Reflected backward stochastic differential equation driven by G-Brownian motion with an upper obstacle

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Abstract

In this paper, we study the reflected backward stochastic differential equation driven by G-Brownian motion (reflected G-BSDE for short) with an upper obstacle. The existence is proved by approximation via penalization. By using a variant comparison theorem, we show that the solution we constructed is the largest one.

Key words: G-expectation, reflected backward SDEs, upper obstacle
MSC-classification: 60H10, 60H30

1 Introduction

Linear backward stochastic differential equations (BSDEs for short) were initiated by Bismut [2]. Then Pardoux and Peng [25] studied the general nonlinear case. Roughly speaking, on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$ generated by a Brownian motion B, a solution to a BSDE is a couple (Y, Z) of progressively measurable processes satisfying:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s,\ 0 \leq t \leq T,\ \mathbb{P}\text{-}a.s.,$$

where the generator is progressively measurable and the terminal value ξ is an $\mathcal{F}_T$-measurable random variable. Pardoux and Peng obtained the existence and uniqueness of the above equation when f is uniformly Lipschitz and both f and ξ are square integrable. Because it can be widely applied in mathematical finance, stochastic control, stochastic differential games and probabilistic method for partial differential equations, the BSDE theory has attracted considerable attention.

Reflected backward stochastic differential equations (RBSDEs for short) were firstly studied by El Karoui, Kapoudjian, Pardoux, Peng and Quenez [6]. The solution Y of RBSDE is required to be above a given continuous process S so that an additional non-decreasing process should be added in the equation. This non-decreasing process should satisfy the Skorohod condition, which ensures that it behaves in a minimal way, i.e., it only acts when Y reaches the obstacle S. This theory provides a useful method for pricing American contingent claims, see [7]. It also gives a probabilistic representation for the solution of an obstacle problem for nonlinear parabolic PDE, which establishes the connection with variational equalities, see [1] and [6].

Motivated by probabilistic interpretation for fully nonlinear PDEs and applications in financial markets in the uncertainty volatility model (UVM for short), Peng [26, 27] systemically established

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a time-consistent fully nonlinear expectation theory. As a typical case, Peng introduced the G-expectation (see [30] and the reference therein). Under G-expectation framework, a new type of Brownian motion $\{B_t\}_{t\geq 0}$, called G-Brownian motion, was constructed. Different from the classical case, its quadratic variation process $\langle B\rangle$ is not deterministic. The stochastic integrals with respect to B and $\langle B\rangle$ were also established. Similar with the classical SDE theory, the existence and uniqueness of solution of a stochastic differential equation driven by G-Brownian motion (G-SDE) can be proved by the contracting mapping theorem. The challenging and fascinating problem of wellposedness for BSDE driven by G-Brownian motion has been solved by Hu et al. [10]. In their paper, they showed that there exists a unique triple (Y, Z, K) in proper Banach spaces satisfying the following equation:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\langle B\rangle_s - \int_t^T Z_s dB_s - (K_T - K_t).$$

In the accompanying paper [11], the comparison theorem, nonlinear Feymann-Kac formula and Girsanov transformation were established. We should point out that the equation above holds P-$a.s.$ for every probability measure P that belongs to a non-dominated class of mutually singular measures. Therefore, the G-BSDE is highly related to the second order BSDEs (2BSDEs for short) developed by Cheridito, Soner, Touzi and Victoir [3] and Soner, Touzi and Zhang [32]. It is worth pointing out that the the terminal value ξ and the generators (f, g) in G-BSDEs should be quasi-continuous in ω and in (t, ω) repsectively. However, it does not mean that they are uniformly continuous. For example, if φ is a Borel measurable function with polynomial growth, then the process $\{\varphi(B_t)\}_{t\in[0,T]}$ satisfies the quasi-continuous property (for more examples, we may refer to the paper [14]). The advantage of studying BSDE in the G-framework is that the solution (Y, Z, K) is universally defined and the processes have strong regularity property.

In the past two decades, a great deal of effort have been devoted to the study of various types of RBSDEs. Cvitanic and Karaztas [4] and Hamadene and Lepeltier [9] generalized the results above to the case of two reflecting obstacles and established the connection between this problem and Dynkin games. Hamadene [8] and Lepeltier and Xu [15] gave a generalized Skorohod condition and obtained a wellposedness theory when the obstacle process has càdlàg paths.

Recently, Li, Peng and Soumana Hima [18] introduced the notion of reflected G-BSDE with a lower obstacle. In order to make sure that the solution Y can be pushed upward so that it is above the given continuous process S, called lower obstacle, a non-decreasing process L will be added in this equation. Due to the appearance of the non-increasing G-martingale K in G-BSDE, if we expect the solution to reflected G-BSDE with a lower obstacle is a 4-tuples of processes (Y, Z, K, L), the solution is not unique (we may refer to Remark 3.7 in [18]). To overcome this shortcoming, we need to put the two processes K and L together as a non-decreasing process A, i.e., $A = L - K$. Since L is minimal in the sense that it satisfies the Skorohod condition, for any t, we have

$$-\int_0^t (Y_s - S_s)dA_s = \int_0^t (Y_s - S_s)dK_s.$$

It follows from the fact $Y \geq S$ that the process $\{-\int_0^t (Y_s - S_s)dA_s\}$ is a non-increasing G-martingale. Therefore, we will use the martingale condition instead of the Skorohod condition. Then, the uniqueness can be derived from a priori estimates and we use the approximation via penalization to solve the existence. More precisely, consider the following G-BSDEs parameterized by $n = 1, 2, \cdots$,

$$\begin{aligned} Y_t^n =& \xi + \int_t^T f(s, Y_s^n, Z_s^n)ds + \int_t^T g(s, Y_s^n, Z_s^n)d\langle B\rangle_s \\ &- \int_t^T Z_s^n dB_s - (K_T^n - K_t^n) + (L_T^n - L_t^n), \end{aligned} \tag{1.1}$$

where $L_t^n = n\int_0^t (Y_s^n - S_s)^- ds$. We claim that the solution (Y, Z, A) of the reflected G-BSDE with parameters (ξ, f, g, S) is the limit of $(Y^n, Z^n, L^n - K^n)$. The proof of convergence in appropriate spaces becomes delicate and challenging. The difficulty lies in the fact that the Fatou's lemma cannot be directly and automatically used in this sublinear expectation framework. Besides, any bounded sequence in $M_G^\beta(0,T)$ is not weakly compact. It is worth mentioning that the uniformly continuous property of the elements in $S_G^p(0,T)$ plays a key role in overcoming this problem (see Lemma 4.3 in [18]).

In the classical situation, the solution (Y, Z, L) of reflected BSDE with terminal value ξ, generator f and upper obstacle S corresponds to $(-\tilde{Y}, -\tilde{Z}, -\tilde{L})$. Here $(\tilde{Y}, \tilde{Z}, \tilde{L})$ is the solution of reflected BSDE with data $(-\xi, \tilde{f}, -S)$, where $\tilde{f}(s, y, z) = -f(s, -y, -z)$. To obtain the existence result for reflected G-BSDE with an upper obstacle, applying the penalization method, we need to replace the non-decreasing process $\{L_t^n\}$ in the penalized G-BSDE (1.1) by a non-increasing one $\{\widetilde{L}_t^n\} = \{-n\int_0^t (Y_s^n - S_s)^+ ds\}$ such that the solution can be pulled downward to be below the given continuous obstacle process. Since there will be a non-increasing G-martingale $\{K_t^n\}$, these two cases are significantly different under the G-framework: $\{L_t^n - K_t^n\}$ is a non-decreasing process while $\{\widetilde{L}_t^n - K_t^n\}$ is a finite variation process. Due to the difficulity of the analysis caused by this finite variation process, we need to put stronger assumptions on the parameters of reflected G-BSDE with an upper obstacle. In the lower obstacle case, we prove the uniform bounded property of sequences $\{Y^n\}$, $\{L^n\}$, $\{K^n\}$ simultaneously by using G-Itô's formula and then get the uniform convergence of $\{(Y^n - S)^-\}$. However, for the upper obstacle case, we will show the rate of convergence of $\{(Y^n - S)^+\}$ in order to derive the uniform bounded property for $\{L^n\}$ and $\{K^n\}$ respectively. Furthermore, the solution to this problem by our construction is proved to be the largest one using a variant comparison theorem.

Due to the connection of G-BSDEs and 2BSDEs, our problem is closely related to the reflected 2BSDE theory. To our best knowledge, Matoussi, Possamaï and Zhou [21] first consider reflected BSDEs under a non-dominated family of probability measures $\mathcal{P}_H^\kappa$. In this paper, they only consider the lower obstacle case and the method cannot be applied to the upper obstacle case directly. Besides, in the original version of [21], the minimal condition for the non-decreasing process of the solution turns out to be wrong and it is corrected as a new one in [22] which relies on the solution of reflected BSDE under each probability $P \in \mathcal{P}_H^\kappa$ and a new process M generated by the parameters of the reflected 2BSDEs. In order to guarantee the existence of the solution, they need to put some restrictions on the oscillations or the variations of the obstacle process (see Assumption 2.1 in [22]). Compared with their results, [18] proposed a so-called martingale condition which can be easliy derived from the classical Skorohod condition. They obtained the existence result under the assumption about the uniform continuity of the obstacle process, which implies Assumption 2.1 in [22]. Due to the stronger regularity of the obstacle process, the solutions of reflected G-BSDEs also inherit stronger regularity and all the processes of the solutions are defined universally. This result can also be applied to the pricing problem of American options under Knightian uncertainty as well as the stochastic representation for the obstacle problem of fully nonlinear partial differential equations (PDEs).

Now let us go back to the upper obstacle case. In the second order BSDE framework, Matoussi, Piozin and Possamaï [20] consider a more general problem, namely the doubly reflected 2BSDEs, where the solution is required to stay between a lower obstacle L and an upper obstacle S. Intuitively, this problem will reduce to the single upper obstacle case by setting $L = -\infty$. As is pointed out in [21], there will be a non-increasing process to pull the solution downwards and a non-decreasing process due to the formulation of second order, which will end up with a finite variation process. In order to derive the uniqueness result, [22] proposed the following conditions: the process which aims to pull the solution downwards should satisfy the Skorohod condition while the finite variation process satisfies the minimal condition similar to the lower obstacle case. Then, applying some a prior estimates yields that the solution is unique. The proof of existence relies on the theory of regular conditional probability distributions and the Doob-Meyer decomposition of reflected nonlinear supermartingales. Roughly speaking, the solutions of reflected 2BSDEs are constructed by the essential supremum of the solutions to classical reflected BSDEs under P over all $P \in \mathcal{P}_H^\kappa$. Since there are infinitely many probabilities in $\mathcal{P}_H^\kappa$, the calculation may turn into a tedious task. Besides, it is worth noting that after the correction about the defintion of solutions (see Definition 3.1 in [22]), the proof of existence given in [20] only holds for uniformly bounded terminal values. Fortunately, applying some recently obtained results in [23] and [31], the regularity condition on the terminal value can be further relaxed. Compared with their results, the definition of solutions to reflected G-BSDE with an upper obstacle is more concise which has almost a one-to-one correspondence with the lower obstacle case except that the third component of the solution is a finite variation process. The proof of existence and uniqueness is simpler. Similar with the classical case, the existence can be derived by a penalization method. With the help of the variant comparison theorem, we can show that the solution by our construction is the maximal one, which implies that the solution is unique. In the G-framework, we need to assume that the upper obstacle is a G-Itô process which is stronger than the condition in [20]. The advantage is that the solutions can be defined universally. Although we do not indicate explicitly in this paper, applying a similar method as in [18], we can also establish a stochastic interpretation for the solution of the obstacle problems for fully nonlinear PDEs.

The rest of paper is organized as follows. Section 2 is devoted to listing some notations and results as preliminaries for the later proofs. In Section 3 we prove a variant comparison theorem for G-BSDEs. The problem is formulated in detail in Section 4 and we present the technics of approximation via penalization to prove the existence. Furthermore, we state that the solution by our construction is the largest one using the variant comparison theorem.

2 Preliminaries

The main purpose of this section is to recall some basic notions and results of G-expectation, which are needed in the sequel. The readers may refer to [10], [11], [28], [29], [30] for more details.

2.1 G-expectation

Definition 2.1 *Let Ω be a given set and let $\mathcal{H}$ be a vector lattice of real valued functions defined on Ω, namely $c \in \mathcal{H}$ for each constant c and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. $\mathcal{H}$ is considered as the space of random variables. A sublinear expectation $\hat{\mathbb{E}}$ on $\mathcal{H}$ is a functional $\hat{\mathbb{E}} : \mathcal{H} \to \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have*

(i) *Monotonicity: If $X \geq Y$, then $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$;*

(ii) *Constant preserving: $\hat{\mathbb{E}}[c] = c$;*

(iii) *Sub-additivity: $\hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$;*

(iv) *Positive homogeneity: $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$ for each $\lambda \geq 0$.*

Definition 2.2 *Let X_1 and X_2 be two n-dimensional random vectors defined respectively in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$. They are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$, if $\hat{\mathbb{E}}_1[\varphi(X_1)] = \hat{\mathbb{E}}_2[\varphi(X_2)]$, for all $\varphi \in C_{Lip}(\mathbb{R}^n)$, where $C_{Lip}(\mathbb{R}^n)$ is the space of real valued Lipschitz continuous functions defined on $\mathbb{R}^n$.*

Definition 2.3 *In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, a random vector $Y = (Y_1, \cdots, Y_n)$, $Y_i \in \mathcal{H}$, is said to be independent from another random vector $X = (X_1, \cdots, X_m)$, $X_i \in \mathcal{H}$ under $\hat{\mathbb{E}}[\cdot]$, denoted it by $Y \perp X$, if for every test function $\varphi \in C_{Lip}(\mathbb{R}^m \times \mathbb{R}^n)$ we have $\hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x, Y)]_{x=X}]$.*

Definition 2.4 *(G-normal distribution) A d-dimensional random vector $X = (X_1, \cdots, X_d)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called G-normally distributed if for each $a, b \geq 0$ we have*

$$aX + b\bar{X} \stackrel{d}{=} \sqrt{a^2 + b^2} X,$$

where $\bar{X}$ is an independent copy of X, i.e., $\bar{X} \stackrel{d}{=} X$ and $\bar{X} \perp X$. Here, the letter G denotes the function

$$G(A) := \frac{1}{2}\hat{\mathbb{E}}[\langle AX, X\rangle] : \mathbb{S}_d \to \mathbb{R},$$

where $\mathbb{S}_d$ denotes the collection of $d \times d$ symmetric matrices.

The function $G(\cdot) : \mathbb{S}^d \to \mathbb{R}$ is a monotonic and sublinear mapping on $\mathbb{S}^d$. In this paper, we suppose that G is non-degenerate, i.e., there exists some $\underline{\sigma}^2 > 0$ such that $G(A) - G(B) \geq \frac{1}{2}\underline{\sigma}^2 tr[A - B]$ for any $A \geq B$.

Let $\Omega_T = C_0([0, T]; \mathbb{R}^d)$, the space of $\mathbb{R}^d$-valued continuous functions on $[0, T]$ with $\omega_0 = 0$, be endowed with the supremum norm, and $B = (B^i)_{i=1}^d$ be the canonical process. Set

$$L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, \cdots, B_{t_n}) : n \geq 1, t_1, \cdots, t_n \in [0, T], \varphi \in C_{Lip}(\mathbb{R}^{d\times n})\}.$$

Definition 2.5 *For all random variable $X \in L_{ip}(\Omega_T)$ of the following form:*

$$\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}}),\ \varphi \in C_{Lip}(\mathbb{R}^{d\times n}),$$

the G-expectation is defined as

$$\hat{\mathbb{E}}[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})] = \tilde{\mathbb{E}}[\varphi(\sqrt{t_1}\xi_1, \cdots, \sqrt{t_n - t_{n-1}}\xi_n)],$$

where $\xi_1, \cdots, \xi_n$ are identically distributed d-dimensional G-normally distributed random vectors in a sublinear expectation space $(\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{\mathbb{E}})$ such that ξ_{i+1} is independent of $(\xi_1, \cdots, \xi_i)$ for each $i = 1, \cdots, n-1$. $(\Omega_T, L_{ip}(\Omega_T), \hat{\mathbb{E}})$ is called a G-expectation space. The conditional G-expectation $\hat{\mathbb{E}}_{t_i}[\cdot]$, $i = 1, \cdots, n$, is defined as follows

$$\hat{\mathbb{E}}_{t_i}[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})] = \tilde{\varphi}(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_i} - B_{t_{i-1}}),$$

where

$$\tilde{\varphi}(x_1, \cdots, x_i) = \hat{\mathbb{E}}[\varphi(x_1, \cdots, x_i, B_{t_{i+1}} - B_{t_i}, \cdots, B_{t_n} - B_{t_{n-1}})].$$

If $t \in (t_i, t_{i+1})$, the conditional G-expectation $\hat{\mathbb{E}}_t[X]$ could be defined by reformulating X as

$$X = \hat{\varphi}(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_t - B_{t_i}, B_{t_{i+1}} - B_t, \cdots, B_{t_n} - B_{t_{n-1}}),\quad \hat{\varphi} \in C_{Lip}(\mathbb{R}^{d\times(n+1)}).$$

Denote by $L_G^p(\Omega_T)$ the completion of $L_{ip}(\Omega_T)$ under the norm $\|\xi\|_{L_G^p} := (\hat{\mathbb{E}}[|\xi|^p])^{1/p}$ for $p \geq 1$. It is easy to check that the conditional G-expectation is a continuous mapping on $L_{ip}(\Omega_T)$ endowed with the norm $\|\cdot\|_{L_G^p}$ and thus can be extended to $L_G^p(\Omega_T)$. Denis et al. [5] proved the following representation theorem of G-expectation on $L_G^1(\Omega_T)$.

Theorem 2.6 ([5, 12]) *There exists a weakly compact set $\mathcal{P} \subset \mathcal{M}_1(\Omega_T)$, the set of all probability measures on $(\Omega_T, \mathcal{B}(\Omega_T))$, such that*

$$\hat{\mathbb{E}}[\xi] = \sup_{P\in\mathcal{P}} E_P[\xi] \quad for \quad all\ \xi \in L_G^1(\Omega_T).$$

$\mathcal{P}$ is called a set that represents $\hat{\mathbb{E}}$.

Let $\mathcal{P}$ be a weakly compact set that represents $\hat{\mathbb{E}}$. For this $\mathcal{P}$, we define capacity

$$c(A) := \sup_{P\in\mathcal{P}} P(A),\ A \in \mathcal{B}(\Omega_T).$$

A set $A \subset \mathcal{B}(\Omega_T)$ is polar if $c(A) = 0$. A property holds "*quasi-surely*" (q.s.) if it holds outside a polar set. In the following, we do not distinguish two random variables X and Y if $X = Y$ q.s..

For $\xi \in L_{ip}(\Omega_T)$, let $\mathcal{E}(\xi) = \hat{\mathbb{E}}[\sup_{t\in[0,T]} \hat{\mathbb{E}}_t[\xi]]$, where $\hat{\mathbb{E}}$ is the G-expectation. For convenience, we call $\mathcal{E}$ G-evaluation. For $p \geq 1$ and $\xi \in L_{ip}(\Omega_T)$, define $\|\xi\|_{p,\mathcal{E}} = [\mathcal{E}(|\xi|^p)]^{1/p}$ and denote by $L^p_{\mathcal{E}}(\Omega_T)$ the completion of $L_{ip}(\Omega_T)$ under $\|\cdot\|_{p,\mathcal{E}}$. We shall give an estimate between the two norms $\|\cdot\|_{L^p_G}$ and $\|\cdot\|_{p,\mathcal{E}}$.

Theorem 2.7 ([33]) *For any $\alpha \geq 1$ and $\delta > 0$, $L^{\alpha+\delta}_G(\Omega_T) \subset L^{\alpha}_{\mathcal{E}}(\Omega_T)$. More precisely, for any $1 < \gamma < \beta := (\alpha+\delta)/\alpha$, $\gamma \leq 2$, we have*

$$\|\xi\|^{\alpha}_{\alpha,\mathcal{E}} \leq \gamma^*\{\|\xi\|^{\alpha}_{L^{\alpha+\delta}_G} + 14^{1/\gamma}C_{\beta/\gamma}\|\xi\|^{(\alpha+\delta)/\gamma}_{L^{\alpha+\delta}_G}\}, \quad \forall \xi \in L_{ip}(\Omega_T).$$

where $C_{\beta/\gamma} = \sum_{i=1}^{\infty} i^{-\beta/\gamma}$,$\gamma^* = \gamma/(\gamma-1)$.

Similar with the classical case, the G-martingale (-sub, -supermartingale) is one of the fundamental concepts under G-expectation framework.

Definition 2.8 *The process $\{M_t\}_{t\in[0,T]}$ is called a G-martingale (-sub, -supermartingale), if for any $t \in [0,T]$, $M_t \in L^1_G(\Omega_t)$ and $\hat{\mathbb{E}}_s[M_t] = M_s$, $(\geq, \leq)$, for any $0 \leq s \leq t \leq T$.*

2.2 G-Itô calculus

For simplicity, we only give the definition of G-Itô's integral with respect to 1-dimensional G-Brownian motion and its quadratic variation. However, our results in the following sections still hold for the multidimensional case unless otherwise stated.

Definition 2.9 *Let $M^0_G(0,T)$ be the collection of processes in the following form: for a given partition $\{t_0,\cdots,t_N\} = \pi_T$ of $[0,T]$,*

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega)\mathbf{1}_{[t_j,t_{j+1})}(t), \tag{2.1}$$

where $\xi_i \in L_{ip}(\Omega_{t_i})$, $i = 0,1,2,\cdots,N-1$. For each $p \geq 1$ and $\eta \in M^0_G(0,T)$ let $\|\eta\|_{H^p_G} := \{\hat{\mathbb{E}}[(\int_0^T |\eta_s|^2 ds)^{p/2}]\}^{1/p}$, $\|\eta\|_{M^p_G} := (\hat{\mathbb{E}}[\int_0^T |\eta_s|^p ds])^{1/p}$ and denote by $H^p_G(0,T)$, $M^p_G(0,T)$ the completion of $M^0_G(0,T)$ under the norm $\|\cdot\|_{H^p_G}$, $\|\cdot\|_{M^p_G}$ respectively.

Definition 2.10 *For each $\eta \in M^0_G(0,T)$ of the form (2.1), we define the linear mappings $I, L : M^0_G(0,T) \to L^p_G(\Omega_T)$ as the following:*

$$I(\eta) := \int_0^T \eta_s dB_s = \sum_{j=0}^{N-1} \xi_j(B_{t_{j+1}} - B_{t_j}),$$

$$L(\eta) := \int_0^T \eta_s d\langle B\rangle_s = \sum_{j=0}^{N-1} \xi_j(\langle B\rangle_{t_{j+1}} - \langle B\rangle_{t_j}).$$

Then I, L can be continuously extended to $H^p_G(0,T)$ and $M^p_G(0,T)$ respectively.

By Proposition 2.10 in [19] and classical Burkholder-Davis-Gundy inequality, we have the following estimate for G-Itô's integral.

Proposition 2.11 ([11]) *If $\eta \in H_G^\alpha(0,T)$ with $\alpha \geq 1$ and $p \in (0,\alpha]$, then we get $\sup_{u\in[t,T]} |\int_t^u \eta_s dB_s|^p \in L_G^1(\Omega_T)$ and*

$$\underline{\sigma}^p c_p \hat{\mathbb{E}}_t[(\int_t^T |\eta_s|^2 ds)^{p/2}] \leq \hat{\mathbb{E}}_t[\sup_{u\in[t,T]} |\int_t^u \eta_s dB_s|^p] \leq \bar{\sigma}^p C_p \hat{\mathbb{E}}_t[(\int_t^T |\eta_s|^2 ds)^{p/2}].$$

Let $S_G^0(0,T) = \{h(t, B_{t_1\wedge t}, \ldots, B_{t_n\wedge t}) : t_1, \ldots, t_n \in [0,T], h \in C_{b,Lip}(\mathbb{R}^{n+1})\}$. For $p \geq 1$ and $\eta \in S_G^0(0,T)$, set $\|\eta\|_{S_G^p} = \{\hat{\mathbb{E}}[\sup_{t\in[0,T]} |\eta_t|^p]\}^{1/p}$. Denote by $S_G^p(0,T)$ the completion of $S_G^0(0,T)$ under the norm $\|\cdot\|_{S_G^p}$.

We consider the following type of G-BSDEs:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\langle B\rangle_s - \int_t^T Z_s dB_s - (K_T - K_t), \tag{2.2}$$

where

$$f(t,\omega,y,z), g(t,\omega,y,z) : [0,T] \times \Omega_T \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$

satisfy the following properties:

(H1') There exists some $\beta > 1$ such that for any y, z, $f(\cdot,\cdot,y,z), g(\cdot,\cdot,y,z) \in M_G^\beta(0,T)$,

(H2) There exists some $L > 0$ such that

$$|f(t,y,z) - f(t,y',z')| + |g(t,y,z) - g(t,y',z')| \leq L(|y-y'| + |z-z'|).$$

For simplicity, we denote by $\mathfrak{S}_G^\alpha(0,T)$ the collection of process (Y,Z,K) such that $Y \in S_G^\alpha(0,T)$, $Z \in H_G^\alpha(0,T)$, K is a non-increasing G-martingale with $K_0 = 0$ and $K_T \in L_G^\alpha(\Omega_T)$.

Definition 2.12 *Let $\xi \in L_G^\beta(\Omega_T)$ and f, g satisfies (H1') and (H2) for some $\beta > 1$. A triplet of processes (Y,Z,K) is called a solution of equation (2.2) if for some $1 < \alpha \leq \beta$ the following properties hold:*

(a) $(Y,Z,K) \in \mathfrak{S}_G^\alpha(0,T)$;

(b) $Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + \int_t^T g(s,Y_s,Z_s)d\langle B\rangle_s - \int_t^T Z_s dB_s - (K_T - K_t)$.

Theorem 2.13 ([10]) *Assume that $\xi \in L_G^\beta(\Omega_T)$ and f, g satisfy (H1') and (H2) for some $\beta > 1$. Then equation (2.2) has a unique solution (Y,Z,K). Moreover, for any $1 < \alpha < \beta$, we have $Z \in H_G^\alpha(0,T)$, $K_T \in L_G^\alpha(\Omega_T)$ and*

$$|Y_t|^\alpha \leq C\hat{\mathbb{E}}_t[|\xi|^\alpha + \int_t^T |f(s,0,0)|^\alpha + |g(s,0,0)|^\alpha ds],$$

where the constant C depends on α, T, $\underline{\sigma}$ and L.

Theorem 2.14 ([11]) *Let $(Y_t^l, Z_t^l, K_t^l)_{t\leq T}$, $l = 1,2$, be the solutions of the following G-BSDEs:*

$$Y_t^l = \xi^l + \int_t^T f^l(s, Y_s^l, Z_s^l)ds + \int_t^T g^l(s, Y_s^l, Z_s^l)d\langle B\rangle_s + V_T^l - V_t^l - \int_t^T Z_s^l dB_s - (K_T^l - K_t^l),$$

where $\{V_t^l\}_{0\leq t\leq T}$ are RCLL processes such that $\hat{\mathbb{E}}[\sup_{t\in[0,T]} |V_t^l|^\beta] < \infty$, f^l, g^l satisfy (H1') and (H2), $\xi^l \in L_G^\beta(\Omega_T)$ with $\beta > 1$. If $\xi^1 \geq \xi^2$, $f^1 \geq f^2$, $g^1 \geq g^2$, for $i,j = 1,\cdots,d$, $V_t^1 - V_t^2$ is a non-decreasing process, then $Y_t^1 \geq Y_t^2$.

3 A variant comparison theorem

In this section, we introduce a variant comparison theorem for solutions to G-BSDEs. First, we state some basic properties as preliminaries.

Lemma 3.1 *Let $X_t \in S_G^\alpha(0,T)$, where $\alpha > 1$. Set $X_t^n = \sum_{i=0}^{n-1} X_{t_i^n} I_{[t_i^n, t_{i+1}^n)}(t)$, where $t_i^n = \frac{iT}{n}$, $i = 0, \cdots, n$, $1/\alpha + 1/\alpha^* = 1$. Suppose that K is a G-submartingale with finite variation satisfying $K_0 = 0$ and $\hat{\mathbb{E}}[|Var(K)|^{\alpha*}] < \infty$, where $Var(K)$ is the total variation of K on $[0,T]$, then*

$$\hat{\mathbb{E}}[\sup_{t\in[0,T]} |\int_0^t (X_s^n - X_s)dK_s|] \to 0.$$

Proof. It is easy to check that

$$\sup_{t\in[0,T]} |\int_0^t (X_s^n - X_s)dK_s| \le \sup_{t\in[0,T]} |X_t^n - X_t||Var(K)|.$$

By applying Lemma 3.2 in [10], we have

$$\hat{\mathbb{E}}[\sup_{t\in[0,T]} |\int_0^t (X_s^n - X_s)dK_s|] \le \| \sup_{t\in[0,T]} |X_t^n - X_t|\|_{L_G^\alpha} \|Var(K)\|_{L_G^{\alpha^*}} \to 0.$$

■

Lemma 3.2 *Let $X_t \in S_G^\alpha(0,T)$, where $\alpha > 1$, $1/\alpha + 1/\alpha^* = 1$. Suppose that K^j is a G-submartingale with finite variation satisfying $K_0^j = 0$ and $\hat{\mathbb{E}}[|Var(K^j)|^{\alpha*}] < \infty$, $j = 1, 2$, then*

$$\int_0^t X_s^+ dK_s^1 + \int_0^t X_s^- dK_s^2,$$

is a G-submartingale.

Proof. It suffices to prove that the process $\int_0^t (X_s^n)^+ dK_s^1 + \int_0^t (X_s^n)^- dK_s^2$ is a G-submartingale, where X^n is the same as Lemma 3.1. Then for any $t \in [t_i^n, t_{i+1}^n)$,

$$\begin{aligned}
&\hat{\mathbb{E}}_t[X_{t_i^n}^+ (K_{t_{i+1}^n}^1 - K_{t_i^n}^1) + X_{t_i^n}^- (K_{t_{i+1}^n}^2 - K_{t_i^n}^2)] \\
&= X_{t_i^n}^+ \hat{\mathbb{E}}_t[(K_{t_{i+1}^n}^1 - K_{t_i^n}^1)] + X_{t_i^n}^- \hat{\mathbb{E}}_t[(K_{t_{i+1}^n}^2 - K_{t_i^n}^2)] \\
&\ge X_{t_i^n}^+ (K_t^1 - K_{t_i^n}^1) + X_{t_i^n}^- (K_t^2 - K_{t_i^n}^2).
\end{aligned}$$

From this we have the desired result. ■

In order to obtain the variant comparison theorem, we need to construct an auxiliary extended $\tilde{G}$-expectation space $(\tilde{\Omega}_T, L_{\tilde{G}}^1(\tilde{\Omega}_T), \hat{\mathbb{E}}^{\tilde{G}})$ with $\tilde{\Omega}_T = C_0([0,T], \mathbb{R}^2)$ and

$$\tilde{G}(A) = \frac{1}{2} \sup_{\underline{\sigma}^2 \le v \le \overline{\sigma}^2} tr\Big[A \begin{bmatrix} v & 1 \\ 1 & v^{-1} \end{bmatrix}\Big], A \in \mathbb{S}^2.$$

Let $\{(B_t, \tilde{B}_t)\}$ be the canonical process in the extended space.

Remark 3.3 *It is easy to check that $\langle B, \tilde{B}\rangle_t = t$. In particular, if $\underline{\sigma}^2 = \overline{\sigma}^2$, we can further get $\tilde{B}_t = \overline{\sigma}^{-2} B_t$.*

Lemma 3.4 *Consider the following bounded processes* $\{a_t\}$, $\{b_t\}$, $\{c_t\}$ *and* $\{d_t\} \in M_G^\beta(0,T)$, *where* $\beta > 1$. *Assume that* K *is a* G*-submartingale with* $\hat{\mathbb{E}}[|Var(K)|^\beta] < \infty$ *and* K_0. *Then we have*

$$K_t \leq (X_t)^{-1}\hat{\mathbb{E}}_t^{\tilde{G}}[X_T K_T - \int_t^T a_s K_s X_s ds - \int_t^T c_s K_s X_s d\langle B\rangle_s],$$

where X *is the solution of the following* $\tilde{G}$*-SDE:*

$$X_t = 1 + \int_0^t a_s X_s ds + \int_0^t c_s X_s d\langle B\rangle_s + \int_0^t d_s X_s dB_s + \int_0^t b_s X_s d\tilde{B}_s.$$

Proof. Since the $\tilde{G}$-SDE is linear, we may solve it explicitly and get that

$$X_t = \exp(\int_0^t (a_s - b_s d_s) ds + \int_0^t c_s d\langle B\rangle_s)\mathcal{E}_t^B \mathcal{E}_t^{\tilde{B}},$$

where $\mathcal{E}_t^B = \exp(\int_0^t d_s dB_s - \frac{1}{2}\int_0^t d_s^2 d\langle B\rangle_s)$, $\mathcal{E}_t^{\tilde{B}} = \exp(\int_0^t b_s d\tilde{B}_s - \frac{1}{2}\int_0^t b_s^2 d\langle \tilde{B}\rangle_s)$. Consider the following equation

$$Y_t = \xi + \int_t^T f_s ds + \int_t^T g_s d\langle B\rangle_s - \int_t^T Z_s dB_s - (K_T - K_t), \tag{3.1}$$

where $f_s = a_s Y_s + b_s Z_s + m_s$, $g_s = c_s Y_s + d_s Z_s + n_s$, $\{m_t\}$, $\{n_t\} \in M_G^\beta(0,T)$. Then applying G-Itô's formula to $X_t Y_t$, we derive that

$$\begin{aligned} &X_t Y_t + \int_t^T (X_s Z_s + d_s X_s Y_s) dB_s + \int_t^T b_s X_s Y_s d\tilde{B}_s + \int_t^T X_s dK_s \\ &= X_T \xi + \int_t^T m_s X_s ds + \int_t^T n_s X_s d\langle B\rangle_s. \end{aligned}$$

From Lemma 3.2, we have $\hat{\mathbb{E}}_t^{\tilde{G}}[\int_t^T X_s dK_s] \geq 0$. Taking $\hat{\mathbb{E}}_t^{\tilde{G}}$ conditional expectations on both sides of the above equality, it follows that

$$Y_t \leq (X_t)^{-1}\hat{\mathbb{E}}_t^{\tilde{G}}[X_T \xi + \int_t^T m_s X_s ds + \int_t^T n_s X_s d\langle B\rangle_s].$$

Consider a special case of Equation (3.1)

$$\begin{aligned} Y_t = &K_T + \int_t^T (a_s Y_s + b_s Z_s - a_s K_s) ds + \int_t^T (c_s Y_s + d_s Z_s - c_s K_s) d\langle B\rangle_s \\ &- \int_t^T Z_s dB_s - (K_T - K_t). \end{aligned}$$

It is easy to check that $Y_t = K_t$, $Z_t = 0$ is the solution of the above equation. Applying the analysis above, we have

$$K_t = Y_t \leq (X_t)^{-1}\hat{\mathbb{E}}_t^{\tilde{G}}[X_T K_T - \int_t^T a_s K_s X_s ds - \int_t^T c_s K_s X_s d\langle B\rangle_s]. \tag{3.2}$$

■

Remark 3.5 *It is important to note that there may not exist a pair of processes* (Y, Z) *satisfying Equation* (3.1) *for a given* G*-submartingale* K *with finite variation. Especially, if* K *is a non-increasing* G*-martingale, then the two sides of the Equation* (3.2) *are equal.*

Theorem 3.6 *Assume that $\xi^i \in L_G^\beta(\Omega_T)$, f_i, g_i satisfy (H1') and (H2) in Section 2 with $\beta > 1$, $i = 1, 2$. Let (Y_t^2, Z_t^2, K_t^2) be the solution of G-BSDE with generators f_2, g_2 and terminal value ξ^2, (Y_t^1, Z_t^1) satisfy the following equation*

$$Y_t^1 = \xi^1 + \int_t^T f_1(s, Y_s^1, Z_s^1)ds + \int_t^T g_1(s, Y_s^1, Z_s^1)d\langle B\rangle_s - \int_t^T Z_s^1 dB_s - (K_T^1 - K_t^1),$$

where K^1 is a G-submartingale with finite variation satisfying $\hat{\mathbb{E}}[|Var(K^1)|^\beta] < \infty$ and $K_0 = 0$. If $\xi^1 \leq \xi^2$, $f_1 \leq f_2$, $g_1 \leq g_2$, then $Y_t^1 \leq Y_t^2$.

Proof. Let $\hat{Y}_t = Y_t^2 - Y_t^1$, $\hat{Z}_t = Z_t^2 - Z_t^1$, $\hat{f}_s = f_2(s, Y_s^2, Z_s^2) - f_1(s, Y_s^1, Z_s^1)$, $\hat{g}_s = g_2(s, Y_s^2, Z_s^2) - g_1(s, Y_s^1, Z_s^1)$, $\hat{\xi} = \xi^2 - \xi^1$. Then we have

$$\hat{Y}_t + K_t^1 = \hat{\xi} + K_T^1 + \int_t^T \hat{f}_s ds + \int_t^T \hat{g}_s d\langle B\rangle_s - \int_t^T \hat{Z}_s dB_s - (K_T^2 - K_t^2). \tag{3.3}$$

For each fixed $\varepsilon > 0$, by the proof of Theorem 3.6 in [11], we can get

$$\hat{f}_s = a_s^\varepsilon \hat{Y}_s + b_s^\varepsilon \hat{Z}_s + m_s - m_s^\varepsilon, \hat{g}_s = c_s^\varepsilon \hat{Y}_s + d_s^\varepsilon \hat{Z}_s + n_s - n_s^\varepsilon,$$

where $|m_s^\varepsilon| \leq 4L\varepsilon$, $|n_s^\varepsilon| \leq 4L\varepsilon$, $m_s = f_2(s, Y_s^1, Z_s^1) - f_1(s, Y_s^1, Z_s^1) \geq 0$, $n_s = g_2(s, Y_s^1, Z_s^1) - g_1(s, Y_s^1, Z_s^1) \geq 0$, $\psi^\varepsilon \in M_G^2(0,T)$ and $|\psi^\varepsilon| \leq L$ for $\psi = a, b, c, d$.

Recalling (3.2), we can solve (3.3) to get

$$\begin{aligned}
\hat{Y}_t + K_t^1 =& (X_t^\varepsilon)^{-1}\hat{\mathbb{E}}_t^{\tilde{G}}[X_T^\varepsilon(\hat{\xi} + K_T^1) + \int_t^T (\widetilde{m}_s^\varepsilon - a_s^\varepsilon K_s^1)X_s^\varepsilon ds + \int_t^T (\widetilde{n}_s^\varepsilon - c_s^\varepsilon K_s^1)X_s^\varepsilon d\langle B\rangle_s] \\
\geq& (X_t^\varepsilon)^{-1}\hat{\mathbb{E}}_t^{\tilde{G}}[X_T^\varepsilon K_T^1 + \int_t^T (-m_s^\varepsilon - a_s^\varepsilon K_s^1)X_s^\varepsilon ds + \int_t^T (-n_s^\varepsilon - c_s^\varepsilon K_s^1)X_s^\varepsilon d\langle B\rangle_s] \\
\geq& (X_t^\varepsilon)^{-1}\hat{\mathbb{E}}_t^{\tilde{G}}[X_T^\varepsilon K_T^1 - \int_t^T a_s^\varepsilon K_s^1 X_s^\varepsilon ds - \int_t^T c_s^\varepsilon K_s^1 X_s^\varepsilon d\langle B\rangle_s] \\
& - (X_t^\varepsilon)^{-1}\hat{\mathbb{E}}_t^{\tilde{G}}[\int_t^T m_s^\varepsilon X_s^\varepsilon ds + \int_t^T n_s^\varepsilon X_s^\varepsilon d\langle B\rangle_s] \\
\geq& K_t^1 - 4L\varepsilon(X_t^\varepsilon)^{-1}\hat{\mathbb{E}}_t^{\tilde{G}}[\int_t^T |X_s^\varepsilon| ds + \int_t^T |X_s^\varepsilon| d\langle B\rangle_s],
\end{aligned}$$

where $\widetilde{m}_s^\varepsilon = m_s - m_s^\varepsilon$, $\widetilde{n}_s^\varepsilon = n_s - n_s^\varepsilon$, $\{X_t^\varepsilon\}_{t\in[0,T]}$ is the solution of the following equation

$$X_t^\varepsilon = 1 + \int_0^t a_s^\varepsilon X_s^\varepsilon ds + \int_0^t c_s^\varepsilon X_s^\varepsilon d\langle B\rangle_s + \int_0^t d_s^\varepsilon X_s^\varepsilon dB_s + \int_0^t b_s^\varepsilon X_s^\varepsilon d\tilde{B}_s.$$

Then by letting $\varepsilon \to 0$, we can derive the desired result. ■

4 Reflected G-BSDE with an upper obstacle

El Karoui, Kapoudjian, Pardoux, Peng and Quenez [6] introduced the reflected BSDE with a lower obstacle. An additional non-decreasing process should be added in this equation to keep the solution above the given obstacle. Substituting the non-increasing process for an non-decreasing one, we can use the same method to deal with the reflected BSDE with an upper obstacle. However, under the G-framework, due to the appearance of the non-increasing G-martingale in the penalized G-BSDEs, these two cases are significantly different. Now we reformulate this problems as follows.

We are given these parameters: the generators f and g, the obstacle process $\{S_t\}_{t\in[0,T]}$ and the terminal value ξ, where f and g are maps

$$f(t,\omega,y,z), g(t,\omega,y,z):[0,T]\times\Omega_T\times\mathbb{R}^2\to\mathbb{R}.$$

The following assumptions will be needed throughout this section. There exists some $\beta>2$ such that

(A1) for any y,z, $f(\cdot,\cdot,y,z)$, $g(\cdot,\cdot,y,z)\in M_G^\beta(0,T)$ and $\hat{\mathbb{E}}[\sup_{t\in[0,T]}(|f(t,0,0)|^\beta+|g(t,0,0)|^\beta)]<\infty$;

(A2) $|f(t,\omega,y,z)-f(t,\omega,y',z')|+|g(t,\omega,y,z)-g(t,\omega,y',z')|\leq L(|y-y'|+|z-z'|)$ for some $L>0$;

(A3) $\{S_t\}_{t\in[0,T]}\in S_G^\beta(0,T)$ is of the following form

$$S_t=S_0+\int_0^t b(s)ds+\int_0^t l(s)d\langle B\rangle_s+\int_0^t\sigma(s)dB_s, \tag{4.1}$$

where $\{b(t)\}_{t\in[0,T]}$, $\{l(t)\}_{t\in[0,T]}$ belong to $M_G^\beta(0,T)$ and $\{\sigma(t)\}_{t\in[0,T]}$ belongs to $H_G^\beta(0,T)$. Furthermore, $\hat{\mathbb{E}}[\sup_{t\in[0,T]}\{|b(t)|^\beta+|l(t)|^\beta+|\sigma(t)|^\beta\}]<\infty$;

(A4) $\xi\in L_G^\beta(\Omega_T)$ and $\xi\leq S_T$, *q.s.*.

Remark 4.1 *Compared with the reflected G-BSDE with a lower obstacle (see [18]), the conditions (A1)-(A4) on the parameters are more restrictive. For simplicity, assume that the time horizon is* $[0,1]$ *and consider the generator f defined as the following:*

$$f(t)=\sum_{n=1}^{\infty}n^{\frac{1}{\beta+1}}I_{(\frac{1}{n+1},\frac{1}{n}]}(t).$$

It is easy to check that $f\in M_G^\beta(0,1)$ while $\sup_{t\in[0,1]}|f(t)|^\beta=\infty$. Hence, f satisfies the condition (H1) in [18] but does not satisfy the condition (A1) in this paper. The requirment that S is a G-Itô process is to ensure that we can derive the convergence property for the penalized G-BSDEs (see Remark 4.6 below).

Then we can introduce our reflected G-BSDE with an upper obstacle. A triple of processes (Y,Z,A) is called a solution of reflected G-BSDE if for some $2\leq\alpha<\beta$ the following properties are satisfied:

(i) $(Y,Z,A)\in\mathbb{S}_G^\alpha(0,T)$ and $Y_t\leq S_t$, $0\leq t\leq T$;

(ii) $Y_t=\xi+\int_t^T f(s,Y_s,Z_s)ds+\int_t^T g(s,Y_s,Z_s)d\langle B\rangle_s-\int_t^T Z_s dB_s+(A_T-A_t)$;

(iii) $\{-\int_0^t(S_s-Y_s)dA_s\}_{t\in[0,T]}$ is a non-increasing G-martingale.

Here we denote by $\mathbb{S}_G^\alpha(0,T)$ the collection of process (Y,Z,A) such that $Y\in S_G^\alpha(0,T)$, $Z\in H_G^\alpha(0,T)$, $A\in S_G^\alpha(0,T)$ is a continuous process with finite variation satisfying $A_0=0$ and $-A$ is a G-submartingale.

For notational simplification, in this paper we only consider the case $g\equiv 0$ and $l\equiv 0$. But the results still hold for the other cases.

Theorem 4.2 *Under the above assumptions, in particular (A1)-(A4), the reflected G-BSDE with parameters (ξ,f,S) has a maximal solution (Y,Z,A), which means that, if (Y',Z',A') is another solution, then $Y_t\geq Y'_t$, for all $t\in[0,T]$.*

The proof will be divided into a sequence of lemmas. For f, $\{S_t\}_{t\in[0,T]}$ and ξ satisfy (A1)-(A4) with $\beta > 2$. We now consider the following family of G-BSDEs parameterized by $n = 1, 2, \ldots$.

$$Y_t^n = \xi + \int_t^T f(s, Y_s^n, Z_s^n)ds - n\int_t^T (Y_s^n - S_s)^+ ds - \int_t^T Z_s^n dB_s - (K_T^n - K_t^n). \tag{4.2}$$

Now let $L_t^n = -n\int_0^t (Y_s^n - S_s)^+ ds$. Then $\{L_t^n\}_{t\in[0,T]}$ is a non-increasing process. We can rewrite G-BSDE (4.2) as

$$Y_t^n = \xi + \int_t^T f(s, Y_s^n, Z_s^n)ds - \int_t^T Z_s^n dB_s - (K_T^n - K_t^n) + (L_T^n - L_t^n). \tag{4.3}$$

Lemma 4.3 *There exists a constant C independent of n, such that for $2 \le \alpha < \beta$, we have*

$$\hat{\mathbb{E}}[\sup_{t\in[0,T]} |Y_t^n|^\alpha] \le C.$$

Proof. For simplicity, first we consider the case where $S \equiv 0$. For the case that S is a G-Itô process, we may refer to Remark 4.4. For any $r > 0$, set $\tilde{Y}_t = (Y_t^n)^2$. Applying Itô's formula to $\tilde{Y}_t^{\alpha/2} e^{rt}$ yields that

$$\begin{aligned}
&\tilde{Y}_t^{\alpha/2} e^{rt} + \int_t^T r e^{rs} \tilde{Y}_s^{\alpha/2} ds + \int_t^T \frac{\alpha}{2} e^{rs} \tilde{Y}_s^{\alpha/2-1} (Z_s^n)^2 d\langle B\rangle_s \\
&= |\xi|^\alpha e^{rT} + \alpha(1 - \frac{\alpha}{2}) \int_t^T e^{rs} \tilde{Y}_s^{\alpha/2-2} (Y_s^n)^2 (Z_s^n)^2 d\langle B\rangle_s + \int_t^T \alpha e^{rs} \tilde{Y}_s^{\alpha/2-1} Y_s^n dL_s^n \\
&\quad + \int_t^T \alpha e^{rs} \tilde{Y}_s^{\alpha/2-1} Y_s^n f(s, Y_s^n, Z_s^n) ds - \int_t^T \alpha e^{rs} \tilde{Y}_s^{\alpha/2-1} (Y_s^n Z_s^n dB_s + Y_s^n dK_s^n) \\
&\le |\xi|^\alpha e^{rT} + \alpha(1 - \frac{\alpha}{2}) \int_t^T e^{rs} \tilde{Y}_s^{\alpha/2-2} (Y_s^n)^2 (Z_s^n)^2 d\langle B\rangle_s \\
&\quad + \int_t^T \alpha e^{rs} \tilde{Y}_s^{\alpha/2-1/2} |f(s, Y_s^n, Z_s^n)| ds - (M_T - M_t),
\end{aligned}$$

where $M_t = \int_0^t \alpha e^{rs} \tilde{Y}_s^{\alpha/2-1} (Y_s^n Z_s^n dB_s + (Y_s^n)^+ dK_s^n)$ is a G-martingale. In the last inequality, we use the fact that $-y(y)^+ \le 0$ for any $y \in \mathbb{R}$. From the assumption of f and the Young inequality, we have

$$\begin{aligned}
&\int_t^T \alpha e^{rs} \tilde{Y}_s^{\frac{\alpha-1}{2}} |f(s, Y_s^n, Z_s^n)| ds \\
&\le \int_t^T \alpha e^{rs} \tilde{Y}_s^{\frac{\alpha-1}{2}} [|f(s, 0, 0)| + L|Y_s^n| + L|Z_s^n|] ds \\
&\le \int_t^T e^{rs} |f(s, 0, 0)|^\alpha ds + \frac{\alpha(\alpha-1)}{4} \int_t^T e^{rs} \tilde{Y}_s^{\alpha/2-1} (Z_s^n)^2 d\langle B\rangle_s \\
&\quad + (\alpha - 1 + \alpha L + \frac{\alpha L^2}{\underline{\sigma}^2(\alpha-1)}) \int_t^T e^{rs} \tilde{Y}_s^{\alpha/2} ds.
\end{aligned}$$

Setting $r = \alpha + \alpha L + \frac{\alpha L^2}{\underline{\sigma}^2(\alpha-1)}$, we can get

$$\tilde{Y}_t^{\alpha/2} e^{rt} + M_T - M_t \le |\xi|^\alpha e^{rT} + \int_t^T e^{rs} |f(s, 0, 0)|^\alpha ds,$$

Taking conditional expectations on both sides, we have

$$|Y_t^n|^\alpha \le C\hat{\mathbb{E}}_t[|\xi|^\alpha + \int_t^T |f(s,0,0)|^\alpha ds].$$

By Theorem 2.7, we can conclude that for $2 \le \alpha < \beta$, there exists a constant C independent of n such that $\hat{\mathbb{E}}[\sup_{t\in[0,T]} |Y_t^n|^\alpha] \le C$. ■

Remark 4.4 *For the case when the obstacle process S is given as (4.1), let $\widetilde{Y}_t^n = Y_t^n - S_t$ and $\widetilde{Z}_t^n = Z_t^n - \sigma(t)$. We can rewrite (4.2) as the following:*

$$\begin{aligned}\widetilde{Y}_t^n =& \xi - S_T + \int_t^T [f(s, \widetilde{Y}_s^n + S_s, \widetilde{Z}_s^n + \sigma(s)) + b(s)]ds \\ &- n\int_t^T (\widetilde{Y}_s^n)^+ ds - \int_t^T \widetilde{Z}_s^n dB_s - (K_T^n - K_t^n).\end{aligned}$$

Using the same method as the proof of Lemma 4.3, we get that

$$|\widetilde{Y}_t^n|^\alpha \le C\hat{\mathbb{E}}_t[|\xi - S_T|^\alpha + \int_t^T |f(s, S_s, \sigma(s)) + b(s)|^\alpha ds].$$

Thus, we conclude that, for $2 \le \alpha < \beta$, there exists a constant C independent of n such that $\hat{\mathbb{E}}[\sup_{t\in[0,T]} |Y_t^n|^\alpha] \le C$.

Compared with Lemma 4.3 in [18], the following result is sharper. More importantly, this lemma allows us to establish uniform estimates on the sequence (K^n, L^n) and then to obtain the convergence of (Y^n). We apply a nonlinear Girsanov transformation approach to prove this result. First, we consider the following G-BSDE driven by 1-dimensional G-Brownian motion:

$$Y_t^L = \xi + \int_t^T L|Z_s^L|ds - \int_t^T Z_s^L dB_s - (K_T^L - K_t^L).$$

For each $\xi \in L_G^\beta(\Omega_T)$ with $\beta > 2$, we define

$$\tilde{\mathbb{E}}_t^L[\xi] := Y_t^L.$$

By Theorem 5.1 in [11], $\tilde{\mathbb{E}}_t^L[\cdot]$ is a consistent sublinear expectation.

Lemma 4.5 *There exists a constant C independent of n such that for $2 \le \alpha < \beta$,*

$$\hat{\mathbb{E}}[\sup_{t\in[0,T]} |(Y_t^n - S_t)^+|^\alpha] \le \frac{C}{n^\alpha}.$$

Proof. Set $\tilde{Y}_t^n = Y_t^n - S_t$, $\tilde{Z}_t^n = Z_t^n - \sigma(t)$, we can rewrite G-BSDE (4.2) as

$$\tilde{Y}_t^n = \xi - S_T + \int_t^T [f(s, Y_s^n, Z_s^n) + b(s)]ds - \int_t^T n(\tilde{Y}_s^n)^+ ds - \int_t^T \tilde{Z}_s^n dB_s - (K_T^n - K_t^n). \tag{4.4}$$

For each given $\varepsilon > 0$, we can choose a Lipschitz continuous function $h(\cdot)$ such that $I_{[-\varepsilon,\varepsilon]}(x) \le h(x) \le I_{[-2\varepsilon,2\varepsilon]}(x)$. Thus we have

$$f(s, Y_s^n, Z_s^n) - f(s, Y_s^n, 0) = (f(s, Y_s^n, Z_s^n) - f(s, Y_s^n, 0))h(Z_s^n) + a_s^{\varepsilon,n} Z_s^n =: m_s^{\varepsilon,n} + a_s^{\varepsilon,n} Z_s^n,$$

where

$$a_s^{\varepsilon,n} = \begin{cases} (1-h(Z_s^n))(f(s,Y_s^n,Z_s^n)-f(s,Y_s^n,0))(Z_s^n)^{-1}, & \text{if } Z_s^n \neq 0; \\ 0, & \text{otherwise}. \end{cases}$$

It is easy to check that $a^{\varepsilon,n} \in M_G^2(0,T)$, $|a_s^{\varepsilon,n}| \leq L$ and $|m_s^{\varepsilon,n}| \leq 2L\varepsilon$. Then we can get

$$f(s,Y_s^n,Z_s^n) = f(s,Y_s^n,0) + a_s^{\varepsilon,n} Z_s^n + m_s^{\varepsilon,n} = f(s,Y_s^n,0) + a_s^{\varepsilon,n}\sigma(s) + a_s^{\varepsilon,n}\tilde{Z}_s^n + m_s^{\varepsilon,n}.$$

Now we consider the following G-BSDE:

$$Y_t^{\varepsilon,n} = \xi + \int_t^T a_s^{\varepsilon,n} Z_s^{\varepsilon,n} ds - \int_t^T Z_s^{\varepsilon,n} dB_s - (K_T^{\varepsilon,n} - K_t^{\varepsilon,n}).$$

For each $\xi \in L_G^\beta(\Omega_T)$ with $\beta > 2$, we define

$$\tilde{\mathbb{E}}_t^{\varepsilon,n}[\xi] := Y_t^{\varepsilon,n}.$$

Set $\tilde{B}_t^{\varepsilon,n} = B_t - \int_0^t a_s^{\varepsilon,n} ds$. By Theorem 5.2 in [11], $\{\tilde{B}_t^{\varepsilon,n}\}$ is a G-Brownian motion under $\tilde{\mathbb{E}}^{\varepsilon,n}[\cdot]$. Moreover, by Theorem 2.14, we have $\tilde{\mathbb{E}}_t^{\varepsilon,n}[\xi] \leq \tilde{\mathbb{E}}_t^L[\xi]$, $\forall \xi \in L_G^\beta(\Omega_T)$. We can rewrite G-BSDE (4.4) as the following

$$\tilde{Y}_t^n = \xi - S_T + \int_t^T f^{\varepsilon,n}(s) ds - \int_t^T n(\tilde{Y}_s^n)^+ ds - \int_t^T \tilde{Z}_s^n d\tilde{B}_s^{n,\varepsilon} - (K_T^n - K_t^n),$$

where $f^{\varepsilon,n}(s) = f(s,Y_s^n,0) + m_s^{\varepsilon,n} + a_s^{\varepsilon,n}\sigma(s) + b(s)$. Applying G-Itô's formula to $e^{-nt}\tilde{Y}_t^n$, we get

$$\begin{aligned} \tilde{Y}_t^n + \int_t^T e^{n(t-s)} dK_s^n =& (\xi - S_T)e^{n(t-T)} + \int_t^T n e^{n(t-s)}[\tilde{Y}_s^n - (\tilde{Y}_s^n)^+] ds \\ &+ \int_t^T e^{n(t-s)} f^{\varepsilon,n}(s) ds - \int_t^T e^{n(t-s)} \tilde{Z}_s^n dB_s^{\varepsilon,n} \\ \leq& \int_t^T e^{n(t-s)} |f^{\varepsilon,n}(s)| ds - \int_t^T e^{n(t-s)} \tilde{Z}_s^n dB_s^{\varepsilon,n}. \end{aligned}$$

Note that $\tilde{\mathbb{E}}_s^{\varepsilon,n}[K_t^n] = K_s^n$ for any $0 \leq s \leq t \leq T$ by Theorem 5.1 in [11]. Taking $\tilde{\mathbb{E}}_t^{\varepsilon,n}$ conditional expectation on both sides, we have

$$\begin{aligned} \tilde{Y}_t^n \leq& \tilde{\mathbb{E}}_t^{\varepsilon,n}[\int_t^T e^{n(t-s)} |f^{\varepsilon,n}(s)| ds] \leq \tilde{\mathbb{E}}_t^L[\int_t^T e^{n(t-s)} |f^{\varepsilon,n}(s)| ds] \\ \leq& \tilde{\mathbb{E}}_t^L[\int_t^T e^{n(t-s)} \sup_{u \in [0,T]} [|f(u,0,0)| + L|Y_u^n| + |m_u^{\varepsilon,n}| + L|\sigma(u)| + |b(u)|] ds] \\ \leq& \frac{C}{n} \tilde{\mathbb{E}}_t^L[\sup_{u \in [0,T]} [|f(u,0,0)| + |Y_u^n| + |\sigma(u)| + |b(u)|] + \varepsilon]. \end{aligned}$$

By Theorem 2.13, for $2 \leq \alpha < \beta$, it follows that

$$\begin{aligned} |(\tilde{Y}_t^n)^+|^\alpha \leq& \frac{C}{n^\alpha} (\tilde{\mathbb{E}}_t^L[\sup_{u \in [0,T]} [|f(u,0,0)| + |Y_u^n| + |\sigma(u)| + |b(u)|] + \varepsilon])^\alpha \\ \leq& \frac{C}{n^\alpha} \hat{\mathbb{E}}_t[\sup_{u \in [0,T]} [|f(u,0,0)| + |Y_u^n| + |\sigma(u)| + |b(u)| + \varepsilon]^\alpha]. \end{aligned}$$

Then applying Lemma 4.3 and Theorem 2.7, letting $\varepsilon \to \infty$, we get the desired result. ■

Remark 4.6 *It is worth pointing out that the uniform convergence property of $\{(Y_t^n - S_t)^+\}$ ($\{(Y_t^n - S_t)^-\}$) is of vital importance in proving the existence of solutions to the reflected G-BSDE with an upper (lower) obstalce. It is easy to see that conditions on the parameters of reflected G-BSDE with an upper obstacle ((A1)-(A4)) is more restrictive than the ones of the lower obstacle case ((H1)-(H3), (H4) or (H4'), see [18]). If the parameters of reflected G-BSDE with a lower obstacle also satisfy (A1)-(A4), we may use the same technique, i.e., the Girsanov transformation, to prove Lemma 4.3 in [18]. More specificly, we may obtain the convergence rate of $\{(Y_t^n - S_t)^-\}$. However, for the general case, this method does not work since we need the decomposition of the obstacle to apply the G-Itô formula.*

Lemma 4.7 *There exists a constant C independent of n, such that for $2 \leq \alpha < \beta$,*

$$\hat{\mathbb{E}}[|L_T^n|^\alpha] = \hat{\mathbb{E}}[n^\alpha(\int_0^T (Y_s^n - S_s)^+ ds)^\alpha] \leq C, \ \hat{\mathbb{E}}[|K_T^n|^\alpha] \leq C, \ \hat{\mathbb{E}}[(\int_0^T |Z_s^n|^2 ds)^{\frac{\alpha}{2}}] \leq C.$$

Proof. The first estimate can be derived easily from Lemma 4.5. Applying G-Itô's formula to $|Y_t^n|^2$, we have

$$\begin{aligned} |Y_0^n|^2 + \int_0^T |Z_s^n|^2 d\langle B\rangle_s =& |\xi|^2 + \int_0^T 2Y_s^n f(s, Y_s^n, Z_s^n) ds \\ &- \int_0^T 2Y_s^n Z_s^n dB_s - \int_0^T 2Y_s^n dA_s^n, \end{aligned}$$

where $A_t^n = L_t^n - K_t^n$. Consequently

$$\begin{aligned} (\int_0^T |Z_s^n|^2 d\langle B\rangle_s)^{\frac{\alpha}{2}} \leq& C\{|\xi|^\alpha + |\int_0^T Y_s^n f(s, Y_s^n, Z_s^n) ds|^\alpha \\ &+ |\int_0^T Y_s^n Z_s^n dB_s|^\alpha + |\int_0^T 2Y_s^n dA_s^n|^\alpha\}. \end{aligned}$$

By Proposition 2.11 and simple calculation, we obtain

$$\begin{aligned} \hat{\mathbb{E}}[(\int_0^T |Z_s^n|^2 ds)^{\frac{\alpha}{2}}] \leq& C\{\hat{\mathbb{E}}[\sup_{t\in[0,T]} |Y_t^n|^\alpha] + (\hat{\mathbb{E}}[\sup_{t\in[0,T]} |Y_t^n|^\alpha])^{1/2}[(\hat{\mathbb{E}}[|K_T^n|^\alpha])^{1/2} \\ &+ (\hat{\mathbb{E}}[|L_T^n|^\alpha])^{1/2} + (\hat{\mathbb{E}}[(\int_0^T |f(s,0,0)|ds)^\alpha])^{1/2}]\}. \end{aligned} \tag{4.5}$$

On the other hand,

$$K_T^n = \xi - Y_0^n + \int_0^T f(s, Y_s^n, Z_s^n) ds - \int_0^T Z_s^n dB_s + L_T^n.$$

An easy computation shows that

$$\hat{\mathbb{E}}[|K_T^n|^\alpha] \leq C\{\hat{\mathbb{E}}[\sup_{t\in[0,T]} |Y_t^n|^\alpha] + \hat{\mathbb{E}}[|L_T^n|^\alpha] + \hat{\mathbb{E}}[(\int_0^T |f(s,0,0)|ds)^\alpha] + \hat{\mathbb{E}}[(\int_0^T |Z_s^n|^2 ds)^{\frac{\alpha}{2}}]\}. \tag{4.6}$$

Combining inequalities (4.5) and (4.6), we can easily see the desired results. ■

Remark 4.8 *Set $A_t^n = L_t^n - K_t^n$. Then $\{A_t^n\}_{t\in[0,T]}$ is a process with finite variation. Moreover, it is easy to check that $\{-A_t^n\}_{t\in[0,T]}$ is a G-submartingale. We denote by $Var(A^n)$ the total variation for A^n on $[0,T]$. Then there exists a constant C independent of n, such that for $2 \leq \alpha < \beta$*

$$\hat{\mathbb{E}}[|Var(A^n)|^\alpha] \leq C\{\hat{\mathbb{E}}[|L_T^n|^\alpha] + \hat{\mathbb{E}}[|K_T^n|^\alpha]\} \leq C.$$

We now show that the sequences $(Y^n)_{n=1}^{\infty}$, $(Z^n)_{n=1}^{\infty}$ and $(A^n)_{n=1}^{\infty}$ are convergent.

Lemma 4.9 *For $m, n \in \mathbb{N}$, set $\hat{Y}_t = Y_t^n - Y_t^m$, $\hat{Z}_t = Z_t^n - Z_t^m$ and $\hat{A}_t = A_t^n - A_t^m$. Then for any $2 \leq \alpha < \beta$, we have*

$$\lim_{m,n\to\infty} \hat{\mathbb{E}}[\sup_{t\in[0,T]} |\hat{Y}_t|^{\alpha}] = 0, \quad \lim_{m,n\to\infty} \hat{\mathbb{E}}[(\int_0^T |\hat{Z}_s|^2 ds)^{\frac{\alpha}{2}}] = 0, \quad \lim_{m,n\to\infty} \hat{\mathbb{E}}[\sup_{t\in[0,T]} |\hat{A}_t|^{\alpha}] = 0. \tag{4.7}$$

Proof. The convergence property for $(Y^n)_{n=1}^{\infty}$ can be proved in a similar way as the proof of Lemma 4.4 in [18]. For reader's convenience, we give a brief proof here. Without loss of generality, we may assume $S \equiv 0$ in (4.2). Set $\hat{L}_t = L_t^n - L_t^m$, $\hat{K}_t = K_t^n - K_t^m$, $\hat{f}_t = f(t, Y_t^n, Z_t^n) - f(t, Y_t^m, Z_t^m)$ and $\bar{Y}_t = |\hat{Y}_t|^2$. By applying Itô's formula to $\bar{Y}_t^{\alpha/2} e^{rt}$, where r is a constant to be determined later, we get

$$\begin{aligned}
&\bar{Y}_t^{\alpha/2} e^{rt} + \int_t^T r e^{rs} \bar{Y}_s^{\alpha/2} ds + \int_t^T \frac{\alpha}{2} e^{rs} \bar{Y}_s^{\alpha/2-1} (\hat{Z}_s)^2 d\langle B\rangle_s \\
&= \alpha(1 - \frac{\alpha}{2}) \int_t^T e^{rs} \bar{Y}_s^{\alpha/2-2} (\hat{Y}_s)^2 (\hat{Z}_s)^2 d\langle B\rangle_s + \int_t^T \alpha e^{rs} \bar{Y}_s^{\alpha/2-1} \hat{Y}_s d\hat{L}_s \\
&\quad + \int_t^T \alpha e^{rs} \bar{Y}_s^{\alpha/2-1} \hat{Y}_s \hat{f}_s ds - \int_t^T \alpha e^{rs} \bar{Y}_s^{\alpha/2-1} (\hat{Y}_s \hat{Z}_s dB_s + \hat{Y}_s d\hat{K}_s) \\
&\leq \alpha(1 - \frac{\alpha}{2}) \int_t^T e^{rs} \bar{Y}_s^{\alpha/2-2} (\hat{Y}_s)^2 (\hat{Z}_s)^2 d\langle B\rangle_s + \int_t^T \alpha e^{rs} \bar{Y}_s^{\frac{\alpha-1}{2}} |\hat{f}_s| ds \\
&\quad - \int_t^T \alpha e^{rs} \bar{Y}_s^{\alpha/2-1} (Y_s^n)^+ dL_s^m - \int_t^T \alpha e^{rs} \bar{Y}_s^{\alpha/2-1} (Y_s^m)^+ dL_s^n - (M_T - M_t),
\end{aligned} \tag{4.8}$$

where $M_t = \int_0^t \alpha e^{rs} \bar{Y}_s^{\alpha/2-1} (\hat{Y}_s \hat{Z}_s dB_s + (\hat{Y}_s)^+ dK_s^m + (\hat{Y}_s)^- dK_s^n)$. By Lemma 3.4 in [10], $\{M_t\}$ is a G-martingale. Applying the assumption on f and the Hölder inequality, we obtain

$$\int_t^T \alpha e^{rs} \bar{Y}_s^{\frac{\alpha-1}{2}} |\hat{f}_s| ds \leq \tilde{L} \int_t^T e^{rs} \bar{Y}_s^{\alpha/2} ds + \frac{\alpha(\alpha-1)}{4} \int_t^T e^{rs} \bar{Y}_s^{\alpha/2-1} (\hat{Z}_s)^2 d\langle B\rangle_s,$$

where $\tilde{L} = \alpha L + \frac{\alpha L^2}{\underline{\sigma}^2(\alpha-1)}$. Let $r = 1 + \tilde{L}$. The above analysis shows that

$$\bar{Y}_t^{\alpha/2} e^{rt} + (M_T - M_t) \leq -\int_t^T \alpha e^{rs} \bar{Y}_s^{\alpha/2-1} (Y_s^n)^+ dL_s^m - \int_t^T \alpha e^{rs} \bar{Y}_s^{\alpha/2-1} (Y_s^m)^+ dL_s^n.$$

Taking conditional expectation on both sides of the above inequality, we conclude that

$$\begin{aligned}
|\hat{Y}_t|^{\alpha} \leq& C\hat{\mathbb{E}}_t[-\int_t^T \bar{Y}_s^{\alpha/2-1} (Y_s^n)^+ dL_s^m - \int_t^T \bar{Y}_s^{\alpha/2-1} (Y_s^m)^+ dL_s^n] \\
\leq& C(n+m)\hat{\mathbb{E}}_t[\int_0^T |(Y_s^n)^+|^{\alpha-1} (Y_s^m)^+ ds + \int_0^T |(Y_s^m)^+|^{\alpha-1} (Y_s^n)^+ ds].
\end{aligned}$$

By Lemma 4.3-4.7, Theorem 2.7 and the Hölder inequality, we have for any $2 \leq \alpha < \beta$,

$$\lim_{n,m\to\infty} \hat{\mathbb{E}}[\sup_{t\in[0,T]} |Y_t^n - Y_t^m|^{\alpha}] = 0.$$

Choosing $\alpha = 2$ and $r = 0$ in (4.8), we get

$$|\hat{Y}_0|^2 + \int_0^T |\hat{Z}_s|^2 d\langle B\rangle_s = \int_0^T 2\hat{Y}_s \hat{f}_s ds - \int_0^T 2\hat{Y}_s d\hat{K}_s + \int_0^T 2\hat{Y}_s d\hat{L}_s - \int_0^T 2\hat{Y}_s \hat{Z}_s dB_s.$$

Observe that

$$\int_0^T 2\hat{Y}_s\hat{f}_s ds \leq 2L\int_0^T (|\hat{Y}_s|^2+|\hat{Y}_s||\hat{Z}_s|)ds \leq (2L+L^2/\varepsilon)\int_0^T |\hat{Y}_s|^2 ds+\varepsilon\int_0^T |\hat{Z}_s|^2 ds,$$

where $\varepsilon<\underline{\sigma}^2$. The above two equations yield that

$$\int_0^T |\hat{Z}_s|^2 ds \leq C(\int_0^T |\hat{Y}_s|^2 ds-\int_0^T \hat{Y}_s d\hat{K}_s+\int_0^T \hat{Y}_s d\hat{L}_s-\int_0^T \hat{Y}_s\hat{Z}_s dB_s).$$

By Lemma 4.3, Lemma 4.7, Proposition 2.11 and the Hölder inequality, we derive that

$$\begin{aligned}\hat{\mathbb{E}}[(\int_0^T |\hat{Z}_s|^2)^{\frac{\alpha}{2}}] \leq & C\{\hat{\mathbb{E}}[\sup_{t\in[0,T]}|\hat{Y}_t|^\alpha+\sup_{t\in[0,T]}|\hat{Y}_t|^{\frac{\alpha}{2}}(\lambda_T^{n,m})^{\frac{\alpha}{2}}]+\hat{\mathbb{E}}[(\int_0^T \hat{Y}_s^2\hat{Z}_s^2 ds)^{\frac{\alpha}{4}}]\} \\ \leq & C\{\hat{\mathbb{E}}[\sup_{t\in[0,T]}|\hat{Y}_t|^\alpha]+(\hat{\mathbb{E}}[\sup_{t\in[0,T]}|\hat{Y}_t|^\alpha])^{\frac{1}{2}}\} \\ & +\frac{C^2}{2}\hat{\mathbb{E}}[\sup_{t\in[0,T]}|\hat{Y}_t|^\alpha]+\frac{1}{2}\hat{\mathbb{E}}[(\int_0^T |\hat{Z}_s|^2)^{\frac{\alpha}{2}}],\end{aligned}$$

where $\lambda_T^{n,m}=|L_T^n|+|L_T^m|+|K_T^n|+|K_T^m|$. It follows that

$$\lim_{n,m\to\infty}\hat{\mathbb{E}}[(\int_0^T |Z_s^n-Z_s^m|^2 ds)^{\frac{\alpha}{2}}]=0.$$

From Proposition 2.11 and the assumption of f, we have

$$\begin{aligned}\hat{\mathbb{E}}[\sup_{t\in[0,T]}|A_t^n-A_t^m|^\alpha] \leq & C\hat{\mathbb{E}}[\sup_{t\in[0,T]}|\hat{Y}_t|^\alpha+(\int_0^T |\hat{f}_s|ds)^\alpha+\sup_{t\in[0,T]}|\int_0^t \hat{Z}_s dB_s|^\alpha] \\ \leq & C\{\hat{\mathbb{E}}[\sup_{t\in[0,T]}|\hat{Y}_t|^\alpha]+\hat{\mathbb{E}}[(\int_0^T |\hat{Z}_s|^2 ds)^{\alpha/2}]\}\to 0.\end{aligned}$$

Using the convergence property of $(Y^n)_{n=1}^\infty$ and $(Z^n)_{n=1}^\infty$, it is easy to check that

$$\lim_{n,m\to\infty}\hat{\mathbb{E}}[\sup_{t\in[0,T]}|A_t^n-A_t^m|^\alpha]=0.$$

■

We now turn to the proof of Theorem 4.2.

Proof. According to Lemma 4.9, for any $2\leq\alpha<\beta$, there exists a triple $(Y,Z,A)\in\mathbb{S}_G^\alpha(0,T)$, such that

$$\hat{\mathbb{E}}[\sup_{t\in[0,T]}|Y_t^n-Y_t|^\alpha]\to 0,\ \hat{\mathbb{E}}[(\int_0^T |Z_s^n-Z_s|^2 ds)^{\frac{\alpha}{2}}]\to 0,\ \hat{\mathbb{E}}[\sup_{t\in[0,T]}|A_t^n-A_t|^\alpha]\to 0,\ \text{as } n\to\infty.$$

We then show that (Y,Z,A) is a solution of the reflected G-BSDE with an upper obstacle. The fact that Y is below the obstacle process S can be derived easily from Lemma 4.5. Besides, since $-A^n$ is a G-submartingale and A^n converges to A uniformly, $-A$ is also a G-submartingale. It remains to check that $\{-\int_0^t(S_s-Y_s)dA_s\}_{t\in[0,T]}$ is a non-increasing G-martingale. Set

$$\widetilde{K}_t^n:=\int_0^t (S_s-Y_s)dK_s^n.$$

Since $S-Y$ is a nonnegative process in $S_G^\alpha(0,T)$, by Lemma 3.4 in [10], $\widetilde{K}^n$ is a non-increasing G-martingale. Note that

$$\begin{aligned}
&\sup_{t\in[0,T]}|-\int_0^t(S_s-Y_s)dA_s-\widetilde{K}_t^n| \\
\leq& \sup_{t\in[0,T]}\{|\int_0^t Y_s dA_s-\int_0^t Y_s dA_s^n|+|\int_0^t(Y_s-Y_s^n)dA_s^n| \\
&+|\int_0^t(Y_s-Y_s^n)dK_s^n|+|\int_0^t -(S_s-Y_s^n)dL_s^n|\} \\
\leq& \sup_{t\in[0,T]}\{|\int_0^t \widetilde{Y}_s^m d(A_s^n-A_s)|+|\int_0^t(Y_s-\widetilde{Y}_s^m)d(A_s^n-A_s)|\} \\
&+\sup_{t\in[0,T]}|Y_s-Y_s^n|[|Var(A^n)|+|K_T^n|]+\sup_{t\in[0,T]}(Y_s^n-S_s)^+|L_T^n|,
\end{aligned}$$

where $\widetilde{Y}_t^m=\sum_{i=0}^{m-1}Y_{t_i^m}I_{[t_i^m,t_{i+1}^m)}(t)$ and $t_i^m=\frac{iT}{m}$, $i=0,1,\cdots,m$. Recalling Lemma 4.3-4.9, by a similar analysis as in the proof of Theorem 5.1 in [18], we have

$$\lim_{n\to\infty}\hat{\mathbb{E}}[\sup_{t\in[0,T]}|-\int_0^t(S_s-Y_s)dA_s-\widetilde{K}_t^n|]\leq C(\hat{\mathbb{E}}[\sup_{t\in[0,T]}|Y_s-\widetilde{Y}_s^m|^2])^{1/2}.$$

Applying Lemma 3.2 in [10] and letting $m\to\infty$, it follows that

$$\lim_{n\to\infty}\hat{\mathbb{E}}[\sup_{t\in[0,T]}|-\int_0^t(S_s-Y_s)dA_s-\widetilde{K}_t^n|]=0,$$

which implies that $\{-\int_0^t(S_s-Y_s)dA_s\}$ is a non-increasing G-martingale.

In the following, we prove that the solution constructed by the penalization procedure is the largest one. Suppose that (Y',Z',A') is the solution of the reflected G-BSDE with parameters (ξ,f,S) and $Y_t'\leq S_t$, $0\leq t\leq T$, we have

$$Y_t'=\xi+\int_t^T f(s,Y_s',Z_s')ds-\int_t^T n(Y_s'-S_s)^+ds-\int_t^T Z_s'dB_s+(A_T'-A_t').$$

Comparing with G-BSDE (4.2) and applying Theorem 3.6, we can easily check that for all $n\in\mathbb{N}$, $Y_t'\leq Y_t^n$. Letting $n\to\infty$, we conclude that $Y_t'\leq Y_t$. ■

Remark 4.10 *The assumption (A3) and (A4) on S and ξ can be weakened in the following sense:*

(A5) *There exist $\{\xi^n\}_{n\in\mathbb{N}}\subset L_G^\beta(\Omega_T)$ and sequence $\{S^n\}_{n\in\mathbb{N}}$ of G-Itô processes*

$$S_t^n=S_0^n+\int_0^t b^n(s)ds+\int_0^t l^n(s)d\langle B\rangle_s+\int_0^t\sigma^n(s)dB_s,$$

with $\{b^n(t)\}$, $\{l^n(t)\}$ belong to $M_G^\beta(0,T)$ and $\{\sigma^n(t)\}$ belong to $H_G^\beta(0,T)$ for all $n\in\mathbb{N}$. Furthermore, $\sup_{n\in\mathbb{N}}\hat{\mathbb{E}}[\sup_{t\in[0,T]}\{|b^n(t)|^\beta+|l^n(t)|^\beta+|\sigma^n(t)|^\beta\}]<\infty$, $\xi^n\leq S_T^n$ and $\xi^n\to\xi$, $\sup_{t\in[0,T]}|S_t^n-S_t|\to 0$ both quasi-surely and in $L_G^\beta(\Omega_T)$ as $n\to\infty$.

Under (A1), (A2) and (A5), we need to consider the following family of G-BSDEs parameterized by $n=1,2,\ldots$.

$$Y_t^n=\xi^n+\int_t^T f(s,Y_s^n,Z_s^n)ds-n\int_t^T(Y_s^n-S_s^n)^+ds-\int_t^T Z_s^n dB_s-(K_T^n-K_t^n).$$

Similar analysis as above, the reflected G-BSDE with parameters (ξ,f,S) has at least one solution.

Remark 4.11 *If we further assume that the process A satisfies the following condition:*

(iv) $A_t = A_t^1 - A_t^2$, $t \in [0,T]$, *where* $A^i \in S_G^\alpha(0,T)$, $i=1,2$, $-A^1$ *is a non-increasing G-martingale,* A^2 *is a non-decreasing process such that* $\int_0^T (S_s - Y_s)dA_s^2 = 0$.

Then the solution satisfying (i), (ii) and (iv) of the reflected G-BSDE with parameters (ξ, f, S) *is unique.*

Assume that (Y,Z,A) *and* $(\tilde{Y}, \tilde{Z}, \tilde{A})$ *are solutions of the reflected G-BSDE satisfying (i), (ii) and (iv). Let* $\hat{Y}_t = Y_t - \tilde{Y}_t$, $\hat{Z}_t = Z_t - \tilde{Z}_t$, $\hat{f}_t = f(t, Y_t, Z_t) - f(t, \tilde{Y}_t, \tilde{Z}_t)$, $\hat{A}_t = A_t - \tilde{A}_t$. *For any* $r, \varepsilon > 0$, *applying Itô's formula to* $\bar{Y}_t^{\frac{\alpha}{2}} e^{rt} = (|\hat{Y}_t|^2 + \varepsilon_\alpha)^{\frac{\alpha}{2}} e^{rt}$, *where* $\varepsilon_\alpha = \varepsilon(1-\alpha/2)^+$, *we get*

$$\begin{aligned}
&\bar{Y}_t^{\alpha/2} e^{rt} + \int_t^T r e^{rs} \bar{Y}_s^{\alpha/2} ds + \int_t^T \frac{\alpha}{2} e^{rs} \bar{Y}_s^{\alpha/2-1} (\hat{Z}_s)^2 d\langle B \rangle_s \\
&= \alpha(1-\frac{\alpha}{2}) \int_t^T e^{rs} \bar{Y}_s^{\alpha/2-2} (\hat{Y}_s)^2 (\hat{Z}_s)^2 d\langle B \rangle_s + \int_t^T \alpha e^{rs} \bar{Y}_s^{\alpha/2-1} \hat{Y}_s \hat{f}_s ds \\
&\quad + \int_t^T \alpha e^{rs} \bar{Y}_s^{\alpha/2-1} \hat{Y}_s d\hat{A}_s - \int_t^T \alpha e^{rs} \bar{Y}_s^{\alpha/2-1} \hat{Y}_s \hat{Z}_s dB_s.
\end{aligned}$$

From the assumption of f, we have

$$\begin{aligned}
&\int_t^T \alpha e^{rs} \bar{Y}_s^{\alpha/2-1} \hat{Y}_s \hat{f}_s ds \le \int_t^T \alpha e^{rs} \bar{Y}_s^{\frac{\alpha-1}{2}} L(|\hat{Y}_s| + |\hat{Z}_s|) ds \\
&\le (\alpha L + \frac{\alpha^2 L}{\underline{\sigma}^2(\alpha-1)}) \int_t^T e^{rs} \bar{Y}_s^{\frac{\alpha}{2}} ds + \frac{\alpha(\alpha-1)}{4} \int_t^T e^{rs} \bar{Y}_s^{\alpha/2-1} (\hat{Z}_s)^2 d\langle B \rangle_s.
\end{aligned}$$

By condition (iv), it is easy to check that

$$\begin{aligned}
&\int_t^T \alpha e^{rs} \bar{Y}_s^{\alpha/2-1} \hat{Y}_s d\hat{A}_s \\
&= \int_t^T \alpha e^{rs} \bar{Y}_s^{\alpha/2-1} \hat{Y}_s d(A_s^1 - \tilde{A}_s^1) + \int_t^T \alpha e^{rs} \bar{Y}_s^{\alpha/2-1} \hat{Y}_s d(\tilde{A}_s^2 - A_s^2) \\
&\le \int_t^T \alpha e^{rs} \bar{Y}_s^{\alpha/2-1} (\hat{Y}_s)^- d\tilde{A}_s^1 + \int_t^T \alpha e^{rs} \bar{Y}_s^{\alpha/2-1} (\hat{Y}_s)^+ dA_s^1.
\end{aligned}$$

Let $M_t = \int_0^t \alpha e^{rs} \bar{Y}_s^{\alpha/2-1} \hat{Y}_s \hat{Z}_s dB_s - \int_0^t \alpha e^{rs} \bar{Y}_s^{\alpha/2-1} (\hat{Y}_s)^- d\tilde{A}_s^1 - \int_0^t \alpha e^{rs} \bar{Y}_s^{\alpha/2-1} (\hat{Y}_s)^+ dA_s^1$. *Then it is a G-martingale. Let* $r = \alpha L + \frac{\alpha^2 L}{\underline{\sigma}^2(\alpha-1)} + 1$, *we have*

$$\bar{Y}_t^{\alpha/2} e^{rt} + (M_T - M_t) \le 0.$$

Taking conditional expectations on both sides, it follows that $Y \equiv \tilde{Y}$. *By applying Itô's formula to* $(Y_t - \tilde{Y}_t)^2 (\equiv 0)$ *on* $[0,T]$ *and taking expectations, we get*

$$\hat{\mathbb{E}}[\int_0^T (Z_s - \tilde{Z}_s)^2 d\langle B \rangle_s] = 0,$$

which implies $Z \equiv \tilde{Z}$. *Then it is easy to check that* $A \equiv \tilde{A}$.

5 Application to optimal stopping problem under volatility uncertainty

Recall that, in the classical case, the solution of reflected BSDE can be represented as the value function of an optimal stopping problem (see [6]). In fact, [18] establishes a similar relation for the reflected G-BSDE with a lower obstacle (see also [16]). Since the G-expectation is an upper expectation induced by a non-dominated family of probability measures $P \in \mathcal{P}$, the corresponding optimal stopping problem is essentially a "$\sup_\tau \sup_P$" problem, where τ represents a stopping time. It is natural to conjecture that the solution of reflected G-BSDE with an upper obstacle coincides with the value function of an optimal stopping problem. Furthermore, this problem can be interpreted as an "$\inf_\tau \sup_P$" problem, which is more complicated than the cases studied before.

To begin with, let us first introduce some spaces and random times appropriate for the optimal stopping problem under G-expectation framework. For more details, we may refer to the papers [13].

Consider a G-expectation space $(\Omega_T, L_G^1(\Omega_T), \hat{\mathbb{E}})$. Define the following spaces:

$$\begin{aligned} L^0(\Omega_T) &:= \{X : \Omega_T \to [-\infty, \infty] \text{ and } X \text{ is } \mathcal{B}(\Omega_T)\text{-measurable}\}, \\ \mathbb{L}^p(\Omega_T) &:= \{X \in L^0(\Omega_T) : \hat{\mathbb{E}}[|X|^p] < \infty\} \text{ for } p \geq 1, \\ L_G^{1^*}(\Omega_T) &:= \{X \in \mathbb{L}^1(\Omega_T) : \exists \{X_n\} \subset L_G^1(\Omega_T) \text{ such that } X_n \downarrow X, q.s.\}. \end{aligned}$$

Definition 5.1 ([13]) *A random time $\tau : \Omega \to [0, T]$ is called a $*$-stopping time if $I_{\{\tau \geq t\}} \in L_G^{1^*}(\Omega_t)$ for each $t \in [0, T]$.*

Example 5.2 ([13]) *A typical example of the $*$-stopping time is the first exist time for a right continuous process. More precisely, let $X = \{X_t\}_{t\in[0,T]}$ be a 1-dimensional right continuous process such that $X_t \in L_G^1(\Omega_t)$ for any $t \in [0, T]$. Then τ defined below is a $*$-stopping time*

$$\tau = \inf\{t \geq 0 : X_t \notin F\} \wedge T,$$

where $F \subset \mathbb{R}$ is a fixed closed set.

Theorem 5.3 ([13]) *For any $\xi \in L_G^p(\Omega_T)$ with $p > 1$, let $M_t = \hat{\mathbb{E}}_t[\xi]$ for any $t \in [0, T]$ and let σ, τ be two $*$-stopping times with $0 \leq \sigma \leq \tau \leq T$. Then, we have $M_\sigma = \hat{\mathbb{E}}_\sigma[M_\tau]$.*

Let $\mathcal{T}_{0,T}$ be the collection of all random times τ such that there exists a sequence of $*$-stopping times $\{\tau_n\}_{n\in\mathbb{N}}$ such that τ_n converges to τ, q.s.

Lemma 5.4 *Suppose that $X \in S_G^p(0, T)$ is a G-martingale, where $p > 1$. Then, for any $\tau \in \mathcal{T}_{0,T}$, we have $\hat{\mathbb{E}}[X_\tau] = X_0$.*

Proof. By Theorem 5.3, for any $*$-stopping time τ, we have $\hat{\mathbb{E}}[X_\tau] = X_0$. Recall that, for any $X \in S_G^p(0, T)$ with $p > 1$, we have the following uniform continuity property (see [17])

$$\lim_{\varepsilon\to 0} \hat{\mathbb{E}}[\sup_{t\in[0,T]} \sup_{s\in[t,t+\varepsilon]} |X_t - X_s|^p] = 0,$$

where $X_s := X_T$ for any $s > T$. The desired result follows easily from the uniform continuity property. ■

Proposition 5.5 *Suppose that Assumptions (A1)-(A4) hold. Let (Y, Z, A) be the maximal solution to reflected G-BSDE with parameters (ξ, f, S). Then, we have*

$$Y_0 = \inf_{\tau\in\mathcal{T}_{0,T}} \hat{\mathbb{E}}[\int_0^\tau f(s, Y_s, Z_s)ds + \xi I_{\{\tau=T\}} + S_\tau I_{\{\tau<T\}}].$$

Proof. First, by the result in [19], for any $\tau \in \mathcal{T}_{0,T}$, we have

$$E[\int_0^\tau Z_s dB_s] = 0.$$

Let (Y^n, Z^n, A^n) be the approximation sequence for (Y, Z, A), where $A^n = L^n - K^n$ and (Y^n, Z^n, K^n) is the solution to the penalized G-BSDE (4.2). We first show that

$$Y_0 \leq \inf_{\tau \in \mathcal{T}_{0,T}} \hat{\mathbb{E}}[\int_0^\tau f(s, Y_s, Z_s)ds + \xi I_{\{\tau=T\}} + S_\tau I_{\{\tau<T\}}].$$

In fact, noting that $L^n \leq 0$ for any $n \in \mathbb{N}$, it is easy to check that

$$\begin{aligned} Y_0 &= \int_0^\tau f(s, Y_s, Z_s)ds - \int_0^\tau Z_s dB_s + Y_\tau + A_\tau \\ &\leq \int_0^\tau f(s, Y_s, Z_s)ds - \int_0^\tau Z_s dB_s + \xi I_{\{\tau=T\}} + S_\tau I_{\{\tau<T\}} + A_\tau - A^n_\tau - K^n_\tau. \end{aligned}$$

Putting K^n_τ to the left-hand side and taking expectations, by Lemma 5.4, we obtain that

$$Y_0 \leq \hat{\mathbb{E}}[\int_0^\tau f(s, Y_s, Z_s)ds + \xi I_{\{\tau=T\}} + S_\tau I_{\{\tau<T\}}] + \hat{\mathbb{E}}[|A_\tau - A^n_\tau|].$$

Letting n go to infinity, it follows that, for any $\tau \in \mathcal{T}_{0,T}$,

$$Y_0 \leq \hat{\mathbb{E}}[\int_0^\tau f(s, Y_s, Z_s)ds + \xi I_{\{\tau=T\}} + S_\tau I_{\{\tau<T\}}].$$

It remains to prove the reverse inequality. Set

$$D = \inf\{t \in [0, T] : S_t = Y_t\} \wedge T.$$

It suffices to show that $D \in \mathcal{T}_{0,T}$ and

$$Y_0 = \hat{\mathbb{E}}[\int_0^D f(s, Y_s, Z_s)ds + \xi I_{\{D=T\}} + S_D I_{\{D<T\}}].$$

In fact, for any $m \in \mathbb{N}$, let

$$D^m = \inf\{t \in [0, T] : S_t - Y_t < \frac{1}{m}\} \wedge T.$$

By Example 5.2, D^m is a $*$-stopping time. Clearly, D^m converges to D, q.s. Therefore, $D \in \mathcal{T}_{0,T}$. Applying the comparison theorem 2.14 yields that $Y^{n_1} \leq Y^{n_2}$, with $n_1 \leq n_2$. It follows that

$$Y^n - S = Y^n - Y + Y - S \leq Y - S.$$

By simple calculation, for any $n \in \mathbb{N}$, we have

$$|L^n_D| = n \int_0^D (Y^n_s - S_s)^+ ds \leq n \int_0^D (Y_s - S_s)^+ ds = 0.$$

Note that

$$\begin{aligned} Y_0 &= \int_0^D f(s, Y_s, Z_s)ds - \int_0^D Z_s dB_s + Y_D + A_D \\ &= \int_0^D f(s, Y_s, Z_s)ds - \int_0^D Z_s dB_s + \xi I_{\{D=T\}} + S_D I_{\{D<T\}} + A_D - A^n_D - K^n_D. \end{aligned}$$

Putting K^n_D to the left-hand side and then taking expectations, finally, letting $n \to \infty$, we get the desired result. The proof is complete. ■

Remark 5.6 *Especially, if Y is the maximal solution to reflected G-BSDE with parameters $(S_T, 0, S)$, by Proposition 5.5 and Theorem 2.6, we have*

$$Y_0 = \inf_{\tau\in\mathcal{T}_{0,T}} \hat{\mathbb{E}}[S_\tau] = \inf_{\tau\in\mathcal{T}_{0,T}} \sup_{P\in\mathcal{P}} E_P[S_\tau].$$

For a more general setting of the optimal stopping problem under adverse nonlinear expectations, we may refer to the paper [24].

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