Well-posedness of Backward Stochastic Partial Differential Equations with Lyapunov Condition *

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Abstract. In this paper we show the existence and uniqueness of strong solutions for a large class of backward SPDE where the coefficients satisfy a specific type Lyapunov condition instead of the classical coercivity condition. Moreover, based on the generalized variational framework, we also use the local monotonicity condition to replace the standard monotonicity condition, which is applicable to various quasilinear and semilinear BSPDE models.

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1 Introduction

The theory of backward stochastic differential equations (BSDEs) has received extensive investigations in the last few decades. BSDEs have been successfully applied in stochastic control theory, econometrics, mathematical finance, nonlinear partial differential equations and so on, see [7, 8, 25, 41] and more references therein. The study of backward stochastic partial differential equations (BSPDEs) could be traced back to the works [1, 24]. This subject arise in many applications of probability theory and stochastic processes, for instance in nonlinear filtering and stochastic control theory for processes with incomplete information, as an adjoint equation of the Duncan-Mortensen-Zakai filtration (see e.g. [1, 13, 14, 37, 43, 44]). In the dynamic programming theory, some nonlinear BSPDEs as the backward stochastic Hamilton-Jacobi-Bellman equations, are also introduced in the investigation of non-Markovian control problems (see e.g [9, 26]). Recently, there are many papers studying

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backward stochastic partial differential equations (see [6, 28, 29, 36, 38, 42] and the references therein). In [39] a very general system of backward stochastic partial differential equations is studied, and in [29, 36] the authors concentrate on the study of the backward stochastic 2D Navier-Stokes equation (BSNSE).

The main aim of this work is to prove the existence and uniqueness of solutions for a large class of backward stochastic partial differential equations using the variational approach. The variational framework has been used intensively for studying PDE and SPDE where the coefficients satisfy the classical monotonicity and coercivity conditions. In the case of deterministic equations, the theory of monotone operators started from the substantial work of Minty [21, 22], then it was studied systematically by Browder [2, 3] in order to obtain the existence of solutions for quasi-linear elliptic and parabolic partial differential equations. We refer to the monograph [4] for more extensive exposition and references. Concerning the stochastic equations, it was first investigated in the seminal works of Pardoux [23] and Krylov and Rozovskii [15], where they adapted the monotonicity tricks to prove the existence and uniqueness of solutions for a class of semilinear and quasilinear SPDE. Recently, this framework has been substantially extended by the first named author and Röckner in [18, 19] for more general class of SPDEs with coefficients satisfying the generalized coercivity and local monotonicity conditions, hence many fundamental examples such as stochastic Burgers type equations and stochastic 2D Navier-Stokes equations can be included into this framework now (see [17, 20] for more examples).

In this paper we will show the existence and uniqueness of strong solutions for a class of BSPDE where the coefficients satisfy a specific type Lyapunov condition (we call it one-sided linear growth here) instead of the classical coercivity condition. Based on [18], we also use the local monotonicity condition here to replace the standard monotonicity condition. This Lyapunov type condition (see (H3) below) is inspired by the recent work of [17] (see also the references therein), where this type of condition is used to investiage stochastic tamed 3D Navier-Stokes equations and the stochastic curve shortening flow in the plane. Moreover, we should remark that our main result is also applicable to backward stochastic 2D Navier-Stokes equations, stochastic p-Laplace equations, stochastic fast diffusion equations, stochastic Burgers type equations and stochastic reaction-diffusion equations. We refer to Section 3 for the details.

2 Main Result

First we introduce our framework in detail. Let $(H, \langle \cdot, \cdot \rangle_H)$ be a separable Hilbert space and identified with its dual space H^* by the Riesz isomorphism, and let $(V, \langle \cdot, \cdot \rangle_V)$ be a Hilbert space such that it is continuously and densely embedded into H. Then we have the following Gelfand triple

$$V \subset H \equiv H^* \subset V^*,$$

where V^* is the dual space of V (w.r.t. $\langle \cdot, \cdot \rangle_H$).

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a complete filtrated probability space, on which a cylindrical Wiener process $\{W_t\}_{t \geq 0}$ is defined on a separable Hilbert space $(U, \langle \cdot, \cdot \rangle_U)$, whose natural augmented filtration is denoted by $\{\mathcal{F}_t, t \in [0,T]\}$ and $(L_2(U,V), \|\cdot\|_{L_2(U,V)})$ denotes the space of all Hilbert-Schmidt operators from U to V. We denote by $\mathcal{P}$ the σ-algebra of the predictable sets on $\Omega \times [0,T]$ associated with $\{\mathcal{F}_t\}_{t\geq 0}$. For any Banach space $\mathbb{B}$, let $L^p_{\mathcal{F}}(\Omega; L^r([0,T];\mathbb{B})), p, r \in [1,\infty]$ be the set of all predictable $\mathbb{B}$-valued processes in $L^p(\Omega; L^r([0,T];\mathbb{B}))$ and let $L^p_{\mathcal{F}}(\Omega \times [0,T];\mathbb{B})$ to denote all predictable $\mathbb{B}$-valued processes in $L^p(\Omega \times [0,T];\mathbb{B})$. We also use $L^p_{\mathcal{F}_T}(\Omega;\mathbb{B}), p \in [1,\infty]$ to denote the set of all $\mathcal{F}_T$-measurable random variable in $L^p(\Omega;\mathbb{B})$.

We consider the following backward stochastic partial differential equation

$$dX_t = -A(t, X_t, Z_t)dt + Z_t dW_t, \quad t \in [0,T], \quad X(T) = \xi, \tag{2.1}$$

where $A : [0,T] \times V \times L_2(U,H) \times \Omega \to V^*$ and for any $(v,z) \in V \times L_2(U,H)$, $A(\cdot, v, z, \cdot)$ is predictable and V^*-valued process.

We need to suppose the following assumptions concerning the Gelfand triple.

(H0) There exists an orthogonal set $\{e_1, e_2, ...\}$ in $(V, \langle\cdot,\cdot\rangle_V)$ such that it constitute an orthonormal basis of $(H, \langle\cdot,\cdot\rangle_H)$.

Suppose that there exist constants $\varepsilon \in (0, \frac{1}{2})$, K and a positive adapted process $f \in L^\infty_{\mathcal{F}}(\Omega, L^1([0,T]))$ such that the following conditions hold for all $v, v_1, v_2 \in V, \phi, \phi_1, \phi_2 \in L_2(U,V)$ and $(t,\omega) \in [0,T] \times \Omega$:

(H1) (Hemicontinuity) The map $s \mapsto {}_{V^*}\langle A(t, v_1 + s v_2, \phi), v\rangle_V$ is continuous on $\mathbb{R}$.

(H2) (Local monotonicity) There exists a locally bounded measurable function $\rho : V \to [0, +\infty)$ such that

$${}_{V^*}\langle A(t, v_1, \phi_1) - A(t, v_2, \phi_2), v_1 - v_2\rangle_V \leq \rho(v_2)[\|v_1 - v_2\|^2_H + \|v_1 - v_2\|_H \|\phi_1 - \phi_2\|_{L_2(U,H)}].$$

(H3) (One-sided linear growth) For any $n \in \mathbb{N}$, the operator A maps $H_n := \operatorname{span}\{e_1, ..., e_n\}$ into V such that for $v \in H^n$

$$\langle A(t, v, \phi), v\rangle_V \leq f_t + \varepsilon\|\phi\|^2_{L_2(U,V)} + K\|v\|^2_V.$$

(H4) (Growth)

$$\|A(t, v, \phi)\|_{V^*} \leq f_t^{1/2} + \rho(v) + K\|\phi\|_{L_2(U,V)}.$$

Definition 2.1 *For $\xi \in L^\infty_{\mathcal{F}_T}(\Omega, V)$ we say that (X, Z) is a solution to (2.1) if*

$$X \in L^2_{\mathcal{F}}(\Omega; L^\infty([0,T];V)) \cap L^2_{\mathcal{F}}(\Omega; C([0,T];H)),$$

$$Z \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; L_2(U,H))),$$

$$X_t = \xi + \int_t^T A(t, X_s, Z_s)ds - \int_t^T Z_s dW_s, \ \text{in } V^* \quad P-a.s..$$

Now we state the main result of this work.

Theorem 2.2 *Suppose (H0)-(H4) hold. For any* $\xi \in L^\infty_{\mathcal{F}_T}(\Omega; V)$*, BSPDE (2.1) admits a unique adapted solution* $(X, Z) \in L^\infty_{\mathcal{F}}(\Omega \times [0,T]; V) \times L^2_{\mathcal{F}}(\Omega; L^2([0,T], L_2(U,H)))$*. Moreover, it satisfies that*

$$\sup_{t\in[0,T]} \|X_t\|^2_V + \frac{1}{2}E\int_0^T \|Z_s\|^2_{L_2(U,V)} ds \le C(\|f\|_{L^\infty_{\mathcal{F}}(\Omega;L^1([0,T]))} + \|\xi\|^2_{L^\infty_{\mathcal{F}_T}(\Omega;V)}), \quad a.s..$$

Remark 2.3 *(1) In the theorem above we assume that* $\xi \in L^\infty_{\mathcal{F}_T}(\Omega, V)$*, which seems quite different to the condition usually posed on the initial value of stochastic PDEs. The reason for this is that we cannot use any stopping time argument for BSPDEs, therefore, here we have to use stochastic Gronwall-Bellman inequality to deduce an uniform estimate (see (2.5) below) to control the nonlinear term. This is also one of the main differences between BSPDEs and (standard/forward) SPDEs.*

(2) Note that (H1) and (H2) imply that $A(t, v, z)$ *is locally Lipschitz continuous with respect to* z *in the following sense:*

$$\|A(t, v, z_1) - A(t, v, z_2)\|_{V^*} \le C(\|v\|_V)\|z_1 - z_2\|_{L_2(U,H)},$$

for all $t \in [0,T], \omega \in \Omega, v \in V, z_1, z_2 \in L_2(U,H)$*.*

The rest part of this section is devoted to the proof of main result, and we need several lemmas for this purpose.

Recall that $\{e_1, e_2, ...\} \subset V$ is an orthonormal basis of H and $H_n = \mathrm{span}\{e_1, ..., e_n\}$. Let $P_n : V^* \to H_n$ be defined by

$$P_n y = \sum_{i=1}^n {}_{V^*}\langle y, e_i\rangle_V e_i, \quad y \in V^*.$$

Hence we have

$${}_{V^*}\langle P_n A(t,u,z), v\rangle_V = \langle P_n A(t,u,z), v\rangle_H = {}_{V^*}\langle A(t,u,z), v\rangle_V, \quad u \in V, v \in H_n, z \in L_2(U,H).$$

By (H0) we have

$$\langle P_n A(t,u), v\rangle_V = \langle A(t,u), v\rangle_V.$$

Now we consider the following projected approximation:

$$X^N_t = P_N\xi + \int_t^T P_N A(t, X^N_t, Z^N_t)dt - \int_t^T Z^N_t dW_t, \tag{2.2}$$

where $Z^N \in L(U, H_N)$ can be extended to a element in $L_2(U,H)$ (still denoted by Z^N) by setting $Z^N(e_j) = 0, j \ge N+1$.

To solve (2.2) we shall make use of the result in [5]. We fixed the filtration generated by the cylindrical Wiener process. We do not approximate W by its finite dimensional projection. Since W is a cylindrical Wiener process, we cannot apply the results in [5, Theorem 4.2] directly. We need to use the lemma below for the following type BSDE

$$Y_t = \zeta + \int_t^T g(s, Y_s, q_s) ds - \int_t^T q_s dW_s, \tag{2.3}$$

where ζ is an $\mathbb{R}^N$-valued $\mathcal{F}_T$-measurable random vector, the random function $g : \Omega \times [0,T] \times \mathbb{R}^N \times L_2(U;\mathbb{R}^N) \to \mathbb{R}^N$ is $\mathcal{P} \times \mathcal{B}(\mathbb{R}^N) \times \mathcal{B}(L_2(U;\mathbb{R}^N))$-measurable.

Lemma 2.4 *Assume that g and ζ satisfy the following four conditions:*
(C1) For some $p > 1$ we have

$$E[|\zeta|^p + (\int_0^T |g(t,0,0)| dt)^p] < \infty.$$

(C2) There exist constants $\alpha \geq 0$ and $\mu \in \mathbb{R}$ such that almost surely we have for each $t \in [0,T], y, y' \in \mathbb{R}^N, z, z' \in L_2(U;\mathbb{R}^N)$,

$$|g(t,y,z) - g(t,y,z')| \leq \alpha \|z - z'\|_{L_2(U;\mathbb{R}^N)},$$

$$\langle y - y', g(t,y,z) - g(t,y',z) \rangle \leq \mu |y - y'|^2.$$

(C3) The function $y \mapsto g(t,y,z)$ is continuous for every $(t,z) \in [0,T] \times L_2(U;\mathbb{R}^N)$.
(C4) For any $r > 0$, the stochastic process

$$\{\psi_r(t) := \sup_{|y| \leq r} |g(t,y,0) - g(t,0,0)|, t \in [0,T]\}$$

lies in the space $L^1_{\mathcal{F}}(\Omega \times [0,T])$.
Then BSDE (2.3) admits a unique solution

$$(Y,q) \in L^p_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R}^N)) \times L^p_{\mathcal{F}}(\Omega; L^2([0,T]; L_2(U;\mathbb{R}^N))).$$

By the martingale representation theorem in infinite dimensional case in [14], we could prove Lemma 2.4. The method to prove it is standard and is a slight modification of the proof of [5, Theorem 4.2], so we omit it here. For more details we refer to [45, 46]. The following lemma comes from [29, Lemma 4.2].

Lemma 2.5 *For any $M, N \in \mathbb{N}$, define $\varphi_n(z) = \frac{zn}{(\|z\|_{L_2(U,H_N)} \vee n)}$, $z \in L_2(U,H_N)$ and set*

$$A^{N,M,n}(t,y,z) := R_M(\|y\|_V) \frac{n}{h_M(t) \vee n} P_N A(t,y,\varphi_n(z)),$$

where $R_M : \mathbb{R} \to [0,1]$ is a smooth function satisfying $R_M(r) = 1, |r| \le M$, $R_M(r) = 0, |r| > M+1$, $|R'_M| \le 1$ and

$$h_M(t) := f_t^{1/2} + \sup_{\|v\|_V \le M} \rho(v) \in L^1(\Omega \times [0,T]).$$

Then under (H0)-(H4) $A^{N,M,n}$ satisfies the conditions (C2)-(C4) of Lemma 2.4.

Proof Now we verify that there is a uniform constant $C_{N,M,n} > 0$ such that

$$\langle A^{N,M,n}(t, X_1, Z) - A^{N,M,n}(t, X_2, Z), X_1 - X_2 \rangle_H \le C_{N,M,n} \|X_1 - X_2\|_H^2, \quad a.s.,$$

for $X_1, X_2 \in H_N, Z \in L_2(U, H_N)$ and all $t \in [0,T]$. It holds trivially if $\|X_1\|_V > M+1$ and $\|X_2\|_V > M+1$. Thus, it is sufficient to consider the case of $\|X_2\|_V \le M+1$. We have

$$\begin{aligned}
&\langle A^{N,M,n}(t, X_1, Z) - A^{N,M,n}(t, X_2, Z), X_1 - X_2 \rangle \\
=& R_M(\|X_1\|_V) \frac{n}{h_M(t) \vee n} \langle P_N A(t, X_1, \varphi_n(Z)) - P_N A(t, X_2, \varphi_n(Z)), X_1 - X_2 \rangle \\
&+ \frac{n}{h_M(t) \vee n} (R_M(\|X_1\|_V) - R_M(\|X_2\|_V)) \langle P_N A(t, X_2, \varphi_n(Z)), X_1 - X_2 \rangle \\
\le& C_{N,M,n} \|X_1 - X_2\|^2,
\end{aligned}$$

where we used (H2) and (H4) in the last inequality. The other conditions are satisfied obviously also by (H2) and (H4). □

We now recall the stochastic Gronwall-Bellman inequality from [7, Corollary B1]. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered probability space whose filtration $\mathcal{F} = \{\mathcal{F}_t : t \in [0,T]\}$ satisfies the usual conditions. Suppose that $\{Y_s\}$ and $\{X_s\}$ are optional integrable processes and α is a nonnegative constant. If for all t, the map $s \to E[Y_s | \mathcal{F}_t]$ is continuous almost surely and

$$Y_t \le E[\int_t^T (X_s + \alpha Y_s) ds + Y_T | \mathcal{F}_t],$$

then we have almost surely

$$Y_t \le e^{\alpha(T-t)} E[Y_T | \mathcal{F}_t] + E[\int_t^T e^{\alpha(s-t)} X_s ds | \mathcal{F}_t], t \in [0,T].$$

Lemma 2.6 *Suppose that Assumptions (H1)-(H4) hold. For any $\xi \in L^\infty_{\mathcal{F}_T}(\Omega; V)$, the projected problem (2.2) admits a unique adapted solution*

$$(X^N, Z^N) \in L^2_{\mathcal{F}}(\Omega; C([0,T]; H)) \times L^2_{\mathcal{F}}(\Omega; L^2([0,T], L_2(U,H))).$$

Proof [Existence]. By Lemmas 2.4 and 2.5 there exists a unique solution $(X^{N,M,n}, Z^{N,M,n})$ to the following BSDE

$$X_t^{N,M,n} = \xi^N + \int_t^T A^{N,M,n}(s, X_s^{N,M,n}, Z_s^{N,M,n}) ds - \int_t^T Z_s^{N,M,n} dW_s,$$

for $\xi^N = P_N\xi$ and

$$X^{N,M,n} \in L^2_{\mathcal{F}}(\Omega; L^\infty([0,T];V)) \cap L^2_{\mathcal{F}}(\Omega; C([0,T];H)),$$
$$Z^{N,M,n} \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; L_2(U,H))).$$

Now by Itô's formula and using (H3) we have

$$\begin{aligned}
\|X_t^{N,M,n}\|_V^2 =& \|\xi^N\|_V^2 + 2\int_t^T \langle A^{N,M,n}(s, X_s^{N,M,n}, Z_s^{N,M,n}), X_s^{N,M,n}\rangle_V ds \\
&- \int_t^T \|Z_s^{N,M,n}\|^2_{L_2(U,V)} ds - 2\int_t^T \langle X_s^{N,M,n}, Z_s^{N,M,n} dW_s\rangle_V \\
\le& \|\xi^N\|_V^2 + 2\int_t^T (f_s + \varepsilon \|Z_s^{N,M,n}\|^2_{L_2(U,V)} + K\|X_s^{N,M,n}\|_V^2) ds \\
&- \int_t^T \|Z_s^{N,M,n}\|^2_{L_2(U,V)} ds - 2\int_t^T \langle X_s^{N,M,n}, Z_s^{N,M,n} dW_s\rangle_V .
\end{aligned} \tag{2.4}$$

By the Burkholder-Davis-Gundy inequality we have

$$\begin{aligned}
&E\left[\sup_{\tau\in[t,T]} \left| \int_\tau^T \langle X_s^{N,M,n}, Z_s^{N,M,n} dW_s\rangle_V \right| \right] \\
\le& 2E\left[\sup_{\tau\in[t,T]} \left| \int_t^\tau \langle X_s^{N,M,n}, Z_s^{N,M,n} dW_s\rangle_V \right| \right] \\
\le& CE\left[\int_t^T \|X_s^{N,M,n}\|_V^2 \|Z_s^{N,M,n}\|^2_{L_2(U,V)} ds\right]^{1/2} \\
\le& C(N)\left\{E\sup_{s\in[t,T]} \|X_s^{N,M,n}\|_V^2 + E\int_t^T \|Z_s^{N,M,n}\|^2_{L_2(U,H)} ds\right\},
\end{aligned}$$

where we used $\|Z_s^{N,M,n}\|_{L_2(U,V)} \le C(N)\|Z_s^{N,M,n}\|_{L_2(U,H)}$ in the last step. Taking conditional expectation on both sides of (2.4) we obtain

$$\|X_t^{N,M,n}\|_V^2 + \frac{1}{2}E_{\mathcal{F}_t}\int_t^T \|Z_s^{N,M,n}\|^2_{L_2(U,V)} ds \le E_{\mathcal{F}_t}\|\xi^N\|_V^2 + 2E_{\mathcal{F}_t}\int_t^T (f_s + K\|X_s^{N,M,n}\|_V^2) ds.$$

By the stochastic Gronwall-Bellman inequality, we conclude that

$$\sup_{t\in[0,T]} \|X_t^{N,M,n}\|_V^2 + \frac{1}{2}E\int_0^T \|Z_s^{N,M,n}\|^2_{L_2(U,V)} ds \le C(\|f\|_{L^\infty_{\mathcal{F}}(\Omega;L^1([0,T]))} + \|\xi\|^2_{L^\infty_{\mathcal{F}_T}(\Omega;V)}), \quad a.s., \tag{2.5}$$

where C is a constant independent of N, M, n. Now we deduce that there exists a positive constant K_1 independent of N, M and n such that

$$\sup_{t\in[0,T]} \|X_t^{N,M,n}\|_V^2 + \frac{1}{2}E\int_0^T \|Z_s^{N,M,n}\|^2_{L_2(U,V)} ds \le K_1, \quad a.s..$$

Then letting $M = K_1+1$ be fixed, we have $R_M(\|X_s^{N,M,n}\|_V) \equiv 1$. Now we write $(X^{N,n}, Z^{N,n})$ instead of $(X^{N,M,n}, Z^{N,M,n})$ below. Then there exists a positive constant K_2 independent of N, M and n such that

$$\|A(t, X_t^{N,n}, z_1) - A(t, X_t^{N,n}, z_2)\|_{V^*} \leq K_2\|z_1 - z_2\|_{L_2(U,H)},$$

$$\rho(X_t^{N,n}) + \rho^2(X_t^{N,n}) \leq K_2.$$

For $j \in \mathbb{N}$, set $(\bar{X}^N, \bar{Z}^N) = (X^{N,n+j} - X^{N,n}, Z^{N,n+j} - Z^{N,n})$. Applying Itô's formula we get

$$\begin{aligned}
&e^{K_2 t}\|\bar{X}_t^N\|_H^2 + \int_t^T e^{K_2 s}\|\bar{Z}_s^N\|_{L_2(U,H)}^2 ds \\
\leq& 2\int_t^T e^{K_2 s}\langle A^{N,n+j}(s, X_s^{N,n+j}, Z_s^{N,n+j}) - A^{N,n}(s, X_s^{N,n}, Z_s^{N,n}), \bar{X}_s^N\rangle_H ds \\
&- 2\int_t^T e^{K_2 s}\langle \bar{X}_s^N, \bar{Z}_s^N dW_s\rangle_H - K_2\int_t^T e^{K_2 s}\|\bar{X}_s^N\|_H^2 ds \\
\leq& 2\int_t^T e^{K_2 s}\langle A^{N,n+j}(s, X_s^{N,n+j}, Z_s^{N,n+j}) - A^{N,n+j}(s, X_s^{N,n}, Z_s^{N,n}), \bar{X}_s^N\rangle_H ds \\
&+ 2\int_t^T e^{K_2 s}\langle A^{N,n+j}(s, X_s^{N,n}, Z_s^{N,n}) - A^{N,n}(s, X_s^{N,n}, Z_s^{N,n}), \bar{X}_s^N\rangle_H ds \\
&- 2\int_t^T e^{K_2 s}\langle \bar{X}_s^N, \bar{Z}_s^N dW_s\rangle_H - K_2\int_t^T e^{K_2 s}\|\bar{X}_s^N\|_H^2 ds \\
\leq& 2\int_t^T e^{K_2 s}\langle \frac{n+j}{h_{K_1+1}(t) \vee (n+j)} P_N A(t, X^{N,n}, \varphi_{n+j}(Z^{N,n})) \\
&- \frac{n}{h_{K_1+1}(t) \vee n} P_N A(t, X^{N,n}, \varphi_n(Z^{N,n})), \bar{X}_s^N\rangle_H ds - 2\int_t^T e^{K_2 s}\langle \bar{X}_s^N, \bar{Z}_s^N dW_s\rangle_H \\
\leq& 8CK_1K_2\int_t^T e^{K_2 s}\left[\|Z_s^{N,n}\|_{L_2(U,H)} 1_{\{\|Z_s^{N,n}\|_{L_2(U,H)} > n\}} + \|Z_s^{N,n}\|_{L_2(U,V)} 1_{\{h_{K_1+1} > n\}} + 2h_{K_1+1} 1_{\{h_{K_1+1} > n\}}\right] ds \\
&- 2\int_t^T e^{K_2 s}\langle \bar{X}_s^N, \bar{Z}_s^N dW_s\rangle_H,
\end{aligned}$$

where we used $\|Z^{N,n}\|_{L_2(U,H)} \sim \|Z^{N,n}\|_{L_2(U,V)}$.

On the other hand by the BDG inequality we have

$$E[\sup_{\tau \in [t,T]} |\int_\tau^T e^{K_2 s}\langle \bar{X}_s^N, \bar{Z}_s^N dW_s\rangle|] \leq \varepsilon_0 E[\sup_{s \in [t,T]} (e^{K_2 s}\|\bar{X}_s^N\|^2)] + CE[\int_t^T e^{K_2 s}\|\bar{Z}_s^N\|_{L_2(U,H)}^2 ds].$$

By (2.5) and $h_{K_1+1} \in L^1(\Omega \times [0,T])$ we conclude that $(X^{N,n}, Z^{N,n})$ is a Cauchy sequence in $L^2_{\mathcal{F}}(\Omega; C([0,T]; H)) \times L^2_{\mathcal{F}}(\Omega; L^2([0,T], L_2(U,H)))$. Denote the limit by (X^N, Z^N). It is easily checked that (X^N, Z^N) is a solution to (2.2).

[Uniqueness]. Suppose (X_1^N, Z_1^N) and (X_2^N, Z_2^N) are two solutions of the projected equation (2.2). In the proof of uniqueness we use $X(t)$ to denote X_t. Denote $(\tilde{X}^N, \tilde{Z}^N) :=$

$(X_1^N - X_2^N, Z_1^N - Z_2^N)$. By the same arguments as above we obtain (2.5) also holds for $(\tilde{X}^N, \tilde{Z}^N)$. Define

$$r(t) := 2\int_0^t \rho(X_2^N(s)) + \rho^2(X_2^N(s))ds.$$

An application of Itô's formula and (H2) yields that

$$\begin{aligned} e^{r(t)}\|\tilde{X}^N(t)\|_H^2 = & \int_t^T e^{r(s)}[2\langle P_N A(s, X_1^N(s), Z_1^N(s)) - P_N A(s, X_2^N(s), Z_2^N(s)), \tilde{X}^N(s)\rangle_H \\ & - \|\tilde{Z}^N\|_{L_2(U,H)}^2 - 2\|\tilde{X}^N(s)\|_H^2(\rho(X_2^N(s)) + \rho^2(X_2^N(s)))]ds \\ & - 2\int_t^T e^{r(s)}\langle \tilde{X}^N(s), \tilde{Z}^N dW_s\rangle_H \\ \le & \int_t^T e^{r(s)}[2\|\tilde{X}^N(s)\|_H^2\rho(X_2^N(s)) + 2\|\tilde{X}^N(s)\|\rho(X_2^N(s))\|\tilde{Z}^N(s)\|_{L_2(U,H)} \\ & - \|\tilde{Z}^N\|_{L_2(U,H)}^2 - 2\|\tilde{X}^N(s)\|_H^2(\rho(X_2^N(s)) + \rho^2(X_2^N(s)))]ds \\ & - 2\int_t^T e^{r(s)}\langle \tilde{X}^N(s), \tilde{Z}^N dW_s\rangle_H \end{aligned}$$

Taking conditional expectation on both sides we have for any $t \in [0, T]$,

$$e^{r(t)}\|\tilde{X}^N(t)\|_H^2 + \frac{1}{2}E_{\mathcal{F}_t}[\int_t^T e^{r(s)}\|\tilde{Z}^N\|_{L_2(U,H)}^2 ds] \le 0, a.s.,$$

which implies the uniqueness. □

Now we can finish the proof of Theorem 2.2.

Proof [Existence] By the same arguments as in the proof of (2.5) we obtain

$$\sup_{t\in[0,T]} \|X_t^N\|_V^2 + \frac{1}{2}E\int_0^T \|Z_s^N\|_{L_2(U,V)}^2 ds \le C(\|f\|_{L^\infty_{\mathcal{F}}(\Omega;L^1([0,T]))} + \|\xi\|^2_{L^\infty_{\mathcal{F}_T}(\Omega;V)}), \quad a.s., \tag{2.6}$$

where C is independent of N. By (H4) we have

$$E\int_0^T \|A(t, X^N, Z^N)\|_{V^*}^2 dt \le C.$$

Then there exists a subsequence $N_k \to \infty$ such that

(i) $X^{N_k} \to \bar{X}$ weakly in $L^2_{\mathcal{F}}(\Omega; L^2([0, T], V))$ and weakly star in $L^\infty_{\mathcal{F}}((\Omega \times [0, T]), V)$.
(ii) $Y^{N_k} := A(t, X^{N_k}, Z^{N_k}) \to Y$ weakly in $L^2_{\mathcal{F}}(\Omega; L^2([0, T], V^*))$.
(iii) $Z^{N_k} \to Z$ weakly in $L^2_{\mathcal{F}}(\Omega; L^2([0, T], L_2(U, V)))$ and hence

$$\int_\cdot^T Z_s^{N_k} dW_s \to \int_\cdot^T Z_s dW_s$$

weakly in $L^\infty([0,T];L^2(\Omega,H))$. Now we define the following process

$$X_t := \xi + \int_t^T Y_s ds - \int_t^T Z_s dW_s, t\in[0,T],$$

then it is easy to show that $X=\bar{X}, dt\times P$-a.e. By [20, Theorem 4.2.5], we conclude that $X\in L^2_{\mathcal{F}}(\Omega;C([0,T],H))$ and by (2.6) we obtain

$$\sup_{t\in[0,T]}\|X_t\|_V^2+\frac{1}{2}E\int_0^T\|Z_s\|^2_{L_2(U,V)}ds\le C(\|f\|_{L^\infty_{\mathcal{F}}(\Omega;L^1([0,T]))}+\|\xi\|^2_{L^\infty_{\mathcal{F}_T}(\Omega;V)}),\quad a.s.,$$

Now it is sufficient to show that

$$A(\cdot,\bar{X},Z)=Y, dt\times P-a.e..$$

For $v\in L^\infty_{\mathcal{F}}(\Omega\times[0,T];V)$ we define

$$r_t:=\int_0^t\rho(v_s)+\rho^2(v_s)ds.$$

Applying the Itô's formula we have

$$\begin{aligned}
&E[e^{r_t}\|X_t^N\|_H^2-e^{r_T}\|X_T^N\|_H^2]\\
=&E[\int_t^T e^{r_s}(2\langle P_N A(s,X_s^N,Z_s^N),X_s^N\rangle_H-\|Z_s^N\|^2_{L_2(U,H)}\\
&-(\rho(v_s)+\rho^2(v_s))\|X_s^N\|_H^2)ds]\\
=&E[\int_t^T e^{r_s}(2_{V^*}\langle A(s,X_s^N,Z_s^N),X_s^N\rangle_V-\|Z_s^N\|^2_{L_2(U,H)}\\
&-(\rho(v_s)+\rho^2(v_s))\|X_s^N\|_H^2)ds]\\
=&E[\int_t^T e^{r_s}(2_{V^*}\langle A(s,X_s^N,Z_s^N)-A(s,v_s,Z_s),X_s^N-v_s\rangle_V\\
&-\|Z_s^N-Z_s\|^2_{L_2(U,H)}-(\rho(v_s)+\rho^2(v_s))\|X_s^N-v_s\|_H^2)ds]\\
&+E[\int_t^T e^{r_s}(2_{V^*}\langle A(s,X_s^N,Z_s^N)-A(s,v_s,Z_s),v_s\rangle_V\\
&+2_{V^*}\langle A(s,v_s,Z_s),X_s^N\rangle_V-2\langle Z_s^N,Z_s\rangle_{L_2(U,H)}+\|Z_s\|_{L_2(U,H)}\\
&-(\rho(v_s)+\rho^2(v_s))2\langle X_s^N,v_s\rangle_H-\|v_s\|_H^2)ds]\\
\le&E[\int_t^T e^{r_s}(2_{V^*}\langle A(s,X_s^N,Z_s^N)-A(s,v_s,Z_s),v_s\rangle_V\\
&+2_{V^*}\langle A(s,v_s,Z_s),X_s^N\rangle_V-2\langle Z_s^N,Z_s\rangle_{L_2(U,H)}+\|Z_s\|^2_{L_2(U,H)}\\
&-(\rho(v_s)+\rho^2(v_s))(2\langle X_s^N,v_s\rangle_H-\|v_s\|_H^2))ds].
\end{aligned}$$

Letting $N \to \infty$, by (H2) and the lower semicontinuity, we have for any nonnegative $\psi \in L^\infty([0,T])$,

$$
\begin{aligned}
&E[\int_0^T \psi_t(e^{r_t}\|X_t\|_H^2 - e^{r_T}\|X_T\|_H^2)dt] \\
\le &E\int_0^T [\int_t^T e^{r_s}(2{}_{V^*}\langle Y_s - A(s,v_s,Z_s), v_s\rangle_V \\
&+ 2{}_{V^*}\langle A(s,v_s,Z_s), X_s\rangle_V - 2\langle Z_s, Z_s\rangle_{L_2(U,H)} + \|Z_s\|^2_{L_2(U,H)} \\
&- (\rho(v_s)+\rho^2(v_s))(2\langle X_s, v_s\rangle_H - \|v_s\|_H^2))ds]dt.
\end{aligned}
\tag{2.7}
$$

By Itô's formula we have

$$
\begin{aligned}
&E[e^{r_t}\|X_t\|_H^2 - e^{r_T}\|X_T\|_H^2] \\
=&E[\int_t^T e^{r_s}(2{}_{V^*}\langle Y_s, X_s\rangle_V - \|Z_s\|^2_{L_2(U,H)} - (\rho(v_s)+\rho^2(v_s))\|X_s\|_H^2)ds].
\end{aligned}
\tag{2.8}
$$

Combining (2.7) with (2.8) we obtain that

$$
E[\int_0^T \psi_t \int_t^T e^{r_s}(2{}_{V^*}\langle Y_s - A(s,v_s,Z_s), X_s - v_s\rangle_V - (\rho(v_s)+\rho^2(v_s))(\|X_s - v_s\|_H^2))dsdt] \le 0.
\tag{2.9}
$$

Taking $v = X - \varepsilon\phi w$ for $\varepsilon > 0$ and $\phi \in L^\infty_{\mathcal{F}}(\Omega\times[0,T]; dt\times P;\mathbb{R})$ and $w \in V$. Then we divide by ε and letting $\varepsilon \to 0$ to derive that

$$
E[\int_0^T \psi_t \int_t^T e^{r_s} 2{}_{V^*}\langle Y_s - A(s,X_s,Z_s), w\rangle_V] \le 0.
\tag{2.10}
$$

Then $Y = A(\cdot, X, Z)$ follows from the arbitrariness of ψ and w.

[Uniqueness] Suppose that (X_1, Z_1) and (X_2, Z_2) are two solutions of the problem (2.1). In the proof of uniqueness we use $X(t)$ to denote X_t. Denote $(\tilde X, \tilde Z) := (X_1 - X_2, Z_1 - Z_2)$. Define

$$
r_1(t) := 2\int_0^t \rho(X_2(s)) + \rho^2(X_2(s))ds.
$$

An application of Itô's formula yields that

$$
\begin{aligned}
e^{r_1(t)}\|\tilde X(t)\|_H^2 = &\int_t^T e^{r_1(s)}[2{}_{V^*}\langle A(s,X_1(s),Z_1(s)) - A(s,X_2(s),Z_2(s)), \tilde X(s)\rangle_V \\
&- \|\tilde Z\|^2_{L_2(U,H)} - 2\|\tilde X(s)\|_H^2(\rho(X_2(s)) + \rho^2(X_2(s)))]ds \\
&- 2\int_t^T e^{r_1(s)}\langle \tilde X(s), \tilde Z dW_s\rangle_H \\
\le &\int_t^T e^{r_1(s)}[2\|\tilde X(s)\|_H^2\rho(X_2(s)) + 2\|\tilde X(s)\|\rho(X_2(s))\|\tilde Z(s)\|_{L_2(U,H)} \\
&- \|\tilde Z\|^2_{L_2(U,H)} - 2\|\tilde X(s)\|_H^2(\rho(X_2(s)) + \rho^2(X_2(s)))]ds \\
&- 2\int_t^T e^{r(s)}\langle \tilde X(s), \tilde Z dW_s\rangle_H
\end{aligned}
$$

Taking conditional expectations on both sides we have for any $t \in [0, T]$,

$$e^{r_1(t)}\|\tilde{X}(t)\|_H^2 + \frac{1}{2}E_{\mathcal{F}_t}[\int_t^T e^{r_1(s)}\|\tilde{Z}\|_{L_2(U,H)}^2 ds] \leq 0, a.s.,$$

which implies the uniqueness. □

3 Applications

Let Λ be a bounded domain in $\mathbb{R}^d$ with sufficiently smooth boundary and $C_0^\infty(\Lambda, \mathbb{R}^d)$ denote the set of all smooth functions from Λ to $\mathbb{R}^d$ with compact support. For $p \geq 1$, let $\big(L^p(\Lambda, \mathbb{R}^d), \|\cdot\|_{L^p}\big)$ be the vector valued L^p-space. For any integer $m \geq 0$, let $W_0^{m,2}$ denote the standard Sobolev space on Λ with values in $\mathbb{R}^d$, *i.e.* the closure of $C_0^\infty(\Lambda, \mathbb{R}^d)$ with respect to the following norm:

$$\|u\|_{W_0^{m,2}}^2 = \left(\sum_{0\leq|\alpha|\leq m}\int_\Lambda |D^\alpha u|^2 \mathrm{d}x\right)^2.$$

For the reader's convenience, we recall the following Gagliardo-Nirenberg interpolation inequality, which is used very often in the study of PDE theory.

Lemma 3.1. *If $q \in [1, \infty]$ such that*

$$\frac{1}{q} = \frac{1}{2} - \frac{m\gamma}{d}, \ 0 \leq \gamma \leq 1,$$

then there exists a constant $C_{m,q} > 0$ such that for any $u \in W_0^{m,2}$,

$$\|u\|_{L^q} \leq C_{m,q}\|u\|_{W_0^{m,2}}^\gamma \|u\|_{L^2}^{1-\gamma}. \tag{3.1}$$

Now we define

$$H^m := \left\{u \in W_0^{m,2} : \ div(u) = 0\right\}.$$

The norm of $W_0^{m,2}$ restricted to H^m will be denoted by $\|\cdot\|_{H^m}$. Note that H^0 is a closed linear subspace of the Hilbert space $L^2(\Lambda, \mathbb{R}^3)$.

For all the examples in below, $\{W_t\}_{t\geq 0}$ denotes a cylindrical Wiener process on a separable Hilbert space $(U, \langle\cdot,\cdot\rangle_U)$ *w.r.t* a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$.

3.1 Backward stochastic (generalized) curve shortening flow and backward singular stochastic p-Laplace equations

The study of the motion by mean curvature of curves and surfaces attracts more and more attentions in recent years. It not only connects to many interesting mathematical theories such as nonlinear PDEs, geometric measure theory, asymptotic analysis and singular perturbations, but also has important applications in image processing and materials science etc (cf.[35, 47]). The incorporation of stochastic perturbations has also been widely used in these models, where the noise can come from the thermal fluctuations, impurities and the atomistic processes describing the surface motions. However, the mathematical theory for the study of those stochastic models are quite incomplete (cf.[10] and the references therein).

The stochastic curve shortening flow (cf.[10, 11]) is formulated in the following form:

$$dX_t = \frac{\partial_x^2 X_t}{1+(\partial_x X_t)^2}dt + \sigma(X_t)dW_t,$$

where ∂_x, ∂_x^2 denote the first and second (spatial) derivative, and σ satisfies some suitable conditions.

The deterministic part is a simplified model in geometric PDE theory which describes the motion by mean curvature of embedded surfaces (in the present model the surface is just some curve in the 2-dimensional plane), we refer to [10] for more detailed exposition on the model. The random forcing was introduced to refine the model by taking the influence of thermal noise into account.

Based on the crucial observation

$$\frac{\partial_x^2 X_t}{1+(\partial_x X_t)^2} = \partial_x\left(\arctan(\partial_x X_t)\right),$$

this equation has been investigated in [10, 11] using the variational framework with following Gelfand triple:

$$V := W_0^{1,2}([0,1]) \subseteq H := L^2([0,1]) \subseteq V^* = W^{-1,2}([0,1]).$$

The first example here is the equation of backward stochastic curve shortening flow, and we consider the following form of BSPDE, which covers a large class of stochastic evolution equations such as stochastic curve shortening flow (with some nonlinear perturbations), stochastic p-Laplace equations and stochastic reaction-diffusion equations. For simplicity we only formulate the result for 1 dimensional underlying domain $[0,1]$ here.

$$dX_t = -\left[\partial_x\left(\bar{f}(\partial_x X_t)\right) + g(X_t) + h(t, X_t, Z_t)\right]dt + Z_t dW_t,\ X_T = \xi. \tag{3.2}$$

Example 3.2. *Suppose that functions $\bar{f}, g \in C^1(\mathbb{R})$ and there exist constants $C, p \geq 2$ such that*

$$\begin{aligned}
&\bar{f}'(x) \geq 0,\ |\bar{f}(x)| \leq C(1+|x|),\ x \in \mathbb{R};\\
&g'(x) \leq C,\ |g(x)| \leq C(1+|x|^{p-1}),\ x \in \mathbb{R};\\
&(g(x)-g(y))(x-y) \leq C(1+|y|^p)|x-y|^2,\ x, y \in \mathbb{R},
\end{aligned} \tag{3.3}$$

and $h : [0,T] \times V \times L_2(U,H) \times \Omega \to V^*$ satisfies (H0)-(H4). Then for any $\xi \in L^\infty_{\mathcal{F}_T}(\Omega; V)$, (3.2) admits a unique adapted solution $(X,Z) \in L^\infty_{\mathcal{F}}(\Omega\times[0,T]; V)\times L^2_{\mathcal{F}}(\Omega; L^2([0,T], L_2(U,H)))$. Moreover, it satisfies that

$$\sup_{t\in[0,T]} \|X_t\|_V^2 + \frac{1}{2}E\int_0^T \|Z_s\|^2_{L_2(U,V)}ds \leq C(1+\|f\|_{L^\infty_{\mathcal{F}}(\Omega;L^1([0,T]))} + \|\xi\|^2_{L^\infty_{\mathcal{F}_T}(\Omega;V)}), \quad a.s..$$

Proof We consider the following Gelfand triple:

$$V := W_0^{1,2}([0,1]) \subseteq H := L^2([0,1]) \subseteq V^* = W^{-1,2}([0,1]).$$

$(H0)$ holds since all eigenvectors $\{e_i, i = 1, 2, \cdots\}$ of the Laplace operator constitute an orthonormal basis of H and an orthogonal set in V.

By the assumptions on $\bar{f}$ we have

$$\langle \partial_x(\bar{f}(\partial_x v)), v\rangle_V = -\int_0^1 \bar{f}'(\partial_x v)\left(\partial_x^2 v\right)^2 dx \le 0, \ v \in H_n \subseteq V;$$

$$\|\partial_x(\bar{f}(\partial_x v))\|_{V^*} \le \|(\bar{f}(\partial_x v))\|_H \le C(1 + \|v\|_V), \ v \in V;$$

$${}_{V^*}\langle \partial_x(\bar{f}(\partial_x u)) - \partial_x(\bar{f}(\partial_x u)), u - v\rangle_V = -\int_0^1 \left(\bar{f}(\partial_x u) - \bar{f}(\partial_x u)\right)(\partial_x u - \partial_x v)\, dx \le 0, \ u, v \in V.$$

We now show that $(H1)$-$(H4)$ hold for the term g in the drift. By the continuity of g and dominated convergence theorem it is easy to show that $(H1)$ holds.

By (3.3) and Sobolev's inequality we have

$$\begin{aligned}
&{}_{V^*}\langle g(u) - g(v), u - v\rangle_V \\
=&\int_0^1 (g(u) - g(v))(u - v)dx \\
\le&C(1 + \|v\|_{L^\infty}^p)\int_0^1 |u - v|^2 dx \\
\le&C(1 + \|v\|_V^p)\|u - v\|_H^2, \ u, v \in V,
\end{aligned}$$

i.e. $(H2)$ holds with $\rho(v) = \|v\|_V^p$.

$(H3)$ also holds since (3.3) implies that

$$\langle g(v), v\rangle_V = -\langle g(v), \partial_x^2 v\rangle_H = \int_0^1 g'(v)(\partial_x v)^2 dx \le C\|v\|_V^2, \ v \in H_n \subseteq V.$$

$(H4)$ follows from the following estimate:

$$\|g(v)\|_{V^*} \le C\|g(v)\|_{L^1} \le C(1 + \|v\|_{L^\infty}^{p-1}) \le C(1 + \|v\|_V^{p-1}), \ v \in V.$$

Then by the assumptions of h, it is easy to show that $(H1)$-$(H4)$ hold for the term $\partial_x(\bar{f}(\partial_x X_t)) + g(X_t) + h(t, X_t, Z_t)$. Therefore, the conclusion follows from Theorem 2.2.

Remark 3.3. *(1) If we take $\bar{f}(x) = \arctan x$ and $g \equiv 0$, then (3.2) reduces back to the model of backward stochastic curve shortening flow.*

(2) The simple example of g satisfying (3.3) is any polynomial of odd degree with negative leading coefficients. Hence (3.2) also covers backward stochastic reaction-diffusion equations (i.e. $\bar{f}(x) = x$).

(3) If $\bar{f}(x) = |x|^{p-2}x (1 < p \le 2)$, then (3.2) covers the singular backward stochastic p-Laplace equations.

(3) If $\bar{f}(x) = |x|^{p-2}x (p > 2)$, then (3.2) reduces to the degenerate backward stochastic p-Laplace equations and the result above can not be applied to this case.

3.2 Backward stochastic fast diffusion equations

Let Λ be a bounded open domain in $\mathbb{R}^d$ with smooth boundary and Δ be the standard Laplace operator with Dirichlet boundary condition. Stochastic fast diffusion equations with general multiplicative noise has been studied a lot in recent years (see e.g. [20, 30, 15]). In this work, we consider the following backward stochastic fast diffusion equations:

$$dX_t = -\left(\Delta\Psi(X_t) + h(t, X_t, Z_t)\right) dt + Z_t dW_t, \ X_T = \xi, \tag{3.4}$$

where $\Psi : \mathbb{R} \to \mathbb{R}$ is measurable. In particular, if $\Psi(s) = s^r := |s|^{r-1}s$ for some $r \in (0, 1)$, then (3.4) reduces back to the classical backward stochastic fast diffusion equations.

Using the Gelfand triple

$$V := L^2(\Lambda) \subseteq H := W^{-1,2}(\Lambda) \subseteq V^* = (L^2(\Lambda))^*,$$

we obtain the following well-posedness result for equation (3.4).

Example 3.4. *Suppose that $h : [0, T] \times V \times L_2(U, H) \times \Omega \to V^*$ satisfies (H0)-(H4), $\Psi \in C^1(\mathbb{R})$ and there exists a constant $C > 0$ such that*

$$\Psi'(x) \geq 0, \ |\Psi(x)| \leq C(1 + |x|), \ x \in \mathbb{R}.$$

Then for any $\xi \in L^\infty_{\mathcal{F}_T}(\Omega; V)$, (3.4) has a unique adapted solution $(X, Z) \in L^\infty_{\mathcal{F}}(\Omega\times[0, T]; V)\times L^2_{\mathcal{F}}(\Omega; L^2([0, T], L_2(U, H)))$. In particular, we have

$$\sup_{t\in[0,T]} \|X_t\|^2_V + \frac{1}{2}E\int_0^T \|Z_s\|^2_{L_2(U,V)}ds \leq C(1 + \|f\|_{L^\infty_{\mathcal{F}}(\Omega;L^1([0,T]))} + \|\xi\|^2_{L^\infty_{\mathcal{F}_T}(\Omega;V)}), \quad a.s..$$

Proof According to the classical result for (3.4) (cf. [27, Example 4.1.11]), here we only need to verify the one-sided linear growth condition $(H3)$ for (3.4). In fact, we have

$$\langle\Delta\Psi(v), v\rangle_V = -\int_\Lambda \Psi'(v)|\nabla v|^2dx \leq 0, \ v \in H^n.$$

Therefore, the assertions follow directly from Theorem 2.6.

Remark 3.5. *(1) If $\Psi(x) = |x|^{r-1}x(0 < r < 1)$, then (3.4) reduces to the backward stochastic fast diffusion equations.*

(2) If $\Psi(x) = |x|^{r-1}x(r > 1)$, then (3.4) is the backward stochastic porous medium equations, and the result above can not be applied to this case.

3.3 Backward stochastic Burgers type and reaction-diffusion equations

The main result in this paper is also applicable to semilinear type BSPDE which is formulated as follows:

$$dX_t = -\left(\partial_x^2 X_t + \bar{f}(X_t)\partial_x X_t + g(X_t) + h(t, X_t, Z_t)\right) dt + Z_t dW_t, \quad X_T = \xi. \tag{3.5}$$

Consider the Gelfand triple

$$V := W_0^{1,2}([0,1]) \subseteq H := L^2([0,1]) \subseteq V^* = W^{-1,2}([0,1]),$$

we have the following result concerning the well-posedness of equation (3.5).

Example 3.6. *Suppose that $\bar{f}$ is a bounded Lipschitz function on $\mathbb{R}$ and $g \in C^1(\mathbb{R})$ and there exists constants $C, p \geq 2$ such that*

$$\begin{aligned} &(g(x) - g(y))(x - y) \leq C(1 + |y|^p)|x - y|^2, \ x, y \in \mathbb{R}; \\ &|g(x)| \leq C(1 + |x|^{p-1}), \quad x \in \mathbb{R}; \\ &g'(x) \leq C, \ x \in \mathbb{R}, \end{aligned}$$

and $h : [0,T] \times V \times L_2(U,H) \times \Omega \to V^*$ satisfies (H0)-(H4). Then for any $\xi \in L^\infty_{\mathcal{F}_T}(\Omega; V)$, (3.5) has a unique adapted solution $(X, Z) \in L^\infty_{\mathcal{F}}(\Omega \times [0,T]; V) \times L^2_{\mathcal{F}}(\Omega; L^2([0,T], L_2(U,H)))$. In particular, we have

$$\sup_{t \in [0,T]} \|X_t\|_V^2 + \frac{1}{2} E \int_0^T \|Z_s\|^2_{L_2(U,V)} ds \leq C(1 + \|f\|_{L^\infty_{\mathcal{F}}(\Omega; L^1([0,T]))} + \|\xi\|^2_{L^\infty_{\mathcal{F}_T}(\Omega;V)}), \quad a.s..$$

Proof Combining with the result in the previous example, here we only need to show $(H1)$-$(H4)$ hold for the term $\partial_x^2 + \bar{f}(\cdot)\partial_x$.

According to the result showed in [18, Example 3.2], $(H1), (H2)$ and $(H4)$ hold.

Since $\bar{f}$ is bounded, by Hölder's inequality and Young's inequality we have

$$\begin{aligned} &\langle \partial_x^2 v + \bar{f}(v)\partial_x v, v \rangle_V \\ = &- \langle \partial_x^2 v + \bar{f}(v)\partial_x v, \partial_x^2 v \rangle_H \\ = &- \|\partial_x^2 v\|^2_{L^2} - \int_0^1 \bar{f}(v)\partial_x v \partial_x^2 v dx \\ \leq &- \|\partial_x^2 v\|^2_{L^2} + C\|\partial_x^2 v\|_{L^2}\|v\|_V \\ \leq &- \frac{1}{2}\|\partial_x^2 v\|^2_{L^2} + C\|v\|_V^2, \ v \in H_n \subseteq V, \end{aligned}$$

i.e. $(H3)$ also holds.

Therefore, the assertion follows from Theorem 2.2.

Remark 3.7. *(1) If we take $\bar{f} = 0$ and $g(x) = \sum_{i=0}^{2n+1} a_i x^i$ with $a_{2n+1} < 0$ (for some fixed $n \in \mathbb{N}$), then (3.5) reduces to the classical backward stochastic reaction-diffusion equations.*

(2) If $g = 0$, then (3.5) covers the backward stochastic Burgers type equations.

3.4 Backward stochastic tamed 3D Navier-Stokes equation

The last example is a tamed version of backward stochastic 3D Navier-Stokes equation. Stochastic tamed 3D Navier-Stokes equation has been investigated in a series of works of

Röckner et al [31, 32, 33, 34]. The classical 3D Navier-Stokes equations (i.e. $g_N = 0,\ B = 0$ in 3.6) is a standard model to describe the evolution of velocity fields of an incompressible fluid (cf.[12, 16, 40]), the uniqueness and regularity of weak solutions are still open problems up to now.

The authors in [29, 36] have studied the backward stochastic 2D Navier-Stokes equation. To the best of our knowledge, there is no result about backward stochastic 3D Navier-Stokes equation, the backward stochastic tamed 3D Navier-Stokes equation can be viewed as a regularized version of the classical backward stochastic 3D Navier-Stokes equation and it can be formulated as follows:

$$\begin{aligned} &dX_t = -\left[\nu\Delta X_t - (X_t\cdot\nabla)X_t + \nabla p(t) - g_N\left(|X_t|^2\right)X_t + h(t,X_t,Z_t)\right]dt + Z_t dW_t,\\ &div(X_t) = 0, \quad X_T = \xi,\\ &X_t|_{\partial\Lambda} = 0, \end{aligned} \tag{3.6}$$

where $\nu > 0$ is the viscosity constant, p is the (unknown) pressure and the taming function $g_N : \mathbb{R}_+ \to \mathbb{R}_+$ is smooth and satisfies for some $N > 0$,

$$\begin{cases} g_N(r) = 0, & \text{if } r \le N,\\ g_N(r) = (r-N)/\nu, & \text{if } r \ge N+1,\\ 0 \le g_N'(r) \le C, & r \ge 0. \end{cases}$$

The main feature of (3.6) is that if there is a bounded smooth solution to the backward (stochastic) 3D Navier-Stokes equation, then this smooth solution must also satisfy this backward tamed equation for some large enough N.

Let $\mathcal{P}$ be the orthogonal (Helmhotz-Leray) projection from $L^2(\Lambda,\mathbb{R}^3)$ to H^0 (cf.[40, 16, 12]). For any $u \in H^0$ and $v \in L^2(\Lambda,\mathbb{R}^3)$ we have

$$\langle u, v\rangle_{H^0} := \langle u, \mathcal{P}v\rangle_{H^0} = \langle u, v\rangle_{L^2}.$$

We consider the following Gelfand triple:

$$V := H^1 \subseteq H := H^0 \subseteq V^* = (H^1)^*,$$

then it is well known that the following operators

$$A : W^{2,2}(\Lambda,\mathbb{R}^3)\cap V \to H,\ Au = \nu\mathcal{P}\Delta u;$$

$$F : \mathcal{D}_{\mathcal{F}} \subset H\times V \to H;\ F(u,v) = -\mathcal{P}\left[(u\cdot\nabla)v\right],\ F(u) := F(u,u)$$

can be extended to the following well defined operators:

$$A : V \to V^*;\ F : V\times V \to V^*.$$

Moreover, we have

$${}_{V^*}\langle F(u,v), w\rangle_V = -{}_{V^*}\langle F(u,w), v\rangle_V,\ {}_{V^*}\langle F(u,v), v\rangle_V = 0,\ u,v,w \in V. \tag{3.7}$$

Without loss of generality we may assume $\nu = 1$. Now we show the existence and uniqueness of solutions to (3.6).

Example 3.8. *Suppose $\xi \in L^\infty_{\mathcal{F}_T}(\Omega;V)$ and $h : [0,T] \times V \times L_2(U,H) \times \Omega \to V^*$ satisfies (H0)-(H4), then (3.6) has a unique adapted solution $(X,Z) \in L^\infty_{\mathcal{F}}(\Omega \times [0,T];V) \times L^2_{\mathcal{F}}(\Omega; L^2([0,T], L_2(U,H)))$. Moreover, it satisfies that*

$$\sup_{t\in[0,T]} \|X_t\|_V^2 + \frac{1}{2} E \int_0^T \|Z_s\|^2_{L_2(U,V)} ds \le C(1 + \|f\|_{L^\infty_{\mathcal{F}}(\Omega;L^1([0,T]))} + \|\xi\|^2_{L^\infty_{\mathcal{F}_T}(\Omega;V)}), \quad a.s..$$

Proof It is well known that (3.6) can be rewritten into the following variational form:

$$dX_t = -\left[AX_t + F(X_t) - \mathcal{P}\left(g_N\left(|X_t|^2\right)X_t\right) + h(t,X_t,Z_t)\right]dt + Z_t dW_t,\ X_T = \xi.$$

It is easy to see that all eigenvectors $\{e_i, i = 1,2,\cdots\} \subset H^2$ of A constitute an orthonormal basis of H^0 and an orthogonal set in H^1, i.e. $(H0)$ holds.

By Hölder's inequality we have the following estimate:

$$\|\psi\|_{L^3(\Lambda;\mathbb{R}^3)} \le \|\psi\|^{1/2}_{L^2(\Lambda;\mathbb{R}^3)} \|\psi\|^{1/2}_{L^6(\Lambda;\mathbb{R}^3)},\ \psi \in L^6(\Lambda;\mathbb{R}^3).$$

Note that $W^{1,2}_0(\Lambda;\mathbb{R}^3) \subseteq L^6(\Lambda;\mathbb{R}^3)$, then by (3.7) one can show that

$$\begin{aligned}
&{}_{V^*}\langle F(u) - F(v), u - v\rangle_V \\
&= -{}_{V^*}\langle F(u-v), v\rangle_V \\
&\le C\|u-v\|_V \|u-v\|_{L^3(\Lambda;\mathbb{R}^3)} \|v\|_{L^6(\Lambda;\mathbb{R}^3)} \\
&\le C\|u-v\|_V^{3/2} \|u-v\|_H^{1/2} \|v\|_{L^6(\Lambda;\mathbb{R}^3)} \\
&\le \frac{1}{2}\|u-v\|_V^2 + C\|v\|^4_{L^6(\Lambda;\mathbb{R}^3)} \|u-v\|_H^2,\ u,v \in V.
\end{aligned}$$

Hence we have the following estimate (recall that $\nu = 1$):

$${}_{V^*}\langle Au + F(u) - Av - F(v), u - v\rangle_V \le -\frac{1}{2}\|u-v\|_V^2 + C\left(1 + \|v\|^4_{L^6(\Lambda;\mathbb{R}^3)}\right)\|u-v\|_H^2.$$

By the definition of g_N and (3.1) we have

$$\begin{aligned}
&-{}_{V^*}\langle \mathcal{P}(g_N(|u|^2)u) - \mathcal{P}(g_N(|v|^2)v), u - v\rangle_V \\
=&-\langle g_N(|v|^2)(u-v), u-v\rangle_H + \langle (g_N(|v|^2) - g_N(|u|^2))u, u-v\rangle_H \\
\le& \int_{\{|u|>|v|\}} (g_N(|v|^2) - g_N(|u|^2))(|u|^2 - u\cdot v)dx \\
&+ \int_{\{|u|\le|v|\}} (g_N(|v|^2) - g_N(|u|^2))(|u|^2 - u\cdot v)dx \\
\le& C\int_{\{|u|\le|v|\}} \left||v|^2 - |u|^2\right| \cdot |u| \cdot |u-v| dx \\
\le& C\int_{\{|u|\le|v|\}} |u|^2 \cdot |u-v|^2 dx \\
\le& C\|v\|^2_{L^6(\Lambda;\mathbb{R}^3)} \|u-v\|^2_{L^3(\Lambda;\mathbb{R}^3)} \\
\le& C\|v\|^2_{L^6(\Lambda;\mathbb{R}^3)} \|u-v\|_H \|u-v\|_V \\
\le& \frac{1}{4}\|u-v\|_V^2 + C\|v\|^4_{L^6(\Lambda;\mathbb{R}^3)} \|u-v\|_H^2,\ u,v \in V.
\end{aligned}$$

Hence $(H2)$ holds with $\rho(v) = C\|v\|^4_{L^6(\Lambda;\mathbb{R}^3)}$.

We recall the following estimate for $v \in span\{e_1, e_1, \cdots, e_n\}$ (cf.[31, Lemma 2.3]):

$$\begin{aligned}
\langle Av, v\rangle_V &= \langle \mathcal{P}\Delta v, (I-\Delta)v\rangle_H \le -\|v\|^2_{H^2} + \|v\|^2_V;\\
\langle F(v), v\rangle_V &= -\langle \mathcal{P}\left[(v\cdot\nabla)v\right], (I-\Delta)v\rangle_H \le \frac{1}{2}\|v\|^2_{H^2} + \frac{1}{2}\||v|\cdot|\nabla v|\|^2_H;\\
-\langle \mathcal{P}\left(g_N(|v|^2)v\right), v\rangle_V &= -\langle \mathcal{P}\left(g_N(|v|^2)v\right), (I-\Delta)v\rangle_H \le -\||v|\cdot|\nabla v|\|^2_H + CN\|v\|^2_V.
\end{aligned}\tag{3.8}$$

Then it is easy to verify $(H3)$ as follows:

$$\langle Av + F(v) - \mathcal{P}(g_N(|v|^2)v), v\rangle_V \le -\frac{1}{2}\|v\|^2_{H^2} + C(N+1)\|v\|^2_V,\ v \in span\{e_1, e_1, \cdots, e_n\}.$$

Concerning the growth condition, we have that

$$\|F(v)\|_{V^*} \le C\|v\|^2_{L^4(\Lambda;\mathbb{R}^3)} \le C\|v\|^2_V,\ v \in V.$$

By (3.1) we have

$$\|g_N(|v|^2)v\|^2_{V^*} \le C\|v\|^2_{L^6(\Lambda;\mathbb{R}^3)} \le C\|v\|^2_V, v \in V.$$

Hence we know that $(H4)$.

Then the existence of a unique solution to (3.6) follows from Theorem 2.2.

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