OPTIMAL CONTROL OF NONLINEAR STOCHASTIC DIFFERENTIAL EQUATIONS ON HILBERT SPACES

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ABSTRACT. We here consider optimal control problems governed by nonlinear stochastic equations on a Hilbert space H with nonconvex payoff, which is rewritten as a deterministic optimal control problem governed by a Kolmogorov equation in H. We prove the existence and first-order necessary condition of closed loop optimal controls for the above control problem. The strategy is based on solving a deterministic bilinear optimal control problem for the corresponding Kolmogorov equation on the space $L^2(H,\nu)$, where ν is the related infinitesimally invariant measure for the Kolmogorov operator.

1. INTRODUCTION

We are concerned with optimal control problems connected with the informal stochastic differential equation on a Hilbert space H (with norm $|\cdot|_H$, inner product $\langle\cdot,\cdot\rangle$) of type

$$dX(t) = A(X(t))dt + Q^{\frac{1}{2}}Bu(X(t))dt + Q^{\frac{1}{2}}dW(t), \quad t \in (0,T), \qquad X(0) = x \in H. \tag{1.1}$$

Here, the operator A is defined by

$$A : D(F) \mapsto (D(\widetilde{A}))^*, \ \langle A(x), h\rangle := \langle x, \widetilde{A}h\rangle + \langle F(x), h\rangle \tag{1.2}$$

for any $x \in D(F)$ and any $h \in D(\widetilde{A})$, where $\widetilde{A}$ is a self-adjoint m-dissipative linear operator in H to be made precise later on, and F is a (possibly nonlinear) operator from $D(F) \subseteq H$ to H.

The operator B is linear and bounded on $L^\infty(H;H,\nu)$, where ν is an infinitesimally invariant measure for the corresponding Kolmogorov operator when $u \equiv 0$ (see Hypothesis (H1) (ii) below), which actually serves as a substitute for Lebesque measure on H that does not exist on infinite dimensional spaces. The operator Q is a positive definite bounded self-adjoint linear operator on H, satisfying that $Qe_i = q_ie_i$, $q_i > 0$ for all but finitely many $i \geq 1$ and for some orthonormal basis $\{e_i\}_{i\geq 1} \subseteq D(\widetilde{A})$ of H, and W is a cylindrical Wiener process on H defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with normal filtration $(\mathscr{F}_t)$, $t \geq 0$.

The term u is an input controller applied to the stochastic system and is taken in the admissible set

$$\mathcal{U}_{ad} = \{u : H \to H;\ u \ is\ \nu - measurable,\ |u(x)|_H \leq \rho,\ \forall x \in H\},$$

where $\rho \in (0,\infty)$ is fixed.

Equation (1.1) is mainly motivated by a number of stochastic partial differential equations, including singular stochastic equations ([16, 15]), gradient systems, stochastic reaction-diffusion equations ([10, 14]) and stochastic porous media equations ([3], see also [4]).

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In the present work, we are interested in the optimal feedback control problem for (1.1), i.e., find a controller $u^* \in \mathcal{U}_{ad}$ such that

(P_0)

$$\text{Min}\left\{\mathbb{E}\int_0^T \int_H g(X^u(t,x))\nu(dx)dt;\ u \in \mathcal{U}_{ad},\ X^u \text{ solves } (1.1)\right\},$$

where g is a given function in $L^2(H,\nu)$, is attained at u^*.

It should be mentioned that the main difficulty for the existence theory for the optimal control problem (P_0) is that the cost functional $\Phi(u) = \mathbb{E}\int_0^T \int_H g(X^u(t,x))\nu(dx)dt$, $u \in \mathcal{U}_{ad}$, is not weakly lower-semicontinuous on $L^2(H;H,\nu)$, if A is nonlinear and g is not convex.

Another delicate problem in infinite dimensional spaces is that, even if (1.1) has a unique strong solution (in the probabilistic sense) in the uncontrolled case where $u \equiv 0$, it is in general not clear whether it still has strong solutions under bounded perturbations. See, e.g., [11]-[13] for the relevant work.

Here, the key idea is to rewrite the original Problem (P_0) as a deterministic bilinear optimal control problem governed by the Kolmogorov equation corresponding to (1.1).

More precisely, we consider the Kolmogorov equation corresponding to (1.1), i.e.,

$$\begin{aligned} &\frac{d\varphi}{dt} = N_2\varphi + \langle Q^{\frac{1}{2}}Bu, D\varphi\rangle, \quad t > 0, \\ &\varphi(0,x) = g(x), \quad x \in H, \end{aligned} \tag{1.3}$$

where $u \in \mathcal{U}_{ad}$, N_2 is the Kolmogorov operator related to (1.1) (see (2.2) and Remark 2.1 below), and equation (1.3) is taken in the space $L^2(H,\nu)$.

Heuristically, via Itô's formula, one has that the solution φ^u for (1.3) is given by

$$\varphi^u(t,x) = \mathbb{E}g(X^u(t,x)),\ for\ dt \times \nu - a.e.\ (t,x) \in [0,T] \times H.$$

This entails that the original optimal control problem can be reformulated as follows: find $u^* \in \mathcal{U}_{ad}$ such that

(P^*)

$$\text{Min}\left\{\int_0^T \int_H \varphi^u(t,x)\nu(dx)dt;\ u \in \mathcal{U}_{ad},\ \textit{and } \varphi^u \textit{ is the solution to } (1.3)\right\}$$

is attained at u^*.

This idea was recently applied in [1] by the first author to the stochastic reflection problem in finite dimensions. The main advantage of Problem (P^*) is that it is a deterministic bilinear optimal control problem. This feature makes it possible to give a unified treatment of optimal control problems for various stochastic equations on Hilbert spaces through the corresponding Kolomogorov operators, under unusually weak conditions of the nonlinearity and the objective functionals. Actually, the usual continuity or convexity conditions are not assumed here, which can be viewed as a regularization effect of noise on control problems through the corresponding Kolmogorov operators.

As a matter of fact, the optimal feedback controllers for Problem (P) can be formally determined by solving an infinite dimensional second order Hamilton-Jacobi equation (see, e.g., [8, 17]). However, such an equation under quite restrictive conditions has only a viscosity solution which is not sufficiently regular to provide an explicit representation for the optimal controller. We would also like to refer to [18] for the solvability of nonlinear Kolmogorov equations, including Hamilton-Jacobi-Bellman equations, and the applications to optimal feedback controls.

Here, for any objective functions g in $D(N_2)$, where $D(N_2)$ is the domain of the closure in $L^2(H,\nu)$ of the Kolmogorov operator $(N_0, D(N_0))$ defined in (2.2) below, we prove the existence of a closed-loop optimal control for Problem (P^*) under mild conditions on F and g.

Moreover, in the symmetric case (i.e., $N_2^* = N_2$ on $L^2(H,\nu)$, where N_2^* denotes the dual operator of N_2), for more general objective functions $g \in L^2(H,\nu)$, we obtain the existence as well as first-order necessary condition of optimal feedback controllers of Problem (P^*).

Regarding the original control problem of the stochastic equation (1.1), it turns out that the martingale problem serves as an appropriate concept of solutions to stochastic equations on Hilbert spaces. More precisely, we consider the problem

(P)

$$\mathrm{Min}\Big\{ \int_0^T \int_H \mathbb{E}_{\mathbb{P}_x} g(X^u(t))\nu(dx)dt;\ u \in \mathcal{U}_{ad},\ \mathbb{P}_x \circ (X^u)^{-1}\ solves$$
$$the\ martingale\ problem\ of\ (1.1)\ for\ \nu - a.e.\ x \in H \Big\}.$$

(See Definition 2.9 below for the definition of the martingale problem corresponding to (1.1).)

We prove that the optimal controllers to Problem (P^*) obtained above actually coincide with those to the Problem (P), as long as the related martingale problems are well posed. In this sense, the optimal controllers for Problem (P^*) of Kolmogorov equations can be viewed as generalized optimal controllers for the Problem (P) of stochastic equations on Hilbert spaces.

Actually, the solutions to the martingale problem for (1.1) suffice to define the objective functional in Problem (P). More importantly, well-posedness for this type of martingale problem holds in a quite general setting (e.g. in the framework of (generalized) Dirichlet forms), and it is also stable under bounded perturbations and thus enables us to treat optimal control problems of stochastic differential equations on Hilbert spaces, of which the nonlinearity may be not continuous or the operator $Q^{\frac{1}{2}}$ is not necessarily Hilbert-Schmidt (see, e.g., [15]). For such equations, it is known that strong solutions (in the probabilistic sense) do not exist in general.

As we shall see below, the martingale problem is well posed for various stochastic equations on Hilbert spaces, including singular dissipative stochastic equations, stochastic reaction-diffusion equations as well as stochastic porous media equations. Moreover, we also prove that the well-posedness of martingale problems are implied by the m-dissipativity of the corresponding Kolmogorov operators in certain situations, by using the theory of (generalized) Dirichlet forms (see Theorems 4.1 and 4.5 below). The interplay between optimal control problems and (generalized) Dirichlet forms would be of independent interest.

We would also like to mention that, by the argument above, the end point optimal control problem

$$\mathrm{Min}\{\int_H \mathbb{E}_{\mathbb{P}_x} g(X^u(T))\nu(dx):\ u \in \mathcal{U}_{ad},\ \mathbb{P}_x \circ (X^u)^{-1}\ solves$$
$$the\ martingale\ problem\ of\ (1.1)\ for\ \nu - a.e.\ x \in H.\}$$

can be also written as

$$\mathrm{Min}\{\int_H \varphi^u(T,x)\nu(dx):\ u \in \mathcal{U}_{ad},\ \varphi^u\ solves\ (1.3)\}.$$

Notation For $k \in \mathbb{N}$, by $\mathcal{F}C_b^k(H)$ we denote the set of C_b^k-cylindrical functions $\varphi(x) = \phi(\langle x, e_1\rangle, \cdots, \langle x, e_n\rangle)$ for some $n \in \mathbb{N}$ and $\phi \in C_b^k(\mathbb{R}^n)$, where $\{e_k : k \in \mathbb{N}\}$ is the eigenbasis of Q introduced above. Let $B_b(H)$ and $C_b(H)$ denote, respectively, the bounded Borel-measurable and bounded continuous functions from H to $\mathbb{R}$, and let $L(H)$ be the set of all bounded operators on H. The symbols D and D^2 denote the first and second Fréchet derivatives, respectively. We also use the notation Id for the identity operator on H.

For any Borel probability measure ν on H, $\text{supp}(\nu)$ denotes the topological support of ν, and $L^2(H, \nu)$ consists of ν-measurable functions φ on H such that $\int_H |\varphi(x)|^2 \nu(dx) < \infty$. We use the notation $(\ , \ )$ for the inner product in $L^2(H, \nu)$. Similarly, $L^2(H; H, \nu)$ denotes the space of H-valued $L^2(\nu)$-integrable maps.

2. Formulation of the main results

To begin with, let us first introduce the Kolmogorov operator related to (1.1), which is formally given by,

$$N_0^u \varphi(x) := \frac{1}{2}\text{Tr}[QD^2\varphi](x) + \langle A(x), D\varphi(x)\rangle + \langle Bu(x), Q^{\frac{1}{2}} D\varphi(x)\rangle, \tag{2.1}$$

for any $\varphi \in \mathcal{F}C_b^2(H)$. In particular, when $u \equiv 0$, we set

$$N_0\varphi(x) := \frac{1}{2}\text{Tr}[QD^2\varphi](x) + \langle A(x), D\varphi(x)\rangle, \quad \varphi \in D(N_0) := \mathcal{F}C_b^2(H). \tag{2.2}$$

Consider the following assumptions.

(H1) There exists a Borel probability measure ν such that $F : D(F) \subseteq H \to H$ is ν-measurable and the following properties hold:
  (i) $\nu(D(A)) = 1$ and $\int_H (|F(x)|_H^2 + |x|_H^2)\nu(dx) < \infty$.
  (ii) ν is the infinitesimally invariant measure for $(N_0, D(N_0))$, i.e.,
  $$\int_H N_0\varphi d\nu = 0, \quad \forall \varphi \in \mathcal{F}C_b^2(H).$$
  (iii) $(N_0, \mathcal{F}C_b^2(H))$ is essentially m-dissipative on $L^2(H, \nu)$, i.e., $(1 - N_0)(\mathcal{F}C_b^2(H))$ is dense in $L^2(H, \nu)$.

(H2) The operator $Q^{\frac{1}{2}}B$ with domain $D(Q^{\frac{1}{2}}B) := \mathcal{U}_{ad}$ and defined by $(Q^{\frac{1}{2}}B)\ (u)(x) := Q^{\frac{1}{2}}(Bu(x))$, $x \in H$, is compact as an operator from $L^\infty(H; H, \nu)$ to $L^2(H; H, \nu)$, i.e., if $u_n, u \in D(Q^{\frac{1}{2}}B)$, $n \in \mathbb{N}$, such that $u_n \to u$ weakly-star in $L^\infty(H; H, \nu)$ as $n \to \infty$, then $Q^{\frac{1}{2}}Bu_n \to Q^{\frac{1}{2}}Bu$ in $L^2(H; H, \nu)$.

(H3) The operator $Q^{\frac{1}{2}}D$ with domain $\mathcal{F}C_b^1(H)$ is closable from $L^2(H, \nu)$ to $L^2(H; H, \nu)$, and the embedding $W^{1,2}(H, \nu)$ into $L^2(H, \nu)$ is compact.

Here $W^{1,2}(H, \nu)$ is the Sobolev space defined as the completion of $\mathcal{F}C_b^2(H)$ under the norm $\|\varphi\|_{W^{1,2}(H,\nu)} = (\int_H (|\varphi|^2 + |Q^{\frac{1}{2}}D\varphi|_H^2)d\nu)^{\frac{1}{2}}$. Note that $W^{1,2}(H, \nu)$ is a subspace of $L^2(H, \nu)$ if and only if $(Q^{\frac{1}{2}}D, \mathcal{F}C_b^1(H))$ is closable, as an operator from $L^2(H, \nu)$ to $L^2(H; H, \nu)$. In this case we denote its closure again by $Q^{\frac{1}{2}}D$ and by construction its domain is $W^{1,2}(H, \nu)$.

Remark 2.1. *As is well-known* $(H1)$ (ii) *implies that* $(N_0, D(N_0))$ *is dissipative on* $L^2(H, \nu)$*, so by* $(H1)$ (iii) *and the Lumer-Phillips Theorem its closure* $(N_2, D(N_2))$ *generates a* C_0*-semigroup* $P_t^\nu = e^{tN_2}$*,* $t > 0$*, of contractions on* $L^2(H, \nu)$*. Furthermore,* $D(N_0)$ *is dense in* $D(N_2)$ *with respect to the graph norm given by* N_2*.*

Remark 2.2. *The compactness of the embedding of* $W^{1,2}(H,\nu)$ *into* $L^2(H,\nu)$ *is equivalent to the compactness of the semigroup* P_t^ν *for some (equivalently, all)* $t>0$. *See, e.g.,* [19, Theorem 1.2], [24, Theorems 1.1 and 3.1] *and* [25, p.3250]. *The above compact embedding can be also deduced from the Logarithmic-Sobolev inequality, see, e.g.,* [10]. *In particular, Hypothesis* $(H3)$ *holds for the Gaussian invariant measures of the Ornstein-Uhlenbeck process (see* [7]*).*

Below we give one specific example satisfying Hypothesis $(H2)$.

Example Let $Q=Id$, and let $f_j\in L^1(H;H,\nu)$, $g_j\in L^\infty(H;H,\nu)$, $j\geq 1$, be such that

$$B(u)=\sum_{j=1}^{\infty}\int_H\langle u,f_j\rangle d\nu g_j,\quad \forall u\in\mathcal{U}_{ad}, \tag{2.3}$$

and

$$C_B:=\sum_{j=1}^{\infty}\|f_j\|_{L^1(H;H,\nu)}\|g_j\|_{L^\infty(H;H,\nu)}<\infty. \tag{2.4}$$

Then, B satisfies $(H2)$.

In fact, let $u_n,u\in D(B)$, $n\in\mathbb{N}$, be such that $u_n\to u$ weakly-star in $L^\infty(H;H,\nu)$ as $n\to\infty$. Then, for every $N\in\mathbb{N}$,

$$\sum_{j=1}^{N}\int_H\langle u-u_n,f_j\rangle\, d\nu g_j\to 0,\quad in\ L^\infty(H;H,\nu).$$

So, let $\varepsilon>0$. Then, by (2.4) there exists $N\in\mathbb{N}$ such that

$$\sum_{j=N+1}^{\infty}\|f_j\|_{L^1(H;H,\nu)}\|g_j\|_{L^\infty(H;H,\nu)}<\varepsilon.$$

Hence for $\varepsilon>0$,

$$\begin{aligned}\limsup_{n\to\infty}\|Bu-Bu_n\|_{L^\infty(H;H,\nu)}\leq&\limsup_{n\to\infty}\|\sum_{j=1}^{N}\int_H\langle u-u_n,f_j\rangle\, d\nu g_j\|_{L^\infty(H;H,\nu)}\\&+2\rho\sum_{j=N+1}^{\infty}\|f_j\|_{L^1(H;H,\nu)}\|g_j\|_{L^\infty(H;H,\nu)}\\\leq&2\rho\varepsilon.\end{aligned}$$

So, we even have $Bu_n\to Bu$ in $L^\infty(H;H,\nu)$ as $n\to\infty$.

Remark 2.3. *A specific example is where* $B(\mathcal{U}_{ad})$ *is in a finite dimensional subspace of* $L^\infty(H;H,\nu)$.

Actually, in this case, there exists linear independent $g_1,\cdots,g_j\in L^\infty(H;H,\nu)$ *such that* $\|g_i\|_{L^2(H;H,\nu)}=1$, $1\leq i\leq j$ *and* $\{B(\mathcal{U}_{ad})\}\subseteq span\{g_1,\cdots,g_n\}$, *and so, for any* $u\in\mathcal{U}_{ad}$, $B(u)=\sum_{j=1}^n c_jg_j$ *for some* $c_j\in\mathbb{R}$, $1\leq j\leq n$. *Then, we take* $\{\widetilde{g}_j\}_{j=1}^n\subseteq L^2(H;H,\nu)$ *such that* $\langle\widetilde{g}_j,g_k\rangle_{L^2(H;H,\nu)}=\delta_{jk}$, $1\leq j,k\leq n$. *This yields that* $c_j=\langle B(u),\widetilde{g}_j\rangle_{L^2(H;H,\nu)}=\langle u,B^*\widetilde{g}_j\rangle_{L^2(H;H,\nu)}$, *where* B^* *is the dual operator of* B *in* $L^2(H;H,\nu)$. *This implies* (2.3) *with* $f_j=B^*\widetilde{g}_j$.

Under Hypothesis $(H1)$, let $(N_2, D(N_2))$ be the closure of $(N_0, \mathcal{F}C_b^2(H))$ in $L^2(H,\nu)$. Then, ν is an invariant measure for $P_t^\nu = e^{tN_2}$, $t > 0$, i.e.,

$$\int_H P_t^\nu f(x)\nu(dx) = \int_H f(x)\nu(dx), \quad \forall t \geq 0, \ \forall f \in \mathcal{B}_b(H). \tag{2.5}$$

(See, e.g., the proof of [15, Corollary 5.3].)

The essential m-dissipativity of $(N_0^u, \mathcal{F}C_b^2(H))$ can be inherited from the uncontrolled case where $u \equiv 0$, more precisely, from $(H1)$ (iii). This is the content of the following theorem to be proved in Section 3 below.

Theorem 2.4. *Assume Hypothesis $(H1)$ to hold. Then, we have the integration by parts formula*

$$\int_H \varphi N_2 \varphi d\nu = -\frac{1}{2}\int_H |\overline{Q^{\frac{1}{2}}D}\varphi|_H^2 d\nu, \quad \forall \varphi \in D(N_2), \tag{2.6}$$

where $\overline{Q^{\frac{1}{2}}D}$ is the continuous extension of the operator

$$D(N_0) \ni \varphi \mapsto Q^{\frac{1}{2}}D\varphi \in L^2(H; H, \nu)$$

with respect to the N_2-graph norm on $D(N_0)$.

Moreover, for each $u \in \mathcal{U}_{ad}$, the operator

$$N_2^u : D(N_2) \mapsto L^2(H,\nu), \quad N_2^u\varphi(x) := N_2\varphi(x) + \langle Bu(x), \overline{Q^{\frac{1}{2}}D}\varphi(x)\rangle$$

has $\mathcal{F}C_b^2(H)$ as a core and generates a C_0-semigroup $e^{tN_2^u}$ on $L^2(H,\nu)$. Furthermore, for some positive constant $C(T,\rho) > 0$,

$$\sup_{u\in\mathcal{U}_{ad}} (\|e^{tN_2^u}g\|_{C([0,T];L^2(H,\nu))} + (\int_0^T \int_H |\overline{Q^{\frac{1}{2}}D}e^{tN_2^u}g|_H^2 d\nu dt)^{\frac{1}{2}}) \leq C(T,\rho)\|g\|_{L^2(H,\nu)}. \tag{2.7}$$

The first result of this paper is concerned with the existence of optimal controllers for Problem (P^*). It will be proved in Section 3 below.

Theorem 2.5. *(Optimal control of Kolmogorov equations: general case)*

Assume that Hypothesis $(H1)$ holds and, in addition, that either $(H2)$ or $(H3)$ holds. Then, for any $g \in D(N_2)$, there exists at least one optimal control to Problem (P^*).

In particular, in the case where $(H1)$ and $(H3)$ hold, one may take $B = Id$.

Remark 2.6. *We would like to mention that, no continuity or convexity of A and g are assumed in Theorem 2.5 which, however, are the usual conditions for optimal feedback controls even in the finite dimensional case.*

Next, we are concerned with the symmetric case (i.e., $N_2^* = N_2$ on $L^2(H,\nu)$) which arises, in particular, in various applications to gradient systems (see, e.g., [13, 14] and the end of Subsection 5.1 below).

In this case, we are able to obtain optimal controllers for more general objective functions $g \in \mathcal{H} := L^2(H,\nu)$. Moreover, we also obtain the first-order necessary condition of the optimal feedback controllers, in terms of the solutions to Kolmogorov equations and adjoint backward equations.

One nice feature here is that the corresponding Kolmogorov operators are defined in the variational form from $\mathcal{V} := W^{1,2}(H,\nu)$ to $\mathcal{V}'$, where $\mathcal{V}'$ is the dual space of $\mathcal{V}$ in the pairing $(\cdot,\cdot)$ with the pivot space $\mathcal{H} := L^2(H,\nu)$. (Note that, $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$ with dense and continuous embeddings.) This fact enables us to analyze the Kolmogorov equations and the adjoint backward equations in the variational setting.

The following result generalizes Theorem 2.5 in the symmetric case for $g \in L^2(H, \nu)$ and is proved in Section 3.

Theorem 2.7. *(Optimal control of Kolmogorov equations: symmetric case)*
Consider the symmetric case $N_2^* = N_2$ on $L^2(H, \nu)$. Assume $(H1)$ and that $(Q^{\frac{1}{2}}D, \mathcal{F}C_b^1(H))$ is closable from $L^2(H, \nu)$ to $L^2(H; H, \nu)$. In addition, assume that either $(H2)$ or $(H3)$ holds.
Then, for any objective function $g \in L^2(H, \nu)$, there exists an optimal control for Problem (P^*) where φ^u solves the equation in the space $\mathcal{V}'$.
In particular, in the case where $(H3)$ holds, we can take $B = Id$.

In order to identify the optimal feedback controllers, we (in the symmetric case) introduce the adjoint-backward equation corresponding to the Kolmogorov equation (1.3)

$$\begin{aligned} \frac{dp}{dt} &= -N_2 p - G^u p - 1, \\ p(T, x) &= 0. \end{aligned} \tag{2.8}$$

Here, G^u is a bounded operator from $\mathcal{H}$ to $\mathcal{V}'$, defined by

$$_{\mathcal{V}}(\varphi, G^u \psi)_{\mathcal{V}'} := \int_H \langle Bu(x), Q^{\frac{1}{2}} D\varphi(x)\rangle \psi(x)\nu(dx), \quad \varphi \in \mathcal{V}, \ \psi \in \mathcal{H}.$$

Note that $\|G^u\psi\|_{\mathcal{V}'} \le \|u\|_{L^\infty(H;H,\nu)}\|B\|\|\psi\|_{\mathcal{H}}$, where $\|B\|$ denotes the operator norm.
The backward equation (2.8) is understood in the variational sense, and its global well-posedness is part of Theorem 3.3 of Subsection 3.2 below.
Now, we are ready to state the first-order necessary condition for the optimal feedback controllers in the symmetric case (see Subsection 3.2).

Theorem 2.8. *(Necessary condition of optimality: the symmetric case)*
Assume that the conditions of Theorem 2.7 hold and let u_* be an optimal controller for Problem (P^*). Then, we have

$$\int_H \langle B(u - u_*), \int_0^T Q^{\frac{1}{2}} D\varphi_* p_* dt\rangle d\nu \ge 0, \quad \forall u \in \mathcal{U}_{ad}, \tag{2.9}$$

where φ_* and p_* are the solutions to (1.3) and (2.8), respectively, with u_* replacing u.

Below we consider the optimal control problem of the original stochastic differential equation (1.1).
As mentioned in the Introduction, the concept of martingale problem is robust under bounded perturbations which, in particular, fits the optimal control problems considered here. Moreover, the martingale problem is well posed in a quite general setting (e.g., the nonlinearity F may be not continuous or the operator $Q^{\frac{1}{2}}$ is not necessary Hilbert-Schmidt) in which case probabilistic strong solutions may not exist.
Following [3, 15], the martingale problem for (1.1) is defined in Definition 2.9, where we use the notion of "ν-martingale problem" to express its dependence on the probability measure ν on H. Later, however, we shall fix ν as in Hypothesis $(H1)$ for the remaining part of the paper, and for simplicity we shall drop the pre-fix ν again.

Definition 2.9. *Let ν be a Borel probability measure on H. A solution to the ν-martingale problem for $(N_0^u, \mathcal{F}C_b^2(H))$ is a conservative Markov process $\mathbb{M}^u = (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\ge 0}, (X^u(t))_{t\ge 0}, (\mathbb{P}_x)_{x\in H_0})$ on $H_0 := \mathrm{supp}(\nu)$ with continuous sample paths $t \mapsto \langle X^u(t), e_i\rangle$, $i \ge 1$, such that $X^u(0) = x$, $\mathbb{P}_x$-a.s., and the following properties hold:*

(i) *There exist* $M, \varepsilon \in (0,\infty)$ *such that*

$$\int_{H_0} (P_t^u f)^2 d\nu \leq M \int_{H_0} f^2 d\nu, \quad \forall f \in C_b(H), \ t \in (0,\varepsilon), \tag{2.10}$$

where P_t^u, $t \geq 0$, *is the transition semigroup of* $\mathbb{M}^u$.

(ii) *For* ν*-a.e.* $x \in H$, $\mathbb{P}_x$*-a.s.,*

$$\int_0^t |\langle A(X^u(s)), e_i\rangle| ds < \infty, \quad for\ every\ t > 0, \tag{2.11}$$

and for all test functions $\varphi \in \mathcal{F}C_b^2(H)$

$$\varphi(X^u(t)) - \int_0^t (N_0^u \varphi(X^u(s))ds, \quad t \geq 0, \tag{2.12}$$

is an $(\mathscr{F}_t)$*-martingale under* $\mathbb{P}_x$.

For simplicity, we also say that X^u *solves the* ν*-martingale problem for* (1.1).

Uniqueness holds if any two Markov processes which are solutions to the ν*-martingale problem for* (1.1) *have the same finite dimensional distributions for* ν*-a.e. starting points* $x \in H$.

The ν*-martingale problem for* (1.1) *is said to be well posed if existence and uniqueness of solutions hold.*

Remark 2.10. (i) *We note that* (2.11) *holds under Hypothesis* $(H1)$, *by the integrability properties of* $|x|_H$ *and* F *in Hypothesis* $(H1)$ (i) *and* (2.10).

(ii) *The uniqueness of solutions to the martingale problem can be derived from the existence of martingale solutions and the m-dissipativity of Kolmogorov operators in Hypothesis* $(H1)$. *See the arguments in the proof of the uniqueness part of Theorem 2.12.*

In order to consider the optimal control problem of the stochastic equation (1.1), we assume that

(H1)' The ν-martingale problem for (1.1) is well posed in the case $u \equiv 0$.

Remark 2.11. (i) *Hypothesis* $(H1)'$ *can be obtained from* $(H1)$ *if the associated generalized Dirichlet form is quasi-regular and has the local property. Actually, under* $(H1)$, *it is known that (see* [22, p.6]*) the closure* $(N_2, D(N_2))$ *of* $(N_0, \mathcal{F}C_b^2(H))$ *induces a generalized Dirichlet form on* $L^2(H,\nu)$ *as follows*

$$\mathcal{E}(\varphi,\psi) := \begin{cases} -(N_2\varphi,\psi), & \varphi \in D(N_2),\ \psi \in L^2(H,\nu); \\ -(N_2^*\psi,\varphi), & \varphi \in L^2(H,\nu),\ \psi \in D(N_2^*), \end{cases}$$

where $(\ ,\ )$ *denotes the inner product in* $L^2(H,\nu)$, *and* N_2^* *is the dual operator of* N_2. *We see that the condition D3 in* [22, p.78] *is satisfied with* $\mathcal{Y} = \mathcal{F}C_b^2(H)$ *and* $\mathcal{F} = D(N_2)$. *Hence, provided* $\mathcal{E}$ *is quasi-regular,* [22, Theorem IV. 2.2] *yields that there exists a sufficiently regular Markov process* $\mathbb{M}$ *(namely, a* ν*-tight special standard process) with transition semigroup* P_t, $t > 0$, *given by* P_t^ν, $t > 0$, *hence satisfying* (i) *of Definition 2.9 with* $M = 1$ *for all* $t \in (0,\infty)$. *In addition, by* [23, Theorem 3.3], *the sample paths of* $\mathbb{M}$ *are continuous* $\mathbb{P}$*-a.s. for* ν*-a.e.* $x \in H_0$ *if* $\mathcal{E}$ *is local. Moreover, Remark 2.10* (i) *and* [5, Proposition 1.4] *yield that* $\mathbb{M}$ *satisfies the property* (ii) *of Definition 2.9. Thus,* $\mathbb{M}$ *solves the martingale problem for* (1.1) *for* $u \equiv 0$. *The uniqueness can be proved as in Section 4 below.*

As specific examples, we show in Theorems 4.1 and 4.5 that Hypothesis $(H1)'$ *can be implied by the m-dissipativity of Kolmogorov operators in certain situations, based on the theory of (generalized) Dirichlet forms.*

(*ii*) *It follows by* (2.12) *and* $(H1)$ *that for all* $\varphi \in \mathcal{F}C_b^2(H)$ *and for all* $t > 0$

$$H_0 \ni x \mapsto P_t^u \varphi(x) := \mathbb{E}_{\mathbb{P}_x}[\varphi(X^u(t))]$$

is a ν*-version of* $e^{tN_2^u}\varphi \in L^2(H,\nu)$. *Below we shall briefly describe this by saying that "*P_t^u *is given by* $e^{tN_2^u}$*".*

Similarly to Theorem 2.4, the well-posedness of the martingale problem for controlled equations can be inherited from that for the uncontrolled equation.

Theorem 2.12. *Assume that Hypotheses* $(H1)$ *and* $(H1)'$ *hold. Then, for each* $u \in \mathcal{U}_{ad}$*, the martingale problem for* (1.1) *is well posed.*

The main result for optimal control problems of the stochastic equation (1.1) is formulated below.

Theorem 2.13. *(Optimal control for stochastic equations on Hilbert spaces)*
(*i*) *(General case) Assume* $(H1)$ *and* $(H1)'$. *Assume in addition that either* $(H2)$ *or* $(H3)$ *holds. Then, for any objective function* $g \in D(N_2)$*, there exists an optimal control to Problem* (P)*.*
(*ii*) *(Symmetric case) Consider the symmetric case. Assume* $(H1)$ *and that* $(Q^{\frac{1}{2}}D, \mathcal{F}C_b^1(H))$ *is closable from* $L^2(H,\nu)$ *to* $L^2(H;H,\nu)$*. Assume additionally* $(H2)$ *or* $(H3)$*. Then, for any objective function* $g \in L^2(H,\nu)$*, there exists an optimal control to Problem* (P) *and the first-order necessary condition* (2.9) *holds.*
In both cases, when $(H3)$ *holds, one can take* $B = Id$*.*

The remainder of this paper is organized as follows. Section 3 is mainly devoted to the optimal control problem of Kolmogorov equations. We first prove Theorem 2.5 in the general case in Subsection 3.1, while Theorems 2.7 and 2.8 are proved later in Subsection 3.2. Then, in Section 4 we study the optimal control problem of stochastic equations on Hilbert spaces. Finally, Section 5 mainly contains the applications to various stochastic partial differential equations, including stochastic equations in H with singular drifts, stochastic reaction-diffusion equations and stochastic porous media equations.

3. Optimal control of Kolmogorov equations

3.1. **General case.** This subsection is mainly devoted to the proof of Theorem 2.5. To begin with, we first prove Theorem 2.4 for the realted Kolmogorov operators.

Proof of Theorem 2.4. For simplicity, we set $\mathcal{H} := L^2(H,\nu)$. Let us first prove the identity (2.6). Actually, by straightforward computations,

$$N_0(\varphi^2) = 2\varphi N_0\varphi + |Q^{\frac{1}{2}}D\varphi|_H^2, \quad \forall \varphi \in \mathcal{F}C_b^2(H),$$

which along with $(H1)$ (ii) implies that

$$\int_H \varphi N_0 \varphi d\nu = -\frac{1}{2}\int_H |Q^{\frac{1}{2}}D\varphi|_H^2 d\nu, \quad \forall \varphi \in \mathcal{F}C_b^2(H). \tag{3.1}$$

To extend (3.1) to all $\varphi \in D(N_2)$ we observe that the map

$$Q^{\frac{1}{2}}D : \mathcal{F}C_b^2(H) \to L^2(H;H,\nu)$$

is linear and by (3.1) continuous with respect to the graph norm of N_2 on $\mathcal{F}C_b^2(H)$. Since $\mathcal{F}C_b^2(H)$ is dense in $D(N_2)$ with respect to the graph norm of N_2, this map extends uniquely by continuity to $D(N_2)$ and then (2.6) follows by continuity.

We also note that

$$D(N_2) \ni \varphi \mapsto \langle Bu, \overline{Q^{\frac{1}{2}}D}\varphi\rangle \in \mathcal{H} \tag{3.2}$$

is a well-defined bounded linear operator. Actually, for any $\varphi \in D(N_2)$, we take $\{\varphi_n\} \subseteq \mathcal{F}C_b^2(H)$ such that $\varphi_n \to \varphi$ in the N_2-graph norm in $\mathcal{H}$. Then, since $\sup_{x\in H} |u(x)|_H \leq \rho$, by (3.1),

$$\begin{aligned} \int |\langle Bu, \overline{Q^{\frac{1}{2}}D}(\varphi_n - \varphi_m)\rangle|^2 d\nu \leq& \rho^2\|B\|^2 \int |\overline{Q^{\frac{1}{2}}D}(\varphi_n - \varphi_m)|^2 d\nu \\ =& -2\rho^2\|B\|^2 \int (\varphi_n - \varphi_m) N_0(\varphi_n - \varphi_m) d\nu \to 0, \end{aligned} \tag{3.3}$$

as $n, m \to \infty$. This implies that $\{\langle Bu, \overline{Q^{\frac{1}{2}}D}\varphi_n\rangle\}$ is a Cauchy sequence in $\mathcal{H}$ and so yields the claim above.

In order to prove that $\mathcal{F}C_b^2(H)$ is a core of $(N_2^u, D(N_2))$, it suffices to prove that the graph norms of N_2 and N_2^u are equivalent, i.e., there exists $C > 0$ such that for any $\varphi \in \mathcal{F}C_b^2(H)$,

$$C^{-1}(\|N_2\varphi\|_{\mathcal{H}}^2 + \|\varphi\|_{\mathcal{H}}^2) \leq (\|N_2^u\varphi\|_{\mathcal{H}}^2 + \|\varphi\|_{\mathcal{H}}^2) \leq C(\|N_2\varphi\|_{\mathcal{H}}^2 + \|\varphi\|_{\mathcal{H}}^2). \tag{3.4}$$

For this purpose, we note that for any $\lambda > 0$, similarly to (3.3),

$$\|\langle Bu, \overline{Q^{\frac{1}{2}}D}\varphi\rangle\|_{\mathcal{H}}^2 \leq -2\rho^2\|B\|^2 \int_H \varphi N_2\varphi d\nu \leq 2\rho^2\|B\|^2(\frac{1}{\lambda}\|N_2\varphi\|_{\mathcal{H}}^2 + \lambda\|\varphi\|_{\mathcal{H}}^2), \tag{3.5}$$

which immediately yields the second inequality of (3.4). Moreover, taking λ large enough such that $2\rho^2\|B\|^2/\lambda \leq 1/4$ we also obtain the first inequality of (3.4).

The fact that $(N_2^u, D(N_2))$ generates a C_0-semigroup $e^{tN_2^u}$ follows from the essential m-dissipativity of $(N_0^u, \mathcal{F}C_b^2(H))$ on $\mathcal{H}$. To this end, we note that for λ large enough, for any $f \in \mathcal{H}$, the equation

$$\lambda\varphi - N_2\varphi - \langle Bu, \overline{Q^{\frac{1}{2}}D}\varphi\rangle = f$$

has the unique solution

$$\varphi = R_\lambda((I - T_\lambda)^{-1}f),$$

where R_λ is the resolvent of N_2, i.e., $R_\lambda = (\lambda - N_2)^{-1}$, and the operator $T_\lambda : L^2(H,\nu) \to L^2(H,\nu)$ is defined by $T_\lambda\psi = \langle Bu, \overline{Q^{\frac{1}{2}}D}R_\lambda\psi\rangle$. (Note that, $\|T_\lambda\psi\|_{\mathcal{H}} \leq \frac{1}{2}\|\psi\|_{\mathcal{H}}$ when λ is large enough, hence $(I - T_\lambda)^{-1}$ is well-defined and $(I - T_\lambda)^{-1} \in L(\mathcal{H})$.) By the essential m-dissipativity of $(N_0, \mathcal{F}C_b^2(H))$, there exists a sequence $(\varphi_n) \subseteq \mathcal{F}C_b^2(H)$ such that $(\lambda - N_2)\varphi_n \to (I - T_\lambda)^{-1}f$, as $n \to \infty$. This yields

$$\begin{aligned} (\lambda - N_2)\varphi_n - \langle Bu(x), \overline{Q^{\frac{1}{2}}D}\varphi_n\rangle =& (\lambda - N_2)\varphi_n - T_\lambda(\lambda - N_2)\varphi_n \\ \to& (I - T_\lambda)^{-1}f - T_\lambda(I - T_\lambda)^{-1}f = f, \end{aligned}$$

as $n \to \infty$, which implies that the image of $\lambda - N_0^u$ is dense in $\mathcal{H}$. Thus, $(N_0^u, \mathcal{F}C_b^2(H))$ is essentially m-dissipative on $\mathcal{H}$ and so generates a semigroup $e^{tN_2^u}$ on $\mathcal{H}$, $t \geq 0$.

Regarding (2.7), by (2.6) and Cauchy's inequality, for $\varphi := e^{tN_2^u} g$,

$$\begin{aligned}\frac{1}{2}\partial_t\|\varphi\|^2_{\mathcal{H}} =&(N_2\varphi,\varphi) + (\langle Bu, \overline{Q^{\frac{1}{2}}D}\varphi\rangle, \varphi)\\ \leq& -\frac{1}{2}\int_H |\overline{Q^{\frac{1}{2}}D}\varphi|^2_H d\nu + \rho\|B\|\int_H |\overline{Q^{\frac{1}{2}}D}\varphi|_H|\varphi|d\nu\\ \leq& -\frac{1}{4}\int_H |\overline{Q^{\frac{1}{2}}D}\varphi|^2_H d\nu + 4\rho^2\|B\|^2\|\varphi\|^2_{\mathcal{H}},\end{aligned}$$

which along with Gronwall's inequality implies (2.7).

Therefore, the proof of Theorem 2.4 is complete. □

Proof of Theorem 2.5. We set $\mathcal{H} := L^2(H,\nu)$ and

$$\Phi(u) := \int_0^T\int_H \psi^u(t,x)\nu(dx)dt,$$

where ψ^u is the solution to (1.3) corresponding to $u \in \mathcal{U}_{ad}$.

Let $I_* := \inf\{\Phi(u) : u \in \mathcal{U}_{ad}\}$ and take a sequence $\{u_n\} \subseteq \mathcal{U}_{ad}$, such that $I_* \leq \Phi(u_n) \leq I_* + \frac{1}{n}$, $n \geq 1$.

Let $\varphi_n := e^{-(4\rho^2\|B\|^2+1)t}\psi^{u_n}$, $n \geq 1$. Then, we have

$$\Phi(u_n) = \int_0^T\int_H e^{(4\rho^2\|B\|^2+1)t}\varphi_n d\nu dt, \tag{3.6}$$

and by (1.3) φ_n solves

$$\begin{aligned}&\frac{d\varphi_n}{dt} = \mathcal{N}^{u_n}\varphi_n, \quad t \in (0,T),\\ &\varphi_n(0) = g,\end{aligned} \tag{3.7}$$

where the operator $\mathcal{N}^{u_n} : D(N_2) \to \mathcal{H}$ is defined by

$$\mathcal{N}^{u_n}\psi := N_2\psi + \langle Bu_n, \overline{Q^{\frac{1}{2}}D}\psi\rangle - (4\rho^2\|B\|^2+1)\psi, \quad \psi \in D(N_2). \tag{3.8}$$

By Theorem 2.4, we see that also $(\mathcal{N}^{u_n}, D(N_2))$ generates a C_0-semigroup $e^{t\mathcal{N}^{u_n}}$ on $\mathcal{H}$, namely $e^{t\mathcal{N}^{u_n}} = e^{-(4\rho^2\|B\|^2+1)t}e^{tN_2^{u_n}}$, where $e^{tN_2^{u_n}}$ is given by Theorem 2.4, and

$$\sup_{n\geq 1}\|\varphi_n\|^2_{C([0,T];\mathcal{H})} + \sup_{n\geq 1}\int_0^T\int |\overline{Q^{\frac{1}{2}}D}\varphi_n(s)|^2_H d\nu ds \leq C\|g\|^2_{\mathcal{H}}. \tag{3.9}$$

Similarly, by (3.4) and (3.9),

$$\sup_{n\geq 1}\|\mathcal{N}^{u_n}\varphi_n\|_{C[0,T;\mathcal{H}]} = \sup_{n\geq 1}\|e^{t\mathcal{N}^{u_n}}\mathcal{N}^{u_n}g\|_{C[0,T;\mathcal{H}]} \leq C(T,\rho)(\|N_2 g\|_{\mathcal{H}} + \|g\|_{\mathcal{H}}). \tag{3.10}$$

Hence, along a subsequence, again denoted $\{n\}$, we have

$$u_n \to u_*, \ \ weakly-star\ in\ L^\infty(H;H,\nu), \tag{3.11}$$

$$\varphi_n \to \varphi_*, \ \ weakly\ in\ L^2(0,T;\mathcal{H}), \tag{3.12}$$

$$\mathcal{N}^{u_n}\varphi_n \to \eta, \ \ weakly\ in\ L^2(0,T;\mathcal{H}). \tag{3.13}$$

Note that, since $t \mapsto e^{(4\rho^2\|B\|^2+1)t} \in L^2(0,T;\mathcal{H})$, we apply (3.12) to pass to the limit in (3.6) to obtain

$$I_* = \lim_{n\to\infty}\Phi(u_n) = \int_0^T\int_H e^{4\rho^2\|B\|^2 t}\varphi_* d\nu dt. \tag{3.14}$$

Moreover, by (3.7), (3.12) and (3.13), φ_* satisfies the equation

$$\varphi_*(t) = g + \int_0^t \eta(s)ds, \quad for\ dt - a.e.\ t \in [0,T]. \tag{3.15}$$

Now it remains to prove that

$$\eta(t) = \mathcal{N}^{u_*}\varphi_*(t), \ for\ a.e.\ t \in (0,T). \tag{3.16}$$

Note that, by (2.6),

$$(\mathcal{N}^{u_n}v, v)_{\mathcal{H}} \le 0, \quad \forall v \in D(N_2), \tag{3.17}$$

For simplicity, we set $\mathcal{H}_t := L^2(0,t;\mathcal{H})$ with the inner produce $(\cdot,\cdot)_{\mathcal{H}_t}$ below. Then, (3.17) yields that, for any positive function $h \in L^\infty(0,T)$ and any $\psi \in \mathcal{M}$, where $\mathcal{M}$ denotes the space of all linear combinations of functions of the form fv, where $f \in L^\infty(0,T)$ and $v \in \mathcal{F}C_b^2(H)$,

$$\begin{aligned}
0 \ge& \int_0^T h(t)(\mathcal{N}^{u_n}(\varphi_n - \psi), \varphi_n - \psi)_{\mathcal{H}_t}dt \\
=& \int_0^T h(t)(\mathcal{N}^{u_n}\varphi_n, \varphi_n)_{\mathcal{H}_t}dt - \int_0^T h(t)(\mathcal{N}^{u_n}\varphi_n, \psi)_{\mathcal{H}_t}dt \\
&- \int_0^T h(t)(\mathcal{N}^{u_n}\psi, \varphi_n)_{\mathcal{H}_t}dt + \int_0^T h(t)(\mathcal{N}^{u_n}\psi, \psi)_{\mathcal{H}_t}dt \\
=:& K_{1,n} - K_{2,n} - K_{3,n} + K_{4,n}.
\end{aligned} \tag{3.18}$$

Below we treat $K_{i,n}$ separately, $1 \le i \le 4$.

For $K_{1,n}$, by equation (3.7),

$$\frac{1}{2}\|\varphi_n(t)\|^2_{\mathcal{H}} = (\mathcal{N}^{u_n}\varphi_n, \varphi_n)_{\mathcal{H}_t} + \frac{1}{2}\|g\|^2_{\mathcal{H}}, \quad 0 < t < T.$$

Then, multiplying both sides by $h(t)$ and integrating over $[0,T]$ we get

$$\frac{1}{2}\int_0^T h(t)\|\varphi_n(t)\|^2_{\mathcal{H}}dt = \int_0^T h(t)(\mathcal{N}^{u_n}\varphi_n, \varphi_n)_{\mathcal{H}_t}dt + \frac{1}{2}\|g\|^2_{\mathcal{H}}\int_0^T h(t)dt. \tag{3.19}$$

Similarly, we infer from the equation (3.15) that

$$\frac{1}{2}\int_0^T h(t)\|\varphi_*(t)\|^2_{\mathcal{H}}dt = \int_0^T h(t)(\eta, \varphi_*)_{\mathcal{H}_t}dt + \frac{1}{2}\|g\|^2_{\mathcal{H}}\int_0^T h(t)dt. \tag{3.20}$$

Moreover, by (3.12),

$$\varphi_n h^{\frac{1}{2}} \to \varphi_* h^{\frac{1}{2}}, \quad weakly\ in\ L^2(0,T;\mathcal{H}), \tag{3.21}$$

which implies that

$$\liminf_{n\to\infty}\int_0^T h(t)\|\varphi_n(t)\|^2_{\mathcal{H}}dt \ge \int_0^T h(t)\|\varphi_*(t)\|^2_{\mathcal{H}}dt. \tag{3.22}$$

Thus, we obtain from (3.19)-(3.22) that

$$\liminf_{n\to\infty} K_{1,n} \ge \int_0^T h(t)(\eta, \varphi_*)_{\mathcal{H}_t}dt. \tag{3.23}$$

Moreover, in order to pass to the limit in $K_{2,n}$, we note that, by (3.13),

$$(\mathcal{N}^{u_n}\varphi_n, \psi)_{\mathcal{H}_t} \to (\eta, \psi)_{\mathcal{H}_t}, \quad \forall t \in (0,T].$$

Taking into account (3.10), we have for any $t \in (0,T]$,

$$|(\mathcal{N}^{u_n}\varphi_n, \psi)_{\mathcal{H}_t}| \leq \|\mathcal{N}^{u_n}\varphi_n\|_{L^2(0,T;\mathcal{H})}\|\psi\|_{L^2(0,T;\mathcal{H})} \leq C(T,\rho)(\|N_2 g\|_{\mathcal{H}} + \|g\|_{\mathcal{H}})\|\psi\|_{L^2(0,T;\mathcal{H})} < \infty.$$

Then, the dominated convergence theorem yields

$$\lim_{n\to\infty} K_{2,n} = \int_0^T h(t)(\eta,\psi)_{\mathcal{H}_t}dt. \tag{3.24}$$

Now, we treat the term $K_{3,n}$. We expand

$$(\mathcal{N}^{u_n}\psi, \varphi_n)_{\mathcal{H}_t} = (N_2\psi, \varphi_n)_{\mathcal{H}_t} + (\langle Bu_n, \overline{Q^{\frac{1}{2}}D\psi}\rangle, \varphi_n)_{\mathcal{H}_t} - 4\rho^2\|B\|^2(\psi,\varphi_n)_{\mathcal{H}_t}. \tag{3.25}$$

By (3.12),

$$\lim_{n\to\infty}(N_2\psi, \varphi_n)_{\mathcal{H}_t} = (N_2\psi, \varphi_*)_{\mathcal{H}_t}, \quad \lim_{n\to\infty}(\psi,\varphi_n)_{\mathcal{H}_t} = (\psi,\varphi_*)_{\mathcal{H}_t}. \tag{3.26}$$

Concerning the second term on the right-hand side of (3.25), we claim that

$$(\langle Bu_n, \overline{Q^{\frac{1}{2}}D\psi}\rangle, \varphi_n)_{\mathcal{H}_t} \to (\langle Bu_*, \overline{Q^{\frac{1}{2}}D\psi}\rangle, \varphi_*)_{\mathcal{H}_t}. \tag{3.27}$$

In order to prove (3.27), we note that

$$\begin{aligned}(\langle Bu_n, \overline{Q^{\frac{1}{2}}D\psi}\rangle, \varphi_n)_{\mathcal{H}_t} &- (\langle Bu_*, \overline{Q^{\frac{1}{2}}D\psi}\rangle, \varphi_*)_{\mathcal{H}_t}\\ =&(\langle Bu_n - Bu_*, \overline{Q^{\frac{1}{2}}D\psi}\rangle, \varphi_*)_{\mathcal{H}_t} + \int_0^t\int_H \langle Bu_n, (\varphi_n(s)-\varphi_*(s))\overline{Q^{\frac{1}{2}}D\psi(s)}\rangle d\nu ds.\end{aligned} \tag{3.28}$$

For the first term on the right-hand side of (3.28), Fubini's theorem yields

$$\begin{aligned}(\langle Bu_n - Bu_*, \overline{Q^{\frac{1}{2}}D\psi}\rangle, \varphi_*)_{\mathcal{H}_t} &= \int_0^t\int_H \langle Bu_n - Bu_*, \varphi_*(s)\overline{Q^{\frac{1}{2}}D\psi(s)}\rangle d\nu ds\\ &= \int_H \langle Bu_n - Bu_*, \int_0^t \varphi_*(s)\overline{Q^{\frac{1}{2}}D\psi(s)}ds\rangle d\nu.\end{aligned}$$

But the latter term converges to zero, since $B \in L(L^\infty(H;H,\nu))$ and thus $Bu_n \to Bu_*$ as $n\to\infty$ weakly-star in $L^\infty(H;H,\nu)$ and since for all $t\in[0,T]$,

$$\int_0^t \varphi_*(s)\overline{Q^{\frac{1}{2}}D\psi(s)}ds \in L^1(H;H,\nu).$$

Now, let us treat the second term on the right-hand side of (3.28). If $(H2)$ holds, then it follows that $Bu_n \to Bu_*$ in $L^2(H;H,\nu)$ as $n\to\infty$, hence the second term in (3.28) obviously converges to zero as $n\to\infty$.

Now assume that $(H3)$ holds and set $\mathcal{V} := W^{1,2}(H,\nu)$. We claim that

$$\varphi_n \to \varphi_*, \quad \textit{strongly in } L^2(0,T;\mathcal{H}). \tag{3.29}$$

Actually, from equation (1.3) it follows that $\frac{d\varphi_n}{dt}$ is bounded in $L^2(0,T;\mathcal{H})$, and by (3.9), $\{\varphi_n\}$ is also bounded in $L^2(0,T;\mathcal{V})$. Since by Hypothesis $(H3)$, $\mathcal{V}$ is compactly embedded in $\mathcal{H}$, by virtue of the Aubin-Lions compactness theorem (see e.g. [2, Theorem 1.2]) we obtain (3.29), as claimed.

Hence, the second term on the right-hand side of (3.28) also converges if $(H3)$ holds.

Thus, we obtain (3.27), as claimed. (Note that, Hypothesis $(H2)$ or $(H3)$ are only used here. When $(H3)$ holds one may take $B = Id$.)

This along with (3.25) and (3.26) yields

$$\begin{aligned}\lim_{n\to\infty}(\mathcal{N}^{u_n}\psi,\varphi_n)_{\mathcal{H}_t} =&(N_2\psi,\varphi_*)_{\mathcal{H}_t}+(\langle Bu_*,\overline{Q^{\frac{1}{2}}D\psi}\rangle,\varphi_*)_{\mathcal{H}_t}-4\rho^2\|B\|^2(\psi,\varphi_*)_{\mathcal{H}_t}\\ =&(\mathcal{N}^{u_*}\psi,\varphi_*)_{\mathcal{H}_t}.\end{aligned}\tag{3.30}$$

Taking into account that

$$\sup_{t\in[0,T]}|(\mathcal{N}^{u_n}\psi,\varphi_n)_{\mathcal{H}_t}|\leq C\sup_{j\geq1}\|\varphi_n\|_{L^2(0,T;\mathcal{H})}<\infty,$$

we may apply the dominated convergence theorem to obtain

$$\lim_{n\to\infty}K_{3,n}=\int_0^T h(t)(\mathcal{N}^{u_*}\psi,\varphi_*)_{\mathcal{H}_t}dt.\tag{3.31}$$

Finally, arguing as in the proof of (3.31) we have

$$\lim_{n\to\infty}K_{4,n}=\int_0^T h(t)(\mathcal{N}^{u_*}\psi,\psi)_{\mathcal{H}_t}dt.\tag{3.32}$$

Thus, combing together (3.18), (3.23), (3.24), (3.31) and (3.32) we conclude that for any positive functions $h\in L^\infty(0,T)$ and any $\psi\in\mathcal{M}$,

$$0\geq\int_0^T h(t)(\eta-\mathcal{N}^{u_*}\psi,\varphi_*-\psi)_{\mathcal{H}_t}dt.\tag{3.33}$$

Now, consider the Hilbert space $L^2(0,T;D(N_2))$, where $D(N_2)$ is equipped with the graph norm given by $\mathcal{N}^{u_*}$. Then, if $\psi\in L^2(0,T;D(N_2))$ such that

$$\int_0^T\langle\psi,fv\rangle_{D(N_2)}\,dt=0,\quad\forall f\in L^\infty(0,T),\ v\in\mathcal{F}C_b^2(H),$$

since $\mathcal{F}C_b^2(H)$ is dense in $D(N_2)$, it follows that $\psi=0$, i.e., $\mathcal{M}$ is dense in $L^2(0,T;D(N_2))$.

Hence, for every $\psi\in L^2(0,T;D(N_2))$, there exist $\psi_n\in\mathcal{M}$, such that $\psi_n\to\psi$ in $L^2(0,T;D(N_2))$ as $n\to\infty$. Then, we have

$$(\eta-\mathcal{N}^{u_*}\psi,\varphi_*-\psi)_{\mathcal{H}_t}=\lim_{n\to\infty}(\eta-\mathcal{N}^{u_*}\psi_n,\varphi_*-\psi_n)_{\mathcal{H}_t}$$

and for all $t\in[0,T]$,

$$\begin{aligned}|(\eta-\mathcal{N}^{u_*}\psi_n,\varphi_*-\psi_n)_{\mathcal{H}_t}|\leq&\|\eta\|_{L^2(0,T;\mathcal{H})}(\|\varphi_*\|_{L^2(0,T;\mathcal{H})}+\sup_{n\geq1}\|\psi_n\|_{L^2(0,T;\mathcal{H})})\\ &+\sup_{n\geq1}\|\mathcal{N}^{u_*}\psi_n\|_{L^2(0,T;\mathcal{H})}(\|\varphi_*\|_{L^2(0,T;\mathcal{H})}+\sup_{n\geq1}\|\psi_n\|_{L^2(0,T;\mathcal{H})})\end{aligned}$$

Hence, (3.33) extends to all $\psi\in L^2(0,T;D(N_2))$ and, if $\varphi_*\in L^2(0,T;D(N_2))$, we may particularly take $\psi:=\varphi_*(t)-\varepsilon f(t)v$, where $\varepsilon\in(0,1)$, $f\in L^\infty(0,T)$ and $v\in\mathcal{F}C_b^2(H)$. Dividing by ε and then letting ε to 0, we arrive at

$$0\geq\int_0^T h(t)(\eta-\mathcal{N}^{u_*}\varphi_*,fv)_{\mathcal{H}_t}dt.\tag{3.34}$$

Therefore, since $h,f\in L^\infty(0,T)$ and $v\in\mathcal{F}C_b^2(H)$ are arbitrary, we obtain (3.16), if $\varphi_*\in L^2(0,T;D(N_2))$.

To show that $\varphi_*\in L^2(0,T;D(N_2))$, we first note that because of (3.12) we only have to prove that

$$\sup_{n\geq1}\|\mathcal{N}^{u_*}\varphi_n\|_{L^2(0,T;\mathcal{H})}<\infty.\tag{3.35}$$

Indeed, then there exists a subsequence of φ_n, $n \in \mathbb{N}$, along which $\mathcal{N}^{u_*}\varphi_n$, $n \in \mathbb{N}$, converges weakly in $L^2(0,T;\mathcal{H})$. Hence, taking into account (3.12) we can select a subsequence φ_{n_l}, $l \in \mathbb{N}$, such that the following two Cesaro-mean convergence strongly in $L^2(0,T;\mathcal{H})$ (by the Banach-Saks Theorem),

$$\frac{1}{N}\sum_{l=1}^{N}\varphi_{n_l} \to \varphi_*, \quad \textit{in } L^2(0,T;\mathcal{H}), \ \textit{as } N \to \infty, \tag{3.36}$$

and

$$\mathcal{N}^{u_*}(\frac{1}{N}\sum_{l=1}^{N}\varphi_{n_l}), \quad N \in \mathbb{N}, \ \textit{converges in } L^2(0,T;\mathcal{H}), \tag{3.37}$$

so $\varphi_* \in L^2(0,T;D(N_2))$ by completeness.

But we have by (3.8), (3.9), (3.13) and since $\sup_{n\geq 1}\|u_n\|_{L^\infty(H;H,\nu)} \leq \rho$, that

$$\sup_{n\geq 1}\|N_2\varphi_n\|_{L^2(0,T,\mathcal{H})} < \infty.$$

Hence, it follows by (3.9) and (3.8) again that (3.35) holds.

Therefore, the proof of Theorem 2.5 is complete. □

3.2. **Symmetric case.** Throughout this subsection, we assume the self-adjointness of the Kolmogorov operator N_2 and the closability of the operator $Q^{\frac{1}{2}}D$. Precisely, we assume that $N_2^* = N_2$ on $\mathcal{H} := L^2(H,\nu)$, and the operator $Q^{\frac{1}{2}}D$ with domain $\mathcal{F}C_b^1(H)$ is closable from $L^2(H,\nu)$ to $L^2(H;H,\nu)$.

A nice feature in the symmetric case is that the Kolmogorov operators are variational from $\mathcal{V}$ to $\mathcal{V}'$. This enables us to analyze the Kolmogorov equation (1.3) and the backward equation (2.8) in the variational setting.

We first extend the integration by parts formula (2.6) in the symmetric case.

Lemma 3.1. *Consider the symmetric case. Assume (H1) and that $(Q^{\frac{1}{2}}D, \mathcal{F}C_b^1(H))$ is closable from $L^2(H,\nu)$ to $L^2(H;H,\nu)$. Then,*

$$\int_H \psi\overline{N}_2\varphi d\nu = -\frac{1}{2}\int_H \langle Q^{\frac{1}{2}}D\varphi, Q^{\frac{1}{2}}D\psi\rangle d\nu, \quad \varphi,\psi \in \mathcal{V} := W^{1,2}(H,\nu), \tag{3.38}$$

where $\overline{N}_2$ is the continuous extension of the operator

$$\mathcal{F}C_b^2(H) \ni \varphi \mapsto N_2\varphi \in \mathcal{V}' \tag{3.39}$$

with respect to the $\mathcal{V}$-norm on $\mathcal{F}C_b^2(H)$.

Proof. Since $N_2^* = N_2$, using (3.1) and polarization we see that (3.38) is valid for all $\varphi,\psi \in \mathcal{F}C_b^2(H)$. Note that, by (3.38), the map (3.39) is linear and continuous with respect to the $\mathcal{V}$-norm on $\mathcal{F}C_b^2(H)$. Hence, since $\mathcal{F}C_b^2(H)$ is dense in $\mathcal{V}$, the map (3.39) can be extended uniquely by continuity to $\mathcal{V}$ and (3.38) follows by continuity. □

Below we still denote $\overline{N}_2$ by N_2 in the symmetric case for simplicity.

In order to obtain the global well-posedness for the Kolmogorov equation (1.3), we use $\widetilde{\varphi} := e^{-(4\rho^2\|B\|^2+1)t}\varphi$ to rewrite (1.3) as follows

$$\begin{aligned} &\frac{d\widetilde{\varphi}}{dt} = \mathcal{N}^u\widetilde{\varphi}, \\ &\widetilde{\varphi}(0) = g, \end{aligned} \tag{3.40}$$

where $\mathcal{N}^u$ is as in (3.8), i.e., $\mathcal{N}^u = N_2^u - (4\rho^2\|B\|^2+1)$.

Similarly, for $\widetilde{p}(t) := e^{-(4\rho^2\|B\|^2+1)t}p(T-t)$, we have from (2.8) that

$$\begin{aligned}\frac{d\widetilde{p}}{dt} &= \widetilde{\mathcal{N}}^u\widetilde{p} + e^{-(4\rho^2\|B\|^2+1)t},\\ \widetilde{p}(0) &= 0,\end{aligned} \tag{3.41}$$

where $\widetilde{\mathcal{N}}^u := N_2 + G^u - (4\rho^2\|B\|^2+1)$ with G^u as in (2.8).

The properties of operators $\mathcal{N}^u$ and $\widetilde{\mathcal{N}}^u$ are collected in the result below.

Proposition 3.2. *Under the conditions of Lemma 3.1, we have*

$$\sup_{u\in\mathcal{U}_{ad}} (\|\mathcal{N}^u\varphi\|_{\mathcal{V}'} + \|\widetilde{\mathcal{N}}^u\varphi\|_{\mathcal{V}'}) \le C(T,\rho)\|\varphi\|_{\mathcal{V}}, \quad \forall\varphi\in\mathcal{V}, \tag{3.42}$$

for some $C(T,\rho)>0$, and for any $\varphi\in\mathcal{V}$,

$$_{\mathcal{V}}(\varphi, \mathcal{N}^u\varphi)_{\mathcal{V}'} + _{\mathcal{V}}(\varphi, \widetilde{\mathcal{N}}^u\varphi)_{\mathcal{V}'} \le -\frac{1}{2}\|\varphi\|^2_{\mathcal{V}}. \tag{3.43}$$

Proof. Let us first consider the operator $\mathcal{N}^u$. By (3.38), for any $\varphi,\psi\in\mathcal{V}$,

$$\begin{aligned}_{\mathcal{V}}\langle\psi, \mathcal{N}^u\varphi\rangle_{\mathcal{V}'} = &-\frac{1}{2}\int_H \langle Q^{\frac12}D\varphi, Q^{\frac12}D\psi\rangle d\nu + \int_H \langle Bu, Q^{\frac12}D\varphi\rangle\psi d\nu\\ &- 4\rho^2\|B\|^2\int\varphi\psi d\nu,\end{aligned} \tag{3.44}$$

which along with Hölder's inequality implies immediately that for some $C>0$

$$\|\mathcal{N}^u\varphi\|_{\mathcal{V}'} \le C\|\varphi\|_{\mathcal{V}}. \tag{3.45}$$

Moreover, by Cauchy's inequality and $ab\le a^2+b^2$, we have

$$|\int_H \langle Bu, Q^{\frac12}D\varphi\rangle\varphi d\nu| \le \rho\|B\|\int_H |Q^{\frac12}D\varphi|_H|\varphi|d\nu \le \frac14\|\varphi\|^2_{\mathcal{V}} + 4\rho^2\|B\|^2\|\varphi\|^2_{\mathcal{H}}.$$

Plugging this into (3.44) with φ replacing ψ we obtain

$$_{\mathcal{V}}(\varphi, \mathcal{N}^u\varphi)_{\mathcal{V}'} \le -\frac14\|\varphi\|^2_{\mathcal{V}}. \tag{3.46}$$

Concerning the operator $\widetilde{\mathcal{N}}^u$, we first see that

$$\sup_{u\in\mathcal{U}_{ad}} \|G^u\psi\|_{\mathcal{V}'} \le \|u\|_{L^\infty(H;H,\nu)}\|B\|\|\psi\|_{\mathcal{H}}. \tag{3.47}$$

Actually, by Hölder's inequality,

$$\begin{aligned}|_{\mathcal{V}}(\varphi, G^u\psi)_{\mathcal{V}'}| = \left|\int_H \langle Bu, Q^{\frac12}D\varphi\rangle\psi d\nu\right| &\le (\int_H |\langle Bu, Q^{\frac12}D\varphi\rangle|^2 d\nu)^{\frac12}(\int_H |\psi|^2 d\nu)^{\frac12}\\ &\le \|u\|_{L^\infty(H;H,\nu)}\|B\|\|\varphi\|_{\mathcal{V}}\|\psi\|_{\mathcal{H}},\end{aligned}$$

which yields (3.47), as claimed.

Then, arguing as above and using (2.6) and (3.47) we have that, for some $C>0$, for any $\varphi\in\mathcal{V}$,

$$\|\widetilde{\mathcal{N}}^u\varphi\|_{\mathcal{V}'} \le C\|\varphi\|_{\mathcal{V}}, \tag{3.48}$$

and

$$_{\mathcal{V}}(\varphi, \widetilde{\mathcal{N}}^u\varphi)_{\mathcal{V}'} \le -\frac14\|\varphi\|^2_{\mathcal{V}}. \tag{3.49}$$

Thus, putting together the estimates above we obtain (3.42) and (3.43). □

As a consequence of Proposition 3.2 and a classical result due to J.L. Lions (see [20] or [2, Theorem 4.10]), we obtain that there exist unique solutions $\widetilde{\varphi}$ and $\widetilde{p}$ to (3.40) and (3.41), respectively, and so do the equations (1.3) and (2.8).

Theorem 3.3. *Under the condition of Lemma 3.1. Let $u \in \mathcal{U}_{ad}$. Then, we have*

(i) For any objective function $g \in L^2(H,\nu)$, there exists a unique solution φ^u to the Kolmogorov equation (1.3) *such that $\varphi^u \in C([0,T];\mathcal{H}) \cap L^2(0,T;\mathcal{V})$, $\frac{d}{dt}\varphi^u \in L^2(0,T;\mathcal{V}')$, and*

$$\frac{d\varphi^u}{dt}(t) = N_2\varphi^u(t) + \langle Q^{\frac{1}{2}}Bu, D\varphi^u\rangle(t), \quad a.e.\ t \in (0,T),\ \varphi^u(0) = g, \tag{3.50}$$

where $\frac{d}{dt}$ is taken in the strong topology of $\mathcal{V}'$, or equivalently in $\mathscr{D}'(0,T;\mathcal{V}')$.

(ii) There exists a unique solution p^u to the backward equation (2.8) *such that $p^u \in C([0,T];\mathcal{H}) \cap L^2(0,T;\mathcal{V})$, $\frac{d}{dt}p^u \in L^2(0,T;\mathcal{V}')$, and*

$$\frac{dp^u}{dt}(t) = -N_2p^u(t) - G^up^u(t) - 1, \quad a.e.\ t \in (0,T),\ p^u(T) = 0. \tag{3.51}$$

The following result contains the uniform estimates and the continuity dependence on controllers of the solutions to Kolmogorov equations.

Theorem 3.4. *Consider the situations as in Lemma 3.1. We have*

(i) For any two solutions φ_1, φ_2 to (1.3) *corresponding to the initial data g_1 and g_2, respectively, we have*

$$\|\varphi_1 - \varphi_2\|_{C([0,T];\mathcal{H})} + \|\varphi_1 - \varphi_2\|_{L^2(0,T;\mathcal{V})} \le C(\rho,T)\|g_1 - g_2\|_{\mathcal{H}}, \tag{3.52}$$

where $C(\rho,T)$ is independent of $u \in \mathcal{U}_{ad}$. In particular, one has

$$\sup_{u\in\mathcal{U}_{ad}} \|\varphi^u\|^2_{C([0,T];\mathcal{H})} + \sup_{u\in\mathcal{U}_{ad}} \int_0^T \int_H |Q^{\frac{1}{2}}D\varphi^u|^2_H d\nu dt \le C(\rho,T) < \infty. \tag{3.53}$$

(ii) For any $u, \widetilde{u} \in \mathcal{U}_{ad}$ and $\lambda \in [0,1]$, set $v := \widetilde{u} - u$. Then, as $\lambda \to 0$,

$$\|\varphi^{u+\lambda v} - \varphi^u\|_{C([0,T];\mathcal{H})} + \int_0^T \int_H |Q^{\frac{1}{2}}D(\varphi^{u+\lambda v} - \varphi^u)|^2_H d\nu dt \to 0. \tag{3.54}$$

Proof. (i) The estimate (3.52) follows from (1.3) and similar arguments as in the proof of (2.7).

(ii) We replace φ by $\varphi_\lambda := e^{-(4\rho^2\|B\|^2+1)t}(\varphi^{u+\lambda v} - \varphi^u)$ in (1.3) to obtain

$$\frac{d}{dt}\varphi_\lambda = \mathcal{N}^u\varphi_\lambda + \lambda e^{-(4\rho^2\|B\|^2+1)t}\langle Bv, Q^{\frac{1}{2}}D\varphi^{u+\lambda v}\rangle,$$

with $\varphi_\lambda(0) = 0$. This, via (3.43), yields that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\varphi_\lambda\|^2_{\mathcal{H}} &\le -\frac{1}{4}\|\varphi_\lambda\|^2_{\mathcal{V}} + \lambda e^{-(4\rho^2\|B\|^2+1)t}\int_H \langle Bv, Q^{\frac{1}{2}}D\varphi^{u+\lambda v}\rangle\varphi_\lambda d\nu \\ &\le -\frac{1}{4}\|\varphi_\lambda\|^2_{\mathcal{V}} + 2\lambda\rho\|B\|\|\varphi_\lambda\|_{\mathcal{H}}(\int_H |Q^{\frac{1}{2}}D\varphi^{u+\lambda v}|^2_H d\nu)^{\frac{1}{2}}.\end{aligned}$$

Thus, in view of the uniform boundedness (3.53), we obtain for some positive constant $C'(T,\rho)$ independent of λ,

$$\|\varphi_\lambda\|^2_{C([0,T];\mathcal{H})} + \int_0^T \int_H |Q^{\frac{1}{2}}D\varphi_\lambda|^2_H d\nu dt \le C'(T,\rho)\lambda^2 \to 0,\ as\ \lambda \to 0,$$

which implies (3.54), thereby finishing the proof. □

Proof of Theorem 2.7. Let u_n, u_*, φ_n, φ_* and η be as in the proof of Theorem 2.5. Using (3.42) and (3.53) we obtain that along a subsequence $\{n\}$,

$$\begin{aligned} u_n \to u_*, &\ \ weak-star\ in\ L^\infty(H;\nu), \\ \varphi_n \to \varphi_*, &\ \ weak-star\ in\ L^\infty(0,T;\mathcal{H}), \\ &\ \ weakly\ in\ L^2(0,T;\mathcal{V}), \\ \mathcal{N}^{u_n}\varphi_n \to \eta, &\ \ weakly\ in\ L^2(0,T;\mathcal{V}'). \end{aligned}$$

Then, using similar arguments as in the proof of Theorem 2.7 we can pass to the limits in the dual pair ${}_{\mathcal{V}_t}(\cdot,\cdot)_{\mathcal{V}_t'}$ instead of the inner product $(\cdot,\cdot)_{\mathcal{H}_t}$, where $\mathcal{V}_t := L^2(0,t;\mathcal{V})$ and $\mathcal{V}_t'$ is the dual space of $\mathcal{V}_t$.

Hence, similarly to (3.33), we have that

$$0 \geq \int_0^T h(t)_{\mathcal{V}_t'}(\eta - \mathcal{N}^{u_*}\psi, \varphi_* - \psi)_{\mathcal{V}_t} dt \tag{3.55}$$

for any positive functions $h \in L^\infty(0,T)$ and any $\psi \in \mathcal{M}$, where $\mathcal{M}$ is defined as in the proof of Theorem 2.5. Then, since $\mathcal{M}$ is dense in $L^2(0,T;\mathcal{V})$ we can extend (3.55) to all $\psi \in L^2(0,T;\mathcal{V})$.

Hence, taking $\psi = \varphi_* - \varepsilon f v$, $f \in L^\infty(0,T)$, $v \in \mathcal{F}C_b^2(H)$, similarly to (3.34) we obtain

$$0 \geq \int_0^T h(t)_{\mathcal{V}_t'}(\eta - \mathcal{N}^{u_*}\varphi_*, fv)_{\mathcal{V}_t} dt \tag{3.56}$$

for any $f \in L^\infty(0,T)$ and any $v \in \mathcal{F}C_b^2(H)$, which suffices to yield that $\eta = \mathcal{N}^{u_*}\varphi_*$, $dt \times \nu$-a.e., thereby yielding that u_* is an optimal controller for Problem (P^*). Therefore, the proof is complete. □

Proof of Theorem 2.8. For any $u \in \mathcal{U}_{ad}$, let $v := u - u_*$ and $\varphi^{u_*+\lambda v}$ be the solution to (1.3) related to $u_* + \lambda v$, $\lambda \in (0,1)$.

We infer from (1.3) and (2.8) that

$$\frac{d}{dt}(\varphi^{u_*+\lambda v} - \varphi_*, p_*) = \lambda(\langle Bv, Q^{\frac{1}{2}} D\varphi^{u_*+\lambda v}\rangle, p_*) - \int_H (\varphi^{u_*+\lambda v} - \varphi_*) d\nu,$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathcal{H}$. This yields that for any $\lambda \in (0,1)$,

$$\int_0^T \int_H \langle B(u-u_*), Q^{\frac{1}{2}} D\varphi^{u_*+\lambda v}\rangle p_* d\nu dt = \frac{1}{\lambda}\int_0^T \int_H (\varphi^{u_*+\lambda v} - \varphi_*) d\nu dt \geq 0,$$

where the last inequality is due to the optimality of u_*.

Therefore, taking $\lambda \to 0$ and using (3.54), Fubini's theorem we obtain that

$$\int_H \langle B(u-u_*), \int_0^T Q^{\frac{1}{2}} D\varphi_* p_* dt\rangle d\nu \geq 0, \quad \forall u \in \mathcal{U}_{ad},$$

which yields (2.9), thereby finishing the proof. □

4. Optimal control of stochastic equations

4.1. **General case.** In this subsection, we first prove Theorem 2.12 under Hypothesis $(H1)'$. Then, we prove the first assertion (i) of Theorem 2.13. At last, we show that Hypothesis $(H1)'$ can be implied by the m-dissipativity of Kolmogorov operators, i.e., Hypothesis $(H1)$-(iii), by applying the theory of generalized Dirichlet forms.

Proof of Theorem 2.12. (**Existence**) By Hypothesis $(H1)'$, there exists a conservative Markov process $\mathbb{M} = (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, (X(t))_{t\geq 0}, (\mathbb{P}_x)_{x\in H_0})$ such that $X(0) = x$, $\mathbb{P}$-a.s., the sample paths $t \mapsto \langle X(t), e_i\rangle$ are continuous for every $i \geq 1$, and for ν-a.e. $x \in H$,

$$\varphi(X(t)) - \int_0^t N_2\varphi(X(s))ds \tag{4.1}$$

is an $(\mathscr{F}_t)$-martingale under $\mathbb{P}_x$ for all $\varphi \in \mathcal{F}C_b^2(H)$. Its transition semigroup P_t, $t > 0$, is given by e^{tN_2}, $t > 0$ (see Remark 2.1 (iii)), where N_2 denotes the corresponding generator.

Let $\{e_i\} \subseteq D(\widetilde{A})$ be the orthonormal basis of H such that $Qe_i = q_ie_i$ with $q_i = 0$ for $i \in J$ and $q_i > 0$ for $i \notin J$, where J is a set of finitely many indices. Set $X_i(t) := \langle X(t), e_i\rangle$, $b_i(X(t)) := \langle A(X(t)), e_i\rangle$ and $(Bu)_i = \langle Bu, e_i\rangle$, $i \geq 1$.

For every $i \notin J$, we set

$$\beta_i(t) := q_i^{-\frac{1}{2}}(X_i(t) - \int_0^t b_i(X(s))ds), \quad t \geq 0. \tag{4.2}$$

By Definition 2.9, the sample paths $t \mapsto \beta_i(t)$ are continuous for every $i \geq 1$ under $\mathbb{P}_x$ for ν-a.e. $x \in H$.

Now, using standard regularization arguments we infer from (4.1) that

$$X_i(t) = x_i + \int_0^t b_i(X(s))ds, \quad t \in [0, T], \ i \in J, \tag{4.3}$$

while β_i, $i \notin J$, are independent $(\mathscr{F}_t)$-Brownian motions with $\beta_i(0) = q_i^{-\frac{1}{2}}x_i$ under $\mathbb{P}_x$, $x \in \overline{H}$. (See, e.g., the proof of [16, Corollary 1.10]. Note that, unlike in [16], the definition of β_i in (4.2) above is independent of x.) Then, set

$$M^u(t) := \exp(\sum_{i\notin J}\int_0^t (Bu)_i(X(s))d\beta_i(s) - \frac{1}{2}\sum_{i\notin J}\int_0^t |(Bu)_i(X(s))|^2ds),$$

where $t \in [0, T]$. Since $\sup_{x\in H}|Bu(x)|_H \leq \rho\|B\|$, $\{M^u(t)\}$ is an $(\mathscr{F}_t)$-martingale under $\mathbb{P}_x$ satisfying $\mathbb{E}M^u(T) = 1$. Hence, we have a new probability measure

$$\mathbb{Q}_x^u := M^u(T) \cdot \mathbb{P}_x.$$

Then, Girsanov's theorem yields that

$$\widetilde{\beta}_i(t) := \beta_i(t) - \int_0^t (Bu)_i(X(s))ds, \quad t \in [0, T], \ i \notin J,$$

are independent $(\mathscr{F}_t)$-Brownian motions with $\widetilde{\beta}_i(0) = q_i^{-\frac{1}{2}}x_i$ under $\mathbb{Q}_x^u$. Taking into account (4.2) we obtain that

$$X_i(t) = \int_0^t b_i(X(s)) + q_i^{\frac{1}{2}}(Bu)_i(X(s))ds + q_i^{\frac{1}{2}}\widetilde{\beta}_i(t), \quad t \in [0, T], \ i \notin J. \tag{4.4}$$

This together with (4.3) yields that $X(0) = x$, $\mathbb{Q}_x^u$-a.s.. It also implies, via Itô's formula, that under $\mathbb{Q}_x^u$, for any $\varphi \in \mathcal{F}C_b^2(H)$,

$$\varphi(X(t)) - \int_0^t N_0\varphi(X(s)) + \langle Bu(X(s)), Q^{\frac{1}{2}}D\varphi(X(s))\rangle ds \tag{4.5}$$

is an $(\mathscr{F}_t)$-martingale.

Hence, $\mathbb{M}^u = (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, (X(t))_{t\geq 0}, (\mathbb{Q}_x^u)_{x\in H_0})$ satisfies the property (ii) of Definition 2.9.

It is also clear that $\mathbb{M}^u$ satisfies the property (i) of Definition 2.9, since P_t^u is bounded on $L^2(H,\nu)$.

Moreover, we also have the Markov property for $(X(t))$ under $\mathbb{Q}_x^u$, i.e., for any $0 < s, t < \infty$,

$$\mathbb{E}_{\mathbb{Q}_x}(f(X(t+s))|\mathscr{F}_s) = \mathbb{E}_{\mathbb{Q}_{X(s)}}(f(X(t))), \quad \forall f \in \mathcal{B}_b(H). \tag{4.6}$$

To this end, we first see that for any $F \in \mathcal{B}_b(H)$,

$$\mathbb{E}_{\mathbb{Q}_x}(F|\mathscr{F}_s) = \frac{\mathbb{E}_{\mathbb{P}_x}(FM^u(T)|\mathscr{F}_s)}{\mathbb{E}_{\mathbb{P}_x}(M^u(T)|\mathscr{F}_s)}, \quad 0 \le s \le T.$$

Then, since $(M^u(t))$ is an $(\mathscr{F}_t)$-martingale under $\mathbb{P}_x$, we have

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}_x}(f(X(t+s))|\mathscr{F}_s) = (M^u(s))^{-1}\mathbb{E}_{\mathbb{P}_x}(f(X(t+s))M^u(t+s)|\mathscr{F}_s)\\ &=\mathbb{E}_{\mathbb{P}_x}(f(X(t+s))\exp(\sum_{i\notin J}\int_s^{t+s}(Bu)_i(X(r))d\beta_i(r) - \frac{1}{2}\sum_{i\notin J}\int_s^{t+s}|(Bu)_i(X(r))|^2dr)|\mathscr{F}_s),\end{aligned}$$

which along with the Markov property of $(X(t))$ under $\mathbb{P}_x$ yields that

$$\mathbb{E}_{\mathbb{Q}_x}(f(X(t+s))|\mathscr{F}_s) = \mathbb{E}_{\mathbb{P}_{X(s)}}(f(X(t))M^u(t)) = \mathbb{E}_{\mathbb{Q}_{X(s)}}(f(X(t))),$$

where the last step is due to the martingale property of $(M^u(t))$ under $\mathbb{E}_{\mathbb{P}_{X(s)}}$. Thus, we obtain (4.6), as claimed.

Therefore, we conclude that $\mathbb{M}^u$ is a solution to the martingale problem for (1.1) in the sense of Definition 2.9.

(**Uniqueness**) We adapt the arguments as in the proof of [15, Theorem 8.3]. Let $\mathbb{M}' = (\Omega', \mathscr{F}', (\mathscr{F}'_t)_{t\ge 0}, (X'(t))_{t\ge 0}, (\mathbb{P}'_x)_{x\in H_0})$ be another solution to the martingale problem of (1.1), with $(P_t^u)'$ and $(N_2^u)'$ being the corresponding semigroup and generator, respectively. Similarly, let (P_t^u) and N_2^u be the semigroup and generator corresponding to $\mathbb{M}^u$.

We shall prove that for ν-a.e. $x \in H$,

$$(P_t^u)'f(x) = P_t^u f(x), \quad t > 0, \ \forall f \in C_b(H). \tag{4.7}$$

For this purpose, we note that the property (ii) in Definition 2.9 implies that under $\mathbb{P}_\nu := \int_{H_0}\mathbb{P}_x\nu(dx)$, $\varphi(X'(t)) - \int_0^t N_0^u\varphi(X'(s))ds$ is a martingale for any $\varphi \in \mathcal{F}C_b^2(H)$. It follows that, for any $g \in L^2(H,\nu)$,

$$\begin{aligned}&\int_{H_0} g(x)((P_t^u)'\varphi(x) - \varphi(x) - \int_0^t (P_s^u)'N_0^u\varphi(x)ds)\nu(dx)\\ &=\mathbb{E}_v(g(X'(0))(\varphi(X'(t)) - \varphi(X'(0)) - \int_0^t N_0^u\varphi(X'(s))ds)) = 0,\end{aligned}$$

which implies that for any $\varphi \in \mathcal{F}C_b^2(H)$,

$$(P_t^u)'\varphi - \varphi - \int_0^t (P_s^u)'N_0^u\varphi ds = 0, \quad in\ L^2(H,\nu).$$

Hence, $\varphi \in D((N_2^u)')$ and $(N_2^u)'\varphi = N_0^u\varphi$.

Taking into account $\mathcal{F}C_b^2(H)$ is a core of N_2^u, we obtain that $D(N_2^u) \subseteq D((N_2^u)')$ and $(N_2^u)' = N_2^u$ on $D(N_2)$. But, by Theorem 2.4, N_2^u is m-dissipative on $L^2(H,\nu)$. Thus, we obtain $(N_2^u)' = N_2^u$, which implies (4.7) and finishes the proof. □

Proof of Theorem 2.13 (i)**.** For any $u \in \mathcal{U}_{ad}$, let X^u solve the martingale problem for (1.1) and P_t^u be the corresponding transition semigroup, i.e., $P_t^u f(x) = \mathbb{E}_{\mathbb{P}_x} f(X^u(t))$, $f \in L^2(H,\nu)$.

Then, for any $g \in D(N_2)$, since $\mathcal{F}C_b^2(H)$ is dense in $L^2(H,\nu)$, we infer from Remark 2.11 (ii) that for any $t > 0$, $P_t^u g = e^{tN_2^u} g$, ν-a.e. $x \in H$. This yields that

$$\int_0^T \int_H \mathbb{E}_{\mathbb{P}_x} g(X^u(t))\nu(dx)dt = \int_0^T \int_H P_t^u g d\nu dt = \int_0^T \int_H e^{tN_2^u} g d\nu dt. \tag{4.8}$$

Thus, taking into account $\{e^{tN_2^u} g\}$ solves equation (1.3) in the space $L^2(H,\nu)$, we infer that the optimal controllers to Problem (P^*) are also optimal to Problem (P).

Actually, let u_* be an optimal controller to Problem (P^*). Suppose that u_* is not an optimal controller to Problem (P), then there exists $\widetilde{u} \in \mathcal{U}_{ad}$ such that

$$\int_0^T \int_H \mathbb{E}_{\mathbb{P}_x} g(X^{\widetilde{u}}(t))\nu(dx)dt < \int_0^T \int_H \mathbb{E}_{\mathbb{P}_x} g(X^{u_*}(t))\nu(dx)dt,$$

which along with (4.8) yields that, if I_* denotes the minimum of objective functionals in Problem (P^*),

$$I_* \leq \int_0^T \int_H e^{tN_2^{\widetilde{u}}} g d\nu dt < \int_0^T \int_H e^{tN_2^{u_*}} g d\nu dt = I_*,$$

thereby yielding a contradiction.

Therefore, the proof is complete. □

To end this subsection, we show that the well-posedness of martingale problems can be implied by the m-dissipativity of Kolmogorov operators in certain situations.

Theorem 4.1. *Assume* $(H1)$. *Assume additionally that* $TrQ < \infty$ *and* $\int_H |A(x)|_H^2 \nu(dx) < \infty$. *Then,* $(H1)'$ *holds, namely, the martingale problem is well posed for* (1.1) *when* $u \equiv 0$.

In order to prove Theorem 4.1, we construct a nice Markov process using the framework of [5], which extends the generalized Dirichlet form in [22] to L^p spaces, $p \geq 1$.

We first see that, by $(H1)$ (iii), since $L^2(H,\nu) \subseteq L^1(H,\nu)$, $(N_0, \mathcal{F}C_b^2(H))$ is also essentially m-dissipative in $L^1(H,\nu)$.

Then, we denote by $(N_1, D(N_1))$ and $G_\lambda^{(1)} := (\lambda - N_1)^{-1}$ the closure of $(N_0, \mathcal{F}C_b^2(H))$ in $L^1(H,\nu)$ and the corresponding resolvent, respectively, $\lambda > 0$. We say that f is a 1-excessive function if $f \geq 0$ and $\lambda G_{1+\lambda}^{(1)} f \leq f$ for all $\lambda > 0$.

Lemma 4.2. *Consider the situations as in Theorem 4.1. Then,*

(i) *For any* $x \in H$,

$$|x|_H^2 \leq G_1^{(1)}(|x|_H^2 + TrQ + 2|A(x)|_H |x|_H) =: g(x). \tag{4.9}$$

(ii) *The function* $g^{\frac{1}{2}}$ *is 1-excessive.*

Proof. (i) Define the projection operator P_n by $P_n x := \sum_{i=1}^n \langle x, e_i \rangle e_i$, $x \in H$. Using similar regularization procedure as in the proof of [3, Lemma 5.5], we see that $|P_n x|_H^2 \in D(N_1)$, and

$$(1 - N_1)|P_n x|_H^2 = |P_n x|_H^2 - \sum_{i=1}^n q_i - 2\langle A(x), P_n x \rangle .$$

Since $TrQ < \infty$ and $\int_H |A(x)|_H^2 + |x|_H^2 \nu(dx) < \infty$, we obtain

$$|P_n x|_H^2 \leq G_1^{(1)}(|P_n x|_H^2 + TrQ + 2|A(x)|_H |P_n x|_H) \in L^1(H,\nu),$$

which implies (4.9) by passing to the limit.

(ii) By the resolvent equation we have $\lambda G^{(1)}_{1+\lambda} g \le g$. Then, using Jensen's inequality we obtain

$$\lambda G^{(1)}_{1+\lambda} g^{\frac{1}{2}} \le \frac{\lambda}{\lambda+1}((\lambda+1)G^{(1)}_{1+\lambda} g)^{\frac{1}{2}} = \frac{\lambda^{\frac{1}{2}}}{(\lambda+1)^{\frac{1}{2}}}(\lambda G^{(1)}_{1+\lambda} g)^{\frac{1}{2}} \le g^{\frac{1}{2}},$$

which finishes the proof. □

Proof of Theorem 4.1. The proof is based on [5, Theorem 1.1]. We first see that the condition (II) of [5, Theorem 1.1] is satisfied with $\mathcal{A} = \mathcal{F}C^2_b(H)$.

Moreover, $F_n := \{x \in H, |x|_H \le n\}$ is weakly compact in H, $n \ge 1$, and

$$R_1(I_{F_n^c}) \le \frac{1}{n}\int_H g^{\frac{1}{2}} d\nu \to 0, \quad as\ n \to \infty,$$

where R_1 is defined as in [5], $I_{F_n^c}$ denotes the characteristic function of the complement set of F_n, and g is the 1-excessive function as in Lemma 4.2. Then, using [5, Remark 2.2] (with $f_0 = 1$, $V_\beta = G^{(1)}_1$, $\beta = 1$) we have that $\{F_n\}_{n\ge1}$ is a ν-nest of weakly compact sets, and so the condition (I) of [5, Theorem 1.1] is also satisfied.

Thus, by virtue of [5, Theorem 1.1], we obtain a ν-standard right process $\mathbb{M} = (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\ge0}, (X(t))_{t\ge0}, (\mathbb{P}_x)_{x\in H})$ in the state space H with càdlàg sample path in the weak topology of H. (Note that, the life time of $(X(t))$ is infinite since $N_1 1 = 0$.)

To show that in our case the paths $t \mapsto \langle X(t), e_i\rangle$, $i \ge 1$, are continuous we adapt a method developed in [15, Section 6]. So, let $\{\varphi_n, n \in \mathbb{N}\}$ be a countable subset of $\mathcal{F}C^2_b(H)$ separating the points of H. Then, by exactly the same arguments as in the proof of [15, Theorem 6.3] one obtains that for all $n \in \mathbb{N}$, $s < t$,

$$\int_\Omega |\varphi_n(X(t)) - \varphi_n(X(s))|^4 d\nu \le C_n (t-s)^{\frac{3}{2}},$$

where $C_n \in (0,\infty)$. Since we already know that $X(t)$, $t \ge 0$, is weakly càdlàg $\mathbb{P}_x$-a.s. for ν-a.e. $x \in H$, this together with the proof of Kolmogorov's continuity criterion implies that

$$\mathbb{P}_\nu(\Lambda_0) = 1, \tag{4.10}$$

and so

$$\mathbb{P}_x(\Lambda_0) = 1, \quad for\ \nu - a.e.\ x, \tag{4.11}$$

where $\Lambda_0 := \bigcap_{k,l\in\mathbb{N}} A^{(l)}_k$ with

$$\begin{aligned} A^{(l)}_k :=& \big\{w \in \Omega : \exists n_0 \in \mathbb{N}, \forall n \ge n_0, \forall s,t \in D_n \cap [0,l], |s-t| \le 2^{-n_0} : \\ & |\varphi_n(X(t)) - \varphi_n(X(s))| \le 2^{-k}\big\}, \\ D_n :=& \{k2^{-n}, k \in \mathbb{N} \cup \{0\}\}, \quad D := \bigcup_{n\in\mathbb{N}} D_n. \end{aligned}$$

This yields that for $\nu - a.e.\ x$, under $\mathbb{P}_x$ the paths $t \mapsto \varphi_n(X(t))$ are continuous for all $n \ge 1$, and so are the paths $t \mapsto \langle X(t), e_i\rangle$ by density, $i \ge 1$.

Therefore, we conclude that $\mathbb{M}$ solves the martingale problem of (1.1) in the sense of Definition 2.9.

The uniqueness can be proved by using similar arguments as in the proof of Theorem 2.12. The proof is complete. □

4.2. **Symmetric case.** In this case, the nice feature is that the associated Dirichlet forms are coercive closed forms. This enables us to apply the general framework of Dirichlet forms as in [21] to obtain that, the corresponding semigroup is even holomorphic and Hypothesis $(H1)'$ also holds, i.e., the martingale problem is well posed for (1.1) when $u \equiv 0$.

Below we fix $\lambda > 4\rho^2\|B\|^2$. For any $u \in \mathcal{U}_{ad}$, we define the bilinear map $\mathcal{E}^u_\lambda : \mathcal{F}C^2_b(H) \times \mathcal{F}C^2_b(H) \to \mathbb{R}$ as follows

$$\mathcal{E}^u_\lambda(\varphi, \psi) := \frac{1}{2}\int_H \langle Q^{\frac{1}{2}} D\varphi, Q^{\frac{1}{2}} D\psi\rangle d\nu - (\langle Bu, Q^{\frac{1}{2}} D\varphi\rangle, \psi) + \lambda(\varphi, \psi)$$

for any $\varphi, \psi \in \mathcal{F}C^2_b(H)$, where $(\cdot, \cdot)$ is the inner product in $\mathcal{H} := L^2(H, \nu)$. Under the closability assumption of $(Q^{\frac{1}{2}} D, \mathcal{F}C^1_b(H))$, we can extend $(\mathcal{E}^u_\lambda, \mathcal{F}C^2_b(H))$ to the closed form $(\mathcal{E}^u_\lambda, \mathcal{V})$, where $\mathcal{V} := W^{1,2}(H, \nu)$.

Lemma 4.3. *Assume $(H1)$. Assume that N_2 is symmetric and $(Q^{\frac{1}{2}} D, \mathcal{F}C^1_b(H))$ is closable from $L^2(H, \nu)$ to $L^2(H; H, \nu)$. Then, $(\mathcal{E}^u_\lambda, \mathcal{V})$ is a coercive closed form.*

Proof. We need only to check that $(\mathcal{E}^u_\lambda, \mathcal{V})$ satisfies the weak sector condition, namely, for some $K > 0$,

$$\mathcal{E}^u_{1+\lambda}(\varphi, \psi) \le K\mathcal{E}^u_{1+\lambda}(\varphi, \varphi)^{\frac{1}{2}}\mathcal{E}^u_{1+\lambda}(\psi, \psi)^{\frac{1}{2}}, \quad \forall \varphi, \psi \in \mathcal{V}. \tag{4.12}$$

For this purpose, it suffices to prove that for some $c > 0$

$$\mathcal{E}^u_\lambda(\varphi, \varphi) \ge c\|\varphi\|^2_{\mathcal{V}}, \quad \forall \varphi \in \mathcal{V}. \tag{4.13}$$

In order to prove (4.13), since $\lambda > 4\rho^2\|B\|^2$, using Cauchy's inequality we get

$$\begin{aligned}|(\langle Bu, Q^{\frac{1}{2}} D\varphi\rangle, \varphi)| \le& \rho\|B\|\|Q^{\frac{1}{2}} D\varphi\|_{L^2(H;H,\nu)}\|\varphi\|_{\mathcal{H}} \\ \le& \frac{1}{4}\|Q^{\frac{1}{2}} D\varphi\|^2_{L^2(H;H,\nu)} + 4\rho^2\|B\|^2\|\varphi\|^2_{\mathcal{H}},\end{aligned}$$

which implies that

$$\mathcal{E}^u_\lambda(\varphi, \varphi) \ge \frac{1}{4}\|Q^{\frac{1}{2}} D\varphi\|^2_{L^2(H;H,\nu)} + (\lambda - 4\rho^2\|B\|^2)\|\varphi\|^2_{\mathcal{H}},$$

thereby yielding (4.13) with $c = \min\{\frac{1}{4}, (\lambda - 2\rho^2\|B\|^2)\} > 0$, as claimed. □

Now, by virtue of [21, I. Proposition 2.16], we have the one-to-one correspondence between $(\mathcal{E}^u_\lambda, \mathcal{V})$ and the generator $(L^u_\lambda, D(L^u_\lambda))$, where L^u_λ is the unique element in $\mathcal{H}$ such that $(-L^u_\lambda\varphi, \psi) = \mathcal{E}^u_\lambda(\varphi, \psi)$ for all $\psi \in \mathcal{V}$ and $\varphi \in D(L^u_\lambda) := \{\varphi \in \mathcal{V}, \psi \to \mathcal{E}^u_\lambda(\varphi, \psi)$ *is continuous w.r.t.* $(\cdot, \cdot)^{\frac{1}{2}}$ *on* $\mathcal{V}\}$. Since L^u_λ and $N^u_2 - \lambda$ coincides on $\mathcal{F}C^2_b(H)$, it follows that $L^u_\lambda = N^u_2 - \lambda$ and $D(L^u_\lambda) = D(N_2)$.

The following result states that the corresponding semigroup is holomorphic, which enables one to solve equation (1.1) in the space $\mathcal{H}$ and also the optimal control problems even for the objective functions in the space $\mathcal{H}$.

Corollary 4.4. *Consider the situations as in Lemma 4.3. Let $(e^{tN^u_2})$ be the semigroup corresponding to $(N^u_2, D(N_2))$. Then, for all $t > 0$ and for any $g \in \mathcal{H}$, we have $e^{tN^u_2} g \in D(N_2)$. In particular, $e^{tN^u_2} g$ is the unique solution to* (1.3).

Proof. By virtue of I. Corollary 2.21 and I. Theorem 2.20 of [21], we have that L^u_λ generates a holomorphic semigroup $(e^{tL^u_\lambda})$ on some sector in $\mathbb{C}$ such that for all $t > 0$ and $g \in \mathcal{H}$, $e^{tL^u_\lambda} g \in D(L^u_\lambda)$ and so $e^{tN^u_2} g \in D(N_2)$, due to $e^{tL^u_\lambda} = e^{tN^u_2} e^{-\lambda t}$ and $D(L^u_\lambda) = D(N_2)$. This yields that $e^{tN^u_2} g$ solves (1.1) in the space $\mathcal{H}$. Moreover, the uniqueness of solutions to (1.3) follows from the monotonicity of N^u_2. Therefore, the proof is complete. □

Below we show that $(H1)'$ can be implied from $(H1)$ in the symmetric case.

Theorem 4.5. *Consider the situations as in Lemma 4.3. Then, $(H1)'$ holds, namely, the martingale problem is well posed for* (1.1) *when* $u \equiv 0$.

Proof. We construct the Markov process by using the framework of Dirichlet forms in [21]. First we see that, when $u \equiv 0$, $\lambda = 0$,

$$\mathcal{E}(\varphi, \psi)(:= \mathcal{E}_0^0(\varphi, \psi)) = \frac{1}{2}\sum_{k=1}^{\infty} q_k \int_H \partial_k \varphi \partial_k \psi d\nu, \quad \varphi, \psi \in \mathcal{F}C_b^2(H).$$

Since Q is bounded on $\mathcal{H}$, $\sup_{i\geq 1} q_i < \infty$. Then, taking into account $\mathcal{F}C_b^2(H) \subseteq \mathcal{V}$ is dense and separates points of $\mathcal{H}$, and using similar arguments as in the proof of [21, IV. Proposition 4.2] we have the quasi-regularity of $(\mathcal{E}, \mathcal{V})$.

Hence, by virtue of [21, IV. Theorem 3.5], we obtain a ν-tight special standard process $\mathbb{M}$ associated with $(\mathcal{E}, \mathcal{V})$ hence also with $(N_2, D(N_2))$, and its life time $\zeta = \infty$ since $N_2 1 = 0$.

Since the semigroup $e^{tN_2^u}$, $t \geq 0$, is bounded on $\mathcal{H}$, the first property (i) of Definition 2.9 holds.

Moreover, since $(\mathcal{E}, \mathcal{V})$ has the local property (see [21, V. Definition 1.1]), [21, V. Theorem 1.5] yields that the sample path of $\mathbb{M}$ is continuous. In view of [5, Proposition 1.4] and Remark 2.10 (i), we also infer that the property (ii) in Definition 2.9 is satisfied for $\mathbb{M}$.

Thus, $\mathbb{M}$ solves the martingale problem for (1.1) when $u \equiv 0$.

The uniqueness of solutions to martingale problem can be proved similarly as in Theorem 2.12. Therefore, the proof is complete. $\square$

Proof of Theorem 2.13 (ii) For any $u \in \mathcal{U}_{ad}$, let X^u solve the martingale problem for (1.1). Similarly to (4.8), for any $g \in \mathcal{H}$ we have

$$\int_0^T \int_H \mathbb{E}_{\mathbb{P}_x} g(X^u(t)) d\nu dt = \int_0^T \int_H e^{tN_2^u} g d\nu dt. \tag{4.14}$$

Moreover, since by Corollary 4.4 $e^{tN_2^u} g \in D(N_2)$, $t > 0$, we have

$$\frac{d}{dt} e^{tN_2^u} g = N_2^u e^{tN_2^u} g, \quad t \in (0, T], \ in \ \mathcal{H}. \tag{4.15}$$

Let φ^u be the variational solution to (1.1) as in Theorem 3.3. Then, since $D(N_2) \subseteq \mathcal{V}$, using Lemma 3.1 and arguing as in the proof of (2.7) we get

$$\frac{1}{2}\frac{d}{dt}\|e^{tN_2^u} g - \varphi^u(t)\|_{\mathcal{H}}^2 =_{\mathcal{V}} (e^{tN_2^u} g - \varphi^u(t), N_2^u(e^{tN_2^u} g - \varphi^u(t)))_{\mathcal{V}'} \leq C\|e^{tN_2^u} g - \varphi^u(t)\|_{\mathcal{H}}^2,$$

which, via Gronwall's inequality, yields $\|e^{tN_2^u} g - \varphi^u(t)\|_{\mathcal{H}}^2 = 0$ for any $t \in [0, T]$ and so,

$$e^{tN_2^u} g = \varphi^u(t), \quad \nu - a.e. \ x, \ t \in [0, T]. \tag{4.16}$$

This along with (4.14) yields

$$\int_0^T \int_H \mathbb{E}_{\mathbb{P}_x} g(X^u(t)) d\nu dt = \int_0^T \int_H \varphi^u(t) d\nu dt. \tag{4.17}$$

Therefore, the optimal controllers to Problem (P^*) are also optimal to Problem (P). The proof is complete. $\square$

5. Applications

5.1. **Singular dissipative stochastic equations.** We consider the singular dissipative stochastic equation as in [15]

$$
\begin{aligned}
&dX(t) = \widetilde{A}X(t)dt + F(X(t))dt + Q^{\frac{1}{2}}Bu(X(t))dt + Q^{\frac{1}{2}}dW(t), \quad t \in (0,T), \\
&X(0) = x \in H.
\end{aligned}
\tag{5.1}
$$

Here, $\widetilde{A} : (D(\widetilde{A})) \subseteq H \mapsto H$ is m-dissipative linear operator and $F : D(F) \subset H \to H$ is an m-dissipative singular valued operator, i.e., $\langle F(x) - F(y), x - y\rangle \leq 0$, $\forall x, y \in D(F)$, and $\mathrm{Range}(I - F) := \bigcup_{x\in D(F)}(x - F(x)) = H$. The operators B and Q are as in (1.1), with the orthonormal basis $\{e_i\} \subseteq D(\widetilde{A})$, defined by $Qe_i = q_i e_i$, $q_i > 0$, $i \geq 1$.

Let A be defined by (1.2). We have

$$
\langle A(x), D\varphi(x)\rangle = \langle x, \widetilde{A}D\varphi(x)\rangle + \langle F(x), D\varphi(x)\rangle\,, \quad \forall \varphi \in \mathcal{F}C_b^2(H),
$$

and $D(A) = D(F)$.

Let us recall the functional framework in [15]. Let $\mathcal{E}_{\widetilde{A}}(H)$ be the linear span of all (real parts of) functions of the form $\varphi = e^{i\langle h,\cdot\rangle}$, $h \in D(\widetilde{A})$.

In addition, we shall assume that

(A1) (i) There exists $\omega > 0$ such that

$$
\langle \widetilde{A}x, x\rangle \leq -w|x|_H^2, \quad \forall x \in H.
$$

(ii) Q is bounded, self-adjoint and positive definite, $Q^{-1} \in L(H)$ and $\mathrm{Tr}\widetilde{Q} < \infty$, where

$$
\widetilde{Q}x := \int_0^\infty e^{t\widetilde{A}}Qe^{t\widetilde{A}}xdt, \quad x \in H.
$$

(A2) There exists a Borel probability measure ν on H such that

(i) $\int_{D(F)}(|x|_H^{12} + |F(x)|_H^2 + |x|_H^4|F(x)|_H^2)\nu(dx) < \infty$.

(ii) For all $\varphi \in \mathcal{E}_{\widetilde{A}}(H)$ we have $N_0\varphi \in L^2(H,\nu)$ and

$$
\int_H N_0\varphi(x)\nu(dx) = 0.
$$

(iii) $\nu(D(F)) = 1$.

We see that Assumption $(A2)$ implies Hypothesis $(H1)$ (i) and (ii). It also follows from [15, Theorem 2.3] that, under Assumptions $(A1)$ and $(A2)$, $(N_0, \mathcal{E}_{\widetilde{A}}(H))$ is essentially m-dissipative in $L^2(H,\nu)$, and so is $(N_0, \mathcal{F}C_b^2(H))$. Thus Hypothesis $(H1)$ is satisfied.

Moreover, [15, Theorem 7.4] yields that the martingale problem is well posed for (1.1) in the case $u \equiv 0$, which yields $(H1)'$.

Thus, both Hypotheses $(H1)$ and $(H1)'$ are fulfilled.

As regards the closability of D we have

Proposition 5.1. *Assume* $(A1)$ *and* $(A2)$*. Then, D is closable from $L^2(H,\nu)$ to $L^2(H;H,\nu)$.*

Proof. Let $\{\widetilde{e}_k\}_{k\geq 1}$ be an orthonorm basis of H such that $\widetilde{Q}\widetilde{e}_k = \widetilde{q}_k\widetilde{e}_k$, $\widetilde{q}_k > 0$, $k \geq 1$, and set $x_k := \langle x, \widetilde{e}_k\rangle$, $D_k\varphi := \langle D\varphi, \widetilde{e}_k\rangle$, $x \in H$.

Let μ be the Gaussian measure with mean zero and covariance operator $\widetilde{Q}$. We have the integration by parts formula,

$$
\int_H D_k\varphi\psi d\mu = -\int_H \varphi D_k\psi d\mu + \frac{1}{\widetilde{q}_k}\int_H x_k\varphi\psi d\mu, \quad \forall \varphi, \psi \in \mathcal{F}C_b^2(H).
\tag{5.2}
$$

Moreover, for the infinitesimal invariant measure ν, it is known that $\nu = \rho\cdot\mu$ with $\rho^{\frac{1}{2}} \in W^{1,2}(H,\mu)$ (see [15, page 292]). Note that $\rho \in \overline{\mathcal{F}C_b^2(H)}^{W^{1,1}(H,\mu)}$, $D_k(\rho\varphi) = D_k\rho\varphi + \rho D_k\varphi$ for any $\varphi \in \mathcal{F}C_b^2(H)$, $D\rho = 2\rho^{\frac{1}{2}}D(\rho^{\frac{1}{2}})$ in $L^1(H;H,\nu)$, and so $D_k\rho/\rho \in L^2(H,\nu)$.

Now, we take any $(\varphi_n) \subseteq \mathcal{F}C_b^2(H)$ such that

$$\varphi_n \to 0,\ in\ L^2(H,\nu),\quad D_k\varphi_n \to g,\quad in\ L^2(H,\nu). \tag{5.3}$$

We shall prove that

$$g(x) = 0,\quad \nu - a.e.\ x. \tag{5.4}$$

For this purpose, for any $\psi \in \mathcal{F}C_b^2(H)$ we set $\psi_{\varepsilon,k} := (1+\varepsilon|x_k|^2)^{-1}\psi$, $\varepsilon > 0$. Note that, by (5.3),

$$\begin{aligned}\int_H \psi_{\varepsilon,k} g\rho d\mu &= \lim_{n\to\infty}\int_H \psi_{\varepsilon,k}D_k\varphi_n\rho d\mu \\ &= \lim_{n\to\infty}\int_H D_k(\psi_{\varepsilon,k}\varphi_n\rho)d\mu - \lim_{n\to\infty}\int_H \varphi_n(D_k\psi_{\varepsilon,k}\rho + \psi_{\varepsilon,k}D_k\rho)d\mu\end{aligned} \tag{5.5}$$

Since $\psi_{\varepsilon,k} \in \mathcal{F}C_b^2(H)$ and $D_k\rho/\rho \in L^2(H,\nu)$, using (5.3) we see that the last limit on the right hand side above equals to zero. Regarding the remaining limit, we take a sequence $\rho_m \in \mathcal{F}C_b^2(H)$, $m \geq 1$, such that $\rho_m \to \rho$ in $W^{1,1}(H,\mu)$. Then, using the integration by parts formula (5.2) we have

$$\begin{aligned}\int_H D_k(\psi_{\varepsilon,k}\varphi_n\rho)d\mu &= \lim_{m\to\infty}\int_H D_k(\psi_{\varepsilon,k}\varphi_n\rho_m)d\mu \\ &= -\frac{1}{\widetilde{q}_k}\lim_{m\to\infty}\int_H x_k\psi_{\varepsilon,k}\varphi_n\rho_m d\mu \\ &= -\frac{1}{\widetilde{q}_k}\int_H x_k\psi_{\varepsilon,k}\varphi_n\rho d\mu,\quad as\ m\to\infty,\end{aligned}$$

where in the last step we used the fact that $\sup_{x\in H}|x_k\psi_{\varepsilon,k}(x)| \leq C_\varepsilon < \infty$. Hence, using again (5.3) and the boundedness of $\sup_{x\in H}|x_k\psi_{\varepsilon,k}(x)|$ we get

$$\lim_{n\to\infty}\int_H D_k(\psi_{\varepsilon,k}\varphi_n\rho)d\mu = -\frac{1}{\widetilde{q}_k}\lim_{n\to\infty}\int_H x_k\psi_{\varepsilon,k}\varphi_n\rho d\mu = 0.$$

Thus, combing back to (5.5) we obtain

$$\int_H \psi_{\varepsilon,k}g\rho d\mu = 0.$$

Taking the limit $\varepsilon \to 0$ yields that, for any $\psi \in \mathcal{F}C_b^2(H)$,

$$\int_H \psi g\rho d\mu = 0,$$

which yields (5.4) and finishes the proof. □

The compact embedding of $W^{1,2}(H,\nu)$ to $L^2(H,\nu)$ also holds in certain situations. Following [16] we assume additionally that

(A3) (i) $\int_0^\infty (1+t^{-\alpha})\|e^{t\widetilde{A}}\sqrt{Q}\|_{HS}^2 dt < \infty$ for some $\alpha > 0$, where $\|\cdot\|_{HS}$ denotes the norm on the space of all Hilbert-Schmidt operators on H, and $Q^{-\frac{1}{2}} \in L(H)$.

(ii) $(1+w-\widetilde{A}, D(\widetilde{A}))$ satisfies the weak sector condition, i.e., for some $K > 0$, for any $x,y \in D(\widetilde{A})$,

$$\langle(1+w-\widetilde{A})x,y\rangle \leq K\langle(1+w-\widetilde{A})x,x\rangle^{\frac{1}{2}}\langle(1+w-\widetilde{A})y,y\rangle^{\frac{1}{2}}.$$

(*iii*) There exists a sequence of $\widetilde{A}$-invariant finite dimensional subspace $H_n \subseteq D(\widetilde{A})$ such that $\cup_{n=1}^{\infty} H_n$ is dense in H.

It follows from [16, Theorem 1.6] that the Harnack inequality holds for the ν-version transition semigroup p_t^ν corresponding to (5.1) when $u \equiv 0$. In particular, by [16, Corollary 1.9], p_t^ν has a density with respect to ν and is even hyperbounded, i.e., $\|p_t^\nu\|_{L^2(H,\nu)\to L^4(H,\nu)} < \infty$ for some $t > 0$.

Thus, by virtue of [19, Theorem 1.2], we obtain the compactness of p_t^ν and also the compact embedding of $W^{1,2}(H,\nu)$ into $L^2(H,\nu)$. Hence, Hypothesis $(H3)$ is satisfied.

Now, we conclude from Theorem 2.13 in Section 2 that

Theorem 5.2. *Consider the controlled stochastic singular differential equation* (5.1). *Assume Hypotheses* $(A1)$ *and* $(A2)$. *Assume additionally* $(H2)$ *or* $(A3)$. *Then, for any* $g \in D(N_2)$, *there exists an optimal control* u_* *for the optimal control problem below*

$$\mathrm{Min}\left\{ \int_0^T \int_H \mathbb{E}_{\mathbb{P}_x} g(X^u(t))\nu(dx)dt;\ u \in \mathcal{U}_{ad},\ \mathbb{P}_x \circ (X^u)^{-1} \ solves \right.$$
$$\left. the\ martingale\ problem\ for\ (5.1)\ for\ \nu - a.e.\ x \in H \right\}.$$

In particular, under Hypotheses $(A1)$, $(A2)$ *and* $(A3)$, *we can take* $B = I_d$.

As a specific example of (5.1), we consider the controlled gradient system

$$\begin{aligned} dX &= \widetilde{A}Xdt + \partial U(X)dt + Bu(X) + dW(t), \\ X(0) &= x \in H. \end{aligned} \tag{5.6}$$

Here, we take $Q = Id$, $\widetilde{A}, B$ are the operators as in (5.1) satisfying additionally that $\widetilde{A}^{-1}$ is of trace class. and ∂U denotes the subdifferential of a convex and lower semicontinuous function $U : H \to (-\infty, \infty]$, satisfying that $\{U < \infty\}$ is open, $\mu(\{U < \infty\}) > 0$ and

$$\rho := Z^{-1}e^{-2U(x)} \in L^1(H,\mu),$$

where μ is the Gaussian measure of mean zero and covariance operator $-\frac{1}{2}\widetilde{A}^{-1}$ and $Z := \int_H e^{-2U(x)}\mu(dx)$.

We know from [15, Section 9.1] that Assumptions $(A1)$ and $(A2)$ are fulfilled and, in particular, the Kolmogorov operator N_2 is symmetric.

Therefore, by virtue of Theorem 2.13, for more general objective functions $g \in L^2(H,\nu)$ we have the existence as well as first-order necessary condition (2.9) of the feedback control problem (P) for the gradient system (5.6).

5.2. **Stochastic reaction-diffusion equation.** Consider the controlled stochastic reaction-diffusion equation below as in [9]

$$\begin{aligned} dX =& \Delta X dt - p(X)dt + C^{\frac{1}{2}}Bu(X)dt + C^{\frac{1}{2}}dW, \\ X(0) =& x \in H, \end{aligned} \tag{5.7}$$

where $H = L^2(\mathcal{O})$, $\mathcal{O} = [0,1]$, Δ is the realization of the Laplace operator with Dirichlet boundary condition, i.e., $D(\Delta) = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$, B is a bounded operator on H, and W is a cylindrical Wiener process on H, $W(t) = \sum_{k=1}^{\infty} e_k\beta_k(t)$ is a cylindrical Wiener process on a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$, where e_k are the eigenbasis of $-\Delta$, such that $-\Delta e_k = \lambda_k e_k$, $\lambda_k \geq 0$, $k \geq 1$.

Concerning p and C we assume that

(B) (i) p is a polynomial of degree $d > 1$, its derivative $p'(\xi) \geq 0$, $\forall \xi \in \mathbb{R}$.
(ii) $C = (-\Delta)^{-\gamma}$, $\gamma > -\frac{1}{2}$.

In this case, we have $A(x) = \Delta x - p(x)$ and $D(A) = \{x \in L^{2d}(\mathcal{O})\}$.

When $u \equiv 0$, it is known (see Theorem 4.8 of [9]) that, for each $x \in H$, there exists a unique generalized solution $X(\cdot, x)$ to (5.7).

Moreover, by [9, Theorem 4.16]), the transition semigroup $P_t : \mathcal{B}_b(H) \to \mathcal{B}_b(H)$ defined by $(P_t\varphi)(x) = \mathbb{E}\varphi(X(t,x))$, $x \in H$, $\varphi \in C_b(H)$, has a unique invariant measure ν satisfying that

$$\lim_{t\to\infty} P_t\varphi(x) = \int_H \varphi(y)\nu(dy) \tag{5.8}$$

and (see [9, Proposition 4.20])

$$\int_H |x|^{2d}_{L^{2d}(\mathcal{O})}\nu(dx) < \infty. \tag{5.9}$$

Furthermore, from [9, Section 4.6] we have that P_t can be uniquely extended to a C_0-semigroup of contractions on $L^2(H;\nu)$. By Theorem 4.23 of [9], the infinitesimal generator N_2 of P_t is the closure in $L^2(H,\nu)$ of the operator

$$(N_0\varphi)(x) := \frac{1}{2}Tr[(-\Delta)^{-\gamma}D^2\varphi](x) + \langle x, \Delta D\varphi\rangle - \langle p(x), D\varphi\rangle\,, \tag{5.10}$$

where $x \in H$, $\varphi \in \mathcal{E}_\Delta(H)$ with $\mathcal{E}_\Delta(H)$ defined similarly as $\mathcal{E}_{\widetilde{A}}(H)$ in the previous subsection.

Now, let us check the Hypothesis $(H1)$. We first infer from (5.9) that Hypothesis $(H1)$ (i) is satisfied. Since for any $\varphi \in \mathcal{F}C_b^2(H)$, $t \geq 0$, $x \in H$,

$$\int_H P_t\varphi(y)\nu(dy) = \lim_{s\to\infty} P_s(P_t\varphi)(x) = \lim_{s\to\infty} P_{s+t}\varphi(x) = \int_H \varphi(y)\nu(dy),$$

we have

$$\int_H N_0\varphi(y)\nu(dy) = \frac{d}{dt}(\int_H P_t\varphi(y)\nu(dy))|_{t=0} = 0, \quad \forall \varphi \in \mathcal{F}C_b^2(H), \tag{5.11}$$

which implies $(H1)$ (ii). Moreover, the results of [9, Section 4.6] presented above show that $(N_0, \mathcal{E}_\Delta(H))$ is essentially m-dissipative, and so is $(N_0\varphi, \mathcal{F}C_b^2(H))$, thereby yielding $(H1)$ (iii). Hence, Hypothesis $(H1)$ is fulfilled.

Concerning Hypothesis $(H1)'$ we have

Proposition 5.3. *Assume* (B). *Then, Hypothesis* $(H1)'$ *is satisfied, i.e., the martingale problem for* (5.7) *is well posed in the case* $u \equiv 0$.

Proof. Set $\overline{H} := L^{2d}(\mathcal{O})$. We have $\nu(\overline{H}) = 1$. For each $x \in \overline{H}$, by Theorem 4.8 of [9], there exists a unique $(\mathscr{F}_t)$-adapted process $X(\cdot, x)$, such that $X(t,x) \in \overline{H}$ for all $t \geq 0$, $X \in C([0,T]; L^2(\Omega; H))$, $\mathbb{E}\|X(t,x)\|^{2d}_{L^{2d}(\mathcal{O})} \leq C_{m,p,T}(1 + \|x\|^{2d}_{L^{2d}})$ for any $m \geq 1$, and X solves (5.7) in the mild sense, i.e., for each $t \in [0,T]$,

$$X(t,x) = e^{t\Delta}x + \int_0^t e^{(t-s)\Delta}F(X(s,x))ds + W_\Delta(t), \quad \mathbb{P} - a.s., \tag{5.12}$$

where $F(X(s,x)) = -p(X(s,x))$,

$$W_\Delta(t) = \int_0^t e^{(t-s)\Delta}(-\Delta)^{-\frac{\gamma}{2}}dW(s) = \sum_{k=1}^\infty \int_0^t e^{(t-s)\Delta}(-\Delta)^{-\frac{\gamma}{2}}e_k d\beta_k(s),\ t \geq 0,$$

and $\{e_k\}$ is the eigenbasis of $-\Delta$, i.e., $-\Delta e_k = \lambda_k e_k$, $k \geq 1$. Moreover, X is a Feller process (see [9, Proposition 4.9]).

Note that, by the integrabilities of X above, we have that $\mathbb{P}$-a.s. $F(X(t,x)) \in H$ and $t \mapsto \int_0^t F(X(s,x))ds$ is continuous in H. Taking into account the continuity of W_Δ in H implied by [9, Proposition 4.3], we can take a $\mathbb{P}$-version of the process X (still denoted by X), such that $X \in C([0,T];H)$, $\mathbb{P}$-a.s., and X satisfies (5.12) for all $t \in [0,T]$ outside a common $\mathbb{P}$-null set. Below we consider this $\mathbb{P}$-version process X.

Next, let $x_k := \langle x, e_k \rangle$, $x \in H$, $k \geq 1$. We claim that $\mathbb{P}$-a.s. for each $k \geq 1$ and for all $t \in [0,T]$,

$$X_k(t,x) = e^{-\lambda_k t} x_k + \int_0^t e^{-\lambda_k (t-s)} (F(X(s,x)))_k ds + \int_0^t e^{-\lambda_k (t-s)} \lambda_k^{-\frac{\gamma}{2}} d\beta_k(s). \tag{5.13}$$

For this purpose, we first infer from (5.12) that $\mathbb{P}$-a.s. for each $k \geq 1$

$$X_k(t,x) = e^{-\lambda_k t} x_k + \langle \int_0^t e^{(t-s)\Delta} F(X(s,x))ds, e_k \rangle + \langle W_\Delta(t), e_k \rangle, \ t \in [0,T]. \tag{5.14}$$

Since $e^{(t-\cdot)\Delta} F(X(\cdot,x))$ is Bochner integrable on H and $y \mapsto \langle y, e_k \rangle$ is a linear bounded operator on H, we get

$$\begin{aligned} \langle \int_0^t e^{(t-s)\Delta} F(X(s,x))ds, e_k \rangle &= \int_0^t \langle e^{(t-s)\Delta} F(X(s,x)), e_k \rangle ds \\ &= \int_0^t e^{-\lambda_k (t-s)} (F(X(s,x)))_k ds \end{aligned} \tag{5.15}$$

Moreover, since

$$\mathbb{E} \sum_{j=1}^\infty |\langle \int_0^t e^{(t-s)\Delta} (-\Delta)^{-\frac{\gamma}{2}} e_j d\beta_j(s), e_k \rangle|^2 < \infty, \tag{5.16}$$

using Fubini's theorem to exchange the integration with sum we get

$$\begin{aligned} \langle \sum_{j=1}^\infty \int_0^t e^{(t-s)\Delta} (-\Delta)^{-\frac{\gamma}{2}} e_j d\beta_j(s), e_k \rangle &= \sum_{j=1}^\infty \langle \int_0^t e^{(t-s)\Delta} (-\Delta)^{-\frac{\gamma}{2}} e_j d\beta_j(s), e_k \rangle \\ &= \int_0^t e^{-\lambda_k (t-s)} \lambda_k^{-\frac{\gamma}{2}} d\beta_k(s). \end{aligned} \tag{5.17}$$

Thus, plugging (5.15) and (5.17) into (5.14) we obtain (5.13), as claimed.

Hence, we infer from (5.13) that $\mathbb{P}$-a.s.

$$dX_k(t,x) = -\lambda_k X_k(t,x)dt + (F(X(t,x)))_k dt + \lambda_k^{-\frac{\gamma}{2}} d\beta_k(t) \tag{5.18}$$

with $X_k(0,x) = x_k$. Since for each $\varphi \in \mathcal{F}C_b^2(H)$, there exists $\phi \in C_b^2(\mathbb{R}^n)$ such that $\varphi(x) = \phi(\langle x, e_1 \rangle, \cdots, \langle x, e_n \rangle)$ for some $n \in \mathbb{N}$, using Itô's formula we obtain that, if $X^n := (X_1, \cdots, X_n)$,

$$\begin{aligned} d\varphi(X(t,x)) =& \sum_{k=1}^n (-\lambda_k X_k + (F(X(t,x)))_k) \partial_k \phi(X^n(t,x))dt \\ &+ \frac{1}{2} \sum_{k=1}^n \lambda_k^{-\gamma} \partial_{kk} \phi(X^n(t,x))dt + \sum_{k=1}^n \lambda_k^{-\frac{\gamma}{2}} \partial_k \phi(X^n(t,x)) d\beta_k(t) \\ =& N_0 \varphi(X(t,x))dt + \sum_{k=1}^n \lambda_k^{-\frac{\gamma}{2}} \partial_k \phi(X^n(t,x)) d\beta_k(t), \end{aligned}$$

This yields that $\varphi(X(t,x)) - \int_0^t N_0\varphi(X(s,x))ds$ is an $(\mathscr{F}_t)$-martingale under $\mathbb{P}$, so the property (ii) in Definition 2.9 is fulfilled.

Therefore, let $\widetilde{\Omega} := C([0,T];H)$, $\widetilde{\mathscr{F}} := \sigma(\mathscr{F}^{X(\cdot,x)}, x \in \overline{H})$ and $\widetilde{\mathscr{F}_t} := \sigma(\mathscr{F}_t^{X(\cdot,x)}, x \in \overline{H})$, $0 \le t \le T$, where $\mathscr{F}^{X(\cdot,x)}$ and $\mathscr{F}_t^{X(\cdot,x)}$ denote the image σ-algebras under $X(\cdot,x)$ of $\mathscr{F}$ and $\mathscr{F}_t$, respectively. Set $\pi_t(\omega) := \omega(t)$ and $\widetilde{\mathbb{P}}_x := \mathbb{P} \circ X(\cdot,x)^{-1}$, $\omega \in \widetilde{\Omega}$, $0 \le t \le T$, $x \in \overline{H}$. Then, $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, (\widetilde{\mathscr{F}_t})_{t\ge 0}, (\pi_t)_{t\ge 0}, (\widetilde{\mathbb{P}}_x)_{x\in\overline{H}})$ solves the martingale problem for (5.7). Taking into account Remark 2.10 (ii) we finish the proof of Proposition 5.3. □

In the case where $\gamma = 0$, Hypothesis $(H3)$ holds in certain situations. Actually, [9, Theorem 4.26] yields that D is closable from $L^2(H,\nu)$ to $L^2(H;H,\nu)$, and it also follows from [9, Theorem 4.34] that the invariant measure ν has the density $\rho = \frac{d\nu}{d\mu}$ with respect to the Gaussian measure μ with mean zero and covariance operator $-\frac{1}{2}A^{-1}$. If, in addition, for some $\varepsilon \in (0,1)$,

$$\int_H |D\log\rho|_H^{2+\varepsilon} d\nu < \infty, \tag{5.19}$$

then, by [9, Theorem 4.35], $W^{1,2}(H,\nu)$ is compactly embedded into $L^2(H,\nu)$, and so Hypothesis $(H3)$ is satisfied.

In conclusion, we have from Theorem 2.13 that

Theorem 5.4. *Consider the controlled stochastic reaction-diffusion equation* (5.7). *Assume* (B). *Assume also* $(H2)$ *or* $(H3)$. *Then, for any* $g \in D(N_2)$, *there exists an optimal control* u_* *for the optimal control problem below*

$$\mathrm{Min}\Big\{ \int_0^T \int_H \mathbb{E}_{\mathbb{P}_x} g(X^u(t))\nu(dx)dt;\ u \in \mathcal{U}_{ad},\quad \mathbb{P}_x \circ (X^u)^{-1}\ solves$$
$$the\ martingale\ problem\ for\ (5.7)\ for\ \nu - a.e.\ x \in H \Big\}.$$

In particular, in the case where $\gamma = 0$ *and that Assumption* (B) *and* (5.19) *hold, we can take* $B = Id$.

5.3. **Stochastic porous media equations.** In this subsection, we are concerned with the optimal control problems for stochastic porous media equations. Precisely, we consider the controlled stochastic low diffusion equation as in [3]

$$\begin{aligned} dX(t) &= \Delta(\Psi(X(t)))dt + Q^{\frac{1}{2}}Bu(X(t))dt + Q^{\frac{1}{2}}dW(t), \\ X(0) &= x \in H. \end{aligned} \tag{5.20}$$

Here $H = H^{-1}(\mathcal{O})$, which is the dual space of $H_0^1(\mathcal{O})$ equipped with the inner product $\langle x,y\rangle := \int_{\mathcal{O}}((-\Delta)^{-1}x)(\xi)y(\xi)d\xi$, $\mathcal{O} \subseteq \mathbb{R}^d$ is a bounded open set with Dirichlet boundary conditions for the Laplacian Δ, B and Q are as in (1.1), and Ψ is a dissipative nonlinearity.

In this case, $A(x) = \Delta(\Psi(x))$ and $D(A) = \{x \in L^2(\mathcal{O}), \Psi(x) \in H_0^1(\mathcal{O})\}$.

Following [3], we assume

- (C1) There exist $q_k \in [0,\infty)$, $k \in \mathbb{N}$, such that for the eigenbasis $\{e_k\}$ of Δ in H, $Qe_k = q_k e_k$, $k \in \mathbb{N}$.
- (C2) $\sum_{k=1}^\infty \sup_{\xi\in D} |e_k(\xi)|^2 q_k < \infty$.
- (C3) $\Psi \in C^1(\mathbb{R})$, $\Psi(0) = 0$, and there exist $r \in (1,\infty)$ and $\kappa_0, \kappa_1, C_1 > 0$ such that

$$\kappa_0|s|^{r-1} \le \Psi'(s) \le \kappa_1|s|^{r-1} + C_1, \quad \forall s \in \mathbb{R}.$$

It is known ([3, Proposition 3.1]) that, under Assumption $(C3)$, A is m-dissipative on H.

The corresponding Kolmogorov operator is formally given by

$$N_0\varphi(x) := \frac{1}{2}\sum_{k=1}^{\infty} q_k \partial_{kk}\varphi(e_k, e_k) + \langle \Delta\Psi(x), D\varphi(x)\rangle\,, \quad x \in H, \ \varphi \in \mathcal{F}C_b^2(H).$$

As in [3], let $\mathcal{M}$ be the set of infinitesimally excessive measures ν, which are infinitesimally invariant measures for N_0 satisfying $(H1)$ (ii),

$$\int_H \int_D |\nabla(\Psi(x))(\xi)|^2 d\xi \nu(dx) < \infty, \tag{5.21}$$

and for some $\lambda_\nu \in (0, \infty)$

$$\int_H N_0\varphi(x)\nu(dx) \leq \lambda_\nu \int_H \varphi\nu(dx), \quad \forall \varphi \in \mathcal{F}C_b^2(H) \ with \ \varphi \geq 0, \ \nu - a.e.. \tag{5.22}$$

We see that $(H1)$ (i) is satisfied for each $\nu \in \mathcal{M}$. Actually, by Poincaré's inequality and (5.21),

$$\int_H \|\Psi(x)\|^2_{L^2(\mathcal{O})}\nu(dx) \leq C \int_H \|\nabla\Psi(x)\|^2_{L^2(\mathcal{O})}\nu(dx) < \infty,$$

which along with Assumption $(C3)$ above yields that

$$\int_H |x|_H^{2r}\nu(dx) \leq \int_H \|x\|^{2r}_{L^2(\mathcal{O})}\nu(dx) \leq C(1 + \int_H \|\Psi(x)\|^2_{L^2(\mathcal{O})}\nu(dx)) < \infty. \tag{5.23}$$

Moreover, by (5.21),

$$\int_H |\Delta\Psi(x)|_H^2\nu(dx) = \int_H \|\nabla\Psi(x)\|^2_{L^2(\mathcal{O})}\nu(dx) < \infty. \tag{5.24}$$

Hence, Hypothesis $(H1)$ (i) follows.

Moreover, under Assumptions $(C1)$-$(C3)$, it follows from [3, Theorem 4.1] that $(N_0, C_b^2(H))$ is essentially m-dissipative on $L^2(H, \nu)$ for each $\nu \in \mathcal{M}$, which implies $(H1)$ (iii), and so Hypothesis $(H1)$ holds.

Furthermore, if in addition $r \geq 2$, [3, Theorems 5.1] yields that the martingale problem for (5.20) has a solution in the case $u \equiv 0$. Then, taking into account Remark 2.10 (ii) on uniqueness we infer that Hypothesis $(H1)'$ holds.

Therefore, in view of Theorems 2.5 and 2.13, we obtain

Theorem 5.5. *Consider the controlled stochastic low diffusion equation* (5.20)*. Assume Hypotheses* $(C1)$, $(C2)$ *and* $(C3)$*. Assume additionally* $(H2)$*. Then, for any* $g \in D(N_2)$*, there exists an optimal control* u_* *for the optimal control problem* (P^*)*.*

Moreover, if in addition $r \geq 2$*, we have the optimal controllers for the problem below*

$$\mathrm{Min}\Big\{ \int_0^T \int_H \mathbb{E}_{\mathbb{P}_x} g(X^u(t))\nu(dx)dt; \ u \in \mathcal{U}_{ad}, \quad \mathbb{P}_x \circ (X^u)^{-1} \ solves$$
$$the \ martingale \ problem \ for \ (5.20) \ for \ \nu - a.e. \ x \in H\Big\}.$$

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References

[1] V. Barbu, Optimal feedback controllers for the stochastic reflection problem in $\mathbb{R}^d$, preprint.

[2] V. Barbu, Nonlinear differential equations of monotone types in Banach spaces. *Springer Monographs in Mathematics*. Springer, New York, 2010. x+272 pp.

[3] V. Barbu, V.I. Bogachev, G. Da Prato, M. Röckner, Weak solutions to the stochastic porous media equation via Kolmogorov equations: the degenerate case. *J. Funct. Anal.* **237** (2006), no. 1, 54–75.

[4] V. Barbu, G. Da Prato, M. Röckner, Stochastic porous media equations. *Lecture Notes in Mathematics*, **2163**. Springer, 2016. ix+202 pp.

[5] L. Beznea, N. Boboc, M. Röckner, Markov processes associated with L^p-resolvents and applications to stochastic differential equations on Hilbert space. *J. Evol. Equ.* **6** (2006), no. 4, 745–772.

[6] L. Beznea, I. Cimpean, M. Röckner, Treatment of noit allowed starting points for solutions to singular SDEs on Hilbert spaces: a general approach, arXiv: 1904.01607v1.

[7] A. Chojnowska-Michalik, B. Goldys, Existence, uniqueness and invariant measures for stochastic semilinear equations on Hilbert spaces. *Probab. Theory Related Fields* **102** (1995), no. 3, 331–356.

[8] M.G. Crandall, H. Ishii, P.L. Lions, User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc.* **27** (1992), no. 1, 1–67.

[9] G. Da Prato, Kolmogorov equations for stochastic PDEs. *Advanced Courses in Mathematics*. CRM Barcelona. Birkhäuser Verlag, Basel, 2004. x+182 pp.

[10] G. Da Prato, A. Debussche, B. Goldys, Some properties of invariant measures of non symmetric dissipative stochastic systems. *Probab. Theory Related Fields* **123** (2002), no. 3, 355–380.

[11] G. Da Prato, F. Flandoli, E. Priola, M. Röckner, Strong uniqueness for stochastic evolution equations in Hilbert spaces perturbed by a bounded measurable drift. *Ann. Probab.* **41** (2013), no. 5, 3306–3344.

[12] G. Da Prato, F. Flandoli, E. Priola, M. Röckner, Strong uniqueness for stochastic evolution equations with unbounded measurable drift term. *J. Theoret. Probab.* **28** (2015), no. 4, 1571–1600.

[13] G. Da Prato, F. Flandoli, M. Röckner, A.Yu Veretennikov, Strong uniqueness for SDEs in Hilbert spaces with nonregular drift. *Ann. Probab.* **44** (2016), no. 3, 1985–2023.

[14] G. Da Prato, A. Lunardi, Sobolev regularity for a class of second order elliptic PDE's in infinite dimension. *Ann. Probab.* **42** (2014), no. 5, 2113–2160.

[15] G. Da Prato, M. Röckner, Singular dissipative stochastic equations in Hilbert spaces. *Probab. Theory Related Fields.* **124** (2002), no. 2, 261–303.

[16] G. Da Prato, M. Röckner, F.Y. Wang, Singular stochastic equations on Hilbert spaces: Harnack inequalities for their transition semigroups. *J. Funct. Anal.* **257** (2009), no. 4, 992–1017.

[17] G. Fabbri, F. Gozzi, A. Świech, Stochastic optimal control in infinite dimension. Dynamic programming and HJB equations. With a contribution by Marco Fuhrman and Gianmario Tessitore. Probability Theory and Stochastic Modelling, 82. Springer, Cham, 2017. xxiii+916 pp.

[18] M. Fuhrman, G. Tessitore, Nonlinear Kolmogorov equations in infinite dimensional spaces: the backward stochastic differential equations approach and applications to optimal control. *Ann. Probab.* **30** (2002), no. 3, 1397–1465.

[19] F.Z. Gong, F.Y. Wang, Functional inequalities for uniformly integrable semigroups and application to essential spectrums. *Forum Math.* **14** (2002), no. 2, 293–313.

[20] J.L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires. (French) Dunod; Gauthier-Villars, Paris 1969 xx+554 pp.

[21] Z.M. Ma, M. Röckner, Introduction to the theory of (nonsymmetric) Dirichlet forms. *Universitext.* Springer-Verlag, Berlin, 1992. vi+209 pp.

[22] W. Stannat, The theory of generalized Dirichlet forms and its applications in analysis and stochastics. *Mem. Amer. Math. Soc.* **142** (1999), no. 678, viii+101 pp.

[23] G. Trutnau, On a class of non-symmetric diffusions containing fully nonsymmetric distorted Brownian motions. *Forum Math.* **15** (2003), no. 3, 409–437.

[24] F.Y. Wang, Functional inequalities, semigroup properties and spectrum estimates. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **3** (2000), no. 2, 263–295.

[25] F.Y. Wang, Integrability conditions for SDEs and semilinear SPDEs. *Ann. Probab.* **45** (2017), no. 5, 3223–3265.

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