beta \lesssim \frac{(\log N)^2}{N^\alpha}\|Y_s\|_{\beta+\alpha} + \|P_N Y_s - Y_s^N\|_\beta.$$

Following (4.16), we further have

$$\begin{aligned}\|P_N Y_t - Y_t^N\|_\beta \lesssim & \int_0^t \left\{(t-s)^{-\frac{\alpha+\beta+\kappa}{2}} s^{-\gamma'} + (t-s)^{-\frac{\kappa}{2}} s^{-2\gamma'}\right\}\|P_N Y_s - Y_s^N\|_\beta ds \\ & + \int_0^t \Big\{ \frac{(\log N)^2}{N^\alpha}\big((t-s)^{-\frac{\alpha+\beta+\kappa}{2}} s^{-\gamma'} + (t-s)^{-\frac{\kappa}{2}} s^{-2\gamma'}\big)\|Y_s\|_{\beta+\alpha} + (t-s)^{-\frac{\alpha+\beta+\kappa}{2}} \\ & \cdot \big(s^{-2\gamma'}\|\bar Z_s - \bar Z_s^N\|_{-\alpha} + s^{-\gamma'}\|\bar Z_s^{:2:} - (\bar Z_s^N)^{:2:}\|_{-\alpha} + \|\bar Z_s^{:3:} - (\bar Z_s^N)^{:3:}\|_{-\alpha}\big)\Big\} ds.\end{aligned}$$

Similar to Theorem 4.1, we deduce that $\mathbb{E}\sup_{0\le t\le T} t^{\gamma' p}\|Y_t\|_{\beta+\alpha}^p < \infty$ since by the setting in Section 4.1 that $\frac{2\alpha+\beta}{2} < \gamma' < \frac{1}{3}$, $\bar\beta := \beta+\alpha$ satisfies the conditions on β in Theorem 4.1. Moreover, for α, β, γ' satisfying $\frac{2\alpha+\beta}{2} < \gamma' < \frac{1}{3}$, it also holds that $1 - \frac{n(2\alpha+\beta)}{2} - (3-n)\gamma' > 0$, $n = 1, 2, 3$. Then using Gronwall's inequality and (4.9), we have

$$\begin{aligned}\mathbb{E} \sup_{0\le t\le \tau^M\wedge\nu_N^{M,\epsilon}} t^{\gamma' p}\|P_N Y_t - Y_t^N\|_\beta^p \lesssim_M & \ T^{(1-\frac{\kappa}{2}-2\gamma')p}\frac{(\log N)^{2p}}{N^{\alpha p}}\mathbb{E}\sup_{0\le t\le T} t^{\gamma' p}\|Y_t\|_{\beta+\alpha}^p \\ & + T^{(1-\frac{n(2\alpha+\beta+\kappa)}{2}-(2-n)\gamma')p}\mathbb{E}\sup_{0\le t\le T} t^{\frac{(\alpha+\beta)(n-1)+n\alpha}{2}p}\|\bar Z_t^{:n:} - (\bar Z_t^N)^{:n:}\|_{-\alpha}^p\end{aligned}$$

with $\kappa > 0$ sufficiently small. By (3.24) we also have

$$\mathbb{E}\sup_{0\le t\le T} t^{\frac{(\alpha+\beta)(n-1)+n\alpha}{2}p}\|\bar Z_t^{:n:} - (\bar Z_t^N)^{:n:}\|_{-\alpha}^p \lesssim \frac{(\log N)^{2p}}{N^{\alpha p}} + (1+N^2)^{-\frac{p\alpha^-}{2}}.$$

Hence (4.15) follows.

$\square$

4.2 Rates of convergence for stochastic Allen-Cahn equations

Now we present rates of convergence for stochastic Allen-Cahn equations, which is the main result throughout our paper.

Theorem 4.4. *Assume the setting in Section 4.1 and let $X_0 \in \mathcal{C}^{-\alpha}$ with $\alpha \in (0, 2/9)$. Let X, X^N denote the solutions to equations (1.1) and (1.9) on $[0,T]$ with initial values X_0 and $P_N X_0$ respectively. Then for any $\delta > 0$ and $\gamma' > \frac{3\alpha}{2}$*

$$\lim_{N\to\infty}\mathbb{P}\Big(\sup_{t\in[0,T]} t^{\gamma'}\|P_N X_t - X_t^N\|_{-\alpha} \gtrsim N^{\delta-\alpha}\Big) = 0, \tag{4.17}$$

$$\lim_{N\to\infty}\mathbb{P}\Big(\sup_{t\in[0,T]} t^{\gamma'}\|X_t - X_t^N\|_{-\alpha} \gtrsim N^{\delta-\alpha}\Big) = 0. \tag{4.18}$$

Proof. By similar discussion as in Remark 4.2, it is enough to prove (4.17) and (4.18) for every $\gamma' \in (3\alpha/2, 1/3)$. Taking the stopping times in Section 4.1 with $\epsilon > 0$ fixed, we see that

$$\mathbb{P}(\sup_{t\in[0,T]} t^{\gamma'}\|P_N X_t - X_t^N\|_{-\alpha} \gtrsim N^{\delta-\alpha}) \leq \mathbb{P}(\sup_{t\in[0,\tau^M\wedge\nu_N^{M,\epsilon}]} t^{\gamma'}\|P_N X_t - X_t^N\|_{-\alpha} \gtrsim N^{\delta-\alpha})$$
$$+\mathbb{P}(\sup_{t\in(\tau^M\wedge\nu_N^{M,\epsilon},T]} t^{\gamma'}\|P_N X_t - X_t^N\|_{-\alpha} \gtrsim N^{\delta-\alpha}). \tag{4.19}$$

On the one hand, by (4.14) for any $\varepsilon > 0$ we have $M > 0$ sufficiently large such that $P(\tau^M < T) < \varepsilon$, and for such M fixed, by (4.13) there exists $N_1 = N(M)$ ($N_1 > N_0$ with N_0 given in (4.15)) such that $P(\nu_N^{M,\epsilon} < T) < \varepsilon$ for all $N \geq N_1$. Then for all $N \geq N_1$,

$$\mathbb{P}(\sup_{t\in(\tau^M\wedge\nu_N^{M,\epsilon},T]} t^{\gamma'}\|P_N X_t - X_t^N\|_{-\alpha} \gtrsim N^{\delta-\alpha}) \leq \mathbb{P}(\tau^M \wedge \nu_N^{M,\epsilon} < T) < \varepsilon.$$

On the other hand, by (4.15) and letting β close to α ($\beta > \alpha$) and by using the embedding $\|\cdot\|_{-\alpha} \lesssim \|\cdot\|_\beta$ and Markov's inequality, for the above M, the first term on the right hand of (4.19) tends to zero when N tends to infinity. Then (4.17) follows.

Moreover, we have

$$\|X_t - X_t^N\|_{-\alpha} \lesssim \|X_t - P_N X_t\|_{-\alpha} + \|P_N X_t - X_t^N\|_{-\alpha},$$

which by (2.11) for any $\lambda < \alpha$ we get

$$\|X_t - P_N X_t\|_{-\alpha} \lesssim \frac{(\log N)^{2p}}{N^{\lambda p}}\|X_t\|_{-\alpha+\lambda},$$

and similar to (4.3) we have $\mathbb{E}\sup_{t\in[0,T]} t^{\gamma' p}\|X_t\|_{-\alpha+\lambda}^p < \infty$. (4.18) holds by choosing λ close to α.

□

A Appendix

Definition A.1. Let $\{\xi(\phi)\}_{\phi\in L^2(\mathbb{R}\times\mathbb{T}^d)}$ be a family of centered Gaussian random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\mathbb{E}(\xi(\phi)\xi(\psi)) = \langle\phi,\psi\rangle_{L^2(\mathbb{R}\times\mathbb{T}^d)},$$

for all $\psi, \phi \in L^2(\mathbb{R}\times\mathbb{T}^d)$. Then ξ is called a space-time white noise on $\mathbb{R}\times\mathbb{T}^d$. We interpret $\xi(\phi)$ as a stochastic integral and write

$$\int_{\mathbb{R}\times\mathbb{T}^d} \psi(t,x)\xi(dt,dx) := \xi(\psi), \quad \psi \in L^2(\mathbb{R}\times\mathbb{T}^d).$$

For any $n \in \mathbb{N}$, the multiple stochastic integrals (see [28, Chapter 1]) on $\mathbb{R}\times\mathbb{T}^d$ are defined for all symmetric functions f in $L^2(\mathbb{R}\times\mathbb{T}^d)$, i.e. functions such that

$$f(z_1,\dots,z_n) = f(z_{\sigma(1)},\dots,z_{\sigma(n)}), \quad z_i \in \mathbb{R}\times\mathbb{T}^d, j = 1,2,\dots,n,$$

for any permutation $(\sigma(1),\dots,\sigma(n))$ of $(1,\dots,n)$. For such a symmetric function f we denote its n-th interated stochastic integral by

$$I_n(f) := \int_{(\mathbb{R}\times\mathbb{T}^d)^n} f(z_1,\dots,z_n)\xi(\otimes_{i=1}^n ds_i, \otimes_{i=1}^n dx_i),\ \ z_i = (t_i,x_i) \in \mathbb{R}\times\mathbb{T}^d.$$

Theorem A.2. *[28, Theorem 1.1.2, Section 1.4] Let f be any symmetric function in $L^2((\mathbb{R}\times\mathbb{T}^d)^n)$. Then*

$$\mathbb{E}(I_n(f))^2 = n!\|f\|^2_{L^2((\mathbb{R}\times\mathbb{T}^d)^n)} \tag{A.1}$$

and

$$\mathbb{E}|I_n(f)|^p \leq (p-1)^{\frac{np}{2}}(\mathbb{E}|I_n(f)|^2)^{\frac{p}{2}} \tag{A.2}$$

for every $p \geq 2$.

For symmetric kernels $K_1, K_2 : \mathbb{Z}^2 \to (0,\infty)$, we denote its convolution

$$K_1 \star K_2(m) := \sum_{l\in\mathbb{Z}^2} K_1(m-l)K_2(l)$$

and for $N \in \mathbb{N}$ we set

$$K_1 \star_{\leq N} K_2(m) := \sum_{|l|\leq N} K_1(m-l)K_2(l), \quad K_1 \star_{>N} K_2(m) := K_1 \star K_2 - K_1 \star_{\leq N} K_2.$$

For convolutions of the same kernel, we introduce

$$K \star^1 K := K,\ K \star^n K := K \star (K \star^{n-1} K), \tag{A.3}$$

$$K{\star^1}_{\leq N}K := K,\ K{\star^n}_{\leq N}K := K \star_{\leq N} (K{\star^{n-1}}_{\leq N}K), \tag{A.4}$$

where by simple calculation we actually obtain

$$K{\star^n}_{\leq N}K(m) = \sum_{|l_i|\leq N, i=1,\dots,n-1} K(m-l_{n-1})\prod_{i=1}^{n-1} K(l_i - l_{i-1}),$$

with the convention that $l_0 = 0$. Similarly for every $n \geq 2$, we denote

$$K{\star^n}_{>N}K(m) := K \star^n K - K{\star^n}_{\leq N}K. \tag{A.5}$$

Following the technique of [18, Lemma 10.14], we have the following estimates.

Lemma A.3. *Let $K^\gamma : \mathbb{Z}^2 \to (0,\infty)$ be a symmetric kernel such that $K^\gamma(m) \lesssim \frac{1}{(1+|m|^2)^{1-\gamma}}$, $\gamma \in [0,\frac{1}{n})$ for $n \in \mathbb{N}$.* (i) *If $\gamma > 0$ then*

$$\max\left\{K^\gamma \star^n K^\gamma(m), \quad \sup_{N\geq 1} K^\gamma \star^n_{\leq N} K^\gamma(m)\right\} \lesssim \frac{1}{(1+|m|^2)^{1-n\gamma}},$$

$$K^\gamma \star^n_{>N} K^\gamma(m) \lesssim \begin{cases} \dfrac{1}{(1+|m|^2)^{1-n\gamma}}, & \text{if } |m| \geq N, \\ \dfrac{1}{(1+|N|^2)^{1-n\gamma}}, & \text{if } |m| < N. \end{cases}$$

(ii) *If $\gamma = 0$ then*

$$\max\left\{K^0 \star^n K^0(m), \quad \sup_{N\geq 1} K^0 \star^n_{\leq N} K^0(m)\right\} \lesssim \frac{1}{(1+|m|^2)^{1-\epsilon}},$$

$$K^0 \star^n_{>N} K^0(m) \lesssim \begin{cases} \dfrac{1}{(1+|m|^2)^{1-\epsilon}}, & \text{if } |m| \geq N \\ \dfrac{1}{(1+|N|^2)^{1-\epsilon}}, & \text{if } |m| < N, \end{cases}$$

for every $\epsilon \in (0,1)$.

Proof. The estimates for $K^\gamma \star^n K^\gamma$ and $K^\gamma \star^n_{>N} K^\gamma$ with $\gamma \in [0,\frac{1}{n})$ were obtained in [38, Corollary C.3]. The term $K^\gamma \star^n_{\leq N} K^\gamma$ can be similarly considered following the same procedure. Now let two symmetric kernels $K_1, K_2 : \mathbb{Z}^2 \to (0,\infty)$ be such that $K_1(m) \lesssim \frac{1}{(1+|m|^2)^\alpha}$ and $K_2(m) \lesssim \frac{1}{(1+|m|^2)^\beta}$ with any $\alpha, \beta \in (0,1]$ and $\alpha + \beta - 1 > 0$. We consider the following regions of $\mathbb{Z}^2$,

$$\begin{aligned} A_1 &= \{l : \ |l| \leq \frac{|m|}{2}\}, \quad A_2 = \{l : \ |l-m| \leq \frac{|m|}{2}\}, \\ A_3 &= \{l : \ \frac{|m|}{2} \leq |l| \leq 2|m|, |l-m| \geq \frac{|m|}{2}\}, \quad A_4 = \{l : \ |l| \geq 2|m|\}. \end{aligned}$$

Since for every $l \in A_1$ we have $|m-l| \geq |m| - |l| \geq \frac{|m|}{2}$, then for uniform $N \geq 1$

$$\begin{aligned} \sum_{l\in A_1, |l|\leq N} K_1(m-l)K_2(l) &\lesssim \frac{1}{(1+|m|^2)^\alpha} \sum_{l\in A_1} \frac{1}{(1+|l|^2)^\beta} \\ &\lesssim \begin{cases} \dfrac{(1+|m|^2)^{1-\beta}}{(1+|m|^2)^\alpha}, & \text{if } \beta < 1, \\ \dfrac{\log|m| \vee 1}{(1+|m|^2)^\alpha}, & \text{if } \beta = 1. \end{cases} \end{aligned}$$

For $l \in A_2$, by symmetry we get that for uniform $N \geq 1$

$$\sum_{l\in A_2, |l|\leq N} K_1(m-l)K_2(l) \lesssim \begin{cases} \dfrac{(1+|m|^2)^{1-\alpha}}{(1+|m|^2)^\beta}, & \text{if } \alpha < 1, \\ \dfrac{\log|m| \vee 1}{(1+|m|^2)^\alpha}, & \text{if } \beta = 1. \end{cases}$$

For $l \in A_3$ we notice that by the definition of A_3, for uniform $N \geq 1$ we have

$$\sum_{l \in A_3, |l| \leq N} K_1(m-l)K_2(l) \lesssim \sum_{|l| \leq 2|m|} \frac{1}{(1+|m|^2)^{\alpha+\beta}} \lesssim \frac{1}{(1+|m|^2)^{\alpha+\beta-1}}.$$

For $l \in A_4$, we have $|m| < \frac{|l|}{2}$ and then $|m-l| \geq |l| - |m| \geq \frac{|l|}{2}$, which implies that for uniform $N \geq 1$

$$\sum_{l \in A_4, |l| \leq N} K_1(m-l)K_2(l) \lesssim \sum_{|l| > 2|m|} \frac{1}{(1+|l|^2)^{\alpha+\beta}} \lesssim \frac{1}{(1+|m|^2)^{\alpha+\beta-1}},$$

where the second inequality comes from $\alpha + \beta - 1 > 0$.

Combining all the above and considering that $\log |m| \lesssim (1+|m|^2)^{\epsilon}$ for $\epsilon > 0$ arbitrarily small, we thus obtain that for uniform $N \geq 1$

$$K_1 \star_{\leq N} K_2(m) \lesssim \begin{cases} \dfrac{1}{(1+|m|^2)^{1-\epsilon}}, & \text{if } \alpha, \beta = 1, \\ \dfrac{1}{(1+|m|^2)^{\alpha+\beta-1}}, & \text{if } \beta < 1 \text{ or } \alpha < 1, \end{cases}$$

for $\epsilon > 0$ arbitrarily small. We prove Lemma A.3 immediately.

□

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