A MARKOV PROCESS FOR AN INFINITE INTERACTING PARTICLE SYSTEM IN THE CONTINUUM

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Abstract. An infinite system of point particles placed in $\mathbb{R}^d$ is studied. Its constituents perform random jumps with mutual repulsion described by a translation-invariant jump kernel and interaction potential, respectively. The pure states of the system are locally finite subsets of $\mathbb{R}^d$, which can also be interpreted as locally finite Radon measures. The set of all such measures Γ is equipped with the vague topology and the corresponding Borel σ-field. For a special class $\mathcal{P}_{\rm exp}$ of (sub-Poissonian) probability measures on Γ, we prove the existence of a unique family $\{P_{t,\mu} : t \geq 0,\ \mu \in \mathcal{P}_{\rm exp}\}$ of probability measures on the space of cadlag paths with values in Γ that solves a restricted initial-value martingale problem for the mentioned system. Thereby, a Markov process with cadlag paths is specified which describes the stochastic dynamics of this particle system.

1. Introduction

Measure-valued Markov processes [11] have become popular on their own – as a challenging object of probability theory – as well as due to their applications in mathematical physics, biology, ecology, etc. Among such applications one can distinguish those describing stochastic evolution of large (infinite) systems of point particles dwelling in continuous habitats, e.g., $\mathbb{R}^d$. In this case, the state space of the system is the set of all locally finite configurations of particles, which can also be interpreted as random counting Radon measures. In the case of finite particle systems, the construction of the corresponding Markov processes is now quite standard. For infinite systems, however, the list of results reduces mostly to free (noninteracting) systems [22] or to birth-and-death dynamics with generators obeying essential restrictions [16, 17, 23, 30]. In this context, one can also mention models with interactions of Curie-Weiss (mean-field) type (e.g., [25]) where one starts with a system of N particles interacting with a strength proportional to $1/N$, and then passes to the limit $N \to +\infty$. In this paper, we prove the existence and uniqueness of a Markov process with cadlag paths for an infinite system of point particles performing random jumps in $\mathbb{R}^d$ with a translation-invariant jump kernel and mutual repulsion, which appears to be the first result of this kind known in the literature.

The state space of the considered model is the set

$$\Gamma = \{\gamma \subset \mathbb{R}^d : |\gamma \cap \Lambda| < \infty \text{ for all compact } \Lambda \subset \mathbb{R}^d\}, \tag{1.1}$$

where γ is a *configuration* and $|\cdot|$ stands for cardinality. An account of mathematical properties of Γ can be found in [10, 20], see also Sect. 2 below where we outline those of them which are relevant to the present study. The space (1.1) is equipped with the vague (weak-hash) topology – the weakest topology that makes continuous all the maps $\gamma \mapsto \sum_{x\in\gamma} g(x)$, $g \in C_{\rm cs}(\mathbb{R}^d)$, where $C_{\rm cs}(\mathbb{R}^d)$ denotes the set of all compactly supported continuous functions $g : \mathbb{R}^d \to \mathbb{R}$. The vague topology is metrizable in such a way that the corresponding metric space is complete and separable. Then the states of the considered system are probability measures on Γ, the set of which is denoted by $\mathcal{P}$. The point states γ are associated to the Dirac measures δ_γ. The evolution of the system which we consider is described by the (backward) Kolmogorov equation

$$\frac{d}{dt} F_t = L F_t, \tag{1.2}$$

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where $F_t : \Gamma \to \mathbb{R}$, $t \geq 0$ and

$$(LF)(\gamma) = \sum_{x\in\gamma} \int_{\mathbb{R}^d} a(x-y) \exp\left(- \sum_{z\in\gamma\setminus x} \phi(z-y) \right) [F(\gamma \setminus x \cup y) - F(\gamma)]\, dy. \tag{1.3}$$

Originally, models of this kind were introduced and (heuristically) studied in physics [18], where they are known under a common name *Kawasaki model.* In the rigorous setting, the model described by (1.2), (1.3) was studied in [3] (see also [5] for a preliminary investigation). In [3], the following result was obtained. For a special class of states $\mathcal{P}_{\rm exp} \subset \mathcal{P}$ and each $\mu_0 \in \mathcal{P}_{\rm exp}$, there was constructed a map $[0, +\infty) \ni t \mapsto \mu_t \in \mathcal{P}_{\rm exp}$ that can be interpreted as the evolution of states related to (1.2). In the present work, we construct a Markov process with cadlag paths in such a way that μ_t is the law at time t. Let us outline some of the aspects of this construction. As we show here, for a sufficiently large set of functions $F : \Gamma \to \mathbb{R}$, the map $[0, +\infty) \ni t \mapsto \mu_t \in \mathcal{P}_{\rm exp}$ constructed in [3] is the unique solution of the Fokker-Planck equation

$$\mu_t(F) = \mu_s(F) + \int_s^t \mu_u(LF) du, \qquad \mu(F) := \int F d\mu, \tag{1.4}$$

holding for all $0 \leq s < t < \infty$, see [6] for the general theory of the Fokker-Planck equation. The peculiarity of $\mathcal{P}_{\rm exp}$ is that the Dirac measures δ_γ are not in this set for each $\gamma \in \Gamma$. Therefore, one cannot directly construct a transition function (and hence the corresponding Markov process) just by setting $\mu_0 = \delta_\gamma$. In view of this, in constructing the process we take a version of the martingale approach suggested in [31], see also [11, Sect. 5.1], [13, Chapter 4]. The main aspects of our construction can be outlined as follows. When dealing with measures $\mu \in \mathcal{P}_{\rm exp}$, it is natural to use a subset $\Gamma_* \subset \Gamma$ such that $\mu(\Gamma_*) = 1$ for all $\mu \in \mathcal{P}_{\rm exp}$. This set Γ_* is equipped with a metric which turns it into a Polish space, continuously embedded in Γ. Then the measures of interest are redefined as measures on Γ_*. Let $\mathfrak{D}_{\mathbb{R}_+}(\Gamma_*)$ stand for the space of all cadlag maps $\mathbb{R}_+ \ni t \mapsto \gamma_t \in \Gamma_*$, equipped with the usual Skorohod topology, see [13, page 118]. Let also $\varpi_t : \mathfrak{D}_{\mathbb{R}_+}(\Gamma_*) \to \Gamma_*$, $t \geq 0$ be the evaluation map, i.e., $\varpi_t(\gamma) = \gamma_t$. For $\mathbb{A} \in \mathcal{B}(\Gamma_*)$, set

$$\varpi_t^{-1}(\mathbb{A}) = \{\gamma \in \mathfrak{D}_{\mathbb{R}_+}(\Gamma_*) : \varpi_t(\gamma) \in \mathbb{A}\}. \tag{1.5}$$

The principal result of this work (Theorem 3.6) can be characterized as follows. We prove that there exists a family of probability measures, $\{P_{s,\mu} : s \geq 0,\ \mu \in \mathcal{P}_{\rm exp}\}$, on $\mathfrak{D}_{\mathbb{R}_+}(\Gamma_*)$ which is a unique solution of the restricted initial-value martingale problem corresponding to (1.3). For such measures, their one-dimensional marginals $\mu_t = P_{s,\mu} \circ \varpi_t^{-1}$, $t \geq s$ belong to $\mathcal{P}_{\rm exp}$ and satisfy the corresponding version of the Fokker-Planck equation (1.4). In view of this, one may say that these path measures are obtained as 'superpositions' of the measures constructed in [3]. This might be interpreted as that an infinite dimensional analog of the theory developed in [14, 33]. Here, however, the techniques are essentially different from those used in the latter works.

In [4], there was studied a model in which point particles of two types perform random jumps over $\mathbb{R}^d$. Their common dynamics is described by the corresponding analog of the Kolmogorov operator (1.3) in which particles of different types repel each other, whereas those of the same type do not interact. This kind of interaction is typical for the classical Widom-Rowlinson model (see [8] and the literature quoted therein), for which the states of thermal equilibrium can be multiple [8, 24]. The latter fact has essential impact on the stochastic dynamics of such models, cf. [19], which further stimulates the efforts in this direction. The results of [4] are pretty analogous to those of [3], which means that – after proper modification – the approach developed in the present work can be applied also to the model of [4], which we will realize in a subsequent paper.

The rest of the paper is organized as follows. In Sect. 2, we introduce all necessary facts and notions, among which are sub-Poissonian measures and the above-mentioned set $\Gamma_* \subset \Gamma$. Here we also introduce and study two classes of functions $F : \Gamma_* \to \mathbb{R}$, which play a crucial role in defining the Kolmogorov operator L introduced in (1.3). In Sect. 3, we impose standard assumptions on a and ϕ and then make precise the domain of L, i.e., the class of functions $F : \Gamma_* \to \mathbb{R}$ for which the Fokker-Planck equation is solved. Thereafter, in Theorem 3.6 we formulate the result – the statement that the restricted initial value martingale problem for our model has precisely one solution. Then we outline our strategy of proving this statement. In Sect. 4, we present and employ the results of [3] where the evolution of states $t \to \mu_t \in \mathcal{P}_{\rm exp}$ was constructed. In Sect. 5, we prove that the restricted initial value martingale problem for our model has at most one solution. This is done by proving that the Fokker-Planck equation (1.4) has a unique solution in the class of sub-Poissonian measures. Since the one-dimensional marginals of the constructed path measures should solve (1.4), this yields a tool of proving the desired uniqueness. In Sect. 6 and 7, we prove the existence of the path measures in question by employing auxiliary models (Sect. 6) for which one can construct the processes directly (by means of transition functions), and then by proving (Sect. 7) that these models approximate the main model. Their Markov property is then obtained similarly as in [11, Sect. 5.1, pages 78, 79].

2. Preliminaries

Throughout this work we use the following notations: Λ – a compact subset of $\mathbb{R}^d$; $\mathbb{R}_+ = [0, +\infty)$; $\mathbb{N}$ – the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$; $C_{\rm cs}(\mathbb{R}^d)$ - the set of all compactly supported continuous functions $g : \mathbb{R}^d \to \mathbb{R}$, $B_r(y) = \{x \in \mathbb{R}^d : |x - y| \leq r\}$, $B_r = B_r(0)$, $r > 0$ and $y \in \mathbb{R}^d$.

2.1. **The configuration spaces.** There exists a metric, $\upsilon^{\#}$, on Γ that makes it a Polish space, see [10, Sects. A25, A26] and a recent work [26]. It is known that

- The metric space $(\Gamma, \upsilon^{\#})$ is complete and separable.
- The metric topology of $(\Gamma, \upsilon^{\#})$ is the weakest topology that makes continuous all the maps $\Gamma \ni \gamma \mapsto \sum_{x\in\gamma} f(x)$ with $f \in C_{\rm cs}(\mathbb{R}^d)$.
- The Borel σ-field $\mathcal{B}(\Gamma)$ is the smallest σ-field of subsets of Γ that makes measurable all the maps $\gamma \mapsto N_\Lambda(\gamma) := |\gamma \cap \Lambda|$ with compact $\Lambda \subset \mathbb{R}^d$.

Set

$$\Gamma_0 = \bigcup_{n\in\mathbb{N}_0} \Gamma^{(n)}, \qquad \Gamma^{(n)} = \{\gamma \in \Gamma : |\gamma| = n\}.$$

Obviously, each $\Gamma^{(n)}$ – and hence the set of finite configurations Γ_0 – belong to $\mathcal{B}(\Gamma)$. The topology induced on Γ_0 by the vague topology of Γ coincides with the weak topology determined by bounded continuous functions $f \in C_{\rm b}(\mathbb{R}^d)$. Then the corresponding Borel σ-field $\mathcal{B}(\Gamma_0)$ is a sub-field of $\mathcal{B}(\Gamma)$. It is possible to show that a function $G : \Gamma_0 \to \mathbb{R}$ is measurable if and only if there exists a family of symmetric Borel functions $G^{(n)} : (\mathbb{R}^d)^n \to \mathbb{R}$, $n \in \mathbb{N}$ such that

$$G(\{x_1, \ldots, x_n\}) = G^{(n)}(x_1, \ldots, x_n). \tag{2.1}$$

In this case, we also write $G^{(0)} = G(\varnothing)$.

Definition 2.1. *A measurable function, $G : \Gamma_0 \to \mathbb{R}$, is said to have bounded support if there exist $N \in \mathbb{N}$ and a compact Λ such that: (a) $G^{(n)} \equiv 0$ for all $n > N$; (b) $G(\eta) = 0$ whenever η is not a subset of Λ. By $B_{\rm bs}$ we will denote the set of all bounded functions with bounded support. For $G \in B_{\rm bs}$, N_G and Λ_G will denote the least N and Λ as in (a) and (b), respectively. We also set $C_G = \sup_{\eta\in\Gamma_0} |G(\eta)|$.*

The Lebesgue-Poisson measure λ is defined on Γ_0 by the integrals

$$\int_{\Gamma_0} G(\eta)\lambda(d\eta) = G(\varnothing) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} G^{(n)}(x_1, \ldots, x_n)dx_1 \cdots dx_n, \tag{2.2}$$

holding for all $G \in B_{\rm bs}$. For $G \in B_{\rm bs}$, set

$$(KG)(\gamma) = \sum_{\eta\subset\gamma} G(\eta), \qquad \gamma \in \Gamma. \tag{2.3}$$

Remark 2.2. [20, Proposition 3.1] *For each $G \in B_{\rm bs}$, KG is measurable and such that $|(KG)(\gamma)| \leq C_G(1 + |\gamma \cap \Lambda_G|^{N_G})$ with C_G, Λ_G and N_G as in Definition 2.1.*

2.2. **Sub-Poissonian measures.** In this subsection, we introduce a set of probability measures on $(\Gamma, \mathcal{B}(\Gamma))$, which plays a key role in this research. Set $\Theta_0 := \{\theta \in C_{\rm cs}(\mathbb{R}^d) : \theta(x) \in (-1, 0]\}$. Then, for each $\mu \in \mathcal{P}$ and $\theta \in \Theta_0$, the function

$$\Gamma \ni \gamma \mapsto F^\theta(\gamma) = \prod_{x\in\gamma}(1+\theta(x)) = \exp\left(\sum_{x\in\gamma}\log\left(1+\theta(x)\right)\right) \tag{2.4}$$

is continuous and bounded.

Definition 2.3. *By $\mathcal{P}_{\rm exp}$ we denote the set of all those $\mu \in \mathcal{P}$ for each of which $\widehat{\mu}(\theta) := \mu(F^\theta)$ can be defined as a real entire function of exponential type of $\theta \in L^1(\mathbb{R}^d)$.*

For $\theta \in \Theta_0$ and $n \in \mathbb{N}_0$, we set $e_n(\theta;\eta) = \prod_{x\in\eta}\theta(x)$ for $|\eta| = n$ and $e_n(\theta;\eta) = 0$ otherwise. Here and in the sequel, we assume that $\prod_{x\in\varnothing} = 1$. Next, set

$$\begin{aligned}(Ke_n(\theta,\cdot))(\gamma) &:= \sum_{\eta\subset\gamma} e_n(\theta;\eta), \qquad n \in \mathbb{N}_0, \\ (Ke(\theta,\cdot))(\gamma) &:= \sum_{\eta\subset\gamma} e(\theta;\eta), \qquad e(\theta;\eta) := \prod_{x\in\eta}\theta(x), \qquad \eta \in \Gamma_0.\end{aligned} \tag{2.5}$$

If $\mu \in \mathcal{P}_{\rm exp}$, then the following holds, see [21, Corollary 8],

$$\widehat{\mu}(\theta) = 1 + \sum_{n=1}^\infty \frac{1}{n!}\int_{(\mathbb{R}^d)^n} k_\mu^{(n)}(x_1,\dots,x_n)\theta(x_1)\cdots\theta(x_n)dx_1\cdots dx_n, \tag{2.6}$$

where each $k_\mu^{(n)}$ is a symmetric element of $L^\infty((\mathbb{R}^d)^n)$ satisfying

$$0 \le k_\mu^{(n)}(x_1,\dots,x_n) \le \varkappa^n, \tag{2.7}$$

for some $\varkappa > 0$ and (Lebesgue) almost all $x_1,\dots,x_n$. Clearly, the converse is also true. The least $\varkappa$ in (2.7) will be called the *type* of μ. Note that the upper estimate in (2.7) is known as Ruelle's bound, cf. [28] and [21, eq. (13), page 388]. Let $k_\mu : \Gamma_0 \to \mathbb{R}$ be defined by $k_\mu^{(n)}$, $n \in \mathbb{N}_0$ in the sense of (2.1). Then (2.6) can be written in the form, cf. (2.2),

$$\widehat{\mu}(\theta) = \int_{\Gamma_0} k_\mu(\eta)e(\theta;\eta)\lambda(d\eta) =: \langle\!\langle k_\mu, e(\theta;\cdot)\rangle\!\rangle, \tag{2.8}$$

where e is as in (2.5). By [20, Corolary 4.1] we also have

$$\mu(KG) = \int_{\Gamma_0} k_\mu(\eta)G(\eta)\lambda(d\eta) =: \langle\!\langle k_\mu, G\rangle\!\rangle, \tag{2.9}$$

holding for all $\mu \in \mathcal{P}_{\rm exp}$ and $G \in B_{\rm bs}$. The following statement provides a characterization of $\mathcal{P}_{\rm exp}$.

Proposition 2.4. *A given $\mu \in \mathcal{P}$ belongs to $\mathcal{P}_{\rm exp}$ if and only if, for each $n \in \mathbb{N}$, the map $\theta \mapsto \mu((Ke_n)(\theta;\cdot))$ can be defined as a continuous monomial of order n on $L^1(\mathbb{R}^d)$. In particular, this means that it satisfies*

$$n!\,|\mu((Ke_n)(\theta;\cdot))| \le \varkappa^n \|\theta\|^n_{L^1(\mathbb{R}^d)}. \tag{2.10}$$

The least $\varkappa > 0$, for which this estimate holds, is then the type of μ.

Proof. For $\mu \in \mathcal{P}_{\rm exp}$, by (2.5) and (2.9) we have

$$\mu((Ke_n)(\theta;\cdot)) = \frac{1}{n!}\int_{(\mathbb{R}^d)^n} k_\mu^{(n)}(x_1,\dots,x_n)\theta(x_1)\cdots\theta(x_n)dx_1\cdots dx_n =: \frac{1}{n!}\langle k_\mu^{(n)}, \theta^{\otimes n}\rangle.$$

Then (2.10) follows by Ruelle's bound (2.7). To prove the converse, by polarization formulas, see [27, Sect. 3], we conclude that the continuous monomial – the left-hand side of (2.10) – defines a continuous linear functional on $L^1((\mathbb{R}^d)^n)$. Hence, its standard form is

$$n!\mu((Ke_n)(\theta;\cdot)) = \langle k_\mu^{(n)}, \theta^{\otimes n}\rangle,$$

where $\theta \in L^1(\mathbb{R}^d)$ and $k_\mu^{(n)}$ is a symmetric element of $L^\infty((\mathbb{R}^d)^n)$ – not necessarily positive. By inclusion-exclusion formulas, see also (2.12) below, one gets from (2.10) that, for each compact $\Lambda \subset \mathbb{R}^d$ and $n \in \mathbb{N}$, the measure satisfies

$$\mu(N_\Lambda^n) < \infty, \tag{2.11}$$

where $N_\Lambda(\gamma) := |\gamma \cap \Lambda|$. That is, μ has all *local moments*, cf. [20, Proposition 4.1]. By [20, Proposition 4.2] it then follows that $k_\mu^{(n)}$ is positive and hence satisfies (2.7). Therefore, $\widehat{\mu}$ can be written in the form of (2.6) with $\theta \in L^1(\mathbb{R}^d)$, by which it is an exponential type entire function. $\square$

For the homogeneous Poisson measure $\pi_\varkappa$ with density $\varkappa > 0$, it follows that is $k_{\pi_\varkappa}^{(n)}(x_1, \dots, x_n) = \varkappa^n$, $n \in \mathbb{N}_0$, and hence

$$\widehat{\pi}_\varkappa(\theta) = \exp\left(\varkappa \int_{\mathbb{R}^d} \theta(x) dx\right),$$

which means that $\pi_\varkappa \in \mathcal{P}_{\rm exp}$ and $\varkappa$ is its type.

Remark 2.5. *Let G in (2.9) be such that $G(\eta) \geq 0$ for all $\eta \in \Gamma_0$. Then by (2.7) it follows that $\mu(KG) \leq \pi_\varkappa(KG)$, where $\varkappa$ is the type of μ. In view of this, the elements of $\mathcal{P}_{\rm exp}$ are called sub-Poissonian measures.*

For a compact Λ, let $\mathbb{1}_\Lambda(x)$, $x \in \mathbb{R}^d$ denote the indicator of Λ. Then $N_\Lambda(\gamma) = \sum_{x\in\gamma} \mathbb{1}_\Lambda(x)$, and thus

$$\begin{aligned} N_\Lambda^n(\gamma) &= \sum_{l=1}^n S(n,l) \sum_{x_1\in\gamma} \sum_{x_2\in\gamma\setminus x_1} \cdots \sum_{x_l\in\gamma\setminus\{x_1,\dots,x_{l-1}\}} \mathbb{1}_\Lambda(x_1)\cdots \mathbb{1}_\Lambda(x_l) \\ &= \sum_{l=1}^n l! S(n,l) \sum_{\{x_1,\dots,x_l\}\subset\gamma} \mathbb{1}_\Lambda(x_1)\cdots\mathbb{1}_\Lambda(x_l), \qquad n \in \mathbb{N}, \end{aligned}$$

where $S(n,l)$ is Stirling's number of second kind – the number of ways to divide n labeled items into l unlabeled groups. By (2.9) this yields

$$\pi_\varkappa(N_\Lambda^n) = \sum_{l=1}^n S(n,l) \left(\varkappa|\Lambda|\right)^l, \qquad n \in \mathbb{N},$$

where $|\Lambda|$ is the Lebesgue measure (volume) of Λ. Then for $\mu \in \mathcal{P}_{\rm exp}$, we obtain, cf. (2.11) and Remark 2.5,

$$\mu(N_\Lambda^n) \leq \sum_{l=1}^n S(n,l)(\varkappa|\Lambda|)^l, \qquad n \in \mathbb{N}. \tag{2.12}$$

In the sequel, the following functions will be extensively used

$$\psi(x) = \frac{1}{1+|x|^{d+1}}, \quad \Psi_0(\gamma) = \sum_{x\in\gamma}\psi(x), \quad \Psi_1(\gamma) = 1 + \Psi_0(\gamma), \qquad \gamma \in \Gamma. \tag{2.13}$$

Similarly as in (2.12), for all $n \in \mathbb{N}$ and each $\mu \in \mathcal{P}_{\rm exp}$, one obtains

$$\mu(\Psi_0^n) \leq \sum_{l=1}^n S(n,l)(\varkappa\langle\psi\rangle)^l, \tag{2.14}$$

where we have taken into account that $\psi^n(x) \leq \psi(x)$ for all $n \geq 1$ and x, $\varkappa$ is the type of μ and

$$\langle\psi\rangle := \int_{\mathbb{R}^d} \psi(x) dx. \tag{2.15}$$

By (2.14) it follows that

$$\int_\Gamma \exp\left(\beta\Psi_0(\gamma)\right)\mu(d\gamma) \leq \exp\left(\varkappa\langle\psi\rangle(e^\beta - 1)\right), \tag{2.16}$$

holding for all $\beta > 0$. Next, we define

$$\Gamma_{*,n} = \{\gamma : \Psi_0(\gamma) \leq n\}, \qquad \Gamma_* = \{\gamma : \Psi_0(\gamma) < \infty\} = \bigcup_{n\in\mathbb{N}} \Gamma_{*,n}. \tag{2.17}$$

Clearly, $\Gamma_* \in \mathcal{B}(\Gamma)$ since $\Gamma_{*,n} \in \mathcal{B}(\Gamma)$ for all $n \in \mathbb{N}$. Moreover, each $\Gamma_{*,n}$ is a relatively compact subset of Γ. Indeed, for a given $r > 0$ and $\gamma \in \Gamma_{*,n}$, we have

$$N_{B_r}(\gamma) = \sum_{x\in\gamma} \mathbb{1}_{B_r}(x) \leq (1 + r^{d+1}) \sum_{x\in\gamma} \mathbb{1}_{B_r}(x)\psi(x) \leq (1 + r^{d+1})n.$$

Then, for each compact Λ, there exists $b_\Lambda > 0$ such that, for each $\gamma \in \Gamma_{*,n}$, one has $N_\Lambda(\gamma) \leq b_\Lambda$, which yields the property in question, see [10, Corollary A2.6V, page 406]. By (2.14) it follows that

$$\forall \mu \in \mathcal{P}_{\rm exp} \qquad \mu(\Gamma_*) = 1. \tag{2.18}$$

This crucial property of the elements of $\mathcal{P}_{\rm exp}$ will allow us to consider only configurations belonging to Γ_*. In particular, this means that we will use the following sub-field of $\mathcal{B}(\Gamma)$:

$$\mathcal{A}_* = \{\mathbb{A} \in \mathcal{B}(\Gamma) : \mathbb{A} \subset \Gamma_*\}. \tag{2.19}$$

2.3. **Functions and measures on Γ_*.** Let $C_{\rm b}(\mathbb{R}^d)$ (resp. $B_{\rm b}(\mathbb{R}^d)$) stand for the set of all bounded continuous (resp. measurable) functions $g : \mathbb{R}^d \to \mathbb{R}$. For ψ defined in (2.13), we set

$$\begin{aligned} \varTheta_\psi &= \{\theta(x) = g(x)\psi(x) : g \in C_{\rm b}(\mathbb{R}^d), \quad \theta(x) \geq 0\}, \\ \varTheta_\psi^+ &= \{\theta \in \varTheta_\psi : \theta(x) > 0 \quad \forall x \in \mathbb{R}^d\}. \end{aligned} \tag{2.20}$$

Clearly, each $\theta \in \varTheta_\psi$ is integrable. For such θ, we also define

$$c_\theta = \sup_{x\in\mathbb{R}^d} \frac{1}{\psi(x)} \log\left(1 + \theta(x)\right), \qquad \bar{c}_\theta := e^{c_\theta} - 1. \tag{2.21}$$

Then

$$0 \leq \theta(x) \leq \bar{c}_\theta \psi(x), \qquad \theta \in \varTheta_\psi. \tag{2.22}$$

By Remark 2.5, (2.15) and (2.22) we then have

$$\mu(F^\theta) \leq \pi_\varkappa(F^\theta) \leq \exp\left(\varkappa\langle\psi\rangle\bar{c}_\theta\right), \qquad \theta \in \varTheta_\psi,$$

whenever $\mu \in \mathcal{P}_{\rm exp}$ and $\varkappa$ is its type. For $\gamma \in \Gamma_*$, the measure on $\mathbb{R}^d$

$$\nu_\gamma(dx) = \psi(x) \sum_{y\in\gamma} \delta_y(dx) = \sum_{y\in\gamma} \psi(y)\delta_y(dx) \tag{2.23}$$

is finite since $\nu_\gamma(\mathbb{R}^d) = \Psi_0(\gamma)$, see (2.13). The set

$$\mathcal{N}_* := \{\nu_\gamma : \gamma \in \Gamma_*\} \tag{2.24}$$

of finite positive measures on $\mathbb{R}^d$ can be metrized in the following way. Consider

$$C_{\rm b}^L(\mathbb{R}^d) = \{g \in C_{\rm b}(\mathbb{R}^d) : \|g\|_L < \infty\}, \quad \|g\|_L := \sup_{x,y\in\mathbb{R}^d,\ x\neq y} \frac{|g(x) - g(y)|}{|x - y|},$$

and then define

$$\|g\|_{BL} = \|g\|_L + \sup_{x\in\mathbb{R}^d} |g(x)|, \qquad g \in C_{\rm b}^L(\mathbb{R}^d),$$

and also

$$\upsilon(\nu, \nu') = \sup_{g:\|g\|_{BL}\leq 1} \left|\nu(g) - \nu'(g)\right|, \qquad \nu, \nu' \in \mathcal{N}_*. \tag{2.25}$$

Proposition 2.6. [12, Theorem 18] *The following three types of the convergence of a sequence $\{\nu_n\} \subset \mathcal{N}_*$ to a positive finite measure ν on $\mathbb{R}^d$ are equivalent:*

(i) $\nu_n(g) \to \nu(g)$ *for all* $g \in C_{\rm b}(\mathbb{R}^d)$*;*
(ii) $\nu_n(g) \to \nu(g)$ *for all* $g \in C_{\rm b}^L(\mathbb{R}^d)$*;*
(iii) $\upsilon(\nu_n, \nu) \to 0$.

That is, υ metrizes the usual weak convergence of the elements of $\mathcal{N}_*$. Set

$$\Theta_\psi^{BL} = \{\theta(x) = g(x)\psi(x) : \|g\|_{BL} \leq 1\}, \tag{2.26}$$

and then define, cf. (2.23) and (2.25),

$$\upsilon_*(\gamma,\gamma') = \upsilon(\nu_\gamma,\nu_{\gamma'}) = \sup_{\theta\in\Theta_\psi^{BL}} \left| \sum_{x\in\gamma}\theta(x) - \sum_{x\in\gamma'}\theta(x) \right|. \tag{2.27}$$

Proposition 2.7. *The metric space (Γ_*, υ_*) is complete and separable. Its metric topology is the weakest topology that makes continuous all the maps $\Gamma_* \ni \gamma \mapsto \sum_{x\in\gamma}\theta(x)$, $\theta \in \Theta_\psi$.*

Proof. Let $\{\gamma_n\} \subset \Gamma_*$ be a Cauchy sequence in metric υ_*. By (2.27) the sequence $\{\nu_{\gamma_n}\} \subset \mathcal{M}^+(\mathbb{R}^d)$ converges in υ to a positive finite measure, ν, on $\mathbb{R}^d$, as the metric space $(\mathcal{M}^+(\mathbb{R}^d), \upsilon)$ of such measures is complete. By Proposition 2.6 this yields $\nu_{\gamma_n}(g) \to \nu(g)$ for all $g \in C_{\rm b}(\mathbb{R}^d)$, also for all $g(x) = \theta(x)/\psi(x)$ with $\theta \in C_{\rm cs}(\mathbb{R}^d)$. Then, for each $\theta \in C_{\rm cs}(\mathbb{R}^d)$, $\{\gamma_n(\theta)\}$ is a Cauchy sequence, that implies the convergence $\gamma_n \to \gamma \in \Gamma$ in the vague topology, since $(\Gamma, \upsilon^{\#})$ is complete. Therefore, for such ν, γ, g and θ, we have that

$$\left(\lim_{n\to+\infty} \nu_{\gamma_n}(g) = \right) \nu(g) = \sum_{x\in\gamma}\theta(x) \left(= \lim_{n\to+\infty}\gamma_n(\theta) \right). \tag{2.28}$$

For $m \in \mathbb{N}$, let $\theta_m \in C_{\rm cs}(\mathbb{R}^d)$ be such that $\theta_m(x) = \psi(x)$ for $|x| \leq m$, and $\theta_m(x) = 0$ for $|x| \geq m+1$. Then $\theta_m \to \psi$ and $g_m \to 1$, point-wise. At the same time, by (2.28) we get

$$\sum_{x\in\gamma}\theta_m(x) = \nu(g_m) \leq \nu(1) = \lim_{n\to+\infty}\nu_n(1). \tag{2.29}$$

By the monotone convergence theorem and (2.29) it follows that $\sum_{x\in\gamma}\psi(x) < \infty$, and thus $\gamma \in \Gamma_*$ which yields the completeness in question. Hence, $\mathcal{N}_*$ defined in (2.24) is a closed subset of $\mathcal{M}^+(\mathbb{R}^d)$, that implies its separability and also the separability of (Γ_*, υ_*). Since the map $\Gamma_* \ni \gamma \mapsto \nu_\gamma \in \mathcal{N}_*$ is an isometry, the second statement follows by Proposition 2.6. $\square$

Let $\mathcal{B}(\Gamma_*)$ be the Borel σ-field of subsets of Γ_* corresponding to the metric topology of (Γ, υ_*). Recall that we have another σ-field, $\mathcal{A}_*$, defined in (2.19).

Corollary 2.8. *The embedding of the metric space (Γ_*, υ_*) into $(\Gamma, \upsilon^{\#})$ is continuous, which implies $\mathcal{A}_* = \mathcal{B}(\Gamma_*)$.*

Proof. The stated continuity follows by the fact that nonnegative $\theta \in C_{\rm cs}(\mathbb{R}^d)$ belong also to Θ_ψ, and by Propositions 2.6 and 2.7. The equality of the σ-fields then follows by Kuratowski's theorem, see [29, Theorem 3.9, page 21]. $\square$

The latter statement allows one to redefine each $\mu \in \mathcal{P}$ with the property $\mu(\Gamma_*) = 1$ as a measure on the measurable space $(\Gamma_*, \mathcal{B}(\Gamma_*))$. Clearly, the class of all such measures, $\mathcal{P}(\Gamma_*)$, includes $\mathcal{P}_{\rm exp}$.

Remark 2.9. *In general, for $\theta \in \Theta_\psi$ the map $\Gamma_* \ni \gamma \mapsto \sum_{x\in\gamma}\theta(x)$ need not be $\upsilon^{\#}$-continuous. But it is υ_*-continuous for all such θ. In particular, the map $\gamma \mapsto \Psi_0(\gamma)$ is υ_*-continuous, that is one of the advantages of passing to the metric space (Γ_*, υ_*).*

For $\theta \in \Theta_\psi$, we set, see (2.20) and (2.21),

$$v_\tau^\theta(x) = \tau - \frac{1}{\psi(x)}\log\left(1+\theta(x)\right), \qquad V = \left\{ v_\tau^\theta : \theta \in \Theta_\psi,\ \tau > c_\theta \right\}.$$

Note that $V \subset C_{\rm b}(\mathbb{R}^d)$ is closed with respect to the pointwise addition and its elements are separated away from zero. The former follows by the fact that $\theta + \theta' + \theta\theta'$ belongs to Θ_ψ for each $\theta, \theta' \in \Theta_\psi$. Next, define

$$\widetilde{F}_\tau^\theta(\gamma) = \prod_{x\in\gamma}\left(1+\theta(x)\right)e^{-\tau\psi(x)} = \exp\left(-\nu_\gamma(v_\tau^\theta)\right), \qquad \gamma \in \Gamma_*. \tag{2.30}$$

For $\tau = 0$ and $\theta(x) \equiv 0$, we also set $\widetilde{F}^0_0(\gamma) \equiv 1$ and include this function in the set

$$\widetilde{\mathcal{F}} := \{\widetilde{F}^\theta_\tau : \theta \in \Theta_\psi, \ \tau > c_\theta\} \subset C_{\rm b}(\Gamma_*). \tag{2.31}$$

By $C_{\rm b}(\Gamma_*)$ (resp. $B_{\rm b}(\Gamma_*)$) we denote the set of all bounded continuous (resp. measurable) functions $F : \Gamma_* \to \mathbb{R}$. Similarly as in [11, Sect. 3.2, page 41], see also [13, page 111], we introduce the following notion.

Definition 2.10. *A sequence of bounded measurable functions $F_n : \Gamma_* \to \mathbb{R}$, $n \in \mathbb{N}$ is said to boundedly and pointwise (bp-) converge to a given $F : \Gamma_* \to \mathbb{R}$ if: (a) $F_n(\gamma) \to F(\gamma)$ for all $\gamma \in \Gamma_*$; (b) $\sup_{n\in\mathbb{N}} \sup_{\gamma\in\Gamma_*} |F_n(\gamma)| < \infty$. The bp-closure of a set $\mathcal{H} \subset B_{\rm b}(\Gamma_*)$ is the smallest subset of $B_{\rm b}(\Gamma_*)$ that contains $\mathcal{H}$ and is closed under the bp-convergence. In a similar way, one defines also the bp-convergence of sequences of functions $g : \mathbb{R}^d \to \mathbb{R}$.*

It is well-known that $C_{\rm b}(\mathbb{R}^d)$ contains a countable family of nonnegative functions, $\{g_i\}_{i\in\mathbb{N}}$, which is *convergence determining* and such that its linear span is bp-dense in $B_{\rm b}(\mathbb{R}^d)$, see [13, Proposition 4.2, page 111] and [11, Lemma 3.2.1, page 41]. This means that a sequence of finite positive measures, $\{\nu_n\}$, on $\mathbb{R}^d$ weakly converges to a certain ν if and only if $\nu_n(g_i) \to \nu(g_i)$, $n \to +\infty$ for all $i \in \mathbb{N}$. One may take such a family containing the constant function $g(x) \equiv 1$ and closed with respect to the pointwise addition. Moreover, one may assume that

$$\forall i \in \mathbb{N} \qquad \inf_{x\in\mathbb{R}^d} g_i(x) =: \varsigma_i > 0. \tag{2.32}$$

If this is not the case for a given g_i, in place of it one may take $\tilde{g}_i(x) = g_i(x) + \varsigma_i$ with some $\varsigma_i > 0$. The new set, $\{\tilde{g}_i\}$, has both mentioned properties and also satisfies (2.32). Then assuming the latter we conclude that

$$V_0 := \{g_i\}_{i\in\mathbb{N}} \subset V. \tag{2.33}$$

To see this, for a given g_i, take $\tau_i \geq \sup_x g_i(x)$ and then set

$$\theta_i(x) = \exp\Big([\tau_i - g_i(x)]\psi(x)\Big) - 1. \tag{2.34}$$

Clearly, $\theta_i(x) \geq 0$. Since $\psi^n(x) \leq \psi(x)$, $n \in \mathbb{N}$, we have that $\theta_i(x) \leq e^{\tau_i}\psi(x)$, and hence $\{\theta_i\}_{i\in\mathbb{N}} \subset \Theta_\psi$, see (2.20). At the same time, $v^{\theta_i}_{\tau_i} = g_i$ and $c_{\theta_i} = \sup_x(\tau_i - g_i(x)) < \tau_i$ in view of (2.32). By (2.34) and (2.33), for all $i \in \mathbb{N}$, it follows that

$$F_i \in \widetilde{\mathcal{F}}, \qquad F_i(\gamma) := \exp\left(-\nu_\gamma(g_i)\right).$$

Proposition 2.11. *The set $\widetilde{\mathcal{F}}$ defined in (2.31) is closed with respect to the poinwise multiplication. Moreover, it has the following properties:*

(i) *It is separating: $\mu_1(F) = \mu_2(F)$, holding for all $F \in \widetilde{\mathcal{F}}$, implies $\mu_1 = \mu_2$ for all $\mu_1, \mu_2 \in \mathcal{P}(\Gamma_*)$.*
(ii) *It is convergence determining: if a sequence $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(\Gamma_*)$ is such that $\mu_n(F) \to \mu(F)$, $n \to +\infty$ for all $F \in \widetilde{\mathcal{F}}$ and some $\mu \in \mathcal{P}(\Gamma_*)$, then $\mu_n(F) \to \mu(F)$ for all $F \in C_{\rm b}(\Gamma_*)$.*
(iii) *The set $B_{\rm b}(\Gamma_*)$ is the bp-closure of the linear span of $\widetilde{\mathcal{F}}$.*

Proof. The closedness of $\widetilde{\mathcal{F}}$ under multiplication follows directly by (2.30) and the fact that $\theta_1 + \theta_2 + \theta_1\theta_2 \in \Theta_\psi$ for each $\theta_1, \theta_2 \in \Theta_\psi$. It is clear that $\widetilde{\mathcal{F}}$ separates points of Γ_*, i.e., one finds $F \in \widetilde{\mathcal{F}}$ such that $F(\gamma_1) \neq F(\gamma_2)$ whenever $\gamma_1 \neq \gamma_2$, that holds for each pair $\gamma_1, \gamma_2 \in \Gamma_*$. Then claim (i) follows by [13, claim (a) Theorem 4.5, page 113]. Claim (ii) follows by the fact that $\{F_i\}_{i\in\mathbb{N}} \subset \widetilde{\mathcal{F}}$ has the property in question, which in turn follows by [11, Theorem 3.2.6, page 43]. Likewise, claim (iii) follows by [11, Lemma 3.2.5, page 43]. □

Note that each function as in (2.30) can be written in the form

$$\widetilde{F}^\theta_\tau(\gamma) = \exp\left(-\tau\Psi_0(\gamma)\right) F^\theta(\gamma), \tag{2.35}$$

where F^θ is as in (2.4), which is a υ_*-continuous function for each $\theta \in \Theta_\psi$.

For $m \in \mathbb{N}$, $\theta_1, \ldots, \theta_m \in \Theta_\psi^+$, see (2.20), we set

$$\begin{aligned}
&\widehat{F}_\tau^{\theta_1,\ldots,\theta_m}(\gamma) \qquad (2.36)\\
&= \sum_{x_1\in\gamma} \theta_1(x_1) \sum_{x_2\in\gamma\setminus x_1} \theta_2(x_2) \cdots \sum_{x_n\in\gamma\setminus\{x_1,\ldots,x_{m-1}\}} \theta_m(x_m) \widetilde{F}_\tau^0(\gamma \setminus \{x_1,\ldots,x_m\})\\
&= \sum_{\{x_1,\ldots,x_m\}\subset\gamma} \sum_{\sigma\in S_m} \theta_1(x_{\sigma(1)}) \cdots \theta_m(x_{\sigma(m)}) \widetilde{F}_\tau^0(\gamma \setminus \{x_1,\ldots,x_m\}),
\end{aligned}$$

where S_m is the symmetric group and $\widetilde{F}_\tau^0(\gamma) = \exp(-\tau\Psi_0(\gamma))$, see (2.35).

Proposition 2.12. *For each $\tau > 0$, $m \in \mathbb{N}$ and $\theta_1, \ldots, \theta_m \in \Theta_\psi^+$, it follows that $\widehat{F}_\tau^{\theta_1,\ldots,\theta_m} \in C_{\mathrm{b}}(\Gamma_*)$.*

Proof. To prove the continuity of $\widehat{F}_\tau^{\theta_1,\ldots,\theta_m}$ we rewrite (2.36) in the form

$$\begin{aligned}
\widehat{F}_\tau^{\theta_1,\ldots,\theta_m}(\gamma) &= \exp(-\tau\Psi_0(\gamma)) \qquad (2.37)\\
&\times \sum_{\{x_1,\ldots,x_m\}\subset\gamma} \sum_{\sigma\in S_m} \varphi_{\sigma(1)}(x_1) \cdots \varphi_{\sigma(m)}(x_m),
\end{aligned}$$

with $\varphi_j(x) := \theta_j(x) e^{\tau\psi(x)}$, $j = 1, \ldots, m$. Clearly, all φ_j belong to Θ_ψ^+. By an inclusion-exclusion formula the right-hand side of (2.37) can be written as a linear combination of the products of the following terms

$$\varPhi_{i_1,\ldots,i_s}(\gamma) = \sum_{x\in\gamma} \varphi_{i_1}(x) \cdots \varphi_{i_s}(x),$$

multiplied by a continuous function, $\gamma \mapsto \exp(-\tau\Psi_0(\gamma))$. Since Θ_ψ^+ is closed with respect to the pointwise multiplication, such terms are continuous that yields the continuity of $\widehat{F}_\tau^{\theta_1,\ldots,\theta_m}$. To prove the boundedness we estimate each $\varphi_j(x) \le \varphi(x) := c\psi(x)e^{\tau\psi(x)} \le c\psi(x)e^\tau$. Then

$$\begin{aligned}
\widehat{F}_\tau^{\theta_1,\ldots,\theta_m}(\gamma) &\le \exp(-\tau\Psi_0(\gamma)) \sum_{x_1\in\gamma} \varphi(x_1) \sum_{x_2\in\gamma\setminus x_1} \varphi(x_2) \cdots \sum_{x_m\in\gamma\setminus\{x_1,\ldots,x_{m-1}\}} \varphi(x_m)\\
&\le \exp(-\tau\Psi_0(\gamma)) \left(\sum_{x\in\gamma} \varphi(x)\right)^m \le c^m \Psi_0^m(\gamma) \exp(-\tau[\Psi_0(\gamma) - m])\\
&\le \left(\frac{cm}{\tau}\right)^m \exp(m(\tau - 1)),
\end{aligned}$$

which completes the proof. □

3. The Result

3.1. **The domain of L.** First we make precise the conditions imposed on the model defined by L given in (1.3). The positive measurable functions a and ϕ are supposed to satisfy the following:

$$\sup_x a(x) = \bar{a} < \infty, \qquad \sup_x \phi(x) = \bar{\phi}, \tag{3.1}$$

$$\int_{\mathbb{R}^d} \phi(x)dx =: \langle\phi\rangle < \infty, \qquad \int_{\mathbb{R}^d} a(x)dx = 1,$$

and

$$\int_{\mathbb{R}^d} |x|^l a(x)dx =: m_l^a < \infty, \qquad \text{for } \ l = 1, \ldots, d+1. \tag{3.2}$$

The conditions in (3.1) are the same as in [3]. We impose them to be able to use the results of this work here. Note that the assumed boubedness of ϕ excludes a hard-core repulsion. The condition in (3.2) was not used in [3].

As mentioned in Introduction, we are going to construct the process as a solution of a *restricted initial value martingale problem.* In this case, the domain of the operator introduced in (1.3) plays a key role, cf. [11, page 79]. Along with the set defined in (2.31), we define

$$\widehat{\mathcal{F}} = \{\widehat{F}_\tau^{\theta_1,\dots,\theta_m} : m \in \mathbb{N},\ \theta_1,\dots,\theta_m \in \Theta_\psi^+,\ \tau > 0\}, \tag{3.3}$$

where $\widehat{F}_\tau^{\theta_1,\dots,\theta_m}$ is the function introduced in (2.36).

Definition 3.1. *By $\mathcal{D}(L)$ we denote the linear span of the set $\widetilde{\mathcal{F}} \cup \widehat{\mathcal{F}}$.*

By (2.31) and Proposition 2.12 one concludes that $\mathcal{D}(L) \subset C_{\rm b}(\Gamma_*)$. Let us show that $L\widehat{F}_\tau^{\theta_1,\dots,\theta_m} \in B_{\rm b}(\Gamma_*)$. For $\gamma \in \Gamma_*$, $x \in \gamma$, $y \in \mathbb{R}^d$ and a suitable $F : \Gamma_* \to \mathbb{R}$, define, cf. (1.3),

$$\nabla^{y,x} F(\gamma) = F(\gamma \setminus x \cup y) - F(\gamma).$$

By (2.36) we have

$$\widehat{F}_\tau^{\theta_1,\dots,\theta_m}(\gamma) = \sum_{x_1\in\gamma} \theta_1(x_1)\widehat{F}_\tau^{\theta_2,\dots,\theta_m}(\gamma \setminus x_1).$$

Then

$$\nabla^{y,x}\widehat{F}_\tau^{\theta_1,\dots,\theta_m}(\gamma) = [\theta_1(y) - \theta_1(x)]\widehat{F}_\tau^{\theta_2,\dots,\theta_m}(\gamma \setminus x) + \sum_{x_1\in\gamma\setminus x} \theta_1(x_1)\nabla^{y,x}\widehat{F}_\tau^{\theta_2,\dots,\theta_m}(\gamma \setminus x_1).$$

By iterating the latter we get

$$\begin{aligned} \nabla^{y,x}\widehat{F}_\tau^{\theta_1,\dots,\theta_m}(\gamma) &= \sum_{j=1}^m [\theta_j(y) - \theta_j(x)]\widehat{F}_\tau^{\theta_1,\dots,\theta_{j-1},\theta_{j+1},\dots,\theta_m}(\gamma \setminus x) \\ &+ \Big(\exp(-\tau\psi(y)) - \exp(-\tau\psi(x))\Big)\widehat{F}_\tau^{\theta_1,\dots,\theta_m}(\gamma \setminus x). \end{aligned} \tag{3.4}$$

For $\theta \in \Theta_\psi$ and a as in (3.1), we set

$$(a * \theta)(x) = \int_{\mathbb{R}^d} a(x-y)\theta(y)dy = \int_{\mathbb{R}^d} \theta(x-y)a(y)dy. \tag{3.5}$$

Then $a * \theta \in C_{\rm b}(\mathbb{R}^d)$, where the continuity follows by the dominated convergence theorem and the latter equality in (3.5). Moreover, by (2.22) we have

$$\begin{aligned} (a*\theta)(x) &\le \bar{c}_\theta\psi(x)\int_{\mathbb{R}^d}(1+|x|^{d+1})a(x-y)\psi(y)dy \\ &\le \bar{c}_\theta\psi(x)\left[1 + \int_{\mathbb{R}^d}(|x-y|+|y|)^{d+1}a(x-y)\psi(y)dy\right] \\ &= \bar{c}_\theta\psi(x)\left[1 + \sum_{l=0}^{d+1}\binom{d+1}{l}\int_{\mathbb{R}^d}|x-y|^{d+1-l}|y|^l\psi(y)a(x-y)dy\right] \\ &\le \bar{c}_\theta\psi(x)\left[1 + \sum_{l=0}^{d+1}\binom{d+1}{l}m_l^a\right], \end{aligned} \tag{3.6}$$

where we have used (3.1), (3.2) and the fact that $|y|^l\psi(y) \le 1$ holding for all y and $0 \le l \le d+1$. Therefore,

$$\theta_j^1(x) := (a*\theta_j)(x) + \theta_j(x) \le c_a\bar{c}_{\theta_j}\psi(x). \tag{3.7}$$

Since $\theta_j \in \Theta_\psi^+$, we then get by the latter that also $\theta_j^1 \in \Theta_\psi^+$, $j = 1,\dots,m$. Here

$$c_a := 2 + \sum_{l=0}^{d+1}\binom{d+1}{l}m_l^a. \tag{3.8}$$

At the same time

$$|\exp(-\tau\psi(y)) - \exp(-\tau\psi(x))| \le \tau\psi(y)\psi(x)||x|^{d+1} - |y|^{d+1}|. \tag{3.9}$$

Then proceeding as in (3.6) we get

$$\int_{\mathbb{R}^d} a(x-y)|\exp(-\tau\psi(y)) - \exp(-\tau\psi(x))|dy \le \tau c_a \psi(x). \tag{3.10}$$

Thereafter, by (3.4), (3.5), (3.7), and (1.3) we obtain

$$\begin{aligned} \left|L\widehat{F}_\tau^{\theta_1,\dots,\theta_m}(\gamma)\right| &= \left|\sum_{x\in\gamma}\int_{\mathbb{R}^d} a(x-y)\exp\left(-\sum_{z\in\gamma\setminus x}\phi(z-y)\right)\nabla^{y,x}\widehat{F}_\tau^{\theta_1,\dots,\theta_m}(\gamma)dy\right| \\ &\le \sum_{j=1}^m \widehat{F}_\tau^{\theta_1,\dots,\theta_{j-1},\theta_j^1,\theta_{j+1},\dots,\theta_m}(\gamma) + \tau c_a\left(\prod_{j=1}^m \bar{c}_{\theta_j}\right)\widehat{F}_\tau^{m+1}(\gamma). \end{aligned} \tag{3.11}$$

where, cf. (2.36),

$$\widehat{F}_\tau^m(\gamma) = \sum_{x_1\in\gamma}\psi(x_1)\sum_{x_2\in\gamma\setminus x_1}\psi(x_2)\cdots\sum_{x_m\in\gamma\setminus\{x_1,\dots,x_{m-1}\}}\psi(x_m)\widetilde{F}_\tau^0(\gamma\setminus\{x_1,\dots,x_m\}). \tag{3.12}$$

Then the boundedness of $L\widehat{F}_\tau^{\theta_1,\dots,\theta_m}$ follows by Proposition 2.12.

Now let us show that $L\widetilde{F}_\tau^\theta \in B_{\mathrm{b}}(\Gamma_*)$ for all $\theta\in\Theta_\psi$ and $\tau > c_\theta$. Similarly as in (3.4) we get

$$\nabla^{y,x}\widetilde{F}_\tau^\theta(\gamma) = \left[e^{-\tau\psi(y)} - e^{-\tau\psi(x)}\right]\widetilde{F}_\tau^\theta(\gamma\setminus x) + \left[\theta(y)e^{-\tau\psi(y)} - \theta(x)e^{-\tau\psi(x)}\right]\widetilde{F}_\tau^\theta(\gamma\setminus x).$$

Then by (3.7), (3.8), (3.9) and (3.10) we arrive at

$$\left|L\widetilde{F}_\tau^\theta(\gamma)\right| \le e^\tau(1+\tau)c_a\Psi_0(\gamma)\exp(-\tau_0\Psi_0(\gamma))\prod_{x\in\gamma}(1+\theta(x))\,e^{-(\tau-\tau_0)\psi(x)},$$

where $\tau_0 > 0$ is such that $\tau - \tau_0 > c_\theta$ which is possible for each $\tau > c_\theta$. Then the boundedness in question follows similarly as in Proposition 2.12. The next statement summarizes the properties of $\mathcal{D}(L)$.

Proposition 3.2. *The set of functions introduced in Definition 3.1 has the following properties:*

(i) $\mathcal{D}(L) \subset C_{\mathrm{b}}(\Gamma_*)$ *and* $L : \mathcal{D}(L) \to B_{\mathrm{b}}(\Gamma_*)$.

(ii) *The set* $B_{\mathrm{b}}(\Gamma_*)$ *is the bp-closure of* $\mathcal{D}(L)$.

(iii) $\mathcal{D}(L)$ *is separating. That is, if* $\mu_1, \mu_2 \in \mathcal{P}_{\exp}$ *satisfy* $\mu_1(F) = \mu_2(F)$ *for all* $F \in \mathcal{D}(L)$, *then* $\mu_1 = \mu_2$.

(iv) *For each* $F \in \widetilde{\mathcal{F}}$, *see (3.3), and* $\mu \in \mathcal{P}_{\exp}$, *the measure* $F\mu/\mu(F)$ *belongs to* $\mathcal{P}_{\exp}$.

Proof. Claim (i) has been just proved. Claims (ii) and (iii) follow by Proposition 2.11 and the fact $\widetilde{\mathcal{F}} \subset \mathcal{D}(L)$. It remains to prove (iv). Take $F\in\widetilde{\mathcal{F}}$, $\mu\in\mathcal{P}_{\exp}$ and denote $\mu_F = F\mu/\mu(F)$. Certainly, $\mu(F) > 0$ and $\mu_F \in \mathcal{P}(\Gamma_*)$. Thus, to prove $\mu_F \in \mathcal{P}_{\exp}$ we have to show that $\widehat{\mu}_F(\theta)$ can be continued to an exponential type entire function of $\theta \in L^1(\mathbb{R}^d)$, see Definition 2.3. Take $\theta \in \Theta_0$ and consider

$$F^\theta(\gamma)\widetilde{F}_\tau^{\theta'}(\gamma) = \prod_{x\in\gamma}(1+\theta''(x)),$$

$$\theta''(x) := \theta(x)e^{-\tau\psi(x)} - (1-e^{-\tau\psi(x)}) + \theta'(x)e^{-\tau\psi(x)}(1+\theta(x)).$$

Since $\mu\in\mathcal{P}_{\exp}$,

$$\mu(F^\theta\widetilde{F}_\tau^{\theta'}) = \mu(F^{\theta''}),$$

is an exponential type entire function of $\theta'' \in L^1(\mathbb{R}^d)$, and hence can be written as in (2.6) with $k_\mu^{(n)} \in L^\infty((\mathbb{R}^d)^n)$. Now we rewrite θ'' as

$$\theta''(x) = \theta(x)e^{-\tau\psi(x)}(1+\theta'(x)) + \theta'(x)e^{-\tau\psi(x)} - (1-e^{-\tau\psi(x)}),$$

where the second and third terms are in $L^1(\mathbb{R}^d)$, and the coefficient at θ is continuous and bounded. Then we plug θ'' in this form into (2.6) and obtain that $\mu(F^\theta\widetilde{F}_\tau^{\theta'})$ can also be written in the form of (2.6) – i.e., as a series in $\theta^{\otimes n}$ – with the coefficients belonging to the corresponding $L^\infty(\mathbb{R}^d)$ and satisfying (2.7) with some $\varkappa$. This yields the proof. $\square$

3.2. **Formulating the result.** As mentioned in Introduction, following [11, Chapter 5] we are going to obtain the process by solving a *restricted initial value martingale problem.* Recall that $\mathfrak{D}_{\mathbb{R}_+}(\Gamma_*)$ stands for the space of all cadlag maps $[0,+\infty) =: \mathbb{R}_+ \ni t \mapsto \gamma_t \in \Gamma_*$, and the evaluation maps ϖ_t are defined in (1.5). In a similar way, one defines also the spaces $\mathfrak{D}_{[s,+\infty)}(\Gamma_*)$, $s > 0$. For $s,t \geq 0$, $s < t$, by $\mathfrak{F}^0_{s,t}$ we denote the σ-field of subsets of $\mathfrak{D}_{\mathbb{R}_+}(\Gamma)$ generated by the family $\{\varpi_u : u \in [s,t]\}$. Then we set

$$\mathfrak{F}_{s,t} = \bigcap_{\varepsilon>0} \mathfrak{F}^0_{s,t+\varepsilon}, \qquad \mathfrak{F}_{s,+\infty} = \bigvee_{n\in\mathbb{N}} \mathfrak{F}_{s,s+n}.$$

That is, $\mathfrak{F}_{s,+\infty}$ is the smallest σ-field which contains all $\mathfrak{F}_{s,s+n}$. Given $s \geq 0$ and $\mu \in \mathcal{P}_{\rm exp}$, in the definition below – which is an adaptation of the definition in [11, Section 5.1, pages 78, 79]) – we deal with probability measures $P_{s,\mu}$ on $(\mathfrak{D}_{[s,+\infty)}(\Gamma_*), \mathfrak{F}_{s,+\infty})$.

Definition 3.3. *A family of probability measures $\{P_{s,\mu} : s \geq 0,\ \mu \in \mathcal{P}_{\rm exp}\}$ is said to be a solution of the restricted initial value martingale problem for our model if, for all $s \geq 0$ and $\mu \in \mathcal{P}_{\rm exp}$, the following holds: (a) $P_{s,\mu} \circ \varpi_s^{-1} = \mu$; (b) $P_{s,\mu} \circ \varpi_t^{-1} \in \mathcal{P}_{\rm exp}$ for all $t > s$; (c) for each $F \in \mathcal{D}(L)$ (Definition 3.1), $t_2 \geq t_1 \geq s$ and any bounded function $\mathsf{G} : \mathfrak{D}_{[s,+\infty)}(\Gamma) \to \mathbb{R}$ which is $\mathfrak{F}_{s,t_1}$-measurable, the function*

$$\mathsf{H}(\gamma) := \left[F(\varpi_{t_2}(\gamma)) - F(\varpi_{t_1}(\gamma)) - \int_{t_1}^{t_2} (LF)(\varpi_u(\gamma))du\right] \mathsf{G}(\gamma) \tag{3.13}$$

is such that

$$\int_{\mathfrak{D}_{[s,+\infty)}} \mathsf{H}(\gamma) P_{s,\mu}(d\gamma) = 0. \tag{3.14}$$

The restricted initial value martingale problem is said to be well-posed if, for each $s \geq 0$ and $\mu \in \mathcal{P}_{\rm exp}$, there exists a unique $P_{s,\mu}$ satisfying all the conditions mentioned above.

Here by saying "for our model" along with the Kolmogorov operator L given in (1.3) we mean also its domain $\mathcal{D}(L)$ (Definition 3.1) and the class $\mathcal{P}_{\rm exp} \subset \mathcal{P}(\Gamma_*)$ defined by the property (2.6), see also Proposition 2.4. Note that H defined in (3.13) is $P_{s,\mu}$-integrable, that follows by claim (i) of Proposition 3.2. Note also that the functions G in (3.13) can be taken in the form

$$\mathsf{G}(\gamma) = F_1(\varpi_{s_1}(\gamma)) \cdots F_m(\varpi_{s_m}(\gamma)), \tag{3.15}$$

with all possible choices $m \in \mathbb{N}$, $F_1, \dots, F_m \in \widetilde{\mathcal{F}}$ (see Proposition 2.11), and $s \leq s_1 < s_2 < \cdots < s_m \leq t_1$, see [13, eq. (3.4), page 174].

Definition 3.4. *For a given $s \geq 0$, a map, $[s,+\infty) \ni t \mapsto \mu_t \in \mathcal{P}(\Gamma_*)$, is said to be measurable if the maps $[s,+\infty) \ni t \mapsto \mu_t(\mathbb{A}) \in \mathbb{R}$ are measurable for all $\mathbb{A} \in \mathcal{B}(\Gamma_*)$. Such a map is said to be a solution of the Fokker-Planck equation for our model if, for each $F \in \mathcal{D}(L)$ and any $t_2 > t_1 \geq s$, the following holds*

$$\mu_{t_2}(F) = \mu_{t_1}(F) + \int_{t_1}^{t_2} \mu_u(LF)du. \tag{3.16}$$

Remark 3.5. *By taking $\mathsf{G} \equiv 1$ in (3.13) one comes to the following conclusion. Let $\{P_{s,\mu} : s \geq 0, \mu \in \mathcal{P}_{\rm exp}\}$ be a solution as in Definition 3.3. Then, for each s and $\mu \in \mathcal{P}_{\rm exp}$, the map $[s,+\infty) \ni t \mapsto P_{s,\mu} \circ \varpi_t^{-1}$ solves (3.16) for all $t_2 > t_1 \geq s$.*

Now we can formulate our principal result.

Theorem 3.6. *For the model defined in (1.3) satisfying (3.1) and (3.2), the following is true:*

(a) *The restricted initial value martingale problem is well-posed in the sense of Definition 3.3.*
(b) *The stochastic process related to the family*

$$(\mathfrak{D}_{[s,+\infty)}(\Gamma), \mathfrak{F}_{s,+\infty}, \{\mathfrak{F}_{s,t} : t \geq s\}, \{P_{s,\mu} : \mu \in \mathcal{P}_{\rm exp}\})_{s\geq 0}$$

is Markov. This means that, for all $t > s$ and $\mathbb{B} \in \mathfrak{F}_{t,+\infty}$, the following holds

$$P_{s,\mu}(\mathbb{B}|\mathfrak{F}_{s,t}) = P_{s,\mu}(\mathbb{B}|\mathfrak{F}_t), \qquad P_{s,\mu} \ - \ \text{almost surely}.$$

Here $\mathfrak{F}_t$ is the smallest σ-field of subsets of $\mathfrak{D}_{[s,+\infty)}$ that contains all $\varpi_t^{-1}(\mathbb{A})$, $\mathbb{A} \in \mathcal{B}(\Gamma)$.

The proof of this statement will be done in the following two steps. First we prove that the restricted initial value martingale problem as in Definition 3.3 has at most one solution. Thereafter, we construct a solution by 'superposing' the collection of measures constructed in [3].

3.3. **Strategy of the proof and some comments.** Our approach is essentially based on the Fokker-Planck equation (1.4), (3.16) for which a solution, $t \to \mu_t \in \mathcal{P}_{\rm exp}$, $\mu_0 \in \mathcal{P}_{\rm exp}$ was constructed in [3]. In Sect. 6, we introduce approximating models by modifying the jump kernel in such the way that allows one to solve the Fokker-Planck equation directly by constructing stochastic semigroups in a Banach space of signed measures, with the possibility to take Dirac measures δ_γ, $\gamma \in \Gamma_*$ as the initial conditions. This allows in turn for introducing finite-dimensional marginals of the presumed law of the processes corresponding to these approximating models by means of the transition functions obtained in that way. Then we prove that these marginals satisfy a Chentsov-like condition (see [13, Theorem 3.8.8, page 139]) – the same for all approximating models. This yields the existence of cadlag versions of the approximating processes and is used in Sect. 7 to prove that their distributions have accumulating points – possible distributions of the process in question. Then we prove that such accumulation points solve the martingale problem in the sense of Definition 3.3. To prove uniqueness we again use the Fokker-Planck equation and the construction made in [3]. At this stage – realized in Sect. 5 – we show that this equation has a unique solution, which implies that the mentioned accumulation points have coinciding one-dimensional marginals. A classical result (see [13, claim (a) of Theorem 4.4.2, page 184]) is that one would have uniqueness if the one-dimensional marginals were equal for *all* initial $\mu \in \mathcal{P}(\Gamma_*)$. Since we have such an equality *only* for μ from a subset of $\mathcal{P}(\Gamma_*)$, we turn to the restricted version of the martingale problem [11, Chapter 5]. A crucial element of this version is Lemma 5.1 that states that a solution of the Fokker-Planck equation with $\mu_0 \in \mathcal{P}_{\rm exp}$ is also in $\mathcal{P}_{\rm exp}$, and its type satisfies $\varkappa_t \leq \varkappa_T$ for $t \leq T$, where $\varkappa_T$ depends on T and $\varkappa_0$ only. The proof of Lemma 5.1 is the most technical element of this part, based on a number of combinatorial results (see also Appendix). By means of Lemma 5.1 we then prove (Theorem 5.3) that (1.4) with $\mu_0 \in \mathcal{P}_{\rm exp}$ has a unique solution coinciding with the map $t \to \mu_t$ constructed in [3]. This finally yields the uniqueness of the solution.

4. The Evolution of States on Γ_*

As mentioned above, in the proof of Theorem 3.6 we essentially use the construction of the family of measures $\{\mu_t\}_{t\geq 0}$ performed in [3]. Thus, we begin by describing this family in a way adapted to the present context.

4.1. **Spaces of functions on Γ_0.** By (2.16) it follows that each measurable F satisfying $|F(\gamma)| \leq C\exp(\beta\Psi_0(\gamma))$ for some positive β and C is μ-absolutely integrable for each $\mu \in \mathcal{P}_{\rm exp}$. This obviously relates to $F = KG$ with $G \in B_{\rm bs}$, see Remark 2.2 and (2.9). For a and ϕ as in (3.1) and $G \in B_{\rm bs}$, let us consider

$$(\widehat{L}G)(\eta) = \sum_{\xi\subset\eta}\sum_{x\in\xi}\int_{\mathbb{R}^d} a(x-y)e(\tau_y;\xi)e(t_y;\eta\setminus\xi)\left[G(\xi\setminus x\cup y) - G(\xi)\right]dy, \tag{4.1}$$

$$\tau_y(x) := e^{-\phi(x-y)}, \qquad t_y(x) := \tau_y(x) - 1, \quad x,y\in\mathbb{R}^d.$$

In (4.1), the sums are finite and the integral is convergent in view of the integrability of the jump kernel a. It turns out that

$$LKG = K\widehat{L}G,$$

holding for all $G \in B_{\rm bs}$, see [15, Corollary 4.3 and eq. (4.7)]. By (2.9) this yields

$$\mu(LKG) = \langle\!\langle k_\mu, \widehat{L}G\rangle\!\rangle, \tag{4.2}$$

which by (2.7) points to the possibility to extend $\widehat{L}$ from $B_{\rm bs}$ to integrable functions. For a given $\vartheta \in \mathbb{R}$, let $\mathcal{G}_\vartheta$ stand for the weighted L^1-space equipped with the norm

$$\begin{aligned} |G|_\vartheta &= \int_{\Gamma_0} |G(\eta)| \exp(\vartheta|\eta|)\, \lambda(d\eta) \\ &= |G(\varnothing)| + \sum_{n=1}^{\infty} \frac{e^{\vartheta n}}{n!} \int_{(\mathbb{R}^d)^n} |G^{(n)}(x_1, \dots, x_n)| dx_1 \cdots dx_n. \end{aligned} \tag{4.3}$$

In fact, we have a descending scale $\{\mathcal{G}_\vartheta : \vartheta \in \mathbb{R}\}$ such that

$$\mathcal{G}_{\vartheta'} \hookrightarrow \mathcal{G}_\vartheta, \qquad \vartheta' > \vartheta, \tag{4.4}$$

where by $\hookrightarrow$ we mean continuous embedding. For a given $\vartheta \in \mathbb{R}$ and $G \in B_{\rm bs}$, let us estimate $|\widehat{L}G|_\vartheta$. By means of [15, Lemma 2.3], see also [3, Lemma 3.1], by (3.1) we get

$$\begin{aligned} |\widehat{L}G|_\vartheta &\leq \int_{\Gamma_0} e^{\vartheta|\eta|} \Bigg(\sum_{\xi \subset \eta} \sum_{x \in \eta \setminus \xi} \int_{\mathbb{R}^d} a(x-y) \left(|G(\eta \setminus \xi \setminus x \cup y)| \right. \\ &+ \left. |G(\eta \setminus \xi)| \right) e(|t_y|; \xi) dy \Bigg) \lambda(d\eta) \\ &= \int_{\Gamma_0} \Bigg(\int_{\mathbb{R}^d} e^{\vartheta|\eta|} \sum_{x \in \eta} a(x-y) \left(|G(\eta \setminus x \cup y)| + |G(\eta)| \right) \\ &\times \left(\int_{\Gamma_0} e(e^\vartheta |t_y|; \xi) \lambda(d\xi) \right) dy \Bigg) \lambda(d\eta) \\ &\leq 2 \exp\left(e^\vartheta \langle \phi \rangle \right) \int_{\Gamma_0} e^{\vartheta|\eta|} |\eta| |G(\eta)| \lambda(d\eta). \end{aligned}$$

To estimate the last line in the latter formula we use the inequality $xe^{-\alpha x} \leq 1/e\alpha$, both x, α positive, and the fact that $B_{\rm bs} \subset \mathcal{G}_{\vartheta'}$ for each $\vartheta' > \vartheta$. Thereafter, we obtain

$$|\widehat{L}G|_\vartheta \leq \frac{2}{e(\vartheta' - \vartheta)} \exp\left(e^\vartheta \langle \phi \rangle \right) |G|_{\vartheta'}. \tag{4.5}$$

Below by means of this estimate we extend $\widehat{L}$ to operators acting in the scale $\{\mathcal{G}_\vartheta\}_{\vartheta \in \mathbb{R}}$, cf. (4.4).

Along with $\mathcal{G}_\vartheta$ we introduce the following Banach spaces. For symmetric $k^{(n)} \in L^\infty((\mathbb{R}^d)^n)$, $n \in \mathbb{N}$, let k be defined by $k^{(n)}$ as in (2.1), that includes also some constant $k(\varnothing) = k^{(0)}$. Such k constitute a real linear space and can be considered as essentially bounded functions $k : \Gamma_0 \to \mathbb{R}$. Note that the correlation functions k_μ, cf. (2.7), are such functions. Then for $\vartheta \in \mathbb{R}$, we define

$$\|k\|_\vartheta = \sup_{n \geq 0} \left(\|k^{(n)}\|_{L^\infty((\mathbb{R}^d)^n)} e^{-\vartheta n} \right) = \operatorname*{ess\,sup}_{\eta \in \Gamma_0} \Big(|k(\eta)| \exp(-\vartheta|\eta|) \Big).$$

The linear space $\mathcal{K}_\vartheta$ equipped with this norm is the Banach space in question. Clearly, cf. (4.4),

$$\mathcal{K}_\vartheta \hookrightarrow \mathcal{K}_{\vartheta'}, \quad \text{for } \vartheta < \vartheta'. \tag{4.6}$$

Note that $\mathcal{K}_\vartheta$ is the topological dual to $\mathcal{G}_\vartheta$ as the value of k on G is given by the formula

$$\langle\!\langle k, G \rangle\!\rangle = \int_{\Gamma_0} k(\eta) G(\eta) \lambda(d\eta).$$

Let us now define L^Δ by the condition, cf. (4.2),

$$\langle\!\langle L^\Delta k_\mu, G \rangle\!\rangle = \langle\!\langle k_\mu, \widehat{L}G \rangle\!\rangle. \tag{4.7}$$

By (4.1) it is obtained in the following form, see [3, eqs. (2.21), (2.22)],

$$
\begin{aligned}
(L^{\Delta}k)(\eta) &= \sum_{y\in\eta}\int_{\mathbb{R}^d} a(x-y)e(\tau_y;\eta\setminus y)(W_y k)(\eta\setminus y\cup x)dx \qquad (4.8)\\
&- \sum_{x\in\eta}\int_{\mathbb{R}^d} a(x-y)e(\tau_y;\eta\setminus x)(W_y k)(\eta)dy,
\end{aligned}
$$

where

$$
(W_y k)(\eta) = \int_{\Gamma_0} k(\eta\cup\xi)e(t_y;\xi)\lambda(d\xi). \qquad (4.9)
$$

Proceeding similarly as in obtaining (4.5), for all $\vartheta\in\mathbb{R}$ and $\vartheta'>\vartheta$, we get

$$
\|L^{\Delta}k\|_{\vartheta'} \le \frac{2}{e(\vartheta'-\vartheta)}\exp\left(e^{\vartheta}\langle\phi\rangle\right)\|k\|_{\vartheta}, \qquad (4.10)
$$

where we have taken into account that $\langle a\rangle = 1$, see (3.1).

4.2. **The evolution in spaces of functions on Γ_0.** By combining (4.2) with (4.7) we introduce the following versions of the Kolmogorov equation (1.2)

$$
\frac{d}{dt}G_t = \widehat{L}G_t, \qquad G_t|_{t=0} = G_0, \qquad (4.11)
$$

$$
\frac{d}{dt}k_t = L^{\Delta}k_t, \qquad k_t|_{t=0} = k_0, \qquad (4.12)
$$

which we will solve in the scales $\{\mathcal{G}_\vartheta : \vartheta\in\mathbb{R}\}$ and $\{\mathcal{K}_\vartheta : \vartheta\in\mathbb{R}\}$, respectively, see (4.4) and (4.6).

Let us first consider (4.12). By (4.10) we see that L^{Δ} maps each $\mathcal{K}_\vartheta$ in each $\mathcal{K}_{\vartheta'}$, cf. (4.6), and the corresponding map is linear and bounded. Likewise, one can define the linear maps $(L^{\Delta})^n : \mathcal{K}_\vartheta \to \mathcal{K}_{\vartheta'}$, $n\in\mathbb{N}$ the norm of which can be estimated by means of the inequality

$$
\|(L^{\Delta})^n k\|_{\vartheta'} \le \frac{n^n}{(eT(\vartheta',\vartheta))^n}\|k\|_{\vartheta}, \qquad (4.13)
$$

where, cf. [3, eq. (4.2)],

$$
T(\vartheta_2,\vartheta_1) = \frac{\vartheta_2-\vartheta_1}{2}\exp\left(-\langle\phi\rangle e^{\vartheta_2}\right), \qquad \vartheta_2>\vartheta_1. \qquad (4.14)
$$

It is known, see [3, eqs. (4.3), (4.4)] that

$$
\sup_{\vartheta'>\vartheta} T(\vartheta',\vartheta) = \frac{\delta(\vartheta)}{2}\exp\left(-\frac{1}{\delta(\vartheta)}\right) =: \tau(\vartheta), \qquad (4.15)
$$

where $\delta(\vartheta)$ is a unique solution of $\delta e^{\delta} = e^{-\vartheta}/\langle\phi\rangle$. The supremum in (4.15) is attained at $\vartheta' = \vartheta+\delta(\vartheta)$. Then the expression in (4.8), (4.9) can be used to define: (a) bounded linear operators $(L^{\Delta})^n_{\vartheta'\vartheta} : \mathcal{K}_\vartheta\to\mathcal{K}_{\vartheta'}$, $n\in\mathbb{N}$ the norm of which can be estimated by means of (4.13); (b) unbounded linear operators $L^{\Delta}_{\vartheta'}$ with domains, cf. [3, eq. 3.19)],

$$
\mathrm{Dom}L^{\Delta}_{\vartheta'} = \{k\in\mathcal{K}_{\vartheta'} : L^{\Delta}k\in\mathcal{K}_{\vartheta'}\}..
$$

Now we turn to (4.11). In a similar way, by means of (4.5) one defines: (a) bounded linear operators $(\widehat{L})^n_{\vartheta\vartheta'} : \mathcal{G}_{\vartheta'}\to\mathcal{G}_\vartheta$, $n\in\mathbb{N}$, the norm of which satisfies

$$
\|(\widehat{L})^n_{\vartheta\vartheta'}\| = \|(L^{\Delta})^n_{\vartheta'\vartheta}\|, \qquad n\in\mathbb{N}; \qquad (4.16)
$$

(b) unbounded operators $\widehat{L}_\vartheta$ with domains

$$
\mathrm{Dom}\widehat{L}_\vartheta = \{G\in\mathcal{G}_\vartheta : \widehat{L}G\in\mathcal{G}_\vartheta\}..
$$

It can be shown, see [3, Lemma 3.1], that, for each $\vartheta\in\mathbb{R}$ and $\vartheta'>\vartheta$, the following is true

$$
\mathcal{K}_\vartheta \subset \mathrm{Dom}L^{\Delta}_{\vartheta'}, \qquad \mathcal{G}_{\vartheta'}\subset\mathrm{Dom}\widehat{L}_\vartheta,
$$

by which one readily obtains that, for all ϑ,ϑ', $\vartheta'>\vartheta$, the following holds

$$
\forall k\in\mathcal{K}_\vartheta \qquad L^{\Delta}_{\vartheta'\vartheta}k = L^{\Delta}_{\vartheta'}k. \qquad (4.17)
$$

Furthermore, up to the embedding (4.6) we have that

$$L^{\Delta}_{\vartheta'\vartheta} k = L^{\Delta}_{\vartheta''\vartheta} k,$$

holding for all $\vartheta'' \in (\vartheta, \vartheta')$. By (4.13) the series

$$Q_{\vartheta'\vartheta}(t) = 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} (L^{\Delta})^n_{\vartheta'\vartheta} \tag{4.18}$$

converges in the operator norm topology – uniformly on compact subsets of $[0, T(\vartheta'\vartheta))$ – to a bounded linear operator

$$Q_{\vartheta'\vartheta}(t) : \mathcal{K}_{\vartheta} \to \mathrm{Dom} L^{\Delta}_{\vartheta'} \subset \mathcal{K}_{\vartheta'},$$

the norm of which satisfies

$$\|Q_{\vartheta'\vartheta}(t)\| \leq \frac{T(\vartheta', \vartheta)}{T(\vartheta', \vartheta) - t}. \tag{4.19}$$

Moreover, the map $[0, T(\vartheta', \vartheta)) \ni t \mapsto Q_{\vartheta'\vartheta}(t)$ is differentiable and the following holds

$$\frac{d}{dt} Q_{\vartheta'\vartheta}(t) = L^{\Delta}_{\vartheta'} Q_{\vartheta'\vartheta}(t) = L^{\Delta}_{\vartheta'\vartheta''} Q_{\vartheta''\vartheta}(t) = Q_{\vartheta'\vartheta''}(t) L^{\Delta}_{\vartheta''\vartheta}, \tag{4.20}$$

with an arbitrary $\vartheta'' \in (\vartheta, \vartheta')$ provided t satisfies $t < T(\vartheta'', \vartheta)$ and $t < T(\vartheta', \vartheta'')$ in the latter two terms, respectively, cf. (4.19). By (4.20) one readily obtains that the Cauchy problem in (4.12) with $k_0 \in \mathcal{K}_{\vartheta}$ has a unique classical solution in $\mathcal{K}_{\vartheta'}$, on the time interval $[0, T(\vartheta', \vartheta))$, cf. [3, Lemma 4.1]. It is

$$k_t = Q_{\vartheta'\vartheta}(t) k_0. \tag{4.21}$$

In a similar way, one shows that the Cauchy problem in (4.11) has a unique classical solution in $\mathcal{G}_{\vartheta}$, on the time interval $[0, T(\vartheta', \vartheta))$, given by the formula

$$G_t = H_{\vartheta\vartheta'}(t) G_0 =: \left(1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} (\widehat{L})^n_{\vartheta\vartheta'} \right) G_0, \qquad G_0 \in \mathcal{G}_{\vartheta'}. \tag{4.22}$$

By construction these solutions of (4.12) and (4.11) satisfy

$$\langle\!\langle k_t, G_0 \rangle\!\rangle = \langle\!\langle k_0, G_t \rangle\!\rangle, \qquad t < T(\vartheta', \vartheta). \tag{4.23}$$

4.3. **The evolution of states.** A priori, the solution given in (4.21) need not be the correlation function for any measure. Moreover, it may not even be positive, cf. (2.7). To check whether a given $k : \Gamma_0 \to \mathbb{R}_+$ is the correlation function of a certain $\mu \in \mathcal{P}_{\rm exp}$ we introduce the following set

$$B^{\star}_{\rm bs} = \{G \in B_{\rm bs} : \sum_{\xi \subset \eta} G(\xi) \geq 0, \ \text{for all } \eta \in \Gamma_0\}. \tag{4.24}$$

Note that some of its members can take also negative values. By [20, Theorems 6.1 and 6.2 and Remark 6.3] one proves the following statement.

Proposition 4.1. *Let a measurable function, $k : \Gamma_0 \to \mathbb{R}$, have the following properties:*

(a) $\langle\!\langle k, G \rangle\!\rangle \geq 0$ *for all* $G \in B^{\star}_{\rm bs}$*, see (4.24);*
(b) $k(\varnothing) = 1$*;*
(c) $k(\eta) \leq \varkappa^{|\eta|}$ *for some* $\varkappa > 0$*, cf. (2.7).*

Then k is the correlation function for a unique $\mu \in \mathcal{P}_{\rm exp}$.

Recall that the least $\varkappa$ as in item (c) above is the type of μ of which k is then the correlation function. Set

$$\mathcal{P}^{\vartheta}_{\rm exp} = \{\mu \in \mathcal{P}_{\rm exp} : \mu \text{ is of type} \leq e^{\vartheta}\}. \tag{4.25}$$

Let $\mathcal{K}^{\star}$ be the set of all $k : \Gamma_0 \to \mathbb{R}$ that possess the properties listed in Proposition 4.1. In [3, Theorem 3.3], it was shown that k_t as in (4.21) belongs to $\mathcal{K}^{\star}$ whenever k_0 is the correlation function of a certain $\mu \in \mathcal{P}_{\rm exp}$. In the context of the present study, the relevant results of [3] can be formulated as follows.

Proposition 4.2. *Given $\vartheta_0 \in \mathbb{R}$, let μ be an arbitrary element of $\mathcal{P}^{\vartheta_0}_{\rm exp}$. For this ϑ_0, set $\vartheta_t = \vartheta_0 + t$, $t \geq 0$. Then there exists a unique map, $[0, +\infty) \ni t \mapsto k_t \in \mathcal{K}^{\star}$, such that $k_0 = k_\mu$ and the following holds:*

(a) *for each* $t > 0$,

$$0 \leq k_t(\eta) \leq e^{\vartheta_t |\eta|}, \qquad \eta \in \Gamma_0,$$

by which $k_t \in \mathcal{K}_{\vartheta_t}$.

(b) *For each* $T > 0$ *and* $t \in [0, T)$, *the map* $t \mapsto k_t \in \mathcal{K}_{\vartheta_t} \subset \mathrm{Dom} L^{\Delta}_{\vartheta_T}$ *is continuous on* $[0, T)$ *and continuously differentiable on* $(0, T)$ *in* $\mathcal{K}_{\vartheta_T}$ *and the following holds:*

$$\frac{d}{dt} k_t = L^{\Delta}_{\vartheta_T} k_t. \tag{4.26}$$

5. The Uniqueness

In this section, we prove that the restricted initial value martingale problem has at most one solution. To this end we use the properties of $\mathcal{D}(L)$ stated in Proposition 3.2. In view of Remark 3.5, see also Lemma 5.5 below, the proof of the uniqueness in question amounts to proving that, for each $\mu \in \mathcal{P}_{\rm exp}$, the Fokker-Planck equation (3.16) has at most one solution $\mu_t \in \mathcal{P}_{\rm exp}$ satisfying $\mu_0 = \mu$. The main tool for this is based on controlling the type of μ_t by a method based on the use of the concrete form of the elements of $\mathcal{D}(L)$, see Definition 3.1.

5.1. **Solving the Fokker-Planck equation.** We begin by pointing out that in Definition 3.4 we do not assume that $\mu_t \in \mathcal{P}_{\rm exp}$ for $t > 0$.

Lemma 5.1. *Let* $[0, +\infty) \ni t \mapsto \mu_t \in \mathcal{P}(\Gamma_*)$ *be a solution of (3.16) with all* F *belonging to the linear span of* $\widehat{\mathcal{F}}$ *and a given* $\mu_0 \in \mathcal{P}^{\vartheta_0}_{\rm exp}$. *Then, for each* $T > 0$, *there exists* $\vartheta_T \in \mathbb{R}$ *such that, for all* $t \in [0, T]$, $\mu_t \in \mathcal{P}^{\vartheta_t}_{\rm exp}$ *with some* $\vartheta_t < \vartheta_T$.

Note that here we assume that *only* the initial state μ_0 belongs to $\mathcal{P}_{\rm exp}$. Also, we assume that μ_t solves (3.16) with F belonging *only* to a subset of $\mathcal{D}(L)$. It turns out that this is enough to solve it for all $\mathcal{D}(L)$, and even more. Set

$$\mathcal{F} = \left\{ F \in B_{\rm b}(\Gamma_*) : F = KG, \ G \in \bigcap_{\vartheta \in \mathbb{R}} \mathcal{G}_\vartheta \right\}, \tag{5.1}$$

where K is defined in (2.3) and G is supposed to be such that $|G|_\vartheta$ is finite for all ϑ, see (4.3). Let us show that $\mathcal{D}(L) \subset \mathcal{F}$. Since K is linear, this will follow from the fact that

$$\widetilde{\mathcal{F}} \cup \widehat{\mathcal{F}} \subset \mathcal{F}. \tag{5.2}$$

By (2.30) we have

$$\begin{aligned} \widetilde{F}^\theta_\tau(\gamma) &= \prod_{x \in \gamma} (1 + \theta(x)) e^{-\tau \psi(x)} = \sum_{\eta \subset \gamma} e(\theta_\tau; \eta) =: (K \widetilde{G}^\theta_\tau)(\gamma), \\ \theta_\tau(x) &:= \theta(x) e^{-\tau\psi(x)} + \psi_\tau(x), \qquad \psi_\tau(x) = -1 + e^{-\tau\psi(x)}. \end{aligned} \tag{5.3}$$

Clearly, $\theta_\tau \in L^1(\mathbb{R}^d)$ for each $\tau \geq 0$ and $\theta \in \varTheta_\psi$, cf. Definition 3.1. Then $\widetilde{G}^\theta_\tau = e(\theta_\tau; \cdot) \in \mathcal{G}_\vartheta$ for any $\vartheta \in \mathbb{R}$, which yields $\widetilde{\mathcal{F}} \subset \mathcal{F}$.

In the case of F given in (2.36), (2.37), we write

$$\begin{aligned} \widehat{F}^{\theta_1, \dots, \theta_m}_\tau(\gamma) &= \sum_{\xi \subset \gamma} g_m(\xi) \prod_{x \in \gamma \setminus \xi} (1 + \psi_\tau(x)) \\ &= \sum_{\eta \subset \gamma} \left(\sum_{\xi \subset \eta} g_m(\xi) e(\psi_\tau; \eta \setminus \xi) \right) =: \sum_{\eta \subset \gamma} \widehat{G}^{\theta_1, \dots, \theta_m}_\tau(\eta), \end{aligned} \tag{5.4}$$

where $\psi_\tau(x)$ is as in (5.3) and

$$g_m(\xi) = \begin{cases} \sum_{\sigma \in S_m} \theta_1(x_{\sigma(1)}) \cdots \theta_m(x_{\sigma(m)}), & \text{if } \xi = \{x_1, \dots, x_m\}; \\ 0 & \text{otherwise.} \end{cases}$$

Let us estimate $\widehat{G}_\tau^{\theta_1,\dots,\theta_m}$ with $\theta_1,\dots,\theta_m \in \Theta_\psi^+$. For $\tau \in (0,1]$, we have $|\psi_\tau(x)| \le \psi(x)$, and hence

$$\left|\widehat{G}_\tau^{\theta_1,\dots,\theta_m}(\eta)\right| \le \sum_{\xi\subset\eta} g_m(\xi) e(\psi;\eta\setminus\xi). \tag{5.5}$$

At the same time, for each $\eta \in \Gamma_0$, it follows that

$$\widehat{G}_\tau^{\theta_1,\dots,\theta_m}(\eta) \to g_m(\eta), \qquad \tau \to 0^+. \tag{5.6}$$

By (5.5) let us show that $\widehat{G}_\tau^{\theta_1,\dots,\theta_m}$ belongs to $\mathcal{G}_\vartheta$, $\vartheta\in\mathbb{R}$ Indeed, by (4.3) we have

$$\begin{aligned} |\widehat{G}_\tau^{\theta_1,\dots,\theta_m}|_\vartheta &= \int_{\Gamma_0} |\widehat{G}_\tau^{\theta_1,\dots,\theta_m}(\eta)| e^{\vartheta|\eta|}\lambda(d\eta) \\ &\le \int_{\Gamma_0}\int_{\Gamma_0} e^{\vartheta|\xi|} g_m(\xi) e^{\vartheta|\eta|} e(\psi;\eta)\lambda(d\xi)\lambda(d\eta) \\ &\le e^{m\vartheta}\langle\theta_1\rangle\cdots\langle\theta_m\rangle \exp\left(e^\vartheta\langle\psi\rangle\right) \\ &=: \delta_m^{\theta_1,\dots,\theta_m}(\vartheta), \end{aligned} \tag{5.7}$$

where $\langle\psi\rangle$, $\langle\theta_i\rangle$, $i=1,\dots,m$ are the corresponding L^1-norms, cf. (2.15). This completes the proof of (5.2).

Lemma 5.2. *Let $t\mapsto\mu_t$ satisfy the assumptions of Lemma 5.1. Then it solves (3.16) with all $F\in\mathcal{F}$ defined in (5.1).*

Both Lemmas 5.1 and 5.2 are proved below. Now assuming that their claims hold true, we prove the next statement – one of the two basic tools of proving Theorem 3.6.

Theorem 5.3. *For each $\mu_0 \in \mathcal{P}_{\rm exp}$, the solution of the Fokker-Planck equation in the sense of Definition 3.4 is unique.*

Proof. We begin by showing that (3.16) has a solution. Take $G = \widehat{G}_\tau^{\theta_1,\dots,\theta_m}$ and let k_t be as in Proposition 4.2 with k_0 being the correlation function of the initial state μ_0. Since k_t is in $\mathcal{K}^\star$, by Proposition 4.1 it determines a unique $\mu_t \in \mathcal{P}_{\rm exp}^{\vartheta_t}$, see (4.25), for which

$$\mu_t(F) = \mu_t(KG) = \langle\!\langle k_t, G\rangle\!\rangle, \qquad t\ge 0, \tag{5.8}$$

holding for all $F\in\widehat{\mathcal{F}}$. By claim (a) of Proposition 4.2 and (5.1), (5.2) the integral in the right-hand side of (5.8) is absolutely convergent for each $t\ge0$. Moreover, by claim (b) of Proposition 4.2 we have that

$$k_{t_2} - k_{t_1} = \int_{t_1}^{t_2} L^\Delta_{\vartheta_T} k_u du$$

holding for all $t_2 > t_1 \ge 0$ and $T > t_2$. We multiply both parts of the latter equality by an arbitrary $G\in\cap_{\vartheta\in\mathbb{R}}\mathcal{G}_\vartheta$ – also corresponding to $F\in\mathcal{F}$ – and then integrate with respect to λ. By claim (b) of Proposition 4.2 this integration and that over $[t_1,t_2]$ can be interchanged, that implies

$$\mu_{t_2}(F) - \mu_{t_1}(F) = \int_{t_1}^{t_2}\langle\!\langle L^\Delta_{\vartheta_T}k_u, G\rangle\!\rangle du = \int_{t_1}^{t_2}\langle\!\langle k_u, \widehat{L}G\rangle\!\rangle du = \int_{t_1}^{t_2}\mu_u(LF)du, \tag{5.9}$$

where we have used (4.2), (4.7) and the fact that $G\in\cap_{\vartheta\in\mathbb{R}}\mathcal{G}_\vartheta$. This yields (3.16). By Lemma 5.2 and (5.2) we then get that μ_t corresponding to k_t is a solution.

Assume now that there exists another solution, say $\{\tilde\mu_t\}_{t\ge0}\subset\mathcal{P}(\Gamma_*)$, such that $\tilde\mu_0 = \mu_0$. By Lemma 5.1 we have that $\tilde\mu_t \in \mathcal{P}_{\rm exp}^{\tilde\vartheta_t}$ and $\tilde\vartheta_t\in(\vartheta_0,\tilde\vartheta_T)$ for some $\tilde\vartheta_T$ and all $t\le T$. This means that the corresponding correlation functions, $\tilde k_t$, $t\le T$ belong to $\mathcal{K}_{\tilde\vartheta_t}$. Then the vector $q_u = L^\Delta_{\tilde\vartheta_T}\tilde k_u = L^\Delta_{\tilde\vartheta_T\tilde\vartheta_u}\tilde k_u$, see (4.17), lies in $\mathcal{K}_{\tilde\vartheta_T}$, and hence in $\mathcal{K}_{\tilde\vartheta_T+\varepsilon}$ for each $\varepsilon>0$, see (4.6). Then, for a fixed ε, by (4.13) and (2.7) we have

$$\|q_u\|_{\tilde\vartheta_T+\varepsilon} \le C(T,\varepsilon)e^{\tilde\vartheta_T}, \qquad u\in[0,t], \tag{5.10}$$

with $C(T,\varepsilon) = 1/eT(\tilde{\vartheta}_T + \varepsilon, \tilde{\vartheta}_T)$, see (4.14). Let us prove that the following holds

$$\forall G \in \bigcap_{\vartheta \in \mathbb{R}} \mathcal{G}_\vartheta \qquad \langle\!\langle \tilde{k}_t - k_0, G \rangle\!\rangle = \int_0^t \langle\!\langle q_u, G \rangle\!\rangle du. \tag{5.11}$$

A priori, the equality in (5.11) holds for only for G corresponding to $F \in \mathcal{D}(L)$, that includes $G = \widehat{G}_\tau^{\theta_1,\dots,\theta_m}$, see (5.4). For $\tau \in (0,1]$, by (5.7) and (5.10) we then have

$$\left| \langle\!\langle \tilde{k}_t - k_0, \widehat{G}_\tau^{\theta_1,\dots,\theta_m} \rangle\!\rangle \right| \leq \langle\!\langle \left| \tilde{k}_t - k_0 \right|, \left| \widehat{G}_\tau^{\theta_1,\dots,\theta_m} \right| \rangle\!\rangle \leq 2 e^{\tilde{\vartheta}_T} \delta_m^{\theta_1,\dots,\theta_m} (\tilde{\vartheta}_T + \varepsilon),$$

$$\left| \int_0^t \langle\!\langle q_u, \widehat{G}_\tau^{\theta_1,\dots,\theta_m} \rangle\!\rangle du \right| \leq \int_0^t \langle\!\langle |q_u|, \left| \widehat{G}_\tau^{\theta_1,\dots,\theta_m} \right| \rangle\!\rangle du \leq t C(T,\varepsilon) e^{\tilde{\vartheta}_T} \delta_m^{\theta_1,\dots,\theta_m} (\tilde{\vartheta}_T + \varepsilon).$$

Now we write (5.11) for $G = \widehat{G}_\tau^{\theta_1,\dots,\theta_m}$ and pass to the limit $\tau \to 0^+$. By the dominated convergence theorem and (5.6) we then obtain

$$\int_{(\mathbb{R}^d)^m} \left[\tilde{k}_t^{(m)}(x_1,\dots,x_m) - k_0^{(m)}(x_1,\dots,x_m) \right] \theta_1(x_1)\cdots\theta_m(x_m) dx_1 \cdots dx_m \tag{5.12}$$

$$= \int_0^t \left(\int_{(\mathbb{R}^d)^m} q_u^{(m)}(x_1,\dots,x_m) \theta_1(x_1)\cdots\theta_m(x_m) dx_1\cdots dx_m \right) du,$$

that holds for all $m \in \mathbb{N}$ and $\theta_1,\dots,\theta_m \in \Theta_\psi^+$, see (2.20). For a fixed $m \in \mathbb{N}$, the set of functions $(x_1,\dots,x_m) \mapsto \theta_1(x_1)\cdots\theta_m(x_m)$ with $\theta_1\dots,\theta_m \in \Theta_\psi^+$ is closed with respect to the pointwise multiplication and separates points of $(\mathbb{R}^d)^m$. Such functions vanish at infinity and are everywhere positive. Then by the corresponding version of the Stone-Weierstrass theorem [7] the linear span of this set is dense (in the supremum norm) in the algebra $C_0((\mathbb{R}^d)^m)$ of all continuous functions that vanish at infinity. At the same time, $C_0((\mathbb{R}^d)^m) \cap L^1((\mathbb{R}^d)^m)$ is dense in $L^1((\mathbb{R}^d)^m)$. For its subset $C_{\rm cs}((\mathbb{R}^d)^m)$ has this property. This allows us to extend the equality in (5.12) to the following

$$\int_{(\mathbb{R}^d)^m} \left[\tilde{k}_t^{(m)}(x_1,\dots,x_m) - k_0^{(m)}(x_1,\dots,x_m) \right] G^{(m)}(x_1,\dots,x_m) dx_1 \cdots dx_m$$

$$= \int_0^t \left(\int_{(\mathbb{R}^d)^m} q_u^{(m)}(x_1,\dots,x_m) G^{(m)}(x_1,\dots,x_m) dx_1\cdots dx_m \right) du,$$

holding for all $G^{(m)} \in L^1((\mathbb{R}^d)^m)$. Then the passage from this equality to that in (5.11) follows by the fact that G belongs to each $\mathcal{G}_\vartheta$, $\vartheta \in \mathbb{R}$.

By (4.7) the equality in (5.11) yields

$$\langle\!\langle \tilde{k}_t, G \rangle\!\rangle = \langle\!\langle k_0, G \rangle\!\rangle + \int_0^t \langle\!\langle \tilde{k}_u, \widehat{L}_{\tilde{\vartheta}_t \vartheta_0} G \rangle\!\rangle du, \tag{5.13}$$

in which $\widehat{L}_{\tilde{\vartheta}_t\vartheta_0} G =: G_1 \in \cap_{\vartheta\in\mathbb{R}} \mathcal{G}_\vartheta$. In view of (5.11), we can repeat (5.13) with G_1 instead of G, and repeat this procedure again by employing the same arguments. After repeating n times we arrive at

$$\langle\!\langle \tilde{k}_t, G \rangle\!\rangle = \langle\!\langle k_0, G \rangle\!\rangle + t \langle\!\langle k_0, \widehat{L}_{\tilde{\vartheta}_t\vartheta} G \rangle\!\rangle + \frac{t^2}{2} \langle\!\langle k_0, (\widehat{L}_{\tilde{\vartheta}_t\vartheta})^2 G \rangle\!\rangle$$

$$+ \cdots + \frac{t^{n-1}}{(n-1)!} \langle\!\langle k_0, (\widehat{L}_{\tilde{\vartheta}_t\vartheta})^{n-1} G \rangle\!\rangle + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \langle\!\langle \tilde{k}_{t_n}, (\widehat{L}_{\tilde{\vartheta}_t\vartheta})^n G \rangle\!\rangle dt_1 \cdots dt_n.$$

Assume now that $\tilde{\vartheta}_T > \vartheta_0 + T$, see Proposition 4.2, that is clearly possible by (4.6). Then we write down the same formula – in the same spaces – for k_t considered in (5.8), i.e., described in Proposition 4.2. This yields

$$\langle\!\langle \tilde{k}_t - k_t, G \rangle\!\rangle = \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \langle\!\langle \tilde{k}_{t_n} - k_{t_n}, (\widehat{L}_{\tilde{\vartheta}_t\vartheta})^n G \rangle\!\rangle dt_1 \cdots dt_n.$$

Now we take $\vartheta = \tilde{\vartheta}_t + \delta(\tilde{\vartheta}_t)$, see (4.15). Then by (4.13), (4.16) and (4.15) we have from the latter

$$\left|\langle\!\langle \tilde{k}_t - k_t, G\rangle\!\rangle\right| \leq \frac{n^n}{n! e^n}\left(\frac{t}{\tau(\tilde{\vartheta}_t)}\right)^n |G|_\vartheta \sup_{u\in[0,t]}\left(\|\tilde{k}_u\|_{\tilde{\vartheta}_t} + \|k_u\|_{\tilde{\vartheta}_t}\right). \tag{5.14}$$

Note that here $\tau(\tilde{\vartheta}_t) \geq \tau(\tilde{\vartheta}_T)$. Then for $t < \tau(\tilde{\vartheta}_T)$, the right-hand side of (5.14) can be made as small as one wants by taking big enough n. Since $G \in \mathcal{G}_\vartheta$ is arbitrary, this yields $\tilde{k}_t = k_t$ for all such t. The latter implies $\tilde{\mu}_t = \mu_t$, see Proposition 4.2. The continuation to bigger values of t is made by repeating the same procedure. The proof that these continuations cover the whole $\mathbb{R}_+$ can be done similarly as in the proof of Theorem 3.3 [3]. □

Proof of Lemma 5.2. By Lemma 5.1 a solution μ_t is in $\mathcal{P}_{\rm exp}^{\vartheta_T}$ for $t \leq T$. Let k_t be its correlation function, which satisfies the equality in (5.11) with $G = \widehat{G}_\tau^{\theta_1,\dots,\theta_m}$. As we have shown in the proof of Theorem 5.3 it satisfies this equality for all G such that $F = KG \in \mathcal{F}$, see (5.1). Then we apply (5.9) with this F and get the proof. □

Remark 5.4. *It follows by (5.11) that the claim of Lemma 5.2 holds true for all $F = KG$ with $G \in \cap_{\vartheta\in\mathbb{R}}\mathcal{G}_\vartheta$, also for unbounded ones.*

5.2. **Further properties of the solutions.** In this subsection, we prepare proving Lemma 5.1. Our ultimate goal here is to estimate the integrals of the solutions of (3.16) taken with the functions

$$F_m^\theta(\gamma) = \sum_{x_1\in\gamma}\theta(x_1)\sum_{x_2\in\gamma\setminus x_1}\theta(x_2)\cdots\sum_{x_m\in\gamma\setminus\{x_1,\dots,x_{m-1}\}}\theta(x_m), \quad \theta\in\Theta_\psi^+, \tag{5.15}$$

which can be obtained from the functions defined in (2.36) by setting $\theta_1 = \cdots = \theta_m = \theta$ and $\tau = 0$. Note that F_m^θ is unbounded, but integrable for each $\mu \in \mathcal{P}_{\rm exp}$, cf. Proposition 2.4. Moreover, for $\mu \in \mathcal{P}_{\rm exp}$,

$$\mu(F_m^\theta) = \int_{\mathbb{R}^d} k_\mu^{(m)}(x_1,\dots,x_m)\theta(x_1)\cdots\theta(x_m)dx_1\cdots dx_m. \tag{5.16}$$

Then by estimating $\mu_t(F_m^\theta)$ and then applying Proposition 2.4 we will prove the mentioned lemma.

To simplify notations by Φ_τ^m we denote a particular case of the function defined in (2.36), corresponding to the choice $\theta_1 = \cdots = \theta_m = \theta \in \Theta_\psi^+$ with $\bar{c}_\theta = 1$, see (2.22). Namely, for $\theta \in \Theta_\psi^+$, we set, cf. also (5.15),

$$\Phi_\tau^m(\gamma) = \sum_{x_1\in\gamma}\theta(x_1)\sum_{x_2\in\gamma\setminus x_1}\theta(x_2)\cdots\sum_{x_m\in\gamma\setminus\{x_1,\dots,x_{m-1}\}}\theta(x_m)\widetilde{F}_\tau^0(\gamma\setminus\{x_1,\dots,x_m\}), \tag{5.17}$$

and consider such functions with $\tau \in (0,1]$. Note that the function defined in (3.12) is a particular case of $\Phi_\tau^m(\gamma)$ corresponding to the choice $\theta = \psi$. Then by (3.11) we obtain

$$|L\Phi_\tau^m(\gamma)| \leq m\Phi_\tau^{m,\theta^1}(\gamma) + \tau c_a \widehat{F}_\tau^{m+1}(\gamma) =: \Phi_{\tau,1}^m(\gamma). \tag{5.18}$$

Here $\theta^1 = a * \theta + \theta$, see (3.7), and

$$\Phi_\tau^{m,\theta'} = \widehat{F}_\tau^{\theta',\theta_2,\dots,\theta_m}, \qquad \theta_2 = \cdots = \theta_m = \theta. \tag{5.19}$$

Note that $\Phi_{\tau,1}^m$ is a linear combination of the elements of $\widehat{\mathcal{F}}$, see (3.3). Hence, any solution of (3.16) should satisfy it also with this function. Let us then estimate $L\Phi_{\tau,1}^m$. Proceeding as in (3.4) we obtain

$$\begin{aligned}\nabla^{y,x}\Phi_{\tau,1}^m(\gamma) &= m[\theta^1(y) - \theta^1(x)]\Phi_\tau^{m-1}(\gamma\setminus x) + m(m-1)[\theta(y)-\theta(x)]\Phi_\tau^{m-1,\theta^1}(\gamma\setminus x)\\ &+ m\left[e^{-\tau\psi(y)} - e^{-\tau\psi(x)}\right]\Phi_\tau^{m,\theta^1}(\gamma\setminus x) + (m+1)\tau c_a[\psi(y)-\psi(x)]\widehat{F}_\tau^m(\gamma\setminus x)\\ &+ \tau c_a\left[e^{-\tau\psi(y)} - e^{-\tau\psi(x)}\right]\widehat{F}_\tau^{m+1}(\gamma\setminus x).\end{aligned}$$

Now to estimate $L\Phi_{\tau,1}^m$ we perform the same calculations as in passing to the second line in the right-hand side of (3.11), see (3.9), (3.10). In addition, the third term in the right-hand side of the latter is estimated by employing $\theta(x) \leq \psi(x)$, cf. (2.22), and $\theta^1(x) \leq c_a\psi(x)$, cf. (3.7). This yields

$$m\left|e^{-\tau\psi(y)} - e^{-\tau\psi(x)}\right| \varPhi_\tau^{m,\theta^1}(\gamma \setminus x) \leq m\tau c_a|\psi(y) - \psi(x)|\widehat{F}_\tau^m(\gamma \setminus x),$$

see also (3.12). Thereafter, we obtain

$$\begin{aligned} \left|L\varPhi_{\tau,1}^m(\gamma)\right| &\leq m\varPhi_\tau^{m,\theta^2}(\gamma) + m(m-1)\varPhi_\tau^{m,\theta^1,\theta^1}(\gamma) \\ &+ (2m+1)\tau c_a^2 \widehat{F}_\tau^{m+1}(\gamma) + \tau^2 c_a^2 \widehat{F}_\tau^{m+2}(\gamma) \\ &=: \varPhi_{\tau,2}^m(\gamma). \end{aligned} \tag{5.20}$$

Here and below we denote $\theta^0 = \theta$ and

$$\theta^k = a * \theta^{k-1} + \theta^{k-1}, \qquad k = 2, 3, \dots, \tag{5.21}$$

$\varPhi_\tau^{m,\theta^2}$ is obtained according to (5.19), and

$$\varPhi_\tau^{m,\theta',\theta''} = \widehat{F}_\tau^{\theta',\theta'',\theta_3,\dots,\theta_m}, \qquad \theta_3 = \cdots = \theta_m = \theta.$$

Note that by (3.7) we have $\theta^k(x) \leq c_a^k\psi(x)$ (recall that $\bar{c}_\theta = 1$).

To proceed further we introduce the following notations. For $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$, by $\mathcal{C}_{m,n}$ we denote the set of all sequences $c = \{c_k\}_{k\in\mathbb{N}_0} \subset \mathbb{N}_0$ such that the following holds:

$$c_0 + c_1 + \cdots + c_k + \cdots = m, \qquad c_1 + 2c_2 + \cdots + kc_k + \cdots = n. \tag{5.22}$$

Since all c_k are nonnegative integers, for $c \in \mathcal{C}_{m,n}$ by (5.22) we have that $c_{n+j} = 0$ for all $j \geq 1$, $c_n \leq 1$, and $c_j = 0$ for all $j = 1, 2, \dots, n-1$ whenever $c_n = 1$. For example, $\mathcal{C}_{m,0}$ and $\mathcal{C}_{m,1}$ are singletons, consisting of $c = (m, 0, 0 \dots)$ and $c = (m-1, 1, 0 \dots)$, respectively. $\mathcal{C}_{m,2}$ consists of $c = (m-1, 0, 1, 0, \dots,)$ and $c = (m-2, 2, 0, \dots)$. For $c \in \mathcal{C}_{m,n}$, $\tau \in (0, 1]$ and $\gamma \in \Gamma_*$, we set

$$V_\tau(c;\gamma) = \widehat{F}_\tau^{\theta^{q_1},\dots,\theta^{q_m}}(\gamma), \tag{5.23}$$

where c_0 members of the family $\{\theta^{q_1}, \dots, \theta^{q_m}\}$ are equal to $\theta^0 = \theta$, c_1 of them are equal to θ^1, etc. In particular, $\varPhi_\tau^{m,\theta^2}$ and $\varPhi_\tau^{m,\theta^1,\theta^1}$ can be written as in (5.23) with $c = (m-1, 0, 1, 0, \dots,)$ and $c = (m-2, 2, 0, \dots)$, respectively. In Appendix below, we prove the following estimates

$$\forall \gamma \in \Gamma_* \qquad \left|L\varPhi_{\tau,n-1}^m(\gamma)\right| \leq \varPhi_{\tau,n}^m(\gamma), \quad n \in \mathbb{N}, \tag{5.24}$$

holding with $\varPhi_{\tau,n}^m$ given by the following formula

$$\begin{aligned} \varPhi_{\tau,n}^m(\gamma) &= \sum_{c\in\mathcal{C}_{m,n}} C_{m,n}(c)V_\tau(c;\gamma) + c_a^n \sum_{k=1}^m \tau^k w_k(m,n)\widehat{F}_\tau^{m+k}(\gamma), \\ C_{m,n}(c) &= \frac{m!n!}{c_0!c_1!\cdots c_k!\cdots(0!)^{c_0}(1!)^{c_1}\cdots(k!)^{c_k}\cdots}. \end{aligned} \tag{5.25}$$

We also prove that

$$\sum_{c\in\mathcal{C}_{m,n}} C_{m,n}(c) = m^n. \tag{5.26}$$

The coefficients in the second summand of the first line in (5.25) are subject to the following recurrence relations

$$\begin{aligned} w_1(m,n+1) &= m^n + (m+1)w_1(m,n), \\ w_k(m,n+1) &= w_{k-1}(m,n) + (m+k)w_k(m,n), \qquad k = 2, \dots n, \\ w_{n+1}(m,n+1) &= w_n(m,n) = 1, \end{aligned} \tag{5.27}$$

that can be deduced in the same way as we obtained the estimate in (5.20). In the first line of (5.27) we take into account also (5.26). The initial condition $w_1(m,1) = 1$ can easily be derived from (5.18). Then iterating back to $n = 1$ in the first line of (5.27) yields $w_1(m,n) = (m+1)^n - m^n$. It turns out that the complete solution of (5.27) has the following simple form

$$w_k(m,n) = \varDelta^k m^n = \frac{1}{k!}\sum_{s=0}^{k}\binom{k}{s}(-1)^{k-s}(m+s)^n, \tag{5.28}$$

where $\varDelta$ is the forward difference operator – a standard combinatorial object. Note that the right-hand side of (5.28) makes sense for all $k \in \mathbb{N}_0$: $w_0(m,n) = m^n$, $w_k(m,n) = 0$ for all $k > n$.

In view of (5.23) and Proposition 2.12, all the terms of the linear combination in the first line in (5.25) are continuous bounded functions of γ. Hence, the same is $\varPhi^m_{\tau,n}$. However, its bound may depend on n, and our aim now is to control this dependence. For $\rho > 0$, set

$$\varUpsilon^m_{\tau,\rho}(\gamma) = \sum_{n=0}^{+\infty}\frac{\rho^n}{n!}\varPhi^m_{\tau,n}(\gamma), \qquad \tau \in (0,1]. \tag{5.29}$$

To get an upper bound for $\varUpsilon^m_{\tau,\rho}$ we estimate each θ^q in the first line of (5.25) as $\theta^q \le c_a^q \psi$, $q \ge 0$, see (5.21), which by (5.23) and (5.22) yields

$$V_\tau(c;\gamma) \le c_a^{q_1+\cdots+q_m}\widehat{F}^m_\tau(\gamma) = c_a^n \widehat{F}^m_\tau(\gamma),$$

where we have taken into account that $q_1 + \cdots + q_m = c_1 + 2c_2 + \cdots + kc_k + \cdots = n$. In view of (5.26), this leads to the following

$$\begin{aligned}
\varUpsilon^m_{\tau,\rho}(\gamma) &\le \sum_{n=0}^{+\infty}\frac{(c_a\rho)^n}{n!}\sum_{k=0}^{n}\tau^k w_k(m,n)\widehat{F}^{m+k}_\tau(\gamma) \qquad (5.30)\\
&= \sum_{k=0}^{+\infty}\tau^k\left(\sum_{n=k}^{+\infty}\frac{(c_a\rho)^n}{n!}w_k(m,n)\right)\widehat{F}^{m+k}_\tau(\gamma)\\
&= \sum_{k=0}^{+\infty}\tau^k\left(\sum_{n=0}^{+\infty}\frac{(c_a\rho)^n}{n!}w_k(m,n)\right)\widehat{F}^{m+k}_\tau(\gamma)\\
&= \sum_{k=0}^{+\infty}\frac{\tau^k}{k!}\sum_{s=0}^{k}\binom{k}{s}(-1)^{k-s}\left(\sum_{n=0}^{+\infty}\frac{(c_a\rho(m+s))^n}{n!}\right)\widehat{F}^{m+k}_\tau(\gamma)\\
&= e^{c_a\rho m}\sum_{k=0}^{+\infty}\frac{\tau^k}{k!}\left(e^{c_a\rho}-1\right)^k\widehat{F}^{m+k}_\tau(\gamma).
\end{aligned}$$

Here we used the fact that $\varDelta^k m^n = 0$ for $k > n$, see (5.28). To proceed further we use Proposition 2.12 and (3.12) and then obtain

$$\widehat{F}^{m+k}_\tau(\gamma) \le e^{\tau(m+k)}\Psi_0^{m+k}(\gamma)\exp\left(-\tau\Psi_0(\gamma)\right),$$

which in turn yields in the last line of (5.30) the following estimate

$$\begin{aligned}
\varUpsilon^m_{\tau,\rho}(\gamma) &\le e^{m(c_a\rho+\tau)}\Psi_0^m(\gamma)\exp\Big(-\tau\Psi_0(\gamma)\left[1-e^\tau(e^{c_a\rho}-1)\right]\Big)\\
&\le e^{m(c_a\rho+\tau)}\Psi_0^m(\gamma)\exp\left(-\tau\varepsilon\Psi_0(\gamma)\right)\\
&\le \left(\frac{m}{e\tau\varepsilon}\right)^m(e+1-\varepsilon)^m =: \delta_m(\tau),
\end{aligned}$$

holding for some fixed $\varepsilon \in (0,1)$ and all

$$\rho \le \rho_\varepsilon := \frac{1}{c_a}\left[\log(1+e-\varepsilon)-1\right]. \tag{5.31}$$

By (5.29) this yields the estimate in question in the following form

$$\varPhi^m_{\tau,n}(\gamma) \leq \frac{n!}{\rho^n_\varepsilon}\delta_m(\tau), \qquad \tau \in (0,1]. \tag{5.32}$$

5.3. **Proof of Lemma 5.1.** According to Definition 3.1 and (5.17), we have that $\varPhi^m_\tau$ lies in the linear span of $\widehat{\mathcal{F}}$ for each $\tau > 0$ and $m \in \mathbb{N}$. If $\{\mu_t\}_{t\geq 0} \subset \mathcal{P}(\Gamma_*)$ solves (3.16), then

$$\mu_t(\varPhi^m_\tau) = \mu_0(\varPhi^m_\tau) + \int_0^t \mu_u(L\varPhi^m_\tau)du \leq \mu_0(\varPhi^m_\tau) + \int_0^t \mu_u(\varPhi^m_{\tau,1})du, \tag{5.33}$$

where we have used (5.18). Since $\varPhi^m_{\tau,1}$ is a linear combination of the elements of $\widehat{\mathcal{F}}$, we can repeat (5.33) with this function and obtain

$$\mu_t(\varPhi^m_{\tau,1}) \leq \mu_0(\varPhi^m_{\tau,1}) + \int_0^t \mu_u(\varPhi^m_{\tau,2})du,$$

which then can be used in (5.33). In view of (5.24), we can repeat this procedure due times and thereby get the following estimate

$$\begin{aligned} \mu_t(\varPhi^m_\tau) &\leq \sum_{k=0}^{n-1} \frac{t^k}{k!}\mu_0(\varPhi^m_{\tau,k}) + \int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}} \mu_{t_n}(\varPhi^m_{\tau,n})dt_n dt_{n-1}\cdots dt_1 \qquad (5.34)\\ &\leq \sum_{k=0}^{n-1} \frac{t^k}{k!}\mu_0(\varPhi^m_{\tau,k}) + \left(\frac{t}{\rho_\varepsilon}\right)^n \delta_m(\tau), \end{aligned}$$

where we have used (5.32) and the fact that μ_t is a probability measure. For $t < \rho_\varepsilon$, the last summand in the right-hand side of (5.34) vanishes as $n \to +\infty$. Hence,

$$\mu_t(\varPhi^m_\tau) \leq \sum_{n=0}^{+\infty} \frac{t^n}{n!}\mu_0(\varPhi^m_{\tau,n}), \qquad t<\rho_\varepsilon, \quad \tau\in(0,1]. \tag{5.35}$$

By (2.36) and (5.23) it follows that the element of $\widehat{\mathcal{F}}$ in the first summand in the first line in (5.25) satisfies

$$V_\tau(c;\gamma) \leq V_0(c;\gamma) := \sum_{x_1\in\gamma}\theta^{q_1}(x_1)\sum_{x_2\in\gamma\setminus x_1}\theta^{q_2}(x_2)\cdots\sum_{x_m\in\gamma\setminus\{x_1,\dots,x_{m-1}\}}\theta^{q_m}(x_m), \qquad \theta\in\Theta^+_\psi.$$

$V_0(c;\cdot)$ is an unbounded function, which, however, is μ_0-integrable. Let $\varkappa_0$ be the type of μ_0. As in Remark 2.5, we then have

$$\mu_0(V_\tau(c;\cdot)) \leq \pi_{\varkappa_0}(V_0(c;\cdot)) = \varkappa_0^m\langle\theta^{q_1}\rangle\cdots\langle\theta^{q_m}\rangle = 2^n\left(\varkappa_0\langle\theta\rangle\right)^m, \tag{5.36}$$

where

$$\langle\theta^{q_j}\rangle := \int_{\mathbb{R}^d}\theta^{q_j}(x)dx = 2^{q_j}\int_{\mathbb{R}^d}\theta(x)dx = 2^{q_j}\langle\theta\rangle,$$

see (5.21) and (3.1). Here we have taken into account that $q_1+\cdots+q_m = n$. By (3.12) we have

$$\widehat{F}^{m+k}_\tau(\gamma) \leq \sum_{x_1\in\gamma}\psi(x_1)\sum_{x_2\in\gamma\setminus x_1}\psi(x_2)\cdots\sum_{x_{m+k}\in\gamma\setminus\{x_1,\dots,x_{m+k-1}\}}\psi(x_{m+k}).$$

Then similarly as in (5.36) we obtain

$$\mu_0(\widehat{F}^{m+k}_\tau) \leq \left(\varkappa_0\langle\psi\rangle\right)^{m+k}. \tag{5.37}$$

We use (5.36) and (5.37) in (5.25) and then in (5.35) and arrive at the following estimate

$$\begin{aligned} \mu_t(\varPhi^m_\tau) &\leq \left(e^{2t}\varkappa_0\langle\theta\rangle\right)^m + \left(\varkappa_0\langle\psi\rangle\right)^m e^{c_a tm}\sum_{k=1}^{\infty}\frac{\tau^k}{k!}\left(\varkappa_0\langle\psi\rangle\right)^k\left(e^{c_a t}-1\right)^k \\ &\leq \left(e^{2t}\varkappa_0\langle\theta\rangle\right)^m + \tau\left(\varkappa_0\langle\psi\rangle\right)^{m+1} e^{c_a tm}\left(e^{c_a t}-1\right)\exp\left(\varkappa_0\langle\psi\rangle\left(e^{c_a t}-1\right)\right), \end{aligned}$$

where we have applied the same approach as in obtaining (5.30) and the fact that $\tau \leq 1$. Since, for each $\gamma\in\Gamma_*$ and an arbitrary sequence $\tau_n \to 0$, $\{\widetilde{F}^0_{\tau_n}(\gamma)\}_{n\in\mathbb{N}}$ is a nondegreasing sequence, by (5.17) and Beppo Levi's monotone convergence theorem we then get from the latter that, cf. (5.16),

$$\begin{aligned}\lim_{\tau\to 0}\mu_t(\Phi^m_\tau) &= \mu_t(F^\theta_m) = \langle k^{(m)}_{\mu_t}, \theta^{\otimes m}\rangle \\ &:= \int_{\mathbb{R}^d} k^{(m)}_{\mu_t}(x_1,\dots,x_m)\theta(x_1)\cdots\theta(x_m)dx_1\cdots dx_m \\ &\leq \left(e^{2t}\varkappa_0\langle\theta\rangle\right)^m,\end{aligned}$$

holding for all $m \in \mathbb{N}$ and $t < \rho_\varepsilon$, see (5.31). Since $\theta \in \Theta^+_\psi$, we have $\langle\theta\rangle = \|\theta\|_{L^1(\mathbb{R}^d)}$, and the latter estimate can be rewritten in the form, cf. (2.10),

$$\forall m \in \mathbb{N} \qquad \langle k^{(m)}_{\mu_t}, \theta^{\otimes m}\rangle \leq (2e^t\varkappa_0)^m \|\theta\|^m_{L^1(\mathbb{R}^d)}, \qquad \theta \in \Theta^+_\psi. \tag{5.38}$$

The set of functions Θ^+_ψ defined in (2.20) is closed with respect to the pointwise multiplication and separates points of $\mathbb{R}^d$. Such functions vanish at infinity and are everywhere positive. Then by the aforementioned version of the Stone-Weierstrass theorem [7] the linear span of this set is dense (in the supremum norm) in the algebra $C_0(\mathbb{R}^d)$ of all continuous functions that vanish at infinity. At the same time, $C_0(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)$ is dense in $L^1(\mathbb{R}^d)$. Therefore, by (5.38) the maps $\theta \mapsto \langle k^{(m)}_{\mu_t}, \theta^{\otimes m}\rangle$, $m \in \mathbb{N}$ can be extended to homogeneous continuous monomials on $L^1(\mathbb{R}^d)$. By Proposition 2.4 this yields the proof of the considered statement for $t < \rho_\varepsilon$. Since ρ_ε is independent of $\varkappa_0$, the continuation to all $t > 0$ can be made by the repetition of the same arguments.

5.4. **Proof of the uniqueness.** By employing Lemmas 5.1 and 5.2, see also Remark 3.5, we prove the following statement.

Lemma 5.5. *Assume that two solutions* $\{P^{(i)}_{s,\mu} : s \geq 0,\ \mu \in \mathcal{P}_{\rm exp}\}$, $i = 1, 2$, *see Definition 3.3, satisfy* $P^1_{s,\mu}\circ\varpi_t^{-1} = P^2_{s,\mu}\circ\varpi_t^{-1}$ *for all* $t \geq s$, $s \geq 0$ *and* $\mu \in \mathcal{P}_{\rm exp}$. *Then* $P^1_{s,\mu} = P^2_{s,\mu}$ *for all* s *and* μ.

Proof. By Kolmogorov's extension theorem it is enough to prove that all finite-dimensional marginals of both path measures coincide. In view of claim (i) of Proposition 2.11, to this end we have to show that the following holds

$$P^1_{s,\mu}\left(\mathsf{F}_{t_1}\cdots\mathsf{F}_{t_n}\right) = P^2_{s,\mu}\left(\mathsf{F}_{t_1}\cdots\mathsf{F}_{t_n}\right), \tag{5.39}$$

where $\mathsf{F}_{t_i}(\gamma) = \widetilde{F}^{\theta_i}_{\tau_i}(\varpi_{t_i}(\gamma))$, $i = 1,\dots,n$, see (3.3), ought to be taken with all possible $\theta_i \in \Theta_\psi$, $\tau_i > c_{\theta_i}$ and t_i satisfying $s \leq t_1 \leq \cdots \leq t_n$. Assume that (5.39) holds with a given n and prove its validity for $n+1$. Since $\mathsf{F}_{t_i}(\gamma) > 0$, see (2.30), we may set

$$C^{-1} = P^1_{s,\mu}\left(\mathsf{F}_{t_1}\cdots\mathsf{F}_{t_n}\right),$$

and then define two path measures on $(\mathfrak{D}_{[t_n,+\infty)}, \mathfrak{F}_{t_n,+\infty})$

$$Q^i(\mathbb{B}) = CP^i_{s,\mu}\left(\mathsf{F}_{t_1}\cdots\mathsf{F}_{t_n}\mathbb{1}_{\mathbb{B}}\right), \qquad i = 1, 2.$$

Since both P^i satisfy (3.14), we have also

$$\int_{\mathfrak{D}_{[t_n,+\infty)}} \mathsf{H}(\gamma)Q^i(d\gamma) = 0, \qquad i = 1, 2.$$

Hence, both maps $[t_n, +\infty) \ni t \mapsto Q^i\circ\varpi_t^{-1} =: \mu^i_t \in \mathcal{P}(\Gamma_*)$, $i = 1, 2$ solve

$$\mu^i_{u_2}(F) = \mu^i_{u_1}(F) + \int_{u_1}^{u_2}\mu^i_v(LF)dv, \qquad F \in \mathcal{D}(L),$$

for all $u_2 > u_1 \geq t_n$, see Remark 3.5. By the inductive assumption and claim (iv) of Proposition 3.2 it follows that $\mu^1_{t_n} = \mu^2_{t_n} =: \mu \in \mathcal{P}_{\rm exp}$. By Lemma 5.1 we then conclude that $\mu^i_t \in \mathcal{P}_{\rm exp}$, $i = 1, 2$ for all $t > t_n$. That is, both Q^i satisfy all the three conditions of Definition 3.3 and thus belong to solutions of the restricted initial value martingale problem. Hence, $\mu_t^1 = \mu_t^2$ by the assumption of the lemma. In particular,

$$\mu^1_{t_{n+1}}(\widetilde{F}^{\theta_{n+1}}_{\tau_{n+1}}) = \mu^2_{t_{n+1}}(\widetilde{F}^{\theta_{n+1}}_{\tau_{n+1}}),$$

which completes the proof. □

Theorem 5.6. *Let* $\{P^{(i)}_{s,\mu} : s \geq 0,\ \mu \in \mathcal{P}_{\rm exp}\}$, $i = 1, 2$ *be two solution of the restricted initial value martingale problem in the sense of Definition 3.3. Then* $P^{(1)}_{s,\mu} = P^{(2)}_{s,\mu}$ *for all* $s \geq 0$ *and* $\mu \in \mathcal{P}_{\rm exp}$.

Proof. By Remark 3.5 both $P^{(i)}_{s,\mu} \circ \varpi_t$, $t \geq s$ solve (3.16), which by Theorem 5.3 yields $P^{(1)}_{s,\mu} \circ \varpi_t^{-1} = P^{(2)}_{s,\mu} \circ \varpi_t^{-1}$, holding for all $t \geq s$ and $\mu \in \mathcal{P}_{\rm exp}$. Then the proof follows by Lemma 5.5. □

6. The Existence: Approximating Models

The aim of this and the subsequent sections is to prove the following statement which is the second corner stone in the proof of Theorem 3.6.

Theorem 6.1. *There exists a family of probability measures which solves the restricted initial value martingale problem for our model in the sense of Definition 3.3.*

The basic idea is to approximate the model by auxiliary models described by L^α, $\alpha \in [0,1]$ with L^0 coinciding with L defined in (1.3). For $\alpha \in (0,1]$, the solution $\{P^\alpha_{s,\mu} : s \geq 0, \mu \in \mathcal{P}_{\rm exp}\}$ of the corresponding restricted initial value martingale problem for L^α will be constructed in a direct way. Then the proof of Theorem 6.1 will be done by showing the weak convergence $P^\alpha_{s,\mu} \Rightarrow P_{s,\mu}$ as $\alpha \to 0$, and then by proving that $\{P_{s,\mu} : s \geq 0, \mu \in \mathcal{P}_{\rm exp}\}$ is a solution in question. In the current section, we introduce the auxiliary models and study their relations with the basic model. The construction of the path measures $P^\alpha_{s,\mu}$ will be preformed in the subsequent section.

6.1. The approximating models.

Recall that the functions ψ, Ψ_0 and Ψ_1 were introduced in (2.13). Along with them we will use

$$\psi_\alpha(x) = \frac{1}{1+\alpha|x|^{d+1}}, \qquad \alpha \in [0,1]. \tag{6.1}$$

Set

$$a_\alpha(x,y) = a(x-y)\psi_\alpha(x), \qquad x, y \in \mathbb{R}^d. \tag{6.2}$$

Note that $a_0(x,y) = a(x-y)$ and $a_\alpha(x,y) \neq a_\alpha(y,x)$ for $\alpha \in (0,1]$. Now let L^α be defined as in (1.3) with a replaced by a_α. That is,

$$(L^\alpha F)(\gamma) = \sum_{x\in\gamma} \int_{\mathbb{R}^d} \psi_\alpha(x) a(x-y) \exp\left(- \sum_{z \in \gamma\setminus x} \phi(z-y) \right) [F(\gamma \setminus x \cup y) - F(\gamma)]\, dy. \tag{6.3}$$

Then keeping in mind (2.8), (4.2) and (4.7) we define $L^{\Delta,\alpha}$ by the following expression

$$\mu(L^\alpha F^\theta) = \langle\!\langle L^{\Delta,\alpha} k_\mu, e(\theta;\cdot)\rangle\!\rangle, \qquad \alpha \in [0,1].$$

One observes that $L^{\Delta,0}$ coincides with the operator introduced in (4.8). For $\alpha \in (0,1]$, $L^{\Delta,\alpha}$ is then obtained by replacing in (4.8) $a(x-y)$ by $a_\alpha(x,y) \leq a(x-y)$. Hence, $L^{\Delta,\alpha}$ clearly satisfies (4.13) and similar estimates. Then by repeating the construction realized in subsection 4.2 we obtain the family of bounded operators $\{Q^\alpha_{\vartheta'\vartheta}(t) : t \in [0, T(\vartheta',\vartheta))\}$ (resp. $\{H^\alpha_{\vartheta\vartheta'}(t) : t \in [0, T(\vartheta',\vartheta))\}$), $\vartheta' > \vartheta$ acting from $\mathcal{K}_\vartheta$ to $\mathcal{K}_{\vartheta'}$ (resp. from $\mathcal{G}_{\vartheta'}$ to $\mathcal{G}_\vartheta$). By employing these families we then set

$$k^\alpha_t = Q^\alpha_{\vartheta'\vartheta}(t) k_0, \qquad G^\alpha_t = H^\alpha_{\vartheta\vartheta'}(t) G_0, \tag{6.4}$$

with $k_0 \in \mathcal{K}_\vartheta$ and $G_0 \in \mathcal{G}_{\vartheta'}$. Note that, for $\alpha = 0$, these vectors coincide with those introduced in (4.21) and (4.22), respectively, and thus they satisfy (4.23) for all $\alpha \in [0,1]$. Moreover, as in Proposition 4.2, for each $\vartheta_0 \in \mathbb{R}$ and $\mu \in \mathcal{P}^{\vartheta_0}_{\rm exp}$, by (6.4) with $k_0 = k_\mu$ we obtain a family, $\{\mu^\alpha_t : t \geq 0, \mu_0 = \mu\} \subset \mathcal{P}_{\rm exp}$, $\mu^\alpha_t \in \mathcal{P}^{\vartheta_t}_{\rm exp}$ such that

$$\mu^\alpha_t(F^\theta) = \langle\!\langle k^\alpha_t, e(\theta,\cdot)\rangle\!\rangle, \qquad \theta \in L^1(\mathbb{R}^d). \tag{6.5}$$

Next, by repeating the construction used in the proof of Theorem 5.3 one obtains that the map $t \mapsto \mu_t^\alpha$ is a unique solution of the equation

$$\mu_{t_2}^\alpha(F) = \mu_{t_1}^\alpha(F) + \int_{t_1}^{t_2} \mu_u^\alpha(L^\alpha F)du, \qquad t_2 > t_1 \geq 0,$$

holding for all $F : \Gamma_* \to \mathbb{R}$ which can be written as $F = KG$ with $G \in \cap_{\vartheta \in \mathbb{R}} \mathcal{G}_\vartheta$, see Remark 5.4. Here and below we set

$$\mathcal{D}(L^\alpha) = \mathcal{D}(L), \qquad \alpha \in (0, 1],$$

with $\mathcal{D}(L)$ as in Definition 3.1.

6.2. **The weak convergence.** Our aim now is to prove that the families $\{\mu_t^\alpha : t \geq 0, \mu_0 = \mu\} \subset \mathcal{P}_{\rm exp}$, $\alpha \in [0, 1]$ constructed above have the following property.

Lemma 6.2. *For each $t > 0$, it follows that $\mu_t^\alpha \Rightarrow \mu_t$ as $\alpha \to 0$, where we mean the weak convergence of measures on the Polish space Γ_*.*

We begin by proving the convergence of the corresponding correlation functions.

Lemma 6.3. *For each $t > 0$, one finds $\tilde{\vartheta}_t > \vartheta_t$ such that the following holds*

$$\forall G \in \mathcal{G}_{\tilde{\vartheta}_t} \qquad \langle\!\langle k_t^\alpha, G\rangle\!\rangle \to \langle\!\langle k_t, G\rangle\!\rangle, \qquad \text{as} \quad \alpha \to 0. \tag{6.6}$$

Proof. We recall that k_t satisfies (4.26) with $L^\Delta_{\vartheta_T}$ corresponding to $\alpha = 0$. Note that the domains of $L^{\Delta,\alpha}_\vartheta$ are the same for all $\alpha \in [0, 1]$.

Assume now that the convergence stated in (6.6) holds for a given $t \geq 0$. Note that $k_0 = k_0^\alpha = k_{\mu_0}$; hence, this assumption is valid for at least $t = 0$. Let us prove that there exists $s_0 > 0$ – possibly dependent on t – such that this convergence holds for all $t + s$, $s \leq s_0$. Keeping in mind that Q^α and k_t^α satisfy the corresponding analogs of (4.20) and (4.26), respectively, we write

$$k_{t+s} - k_{t+s}^\alpha = Q_{\bar{\vartheta}_t\vartheta_t}(s)k_t - Q^\alpha_{\bar{\vartheta}_t\vartheta_t}(s)k_t^\alpha, \tag{6.7}$$

where $\bar{\vartheta}_t = \vartheta_t + \delta(\vartheta_t)$ and $\vartheta_t = \vartheta_0 + t$. Note that the left-hand side of (6.7) is considered as a vector in $\mathcal{K}_{\bar{\vartheta}_t}$. Both $Q_{\bar{\vartheta}_t\vartheta_t}(s)$ and $Q^\alpha_{\bar{\vartheta}_t\vartheta_t}(s)$ are defined only for $s < \tau(\vartheta_t)$, see (4.15). At the same time, for each $\vartheta' > \vartheta$, $Q_{\vartheta'\vartheta}(0) = Q^\alpha_{\vartheta'\vartheta}(0) = I_{\vartheta'\vartheta}$, where the latter is the embedding operator, see (4.6). Keeping this and (4.20) in mind we rewrite (6.7) as follows

$$\begin{aligned} k_{t+s} - k_{t+s}^\alpha &= Q_{\bar{\vartheta}_t\vartheta_t}(s)(k_t - k_t^\alpha) - \left(\int_0^s \frac{d}{du}[Q_{\bar{\vartheta}_t\vartheta_1}(s-u)Q^\alpha_{\vartheta_1\vartheta_t}(u)]du\right)k_t^\alpha \\ &= Q_{\bar{\vartheta}_t\vartheta_t}(s)(k_t - k_t^\alpha) + \int_0^s Q_{\bar{\vartheta}_t\vartheta_2}(s-u)L^\Delta_{\vartheta_2\vartheta_1}Q^\alpha_{\vartheta_1\vartheta_t}(u)k_t^\alpha du \\ &- \int_0^s Q_{\bar{\vartheta}_t\vartheta_2}(s-u)L^{\Delta,\alpha}_{\vartheta_2\vartheta_1}Q^\alpha_{\vartheta_1\vartheta_t}(u)k_t^\alpha du \\ &= Q_{\bar{\vartheta}_t\vartheta_t}(s)(k_t - k_t^\alpha) + \int_0^s Q_{\bar{\vartheta}_t\vartheta_2}(s-u)\widetilde{L}^{\Delta,\alpha}_{\vartheta_2\vartheta_1}k_{t+u}^\alpha du, \end{aligned} \tag{6.8}$$

where $\widetilde{L}^{\Delta,\alpha}$ is given in (4.8) with $a(x-y)$ replaced by $\tilde{a}_\alpha(x,y) = a(x-y)(1-\psi_\alpha(x))$, see (2.24). The choice of s and ϑ_1, ϑ_2 should be made in such a way that the series as in (4.18) converge for the corresponding operators. Set $\vartheta_1 = \vartheta_t + \delta(\vartheta_t)/2$. We use this in (4.14) and obtain that

$$T(\bar{\vartheta}_t, \vartheta_1) = \frac{\tau(\vartheta_t)}{2} < T(\vartheta_1, \vartheta_t). \tag{6.9}$$

Then for some $\epsilon \in (0, 1)$, we set

$$s_0 = \epsilon\tau(\vartheta_t)/2 = \epsilon T(\bar{\vartheta}_t, \vartheta_1). \tag{6.10}$$

Since the map $\vartheta \mapsto T(\bar{\vartheta}_t, \vartheta)$ is continuous, one can find $\vartheta_2 \in (\vartheta_1, \vartheta_t)$ such that $s_0 < T(\bar{\vartheta}_t, \vartheta_2)$, cf. (6.10), which together with (6.9) yields that all the three $Q_{\bar{\vartheta}_t\vartheta_1}(s-u)$, $Q_{\bar{\vartheta}_t\vartheta_2}(s-u)$ and $Q^\alpha_{\vartheta_1\vartheta_t}(u)$

in (6.8) are defined for all $s \le s_0$ and $u \in [0, s]$. Now we take $G \in \mathcal{G}_{\bar{\vartheta}_t}$ and set $G_s = H_{\vartheta_2\bar{\vartheta}_t}(s)G$, $s \le s_0$. Then $G_s \in \mathcal{G}_{\vartheta_2} \subset \mathcal{G}_{\vartheta_t}$, which yields by (6.8) the following

$$\langle\!\langle k_{t+s} - k^\alpha_{t+s}, G\rangle\!\rangle = \langle\!\langle k_t - k^\alpha_t, G_s\rangle\!\rangle + Y_\alpha(s), \tag{6.11}$$

$$Y_\alpha(s) := \int_0^s \langle\!\langle \widetilde{L}^{\Delta,\alpha}_{\vartheta_2\vartheta_1} k^\alpha_{t+u}, G_{s-u}\rangle\!\rangle du.$$

Thus, we have to prove that $Y_\alpha(s) \to 0$ as $\alpha \to 0$. Since $L^{\Delta,\alpha}$ consists of two terms, see (4.8), it is convenient for us to write $Y_\alpha(s) = Y^{(1)}_\alpha(s) + Y^{(2)}_\alpha(s)$, where

$$\begin{aligned} Y^{(1)}_\alpha(s) = & \int_0^s \int_{\Gamma_0} \left(\sum_{y\in\eta} \int_{\mathbb{R}^d} \tilde{a}_\alpha(x,y) e(\tau_y; \eta\setminus y)(W_y k^\alpha_{t+u})(\eta\setminus y \cup x) dx \right) \\ & \times G_{s-u}(\eta)\lambda(d\eta)du \\ = & \int_0^s \int_{\Gamma_0} \left(\int_{(\mathbb{R}^d)^2} \tilde{a}_\alpha(x,y) e(\tau_y;\eta)(W_y k^\alpha_{t+u})(\eta\cup x) G_{s-u}(\eta\cup y) dx dy \right) \lambda(d\eta) du, \end{aligned} \tag{6.12}$$

and

$$\begin{aligned} Y^{(2)}_\alpha(s) = & -\int_0^s \int_{\Gamma_0} \left(\sum_{x\in\eta} \int_{\mathbb{R}^d} \tilde{a}_\alpha(x,y) e(\tau_y; \eta\setminus x)(W_y k^\alpha_{t+u})(\eta) dy \right) \\ & \times G_{s-u}(\eta)\lambda(d\eta)du \\ = & -\int_0^s \int_{\Gamma_0} \left(\int_{(\mathbb{R}^d)^2} \tilde{a}_\alpha(x,y) e(\tau_y;\eta)(W_y k^\alpha_{t+u})(\eta\cup x) G_{s-u}(\eta\cup x) dx dy \right) \lambda(d\eta) du. \end{aligned} \tag{6.13}$$

To estimate both terms we take into account that $e(\tau_y;\eta) \le 1$ and

$$|(W_y k^\alpha_{t+u})(\eta\cup x)| \le \exp\left(\vartheta_1 + \vartheta_1|\eta| + \langle\phi\rangle e^{\vartheta_1}\right),$$

where the latter estimate follows by the fact that $k^\alpha_{t+u}(\eta) \le \exp(\vartheta_{t+u}|\eta|) \le \exp(\vartheta_1|\eta|)$, see claim (a) of Proposition 4.2. By these estimates we obtain from (6.12) and (6.13) the following

$$\left|Y^{(i)}_\alpha(s)\right| \le \int_{\mathbb{R}^d} h^{(i)}_\alpha(y) g^{(i)}_s(y) dy, \qquad i = 1, 2, \tag{6.14}$$

where

$$h^{(1)}_\alpha(y) = \int_{\mathbb{R}^d} \tilde{a}_\alpha(x,y)dx = \int_{\mathbb{R}^d} (1 - \bar{\psi}_\alpha(|x|)) a(x-y) dx, \tag{6.15}$$

$\bar{\psi}_\alpha(r) := (1 + \alpha r^{d+1})^{-1}$, cf. (6.1), and

$$g^{(1)}_s(y) = c(\vartheta_1) \int_0^s \int_{\Gamma_0} |G_{s-u}(\eta\cup y)|\, e^{\vartheta_1|\eta|}\lambda(d\eta)du,$$

$$c(\vartheta_1) := \exp\left(\vartheta_1 + \langle\phi\rangle e^{\vartheta_1}\right).$$

Let us show that $g^{(1)}_s$ is integrable for all $s \le s_0$. To this end we use the fact that $G_{s-u} \in \mathcal{G}_{\vartheta_2}$ for all $s \le s_0$ and $u \le s$. Then its norm can be estimated

$$|G_{s-u}|_{\vartheta_2} \le \frac{T(\bar{\vartheta}_t, \vartheta_2)}{T(\bar{\vartheta}_t,\vartheta_2) - s_0} |G|_{\bar{\vartheta}_t} =: C_G$$

which is finite by our choice of ϑ_2 and s_0. Then

$$\begin{aligned}\int_{\mathbb{R}^d} g_s^{(1)}(y)dy &= c(\vartheta_1)\int_0^s \int_{\Gamma_0}\int_{\mathbb{R}^d} |G_{s-u}(\eta\cup y)|e^{\vartheta_1|\eta|}dy\lambda(d\eta)du \qquad (6.16)\\ &= c(\vartheta_1)e^{-\vartheta_1}\int_0^s\int_{\Gamma_0}|G_{s-u}(\eta)||\eta|e^{\vartheta_1|\eta|}dy\lambda(d\eta)du\\ &= c(\vartheta_1)e^{-\vartheta_1}\int_0^s\int_{\Gamma_0}\left(|\eta|e^{-|\eta|(\vartheta_2-\vartheta_1)}\right)|G_{s-u}(\eta)|e^{\vartheta_2|\eta|}\lambda(d\eta)du\\ &\le \frac{c(\vartheta_1)s}{e^{1+\vartheta_1}(\vartheta_2-\vartheta_1)}C_G.\end{aligned}$$

Now let us turn to (6.15). First of all, we note that $h_\alpha^{(1)}(y)\le 1$, see (3.1). The function $r\mapsto 1-\bar{\psi}_\alpha(r)$ is increasing. Then, for a certain $r>0$, we have

$$\begin{aligned}h_\alpha^{(1)}(y) &= \int_{\mathbb{R}^d}(1-\bar{\psi}_\alpha(|x+y|))a(x)dx \qquad (6.17)\\ &\le \int_{B_r}(1-\bar{\psi}_\alpha(r+|y|))a(x)dx + \int_{B_r^c}a(x)dx\\ &\le (1-\bar{\psi}_\alpha(r+|y|)) + \frac{m_{d+1}^a}{r^{d+1}},\end{aligned}$$

where the second term of the last line was obtained by Markov's inequality and (3.2) together with the estimate $1-\bar{\psi}_\alpha(r)\le 1$. Now we set in (6.17) $r=\alpha^{-1/(d+2)}$ and obtain

$$h_\alpha^{(1)}(y)\le \frac{\alpha^{1/(d+2)}(1+|y|)^{d+1}}{1+\alpha^{1/(d+2)}(1+|y|)^{d+1}} + m_{d+1}^a\alpha^{\frac{d+1}{d+2}}.$$

Hence, for each y, $h_\alpha^{(1)}(y)\to 0$ as $\alpha\to 0$. Then by Lebesgue's dominated convergence theorem and (6.16) and (6.14) we conclude that $Y_\alpha^{(1)}(s)\to 0$ as $\alpha\to 0$, holding for all $s\le s_0$.

Now we turn to (6.13) by which we get

$$h^{(2)}(y) = \int_{\mathbb{R}^d}\tilde{\psi}_\alpha(y)a(x-y)dx = \tilde{\psi}_\alpha(y) =: 1-\psi_\alpha(y),$$

and $g_s^{(2)}(y)=g_s^{(1)}(y)$. Hence, also $Y_\alpha^{(2)}(s)\to 0$ as $\alpha\to 0$, holding for all $s\le s_0$, which by (6.11) yields the proof of (6.6) for $t+s$ with $s\le s_0$ whenever it holds for t. To complete the proof let us consider the following sequences, cf. (6.10),

$$t_l = t_{l-1}+s_{0l}, \qquad t_0=0, \quad l\in\mathbb{N}, \qquad (6.18)$$
$$s_{0l} = \epsilon\tau(\vartheta_{t_{l-1}})/2.$$

Since $k_0^\alpha = k_0 = k_\mu$, the proof made above yields the stated convergence for $t\le\sup_l t_l = \lim_l t_l$. Thus, our aim is to show that $t_l\to+\infty$ as $l\to+\infty$. Assume that $\sup_l t_l = t_* < \infty$. By the first line in (6.18) we have that $t_l = s_{01}+\cdots s_{0l}$ and hence $s_{0l}\to 0$ in this case. Now we pass in the second line of (6.18) to the limit $l\to+\infty$ (τ is continuous) and get that t_* should satisfy $\tau(\vartheta_{t_*})=\tau(\vartheta_0+t_*)=0$, which is impossible as $\tau(\vartheta)>0$ for all $\vartheta\in\mathbb{R}$. This completes the proof with $\tilde{\vartheta}_t=\bar{\vartheta}_t$. □

Proof of Lemma 6.2. By Lemma 6.3 and (2.9) it follows that $\mu_t^\alpha(F)\to\mu_t(F)$ as $\alpha\to 0$, holding for all $F\in\mathcal{F}$, see (5.1). Then the proof follows by the fact that $\widetilde{\mathcal{F}}\subset\mathcal{F}$, see (5.2), and claim (ii) of Proposition 2.11. □

Below we us the following fact, that can be considered as a complement to Lemma 6.2.

Lemma 6.4. *Assume that a sequence $\{\nu_n\}_{n\in\mathbb{N}} \subset \mathcal{P}^{\vartheta}_{\rm exp}$, $\vartheta \in \mathbb{R}$, cf. (4.25), satisfy $\nu_n \Rightarrow \nu$ as $n \to +\infty$ for some $\nu \in \mathcal{P}(\Gamma_*)$. Then $\nu \in \mathcal{P}^{\vartheta}_{\rm exp}$. Furthermore, for each $G \in \cap_\vartheta \mathcal{G}_\vartheta$, it follows that*

$$\langle\!\langle k_{\nu_n}, G\rangle\!\rangle \to \langle\!\langle k_\nu, G\rangle\!\rangle, \qquad n \to +\infty. \tag{6.19}$$

Proof. By assumption $\nu_n(F) \to \nu(F)$ for each $F \in \widehat{\mathcal{F}}$, see (3.3) and Proposition 2.12. By (2.36), (5.17) and (5.16), for given $m \in \mathbb{N}$, $\theta \in \Theta^+_\psi$ and $\tau \in (0,1]$, we then get

$$\nu(\Phi^m_\tau) \le \sup_{n\in\mathbb{N}} \nu_n \le (\Phi^m_\tau) e^{m\vartheta}\|\theta\|^m.$$

Then the proof of $\nu \in \mathcal{P}^{\vartheta}_{\rm exp}$ follows by the monotone convergence theorem and Proposition 2.4. The validity of (6.19) for G such that $KG \in \widehat{\mathcal{F}}$ follows by the fact just mentioned, i.e., just because ν has a correlation function. The extension of (6.19) to all $G \in \cap_\vartheta \mathcal{G}_\vartheta$ is then made by the same arguments as the proof of (5.11). □

7. The Existence: Approximating Processes

In this section, we prove Theorem 6.1 by constructing path measures for the models described by L^α, $\alpha \in (0.,1]$ introduced in the preceding section. This will be done in a direct way by means of the corresponding Markov transition functions.

7.1. **The Markov transition functions.**

The transition functions in question will be obtained in the form

$$p^\alpha_t(\gamma, \cdot) = S^\alpha(t)\delta_\gamma, \quad t \ge 0, \quad \alpha \in (0,1], \tag{7.1}$$

where δ_γ is the Dirac measure with atom at $\gamma \in \Gamma_*$ and $S^\alpha = \{S^\alpha(t)\}_{t\ge 0}$ is a stochastic semigroup of linear operators, related to the Kolmogorov operator L^α. Hence, we begin by constructing S^α.

7.1.1. *Stochastic semigroups.* A more detailed presentation of the notions and facts which we introduce here can be found in [1, 2, 32].

Let $\mathcal{E}$ be an ordered real Banach space, and $\mathcal{E}^+$ be a generating cone of its positive elements. Set $\mathcal{E}^{+,1} = \{x \in \mathcal{E}^+ : \|x\|_{\mathcal{E}} = 1\}$ and assume that the norm is additive on $\mathcal{E}^+$, i.e., $\|x+y\|_{\mathcal{E}} = \|x\|_{\mathcal{E}} + \|y\|_{\mathcal{E}}$ whenever $x, y \in \mathcal{E}^+$. In such spaces, there exists a positive linear functional, $\varphi_{\mathcal{E}}$, such that

$$\varphi_{\mathcal{E}}(x) = \|x\|_{\mathcal{E}}, \quad x \in \mathcal{E}^+. \tag{7.2}$$

A C_0-semigroup, $S = \{S(t)\}_{t\ge 0}$, of bounded linear operators on $\mathcal{E}$ is said to be stochastic (resp. substochastic) if the following holds $\|S(t)x\|_{\mathcal{E}} = 1$ (resp. $\|S(t)x\|_{\mathcal{E}} \le 1$) for all $t > 0$ and $x \in \mathcal{E}^{+,1}$. Let $\mathcal{D} \subset \mathcal{E}$ be a dense linear subspace, $\mathcal{D}^+ = \mathcal{D} \cap \mathcal{E}^+$ and $(A, \mathcal{D})$, $(B, \mathcal{D})$ be linear operators in $\mathcal{E}$. A paramount question of the theory of stochastic semigroups is under which conditions the closure (resp. an extension) of $(A+B, \mathcal{D})$ is the generator of a stochastic semigroup. Classical works on this subject trace back to Feller, Kato, Miyadera, etc, see [1, 32]. In the present work, we will use a result of [32], which we present now in the form adapted to the context.

To proceed we need to further specify the properties of the space $\mathcal{E}$.

Assumption 7.1. *There exists a linear subspace, $\widetilde{\mathcal{E}} \subset \mathcal{E}$, which has the following properties:*

(i) *$\widetilde{\mathcal{E}}$ is dense in $\mathcal{E}$ in the norm $\|\cdot\|_{\mathcal{E}}$.*
(ii) *There exists a norm, $\|\cdot\|_{\widetilde{\mathcal{E}}}$, on $\widetilde{\mathcal{E}}$ that makes it a Banach space.*
(iii) *$\widetilde{\mathcal{E}}^+ := \widetilde{\mathcal{E}} \cap \mathcal{E}^+$ is a generating cone in $\widetilde{\mathcal{E}}$; $\|\cdot\|_{\widetilde{\mathcal{E}}}$ is additive on $\widetilde{\mathcal{E}}^+$ and hence there exists a linear functional, $\varphi_{\widetilde{\mathcal{E}}}$, on $\widetilde{\mathcal{E}}$, such that $\|x\|_{\widetilde{\mathcal{E}}} = \varphi_{\widetilde{\mathcal{E}}}(x)$ whenever $x \in \widetilde{\mathcal{E}}^+$, cf. (7.2).*
(iv) *The cone $\widetilde{\mathcal{E}}^+$ is dense in $\mathcal{E}^+$.*

For $\mathcal{D}$ as above, set $\widetilde{\mathcal{D}} = \{x \in \mathcal{D} \cap \widetilde{\mathcal{E}} : Ax \in \widetilde{\mathcal{E}}\}$. Then $(A, \widetilde{\mathcal{D}})$ is the *trace* of A in $\widetilde{\mathcal{E}}$. The next statement is an adaptation of [32, Theorem 2.7].

Proposition 7.2 (Thieme-Voigt)**.** *Assume that:*

(i) *$-A : \mathcal{D}^+ \to \mathcal{E}^+$ and $B : \mathcal{D}^+ \to \mathcal{E}^+$;*

(ii) $(A, \mathcal{D})$ *is the generator of a substochastic semigroup,* $S = \{S(t)\}_{t\geq 0}$, *on* $\mathcal{E}$ *such that* $S(t) : \widetilde{\mathcal{E}} \to \widetilde{\mathcal{E}}$ *for all* $t \geq 0$ *and the restrictions* $S(t)|_{\widetilde{\mathcal{E}}}$ *constitute a* C_0*-semigroup on* $\widetilde{\mathcal{E}}$ *generated by* $(A, \widetilde{\mathcal{D}})$*;*

(iii) $B : \widetilde{\mathcal{D}} \to \widetilde{\mathcal{E}}$ *and* $\varphi_{\mathcal{E}}\left((A+B)x\right) = 0$, *for* $x \in \mathcal{D}^+$*;*

(iv) *there exist* $c > 0$ *and* $\varepsilon > 0$ *such that*

$$\varphi_{\widetilde{\mathcal{E}}}\left((A+B)x\right) \leq c\varphi_{\widetilde{\mathcal{E}}}(x) - \varepsilon\|Ax\|_{\mathcal{E}}, \qquad \text{for } \ x \in \widetilde{\mathcal{D}} \cap \mathcal{E}^+.$$

Then the closure of $(A + B, \mathcal{D})$ *in* $\mathcal{E}$ *is the generator of a stochastic semigroup,* $S_{\mathcal{E}} = \{S_{\mathcal{E}}(t)\}_{t\geq 0}$, *on* $\mathcal{E}$ *which leaves* $\widetilde{\mathcal{E}}$ *invariant. The restrictions* $S_{\widetilde{\mathcal{E}}}(t) := S_{\mathcal{E}}(t)|_{\widetilde{\mathcal{E}}}$, $t \geq 0$ *constitute a* C_0*-semigroup,* $S_{\widetilde{\mathcal{E}}}$, *on* $\widetilde{\mathcal{E}}$ *generated by the trace of the generator of* $S_{\mathcal{E}}$ *in* $\widetilde{\mathcal{E}}$.

Remark 7.3. *Without assuming item (iv) above one can only guarantee that an extension of* $(A+B, \mathcal{D})$ *is the generator of a substochastic semigroup on* $\mathcal{E}$*, which corresponds to a dishonesty of the evolution described by this semigroup. More on this item can be found in* [1].

Now we turn to constructing the semigroups S^{α}.

7.1.2. *The Banach spaces of measures.* Let $\mathcal{M}$ be the linear space of finite signed measures on $(\Gamma, \mathcal{B}(\Gamma))$, see [9, Chapter 4]. That is, $\mu \in \mathcal{M}$ is a σ-additive map $\mu : \mathcal{B}(\Gamma) \to \mathbb{R}$ which takes only finite values. By $\mathcal{M}^+$ we denote the set of all such μ that take only nonnegative values. Then the Jordan decomposition of μ is the unique representation $\mu = \mu^+ - \mu^-$ with $\mu^{\pm} \in \mathcal{M}^+$. Thus, $\mathcal{M}^+$ is a generating cone. Set $|\mu| = \mu^+ + \mu^-$. Then

$$\|\mu\| := |\mu|(\Gamma) \tag{7.3}$$

is a norm, that is clearly additive on $\mathcal{M}^+$. By [9, Proposition 4.1.8, page 119] with this norm $\mathcal{M}$ is a Banach space. Let Ψ_1 be the function defined in (2.13). For $n \in \mathbb{N}$, let $\mathcal{M}_n$ be the subset of $\mathcal{M}$ consisting of all those μ for which $\Psi_1^n \mu$ are finite signed measures. We equip $\mathcal{M}_n$ with the norm

$$\|\mu\|_n = \int_{\Gamma} \Psi_1^n(\gamma)|\mu|(d\gamma) =: \varphi_n(|\mu|). \tag{7.4}$$

By the same [9, Proposition 4.1.8, page 119] with this norm $\mathcal{M}_n$ is a Banach space. Now for $\beta > 0$, let $\mathcal{M}_\beta$ be the subset of $\mathcal{M}$ the elements of which remain finite measures being multiplied by $\exp(\beta\Psi_0(\gamma))$. We equip it with the norm

$$\|\mu\|_\beta = \int_{\Gamma} \exp(\beta\Psi_0(\gamma))|\mu|(d\gamma) =: \varphi_\beta(|\mu|).$$

Then also $(\mathcal{M}_\beta, \|\cdot\|_\beta)$ is a Banach space. By (2.17) and (2.18) it follows that

$$\forall \mu \in \mathcal{M}_1 \qquad |\mu|(\Gamma_*) = |\mu|(\Gamma).$$

That is, for each $\mu \in \mathcal{M}_1$, it follows that $|\mu|(\Gamma_*^c) = 0$. Define

$$\mathcal{M}_* = \{\mu \in \mathcal{M} : |\mu|(\Gamma_*^c) = 0\}.$$

Thus, $\mathcal{M}_1 \subset \mathcal{M}_*$. Obviously, also all $\mathcal{M}_n$ and $\mathcal{M}_\beta$ have the same property. For a subset, $\mathcal{M}' \subset \mathcal{M}$, let $\overline{\mathcal{M}'}$ denote its closure in $\|\cdot\|$ defined in (7.3).

Lemma 7.4. *For each* $n \in \mathbb{N}$ *and* $\beta > 0$*, it follows that*

$$\overline{\mathcal{M}_n} = \overline{\mathcal{M}_\beta} = \mathcal{M}_*. \tag{7.5}$$

Proof. Obviously, for each $n \in \mathbb{N}$ and $\beta > 0$, the following holds $\mathcal{M}_\beta \subset \mathcal{M}_n$. Then it is enough to prove the validity of (7.5) for $\mathcal{M}_\beta$. Let us prove the inclusion $\overline{\mathcal{M}_\beta} \subset \mathcal{M}_*$. For a given $\mu \in \overline{\mathcal{M}_\beta}$, let $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{M}_\beta$ be a sequence such that $\|\mu - \mu_n\| \to 0$. Fix n and let then $\Gamma = \mathbb{P} \cup \mathbb{N}$ be the Hahn decomposition for $\mu - \mu_n$, i.e., $\mu(\mathbb{A}) \geq \mu_n(\mathbb{A})$ for each $\mathbb{A} \subset \mathbb{P}$, and $\mu(\mathbb{A}) \leq \mu_n(\mathbb{A})$ for each $\mathbb{A} \subset \mathbb{N}$. Then

$$\begin{aligned}\|\mu - \mu_n\| &= (\mu - \mu_n)(\mathbb{P}) + (\mu_n - \mu)(\mathbb{N}) \geq (\mu - \mu_n)(\mathbb{P} \cap \Gamma_*^c) + (\mu_n - \mu)(\mathbb{N} \cap \Gamma_*^c) \\ &= \mu(\mathbb{P} \cap \Gamma_*^c) - \mu(\mathbb{N} \cap \Gamma_*^c) = \mu^+(\Gamma_*^c) + \mu^-(\Gamma_*^c) = |\mu|(\Gamma_*^c),\end{aligned}$$

where we have taken into account that $|\mu_n|(\Gamma_*^c) = 0$. Then the assumed convergence $\mu_n \to \mu$ yields that $\mu \in \mathcal{M}_*$. To prove the opposite inclusion we take an arbitrary $\mu \in \mathcal{M}_*$ and write its Jordan decomposition $\mu = \mu^+ - \mu^-$. For a given $n \in \mathbb{N}$, let I_n be the indicator of the set $\Gamma_{*,n}$ defined in (2.17). Then both $\mu_n^\pm := I_n \mu^\pm$ are in $\mathcal{M}_\beta$. At the same time, by (2.17) the sequence of function $J_n(\gamma) := 1 - I_n(\gamma)$ converges to zero pointwise on Γ_*. Since $\mu \in \mathcal{M}_*$, we have

$$\|\mu^\pm - \mu_n^\pm\| = \int_\Gamma J_n(\gamma)\mu^\pm(d\gamma) = \int_{\Gamma_*} J_n(\gamma)\mu^\pm(d\gamma) \to 0, \quad \text{as } \ n \to +\infty. \tag{7.6}$$

By the triangle inequality we then obtain that $\|\mu - \mu_n\| \to 0$, where $\mu_n := \mu_n^+ - \mu_n^- \in \mathcal{M}_\beta$. □

By the very definition of the spaces $\mathcal{M}_n$, $\mathcal{M}_\beta$ and $\mathcal{M}_*$, we conclude that they have generating cones of positive elements consisting of those μ that take nonnegative values only.

Corollary 7.5. *The set $\mathcal{M}_*$ equipped with the norm $\|\cdot\|$ defined in (7.3) is a Banach space. Let $\mathcal{M}_*^+$ be its cone of positive elements. Then for each $n \in \mathbb{N}$ and $\beta > 0$, it follows that*

$$\overline{\mathcal{M}_n^+} = \overline{\mathcal{M}_\beta^+} = \mathcal{M}_*^+,$$

where we mean the closure in the norm of $\mathcal{M}_*$.

Proof. The first part of the statement follows directly by (7.5). The second part is obtained by the construction used in (7.6). □

Let $\mathcal{M}_*^{+,1}$ be the subset of $\mathcal{M}_*$ consisting of probability measures, i.e., for which $\|\mu\| = \mu(\Gamma) = \mu(\Gamma_*) = 1$. Then by (2.18) it follows that

$$\mathcal{P}_{\rm exp} \subset \mathcal{M}_1^{+,1} \subset \mathcal{M}_*^{+,1}.$$

By (2.16), for each $\beta > 0$, we also have

$$\mathcal{P}_{\rm exp} \subset \mathcal{M}_\beta^{+,1} := \mathcal{M}_\beta \cap \mathcal{M}_*^{+,1} \subset \mathcal{M}_n \cap \mathcal{M}_*^{+,1}, \quad \text{for all } n \in \mathbb{N}.$$

7.1.3. *The stochastic semigroup.* For a given $\alpha \in (0,1]$, set

$$\Phi_\alpha(\gamma) = \sum_{x\in\gamma} \int_{\mathbb{R}^d} a_\alpha(x,y) \exp\left(- \sum_{z\in\gamma\setminus x} \phi(y-z) \right) dy, \qquad \gamma \in \Gamma_*. \tag{7.7}$$

Since $\psi(x) < \psi_\alpha(x) \le \psi(x)/\alpha$, see (2.13), for all $\alpha \in (0,1]$ we have

$$\Psi_0(\gamma) < \Phi_\alpha(\gamma) \le \Psi_0(\gamma)/\alpha, \tag{7.8}$$

and hence $\Phi_\alpha(\gamma) < \infty$ for $\gamma \in \Gamma_*$. Now let L^α be the corresponding Kolmogorov operator (6.3). Our aim is to define its 'predual', $L^{\dagger,\alpha}$, acting according to the rule

$$\mu(L^\alpha F) = (L^{\dagger,\alpha}\mu)(F), \tag{7.9}$$

for appropriate $\mu \in \mathcal{P}(\Gamma_*)$ and $F : \Gamma_* \to \mathbb{R}$, and then to use it to define the corresponding operators acting in the spaces of measures just introduced. Obviously, we can restrict ourselves to the elements of $\mathcal{M}_*$. By (6.3) and (6.2) we thus obtain it in the form

$$L^{\dagger,\alpha} = A + B \tag{7.10}$$

where A is the multiplication operator by the function $-\Phi_\alpha$ defined in (7.7). In view of (7.8) the domain of A is to be

$$\mathcal{D} = \{\mu \in \mathcal{M}_* : \Phi_\alpha \mu \in \mathcal{M}_*\} = \mathcal{M}_1. \tag{7.11}$$

To define B we introduce the following measure kernel

$$\Omega_\alpha^\gamma(\mathbb{A}) = \sum_{x\in\gamma} \int_{\mathbb{R}^d} a_\alpha(x,y) \exp\left(- \sum_{z\in\gamma\setminus x} \phi(y-z) \right) \mathbb{1}_{\mathbb{A}}(\gamma \setminus x \cup y) dy, \tag{7.12}$$

with $\gamma \in \Gamma_*$ and $\mathbb{A} \in \mathcal{B}(\Gamma_*)$. By (7.7) we then have

$$\Omega_\alpha^\gamma(\Gamma_*) = \Phi_\alpha(\gamma). \tag{7.13}$$

Next, define

$$(B\mu)(\mathbb{A}) = \int_{\Gamma_*} \Omega^\gamma_\alpha(\mathbb{A})\mu(d\gamma). \tag{7.14}$$

Note that

$$B : \mathcal{M}^+_1 \to \mathcal{M}^+_*. \tag{7.15}$$

Moreover, for $\mu \in \mathcal{M}^+_1$, by (7.13) and (7.14) we have

$$\|B\mu\| = (B\mu)(\Gamma_*) = \int_{\Gamma_*} \Phi_\alpha(\gamma)\mu(d\gamma) = -(A\mu)(\Gamma_*). \tag{7.16}$$

Hence, we can take $\mathcal{M}_1$ as the domain of B and then define $L^{\dagger,\alpha}$ by (7.10) with domain $\mathcal{D} = \mathcal{M}_1$, see (7.11).

In the sequel, we will use one more property of B. By (7.12), (7.14) and (7.4) we get

$$\begin{aligned} \varphi_n(B\mu) &= \int_{\Gamma_*} \Psi^n_1(\gamma)(B\mu)(d\gamma) \\ &= \int_{\Gamma_*} \left(\sum_{x\in\gamma} \int_{\mathbb{R}^d} a_\alpha(x,y) \exp\left(- \sum_{z\in\gamma\setminus x} \phi(y-z)\right) \Psi^n_1(\gamma \setminus x \cup y) dy \right) \mu(d\gamma). \end{aligned} \tag{7.17}$$

By (2.13) it follows that

$$\Psi^n_1(\gamma \setminus x \cup y) = (\Psi_1(\gamma) + \psi(y) - \psi(x))^n \leq 2^n \Psi^n_1(\gamma). \tag{7.18}$$

We apply this and (7.8) in (7.17) and obtain

$$\forall \mu \in \mathcal{M}^+_{n+1} \qquad \|B\mu\|_n = \varphi_n(B\mu) \leq (2^n/\alpha)\|\mu\|_{n+1}.$$

This yields the following extension of (7.15)

$$B : \mathcal{M}^+_{n+1} \to \mathcal{M}^+_n, \tag{7.19}$$

holding for all $n \in \mathbb{N}$. Since $\|A\mu\|_n \leq \alpha^{-1}\|\mu\|_{n+1}$, by (7.19) we also get

$$\forall n \in \mathbb{N}_0 \qquad L^{\dagger,\alpha} : \mathcal{M}_{n+1} \to \mathcal{M}_n,$$

that can be used to define the powers of $L^{\dagger,\alpha}$

$$(L^{\dagger,\alpha})^m : \mathcal{M}_{n+m} \to \mathcal{M}_n, \qquad n \in \mathbb{N}_0,\ m \in \mathbb{N}. \tag{7.20}$$

Here – and in the sequel in similar expressions – $\mathcal{M}_0$ (corresponding to $\mathcal{M}_n$ with $n = 0$) is understood as $\mathcal{M}_*$ Let us now define a bounded linear operator $L^{\dagger,\alpha}_{\beta'\beta} : \mathcal{M}_\beta \to \mathcal{M}_{\beta'}$, $\beta' < \beta$, the action of which is the same as that of the unbounded operator $L^{\dagger,\alpha} = A + B$ defined in (7.10) and (7.14). For a given $\mu \in \mathcal{M}_*$, let $\mu = \mu^+ - \mu^-$ be its Jordan decomposition. Then

$$L^{\dagger,\alpha}\mu = \left(B\mu^+ - A\mu^-\right) - \left(B\mu^- - A\mu^+\right) =: \mu^+_1 - \mu^-_1, \qquad \mu^\pm_1 \in \mathcal{M}^+_*.$$

This yields that

$$\|L^{\dagger,\alpha}\mu\|_{\beta'} \leq \|\mu^+_1\|_{\beta'} + \|\mu^-_1\|_{\beta'} = \|B\mu^+\|_{\beta'} + \|B\mu^-\|_{\beta'} + \|A\mu^+\|_{\beta'} + \|A\mu^-\|_{\beta'}, \tag{7.21}$$

holding for all $\mu \in \mathcal{M}_\beta$. Here we have used the additivity of the norms on the positive cone as well as the positivity of B and $-A$. By (7.8) and the following evident inequality $xe^{-\kappa x} \leq 1/e\kappa$ holding for all positive x and κ, we obtain

$$\Phi_\alpha(\gamma) \exp\left(\beta'\Psi_0(\gamma)\right) \leq \frac{1}{\alpha e(\beta - \beta')} \exp\left(\beta\Psi_0(\gamma)\right). \tag{7.22}$$

By (7.22) for $\mu \in \mathcal{M}^+_\beta$, we then get

$$\|A\mu\|_{\beta'} \leq \frac{\|\mu\|_\beta}{\alpha e(\beta - \beta')}. \tag{7.23}$$

Next, similarly as in (7.17) it follows that

$$\int_{\Gamma_*} \exp\left(\beta'\Psi_0(\gamma)\right)(B\mu)(d\gamma)$$

$$= \int_{\Gamma_*}\left(\sum_{x\in\gamma}\int_{\mathbb{R}^d} a_\alpha(x,y)\exp\left(-\sum_{z\in\gamma\setminus x}\phi(y-z)\right)\right.$$

$$\left.\times \exp\left(\beta'\Psi_0(\gamma)\right)\exp\left(\beta'[\psi(y)-\psi(x)]\right)dy\right)\mu(d\gamma)$$

$$\le e^\beta \int_{\Gamma_*}\Phi_\alpha(\gamma)\exp\left(\beta'\Psi_0(\gamma)\right)\mu(d\gamma) \le \frac{e^\beta\|\mu\|_\beta}{\alpha e(\beta-\beta')}.$$

We combine this estimate with (7.23) and (7.21) to obtain

$$\|L^{\dagger,\alpha}_{\beta'\beta}\| \le \frac{e^\beta+1}{\alpha e(\beta-\beta')}.$$

In a similar way, for each $n\in\mathbb{N}$, we also obtain, cf. (4.13),

$$\|(L^{\dagger,\alpha})^n_{\beta'\beta}\| \le \left(\frac{n}{eT_\alpha(\beta,\beta')}\right)^n, \qquad T_\alpha(\beta,\beta') := \frac{\alpha(\beta-\beta')}{e^\beta+1}. \tag{7.24}$$

By (7.20), for each $n\in\mathbb{N}$ and $\mu\in\mathcal{M}_\beta$, we have that $(L^{\dagger,\alpha})^n\mu\in\mathcal{M}_{\beta'}$, $\beta'<\beta$, and the following holds

$$(L^{\dagger,\alpha})^n_{\beta'\beta}\mu = (L^{\dagger,\alpha})^n\mu. \tag{7.25}$$

Lemma 7.6. *For each $\alpha\in(0,1]$, the closure of $(L^{\dagger,\alpha},\mathcal{M}_1)$ in $\mathcal{M}_*$ is the generator of a stochastic semigroup, $S^\alpha = \{S^\alpha(t)\}_{t\ge0}$, in $\mathcal{M}_*$ such that $S^\alpha(t):\mathcal{M}_n\to\mathcal{M}_n$ for each $n\in\mathbb{N}$. The restrictions $S^\alpha(t)|_{\mathcal{M}_n}$ constitute a C_0-semigroup on $\mathcal{M}_n$. Moreover, for each $\beta>0$ and $\beta'\in(0,\beta)$, $S^\alpha(t):\mathcal{M}^+_\beta\to\mathcal{M}^+_{\beta'}$ for $t<T_\alpha(\beta,\beta')$, see (7.24).*

Proof. The construction of the semigroup in question will be made, in particular, by showing that all the conditions of Proposition 7.2 are met. We thus begin by checking whether each of the spaces $\mathcal{M}_n$ and $\mathcal{M}_\beta$ enjoys the properties listed in Assumption 7.1. By Lemma 7.4 the density assumed in (i) is guaranteed. Each of these spaces is a Banach space with the corresponding norm, that was already mentioned in the course of their introduction. The properties assumed in (iii) are evident, whereas (iv) follows by Corollary 7.5. Thus, we can start checking the validity of the conditions imposed in Proposition 7.2. Recall that both A and B are (densely) defined on the domain $\mathcal{D}=\mathcal{M}_1$, see (7.11) and Lemma 7.4, and A is the multiplication operator by the function $(-\Phi_\alpha)$. Hence, condition (i) of Proposition 7.2 is satisfied. Moreover, A generates the semigroup S consisting of the following operators

$$(S(t)\mu)(d\gamma) = \exp\left(-t\Phi_\alpha(\gamma)\right)\mu(d\gamma). \tag{7.26}$$

Then

$$\|S(t)\mu\| \le \|\mu\|, \tag{7.27}$$

which obviously holds for all $\mu\in\mathcal{M}_*$. To check whether S is strongly continuous in $\mathcal{M}_*$, for a given $\mu\in\mathcal{M}_*$ and $\varepsilon>0$, we have to find $\delta>0$ such that $\|\mu-S(t)\mu\|<\varepsilon$ for all $t<\delta$. Since $\mathcal{M}_*$ is the $\|\cdot\|$-closure of $\mathcal{M}_1$ (by Lemma 7.4), for the chosen μ, one finds $\mu'\in\mathcal{M}_1$ such that $\|\mu-\mu'\|<\varepsilon/3$. Then by (7.26) and (7.27) we get

$$\begin{aligned}\|\mu-S(t)\mu\| &\le \|\mu-\mu'\|+\|S(t)(\mu-\mu')\|+\|\mu'-S(t)\mu'\| \\ &\le t\|A\mu'\|+2\varepsilon/3 \le (t/\alpha)\|\mu'\|_1+2\varepsilon/3,\end{aligned} \tag{7.28}$$

which completes the proof for $\mathcal{M}_*$. Clearly, $S(t):\mathcal{M}^+_n\to\mathcal{M}^+_n$, and the domain of the trace of A in $\mathcal{M}_n$ is $\widetilde{\mathcal{D}}_n=\mathcal{M}_{n+1}$. Then the proof that $S(t)|_{\mathcal{M}_n}$ is strongly continuous in $\mathcal{M}_n$ can be performed similarly as in (7.28). Thus, condition (ii) of Proposition 7.2 is met. In view of (7.19) to complete the proof of (iii) we have to show $\varphi((A+B)\mu) = 0$ whenever $\mu \in \mathcal{M}_1^+$, which is obviously the case by (7.16). Then it remains to show that, for a fixed $n \in \mathbb{N}$,

$$\int_{\Gamma_*} \Psi_1^n(\gamma)(L^{\dagger,\alpha}\mu)(d\gamma) \leq c \int_{\Gamma_*} \Psi_1^n(\gamma)\mu(d\gamma) - \varepsilon \int_{\Gamma_*} \Phi_\alpha(\gamma)\mu(d\gamma), \tag{7.29}$$

holding for each $\mu \in \mathcal{M}_{n+1}^+$ and some positive c and ε, possibly dependent on n. In view of the following estimate, cf. (7.8),

$$\alpha\Phi_\alpha(\gamma) \leq 1 + n\sum_{x\in\gamma}\psi(x) \leq \Psi_1^n(\gamma), \qquad n \in \mathbb{N}, \quad \gamma \in \Gamma_*,$$

it is enough to show (7.29) with $\varepsilon = 0$ and sufficiently big c. By (7.9) this amounts to showing

$$L^\alpha \Psi_1^n(\gamma) \leq c\Psi_1^n(\gamma), \qquad \gamma \in \Gamma_*. \tag{7.30}$$

By (7.18) it follows that

$$|\Psi_1^n(\gamma \setminus x \cup y) - \Psi_1^n(\gamma)| \leq 2^n|\psi(y) - \psi(x)|\Psi_1^{n-1}(\gamma). \tag{7.31}$$

Assume that $|x| > |y|$. By (2.13) we have

$$\begin{aligned} |\psi(y) - \psi(x)| &= \left(|x|^{d+1} - |y|^{d+1}\right)\psi(x)\psi(y) \\ &\leq \left[(|x-y| + |y|)^{d+1} - |y|^{d+1}\right]\psi(x)\psi(y) \\ &= \sum_{l=1}^{d+1}\binom{d+1}{l}|x-y|^l|y|^{d+1-l}\psi(x)\psi(y) \\ &\leq \psi(x)\sum_{l=1}^{d+1}\binom{d+1}{l}|x-y|^l \\ &:= \psi(x)v(|x-y|). \end{aligned} \tag{7.32}$$

For $|y| > |x|$, in a similar way we get

$$\begin{aligned} |\psi(y) - \psi(x)| &\leq \sum_{l=1}^{d+1}\binom{d+1}{l}|x-y|^l|x|^{d+1-l}\psi(x)\psi(y) \\ &\leq \sum_{l=1}^{d+1}\binom{d+1}{l}|x-y|^l|y|^{d+1-l}\psi(x)\psi(y) \\ &\leq \psi(x)\sum_{l=1}^{d+1}\binom{d+1}{l}|x-y|^l = \psi(x)v(|x-y|). \end{aligned} \tag{7.33}$$

Now we apply (7.31), (7.32) and (7.33) to obtain

$$\begin{aligned} \mathrm{LHS}(7.30) &\leq 2^n\sum_{x\in\gamma}\int_{\mathbb{R}^d}\psi_\alpha(x)\psi(x)a(x-y)v(|x-y|)\Psi_1^{n-1}(\gamma)dy \\ &\leq 2^n\Psi_1^{n-1}(\gamma)\left(\sum_{x\in\gamma}\psi(x)\right)\int_{\mathbb{R}^d}a(y)v(|y|)dy \\ &\leq \left(2^n\int_{\mathbb{R}^d}a(y)v(|y|)dy\right)\Psi_1^n(\gamma), \end{aligned}$$

that by (3.2) proves (7.30). Thus, all the conditions of Proposition 7.2 are met, which proves the part of the lemma related to the stochastic semigroup S^α acting in $\mathcal{M}_*$ and its restrictions to $\mathcal{M}_n$, $n \in \mathbb{N}$. To prove the second part of the lemma we use the estimate in (7.24) and define bounded operators $S^\alpha_{\beta'\beta}(t) : \mathcal{M}_\beta \to \mathcal{M}_{\beta'}$, $t < T_\alpha(\beta, \beta')$ by the series

$$S^\alpha_{\beta'\beta}(t) = \mathbb{I}_{\beta'\beta} + \sum_{n=1}^{\infty} \frac{t^n}{n!} (L^{\dagger,\alpha})^n_{\beta'\beta},$$

where $\mathbb{I}_{\beta'\beta}$ is the embedding operator. By the latter formula and (7.25) we conclude that

$$\forall \mu \in \mathcal{M}_\beta \qquad S^\alpha(t)\mu = S^\alpha_{\beta'\beta}(t)\mu, \quad t < T_\alpha(\beta, \beta'). \tag{7.34}$$

By (7.24) $S^\alpha_{\beta'\beta}(t) : \mathcal{M}_\beta \to \mathcal{M}_{\beta'}$ is a bounded operator, the norm of which satisfies

$$\|S^\alpha_{\beta'\beta}(t)\| \le \frac{T_\alpha(\beta, \beta')}{T_\alpha(\beta, \beta') - t}.$$

The positivity of $S^\alpha_{\beta'\beta}(t)$ follows by (7.34) and the positivity of $S^\alpha(t)$. This completes the proof. □

In view of (7.1), we conclude that Lemma 7.6 establishes the existence of the transition function corresponding to L^α, with the properties arising from the corresponding properties of the semigroup. Note that $\delta_\gamma \in \mathcal{M}_*$ (and hence in all $\mathcal{M}_n$ and $\mathcal{M}_\beta$) if and only if $\gamma \in \Gamma_*$, that will be assumed below. It is straightforward that $p^\alpha_t(\gamma, \cdot)$ satisfies the corresponding standard conditions and thus determines finite-dimensional distributions of a Markov process, see [13, pages 156, 157]. Our next step is to show that it has cadlag versions.

7.2. **Constructing path measures.** The construction of the families of path measures P^α which solve the martingale problem in the sense of Definition 3.3 can be done by defining their finite dimensional marginals with the help of the transition function (7.1). In this case, however, the one dimensional marginals

$$\Pi^\alpha_t = S^\alpha(t)\mu = \int_{\Gamma_*} p^\alpha_t(\gamma, \cdot)\mu(d\gamma), \tag{7.35}$$

need not be in $\mathcal{P}_{\exp}$, even for $\mu \in \mathcal{P}_{\exp}$. The only fact guaranteed by Lemma 7.6 is that $\Pi^\alpha_t \in \mathcal{M}_n$, for all $n \in \mathbb{N}$, and that $\Pi^\alpha_t \in \mathcal{M}_\beta$ with t belonging to a bounded interval. This obstacle is removed by the following statement.

Lemma 7.7. *For a given $\mu \in \mathcal{P}_{\exp}$, let $\{\mu^\alpha_t : \mu_0 = \mu,\ t \ge 0\} \subset \mathcal{P}_{\exp}$ be the family of measures defined by their correlation functions k^α_t according to (6.5). For the same μ, let Π^α_t, $t \ge 0$ be as in (7.35). Then $\Pi^\alpha_t = \mu^\alpha_t$ for all $t > 0$ and $\alpha \in (0, 1]$.*

Proof. Exactly as in Theorem 5.3 one proves that the Fokker-Planck equation (3.16) with L replaced by L^α has a unique solution, which is μ^α_t. At the same time, by construction Π^α_t also solves this equation. □

Now we can start constructing the path measures in question. To this end we use Chentsov's theorem in the following version, see [13, Theorems 8.6 – 8.8, pages 137–139]. Recall that the metric υ_* of Γ_* was introduced in (2.27). For $\alpha \in (0, 1]$, $\gamma \in \Gamma_*$ and $u, v \ge 0$, set

$$w^\alpha_u(\gamma) = \int_{\Gamma_*} \upsilon_*(\gamma, \gamma') p^\alpha_u(\gamma, d\gamma'), \tag{7.36}$$

$$W^\alpha_{u,v}(\gamma) = \int_{\Gamma_*} \upsilon_*(\gamma, \gamma') w^\alpha_u(\gamma') p^\alpha_v(\gamma, d\gamma').$$

Thereafter, for $t_3 \ge t_2 \ge t_1 \ge 0$ (such sets are called *triples*), let us consider

$$\mathcal{W}^\alpha(t_1, t_2, t_3) = \int_{\Gamma_*} W^\alpha_{t_3 - t_2, t_2 - t_1}(\gamma') \Pi^\alpha_{t_1}(d\gamma') = \int_{\Gamma_*} W^\alpha_{t_3 - t_2, t_2 - t_1}(\gamma') \mu^\alpha_{t_1}(d\gamma'), \tag{7.37}$$

where μ, μ^α_t and Π^α_t are as in (7.35).

Proposition 7.8. *(Chentsov) Assume that there exists $C_\alpha > 0$ and $\delta > 0$ such that, for each triple t_1, t_2, t_3, the following holds*

$$\mathcal{W}^\alpha(t_1, t_2, t_3) \le C_\alpha |t_3 - t_1|^2, \qquad t_3 - t_1 < \delta. \tag{7.38}$$

Then the following is true:

(i) *The transition function (7.1) and $\mu \in \mathcal{P}_{\rm exp}$ determine a probability measure P^α on $\mathfrak{D}_{\mathbb{R}_+}(\Gamma_*)$.*

(ii) *If the estimate in (7.38) holds uniformly in α, i.e., with some $C > 0$ independent of $\alpha \in (0, 1]$, and if the family $\{\Pi^\alpha_t : \alpha \in (0, 1]\} \subset \mathcal{P}(\Gamma_*)$ is tight for each $t > 0$, then the family $\{P^\alpha : \alpha \in (0, 1]\}$ of measures as in (i) is also tight, and hence possesses accumulation point in the weak topology.*

Note that the tightness of the family $\{\Pi^\alpha_t : \alpha \in (0, 1]\}$ follows by Lemmas 7.7 and 6.2.

Lemma 7.9. *For each $\mu \in \mathcal{P}_{\rm exp}$, the estimate in (7.38) holds true for all $\alpha \in (0, 1]$ with one and the same $C > 0$.*

Proof. By (7.1) and standard semigroup formulas, e.g., [13, page 9], we have

$$p^\alpha_u(\gamma, \cdot) = \delta_\gamma + \int_0^u L^{\dagger,\alpha} p^\alpha_s(\gamma, \cdot) ds, \tag{7.39}$$

since $\delta_\gamma \in \mathcal{D} = \mathcal{M}_1$. Then by this formula and (7.36) we obtain

$$\begin{aligned} w^\alpha_u(\gamma) &= w^\alpha_0(\gamma) + \int_0^u \left(\int_{\Gamma_*} \upsilon_*(\gamma, \gamma')(L^{\dagger,\alpha} p^\alpha_s(\gamma, d\gamma') \right) ds \\ &= \int_0^u \left(\int_{\Gamma_*} (L^\alpha \upsilon_*(\gamma, \gamma') p^\alpha_s(\gamma, d\gamma') \right) ds, \end{aligned} \tag{7.40}$$

where we have taken into account that $w^\alpha_0(\gamma) = \upsilon_*(\gamma, \gamma) = 0$ as υ_* is a metric. We apply now L^α to $\upsilon_*(\gamma, \cdot)$ – which is a bounded continuous function of γ', and obtain

$$\begin{aligned} J^\gamma(\gamma') &:= (L^\alpha \upsilon_*)(\gamma, \gamma') \\ &= \sum_{x\in\gamma'} \int_{\mathbb{R}^d} a_\alpha(x, y) \exp\left(- \sum_{z\in\gamma'\setminus x} \phi(y - z) \right) \left[\upsilon_*(\gamma, \gamma' \setminus x \cup y) - \upsilon_*(\gamma, \gamma')\right] dy. \end{aligned}$$

By the triangle inequality for υ_* we then get from the latter

$$|J^\gamma(\gamma')| \le \sum_{x\in\gamma'} \int_{\mathbb{R}^d} a_\alpha(x, y) \upsilon_*(\gamma' \setminus x \cup y, \gamma') dy. \tag{7.41}$$

In view of (2.27) and (2.26), to estimate $\upsilon_*(\gamma' \setminus x \cup y, \gamma')$ we consider $|\theta(y) - \theta(x)|$ with $\theta(x) = g(x)\psi(x)$, $g \in C^L_{\rm b}(\mathbb{R}^d)$, $\|g\|_{BL} \le 1$, for which we obtain, cf. (3.6),

$$\begin{aligned} |\theta(y) - \theta(x)| &= \psi(x)\psi(y) \left| \frac{g(y)}{\psi(x)} - \frac{g(x)}{\psi(y)} \right| \\ &= \psi(x)\psi(y) \left| \frac{g(y) - g(x)}{\psi(y)} + g(y) \left[\frac{1}{\psi(x)} - \frac{1}{\psi(y)} \right] \right| \\ &\le \psi(x)|x - y| + \psi(x)\psi(y) \left| |x|^{d+1} - |x|^{d+1} \right| \\ &\le \psi(x) \left[|x - y| + \sum_{l=1}^{d+1} \binom{d+1}{l} |x - y|^l \right]. \end{aligned} \tag{7.42}$$

Now we use this in (7.41) and arrive at

$$\left|J^\gamma(\gamma')\right| \le C_a \Psi_0(\gamma'), \qquad C_a := m^a_1 + \sum_{l=1}^{d+1} \binom{d+1}{l} m^a_l. \tag{7.43}$$

Then we use (7.43) in (7.40) and obtain

$$w_u^\alpha(\gamma) \leq C_a \int_0^u \chi_s^\alpha(\gamma) ds, \qquad \chi_s^\alpha(\gamma) := \int_{\Gamma_*} \Psi_0(\gamma') p_s^\alpha(\gamma, d\gamma'). \tag{7.44}$$

Note that $\chi_0^\alpha(\gamma) = \Psi_0(\gamma)$. Similarly as in (7.39) we write

$$\chi_s^\alpha(\gamma) = \Psi_0(\gamma) + \int_0^s \left(\int_{\Gamma_*} (L^\alpha \Psi_0)(\gamma') p_v^\alpha(\gamma, d\gamma') \right) dv \tag{7.45}$$

Like in (7.30) one gets

$$(L^\alpha \Psi_0)(\gamma) \leq |(L^\alpha \Psi_0)(\gamma)| \leq 2c_a \Psi_0(\gamma),$$

where c_a is as in (3.8). We use this in (7.45), take also into account the definition of χ_s^α in (7.44) and obtain

$$\chi_s^a(\gamma) \leq \Psi_0(\gamma) + 2c_a \int_0^s \chi_v^a(\gamma) dv,$$

which by the Grönwall inequality and (7.44) yields the following estimate

$$w_u^\alpha(\gamma) \leq C_a u e^{2c_a u} \Psi_0(\gamma).$$

We employ this in the second line of (7.36) and obtain

$$W_{u,v}^\alpha(\gamma) \leq C_a u e^{2c_a u} q_v^\alpha(\gamma), \qquad q_v^\alpha(\gamma) := \int_{\Gamma_*} \Psi_0(\gamma') \upsilon_*(\gamma, \gamma') p_v^\alpha(\gamma, d\gamma'). \tag{7.46}$$

Note that $q_0^\alpha(\gamma) = 0$ as υ_* is a metric. Similarly as in (7.40) we then get

$$q_v^\alpha(\gamma) = \int_0^v \left(\int_{\Gamma_*} \left(L^\alpha \Psi_0 \upsilon_*(\gamma, \cdot)\right)(\gamma') p_s^\alpha(\gamma, d\gamma') \right) ds.$$

Thus, we have to estimate

$$\begin{aligned} \left|\left(L^\alpha \Psi_0 \upsilon_*(\gamma, \cdot)\right)(\gamma')\right| &\leq \sum_{x\in\gamma'} \int_{\mathbb{R}^d} a(x-y) \Big| \Psi_0(\gamma' \setminus x \cup y) \upsilon_*(\gamma, \gamma' \setminus x \cup y) \\ &- \Psi_0(\gamma') \upsilon_*(\gamma, \gamma') \Big| dy \leq \sum_{x\in\gamma'} \int_{\mathbb{R}^d} a(x-y) |\psi(y) - \psi(x)| dy \\ &+ \Psi_0(\gamma') \sum_{x\in\gamma'} \int_{\mathbb{R}^d} a(x-y) \upsilon_*(\gamma', \gamma' \setminus x \cup y) dy \\ &\leq c_a \Psi_0(\gamma') + C_a \Psi_0^2(\gamma'), \end{aligned} \tag{7.47}$$

where we used the same estimate as in as in (7.42). Now we use (7.47) in (7.46) and then plug this into (7.37). In doing so, we will deal with

$$\int_{\Gamma_*} p_s^\alpha(\gamma, d\gamma') \Pi_{t_1}^\alpha(d\gamma) = \Pi_{t_1+s}^\alpha(d\gamma') = \mu_{t_1+s}^\alpha(d\gamma'),$$

that follows by the Chapman-Kolmogorov property of the transition function (7.1), see [13, page 156], and then by Lemma 7.7. Thereafter, we obtain

$$\mathcal{W}^\alpha(t_1, t_1+v, t_1+u+v) \leq C_a u e^{2c_a u} \int_0^v \left(c_a \mu_{t_1+s}^\alpha(\Psi_0) + C_a \mu_{t_1+s}^\alpha(\Psi_0^2) \right) ds. \tag{7.48}$$

We recall that $\mu \in \mathcal{P}_{\exp}^{\vartheta_0}$ and thus the correlation functions of μ_t^α satisfy the estimate in claim (a) of Proposition 4.2 with $\vartheta_t = \vartheta_0 + t$. Then by (2.14) we get

$$\mu_t^\alpha(\Psi_0) \leq \langle\psi\rangle e^{\vartheta_0+t}, \quad \mu_t^\alpha(\Psi_0^2) \leq \langle\psi\rangle e^{\vartheta_0+t} + \langle\psi\rangle^2 e^{2\vartheta_0+2t}.$$

We use this in (7.48) and obtain

$$\mathcal{W}^\alpha(t_1, t_1+v, t_1+u+v) \leq C(u+v)^2,$$

where, for a fixed $\delta > 0$, the independent of α constant can be calculated explicitly for $u+v < \delta$. This yields (7.38) with C independent of α, and hence completes the whole proof. $\square$

7.3. **Proof of Theorems 6.1 and 3.6.** For each $\alpha \in (0,1]$, $s \geq 0$ and $\mu \in \mathcal{P}_{\rm exp}$, by Proposition 7.8 the measure $P^\alpha_{s,\mu}$ on $\mathfrak{D}_{[s,+\infty)}(\Gamma_*)$ is defined by its finite dimensional marginals constructed with the use of the transition function (7.1). Namely, for $s \leq t_1 < t_2 < \cdots < t_m$ and $\mathbb{A}_1, \dots, \mathbb{A}_m \in \mathcal{B}(\Gamma_*)$, we have, cf. [13, eq. (1.10), page 157],

$$P^\alpha_{s,\mu}\left((\mathbb{1}_{\mathbb{A}_1} \circ \varpi_{t_1}) \cdots (\mathbb{1}_{\mathbb{A}_m} \circ \varpi_{t_m})\right) = \int_{\Gamma_*^{m+1}} \mathbb{1}_{\mathbb{A}_m}(\gamma_m) p^\alpha_{t_m - t_{m-1}}(\gamma_{m-1}, d\gamma_m) \tag{7.49}$$

$$\times \mathbb{1}_{\mathbb{A}_{m-1}}(\gamma_{m-1}) p^\alpha_{t_{m-1}-t_{m-2}}(\gamma_{m-2}, d\gamma_{m-1}) \cdots \mathbb{1}_{\mathbb{A}_1}(\gamma_1) p^\alpha_{t_1 - s}(\gamma_0, d\gamma_1)\mu(d\gamma_0).$$

In particular, for $t \geq s$, this yields

$$P^\alpha_{s,\mu} \circ \varpi_t^{-1} = S^\alpha(t-s)\mu. \tag{7.50}$$

Then the validity of conditions (a) and (b) of Definition 3.3 follow by (7.50) and Lemma 7.7. Now we turn to proving the validity of (c). Let G be as in (3.15) with a given $m \in \mathbb{N}$ and $s \leq s_1 < s_2 < \cdots < s_m < t_2$. For a given $F \in \mathcal{D}(L)$ and $u \in [s_m, t_2]$, we set $\mathsf{F}_u = F \circ \varpi_u$, $\mathsf{K}_u = (LF) \circ \varpi_u$ and $\mathsf{K}^\alpha_u = (L^\alpha F) \circ \varpi_u$. Next, we define

$$\chi^\alpha_{s_1}(d\gamma) = C_1 F_1(\gamma) \mu^\alpha_{s_1}(d\gamma) = C_1 F_1(\gamma) \int_{\Gamma_*} p^\alpha_{s_1 - s}(\gamma_0, d\gamma)\mu(d\gamma_0), \quad C_1^{-1} := \int_{\Gamma_*} F_1(\gamma)\mu^\alpha_{s_1}(d\gamma).$$

By (7.35) and Lemma 7.7, and then by claim (iv) of Proposition 3.2, we have that $\chi^\alpha_{s_1} \in \mathcal{P}^{\vartheta_1}_{\rm exp}$ with ϑ_1 dependent on $s_1 - s$ and the type of $\mu \in \mathcal{P}_{\rm exp}$, and independent of α since the norms of $L^{\Delta,\alpha}$ can be estimated uniformly in $\alpha \in [0,1]$. Then we define recursively

$$\chi^\alpha_{s_l}(d\gamma) = C_l F_l(\gamma) \int_{\Gamma_*} p^\alpha_{s_l - s_{l-1}}(\gamma_0, d\gamma) \chi^\alpha_{s_{l-1}}(d\gamma_0), \qquad l = 2, \dots, m, \tag{7.51}$$

and obtain $\chi^\alpha_{s_m} \in \mathcal{P}^{\vartheta_m}_{\rm exp}$ with ϑ_m independent of α. Thereafter, by (7.49) we conclude that

$$P^\alpha_{s,\mu}(\mathsf{F}_u \mathsf{G}) = C P^\alpha_{s_m, \chi_{s_m}}(\mathsf{F}_u) = C P^\alpha_{s_m,\chi_{s_m}}(F \circ \varpi_u), \qquad u \geq s_m, \tag{7.52}$$

where C is a normalizing constant, i.e. $C = P^\alpha_{s,\mu}(\mathsf{G})$. Then $P^\alpha_{s,\mu}(\mathsf{H}) = 0$ follows by the fact that the map $u \mapsto P^\alpha_{s_m,\chi_{s_m}} \circ \varpi_u^{-1}$ solves the Fokker-Planck equation (3.16) with L^α, see (7.50). This proves (c), and hence the family $\{P^\alpha_{s,\mu} : s \geq 0, \mu \in \mathcal{P}_{\rm exp}\}$ is a unique solution of the corresponding restricted martingale problem, see Theorem 5.6.

By Lemma 7.9 and claim (ii) of Proposition 7.8, for each s and μ, the family $\{P^\alpha_{s,\mu} : \alpha \in (0,1]\}$ is relatively weakly compact, and each of its accumulation points has the same one dimensional marginals, that coincide with the measures μ_t, see Lemmas 6.2 and 7.7. Let us show that these accumulation points solve the restricted initial value martingale problem for L. By Lemmas 6.2 and 7.7 one concludes that conditions (a) and (b) of Definition 3.3 are met, and we thus turn to proving (3.14). Given sequence $\{\alpha_n\}_{n\in\mathbb{N}} \subset (0,1]$, $\alpha_n \to 0$ and $s \geq 0$, $\mu \in \mathcal{P}_{\rm exp}$, let $P^{\alpha_n}_{s,\mu} \Rightarrow P_{s,\mu}$. Let also G in (3.13) be as in (3.15) with a given m, $s_1, \dots, s_m$ and $F_j \in \widetilde{\mathcal{F}}$, $j = 1, \dots, m$. Set $C_n = P^{\alpha_n}_{s,\mu}(\mathsf{G})$. Then the measures $\nu_{n,u} \in \mathcal{P}(\Gamma_*)$ defined by

$$\nu_{n,u}(\mathbb{A}) = C_n^{-1} P^{\alpha_n}_{s,\mu}(\mathsf{G} \cdot (\mathbb{1}_{\mathbb{A}} \circ \varpi_u)) = P^{\alpha_n}_{s_m \chi_{s_m}}(\varpi_u^{-1}(\mathbb{A})), \qquad u \in [s_m, t_2], \quad \mathbb{A} \in \mathcal{B}(\Gamma_*).$$

are in $\mathcal{P}^\vartheta_{\rm exp}$ with ϑ independent of n and $u \in [s_m, t_2]$ (see (7.52)). We also let

$$\nu_u(\mathbb{A}) = C^{-1} P_{s,\mu}(\mathsf{G} \cdot (\mathbb{1}_{\mathbb{A}} \circ \varpi_u)), \qquad u \in [s_m, t_2], \quad \mathbb{A} \in \mathcal{B}(\Gamma_*),$$

with $C = P_{s,\mu}(\mathsf{G})$. Then $\nu_{n,u} \Rightarrow \nu_u$ for all $u \in [s_m, t_2]$. By Lemma 6.4 this yields $\nu_u \in \mathcal{P}^\vartheta_{\rm exp}$, and hence the corresponding correlation functions satisfy $k^{\alpha_n}_u, k_u \in \mathcal{K}_\vartheta$ for all $u \in [s_m, t_2]$ and $n \in \mathbb{N}$, see (2.6). To prove $P_{s,\mu}(\mathsf{H}) = 0$ we rewrite it, cf. (3.13),

$$P_{s,\mu}(\mathsf{F}_{t_2}\mathsf{G}) - P_{s,\mu}(\mathsf{F}_{t_1}\mathsf{G}) - \int_{t_1}^{t_2} P_{s,\mu}(\mathsf{K}_u \mathsf{G}) du = 0. \tag{7.53}$$

For $u \in [s_m, t_2]$ and $n \in \mathbb{N}$, we then set

$$a_n(u) = P_{s,\mu}(\mathsf{F}_u\mathsf{G}) - P^{\alpha_n}_{s,\mu}(\mathsf{F}_u\mathsf{G}),$$

$$b_n(u) = P_{s,\mu}(\mathsf{K}_u\mathsf{G}) - P^{\alpha_n}_{s,\mu}(\mathsf{K}_u\mathsf{G}),$$

$$c_n(u) = P^{\alpha_n}_{s,\mu}((\mathsf{K}_u - \mathsf{K}^{\alpha_n}_u)\mathsf{G}).$$

Since $P^{\alpha_n}_{s,\mu}(\mathsf{H}) = 0$, it follows that

$$\mathrm{LHS}(7.53) = [a_n(t_2) - a_n(t_1)] - \int_{t_1}^{t_2} b_n(u)du - \int_{t_1}^{t_2} c_n(u)du =: I_n^{(1)} + I_n^{(2)} + I_n^{(3)}. \tag{7.54}$$

By the assumed weak convergence of $P^{\alpha_n}_{s,\mu}$ one readily gets $a_n(u) \to 0$, which yields $I_n^{(1)} \to 0$ as $n \to +\infty$. At the same time,

$$b_n(u) = C_n^{-1}\left[\nu_u(LF) - \nu_{n,u}(LF)\right] + (C^{-1} - C_n^{-1})\nu_u(LF). \tag{7.55}$$

Since $C_n \to C > 0$, to prove $b_n(u) \to 0$ as $n \to +\infty$ by (7.55) it is enough to show that $\nu_u(LF) - \nu_{n,u}(LF) \to 0$ for $F \in \mathcal{D}(L)$. To this end we recall that $\mathsf{G} = \mathsf{G}_{m-1}(F_m \circ \varpi_{s_m})$, see (3.15). Set

$$\tilde{\nu}_{n,u}(\mathbb{A}) = \tilde{C}_n^{-1} P^{\alpha_n}_{s,\mu}(\mathsf{G}_{m-1}(\mathbb{1}_{\mathbb{A}} \circ \varpi_u)), \quad u \in [s_{m-1}, s_m],$$

$$\tilde{\nu}_u(\mathbb{A}) = \tilde{C}^{-1} P_{s,\mu}(\mathsf{G}_{m-1}(\mathbb{1}_{\mathbb{A}} \circ \varpi_u)).$$

As above, we have that $\tilde{\nu}_{n,u}, \tilde{\nu}_u \in \mathcal{P}^{\tilde{\vartheta}}_{\exp}$ for all n and u as above. Clearly, we may assume that $\vartheta > \tilde{\vartheta}$, and hence their correlation functions, $\tilde{k}_u$ and $\tilde{k}^{\alpha_n}_u$, lie in the corresponding $\mathcal{K}_{\tilde{\vartheta}}$. As in (6.7) we then can write

$$k_u - k^{\alpha_n}_u = Q_{\vartheta\tilde{\vartheta}}(u - s_m)\tilde{k}_{s_m} - Q^{\alpha_n}_{\vartheta\tilde{\vartheta}}(u - s_m)\tilde{k}^{\alpha_n}_{s_m}. \tag{7.56}$$

For $m = 1$, $\tilde{k}_u$ and $\tilde{k}^{\alpha_n}_u$ are the correlation functions of μ_u and $\mu^{\alpha_n}_u$, and hence one may apply Lemma 6.3, which yields

$$\mu_u(LF) - \mu^{\alpha_n}_u(LF) = \langle\!\langle \tilde{k}_u - \tilde{k}^{\alpha_n}_u, \widehat{L}\tilde{G}\rangle\!\rangle = \langle\!\langle \tilde{k}_u - \tilde{k}^{\alpha_n}_u, G\rangle\!\rangle \to 0, \qquad n \to +\infty,$$

where $\tilde{G} \in \cap_\vartheta \mathcal{G}_\vartheta$ is such that $F = K\tilde{G}$, see (5.1) and (5.2). Therefore, we may inductively assume in (7.56) that

$$\langle\!\langle \tilde{k}_{s_m} - \tilde{k}^{\alpha_n}_{s_m}, \widehat{L}\tilde{G}\rangle\!\rangle \to 0,$$

and obtain

$$\nu_u(LF) - \nu_{n,u}(LF) = \langle\!\langle k_u - k^{\alpha_n}_u, \widehat{L}\tilde{G}\rangle\!\rangle \to 0,$$

by repeating the steps made in the proof of Lemma 6.3. This yields $b_n(u) \to 0$. As already mentioned above, both terms of $b_n(u)$ are bounded uniformly in n and u, that yields in (7.54) $I_n^{(2)} \to 0$.

Let us now turn to $I_n^{(3)}$. As above, we have here

$$|c_n(u)| = \left|\langle\!\langle k^{\alpha_n}_u, \widehat{L}_n G\rangle\!\rangle\right| \le e^{\vartheta}|\widehat{L}_n G|_\vartheta,$$

where $G \in \cap_\vartheta \mathcal{G}_\vartheta$ is such that $F = KG$, see (5.1) and (5.2), and $\widehat{L}_n$ is obtained by replacing $a(x-y)$ in (4.1) by

$$a_n(x,y) = a(x-y)(1 - \psi_{\alpha_n})(x) = \alpha_n \frac{a(x-y)|x|^{d+1}}{1 + \alpha_n|x|^{d+1}} \le \alpha_n a(x-y)|x|^{d+1} =: \alpha_n \hat{a}(x,y).$$

Proceeding as in obtaining (4.5) we then get, see (3.2),

$$|\widehat{L}_n G|_\vartheta \le \frac{2\alpha_n m^a_{d+1}}{e(\vartheta' - \vartheta)} \exp\left(e^{\vartheta}\langle\phi\rangle\right)|G|_{\vartheta'}.$$

Here ϑ' can be an arbitrary number since $G \in \cap_{\vartheta'}\mathcal{G}_{\vartheta'}$, see (5.2). This yields $I_n^{(3)} \to 0$ as $n \to +\infty$ (and hence $\alpha_n \to 0$), which by (7.54) implies (7.53). Therefore, the proof of Theorem 6.1 is completed. The uniqueness then follows by Theorem 5.3 and hence the proof of Theorem 3.6 follows by the latter two statements.

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Appendix

Here we prove (5.24), (5.25) and (5.26). For $n = 2$, (5.24) is just (5.20) with $\varPhi^m_{\tau,2}$ given by (5.25) with $w_1(m,2) = (m+1)^2 - m^2 = 2m+1$, $w_2(m,2) = 1$, see (5.28) and (5.27). Assume then that $\varPhi^m_{\tau,n}$ is as in (5.25). By (5.23), similarly as in (3.11), we get

$$|LV_\tau(c;\gamma)| \ \leq \ \sum_{j=1}^m \widehat{F}_\tau^{m,\theta^{q_1},\dots,\theta^{q_{j-1}},\theta^{q_j}+1,\theta^{q_{j+1}},\dots\theta^{q_m}}(\gamma)) + \tau c_a^{n+1}\widehat{F}_\tau^{m+1}(\gamma).$$

Here we have taken into account that $\bar{c}_\theta = 1$, and also $\theta^q(x) \leq c_a^q \psi(x)$, see (3.5) and (3.6). In a similar way, by (3.12) we obtain

$$\left| L\widehat{F}^m_\tau(\gamma)\right| \leq mc_a \widehat{F}^m_\tau(\gamma) + \tau c_a \widehat{F}^{m+1}_\tau(\gamma).$$

Now we use both this estimates in (5.25) and obtain

$$\begin{aligned}
\left|L\varPhi^m_{\tau,n}(\gamma)\right| \ &\leq \ \sum_{c\in\mathcal{C}_{m,n}} C_{m,n}(c)\Big(c_0 V_\tau(c_0-1, c_1+1, c_2,\dots,c_k,\dots;\gamma) \\
&+ \ c_1 V_\tau(c_0, c_1-1, c_2+1,\dots,c_k,\dots;\gamma) + \cdots + \\
&+ \ c_n V_\tau(c_0, c_1,\dots,c_n-1, c_{n+1}+1,\dots;\gamma)\Big) \\
&+ \ \tau c_a^{n+1} m^n \widehat{F}^{m+1}_\tau(\gamma) + c_a^{n+1}\Bigg(\sum_{k=1}^n \tau^k (m+k) w_k(m,n)\widehat{F}^{m+k}_\tau(\gamma) \\
&+ \ \sum_{k=2}^{n+1}\tau^k(m+k)w_{k-1}(m,n)\widehat{F}^{m+k}_\tau(\gamma)\Bigg).
\end{aligned} \tag{7.57}$$

If one takes into account the recurrence formulas in (5.27), the latter two lines of the right-hand side of (7.57) convert into the second term of (5.25) written for $\varPhi^m_{\tau,n+1}$. Thus, it remains to prove that the first three lines of (7.57) yield the first term of (5.25) written for $\varPhi^m_{\tau,n+1}$. Note that therein the summands corresponding to $c_j = 0$ vanish automatically since we multiply them by zero in this case. Assuming that a given $c_j \neq 0$ we can write the corresponding summand in (7.57), denoted S^{n+1}_j, as follows, see the second line in (5.25),

$$\begin{aligned}
S^{n+1}_j \ &= \ \frac{m!n!(j+1)!(c_{j+1}+1)}{c_0!\cdots(c_j-1)!(c_{j+1}+1)!\cdots(0!)^{c_0}\cdots(j!)^{c_j-1}j!((j+1)!)^{c_{j+1}+1}\cdots} \\
&\times \ V_\tau(c_0,c_1,\dots,c_j-1,c_{j+1}+1,\cdots;\gamma) \\
&= \ \frac{j+1}{n+1}c'_{j+1}C_{m,n+1}(c')V_\tau(c';\gamma), \qquad c'\in\mathcal{C}_{m,n+1},
\end{aligned} \tag{7.58}$$

where $c' = (c_0,\dots,c_j-1,c_{j+1}+1,\dots)$. To get convinced that c' is indeed in $\mathcal{C}_{m,n+1}$ one computes the corresponding sums, cf. (5.22), that yields $c_0+\cdots+c_j-1+c_{j+1}+1+\cdots = c_0+\cdots+c_j+c_{j+1}+\cdots = m$, and $c_1+\cdots+j(c_j-1)+(j+1)(c_{j+1}+1)+\cdots = c_1+\cdots+jc_j+(j+1)c_{j+1}+\cdots-j+j+1 = n+1$. Then we rewrite each summand in the first three lines of (7.57) as in (7.58) and observe that the corresponding c' runs over the whole $\mathcal{C}_{m,n+1}$ when c runs through $\mathcal{C}_{m,n}$. Then these three lines, denoted S^{n+1}, take the following form

$$\begin{aligned} S^{n+1} &= \sum_{c' \in \mathcal{C}_{m,n+1}} \left(\frac{1}{n+1} \sum_{j=1}^{n+1} j c'_j \right) C_{m,n+1}(c') V_\tau(c';\gamma) \qquad (7.59) \\ &= \sum_{c' \in \mathcal{C}_{m,n+1}} C_{m,n+1}(c') V_\tau(c';\gamma), \end{aligned}$$

where we have taken into account that $\sum_j j c_j = n+1$, see (5.22). This completes the proof of (5.24) and (5.25). It then remains to prove (5.26). For $n=1$, $\mathcal{C}_{m,1}$ is a singleton consisting of $c = (m-1, 1, 0, \dots)$, which yields

$$\sum_{c \in \mathcal{C}_{m,1}} C_{m,1}(c) = \frac{m!}{(m-1)!1!} = m.$$

Now we set in the second line of (7.59) $V_\tau(c';\gamma) \equiv 1$ and calculate S^{n+1} with this V_τ, which is equal to the first three lines of (7.57). That is,

$$\sum_{c' \in \mathcal{C}_{m,n+1}} C_{m,n+1}(c') = \sum_{c \in \mathcal{C}_{m,n}} C_{m,n}(c) \Big(c_0 + c_1 + \cdots + c_n \Big) = m \sum_{c \in \mathcal{C}_{m,n}} C_{m,n}(c),$$

where we once again have used the first equality in (5.22). Now (5.26) is obtained from the latter by the induction in n.

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