STOCHASTIC LAGRANGIAN FLOWS FOR SDES WITH ROUGH COEFFICIENTS

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Abstract. We prove the existence and uniqueness of Stochastic Lagrangian Flows and almost everywhere Stochastic Flows for non-degenearted SDEs with rough coefficients. As an application of our main result, we show that there exists a unique Stochastic Flow corresponding to each Leray-Hopf solution of 3D Navier-Stokes equation in the DiPerna-Lions sense.

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1. Introduction

The Navier-Stokes equation

$$\begin{cases} \partial_t u = \nu \Delta u + u \cdot \nabla u + \nabla p, \\ \operatorname{div} u = 0, \\ u(0) = \varphi \end{cases} \tag{1.1}$$

in $\mathbb{R}^d$ is the core of Eulerian approach dealing with the time evolution of the velocity fields of Newtonian fluids. Here u represents the velocity, $\nu > 0$ is the viscosity constant and p is the pressure. It is well known that for any divergence free vector field $\varphi \in L^2(\mathbb{R}^d)$, there exists a divergence free Leray-Hopf weak solution to NS equations in

$$V := \left\{ u : \|u\|_{L^\infty([0,T];L^2(\mathbb{R}^d))} + \|\nabla u\|_{L^2([0,T];L^2(\mathbb{R}^d))} < \infty, \quad \forall T > 0 \right\}. \tag{1.2}$$

However, it is still unknown whether the above Leray-Hopf solution is unique and smooth when $d = 3$, which is one of the most famous open problems in the area of partial differential equations.

If one imagine the fluid as being composed of many 'fluid particles', then one can work out the paths followed by these particles, this is the Lagrangian approach to hydrodynamics studies the configuration of the underlying particles, namely the solutions of the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} X_t(x) = u(t, X_t(x)), \quad X_0(x) = x \in \mathbb{R}^d. \tag{1.3}$$

It was first proved by Chemin and Lerner [3] that if $d = 2$, there is a unique trajectories corresponding to each Leray-Hopf solution u with an initial condition that is only L^2. However, when $d = 3$ and u is a Leray-Hopf weak solution of (1.1) with initial data $\varphi \in H^{\frac{1}{2}}$, only local well-posedness of (1.3) is proved. Their proof was later simplified by Dashti and Robinson in [5].

It is also interesting and meaningful to construct the stochastic Lagrangian particle trajectory X_t associated with the velocity field u. More precisely, suppose W is a $d-$dimensional standard

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Brownian motion on some probability space $(\Omega, \mathscr{F}, \mathbf{P})$, u is a Leray-Hopf weak solution to (1.1), we want to investigate the well-posedness of the following SDE

$$\mathrm{d}X_t = u(t, X_t)\mathrm{d}t + \sqrt{2\nu}\mathrm{d}W_t,\ X_0 = \xi, \tag{1.4}$$

in weak or strong sense. If u is smooth in x, then by Constantin and Iyer's representation [4] (see also [20]), u can be reconstructed from $X_t(x)$, the unique solution to (1.4) with $\xi = x$, as follows:

$$u(t,x) = \mathcal{P}\mathbb{E}(\nabla^{\mathrm{t}} X_t^{-1}(x) \cdot \varphi(X_t^{-1}(x))),$$

where $\mathcal{P}$ is the Leray projection and $X_t^{-1}(x)$ is the inverse of stochastic flow $x \mapsto X_t(x)$, and ∇^{t} stands for the transpose of the Jacobian matrix. However, it is well known that even if the initial data is smooth with compact support, the smoothness of u in x can only be proved in short time. When the initial data φ is only square integrable, by (1.2) and Sobolev's embedding,

$$u \in L^q_t L^p_x,\quad p, q \geqslant [2, \infty],\quad \tfrac{d}{p} + \tfrac{2}{q} = \tfrac{d}{2}. \tag{1.5}$$

When $d \geqslant 2$, the classic result of Krylov and Röckner [9] can not be applied in this case, since in their work, the drift term u in (1.4) should satisfy the following Ladyzhenskaya-Prodi-Serrin's type condition (abbreviated as LPS):

$$u \in L^q_t L^p_x,\quad p, q \in [2, \infty),\quad \tfrac{d}{p} + \tfrac{2}{q} < 1.$$

See also [22, 24] for further discussion. Recently, in [26], Zhang and the author of this paper studied the following singular SDE:

$$X_t = x + \int_0^t b_r(X_s)\mathrm{d}s + W_t$$

beyond the LPS condition. Their main result shows that if

$$b, (\mathrm{div} b)^- \in L^q_t L^p_x,\quad p, q \in [2, \infty),\quad \tfrac{d}{p} + \tfrac{2}{q} < 2,$$

then the above SDE admits at least one weak(martingale) solution. This implies that when $d = 3$, for each $x \in \mathbb{R}^3$, (1.4) admits at least one weak solution. Unfortunately, the uniqueness of finite dimensional distribution of solutions to the above SDE starting from each single point was not proved or disproved in that work, and we tend to think it is not true.

Motivated by [4] and [26], in this paper, we want to show a suitable sort of well-posedness of the following Itô's type SDE with rough coefficients:

$$X_t = \xi + \int_0^t b(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_s)\mathrm{d}W_s, \tag{1.6}$$

where $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^m$ are measurable functions and W is an $m-$dimensional standard Brownian motion. For deterministic case, in the celebrated paper [6], DiPerna and Lions studied the connection between the transport equation and the associated ODE

$$X_t(x) = x + \int_0^t b(s, X_s(x))\mathrm{d}s.$$

They showed that the existence and uniqueness for the transport equation is equivalent to a sort of well-posedness of the ODE. Roughly speaking, their result shows the ODE has a unique solution for λ_d-a.e. initial datum(here and below, λ_d denotes the Lebesgue measure in $\mathbb{R}^d$) provided that $b \in L^1_t(W^{1,1}_{loc})$ and $\mathrm{div} b \in L^\infty_{t,x}$. In [1], Ambrosio developed the theory of Regular Lagrangian Flows(Abbreviated as RLF), which relates existence and uniqueness for the continuity equation with well-posedness of the ODE and the well-posedness of the continuity equations in L^∞ is proved in the case of vector fields with BV regularity whose distributional divergence is in L^∞. Later, Figalli [7] studied the stochastic counterpart of RLF and a formalization akin to that of DiPerna-Lions is introduced, the main objects is called Stochastic Lagrangian Flows(see

Definition 2.3). As in the deterministic case, the well-posedness of SDE (1.6) has to be understood "in average" with respect to λ_d-a.e. initial datum. Such a theory provides efficient tools to study stochastic differential equations under low regularity assumptions(see also [18]). Deep connections between well-posedness of Fokker-Planck equation and martingale problems associated with (1.6) are established in their works, in particular for a wide class of diffusions having not necessarily continuous nor elliptic coefficients, provided that some Sobolev regularity holds.

Let us denote

$$a^{ij}(t,x) := \frac{1}{2}\sigma^{ik}(t,x)\sigma^{kj}(t,x), \quad Lf := a^{ij}\partial_{ij}f + b^i\partial_i f,$$

and

$$L^*f := \partial_{ij}(a^{ij}f) - \partial_i(b^if) = \partial_j(a^{ij}\partial_i f) - \partial_i(V^if),$$

where $V^i := b^i - \partial_j a^{ij}$. Formally, the distribution of X_t solves the following Fokker-Planck(or Kolmogorov's forward) in the sense of distribution:

$$\begin{cases} \partial_t\mu_t - L_t^*\mu_t = \partial_t\mu_t - \partial_{ij}(a^{ij}\mu_t) + \partial_i(b^i\mu_t) = 0 \\ \mu_0 = \bar{\mu} \end{cases} \qquad (\mathbf{FPE}_1)$$

where $\bar{\mu} = \mathbf{P}\circ X_0^{-1}$. As showed in [7] and [18] under some mild conditions, the existence and uniqueness of Stochastic Lagrangian Flow associated to L is equivalent with the well-posedness of above Fokker-Planck equation in L^∞ setting. So a good understanding of above equation is crucial for studying of Stochastic Lagrangian Flow associated to L. If μ_t is absolutely continuous with respect to the Lesbesgue measure, and $\mu_t(\mathrm{d}x) = u(t,x)\lambda_d(\mathrm{d}x)$, $\bar{\mu}(\mathrm{d}x) = \phi(x)\lambda_d(\mathrm{d}x)$, then the above equation can be rewritten as

$$\begin{cases} \partial_t u - \nabla\cdot(a\nabla u) + \nabla\cdot(Vu) = 0 \\ u(0) = \phi. \end{cases} \qquad (\mathbf{FPE}_2)$$

Inspired by [7] and [18], in this paper, by studying the above Forker-Planck equation in $L^\infty-$setting, we establish the well-posedness of (1.6) in the sense of DiPerna-Lions(or Stochastic Lagrangian Flow corresponding to (1.6)) under some local integrability assumptions on b, $\partial_j a^{ij}$ and $\mathrm{div}V$ (see ($\mathbf{A}_2$) below), provide that the diffusion coefficient a is uniformly elliptic. Compared with the result in [7] and [18] for bounded elliptic case, we do not need to assume the drift coefficient b is bounded in x. To prove the existence and uniqueness for L^∞ solution of ($\mathbf{FPE}_2$), the key point for us is to establish a priori global maximum estimate. We use the classic energy method and establish the key maximum estimate (3.11) by De Giorgi's iteration. It should be mentioned that similar local maximum principle for homogenous Kolmogorov's backward equation is proved by Nazarov and Ural'tseva in [12] by using Moser's iteration. In [26], global result for general backward equation was established by De Giorgi's method. And when b is divergence-free and smooth, Qian and Xi [13] studied the Aronson's type estimate for the heat kernel of operator $\mathscr{L}_t^b = \Delta + b\cdot\nabla$, where the bound depends only on the norm $\|b\|_{L_t^l L_x^q}$, where $q, l \in (2,\infty)$ satisfies $1 \leqslant \frac{d}{q} + \frac{2}{l} < 2$.

Obviously, the Stochastic Lagrangian Flows are close related to the weak solutions to SDE. In [21] and [23], Zhang proposed the "strong" version of Stochastic Lagrangian Flows, which is called almost everywhere Stochastic Flow. When b and σ satisfy some Sobolev regularity assumption(see ($\mathbf{A}_4$) below), in this paper, a pathwise uniqueness result is proved for particular solutions to the original SDE (1.6). Combine this and a Yamada-Watanabe's type argument, we show that there is a unique almost everywhere Stochastic Flow corresponding to (1.6). Since each Leray-Hopf solution u of 3D-NS equation with L^2 initial datum is in $L_t^2 W_x^{1,2}$, our results imply a sort of strong well-posedness of (1.4). However, we should point out that the weak differentiability of the stochastic flow with respect to the starting point x remains open.

This paper is organized as follows: In Section 2, we give some basic definitions of certain local Sobolev spaces and state our main results. In Section 3, we study the Fokker-Planck equation (**FPE**$_2$) using classic energy method and establish the key maximum estimate (3.11) by De Giorgi's iteration. In Section 4, we prove our main result Theorem 2.4 and Theorem 2.7.

2. Definition and Main Results

Suppose $(E,\mathcal{E})$ is a measurable space, the collection of all $\sigma-$finite measures and probability measures on E are denoted by $\mathscr{M}(E)$ and $\mathscr{P}(E)$, respectively. Given $T>0$, let $C([0,T];\mathbb{R}^d)$ be the continuous function space equipped with the uniform topology, ω_t be the canonical process on it and $\mathcal{B}_t := \sigma\{\omega_s \in C([0,T];\mathbb{R}^d) : 0 \leqslant s \leqslant t\}$.

For $p,q\in[1,\infty]$, we define

$$\mathbb{L}^p_q(T) := L^q([0,T];L^p(\mathbb{R}^d)),$$

and $\mathbb{L}^p(T):=\mathbb{L}^p_p(T)$. For $p,q\in(1,\infty), s\in\mathbb{R}$, we also define

$$\mathbb{H}^{s,p}_q(T) = L^q([0,T];H^{s,p}(\mathbb{R}^d)),$$

where $H^{s,p}$ is the Bessel potential space. The usual energy space is defined as the following way:

$$V(T) := \Big\{ f\in \mathbb{L}^2_\infty(T)\cap L^2([0,T];H^1) : \|f\|_{V(T)} := \|f\|_{\mathbb{L}^2_\infty} + \|\nabla_x f\|_{\mathbb{L}^2(T)} < \infty \Big\}.$$

Throughout this paper we fix a cutoff function

$$\chi\in C^\infty_c(\mathbb{R}^d;[0,1]) \text{ with } \chi|_{B_1}=1 \text{ and } \chi|_{B_2^c}=0,$$

and for $r>0$ and $x\in\mathbb{R}^d$, define

$$\chi_r(x) := \chi(r^{-1}x),\ \chi^y_r(x) := \chi_r(x-y), \quad x\in\mathbb{R}^d. \tag{2.1}$$

Next we introduce the localized Sobolev spaces for later use.

Definition 2.1. *Let $p,q\in[1,\infty]$, we define the Banach space: for fixed $r>0$,*

$$\widetilde{\mathbb{L}}^p_q(T) := \left\{ f\in L^q([0,T];L^p_{loc}(\mathbb{R}^d)) : \|f\|_{\widetilde{\mathbb{L}}^p_q(T)} := \sup_{y\in\mathbb{R}^d} \|f\chi^y_r\|_{\mathbb{L}^p_q(T)} < \infty. \right\}$$

and $\widetilde{\mathbb{L}}^p(T) := \widetilde{\mathbb{L}}^p_p(T)$; For any $p,q\in(1,\infty)$, $s\in\mathbb{R}$,

$$\widetilde{\mathbb{H}}^{s,p}_q(T) := \left\{ f\in L^q([0,T];H^{s,p}_{loc}) : \|f\|_{\widetilde{\mathbb{H}}^{s,p}_q(T)} := \sup_{y\in\mathbb{R}^d} \|f\chi^y_r\|_{\mathbb{H}^{s,p}_q(T)} \right\}.$$

Moreover, we also introduce the localized energy space

$$\widetilde{V}(T) := \left\{ f\in \widetilde{\mathbb{L}}^2_\infty(T)\cap\widetilde{\mathbb{H}}^{1,2}_2(T) : \|f\|_{\widetilde{V}(T)} := \|f\|_{\widetilde{\mathbb{L}}^2_\infty(T)} + \|\nabla_x f\|_{\widetilde{\mathbb{L}}^2(T)} < \infty \right\},$$

$$\widetilde{V}^0(T) := \Big\{ f\in\widetilde{V}(T) : \text{for any } r>0, y\in\mathbb{R}^d, t\mapsto f(t)\chi^y_r \text{ is strong continuous from } [0,T] \text{ to } L^2(\mathbb{R}^d) \Big\}.$$

Let us recall the definition of martingale solutions associated to operator L.

Definition 2.2 (MP)**.** *A continuous process $\{X_t\}_{t\in[0,T]}$ with value in $\mathbb{R}^d$ define on some filtered probability space $(\Omega,\mathscr{F},\mathscr{F}_t,\mathbf{P})$ is a solution of the martingale problem(MP) associated to (L,μ_0) or martingale solution to (1.6), if it holds*

$$\mathbf{P}\circ X_0^{-1}=\mu_0=law(\xi)\in\mathscr{P}(\mathbb{R}^d);$$

$$\mathbf{E}\int_0^T|a(t,X_t)|+|b(t,X_t)|\mathrm{d}t<\infty;$$

and for each $f\in C^{1,2}_{t,x}$, the process

$$t\mapsto M_t^f:=f(t,X_t)-f(0,X_0)-\int_0^t[\partial_s+L_s]f(s,X_s)\mathrm{d}s$$

is a $\mathscr{F}_t-$martingale. Or equivalently, a probability measure $\mathbb{P}$ on $C([0,T];\mathbb{R}^d)$ is a solution to MP associated to (L,μ_0) or martingale solution of (1.6), if the above relations hold for $(C([0,T];\mathbb{R}^d),\mathcal{B},\mathcal{B}_t,\mathbb{P},\omega)$.

The following definition of Stochastic Lagrangian Flow is taken from [7].

Definition 2.3 (SLF)**.** *Given a measure $m_0=\rho_0\lambda_d\in\mathscr{M}(\mathbb{R}^d)$ with $\rho_0\in L^\infty$, we say that a measurable family of probability measures $\{\mathbb{P}_x\}_{x\in\mathbb{R}^d}$ on $C([0,T];\mathbb{R}^d)$ is a m_0-Stochastic Lagrangian Flow (m_0-SLF) associated with L, if:*

(i) for m_0-a.e. x, $\mathbb{P}_x$ is a martingale solution of the SDE (1.6) starting from x;

(ii) for any $t\in[0,T]$

$$m_t:=\int_{\mathbb{R}^d}\mathbb{P}_x\circ\omega_t^{-1}\,m_0(\mathrm{d}x)\ll\lambda_d,$$

and $m_t=\rho_t\lambda_d$ with $\rho_t\in L^\infty$ uniformly in $t\in[0,T]$.

And the λ_d-SLF is abbreviated as SLF.

Our main assumptions on the coefficients a and b are following:

Assumption 1. *There are constants $\Lambda>1$, $\kappa>0$ $p_1,p_2,q_1,q_2\in[2,\infty)$ and $\frac{d}{p_i}+\frac{2}{q_i}<2$ $(i=1,2)$ such that*

$$\Lambda^{-1}|\xi|^2\leqslant a^{ij}\xi_i\xi_j\leqslant\Lambda|\xi|^2; \tag{$\mathbf{A}_1$}$$

$$\|b\|_{\widetilde{\mathbb{L}}^{p_1}_{q_1}(T)}+\|\partial_j a^{ij}\|_{\widetilde{\mathbb{L}}^{p_1}_{q_1}(T)}+\|(\nabla\cdot V)^-\|_{\widetilde{\mathbb{L}}^{p_2}_{q_2}(T)}\leqslant\kappa; \tag{$\mathbf{A}_2$}$$

$$\partial_t a^{ij}\in\mathbb{L}^\infty(T). \tag{$\mathbf{A}_3$}$$

The following Theorem is our first main result:

Theorem 2.4. *Under* **Assumption** *1,*

(1) for any $m_0=\rho_0\lambda_d\in\mathscr{M}(\mathbb{R}^d)$ with $\rho_0\in L^\infty$, then there is a unique μ_0-SLF associated with L;

(2) for any $\mu_0\in\mathscr{P}(\mathbb{R}^d)$ with bounded density with respect to λ_d, there is a unique martingale solution $\mathbb{P}$ associated to (L,μ_0) such that $\mu_t=\mathbb{P}\circ\omega_t^{-1}\ll\lambda_d$ and $\mu_t=\rho_t\lambda_d$ with $\rho_t\in L^\infty$ uniformly in t.

If u is a Leray-Hopf solution to 3D-NS equation with initial condition $u(0)\in L^2(\mathbb{R}^3)$, then $u\in L^\infty([0,T];L^2)\cap L^2([0,T];H^1)$, by Sobolev embedding and interpolation theorem,

$$u\in\mathbb{L}^p_q(T),\quad\tfrac{3}{p}+\tfrac{2}{q}=\tfrac{3}{2}<2,\ p,q\in[2,\infty].$$

Thus, Theorem 2.4 implies the following corollary:

Corollary 2.5. *Suppose u is the Leray-Hopf weak solution to 3D-NS equation with L^2 initial datum, then*

(1) *for $m_0 \in \mathscr{M}(\mathbb{R}^3)$ with a bounded density with respective to λ_3, there is a unique m_0-SLF associated with (1.4);*
(2) *for any $\mu_0 \in \mathscr{P}(\mathbb{R}^3)$ with bounded density with respect to λ_3, (1.4) admits a unique martingale solution $\mathbb{P}$ such that $\mu_t = \mathbb{P} \circ \omega_t^{-1} \ll \lambda_3$ and $\mu_t = \rho_t \lambda_3$ with $\rho_t \in L^\infty$ uniformly in t.*

From the probabilistic view, both results above are about the weak(martingale) solutions of SDE. Notice that a Leray-Hopf solution u of 3D-NS equation with L^2 initial datum is in $\mathbb{H}_2^{1,2}(T)$. Our next main result show that the Sobolev regularity of u leads a sort of well-posedness of (1.4) in strong sense. Before presenting our statement of second theorem, let us give the definition of almost everywhere Stochastic Flow mentioned by Zhang in [23, Definition 2.1], which can be regard as the "strong" version of SLF.

Definition 2.6 (AESF). *Suppose $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbf{P})$ is a filtered probability space satisfying the common conditions and W is a standard $d-$dimensional Brownian motion on it. Given a measure $m_0 = \rho_0 \lambda_d \in \mathscr{M}(\mathbb{R}^d)$ with $\rho_0 \in L^\infty$, we say a $\mathbb{R}^d-$valued measurable stochastic field on $[0,T] \times \Omega \times \mathbb{R}^d$, $X_t(\omega, x)$, is a m_0-**almost everywhere** Stochastic Flow (AESF) of (1.6) if*

(1) *$\{\mathbb{P}_x\}_{x\in\mathbb{R}^d} := \{\mathbf{P} \circ X^{-1}(x)\}_{x\in\mathbb{R}^d}$ is a m_0-SLF corresponding to L;*
(2) *for m_0-almost all $x \in \mathbb{R}^d$, $X_t(x)$is a continuous $\mathscr{F}_t-$adapted process satisfying that*

$$X_t(x) = x + \int_0^t b(s, X_s(x)) \mathrm{d}s + \sigma(s, X_s(x)) \mathrm{d}W_s, \quad \forall t \in [0,T].$$

In order to get the well-posedness of almost everywhere Stochastic Flow, we need a stronger assumption on the coefficients.

Assumption 2. *The coefficients b and σ satisfy*

$$b \in L^1([0,T], W^{1,1}_{loc}(\mathbb{R}^d)),\ \sigma \in L^2([0,T]; W^{1,2}_{loc}(\mathbb{R}^d)). \tag{$\mathbf{A}_4$}$$

Theorem 2.7. *Under* **Assumption** *1 and 2,*

(1) *for any $m_0 = \rho_0 \lambda_d \in \mathscr{M}(\mathbb{R}^d)$ with $\rho_0 \in L^\infty$, equation (1.6) admits a unique m_0-AESF;*
(2) *if $\xi \in \mathscr{F}_0$ is a random variable with bounded density, then equation (1.6) has a unique strong solution X_t such that the density of $\mathbf{P} \circ X_t^{-1}$ is uniformly bounded in t.*

We should emphasize that a similar result had been stated in [11] under the assumptions that $\sigma = \mathbb{I}$, $\nabla \cdot b = 0$, $b \in \mathbb{H}_1^{1,r} \cap \mathbb{L}_q^p$ with $r > 1$, $\frac{d}{p} + \frac{2}{q} < 2$. Their argument essentially follows Zhang [23]. In this paper, we will give a slight different proof based on some techniques from [19], [23] and [2]. Theorem 2.7 implies

Corollary 2.8. *If $d = 3$,*

(1) *for any $m_0 = \rho_0 \lambda_d \in \mathscr{M}(\mathbb{R}^d)$ with $\rho_0 \in L^\infty$, there is a unique m_0-AESF corresponding to each Leray-Hopf solution of 3D-NS equation;*
(2) *for any random variable $\xi \in \mathscr{F}_0$ with bounded density, equation (1.4) admits a unique strong solution X_t satisfying $\mathbf{P} \circ X_t^{-1} \in \mathbb{L}^\infty(T)$.*

3. Kolmogorov and Fokker-Planck Equation

In this section, we study the Fokker-Planck Equation associated to (1.6) and establish the well-posedness of ($\mathbf{FPE}_2$) in L^∞ setting.

Here and in the sequel, we always assume $d \geqslant 2$, $(p_i, q_i, e_i) \in (1,\infty)^2 \times (0,1)$ and

$$\frac{d}{p_i} + \frac{2}{q_i} = 2 - e_i. \tag{3.1}$$

For any (p_i,q_i) given above, we define $p_i^*, q_i^* \in [2,\infty)$ by relations

$$\frac{1}{p_i}+\frac{2}{p_i^*}=1,\quad \frac{1}{q_i}+\frac{2}{q_i^*}=1, \tag{3.2}$$

which implies that

$$\frac{d}{p_i}+\frac{2}{q_i}=2-e_i \Leftrightarrow \frac{d}{p_i^*}+\frac{2}{q_i^*}=\frac{d+e_i}{2}. \tag{3.3}$$

Let I be an open interval of $\mathbb{R}$ and D be a domain in $\mathbb{R}^d$, $Q:=I\times D$. Consider the following PDE:

$$\partial_t u-\nabla\cdot(a\nabla u)+\nabla\cdot(Vu)+cu=f \text{ in } Q. \tag{3.4}$$

Definition 3.1. *We say $u\in \widetilde{V}(Q)$ is a subsolution(supersolution) to (3.4) if for any almost every $t\in I$, $\varphi\in C_c^\infty(Q)$ with $\varphi\geqslant 0$,*

$$\int_D u(t)\varphi(t)+\int_{D_t}\big[-u\partial_t\varphi+(a\nabla u)\cdot\nabla\varphi-uV\cdot\nabla\varphi+cu\varphi\big]\leqslant(\geqslant)\int_{D_t}f\varphi, \tag{3.5}$$

where $D_t=(I\cap(-\infty,t])\times D$.

3.1. **A maximum principle.** We first prove an energy inequality for the subsolution of (3.4), which is crucial for the De-Giorgi iteration technique.

We need the following assumption:

$$\|V\|_{\widetilde{\mathbb{L}}^{p_1}_{q_1}}+\|(\tfrac{1}{2}\nabla\cdot V+c)^-\|_{\widetilde{\mathbb{L}}^{p_2}_{q_2}}+\|(\nabla\cdot V+c)^-\|_{\widetilde{\mathbb{L}}^{p_2}_{q_2}}\leqslant\kappa'. \tag{$\mathbf{A}_2$'}$$

Lemma 3.2 (Energy inequality)**.** *Let $0<\rho<R\leqslant 1$, $k>0$, $I\subseteq\mathbb{R}$, $Q=I\times B_R$. Suppose $u\in V(Q)$ is a locally bounded weak subsolution to (3.4) and a,V,c satisfy ($\mathbf{A}_1$), ($\mathbf{A}_2$'). η is a cut off function in x, compactly supported in B_R, $\eta(x)\equiv 1$ in B_ρ and $|\nabla\eta|\leqslant 2(R-\rho)^{-1}$. Then, for any $u_k:=(u-k)^+$ and almost every $s,t\in I$ with $s<t$, we have*

$$\begin{aligned}&\left(\int_D u_k^2\eta^2\right)(t)-\left(\int_D u_k^2\eta^2\right)(s)+\int_s^t\!\!\int_D|\nabla(u_k\eta)|^2\\ \leqslant&\frac{C}{(R-\rho)^2}\left(\|u_k\|^2_{\mathbb{L}^2(A_s^t(k))}+\sum_{i=1}^3\|u_k\|^2_{\mathbb{L}^{p_i^*}_{q_i^*}(A_s^t(k))}\right)+C\left(k^2+\|f\|^2_{\widetilde{\mathbb{L}}^{p_3}_{q_3}}\right)\sum_{i=2}^3\|\mathbf{1}_{A_s^t(k)}\|^2_{\mathbb{L}^{p_i^*}_{q_i^*}},\end{aligned} \tag{3.6}$$

where $A_s^t(k)=\{u>k\}\cap[s,t]\times B_R$ and the constant C only depends on d,Λ,κ and (p_i,q_i).

Proof. We claim that : for almost every $s,t\in I$ with $s<t$, it holds that

$$\begin{aligned}&\frac{1}{2}\left(\int_D u_k^2\eta^2\right)(t)-\frac{1}{2}\left(\int_D u_k^2\eta^2\right)(s)+\int_s^t\!\!\int_D\nabla u_k\cdot a\nabla(u_k\eta^2)\\ \leqslant&\int_s^t\!\!\int_D(u_k+k)V\cdot\nabla(u_k\eta^2)-\int_s^t\!\!\int_D c(u_k+k)u_k\eta^2+\int_s^t\!\!\int_D fu_k\eta^2.\end{aligned} \tag{3.7}$$

Indeed, if $[t,t+h]\subseteq I$, we define the Steklov's mean of u:

$$u^h(t,x):=\frac{1}{h}\int_0^h u(t+s,x)\mathrm{d}s=\frac{1}{h}\int_t^{t+h}u(s,x)\mathrm{d}s, \tag{3.8}$$

and define $u_k^h:=(u^h-k)^+$. Suppose $\varphi\in C_c^\infty(Q)$ with $\varphi\geqslant 0$, by (3.5) and choosing h sufficiently small, we get

$$\int_{I\times D}-u\partial_t\varphi^{-h}+(a\nabla u)\cdot\nabla\varphi^{-h}-(uV)\cdot\nabla\varphi^{-h}+(cu)\varphi^{-h}\leqslant\int_{I\times D}f\varphi^{-h}.$$

Notice that for sufficiently small $h > 0$, $\partial_t u^h \in \mathbb{L}_2^2(Q')$, by the above inequality, we obtain

$$\int_{I\times D} \partial_t u^h \varphi + (a\nabla u)^h \cdot \nabla\varphi - (uV)^h \cdot \nabla\varphi + (cu)^h \varphi \leqslant \int_{I\times D} f^h \varphi. \tag{3.9}$$

Now let $\varepsilon > 0$ sufficiently small such that $[s-\varepsilon, t+\varepsilon] \Subset I$, define

$$\zeta^\varepsilon_{s,t}(r) = \begin{cases} \varepsilon^{-1}(r+\varepsilon-s), & r \in [s-\varepsilon, s) \\ 1, & r \in [s,t] \\ (1-\varepsilon^{-1}(r-t)), & r \in (t, t+\varepsilon] \\ 0, & I\backslash[s-\varepsilon, t+\varepsilon] \end{cases}$$

Let $\varphi = u^h_k \eta^2 \cdot \zeta^\varepsilon_{s,t}$, integration by parts yields

$$\int_{I\times D} \partial_t u^h \varphi = \frac{1}{2}\int_{I\times D} \partial_t[(u^h_k)^2]\eta^2 \cdot \zeta^\varepsilon_{s,t} = \frac{1}{2}\int_{I\times D} \partial_t[(u^h_k\eta)^2 \cdot \zeta^\varepsilon_{s,t}] - \frac{1}{2}\int_{I\times D} (u^h_k\eta)^2 (\zeta^\varepsilon_{s,t})'$$
$$= \frac{1}{2\varepsilon}\int_t^{t+\varepsilon}\int_D (u^h_k\eta)^2 - \frac{1}{2\varepsilon}\int_{s-\varepsilon}^s\int_D (u^h_k\eta)^2.$$

By standard approximation argument one can see that (3.9) still holds for $\varphi = u^h_k\eta^2 \cdot \zeta^\varepsilon_{s,t}$($h$ is sufficiently small). Thus,

$$\begin{aligned} &\frac{1}{2\varepsilon}\int_t^{t+\varepsilon}\int_D (u^h_k\eta)^2 - \frac{1}{2\varepsilon}\int_{s-\varepsilon}^s\int_D (u^h_k\eta)^2 \\ &+ \int_{I\times D}\Big[(a\nabla u)^h \cdot \nabla(u^h_k\eta^2)\zeta^\varepsilon_{s,t} - (uV)^h \cdot \nabla(u^h_k\eta^2)\zeta^\varepsilon_{s,t} + (cu)^h(u^h_k\eta^2)\zeta^\varepsilon_{s,t}\Big] \\ \leqslant &\int_{I\times D} f^h (u^h_k\eta^2)\zeta^\varepsilon_{s,t}. \end{aligned}$$

Letting $h \to 0$ and then $\varepsilon \to 0$, by Lebesgue's dominated convergence theorem and differentiation theorem, we obtain that for almost every $s, t \in I$,

$$\begin{aligned} &\frac{1}{2}\left(\int_D u_k^2\eta^2\right)(t) - \frac{1}{2}\left(\int_D u_k^2\eta^2\right)(s) + \int_s^t\!\!\int_D \nabla u_k \cdot a\nabla(u_k\eta^2) \\ \leqslant &\int_s^t\!\!\int_D uV \cdot \nabla(u_k\eta^2) - \int_s^t\!\!\int_D cu\, u_k\eta^2 + \int_s^t\!\!\int_D f u_k\eta^2. \end{aligned}$$

Notice that $u \cdot \mathbf{1}_{\{u>k\}} = (u_k + k)\mathbf{1}_{\{u>k\}}$, we complete the proof for (3.7).

For almost every $s, t \in I$, using integration by parts, we get

$$\begin{aligned} \int_s^t\!\!\int_D (u_k+k)V \cdot \nabla(u_k\eta^2) =& \frac{1}{2}\int_s^t\!\!\int_D \eta^2 V \cdot \nabla(u_k^2) + 2\int_s^t\!\!\int_D u_k^2\eta V \cdot \nabla\eta \\ &+ k\int_s^t\!\!\int_D \eta^2 V \cdot \nabla u_k + 2k\int_s^t\!\!\int_D u_k\eta V \cdot \nabla\eta \\ =& -\int_s^t\!\!\int_D u_k^2\eta V \cdot \nabla\eta - \frac{1}{2}\int_s^t\!\!\int_D \nabla \cdot V u_k^2\eta^2 + 2\int_s^t\!\!\int_D u_k^2\eta V \cdot \nabla\eta \\ &- 2k\int_s^t\!\!\int_D u_k\eta V \cdot \nabla\eta - k\int_s^t\!\!\int_D \nabla \cdot V u_k\eta^2 + 2k\int_s^t\!\!\int_D u_k\eta V \cdot \nabla\eta \\ =& \int_s^t\!\!\int_D u_k^2\eta V \cdot \nabla\eta - \frac{1}{2}\int_s^t\!\!\int_D \nabla \cdot V u_k^2\eta^2 - k\int_s^t\!\!\int_D \nabla \cdot V u_k\eta^2. \end{aligned} \tag{3.10}$$

Combing (3.7), (3.10), ($\mathbf{A}_1$) and using Hölder's inequality, we obtain

$$
\begin{aligned}
&\frac{1}{2}\left(\int_D u_k^2\eta^2\right)(t)-\frac{1}{2}\left(\int_D u_k^2\eta^2\right)(s)+\frac{1}{\Lambda}\int_s^t\!\!\int_D|\eta\nabla u_k|^2\\
\overset{(\mathbf{A}_1)}{\leqslant}&\frac{1}{2}\left(\int_D u_k^2\eta^2\right)(t)-\frac{1}{2}\left(\int_D u_k^2\eta^2\right)(s)+\int_s^t\!\!\int_D \eta^2\nabla u_k\cdot a\nabla u_k\\
\overset{(3.7),(3.10)}{\leqslant}&-2\int_s^t\!\!\int_D u_k\eta\nabla\eta\cdot(a\nabla u_k)+\int_s^t\!\!\int_D u_k^2\eta V\cdot\nabla\eta-\int_s^t\!\!\int_D\left(\tfrac{1}{2}\nabla\cdot V+c\right)u_k^2\eta^2\\
&-k\int_s^t\!\!\int_D(\nabla\cdot V+c)u_k\eta^2+\int_s^t\!\!\int_D f u_k\eta^2\\
\overset{(\mathbf{A}_1)}{\leqslant}&2\Lambda\int_s^t\!\!\int_D|\eta\nabla u_k|\cdot|u_k\nabla\eta|+\int_s^t\!\!\int_D u_k^2|V|\cdot|\nabla\eta|+k^2\int_s^t\!\!\int_D(\nabla\cdot V+c)^-\eta^2\\
&+\int_s^t\!\!\int_D\left[(\tfrac{1}{2}\nabla\cdot V+c)^-+(\nabla\cdot V+c)^-\right]u_k^2\eta^2+\int_s^t\!\!\int_D f u_k\eta^2.
\end{aligned}
$$

For any $\delta>0$, by Hölder's inequality, (3.2) and ($\mathbf{A}_2$'), we have

$$
2\Lambda\int_s^t\!\!\int_D|\eta\nabla u_k|\cdot|u_k\nabla\eta|\leqslant\delta\int_s^t\!\!\int_D|\eta\nabla u_k|^2+4\Lambda^2\delta^{-1}(R-\rho)^{-2}\|u_k\|^2_{\mathbb{L}^2(A_s^t(k))},
$$

where $A_s^t(k)=\{u>k\}\cap[s,t]\times B_R$;

$$
\int_s^t\!\!\int_D u_k^2|V|\cdot|\nabla\eta|\leqslant 2(R-\rho)^{-1}\kappa'\|u_k\|^2_{\mathbb{L}^{p_1^*}_{q_1^*}(A_s^t(k))};
$$

$$
k^2\int_s^t\!\!\int_D(\nabla\cdot V+c)^-\eta^2\leqslant k^2\,\kappa'\|\mathbf{1}_{A_s^t(k)}\|^2_{\mathbb{L}^{p_2^*}_{q_2^*}};
$$

$$
\int_s^t\!\!\int_D\left[(\tfrac{1}{2}\nabla\cdot V+c)^-+(\nabla\cdot V+c)^-\right]u_k^2\eta^2\leqslant 2\kappa'\|u_k\|^2_{\mathbb{L}^{p_2^*}_{q_2^*}(A_s^t(k))};
$$

$$
\int_s^t\!\!\int_D f u_k\eta^2\leqslant\|f\|_{\widetilde{\mathbb{L}}^{p_3}_{q_3}}\|u_k\|_{\mathbb{L}^{p_3^*}_{q_3^*}}\|\mathbf{1}_{A_s^t(l)}\|_{\mathbb{L}^{p_3^*}_{q_3^*}}\leqslant\|f\|^2_{\widetilde{\mathbb{L}}^{p_3}_{q_3}}\|\mathbf{1}_{A_s^t(k)}\|^2_{\mathbb{L}^{p_3^*}_{q_3^*}}+\|u_k\|^2_{\mathbb{L}^{p_3^*}_{q_3^*}(A_s^t(k))}.
$$

Choosing $\delta=(2\Lambda)^{-1}$ and combining the above inequalities, we get

$$
\begin{aligned}
&\left(\int_D u_k^2\eta^2\right)(t)-\left(\int_D u_k^2\eta^2\right)(s)+\int_s^t\!\!\int_D|\nabla(u_k\eta)|^2\\
\leqslant&C(R-\rho)^{-2}\left(\|u_k\|^2_{\mathbb{L}^2(A_s^t(k))}+\sum_{i=1}^3\|u_k\|^2_{\mathbb{L}^{p_i^*}_{q_i^*}(A_s^t(k))}\right)+C\left(k^2+\|f\|^2_{\widetilde{\mathbb{L}}^{p_3}_{q_3}}\right)\sum_{i=2}^3\|\mathbf{1}_{A_s^t(k)}\|^2_{\mathbb{L}^{p_i^*}_{q_i^*}},
\end{aligned}
$$

where C only depends on d,Λ,κ' and (p_i,q_i).

□

From now no, we assume $Q=I\times D=(0,T)\times\mathbb{R}^d$. Using De Giorgi iteration, we will prove a global L^∞ estimate for the solutions to (3.4). A similar approach can be found in [10].

We need the following elementary lemma.

Lemma 3.3. *Suppose $\{y_j\}_{j\in\mathbb{N}}$ is a nonnegative nondecreasing real sequence,*

$$
y_{j+1}\leqslant NC^jy_j^{1+\varepsilon}
$$

with $\varepsilon > 0$ and $C > 1$. Assume

$$y_0 \leqslant N^{-1/\varepsilon} C^{-1/\varepsilon^2},$$

then $y_j \to 0$ as $j \to \infty$.

The following maximum principle is crucial.

Theorem 3.4 (Global maximum principle)**.** *Assume $u \in \widetilde{V}^0(T)$ is a locally bounded weak subsolution to (3.4), $u^+(0) \in L^\infty(\mathbb{R}^d)$ and V, c satisfy (**$\mathbf{A}_2$'**), then there is a constant C only depends on d, Λ, κ', T and (p_i, q_i) such that for any $f \in \widetilde{\mathbb{L}}^{p_3}_{q_3}(T)$,*

$$\|u\|_{\widetilde{V}(T)} + \|u^+\|_{\mathbb{L}^\infty(T)} \leqslant C \left(\|u^+(0)\|_{L^\infty} + \|f\|_{\widetilde{\mathbb{L}}^{p_3}_{q_3}(T)} \right). \tag{3.11}$$

Proof. Take $R = 1$, $\rho = \frac{1}{2}$ in Lemma 3.2 and let η be the same function there. Define $\eta_x(\cdot) := \eta(\cdot - x)$ and $Q_{\tau,x} := (0, \tau] \times B_1(x)$.

Step 1: choose $k \geqslant K_0 := \|u^+(0)\|_{L^\infty} + \|f\|_{\widetilde{\mathbb{L}}^{p_3}_{q_3}(T)}$, by (3.6) and letting $s \downarrow 0$, we have

$$\begin{aligned} &\sup_{t\in[0,\tau]} \left(\int_{B_1(x)} u_k^2 \eta_x^2 \right)(t) + \int_0^\tau\!\!\int_{B_1(x)} |\nabla(u_k\eta_x)|^2 \\ \leqslant & C \left(\|u_k\|^2_{\mathbb{L}^2(Q_{\tau,x})} + \sum_{i=1}^3 \|u_k\|^2_{\mathbb{L}^{p_i^*}_{q_i^*}(Q_{\tau,x})} \right) + Ck^2 \sum_{i=2}^3 \|\mathbf{1}_{A(x,k)}\|^2_{\mathbb{L}^{p_i^*}_{q_i^*}}, \end{aligned} \tag{3.12}$$

where $\tau \in (0, T)$ and $A(x, k) := Q_{\tau,x} \cap \{u > k\}$.

Let $\tilde{\eta}_x(\cdot) = \eta(\frac{\cdot - x}{2})$, $\frac{1}{\bar{p}_i} = \frac{1}{p_i^*} - \frac{e_i}{2d+4}$, $\frac{1}{\bar{q}_i} = \frac{1}{q_i^*} - \frac{e_i}{2d+4}$. By (3.3), we have $\frac{d}{\bar{p}_i} + \frac{2}{\bar{q}_i} = \frac{d}{2}$ and $\mathbb{L}^{\bar{p}_i}_{\bar{q}_i}(\tau) \subseteq V(\tau)$, so Hölder's inequality yields,

$$\begin{aligned} \|u_k\|_{\mathbb{L}^{p_i^*}_{q_i^*}(Q_{\tau,x})} \leqslant & C\|u_k\tilde{\eta}_x\|_{\mathbb{L}^{\bar{p}_i}_{\bar{q}_i}(\tau)} |Q_{\tau,x}|^{\frac{e_i}{2d+4}} \\ \leqslant & C\|u_k\tilde{\eta}_x\|_{V(\tau)} \tau^{\frac{e_i}{2d+4}} \leqslant C\tau^{\frac{e_i}{2d+4}} \|u_k\|_{\widetilde{V}(\tau)}. \end{aligned} \tag{3.13}$$

Obviously,

$$\|u_k\|_{\mathbb{L}^2_2(Q_{\tau,x})} \leqslant C\tau^{\frac{1}{2}} \|u_k\|_{\widetilde{V}(\tau)}.$$

By above estimates and (3.12), we get

$$\begin{aligned} \|u_k\|^2_{\widetilde{V}(\tau)} \leqslant & C_d \sup_{x\in\mathbb{R}^d} \|u_k\eta_x\|^2_{V(\tau)} \\ \leqslant & C\tau^\delta \|u_k\|^2_{\widetilde{V}(\tau)} + Ck^2 \sum_{i=2}^3 \sup_{x\in\mathbb{R}^d} \|\mathbf{1}_{A(x,k)}\|^2_{\mathbb{L}^{p_i^*}_{q_i^*}}, \end{aligned}$$

where $\delta = \min_i\{\frac{e_i}{d+2}\}$. By choosing $\tau = (2C)^{-\delta^{-1}}$, we get

$$\|u_k\|^2_{\widetilde{V}(\tau)} \leqslant Ck^2 \sum_{i=2}^3 \sup_{x\in\mathbb{R}^d} \|\mathbf{1}_{A(x,k)}\|^2_{\mathbb{L}^{p_i^*}_{q_i^*}}. \tag{3.14}$$

Now let $\tilde{p}_i = (d + e_i)p_i^*/d$, $\tilde{q}_i = (d + e_i)q_i^*/d$, then by (3.3), $\frac{d}{\tilde{p}_i} + \frac{2}{\tilde{q}_i} = \frac{d}{2}$, so $\mathbb{L}^{\tilde{p}_i}_{\tilde{q}_i}(\tau) \subseteq V(\tau)$. For any $h > k$, since $A(x, h) \subseteq \{u_k > h - k\} \cap Q_{\tau,x}$, by Chebyshev's inequality, Hölder's inequality and (3.14), we get

$$
\begin{aligned}
&\|\mathbf{1}_{A(x,h)}\|_{\mathbb{L}^{p_i^*}_{q_i^*}} \leqslant (h-k)^{-1}\|u_k\|_{\mathbb{L}^{p_i^*}_{q_i^*}(Q_{\tau,x})} \leqslant (h-k)^{-1}\|u_k\|_{\mathbb{L}^{\tilde p_i}_{\tilde q_i}(Q_{\tau,x})}\|\mathbf{1}_{A(x,k)}\|_{\mathbb{L}^{(d+e_i)p_i^*/e_i}_{(d+e_i)q_i^*/e_i}}\\
\leqslant&(h-k)^{-1}\|u_k\tilde\eta_x\|_{\mathbb{L}^{\tilde p_i}_{\tilde q_i}(\tau)}\|\mathbf{1}_{A(x,k)}\|^{\frac{e_i}{d+e_i}}_{\mathbb{L}^{p_i^*}_{q_i^*}} \leqslant C(h-k)^{-1}\|u_k\tilde\eta_x\|_{V(\tau)}\|\mathbf{1}_{A(x,k)}\|^{\frac{e_i}{d+e_i}}_{\mathbb{L}^{p_i^*}_{q_i^*}}\\
\leqslant&C(h-k)^{-1}\|u_k\|_{\widetilde V(\tau)}\|\mathbf{1}_{A(x,k)}\|^{\frac{e_i}{d+e_i}}_{\mathbb{L}^{p_i^*}_{q_i^*}} \overset{(3.14)}{\leqslant} C\ \frac{k}{h-k}\left(\sum_{i=2}^{3}\sup_{x\in\mathbb{R}^d}\|\mathbf{1}_{A(x,k)}\|_{\mathbb{L}^{p_i^*}_{q_i^*}}\right)\|\mathbf{1}_{A(x,k)}\|^{\frac{e_i}{d+e_i}}_{\mathbb{L}^{p_i^*}_{q_i^*}} \qquad (3.15)\\
\leqslant&C_1\frac{k}{h-k}\left(\sum_{i=2}^{3}\sup_{x\in\mathbb{R}^d}\|\mathbf{1}_{A(x,k)}\|_{\mathbb{L}^{p_i^*}_{q_i^*}}\right)^{1+\varepsilon}\quad (\forall x\in\mathbb{R}^d),
\end{aligned}
$$

where $\varepsilon=\min_i\{\frac{e_i}{d+e_i}\}$ and C_1 only depends on d,Λ,κ' and (p_i,q_i). Let $N>1$ be a number will be determined later, define $k_j:=NK_0(2-2^{-j})\,(j\in\mathbb{N})$ and

$$y_j:=\sum_{i=2}^{3}\sup_{x\in\mathbb{R}^d}\|\mathbf{1}_{A(x,k_j)}\|_{\mathbb{L}^{p_i^*}_{q_i^*}}.$$

By (3.15), we have

$$y_{j+1}\leqslant 8C_1 2^j y_j^{1+\varepsilon}.$$

Thus, by Lemma 3.3, if

$$\sum_{i=2}^{3}\sup_{x\in\mathbb{R}^d}\|\mathbf{1}_{A(x,NK_0)}\|_{\mathbb{L}^{p_i^*}_{q_i^*}}=y_0\leqslant(8C_1)^{-1/\varepsilon}2^{-1/\varepsilon^2}, \qquad (3.16)$$

then $\lim_{j\to\infty}y_j=0$, i.e. $u^+\leqslant 2NK_0$ almost everywhere. Indeed, by (3.15), for any $x\in\mathbb{R}^d$,

$$
\begin{aligned}
&\|\mathbf{1}_{A(x,NK_0)}\|_{\mathbb{L}^{p_i^*}_{q_i^*}}\leqslant\frac{C_1}{N-1}\left(\sum_{i=2}^{3}\sup_{x\in\mathbb{R}^d}\|\mathbf{1}_{A(x,K_0)}\|_{\mathbb{L}^{p_i^*}_{q_i^*}}\right)^{1+\varepsilon}\\
\leqslant&\frac{C_1}{N-1}\left(\sum_{i=2}^{3}\sup_{x\in\mathbb{R}^d}|Q_{\tau,x}|^{\frac{1}{q_i}}\right)^{1+\varepsilon}\leqslant 2^{1+\varepsilon}C_1/(N-1),
\end{aligned}
$$

which implies $y_0\leqslant 2^{2+\varepsilon}C_1/(N-1)$. Let $N=1+2^{\frac{100}{\varepsilon^2}}(C_1)^{1+\frac{1}{\varepsilon}}$, then we have (3.16). Thus, there is a constant C_2 depending only on d,Λ,κ' and (p_i,q_i) such that $u^+(t,x)\leqslant C_2K_0=C_2(\|u^+(0)\|_{L^\infty}+\|f\|_{\widetilde{\mathbb{L}}^{p_3}_{q_3}})$ for almost every $(t,x)\in[0,\tau]\times\mathbb{R}^d$. Since C_2 does not depends on the initial value of u, we obtain that $\|u^+\|_{\mathbb{L}^\infty(T)}\leqslant C_2([T/\tau]+1)K_0$.

Step 2: choose $k=0$, by (3.6) and similar argument in Step 1, we can obtain that for any $\tau\in[0,T]$,

$$\|u^+\|^2_{\widetilde{\mathbb{L}}^2_\infty(\tau)}+\sup_{x\in\mathbb{R}^d}\|\nabla(u^+\eta_x)\|^2_{\mathbb{L}^2(\tau)}\leqslant\|u^+(0)\|^2_{\widetilde L^2}+C\tau^\delta\|u^+\|^2_{\widetilde V(\tau)}+C\|f\|^2_{\widetilde{\mathbb{L}}^{p_3}_{q_3}(\tau)},$$

and the constant C only depends on d,Λ,κ' and (p_i,q_i). This yields

$$\|u^+\|_{\widetilde V(T)}\leqslant C\left(\|u^+(0)\|_{L^\infty}+\|f\|_{\widetilde{\mathbb{L}}^{p_3}_{q_3}(T)}\right).$$

So we complete our proof. □

Next we give the precise definition of weak solution to Cauchy problem.

Definition 3.5. *$u\in \widetilde{V}^0(T)$ is called a weak solution of equation*

$$
\begin{cases}\partial_t u-\nabla\cdot(a\nabla u)+\nabla\cdot(Vu)+cu=f\\ u(0)=\phi\end{cases} \tag{3.17}
$$

in $[0,T]\times\mathbb{R}^d$, if for any $\varphi\in C_c^\infty([0,T]\times\mathbb{R}^d)$ and almost every $t\in[0,T]$, it holds that

$$
\begin{aligned}&\int_{\mathbb{R}^d} u(t)\varphi(t)-\int_{\mathbb{R}^d}\phi\varphi(0)\\&+\int_0^t\!\!\int_{\mathbb{R}^d}\big[-u\partial_t\varphi+(a\nabla u)\cdot\nabla\varphi-uV\cdot\nabla\varphi+cu\varphi\big]=\int_0^t\!\!\int_{\mathbb{R}^d}f\varphi.\end{aligned} \tag{3.18}
$$

3.2. **Existence, uniqueness and stability.** In this section, we will use the apriori estimate (3.11) to prove the existence-uniqueness and stability of weak solutions for equation

$$
\begin{cases}\partial_t u-\nabla\cdot(a\nabla u)+\nabla\cdot(Vu)=f\\ u(0)=\phi.\end{cases} \tag{3.19}
$$

Theorem 3.6 (Existence-uniqueness)**.** *Under ($\mathbf{A}_1$) and ($\mathbf{A}_2$), for each $f\in\mathbb{L}^{p_3}_{q_3}$, $\phi\in L^\infty$ there exists a unique weak solution to (3.19) in $\widetilde{V}^0(T)\cap\mathbb{L}^\infty(T)$.*

Proof. The proof is essentially the same as the one of [26, Theorem 2.3]. First of all, the uniqueness is a direct consequence of (3.11). We prove the existence by weak convergence method. Let

$$
\varrho_n(x):=n^d\varrho(nx),
$$

where $0\leqslant\varrho\in C_c^\infty(B_1)$ with $\int\varrho=1$. $a_n(t,x):=a(t,\cdot)*\varrho_n(x)$, $V_n(t,x):=V(t,\cdot)*\varrho_n(x)$, $f_n(t,x):=f(t,\cdot)*\varrho_n(x)$ and $\phi_n=\phi*\varrho_n$. By Proposition 4.1 of [26], we have

$$
V_n\in L^{q_1}([0,T];C_b^\infty(\mathbb{R}^d)),\quad f_n\in L^{q_3}([0,T];C_b^\infty(\mathbb{R}^d)),
$$

and

$$
\sup_n\Big(\|V_n\|_{\widetilde{\mathbb{L}}^{p_1}_{q_1}}+\|(\nabla\cdot V_n)^-\|_{\widetilde{\mathbb{L}}^{p_2}_{q_2}}+\|f_n\|_{\widetilde{\mathbb{L}}^{p_3}_{q_3}(T)}\Big)<\infty. \tag{3.20}
$$

It is well known that the following PDE has a unique smooth solution $u_n\in C([0,T];C_b^\infty(\mathbb{R}^d))$:

$$
\partial_t u_n=\nabla\cdot(a_n\nabla u_n)-\nabla\cdot(V_nu_n)+f_n,\quad u_n(0)=\phi_n
$$

holds in the distributional sense. In particular, for any $\varphi\in C_c^\infty([0,T]\times\mathbb{R}^d)$ and $t\in[0,T]$,

$$
\begin{aligned}\int_{\mathbb{R}^d}u_n(t)\varphi(t)-\int_{\mathbb{R}^d}\phi_n\varphi(0)=&\int_0^t\!\!\int_{\mathbb{R}^d}u_n\partial_t\varphi\\&+\int_0^t\!\!\int_{\mathbb{R}^d}-(a_n\nabla u_n)\cdot\nabla\varphi+u_nV_n\cdot\nabla\varphi+f_n\varphi.\end{aligned} \tag{3.21}
$$

Since

$$
\begin{aligned}\|\partial_t u_n\|_{\widetilde{\mathbb{H}}^{-1,2}_2(T)}&\leqslant\|\nabla\cdot(a_n\nabla u_n)-\nabla\cdot(V_nu_n)+f_n\|_{\widetilde{\mathbb{H}}^{-1,2}_2(T)}\\&\leqslant C\Big(\|a_n\nabla u_n\|_{\widetilde{\mathbb{L}}^2_2(T)}+\|V_nu_n\|_{\widetilde{\mathbb{L}}^2_2(T)}+\|f_n\|_{\widetilde{\mathbb{H}}^{-1,2}_2(T)}\Big)\\&\leqslant C\Big(\|a_n\|_{L^\infty}\|u_n\|_{\widetilde{\mathbb{H}}^{1,2}_2(T)}+\|V_n\|_{\widetilde{\mathbb{L}}^2_2(T)}\|u_n\|_{L^\infty(T)}+\|f_n\|_{\widetilde{\mathbb{L}}^{p_3}_{q_3}}\Big)\\&\leqslant C\Big(\|u_n\|_{\widetilde{\mathbb{H}}^{1,2}_2(T)}+\|u_n\|_{L^\infty(T)}+\|f_n\|_{\widetilde{\mathbb{L}}^{p_3}_{q_3}(T)}\Big).\end{aligned}
$$

By Theorem 3.4, we get for any $T>0$,

$$
\sup_n\Big(\|u_n\|_{\mathbb{L}^\infty(T)}+\|u_n\|_{\widetilde{V}(T)}+\|\partial_t u_n\|_{\widetilde{\mathbb{H}}^{-1,2}_2(T)}\Big)<\infty. \tag{3.22}
$$

Hence, by the fact that every bounded subset of $\widetilde{V}(T)$ is relatively weak compact, there is a subsequence(still be denoted by n) and $\bar{u} \in \widetilde{V}(T) \cap \mathbb{L}^\infty(T)$ such that for any $\varphi \in C_c^\infty([0,T]\times\mathbb{R}^d)$

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^d} u_n\partial_t\varphi + \int_0^t\int_{\mathbb{R}^d} -(a_n\nabla u_n)\cdot\nabla\varphi + u_nV_n\cdot\nabla\varphi + f_n\varphi \\ \to &\int_0^t\int_{\mathbb{R}^d} \bar{u}\partial_t\varphi + \int_0^t\int_{\mathbb{R}^d} -(a\nabla\bar{u})\cdot\nabla\varphi + \bar{u}V\cdot\nabla\varphi + f\varphi\end{aligned} \tag{3.23}$$

and

$$\|\bar{u}\|_{\mathbb{L}^\infty(T)} + \|\bar{u}\|_{\widetilde{V}(T)} + \|\partial_t\bar{u}\|_{\widetilde{\mathbb{H}}_2^{-1,2}(T)} < \infty.$$

By Lions-Magenes lemma(cf. [17, Lemma 1.2, Chapter 3]), we obtain that $\bar{u} \in C([0,T];\widetilde{L}^2(\mathbb{R}^d))$, hence $\bar{u} \in \widetilde{V}^0(T) \cap \mathbb{L}^\infty(T)$. On the other hand, by (3.22) and Aubin-Lions lemma (cf. [14]), there is a subsequence of n(still be denoted by n) such that (3.23) holds and

$$\lim_{n\to\infty} \|u_n - \bar{u}\|_{L^2([0,T]\times B_R)} = 0, \quad \forall R > 0.$$

It holds that for Lebesgue almost all $(t,x) \in [0,T]\times\mathbb{R}^d$,

$$u_n(t,x) \to \bar{u}(t,x),$$

as $n \to \infty$ along an appropriate subsequence. Thus, for almost every $t \in [0,T]$,

$$\int_{\mathbb{R}^d} \bar{u}(t)\varphi(t) - \int_{\mathbb{R}^d} \phi_n\varphi(0) \to \int_{\mathbb{R}^d} \bar{u}(t)\varphi(t) - \int_{\mathbb{R}^d} \phi\varphi(0). \tag{3.24}$$

Combing (3.21),(3.23) and (3.24), we obtain that for all $\varphi \in C_c^\infty([0,T]\times\mathbb{R}^d)$ and almost every $t \in [0,T]$

$$\int_{\mathbb{R}^d} \bar{u}(t)\varphi(t) - \int_{\mathbb{R}^d} \phi\varphi(0) = \int_0^t\int_{\mathbb{R}^d} \bar{u}\partial_t\varphi + \int_0^t\int_{\mathbb{R}^d} -(a\nabla\bar{u})\cdot\nabla\varphi + \bar{u}V\cdot\nabla\varphi + f\varphi,$$

i.e. $\bar{u}$ solves (3.19). □

Theorem 3.7. *(Stability) Let $(p_i,q_i) \in [2,\infty)$ with $\frac{d}{p_i} + \frac{2}{q_i} < 2$, where $i = 1,2,3$, $T > 0$. For any $n \in \mathbb{N}\cup\{\infty\} =: \mathbb{N}_\infty$, let b_n, f_n, ϕ_n satisfy*

$$\sup_{n\in\mathbb{N}_\infty}\Big(\|V_n\|_{\widetilde{\mathbb{L}}_{q_1}^{p_1}} + \|(\nabla\cdot V_n)^-\|_{\widetilde{\mathbb{L}}_{q_2}^{p_2}} + \|f_n\|_{\widetilde{\mathbb{L}}_{q_3}^{p_3}(T)} + \|\phi_n\|_{L^\infty}\Big) < \infty.$$

For $n \in \mathbb{N}_\infty$, let $u_n \in \widetilde{V}^0(T)\cap\mathbb{L}^\infty(T)$ be the unique weak solutions of (3.19) associated with coefficients (V_n, f_n, ϕ_n) with initial value $u_n(0) = \phi_n$. Assume that for any $\varphi \in C_c(\mathbb{R}^d)$,

$$\lim_{n\to\infty}\Big(\|(V_n - V_\infty)\varphi\|_{\mathbb{L}_{q_1}^{p_1}(T)} + \|(f_n - f_\infty)\varphi\|_{\mathbb{L}_{q_3}^{p_3}(T)} + \|\phi_n - \phi_\infty\|_{L^\infty}\Big) = 0.$$

Then it holds that for Lebesgue almost all $(t,x) \in [0,T]\times\mathbb{R}^d$,

$$\lim_{n\to\infty} u_n(t,x) = u_\infty(t,x).$$

The proof of above theorem is essentially same with Theorem 3.6, so we omit its proof here.

Let us also mention the following Kolmogorov's backward equation

$$\begin{cases}\partial_t u - Lu = \partial_t u - a_t^{ij}\partial_{ij}u - b_t^i\partial_i u = f \\ u(0) = \phi,\end{cases}$$

which can be rewritten as

$$\begin{cases}\partial_t u - \nabla\cdot(a\nabla u) - \nabla\cdot(Vu) + \nabla\cdot Vu = f, \\ u(0) = \phi.\end{cases} \tag{\textbf{KE}}$$

If $V \in \widetilde{\mathbb{L}}^{p_1}_{q_1}(T)$, $(\nabla\cdot V)^- \in \widetilde{\mathbb{L}}^{p_2}_{q_2}(T)$, due to Theorem 3.4, any subsolution $u\in \widetilde{V}^0(T)$ satisfies (3.11). Using similar argument in Theorem 3.6(see also [26]), we have

Proposition 3.8. *Assume a, b, V satisfy ($\mathbf{A}_1$) and ($\mathbf{A}_2$), then for each $f\in \widetilde{\mathbb{L}}^{p_3}_{q_3}(T)$ and $\phi\in L^\infty$ equation (**KE**) admits a unique weak solution $u\in \widetilde{V}^0(T)\cap \mathbb{L}^\infty(T)$.*

In order to apply the theory on SLF developed in [7] and [18], we first need to extend the uniqueness result in Theorem 3.6 to larger space $\mathbb{L}^\infty(T)$.

We first give a standard lemma.

Lemma 3.9. *Suppose $F\in \widetilde{\mathbb{L}}^2(T)$, then the following PDE:*

$$\begin{cases} \partial_t u - \nabla\cdot(a\nabla u) = \nabla\cdot F \quad \text{in } (0,T)\times\mathbb{R}^d, \\ u(0)=\phi\in \widetilde{\mathbb{L}}^2. \end{cases} \tag{3.25}$$

admits a unique weak solution $u\in \widetilde{V}^0(T)$ and

$$\|u\|_{\widetilde{V}(T)} \leqslant \|u(0)\|_{\widetilde{L}^2} + C\|F\|_{\widetilde{\mathbb{L}}^2_2(T)}.$$

Proof. The proof is quite standard, here we prove the apriori estimate for reader's convenience. Take test function $\varphi = u\eta_x^2$, where η_x is the same cut off function in the proof of Theorem 3.4. By basic calculations and Hölder's inequality, we obtain that for almost every $s,t\in[0,T]$,

$$\begin{aligned} &\left(\int_{\mathbb{R}^d} u^2\eta_x^2\right)(t) - \left(\int_{\mathbb{R}^d} u^2\eta_x^2\right)(s) + \int_s^t\!\!\int_{\mathbb{R}^d} |\nabla(u\eta_x)|^2 \\ \leqslant & C\int_s^t\!\!\int_{\mathbb{R}^d} (u\nabla\eta_x)^2 + C\int_s^t\!\!\int_{\mathbb{R}^d} F^2(|\eta_x|^2+|\nabla\eta_x|^2). \end{aligned}$$

Thus,

$$\begin{aligned} \|u\|^2_{\widetilde{V}(\tau)} \leqslant & \sup_{x\in\mathbb{R}^d}\left[\sup_{t\in[0,\tau]}\left(\int_{\mathbb{R}^d} u^2\eta_x^2\right)(t) + \int_0^\tau\!\!\int_{\mathbb{R}^d}|\nabla(u\eta_x)|^2\right] \\ \leqslant & \|u(0)\|^2_{\widetilde{L}^2} + C\|F\|^2_{\widetilde{\mathbb{L}}^2_2(\tau)} + C\int_0^\tau \|u\|^2_{\widetilde{\mathbb{L}}^\infty_2(t)}\mathrm{d}t. \end{aligned}$$

Gronwall's inequality yields

$$\|u\|_{\widetilde{V}(T)} \leqslant \|u(0)\|_{\widetilde{L}^2} + C\|F\|_{\widetilde{\mathbb{L}}^2_2(T)}.$$

□

Now we extend the uniqueness result of Theorem 3.6 to larger space $\mathbb{L}^\infty(T)$. Our proof mainly follows [7, Theorem 4.3].

Theorem 3.10. *Suppose a, b satisfy ($\mathbf{A}_1$), ($\mathbf{A}_2$), for any $\phi\in L^\infty$, ($\mathbf{FPE}_2$) has a unique solution $u\in\widetilde{V}^0(T)\cap\mathbb{L}^\infty(T)$. If moreover, a satisfies ($\mathbf{A}_3$), then uniqueness also holds in $\mathbb{L}^\infty(T)$. In particular, any $\mathbb{L}^\infty(T)$ distributional solution of ($\mathbf{FPE}_2$) with bounded initial value belongs to $\widetilde{V}^0(T)\cap\mathbb{L}^\infty(T)$.*

Proof. Suppose $u\in\mathbb{L}^\infty(T)$ is a distributional solution to ($\mathbf{FPE}_2$), then

$$\partial_t u - \nabla\cdot(a\nabla u) = -\nabla(Vu), \quad u(0)\in L^\infty$$

Notice that $Vu\in\widetilde{\mathbb{L}}^2(T)$, by Lemma 3.9, there exists $\bar{u}\in\widetilde{V}^0(T)$ solves the above equation, with the same initial condition. Let us define $g:=\bar{u}-u$, $Ag:=\nabla\cdot(a\nabla g)$. $g\in\widetilde{\mathbb{L}}^2(T)$ is a distributional solution to equation

$$\partial_t g - Ag = \partial_t g - \nabla\cdot(a\nabla g) = 0, \quad g(0)=0. \tag{3.26}$$

Here $\nabla\cdot(a\nabla g)$ should be read by $\partial_{ij}(a^{ij}g)+\partial_i(\partial_j a^{ij}g)$. Assume $w\in\widetilde{\mathbb{H}}^{1,2}_2(T)$ solves

$$\lambda w - Aw = \lambda w - \nabla\cdot(a\nabla w) = g, \quad \lambda>0. \tag{3.27}$$

in $[0,T]\times\mathbb{R}^d$. Multiple the above equation by $w\eta_x^2$, integrate on $[0,t]\times\mathbb{R}^d$ obtaining

$$\begin{aligned}\lambda\int_0^t\!\!\int_{\mathbb{R}^d} w^2\eta_x^2+\frac{1}{\Lambda}\int_0^t\!\!\int_{\mathbb{R}^d}|\nabla w\eta_x|^2 \leqslant& C\int_0^t\!\!\int_{\mathbb{R}^d}(w\nabla\eta_x)(\nabla w\eta_x)+\int_0^t\!\!\int_{\mathbb{R}^d}(g\eta_x)(w\eta_x)\\ \leqslant& C\|w\|^2_{\mathbb{L}^2(t)}+\frac{1}{2\Lambda}\|\nabla w\eta_x\|_{\mathbb{L}^2(t)}+\|g\|^2_{\widetilde{\mathbb{L}}^2(t)},\end{aligned}$$

this yields that there is a constant $\lambda_0>0$ such that for any $\lambda\geqslant\lambda_0$,

$$\lambda\|w\|_{\widetilde{\mathbb{L}}^2(T)}+\|\nabla w\|_{\widetilde{\mathbb{L}}^2(T)}\leqslant C\|g\|_{\widetilde{\mathbb{L}}^2(T)}.$$

This estimate implies that for any $\lambda\geqslant\lambda_0$, there is a unique solution $w=:G_\lambda g\in\widetilde{\mathbb{H}}^{1,2}_2(T)$, here G_λ is the solution map of (3.27). It is also easy to verify that G_λ is also bounded from $\mathbb{L}^2(T)$ to $\mathbb{H}^{1,2}_2(T)$ and

$$\lambda\|G_\lambda g\|_{\mathbb{L}^2(T)}+\|\nabla G_\lambda g\|_{\mathbb{L}^2(T)}\leqslant C\|g\|_{\mathbb{L}^2(T)}. \tag{3.28}$$

By (3.26), we have

$$0=\partial_t G_\lambda^{-1}w - AG_\lambda^{-1}w = G_\lambda^{-1}(\partial_t w - Aw)+[\partial_t, G_\lambda^{-1}]w,$$

thus formally

$$\partial_t w - Aw = G_\lambda\{[G_\lambda^{-1},\partial_t]w\} = G_\lambda[\nabla\cdot(\partial_t a\nabla w)] \tag{3.29}$$

in the sense of distribution. One can find the rigurous proof for (3.29) in [7]. Like before, multiplying (3.29) by $w\eta_x^2$, integrating on $[0,t]\times\mathbb{R}^d$, using Hölder's inequality and (3.28), we obtain

$$\begin{aligned}&\frac12\int_{\mathbb{R}^d}|w(t)\eta_x|^2+\frac1\Lambda\int_0^t\!\!\int_{\mathbb{R}^d}|\nabla w\cdot\eta_x|^2\\ \leqslant&\int_0^t\!\!\int_{\mathbb{R}^d}\nabla\cdot(\partial_t a\nabla w)\,[G_\lambda(w\eta_x^2)] = -\int_0^t\!\!\int_{\mathbb{R}^d}\partial_t a\nabla w\cdot\nabla[G_\lambda(w\eta_x^2)]\\ \leqslant&\|\partial_t a\|_{L^\infty}\int_0^t\sum_{z\in\mathbb{Z}^d/2}\int_{B_{\frac12}(z)}\nabla w\cdot\nabla[G_\lambda(w\eta_x^2)]\\ \leqslant& C\sum_{z\in\mathbb{Z}^d/2}\left(\int_0^t\!\!\int_{B_{\frac12}(z)}|\nabla w\cdot\eta_z|^2\right)^{1/2}\left(\int_0^t\!\!\int_{B_{\frac12}(z)}|\nabla[G_\lambda(w\eta_x^2)]|^2\right)^{1/2}\\ \leqslant& C\left(\sup_{z\in\mathbb{Z}^d/2}\int_0^t\!\!\int_{\mathbb{R}^d}|\nabla w\cdot\eta_z|^2\right)^{1/2}\cdot\left(\int_0^t\!\!\int_{\mathbb{R}^d}|\nabla[G_\lambda(w\eta_x^2)]|^2\right)^{1/2}\\ \leqslant& C\left(\sup_{x\in\mathbb{R}^d}\|\nabla(w\eta_x)\|_{\mathbb{L}^2_2(t)}\right)\|w\eta_x^2\|_{\mathbb{L}^2_2(t)}\\ \leqslant&\frac{1}{2\Lambda}\sup_{x\in\mathbb{R}^d}\|\nabla(w\eta_x)\|^2_{\mathbb{L}^2_2(t)}+C\|w\|^2_{\widetilde{\mathbb{L}}^2_2(t)}.\end{aligned}$$

In the first inequality, we use the fact that G_λ is a symmetric operator in L^2 space. Taking supermum over $x\in\mathbb{R}^d$ on the left side of above inequalities, we get

$$\|w(t)\|_{\widetilde{L}^2}\leqslant C\int_0^t\|w(s)\|^2_{\widetilde{L}^2}\mathrm{d}s,\quad t\in[0,T].$$

Gronwall's inequality yields $w\equiv0$ and hence $g\equiv0$. □

4. Proof of Main results

Before proving our main results, let us list some conclusions in [26] and [18](see also [7]).

Proposition 4.1 (cf. [26])**.** *Assume a, b satisfy ($\mathbf{A}_1$) and ($\mathbf{A}_2$), then for each $\mu_0 \in \mathscr{P}(\mathbb{R}^d)$, there exists at least one martingale solution associated with (L, μ_0), say $\mathbb{P}$, which satisfies the following Krylov's type estimate: for any $p, q \in [2, \infty)$ with $\frac{d}{p} + \frac{2}{q} < 2$, there exist $\theta = \theta(p, q) > 0$ and a constant $C > 0$ such that for all $0 \leqslant t_0 < t_1 \leqslant T$ and $f \in C_c^\infty(\mathbb{R}^{d+1})$,*

$$\mathbb{E}^{\mathbb{P}} \left(\int_{t_0}^{t_1} f(t, \omega_t) \mathrm{d}t \Big| \mathcal{B}_{t_0} \right) \leqslant C(t_1 - t_0)^\theta \|f\|_{\widetilde{\mathbb{L}}_q^p(T)}. \tag{4.1}$$

Define

$$\mathscr{L}_+ := \Big\{ \mu : [0, T] \ni t \mapsto \mu_t \in \mathscr{P}(\mathbb{R}^d) : \int_0^T \int_{\mathbb{R}^d} (|a(t, x)| + |b(t, x)|) \mu_t(\mathrm{d}x) \mathrm{d}t < \infty$$
$$\mu_t = \rho_t \lambda_d, \rho_t \in L^\infty \text{ uniformly for } t \in [0, T],$$
$$\text{and for any } \varphi \in C_b(\mathbb{R}^d), t \mapsto \int_{\mathbb{R}^d} \varphi \mathrm{d}\mu_t \text{ is continuous} \Big\}.$$

The following two Propositions are consequences of [18, Theorem 2.5] and [18, Lemma 2.12] respectively.

Proposition 4.2. *Suppose $\{\mu_t\}_{t \in [0,T]} \in \mathscr{L}_+$, then there exists $\mathbb{P} \in \mathscr{P}(C([0, T]; \mathbb{R}^d))$ which is a solution to the MP associated to the diffusion operator L such that, for every $t \in [0, T]$, it holds $\mu_t = \mathbb{P} \circ \omega_t^{-1}$.*

Proposition 4.3. *Assume that forward uniqueness for the ($\mathbf{FPE}_1$) hold in the class $\mathscr{L}_+$ for any initial time. Then, for any $\mu_0 = \rho_0 \lambda_d \in \mathscr{P}(\mathbb{R}^d)$with $\rho_0 \in L^\infty$, the μ_0-SLF is uniquely determined μ_0-a.e..*

Lemma 4.4. *Under* **Assumption** *1, assume that $\mu_0 = \rho_0 \lambda_d \in \mathscr{P}(\mathbb{R}^d)$ with $\rho_0 \in L^\infty$, then equation ($\mathbf{FPE}_1$) admits a unique solution in $\mu \in \mathscr{L}_+$.*

Proof. The uniqueness follows from Theorem 3.10, so we only need to show the existence. We prove this by using probability method. Let a_n, V_n be the same functions in the proof of Theorem 3.6, then we can find a collection of probability measures $\{\mathbb{P}^n\}_{n \in \mathbb{N}}$ on $C([0, T]; \mathbb{R}^d)$ such that $\mathbb{P}^n$ is the unique martingale solution associated to $L^n := a_n^{ij} \partial_{ij} + b_n^i \partial_i$ with initial data μ_0. For any stopping time τ, $\delta > 0$ with $\tau + \delta \leqslant T$, thanks to (4.1), we have

$$\sup_{n \in \mathbb{N}} \mathbb{E}^{\mathbb{P}^n} \int_\tau^{\tau + \delta} |b^n|(s, \omega_s) \mathrm{d}s \leqslant C \delta^\theta \|b\|_{\widetilde{\mathbb{L}}_{p_1}^{q_1}(T)}.$$

Using above estimate and BDG inequality, we get

$$\mathbb{E}^{\mathbb{P}^n} \left(\sup_{0 \leqslant s \leqslant \delta} |\omega_{\tau+s} - \omega_\tau| \right) \leqslant \mathbb{E}^{\mathbb{P}^n} \int_\tau^{\tau+\delta} |b_n|(t, \omega_t) \mathrm{d}t + \mathbb{E}^{\mathbb{P}^n} \left| \sup_{0 \leqslant s \leqslant \delta} \int_\tau^{\tau+s} \sqrt{2a^n(t, \omega_t)} \mathrm{d}W_t \right|$$
$$\leqslant C(\delta^\theta + \delta^{1/2}),$$

where C is independent of n. Thus by [25, Lemma 2.7], we obtain

$$\sup_n \mathbb{E}^{\mathbb{P}^n} \left(\sup_{|t-s| \leqslant \delta} |\omega_t - \omega_s|^{1/2} \right) \leqslant C(\delta^\theta + \delta^{1/2}).$$

From this, by Chebyshev's inequality, we derive that for any $\varepsilon > 0$,

$$\lim_{\delta \to 0} \sup_n \mathbb{P}^n \left(\sup_{|t-s| \leqslant \delta} |\omega_t - \omega_s| > \varepsilon \right) = 0.$$

Hence, $\{\mathbb{P}^n\}$ is tight in $\mathscr{P}(C([0,T];\mathbb{R}^d))$. Suppose $\mathbb{P}$ is an limit point of $\{\mathbb{P}^n\}$, then for each $t\in[0,T]$, $\mu_t^n := \mathbb{P}^n\circ\omega_t^{-1} \Rightarrow \mathbb{P}\circ\omega_t^{-1} =: \mu_t$, as $n\to\infty$ along an appropriate subsequence. For each $n\in\mathbb{N}$, notice that $\rho_t^n(x) := \frac{\mathrm{d}\mu_t^n}{\mathrm{d}\lambda_d}(x)$ is a distributional solution of (**FPE**$_2$) with a, b, ϕ replaced by a_n, b_n, ρ_0 and

$$\sup_n\left(\|V_n\|_{\widetilde{\mathbb{L}}^{p_1}_{q_1}} + \|(\nabla\cdot V_n)^-\|_{\widetilde{\mathbb{L}}^{p_2}_{q_2}}\right) < \infty, \quad \Lambda^{-1}|\xi|^2 \leqslant a_n^{ij}\xi_i\xi_j \leqslant \Lambda|\xi|^2.$$

By Theorem 3.7, we obtain that $0\leqslant\rho_t^n(x)\to\rho_t(x)$ for almost everywhere $(t,x)\in[0,T]\times\mathbb{R}^d$, where ρ_t is the unique solution to (**FPE**$_2$)(with $\phi=\rho_0$) in class $\mathbb{L}^\infty(T)$(or $\widetilde{V}^0(T)\cap\mathbb{L}^\infty(T)$). Moreover,

$$\|\rho\|_{\mathbb{L}^\infty(T)} \leqslant \sup_{n\in\mathbb{N}}\|\rho_n\|_{\mathbb{L}^\infty(T)} < \infty.$$

Lebesgue's dominated convergence theorem yields that for each $f\in C_b(\mathbb{R}^d)$ and almost every $t\in[0,T]$,

$$\int_{\mathbb{R}^d} f\rho_t = \lim_{n\to\infty}\int_{\mathbb{R}^d} f\rho_t^n = \lim_{n\to\infty}\int_{\mathbb{R}^d} f\mathrm{d}\mu_t^n = \int_{\mathbb{R}^d} f\mathrm{d}\mu_t.$$

Notice that the map $[0,T]\ni t\mapsto\mu_t\in\mathscr{P}(\mathbb{R}^d)$ is continuous, so for any $t\in[0,T]$,

$$\sup_{\substack{\|f\|_{L^1}=1;\\ f\in C_b(\mathbb{R}^d)}}\int_{\mathbb{R}^d} f\mathrm{d}\mu_t \leqslant \sup_{\substack{\|f\|_{L^1}=1;\\ f\in C_b(\mathbb{R}^d)}} \mathrm{esssup}_{t\in[0,T]}\int_{\mathbb{R}^d} f\rho_t \leqslant \|\rho\|_{\mathbb{L}^\infty(T)} \leqslant C.$$

Thus, $\mu_t = \mathbb{P}\circ\omega_t^{-1}\in\mathscr{L}_+$. □

Proof of Theorem 2.4. (1). If m_0 is a probability measure, then the uniqueness of m_0-SLF is a consequence of Lemma 4.4 and Proposition 4.3. For arbitrary $m_0\in\mathscr{M}(\mathbb{R}^d)$, one can find a probability measure μ_0 such that $\mu_0(\mathrm{d}x) = \rho'(x)m_0(\mathrm{d}x)$ and $0<\rho'\leqslant C<\infty$, m_0-a.e.. Notice that each m_0-SLF is a μ_0-SLF, by the uniqueness of μ_0-SLF and the fact $m_0\ll\mu_0$, we obtain the uniqueness of m_0-SLF.

For the existence, we only need to prove the case $m_0=\lambda_d$. Let $\varrho(x)=\mathrm{e}^{-|x|^2/2}$. $m_0^k(\mathrm{d}x) := \varrho(x/k)\mathrm{d}x\in\mathscr{M}(\mathbb{R}^d)$, $\mu_0^k(\mathrm{d}x) := (2\pi k^2)^{-d/2}\varrho(x/k)\mathrm{d}x\in\mathscr{P}(\mathbb{R}^d)$. The existence of μ_0^k-SLF(or m_0^k-SLF) is a consequence of Proposition 4.2 and Lemma 4.4. Suppose $\{\mathbb{P}_x^k\}_{x\in\mathbb{R}^d}$ is a m_0^k-SLF, notice that for any $k,k'\in\mathbb{N}$, $\lambda_d\ll m_0^k\ll m_0^{k'}$, by the uniqueness result proved above, we obtain that $\mathbb{P}_x^k=\mathbb{P}_x$ for all $k\in\mathbb{N}$ and a.e. $x\in\mathbb{R}^d$. Thus, by the definition of m_0^k-SLF, for each k,

$$m_t^k := \int_{\mathbb{R}^d}\mathbb{P}_x\circ\omega_t^{-1}\,m_0^k(\mathrm{d}x)$$

has a bounded density with respect to λ_d, say ρ_t^k. $m_t^k(\mathrm{d}x)=\rho_t^k(x)\mathrm{d}x$ is the unique $\mathscr{L}_+-$solution to (**FPE**$_1$) with initial value $m_0^k(\mathrm{d}x)=\varrho(x/k)\mathrm{d}x$. By Theorem 3.4 and Theorem 3.10,

$$\sup_{k\in\mathbb{N}}\sup_{t\in[0,T]}\|\rho_t^k\|_{L^\infty} \leqslant C\sup_{k\in\mathbb{N}}\|\varrho(\cdot/k)\|_{L^\infty} \leqslant C.$$

Hence, for any $A\in\mathcal{B}(\mathbb{R}^d)$, $t\in[0,T]$,

$$\begin{aligned}\int_{\mathbb{R}^d}\mathbb{P}_x\circ\omega_t^{-1}(A)\,\mathrm{d}x &= \lim_{k\to\infty}\int_{\mathbb{R}^d}\mathbb{P}_x^k\circ\omega_t^{-1}(A)\,\varrho(x/k)\mathrm{d}x\\ &= \lim_{k\to\infty} m_t^k(A) = \lim_{k\to\infty}\int_A\rho_t^k \leqslant C\lambda_d(A),\end{aligned}$$

which implies $\{\mathbb{P}_x\}_{x\in\mathbb{R}^d}$ is also an SLF.

(2). Suppose $\{\mathbb{P}_x\}_{x\in\mathbb{R}^d}$ is the SLF associated with L. For any $\mu_0\in\mathscr{P}(\mathbb{R}^d)$ with $\mu_0\ll\lambda_d$, $\mathbb{P} := \int\mathbb{P}_x\,\mu_0(\mathrm{d}x)$ is a martingale solution associated with (L,μ_0).

Suppose $\mathbb{P}$ is a a martingale solution associated with (L,μ_0) and $\mu_t := \mathbb{P}\circ\omega_t^{-1} = \rho_t\lambda_d$ with $\rho_t \in L^\infty$ uniformly in t. Let $\{Q_x\}_{x\in\mathbb{R}^d} \subseteq \mathscr{P}(C([0,T];\mathbb{R}^d))$ be the regular conditional distribution given by $\omega_0 = x$. By [16, Theorem 6.1.2] for μ_0-a.e. x, Q_x is a martingale solution to corresponding to (L,δ_x). Notice that

$$\int_{\mathbb{R}^d} Q_x\circ\omega_t^{-1}\,\mu_0(\mathrm{d}x) = \left(\int_{\mathbb{R}^d} Q_x\,\mu_0(\mathrm{d}x)\right)\circ\omega_t^{-1} = \mathbb{P}\circ\omega_t^{-1} = \mu_t,$$

we get $\{Q_x\}_{x\in\mathbb{R}^d}$ is a μ_0-SLF. The uniqueness of $\mathbb{P}$ follows by the uniqueness of μ_0-SLF. □

Remark 4.5. *If $m_0(\mathrm{d}x) = \rho_0(x)\mathrm{d}x \in \mathscr{M}(\mathbb{R}^d)$ with $0<\rho_0\leqslant C<\infty$ and $\rho_0\in C(\mathbb{R}^d)$, by the proof of Theorem 2.4, one can see that under Assumption 1, any m_0-SLF is an SLF and vice versa.*

Before we give the proof of Theorem 2.7, we need state a lemma about the maximum functions. One can find its proof in [23, Lemma 3.6] and [15].

Lemma 4.6. *(i) Let $f\in W^{1,1}_{loc}(\mathbb{R}^d)$, $\varrho_n(x) := n^d\varrho(x/n)\in C_c^\infty(\mathbb{R}^d)$ with $\int\varrho = 1$. For almost every $x,y\in\mathbb{R}^d$ with $|x-y|\leqslant\sqrt{\varepsilon}\ll 1$,*

$$\frac{|f(x)-f(y)|}{\sqrt{|x-y|^2+\varepsilon^2}} \leqslant 2^d(F^f_{\varepsilon,n}(x)+F^f_{\varepsilon,n}(y)),$$

where $F^f_{\varepsilon,n}$ is a function depends on f, ϱ, ε, n. And there is a constant $C = C(\rho,d)$,

$$\int_{B_R} F^f_{\varepsilon,n}(x)\mathrm{d}x \leqslant Cn^d\|\nabla f\|_{L^1(B_{R+1})} + \log\varepsilon^{-1}\left\|\nabla\left(f_n - f\right)\right\|_{L^1(B_{R+1})}. \tag{4.2}$$

(ii) For any $p>1$, $r,R>0$,

$$\int_{B_r}(M_R f(x))^p\,\mathrm{d}x \leqslant C_{d,p}\int_{B_{r+R}}|f(x)|^p\mathrm{d}x \tag{4.3}$$

Now we are on the point to prove Theorem 2.7. Instead of proving an stability result for the approximation solutions of (1.6), we first prove the pathwise uniqueness of (1.6) if ξ has a bounded density, then using an Yamada-Watanabe type argument(cf. [19]) we show the existence of AESF.

Lemma 4.7. *Suppose b,σ satisfy Assumption 2, $\xi\in\mathscr{F}_0$ is a random variable with bounded density. Assume X_t, Y_t are two strong solutions of (1.6) whose one dimensional distributions have uniformly bounded densities, then we have $X = Y$ a.s..*

Proof. For any $\varepsilon>0$, let ϕ_ε be a increasing smooth function on $[0,\infty)$,

$$\phi_\varepsilon(s) = \begin{cases} s & s\in[0,\varepsilon/2] \\ \varepsilon & s\in[\varepsilon,\infty)\end{cases}$$

and $\phi'_\varepsilon(s)\leqslant C\mathbf{1}_{[0,\varepsilon]}(s)$, $\phi''_\varepsilon(s)\leqslant C\varepsilon^{-1}\mathbf{1}_{[0,\varepsilon]}(s)$.

$$\Phi_\varepsilon(z) := \log\left(1+\frac{\phi_\varepsilon(|z|^2)}{\varepsilon^2}\right), \quad Z_t := X_t - Y_t.$$

Then,

$$|\partial_i\Phi_\varepsilon(z)| = \left|\frac{2\phi'_\varepsilon(|z|^2)z_i}{\varepsilon^2+\phi_\varepsilon(|z|^2)}\right| \leqslant \frac{C\mathbf{1}_{\{|z|\leqslant\sqrt{\varepsilon}\}}}{\sqrt{\varepsilon^2+|z|^2}},$$

$$|\partial_{ij}\Phi_\varepsilon(z)| = \left|\frac{2\phi'_\varepsilon(|z|^2)\delta_{ij}}{\varepsilon^2+\phi_\varepsilon(|z|^2)} + \frac{4\phi''_\varepsilon(|z|^2)z_iz_j}{\varepsilon^2+\phi_\varepsilon(|z|^2)} - \frac{4\phi'^2_\varepsilon(|z|^2)z_iz_j}{[\varepsilon^2+\phi_\varepsilon(|z|^2)]^2}\right| \leqslant \frac{C\mathbf{1}_{\{|z|\leqslant\sqrt{\varepsilon}\}}}{\varepsilon^2+|z|^2}.$$

Denote $\tau_R := \inf\{t > 0 : |X_t| > R, |Y_t| > R\}$. By Itô's formula and Lemma 4.6,

$$\begin{aligned}\mathbf{E}\Phi_\varepsilon\left(Z_{t\wedge\tau_R}\right) =& \int_0^{t\wedge\tau_R} \mathbf{E}\left[\partial_i\Phi_\varepsilon\left(Z_s\right)\cdot\left(b^i\left(s,X_s\right)-b^i\left(s,Y_s\right)\right)\right]\mathrm{d}s \\ &+\frac{1}{2}\int_0^{t\wedge\tau_R}\mathbf{E}\partial_{ij}\Phi_\varepsilon(Z_s)\left[(\sigma^{ik}(X_s)-\sigma^{ik}(Y_s))\cdot(\sigma^{jk}(X_s)-\sigma^{jk}(Y_s))\right]\mathrm{d}s \\ \leqslant& 2\mathbf{E}\int_0^{t\wedge\tau_R}\frac{\left|b\left(s,X_s\right)-b\left(s,Y_s\right)\right|\mathbf{1}_{\{|X_s-Y_s|\leqslant\sqrt{\varepsilon}\}}}{\sqrt{\varepsilon^2+|X_s-Y_s|^2}}\mathrm{d}s \\ &+C\mathbf{E}\int_0^{t\wedge\tau_R}\frac{|\sigma(X_s)-\sigma(Y_s)|^2}{\varepsilon^2+|X_s-Y_s|^2}\mathrm{d}s \\ \leqslant& C\mathbf{E}\int_0^{t\wedge\tau_R}[F_{\varepsilon,n}(s,X_s)+F_{\varepsilon,n}(s,Y_s)]\mathrm{d}s+ \\ &+C\mathbf{E}\int_0^{t\wedge\tau_R}\left[M|\nabla\sigma|(s,X_s)+M|\nabla\sigma|(s,Y_s)\right]^2\mathrm{d}s =: I_1(\varepsilon)+I_2,\end{aligned}$$

where $F_{\varepsilon,n}(s,x) = F^{b(s)}_{\varepsilon,n}(x)$ in Lemma 4.6. Let ρ^X_t, ρ^Y_t be the density of X_t and Y_t respectively, then

$$\begin{aligned}I_2 \leqslant& C\int_0^{t\wedge\tau_R}\mathbf{E}\left[(M_R|\nabla\sigma|(X_s))^2+(M_R|\nabla\sigma|(Y_s))^2\right]\mathrm{d}s \\ \leqslant& C\int_0^t\int_{B_R}[M_R|\nabla\sigma|(s,x)]^2\left(\rho^X_s(x)+\rho^Y_s(x)\right)\mathrm{d}x\mathrm{d}s \\ \leqslant& C\int_0^t\int_{B_{2R}}|\nabla\sigma|^2(s,x)\mathrm{d}x\mathrm{d}s \leqslant C.\end{aligned}$$

For $I_1(\varepsilon)$, by (4.2),

$$\begin{aligned}I_1(\varepsilon) \leqslant& C\int_0^t\int_{B_R}F_{\varepsilon,n}(s,x)(\rho^X_s+\rho^Y_s)\mathrm{d}x\mathrm{d}s \\ \leqslant& Cn^d\int_0^t\int_{B_{R+1}}\nabla b(s,x)\mathrm{d}x\mathrm{d}s + C|\log\varepsilon|\int_0^t\int_{B_{R+1}}|\nabla b(s,x)-\nabla b_n(s,x)|\mathrm{d}x\mathrm{d}s.\end{aligned}$$

Thus,

$$\mathbf{E}\Phi_\varepsilon(Z_{t\wedge\tau_R}) \leqslant C(1+n^d\|\nabla b\|_{L^1([0,t]\times B_{R+1})}) + C|\log\varepsilon|\|\nabla b-\nabla b_n\|_{L^1([0,t]\times B_{R+1})}.$$

By Chebyshev's inequality,

$$\begin{aligned}&\mathbf{P}\left(|X_t-Y_t|>\sqrt{\varepsilon}; t\leqslant\tau_R\right) \\ \leqslant&\mathbf{E}\log\left(1+\frac{\phi_\varepsilon(|Z_{t\wedge\tau_R}|^2)}{\varepsilon^2}\right)/|\log\varepsilon| \\ \leqslant& C(1+n^d\|\nabla b\|_{L^1([0,t]\times B_{R+1})})/|\log\varepsilon| + C\|\nabla b-\nabla b_n\|_{L^1([0,t]\times B_{R+1})}.\end{aligned}$$

Let $\varepsilon\to 0$, then $n\to\infty$ and then $R\to\infty$, we obtain

$$\mathbf{P}\left(|X_t-Y_t|>0\right)=0.$$

Notice X, Y are both continuous processes, we obtain that $X = Y$ a.s..

□

Proof of Theorem 2.7. Let $\mu_0(\mathrm{d}x) = (2\pi)^{-d/2}\mathrm{e}^{-|x|^2/2}\mathrm{d}x$. By Remark 4.5, we only need to prove the existence and uniqueness of μ_0-AESF associated to SDE (1.6). By Proposition 4.1, there exists at least one weak solution(martingale solution), say (X, W) to (1.6) with $\mathrm{law}(\xi) = \mu_0$ and $\rho_t := \mathrm{d}\mathbf{P}\circ X_t^{-1}/\mathrm{d}\lambda_d$ is uniformly bounded on $[0,T]\times\mathbb{R}^d$. Suppose (X', W') is another weak solution to (1.6) and the one-dimensional distribution of X' is also uniformly bounded. Let $Q(x,w;\mathrm{d}\omega)$ be the regular conditional distribution of X given $(X_0,W)=(x,w)$ and $Q'(x,w;\mathrm{d}\omega')$ is defined as the some way. Denote $\Omega:=C([0,T];\mathbb{R}^d)\times C([0,T];\mathbb{R}^d)\times C([0,T];\mathbb{R}^m)$,

$$\mathbb{Q}(\mathrm{d}\omega,\mathrm{d}\omega',\mathrm{d}w):=\int_{\mathbb{R}^d}Q(x,w;\mathrm{d}\omega)\times Q'(x,w;\mathrm{d}\omega')\mu_0(\mathrm{d}x)\ \boldsymbol{\eta}(\mathrm{d}w),$$

where $\boldsymbol{\eta}$ is the Wiener on $C([0,T];\mathbb{R}^m)$. Let $\mathscr{F}_t^0=\mathcal{B}_t(C([0,T];\mathbb{R}^d))\times\mathcal{B}_t(C([0,T];\mathbb{R}^d))\times\mathcal{B}_t(C([0,T];\mathbb{R}^m))$, $\mathscr{N}$ be the collection of subsets of Ω with zero measure under $\mathbb{Q}$ and $\mathscr{F}:=(\mathscr{F}_T^0\vee\mathscr{N})$, $\mathscr{F}_t:=\bigcap_{s>t}(\mathscr{F}_s^0\bigvee\mathscr{N})$. Suppose (ω,ω',w) is the canonical process on the probability space $(\Omega,\mathscr{F},\mathbb{Q})$, then (ω,w) and (ω',w) have the same distributions as (X,W) and (X',W'), respectively. Moreover, w is an $\mathscr{F}_t-$Brownian motion under $\mathbb{Q}$(see [8, Lemma 1.2, Chapter IV]). In the probability space $(\Omega,\mathscr{F},\mathbb{Q})$, (ω,w) and (ω',w) are two solutions of (1.6), and $\mathbb{Q}\circ\omega_t^{-1}$, $\mathbb{Q}\circ\omega_t'^{-1}$ both enjoy uniformly bounded density. Thus, pathwise uniqueness yields $\mathbb{Q}(\omega=\omega')=1$, which implies

$$Q(x,w;\cdot)\times Q'(x,w;\cdot)(\omega=\omega')=1,\quad \mu_0\times\boldsymbol{\eta}-a.s.\ (x,w).$$

Hence, there exists a measurable map $\psi(x,w)$ such that for $\mu_0\times\boldsymbol{\eta}-$a.s. (x,w),

$$Q(x,w;\{\omega=\psi(x,w)\})=Q'(x,w;\{\omega'=\psi(x,w)\})=1,$$

i.e.

$$Q(x,w;B)=\mathbf{1}_B(\psi(x,w)),\quad \forall B\in\mathcal{B}(C([0,T];\mathbb{R}^d)).$$

Moreover, for a.e. x, the map $w\mapsto\psi(x,w)$ is $\overline{\mathcal{B}_t(C([0,T];\mathbb{R}^m))}^{\boldsymbol{\eta}}/\mathcal{B}_t(C([0,T];\mathbb{R}^d))-$measurable(see [8, Lemma 1.1, Chapter IV]). Recalling that $Q(x,w;\cdot)$ is the the regular conditional probability of ω given $(\omega_0=x,w)$, so $\int Q(x,w;\cdot)\boldsymbol{\eta}(\mathrm{d}w)$ is the regular conditional probability of ω given $\omega_0=x$. Notice that $(\Omega,\mathbb{Q},\omega)$ is a martingale solution to (1.6) with initial distribution μ_0, by [16, Theorem 6.1.2], for a.e. x the probability measure

$$\mathcal{B}(C([0,T];\mathbb{R}^d))\ni B\mapsto\int Q(x,w;B)\boldsymbol{\eta}(\mathrm{d}w)=\int\mathbf{1}_B(\psi(x,w))\boldsymbol{\eta}(\mathrm{d}w)=\boldsymbol{\eta}\circ\psi^{-1}(x,\cdot)(B)$$

is a martingale solution to (1.6) with initial data $\xi=x$. Thus, given a filtered probability space $(\Omega,\mathscr{F},\mathscr{F}_t,\mathbf{P})$ and a standard Brownian motion W on it, for a.e. $x\in\mathbb{R}^d$,

$$(X(x):=\psi(x,W),W)$$

is a strong solution to (1.6) with initial datum $\xi=x$. Moreover, for any $A\in\mathcal{B}(\mathbb{R}^d)$

$$\begin{aligned}\mu_t(A)&:=\int_{\mathbb{R}^d}\mathbf{P}\circ X_t^{-1}(x)(A)\mu_0(\mathrm{d}x)\\&=\int_{\mathbb{R}^d}\mu_0(\mathrm{d}x)\int Q(x,w;\omega_t\in A)\boldsymbol{\eta}(\mathrm{d}w)\leqslant C\lambda_d(A).\end{aligned}$$

Thus, $\{X(x)\}_{x\in\mathbb{R}^d}$ is a μ_0-AESF. The proof for uniqueness of AESF is essentially the same with the one of Lemma 4.7, so we leave it to the reads. □

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