CONVEX MONOTONE SEMIGROUPS ON LATTICES OF CONTINUOUS FUNCTIONS

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Abstract. We consider convex monotone semigroups on a Banach lattice, which is assumed to be a Riesz subspace of a σ-Dedekind complete Banach lattice with an additional assumption on the dual space. As typical examples, we consider the space of bounded uniformly continuous functions and the space of continuous functions vanishing at infinity. We show that the domain of the classical generator for convex monotone C_0-semigroups, which is defined in terms of the time derivative at 0 w.r.t. the supremum norm, is typically not invariant. We thus propose alternative forms of generators and domains, for which we prove the invariance under the semigroup. As a consequence, we obtain the uniqueness of the semigroup in terms of an extended version of the generator. The results are discussed in several examples related to fully nonlinear partial differential equations, such as uncertain shift semigroups and semigroups related to G-heat equations (fully nonlinear versions of the heat equation).

Key words: Convex semigroup, nonlinear Cauchy problem, fully nonlinear PDE, uniqueness, Hamilton-Jacobi-Bellman equation

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1. Introduction

The topic of model uncertainty or ambiguity in the fields of Mathematical Economics and Mathematical Finance has been extensively studied in the past decades. Hereby, a particular focus has been put on parameter uncertainty of stochastic processes describing the evolution of an underling asset. Examples include a Brownian motion with drift uncertainty (cf. Coquet et al. [6]) or volatility uncertainty (cf. Peng [31],[32]), a Black-Scholes model with volatility uncertainty (cf. Avellaneda et al. [2], Epstein and Ji [15], Vorbrink [36]), and Lévy processes with uncertainty in the Lévy triplet (cf. Hu and Peng [20], Neufeld and Nutz [28], Hollender [19], Kühn [23]). The aforementioned examples lead to nonlinear, more precisely, convex Hamilton-Jacobi-Bellman-type partial differential equations. In the case of a Brownian Motion with uncertain volatility within an interval $[\underline{\sigma}, \overline{\sigma}]$ with $0 \leq \underline{\sigma} \leq \overline{\sigma}$ (cf. Section 4.2), this leads, for instance, to the HJB equation

$$\partial_t y(t,u) = \sup_{\sigma \in [\underline{\sigma}, \overline{\sigma}]} \frac{\sigma^2}{2} \partial_{uu} y(t,u) \quad \text{for } t \geq 0 \text{ and } u \in \mathbb{R}. \tag{1.1}$$

The latter is typically referred to as G-heat equation, and their solutions (for different initial values) can be represented by means of the so-called G-expectation, cf. [31, 32]. Moreover, the G-heat equation (1.1) is intimately related to a stochastic optimal control problem with control set $[\underline{\sigma}, \overline{\sigma}]$. We refer to Denis et al. [10] for a detailed illustration of this relation.

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Inspired by a construction of Nisio [29], abstract versions of Hamilton-Jacobi-Bellman equations

$$\partial_t y = \sup_{\lambda\in\Lambda} A_\lambda y, \tag{1.2}$$

for a suitable family $(A_\lambda)_{\lambda\in\Lambda}$ of generators and a nonempty control set Λ, have been studied using a semigroup-theoretic approach, cf. Denk et al. [14] and Nendel and Röckner [27]. We would like to point out, choosing $A_\lambda := \frac{\lambda^2}{2}\partial_{uu}$ for $\lambda \in \Lambda := [\underline{\sigma}, \overline{\sigma}]$, the G-heat equation (1.1) is of the form (1.2).

From a semigroup-theoretic perspective it is therefore very natural to try relate convex differential equations $\partial_t y - Ay = 0$, such as the G-heat equation (1.1) or more general Hamilton-Jacobi-Bellman equations of the form (1.2) with $Ay = \sup_\lambda A_\lambda y$, to convex semigroups, where the semigroup property is the abstract analogon to the dynamic programming principle of the related optimal control problem. One classical approach to treat such fully nonlinear equations uses the theory of maximal monotone or m-accretive operators (see, e.g., Barbu [3], Bénilan and Crandall [4], Brézis [5], Evans [16], Kato [21], and the references therein). To show that an accretive operator is m-accretive, one has to prove that $1 + hA$ is surjective for $h > 0$, and in many cases it is quite delicate to verify this condition (see Example 4.2). Moreover, it is known that m-accretive operators lead to the existence of a mild solution, but the existence of strong solutions is only known under additional assumptions on the underlying Banach space, including reflexivity (see [3, Section 4.1]). In terms of nonlinear semigroups, this means that even if the initial value is smooth, the solution (i.e., the semigroup applied to the initial value) does not belong to the domain of the operator for positive time, so the domain of the operator is not invariant under the semigroup (see, e.g., [8, Section 4], or Example 4.4 below).

One approach to deal with this problem is to consider more general solution concepts, and in fact, this was one of the reasons for the introduction of viscosity solutions, cf. Crandall et al. [7], Crandall and Lions [9], and the discussion in Evans [16, Section 4]. On the other hand, one can construct operators with larger domains, which are invariant under the semigroup, in this way also obtaining regularity of the solution of the Cauchy problem. This is one of the topics of the present paper. We study convex monotone semigroups on spaces of continuous functions and construct invariant domains with a particular interest in the regularity and uniqueness of the solution. Here, the main object and the starting point of our investigation is the nonlinear semigroup. We point out that, in the context of optimal control theory, the uniqueness and regularity of solutions to Hamilton-Jacobi-Bellman equations are fundamental in order to come up with verification theorems; ensuring that the solution to the HJB equation is in fact the value function of an optimal control problem, cf. Fleming and Soner [18], Pham [34], and Yong and Zhou [37].

We consider convex monotone C_0-semigroups $S = \big(S(t)\big)_{t\geq 0}$ on Banach lattices and their generator A defined by

$$Ax := \lim_{h\downarrow 0} \frac{S(h)x - x}{h} \quad \text{for } x \in D(A),$$

where $D(A) := \big\{x \in X : \lim_{h\downarrow 0} \frac{S(h)x-x}{h} \text{ exists}\big\}$. Throughout, we consider the case where X is a Riesz subspace of some Dedekind σ-complete Riesz space $\overline{X}$ with an additional property on the dual space. Typical examples for X are the space BUC of all bounded uniformly continuous functions, the space C_0 of all continuous functions vanishing at infinity, or spaces of uniformly continuous functions with certain growth at infinity. We focus on monotone semigroups that are continuous from above, meaning that $S(t)x_n \downarrow 0$ for all $t \geq 0$, whenever $x_n \downarrow 0$. This additional continuity property allows to extend the semigroup to

$$X_\delta := \{x \in \overline{X} : x_n \downarrow x \text{ for some bounded sequence } (x_n)_n \text{ in } X\}.$$

In general, the domain $D(A)$ is not invariant under the semigroup S. However, the invariance can be achieved by extending the generator. Inspired by results on convex semigroups in L^p-like spaces in Denk et al. [13], we define the domain $D(A_\delta)$ of the monotone generator A_δ as the set of all $x \in X$ such that, for every sequence $(h_n)_n$ in $(0, \infty)$ with $h_n \downarrow 0$, there exists an approximating sequence $(y_n)_n$ in X such that

$$\left\| \frac{S(h_n)x - x}{h_n} - y_n \right\| \to 0 \quad \text{and} \quad y_n \downarrow y =: A_\delta x.$$

The main results in Sections 2 and 3 state that a convex monotone C_0-semigroup leaves the domain $D(A_\delta)$ of its monotone generator invariant (Theorem 2.6), and that the semigroup is uniquely determined by A_δ on $D(A_\delta)$ if, in addition, the semigroup is continuous from above (Corollary 3.2).

We also study even weaker forms of domains requiring only the local Lipschitz continuity of the map $t \mapsto S(t)x$, or, in other words, a weak Sobolev regularity of the map $t \mapsto S(t)x$, i.e., for every continuous linear functional μ, the map $\big(t \mapsto \mu S(t)x\big) \in W^{1,\infty}_{\text{loc}}\big([0,\infty)\big)$. These domains are shown to be invariant as well, and we discuss their relation to one another. In Section 4, we consider the example of the uncertain shift semigroup, which corresponds to the fully nonlinear PDE

$$\partial_t y(t,u) = |\partial_u y(t,u)|, \quad y(0,\cdot) = x. \tag{1.3}$$

Here, the nonlinear operator is given by $Ay = |\partial_u y|$. In that case, it holds $\mathrm{BUC}^1 \subset D(A_\delta) \subset W^{1,\infty}$ and $W^{1,\infty}$ is invariant under the corresponding semigroup. Note that (1.3) is a special case of the Hamilton-Jacobi PDE, where under appropriate conditions on the nonlinearity the viscosity solution is given by the Hopf-Lax formula (see, e.g., [17, Section 3.3], [25, Section 11.1]).

Similarly, for the second-order differential operator $Ay = \frac{1}{2}\max\{\underline{\sigma}\partial_{uu}y, \overline{\sigma}\partial_{uu}y\}$, where $0 \leq \underline{\sigma} \leq \overline{\sigma}$, we derive that $W^{2,\infty}$ is invariant under the respective semigroup, which corresponds to the G-heat equation. We remark that in the parabolic situation $\underline{\sigma} > 0$ many results on the solvability of this second-order fully nonlinear equation in Sobolev and Hölder spaces were obtained by Krylov, see [22, Chapters 12 and 13].

Assumptions and notation. Throughout this article, we assume that X is a real Banach lattice which is a Riesz subspace of a Dedekind σ-complete Riesz space $\overline{X}$. A typical example is the space BUC as a subspace of the space $\mathcal{L}^\infty$ of all bounded measurable functions. We denote by X' the dual space of X, i.e., the space of all continuous linear functionals $X \to \mathbb{R}$. For a sequence $(x_n)_n$ in X, we write $x_n \downarrow x$ if $(x_n)_n$ is decreasing, bounded from below, and $x = \inf_n x_n \in \overline{X}$. We define

$$X_\delta := \left\{x \in \overline{X} \colon x_n \downarrow x \text{ for some sequence } (x_n)_n \text{ in } X\right\}.$$

Let M be the space of all positive linear functionals $\mu \colon X \to \mathbb{R}$ which are continuous from above, i.e. $\mu x_n \downarrow 0$ for every sequence $(x_n)_n$ in X such that $x_n \downarrow 0$. Every $\mu \in M$ has a unique extension $\mu \colon X_\delta \to \mathbb{R}$ which is continuous from above, i.e. $\mu x_n \downarrow \mu x$ for every sequence $(x_n)_n$ in X_δ such that $x_n \downarrow x \in X_\delta$, see e.g. [12, Lemma 3.9]. We assume that the set M separates the points of X_δ, i.e. for every $x, y \in X_\delta$ with $x \neq y$ there exists some $\mu \in M$ with $\mu x \neq \mu y$. For an operator $S\colon X \to X$, we define

$$\|S\|_r := \sup_{x \in B(0,r)} \|Sx\|$$

for all $r > 0$, where $B(x_0, r) := \{x \in X\colon \|x - x_0\| \leq r\}$ for $x_0 \in X$. We say that an operator $S\colon X \to X$ is *convex* if $S\big(\lambda x + (1-\lambda)y\big) \leq \lambda Sx + (1-\lambda)Sy$ for all $\lambda \in [0,1]$, *positive homogeneous* if $S(\lambda x) = \lambda Sx$ for all $\lambda > 0$, *sublinear* if S is convex and positive homogeneous, *monotone* if $x \leq y$ implies $Sx \leq Sy$ for all $x, y \in X$, and *bounded* if $\|S\|_r < \infty$ for all $r > 0$.

We consider a convex C_0-semigroup S on X, i.e., a family $S = \big(S(t)\big)_{t \geq 0}$ of bounded operators $X \to X$ satisfying

- (S1) $S(0)x = x$ for all $x \in X$,
- (S2) $S(t+s)x = S(t)S(s)x$ for all $x \in X$ and $s, t \in [0, \infty)$,
- (S3) $S(t)x \to x$ as $t \downarrow 0$ for all $x \in X$.

We say that S is *monotone*, *convex*, or *sublinear* if $S(t)$ is monotone, convex, or sublinear for all $t \geq 0$, respectively. For $t \geq 0$ and $x \in X$, we define the convex operator $S_x(t)\colon X \to X$ by

$$S_x(t)y := S(t)(x+y) - S(t)x.$$

2. Invariant domains

In this section, we discuss the invariance of various notions of generators and domains. We start with a notion of continuity, which we will require on several occasions.

Definition 2.1. A monotone semigroup S is called *continuous from above* if $S(t)x_n \downarrow S(t)0$ for all $t \in [0, \infty)$ and every sequence $(x_n)_n$ in X with $x_n \downarrow 0$.

As before, let S be a convex semigroup on X. In contrast to [13], where the Banach lattice X is Dedekind σ-complete with order continuous norm, the domain

$$D(A) := \left\{ x \in X\colon \frac{S(h)x - x}{h} \text{ is convergent in } X \text{ for } h \downarrow 0 \right\}$$

is in general not invariant under the semigroup. For instance, for the uncertain semigroup $(S(t))_{t \in [0,\infty)}$ in Section 4.1, there exists some $x \in D(A)$ such that $S(t)x \notin D(A)$ for some $t \in (0, \infty)$. We therefore introduce the following modified versions of the domain.

Definition 2.2. The domain $D(A_\delta)$ of the *monotone generator* A_δ is defined as the set of all $x \in X$ such that, for every $(h_n)_n$ in $(0,\infty)$ with $h_n \downarrow 0$, there exists a sequence $(A_n x)_n$ in X and some $y \in X_\delta$ such that

$$\left\| \frac{S(h_n)x - x}{h_n} - A_n x \right\| \to 0 \quad \text{and} \quad A_n x \downarrow y. \tag{2.1}$$

We define the monotone generator $A_\delta\colon D(A_\delta) \subset X \to X_\delta$ of S by $A_\delta x := y$ for $x \in D(A_\delta)$, where y is the limit in (2.1), which is uniquely determined by Lemma B.1.

Definition 2.3. The *Lipschitz set* of the semigroup S is defined as

$$D_L := \left\{ x \in X\colon \sup_{h \in (0, h_0]} \left\| \frac{S(h)x - x}{h} \right\| < \infty \quad \text{for some } h_0 > 0 \right\}. \tag{2.2}$$

We further define the *symmetric Lipschitz set* of the semigroup S by

$$D_L^s := \big\{ x \in X\colon x, -x \in D_L \big\}.$$

Let $W^{1,\infty}_{\mathrm{loc}}\big([0,\infty)\big)$ denote the space of all functions $f \in L^\infty_{\mathrm{loc}}\big([0,\infty)\big)$ with weak derivative $f' \in L^\infty_{\mathrm{loc}}\big([0,\infty)\big)$. Recall that $W^{1,\infty}_{\mathrm{loc}}\big([0,\infty)\big)$ coincides with space of all locally Lipschitz continuous functions $[0,\infty) \to \mathbb{R}$. The following observation is one of the fundamental ingredients in the proof of Section 3, below.

Remark 2.4. Let $x \in X$. Then, $x \in D(L)$ if and only if

$$\big(t \mapsto \mu S(t)x\big) \in W^{1,\infty}_{\mathrm{loc}}\big([0,\infty)\big) \quad \text{for all } \mu \in X'.$$

In fact, by Proposition A.4, the map $[0,\infty) \to X,\ t \mapsto \mu S(t)x$ is locally Lipschitz for every $x \in D_L$ and $\mu \in X'$, which proves one direction of the equivalence. Now, assume that $\big(t \mapsto \mu S(t)x\big) \in W^{1,\infty}_{\mathrm{loc}}\big([0,\infty)\big)$ for all $\mu \in X'$. Then, for every $\mu \in X'$,

$$\sup_{h\in(0,1]} \left|\mu\left(\frac{S(h)x - x}{h}\right)\right| < \infty.$$

By the Banach-Steinhaus theorem, it follows that $x \in D_L$. If $\sup_{t\geq 0}\|S(t)\|_r < \infty$ for all $r \geq 0$, as, for example, in Section 4.1 and Section 4.2, we obtain that $x \in D(L)$ if and only if

$$\big(t \mapsto \mu S(t)x\big) \in W^{1,\infty}\big([0,\infty)\big) \quad \text{for all } \mu \in X'.$$

We say that the norm $\|\cdot\|$ on X is *σ-order continuous* if $\lim_{n\to\infty}\|x_n\| = 0$ for every decreasing sequence $(x_n)_{n\in\mathbb{N}}$ with $\inf_{n\in\mathbb{N}} x_n = 0$. The prime example for a Banach lattice with σ-order continuous norm is the closure C_0 w.r.t. supremum norm $\|\cdot\|_\infty$ of the space C_c of all continuous functions $\Omega \to \mathbb{R}$ with compact support, where Ω is a locally compact metric space. Moreover, we say that the norm $\|\cdot\|$ on X is *order continuous* if, for every net $(x_\alpha)_\alpha$ with $x_\alpha \downarrow 0$, we have $\|x_\alpha\| \to 0$. Notice that order continuity of the norm is, for example, implied by separability of X together with *Dedekind σ-completeness* of X, i.e., any countable non-empty subset of X, which is bounded above, has a supremum, cf. [26, Exercise 2.4.1] or [35, Corollary to Theorem II.5.14]. Typical examples for Banach lattices with order continuous norm are the spaces $L^p(\mu)$ for $p \in [1,\infty)$ and an arbitrary measure μ, the space c_0 of all sequences vanishing at infinity, and Orlicz spaces. We would like to point out that, due to its strong implications, we avoid order continuity of the norm in the present paper. A detailed study of convex semigroups on Banach lattices with order continuous norm can be found in [13].

We have the following relations between the domains and generators.

Lemma 2.5. *One has $D(A) \subset D(A_\delta) \subset D_L$, and $A_\delta|_{D(A)} = A$. If the norm $\|\cdot\|$ on X is σ-order complete, then $x \in D(A_\delta)$ with $A_\delta x \in X$ implies $x \in D(A)$ and $A_\delta x = Ax$. If the norm $\|\cdot\|$ on X is order complete, then $A_\delta = A$.*

Proof. We first assume that $x \in D(A)$. Then, for every $h_n \downarrow 0$ and $A_n x := Ax$ for all $n \in \mathbb{N}$, one has

$$\left\|\frac{S(h_n)x - x}{h_n} - A_n x\right\| \to 0,$$

which shows that $x \in D(A_\delta)$ with $A_\delta x = Ax$.

We next assume that $x \in D(A_\delta)$. Then, there exists some $h_0 > 0$ such that

$$\sup_{h\in(0,h_0]} \left\|\frac{S(h)x - x}{h}\right\| < \infty.$$

Otherwise, there exists a sequence $h_n \downarrow 0$ such that $\|\frac{S(h_n)x-x}{h_n}\| \geq n$ for all n. Since $x \in D(A_\delta)$ there exists a bounded decreasing sequence $(A_n x)_n$ in X such that $A_n x \downarrow A_\delta x$ and

$$\left\| \frac{S(h_n)x - x}{h_n} - A_n x \right\| \to 0.$$

But then,

$$\sup_n \left\| \frac{S(h_n)x - x}{h_n} \right\| \leq \sup_n \left\| \frac{S(h_n)x - x}{h_n} - A_n x \right\| + \sup_n \|A_n x\| < \infty,$$

which is a contradiction. This shows that $x \in D_L$. If the norm $\|\cdot\|$ on X is order complete and $x \in D(A_\delta)$ with $A_\delta x \in X$, then $\|A_n x - A_\delta x\| \to 0$, so that $\frac{S(h_n)x-x}{h_n} \to A_\delta x$. If, in addition, X is σ-Dedekind complete, then $A_\delta x \in X$ for all $x \in D(A_\delta)$, which shows that $A_\delta = A$. $\square$

For every $x \in X$ and $y \in X_\delta$, the directional derivative is defined as

$$S'_+(t,x)y = \inf_{h>0} \frac{S(t)(x+hy) - S(t)x}{h} \in X_\delta.$$

For further details on the directional derivative we refer to Appendix B. The main result of this subsection is that both, $D(A_\delta)$ and D_L, are invariant under the semigroup, and states regularity properties in the time variable t.

Theorem 2.6. *For every $x \in D_L$ one has*

(i) *$S(t)x \in D_L$ for all $t \in [0,\infty)$,*
(ii) *for every $\mu \in M$ there is a locally bounded measurable function $f_\mu\colon [0,\infty) \to \mathbb{R}$ with $\mu S(t)x = \mu x + \int_0^t f_\mu(s)\,ds$ for all $x \in D(A_\delta)$ and $t \geq 0$.*

For every $x \in D(A)$ it holds

(iii) *$S(t)x \in D(A_\delta)$ for all $t \geq 0$ with $A_\delta S(t)x = S'_+(t,x)A_\delta x$,*
(iv) *$\mu S(t)x = \mu x + \int_0^t \mu S'_+(s,x)A_\delta x\,ds$ for every $\mu \in M$ and all $t \geq 0$. In particular, $f_\mu(s) = \mu S'_+(s,x)A_\delta x$ for almost every $s \in [0,\infty)$.*

Moreover, (iii) *and* (iv) *hold for all $x \in D(A_\delta)$ if, in addition, the semigroup is monotone and continuous from above.*

Proof. (i) Fix $t \geq 0$. By Corollary A.2 there exist $L \geq 0$ and $r > 0$ such that

$$\|S(t)(y+x) - S(t)x\| \leq L\|y\|$$

for all $y \in B(x,r)$. Since $S(h)x \to x$ as $h \downarrow 0$, it follows that

$$\left\| \frac{S(h)S(t)x - S(t)x}{h} \right\| = \left\| \frac{S(t)S(h)x - S(t)x}{h} \right\| \leq L \left\| \frac{S(h)x - x}{h} \right\| < \infty$$

for all $h \in (0, h_0']$ and some $h_0' > 0$.

(ii) Since $x \in D_L$, it follows from Proposition A.4 that the map $[0,\infty) \to X$, $t \mapsto S(t)x$ is locally Lipschitz continuous. Fix $\mu \in M$. Since μ is continuous on X, see e.g. [1, Theorem 9.6], the map $[0,\infty) \to \mathbb{R}$, $t \mapsto \mu S(t)x$ is also locally Lipschitz continuous and is therefore in $W^{1,\infty}_{\mathrm{loc}}\big([0,\infty)\big)$ by Lebesgue's theorem. That is, there exists a locally bounded measurable function $f_\mu\colon [0,\infty) \to \mathbb{R}$ with $\mu S(t)x = \mu x + \int_0^t f_\mu(s)\,\mathrm{d}s$.

(iii) Fix $t > 0$, let $(h_n)_n$ be a sequence in $(0,\infty)$ with $h_n \downarrow 0$, and $x \in D(A)$. By Corollary A.2, there exists some $L > 0$ such that

$$\left\| \frac{S(t+h_n)x - S(t)x}{h_n} - \frac{S(t)(x + h_n Ax) - S(t)x}{h_n} \right\| = \left\| \frac{S(t)S(h_n)x - S(t)(x + h_n Ax)}{h_n} \right\|$$
$$\leq L \left\| \frac{S(h_n)x - x - h_n Ax}{h_n} \right\| = L \left\| \frac{S(h_n)x - x}{h_n} - Ax \right\| \to 0 \quad \text{as } n \to \infty.$$

Moreover, the sequence

$$A_n\big(S(t)x\big) := \frac{S(t)(x + h_n Ax) - S(t)x}{h_n}$$

is decreasing and satisfies $A_n(S(t)x) \downarrow S'_+(t,x)Ax$. This shows that $S(t)x \in D(A_\delta)$ with $A_\delta S(t)x = S'_+(t,x)Ax$. Recall that $Ax = A_\delta x$ for all $x \in D(A)$ by Lemma 2.5.

If in addition, S is monotone, continuous from above, and $x \in D(A_\delta)$, then there exists a bounded decreasing sequence $(A_n x)_n$ in X such that

$$\left\| \frac{S(h_n)x - x}{h_n} - A_n x \right\| \to 0 \quad \text{and} \quad A_n x \downarrow A_\delta x.$$

By Corollary A.2, there exists some $L > 0$ such that

$$\left\| \frac{S(t+h_n)x - S(t)x}{h_n} - \frac{S(t)(x + h_n A_n x) - S(t)x}{h_n} \right\| \leq L \left\| \frac{S(h_n)x - x}{h_n} - A_n x \right\| \to 0$$

as $n \to \infty$. By Lemma B.4, the sequence $(A_n S(t)x)$ given by

$$A_n S(t)x := \frac{S(t)(x + h_n A_n x) - S(t)x}{h_n}$$

is decreasing and satisfies $A_n S(t)x \downarrow S'_+(t,x)A_\delta x$. This shows that $S(t)x \in D(A_\delta)$ with $A_\delta S(t)x = S'_+(t,x)A_\delta x$.

(iv) Since $x \in D(A_\delta)$, it follows from Lemma 2.5 that $x \in D_L$. Fix $\mu \in M$. By (ii) one has

$$\mu S(t)x = \mu x + \int_0^t f_\mu(s)\, ds$$

for all $t \geq 0$. In particular, $t \mapsto \mu S(t)x$ is differentiable almost everywhere. Since μ is continuous from above it follows from the previous step (iii) that the derivative is a.e. given by

$$f_\mu(t) = \lim_{h \downarrow 0} \frac{\mu S(t+h)x - \mu S(t)x}{h} = \mu A_\delta S(t)x = \mu S'_+(t,x)A_\delta x.$$

The proof is complete. $\square$

For the symmetric Lipschitz set of a sublinear monotone semigroup, we have the following proposition.

Proposition 2.7. *Let S be sublinear and monotone. Then, the symmetric Lipschitz set D_L^s is a linear subspace of X. If*

$$-S(s)\big(-S(t)x\big) \geq S(t)\big(-S(s)(-x)\big) \quad \textit{for all } s,t \geq 0 \textit{ and } x \in X, \tag{2.3}$$

then $S(t)x \in D_L^s$ for all $t \geq 0$ and $x \in D_L^s$.

Proof. The sublinearity of S implies that

$$S(t)(x+\lambda y)-(x+\lambda y)\leq S(t)x-x+\lambda\big(S(t)y-y\big)$$

and

$$-S(t)(x+\lambda y)+x+\lambda y\leq S(t)(-x)+x+\lambda\big(S(t)(-y)+y\big)$$

for all $x,y\in X$ and $\lambda>0$. Consequently,

$$\|S(t)(x+\lambda y)-(x+\lambda y)\|\leq\|S(t)x-x\|+\|S(t)(-x)+x\|+\lambda\big(\|S(t)y-y\|+\|S(t)(-y)+y\|\big)$$

for all $x,y\in X$ and $\lambda>0$, which shows that $x+\lambda y\in D_L^s$ for all $x,y\in D_L^s$ and $\lambda>0$. Since $-x\in D_L^s$ for all $x\in D_L^s$, it follows that D_L^s is a linear subspace of X.

Now, let $x\in D_L^s$ and $t\geq 0$. Since $S(t)$ is sublinear and bounded, it is globally Lipschitz with some Lipschitz constant $L>0$ (see Lemma A.1). Therefore,

$$\|S(h)S(t)x-S(t)x\|\leq L\|S(h)x-x\|,$$

i.e. $S(t)x\in D_L$. It remains to show that $-S(t)x\in D_L$. First, observe that

$$-S(t)x-S(h)\big(-S(t)x\big)\leq -S(t)x+S(h)S(t)x\leq S(t)\big(S(h)x-x\big)$$

and, by (2.3),

$$S(h)\big(-S(t)x\big)+S(t)x\leq -S(t)\big(-S(t)(-x)\big)+S(t)x\leq S(t)\big(S(h)(-x)+x\big).$$

Therefore,

$$\big\|S(h)\big(-S(t)x\big)+S(t)x\big\|\leq L\Big(\|S(h)x-x\|+\big\|\big(S(h)(-x)+x\big)\big\|\Big),$$

which shows that $-S(t)x\in D_L$. □

Example 2.8. Let S be a translation-invariant sublinear monotone semigroup on the space $\mathrm{BUC}=\mathrm{BUC}(G)$, where G is an abelian group with a translation invariant metric d such that (G,d) is separable and complete. Here, *translation invariant* means that

$$\big(S(t)x(u+\cdot)\big)(0)=\big(S(t)x\big)(u)\quad\text{for all } x\in\mathrm{BUC},\ u\in G\text{ and } t\geq 0.$$

The space BUC of all bounded uniformly continuous functions $x\colon G\to\mathbb{R}$ is endowed with the supremum norm $\|x\|_\infty:=\sup_{u\in G}|x(u)|$. Under mild continuity assumptions, the semigroup has a dual representation

$$\big(S(t)x\big)(u)=\sup_{\mu\in\mathcal{P}_t}\int_G x(u+v)\,\mathrm{d}\mu_t(v)\quad\text{for all } x\in\mathrm{BUC},\ u\in G\text{ and } t\geq 0.\tag{2.4}$$

where $\mathcal{P}_t$ is a convex set of Borel measures on G for all $t\geq 0$. For further details on dual representations we refer to [12] and, for further examples, we refer to [14]. Notice that, under (2.4),

$$-\big(S(t)(-x)\big)(u)=\inf_{\mu\in\mathcal{P}_t}\int_G x(u+v)\,\mathrm{d}\mu_t(v)\quad\text{for all } x\in\mathrm{BUC},\ u\in G\text{ and } t\geq 0.$$

Then, for $x\in\mathrm{BUC}$, $u\in G$, $\mu_t\in\mathcal{P}_t$ and $\mu_s\in\mathcal{P}_s$, it follows from (2.4) and Fubini's theorem that

$$\begin{aligned}\int_G\big(S(t)x\big)(u+v)\,\mathrm{d}\mu_s(v)&\geq\int_G\int_G x(u+v+w)\,\mathrm{d}\mu_t(w)\,\mathrm{d}\mu_s(v)\\&=\int_G\int_G x(u+v+w)\,\mathrm{d}\mu_s(v)\,\mathrm{d}\mu_t(w)\\&\geq\int_G-\big(S(s)(-x)\big)(u+w)\,\mathrm{d}\mu_t(w).\end{aligned}$$

Taking the infimum over all $\mu_s \in \mathcal{P}_t$ and supremum over all $\mu_t \in \mathcal{P}_s$ yields

$$-S(s)\big(-S(t)x\big) \geq S(t)\big(-S(s)(x)\big).$$

By Proposition 2.7, we thus find that D_L^s is $S(t)$-invariant for all $t \geq 0$.

Remark 2.9. Consider the setup of the previous example. Given $C \geq 0$ and $h_0 > 0$, let $D_L^s(C, h_0)$ denote the set of all $x \in D_L^s$ such that $\|S(h)x - x\|_\infty \leq Ch$ and $\|S(h)(-x) + x\|_\infty \leq Ch$ for all $h \in [0, h_0]$. Let $x \in D_L^s(C, h_0)$ and ν be a Borel probability measure on G. Then, one has $x_\nu \in D_L^s(C, h_0)$, where $x_\nu(u) := \int_G x(u+v)\,\nu(dv)$. In fact, by a Banach space valued version of Jensen's inequality (see e.g. [14] or [27]) and the translation invariance of S,

$$\begin{aligned} S(h)x_\nu - x_\nu &= S(h)\bigg(\int_G x(\,\cdot\,+v)\,\mathrm{d}\nu(v)\bigg) - x_\nu \leq \int_G \big(S(h)x\big)(\,\cdot\,+v)\,\mathrm{d}\nu(v) - x_\nu \\ &= \int_G \big(S(h)x\big)(\,\cdot\,+v) - x(\,\cdot\,+v)\,\mathrm{d}\nu(v) \leq Ch \end{aligned}$$

for all $h \geq 0$. In a similar way, it follows that

$$S(h)(-x_\nu) + x_\nu \leq \int_G \big(S(h)(-x)\big)(\,\cdot\,+v) + x(\,\cdot\,+v)\,\mathrm{d}\nu(v) \leq Ch$$

for all $h \in [0, h_0]$. Combining these two estimates yields that

$$\big\|S(h)x_\nu - x_\nu\big\|_\infty \leq Ch \quad \text{and} \quad \big\|S(h)(-x_\nu) + x_\nu\big\|_\infty \leq Ch$$

for all $h \in [0, h_0]$, i.e. $x_\nu \in D_L^s(C, h_0)$.

3. Uniqueness

We are now ready to state the main result of this paper. Again, we assume that X is a Banach lattice which is a Riesz subspace of a Dedekind σ-complete Riesz space $\bar{X}$ and that the set M separates the points of X_δ (see Assumptions and Notation). We show that a convex semigroup is uniquely determined on $D(A_\delta)$ through its generator A_δ if the semigroup is, in addition, monotone and continuous from above.

Theorem 3.1. *Let S be a convex monotone C_0-semigroup on X which is continuous from above with monotone generator A_δ. Let $y\colon [0,\infty) \to X$ be a continuous function with $y(t) \in D(A_\delta)$ for all $t \geq 0$, and assume that, for all $t \geq 0$ and $(h_n)_n$ in $(0,\infty)$ with $h_n \downarrow 0$, there exists a bounded decreasing sequence $(B_n y(t))_n$ in X such that*

$$\left\|\frac{y(t+h_n) - y(t)}{h_n} - B_n y(t)\right\| \to 0 \quad \text{and} \quad B_n y(t) \downarrow A_\delta y(t).$$

Then, $y(t) = S(t)x$ for all $t \geq 0$, where $x := y(0)$.

Proof. Let $t > 0$ and $g(s) := S(t-s)y(s)$ for all $s \in [0,t]$. Fix $s \in (0,t)$. For every $h > 0$ with $h < t - s$ one has

$$\begin{aligned} \frac{g(s+h) - g(s)}{h} &= \frac{S(t-s-h)y(s+h) - S(t-s)y(s)}{h} \\ &= \frac{S(t-s-h)y(s+h) - S(t-s-h)y(s)}{h} \\ &\quad - \frac{S(t-s-h)S(h)y(s) - S(t-s-h)y(s)}{h}. \end{aligned}$$

Let $(h_n)_n$ in $(0,\infty)$ with $h_n \downarrow 0$ and $\mu \in M$. By assumption, for $y := y(s) \in D(A_\delta)$, there exists a bounded decreasing sequence $(B_n y)_n$ in X with

$$\left\| \frac{y(s+h_n) - y(s)}{h_n} - B_n y \right\| \to 0 \quad \text{and} \quad B_n y \downarrow A_\delta y. \tag{3.1}$$

We define

$$\nu_n z := \frac{\mu S(t-s-h_n)(y+h_n z) - \mu S(t-s-h_n) y}{h_n}$$

for all $z \in X_\delta$ and $n \in \mathbb{N}$ with $t-s-h_n > 0$, where we take the unique extension of S to X_δ given by Lemma B.2. Moreover, let

$$\nu z := \limsup_{n\to\infty} \nu_n z \quad \text{for all } x \in X.$$

We first show that

$$\nu z \le \inf_{h>0} \frac{\mu S(t-s)(y+hz) - \mu S(t-s)y}{h} \quad \text{for all } z \in X. \tag{3.2}$$

Indeed, for every $\varepsilon > 0$, there exists some $h_0 > 0$ and, by Corollary A.3 there exists some $m_0 \in \mathbb{N}$ such that

$$\begin{aligned} &\inf_{h>0} \frac{\mu S(t-s)(y+hz) - \mu S(t-s)y}{h} + 2\varepsilon \ge \frac{\mu S(t-s)(y+h_0 z) - \mu S(t-s)y}{h_0} + \varepsilon \\ &\ge \frac{\mu S(t-s-h_m)(y+h_0 z) - \mu S(t-s-h_m)y}{h_0} \end{aligned}$$

for all $m \ge m_0$. Hence, for all $n \ge m_0$, which satisfy $h_n \le h_0$, one has

$$\begin{aligned} &\inf_{h>0} \frac{\mu S(t-s)(y+hz) - \mu S(t-s)y}{h} + 2\varepsilon \\ &\qquad \ge \frac{\mu S(t-s-h_n)(y+h_n z) - \mu S(t-s-h_n)y}{h_n} = \nu_n z, \end{aligned}$$

which shows (3.2) by taking the limit superior as $n \to \infty$ and letting $\varepsilon \downarrow 0$. As a consequence of (3.2), it follows that ν is continuous from above (on X). Indeed, for every sequence $(z_n)_n$ in X with $z_n \downarrow 0$, one has

$$0 \le \inf_n \nu z_n \le \inf_{h>0} \inf_n \frac{\mu S(t-s)(y+hz_n) - \mu S(t-s)y}{h} = 0$$

so that $\nu z_n \downarrow 0$. Moreover, by definition, $\nu z = \lim_{n\to\infty} \sup_{k\ge n} \nu_k z$ for all $z \in X$, and therefore $\nu\colon X \to \mathbb{R}$ is convex. By [12, Lemma 3.9], ν uniquely extends to a convex monotone functional $\overline{\nu}\colon X_\delta \to \mathbb{R}$, which is continuous from above. We next show that

$$\limsup_{n\to\infty} \nu_n B_n y = \overline{\nu} A_\delta y. \tag{3.3}$$

To that end, let $\varepsilon > 0$. Then, there exist $n_0, m_0 \in \mathbb{N}$ such that

$$\overline{\nu} A_\delta y + 2\varepsilon \ge \overline{\nu} B_{n_0} y + \varepsilon = \nu B_{n_0} y + \varepsilon \ge \nu_m B_{n_0} y \ge \nu_m B_m y$$

for all $m \ge m_0 \vee n_0$, where the last inequality follows by monotonicity of ν_m. This shows that

$$\overline{\nu} A_\delta y \ge \limsup_{n\to\infty} \nu_n B_n y.$$

Further,

$$\begin{aligned}\overline{\nu}A_\delta y &= \inf_{m\in\mathbb{N}} \nu B_m y = \inf_{m\in\mathbb{N}} \inf_{n\in\mathbb{N}} \sup_{k\geq n} \nu_k B_m y = \inf_{n\in\mathbb{N}} \inf_{m\in\mathbb{N}} \sup_{k\geq n} \nu_k B_m y \\ &\leq \inf_{n\in\mathbb{N}} \sup_{k\geq n} \nu_k B_k y = \limsup_{n\to\infty} \nu_n B_n y.\end{aligned}$$

By Lemma A.2, there exists some $L > 0$ such that

$$\left\| \frac{S(t-s-h_n)y(s+h_n) - S(t-s-h_n)\big(y + h_n B_n y\big)}{h_n} \right\| \leq L \left\| \frac{y(s+h_n) - y}{h_n} - B_n y \right\| \to 0$$

as $n \to \infty$. Therefore, we conclude that

$$\limsup_{n\to\infty} \mu\left(\frac{S(t-s-h_n)y(s+h_n) - S(t-s-h_n)y}{h_n} \right) = \limsup_{n\to\infty} \nu_n B_n y = \overline{\nu}A_\delta y. \tag{3.4}$$

Since $y = y(s) \in D(A_\delta)$, it follows from (2.1) that there exists a bounded decreasing sequence $(A_n y)_n$ with

$$\left\| \frac{S(h_n)y - y}{h_n} - A_n y \right\| \to 0 \quad \text{and} \quad A_n y \downarrow A_\delta y.$$

By the same arguments as before, we get

$$\limsup_{n\to\infty} \mu\left(\frac{S(t-s-h_n)S(h_n)y - S(t-s-h_n)y}{h_n} \right) = \limsup_{n\to\infty} \nu_n A_n y = \overline{\nu}A_\delta y. \tag{3.5}$$

Hence, in combination with (3.4), we get

$$\begin{aligned}&\limsup_{n\to\infty} \mu\left(\frac{S(t-s-h_n)y(s+h_n) - S(t-s-h_n)y(s)}{h_n} \right) \\ &\quad = \limsup_{n\to\infty} \mu\left(\frac{S(t-s-h_n)S(h_n)y(s) - S(t-s-h_n)y(s)}{h_n} \right)\end{aligned} \tag{3.6}$$

for every sequence $(h_n)_n$ in $(0,\infty)$ with $h_n \downarrow 0$ and all $\mu \in M$. As a consequence, we conclude that

$$\frac{\mu g(s+h_n) - \mu g(s)}{h_n} \to 0 \tag{3.7}$$

for every sequence $(h_n)_n$ in $(0,\infty)$ with $h_n \downarrow 0$ and all $\mu \in M$. Indeed, by passing to a subsequence $(n_k)_k$, we may assume that

$$\limsup_{n\to\infty} \frac{\mu g(s+h_n) - \mu g(s)}{h_n} = \lim_{k\to\infty} \frac{\mu g(s+h_{n_k}) - \mu g(s)}{h_{n_k}}.$$

By passing to another subsequence, which we still denote by $(n_k)_k$, we can further assume that

$$\begin{aligned}&\liminf_{k\to\infty} \mu\left(\frac{S(t-s-h_{n_k})S(h_{n_k})y(s) - S(t-s-h_{n_k})y(s)}{h_{n_k}} \right) \\ &= \limsup_{k\to\infty} \mu\left(\frac{S(t-s-h_{n_k})S(h_{n_k})y(s) - S(t-s-h_{n_k})y(s)}{h_{n_k}} \right).\end{aligned} \tag{3.8}$$

Then, by applying the equality (3.6) to the subsequence $(h_{n_k})_k$ we obtain

$$\begin{aligned}\limsup_{n\to\infty} \frac{\mu g(s+h_n)-\mu g(s)}{h_n} &= \lim_{k\to\infty} \frac{\mu g(s+h_{n_k})-\mu g(s)}{h_{n_k}}\\ &\le \limsup_{k\to\infty} \mu\Big(\frac{S(t-s-h_{n_k})y(s+h_{n_k})-S(t-s-h_{n_k})y(s)}{h_{n_k}}\Big)\\ &\quad - \liminf_{k\to\infty} \mu\Big(\frac{S(t-s-h_{n_k})S(h_{n_k})y(s)-S(t-s-h_{n_k})y(s)}{h_{n_k}}\Big) = 0,\end{aligned}$$

where the last equality follows from (3.6) and (3.8). With similar arguments, we also obtain $\liminf_{n\to\infty} \frac{\mu g(s+h_n)-\mu g(s)}{h_n} \ge 0$, which shows (3.7).

Since μ is continuous on X, see e.g. [1, Theorem 9.6], it follows by the same arguments as in the proof of [13, Theorem 3.5] that $s \mapsto \mu g(s)$ is continuous on $[0,t]$. By [30, Lemma 1.1, Chapter 2], we conclude that the map $s \mapsto \mu g(s)$ is constant on $[0,t]$, since it is continuous and its right derivative vanishes on $[0,t)$. In particular, $\mu y(t) = \mu g(t) = \mu g(0) = \mu S(t)y(0)$ for all $\mu \in M$. This shows that $y(t) = S(t)y(0)$ as M separates the points of X. □

Corollary 3.2. *Let S be a convex monotone C_0-semigroup on X which is continuous from above with monotone generator A_δ, and let T be a convex C_0-semigroup on X with generator B and monotone generator B_δ such that $B_\delta \subset A_\delta$. If $\overline{D(B)} = X$, then $S(t) = T(t)$ for all $t \ge 0$.*

Proof. For every $x \in D(B)$, the mapping $y\colon [0,\infty) \to X$, $y(t) := T(t)x$ satisfies the assumptions of Theorem 3.1. Indeed, $y(0) = x$ by definition, $t \mapsto y(t)$ is continuous by Corollary A.3, and $y(t) \in D(B_\delta) \subset D(A_\delta)$ by Theorem 2.6 with

$$\Big\|\tfrac{y(t+h_n)-y(t)}{h_n} - B_n y(t)\Big\| \to 0 \quad \text{and} \quad B_n y(t) \downarrow B_\delta y(t) = A_\delta y(t)$$

where $B_n y(t) := \frac{T(t)(x+h_nBx)-T(t)x}{h_n}$ for all $n \in \mathbb{N}$. Hence, by Theorem 3.1, it follows that $T(t)x = y(t) = S(t)x$ for all $t \ge 0$. Since, by Lemma A.1, the bounded convex functions $T(t)$ and $S(t)$ are continuous, and $\overline{D(B)} = X$, it holds $S(t) = T(t)$ for all $t \ge 0$. □

4. Examples

4.1. **The uncertain shift semigroup on** BUC**.** Let G be a convex set endowed with a metric $d\colon G \times G \to [0,\infty)$. We assume that, for every $u, v \in G$ and $\lambda \in (0,1)$, there exists some $\lambda(u,v) \in G$ such that $d(u,\lambda(u,v)) = \lambda d(u,v)$ and $d(\lambda(u,v),v) = (1-\lambda)d(u,v)$. The space of all bounded uniformly continuous functions $x\colon G \to \mathbb{R}$ is denoted by $\mathrm{BUC} = \mathrm{BUC}(G)$ and endowed with the supremum norm $\|x\|_\infty := \sup_{u\in G} |x(u)|$. Notice that BUC is a Riesz subspace of the Dedekind σ-complete Riesz space $\mathcal{L}^\infty$ of all bounded Borel measurable functions $x\colon G \to \mathbb{R}$. On $\mathcal{L}^\infty$, we consider the partial order $x \le y$ whenever $x(u) \le y(u)$ for all $u \in G$.

The *uncertain shift semigroup* S on BUC is defined by

$$\big(S(t)x\big)(u) := \sup_{d(u,v)\le t} x(v) \quad \text{for all } x \in \mathrm{BUC},\ u \in G \text{ and } t \ge 0.$$

Lemma 4.1. *S is a sublinear monotone C_0-semigroup on* BUC*. Moreover,*

$$D_L = D_L^s = \mathrm{Lip}_b,$$

where $\mathrm{Lip}_b = \mathrm{Lip}_b(G)$ is the space of all bounded Lipschitz continuous functions $G \to \mathbb{R}$.

Proof. We first show that $S(t)\colon \mathrm{BUC} \to \mathrm{BUC}$ is well-defined and bounded. To this end, fix $x \in \mathrm{BUC}$. Since

$$|S(t)x(u)| \leq \sup_{d(u,v)\leq t} |x(v)| = \|x\|_\infty \quad \text{for all } u \in G,$$

it follows that $\|S(t)x\|_\infty \leq \|x\|_\infty$. Fix $\varepsilon > 0$ and $\delta > 0$ such that $|x(u) - x(v)| \leq \varepsilon$ for all $u, v \in G$ with $d(u,v) \leq \delta$. Let $u, v \in G$ with $d(u,v) \leq \delta$ and $w \in G$ with $d(u,w) \leq t$. Then, for $\lambda := \frac{t}{t+\delta}$, one has

$$d\big(v, \lambda(v,w)\big) = \lambda d(v,w) \leq \lambda(t+\delta) = t$$

and

$$d\big(w, \lambda(v,w)\big) = (1-\lambda) d(v,w) \leq (1-\lambda)(t+\delta) = \delta$$

Hence,

$$x(w) - \big(S(t)x\big)(v) \leq x(w) - x\big(\lambda(v,w)\big) \leq \varepsilon.$$

Taking the supremum over all $w \in G$ with $d(u,w) \leq t$, it follows that

$$\big(S(t)x\big)(u) - \big(S(t)x\big)(v) \leq \varepsilon.$$

By a symmetry argument, we obtain that $|S(t)x(u) - S(t)x(v)| \leq \varepsilon$, showing that $S(t)x$ is uniformly continuous with the same modulus of continuity as x. We thus have shown that $S(t)\colon \mathrm{BUC} \to \mathrm{BUC}$ is well-defined and bounded. By definition, each $S(t)$ is sublinear and monotone, and $S(0)x = x$ for all $x \in \mathrm{BUC}$. Moreover, for $t \leq \delta$, one has

$$\big|\big(S(t)x\big)(u) - x(u)\big| \leq \sup_{d(u,v)\leq t} |x(v) - x(u)| \leq \varepsilon$$

for all $u \in G$, i.e. $\|S(t)x - x\|_\infty \leq \varepsilon$ for all $t \leq \delta$, which shows that S is strongly continuous. It remains to show that S satisfies the semigroup property. Let $s, t \geq 0$. Further, let $u \in G$ and $w \in G$ with $d(u,w) \leq s + t$. Then, for $\lambda := \frac{t}{s+t}$, it holds

$$d\big(w, \lambda(u,w)\big) = (1-\lambda) d(u,w) \leq s$$

and

$$d\big(u, \lambda(u,w)\big) = \lambda d(u,w) \leq t.$$

Hence,

$$x(w) \leq \sup_{d(\lambda(u,w),v)\leq s} x(v) = \big(S(s)x\big)\big(\lambda(u,w)\big) \leq \sup_{d(u,v)\leq t} \big(S(s)x\big)(v) = \big(S(t)S(s)x\big)(u).$$

Taking the supremum over all $w \in G$ with $d(u,w) \leq s + t$, it follows that

$$\big(S(s+t)x\big)(u) \leq \big(S(t)S(s)x\big)(u).$$

Now, let $w \in G$ with $d(u,w) \leq t$. Then, there exists a sequence $(w_n)_n$ in G with $d(w, w_n) \leq s$ and $x(w_n) \to \big(S(s)x\big)(w)$. Then,

$$\big(S(s)x\big)(w) = \lim_{n\to\infty} x(w_n) \leq \sup_{d(u,v)\leq s+t} x(v) = \big(S(s+t)x\big)(u).$$

Taking the supremum over all $w \in G$ with $d(u,w) \leq t$, yields that

$$\big(S(t)S(s)x\big)(u) \leq \big(S(s+t)x\big)(u).$$

Altogether, we have shown that S is a sublinear monotone C_0-semigroup on BUC.

Now, let $x \in D_L$. Then, there exist $h_0 > 0$ and $C \geq 0$ such that $\|S(h)x - x\|_\infty \leq Ch$ for all $h \in [0, h_0]$. Hence, for all $u, v \in G$ with $d(u,v) =: h \leq h_0$,

$$x(u) - x(v) \leq \big(S(h)x\big)(v) - x(v) \quad \text{and} \quad x(v) - x(u) \leq \big(S(h)x\big)(u) - x(u).$$

This implies that $|x(u)-x(v)|\leq \|S(h)x-x\|_\infty\leq Ch = Cd(u,v)$. Since $x\in \mathrm{BUC}$ is bounded, it follows that $x\in \mathrm{Lip}_b$. On the other hand, if $x\in\mathrm{Lip}_b\subset\mathrm{BUC}$ with Lipschitz constant $C>0$, it follows that

$$\|\big(S(h)x\big)(u)-x(u)\|\leq \sup_{d(u,v)\leq h}|x(v)-x(u)|\leq Cd(u,v)\leq Ch$$

for all $u\in G$ and $h\geq 0$. Therefore $x\in D_L$. Since $-x\in\mathrm{Lip}_b$ for all $x\in\mathrm{Lip}_b$, it follows that $\mathrm{Lip}_b\subset D_L^s$. Since, by definition, $D_L^s\subset D_L$, the assertion follows. □

We now specialize on the case, where $G=\mathbb{R}$ with the Euclidean distance $d(u,v)=|u-v|$. In this case, the uncertain shift semigroup is given by

$$\big(S(t)x\big)(u)=\sup_{|v|\leq t}x(u+v)$$

for all $u\in\mathbb{R}$ and $t\in[0,\infty)$. By Lemma 4.1, it follows that S is a sublinear monotone C_0-semigroup on BUC. In addition, by Dini's lemma, it is continuous from above. Denote by $A_\delta\colon D(A_\delta)\subset\mathrm{BUC}\to\mathrm{BUC}_\delta$ the monotone generator of S. Notice that BUC_δ is the space of all bounded upper semicontinuous functions $\mathbb{R}\to\mathbb{R}$. Moreover, by Lemma 4.1, we have that $D_L=D_L^s=W^{1,\infty}$. Recall that the space of all Lipschitz continuous functions coincides with the space $W^{1,\infty}=W^{1,\infty}(\mathbb{R})$ of all functions with weak derivative $x'\in L^\infty=L^\infty(\mathbb{R})$ (w.r.t. the Lebesgue measure). As usual, we denote by $\mathrm{BUC}^1=\mathrm{BUC}^1(\mathbb{R})$ the set of all $x\in\mathrm{BUC}$ which are differentiable with $x'\in\mathrm{BUC}$. From a PDE point of view, one might consider BUC^1 to be the canonical choice for the domain of the generator of S. However, the following example shows that this does not yield an m-accretive operator.

Example 4.2. Let $X=\mathrm{BUC}$ and $B\colon D(B)\to X$ with $Bx:=|x'|$ for $x\in D(B):=\mathrm{BUC}^1$. Then B is accretive, i.e., for some (equivalently, for any) $h>0$, $1+hB$ is injective and

$$\big\|(1+hB)^{-1}y_1-(1+hB)^{-1}y_2\big\|\leq\|y_1-y_2\|\quad\text{for all } y_1,y_2\in R(1+hB),$$

cf. [3, Proposition 3.1] and [8, Formula (8)]. To see this, let $x_1,x_2\in D(B)$ and $h>0$. We set $y:=x_1-x_2$ and choose a sequence $(u_k)_k$ in $\mathbb{R}$ with $|y(u_k)|\to\|y\|_\infty$ as $k\to\infty$. If $(u_k)_k$ has a finite accumulation point u_0, then we have $|y(u_0)|=\|y\|_\infty$, and the function y has a local extremum at u_0. Consequently, $y'(u_0)=0$ and therefore $x_1'(u_0)=x_2'(u_0)$. We obtain

$$\begin{aligned}\big\|x_1-x_2+&h(|x_1'|-|x_2'|)\big\|_\infty\geq\big|x_1(u_0)-x_2(u_0)+h(|x_1'(u_0)|-|x_2'(u_0)|\big|\\&=|x_1(u_0)-x_2(u_0)|=|y(u_0)|=\|y\|_\infty=\|x_1-x_2\|_\infty.\end{aligned}$$

If $(u_k)_{k\in\mathbb{N}}$ has no finite accumulation point, we may w.l.o.g. assume that $u_k\to\infty$ as $k\to\infty$. Moreover, taking a subsequence we may also assume that $y(u_k)\to\pm\|y\|_\infty$ as $k\to\infty$. Again, w.l.o.g. let $y(u_k)\to\|y\|_\infty$ as $k\to\infty$. Let $\varepsilon>0$, and choose $k_0\in\mathbb{N}$ with

$$\|y\|_\infty-\tilde\varepsilon\leq y(u_k)\leq\|y\|_\infty\quad\text{for all } k\geq k_0,$$

where we have set $\tilde\varepsilon:=\min\{\frac{\varepsilon}{2},\frac{\varepsilon}{2h}\}$. Let $\ell_0>k_0$ with $u_{\ell_0}\geq u_{k_0}+1$. As $y\in\mathrm{BUC}^1$, there exists some $v_0\in(u_{k_0},u_{\ell_0})$ with

$$\tilde\varepsilon\geq|y(u_{\ell_0})-y(u_{k_0})|=|y'(v_0)|\,|u_{\ell_0}-u_{k_0}|\geq|y'(v_0)|.$$

We obtain

$$\begin{aligned}\big\|x_1 - x_2 + h(|x_1'| - |x_2'|)\big\|_\infty &\geq \big|x_1(v_0) - x_2(v_0) + h(|x_1'(v_0)| - |x_2'(v_0)|)\big| \\ &\geq |x_1(v_0) - x_2(v_0)| - h\big||x_1'(v_0)| - |x_2'(v_0)|\big| \\ &\geq |x_1(v_0) - x_2(v_0)| - h|x_1'(v_0) - x_2'(v_0)| \\ &= |y(v_0)| - h|y'(v_0)| \geq \|y\|_\infty - \tfrac{\varepsilon}{2} - h\,\tfrac{\varepsilon}{2h} = \|x_1 - x_2\|_\infty - \varepsilon.\end{aligned}$$

As $\varepsilon > 0$ was arbitrary, we see that also in this case the inequality

$$\big\|x_1 - x_2 + h(|x_1'| - |x_2'|)\big\|_\infty \geq \|x_1 - x_2\|_\infty$$

holds, which shows that B is accretive.

However, the operator B is not m-accretive, i.e., the operator $1+hB$ is not surjective. For this, let $h > 0$, and set $f(u) := (1 - |u|)\mathbf{1}_{[-1,1]}(u)$ for $u \in \mathbb{R}$. Assume that there exists some $x \in D(B)$ with

$$x(u) + h|x'(u)| = f(u) \quad \text{for } u \in \mathbb{R}. \tag{4.1}$$

As f is an even function, we see that the function $\bar{x}$ defined by $\bar{x}(u) := x(-u)$ is also a solution of (4.1). As B is accretive, the operator $1 + hB$ is injective, which shows that $\bar{x} = x$, i.e., the solution x is an even function, too. As $x \in \mathrm{BUC}^1$, we get $x'(0) = 0$ and therefore $x(0) = f(0) = 1$. Now, the differentiability of x leads to a contradiction to $x(u) \leq f(u)$ for all $u \in \mathbb{R}$, which holds by (4.1).

Proposition 4.3. *Let $G = \mathbb{R}$. Then, $\mathrm{BUC}^1 \subset D(A) \subset D(A_\delta) \subset D_L = D_L^s = W^{1,\infty}$. In particular, $S(t)x \in W^{1,\infty}$ for every $x \in W^{1,\infty}$ and all $t \geq 0$. Further, for $x \in D(A_\delta)$, one has $A_\delta x = |x'|$ almost everywhere.*

Proof. If $x \in \mathrm{BUC}^1$, it follows from Taylor's theorem that

$$\left\|\frac{S(h)x - x}{h} - |x'|\right\|_\infty \to 0 \quad \text{as } h \downarrow 0.$$

Hence, by Lemma 2.5 and Lemma 4.1,

$$\mathrm{BUC}^1 \subset D(A) \subset D(A_\delta) \subset D_L = D_L^s = W^{1,\infty}.$$

In particular, $W^{1,\infty}$ is invariant under the uncertain shift semigroup by Theorem 2.6.

Let $x \in W^{1,\infty}$. By Rademacher's theorem the function x is differentiable almost everywhere. If x is differentiable at u, then

$$\begin{aligned}\lim_{h\downarrow 0} \frac{\big(S(h)x\big)(u) - x(u)}{h} &= \lim_{h\downarrow 0} \sup_{|v|\leq h} \frac{x(u+v) - x(u)}{h} = \lim_{h\downarrow 0} \sup_{|v|=h} \frac{x(u+v) - x(u)}{h} \\ &= |x'(u)|.\end{aligned}$$

Since, for $x \in D(A_\delta)$, one has

$$\big(A_\delta x\big)(u) = \lim_{h\downarrow 0} \frac{\big(S(h)x\big)(u) - x(u)}{h}$$

for all $u \in \mathbb{R}^d$, we conclude that $A_\delta x = |x'|$ almost everywhere. Here, x' is understood as the weak derivative in L^∞. ☐

The following example shows that, in general, $D(A)$ is not invariant under the semigroup $(S(t))_{t\geq 0}$.

Example 4.4. Consider the case $G = \mathbb{R}$, and let $x \in \mathrm{BUC}^1$ with

$$x(u) = \begin{cases} u^2, & u \in [0,2], \\ u^4, & u \in [-2,0). \end{cases}$$

Then, by Proposition 4.3, $S(1)x \in D(A_\delta) \subset W^{1,\infty}$ with $A_\delta S(1)x = \big|\big(S(1)x\big)'\big|$. By definition of $S(1)$,

$$\big(S(1)x\big)(u) = \begin{cases} (u+1)^2, & u \in [0,1], \\ (u-1)^4, & u \in [-1,0), \end{cases}$$

which implies that

$$\big(S(1)x\big)'(u) = \begin{cases} 2(u+1), & u \in (0,1), \\ 4(u-1)^3, & u \in (-1,0). \end{cases}$$

Therefore, $A_\delta S(1)x = \big|\big(S(1)x\big)'\big| \notin \mathrm{BUC}$ and, in particular, $S(1)x \notin D(A)$.

4.2. **The symmetric Lipschitz set of the G-expectation.** We consider the G-expectation on $\mathrm{BUC} = \mathrm{BUC}(\mathbb{R})$, which corresponds to the sublinear semigroup

$$\big(S(t)x\big)(u) := \sup_{\underline{\sigma} \le \sigma \le \overline{\sigma}} \mathbb{E}\Big[x\Big(u + \int_0^t \sigma_s \, dW_s\Big)\Big] \quad \text{for } x \in \mathrm{BUC},\ u \in G \text{ and } t \ge 0,$$

where W is a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and the supremum is taken over all progressively measurable processes with values in $[\underline{\sigma}, \overline{\sigma}]$, see e.g. [11] and [33] for an overview on G-expectations. We assume that $0 \le \underline{\sigma} \le \overline{\sigma}$. One can verify that S is a translation invariant sublinear C_0-semigroup on BUC which is continuous from above. Moreover, an application of Itô's formula shows that

$$\lim_{h \downarrow 0} \frac{S(h)x - x}{h} = \frac{1}{2} \max\left\{\underline{\sigma} x'', \overline{\sigma} x''\right\}$$

for all $x \in \mathrm{BUC}^2 = \mathrm{BUC}^2(\mathbb{R})$.

Fix $x \in D_L^s$. By definition of the symmetric Lipschitz set, there exist $C > 0$ and $h_0 > 0$ such that $x \in D_L^s(C, h_0)$. For every $\delta > 0$, define $x_\delta(u) := \int_{\mathbb{R}} x(u+v)\, \nu_\delta(dv)$, where ν_δ is the normal distribution $\mathcal{N}(0,\delta)$ with mean zero and variance δ. Then, $x_\delta \in \mathrm{BUC}^2$ for all $\delta > 0$, and $\|x_\delta - x\|_\infty \to 0$ as $\delta \downarrow 0$. In view of Remark 2.9, one has

$$S(h)x_\delta - x_\delta \le Ch \quad \text{and} \quad -S(h)(-x_\delta) - x_\delta \ge -Ch$$

for all $h \in [0, h_0]$ and $\delta > 0$. Hence, letting $h \downarrow 0$, it follows that

$$\frac{1}{2}\overline{\sigma} x_\delta'' \le C \quad \text{and} \quad \frac{1}{2}\underline{\sigma} x_\delta'' \ge -C.$$

This shows that $\|x_\delta''\|_\infty$ is uniformly bounded in $\delta > 0$. Hence, there exists a sequence $\delta_n \downarrow 0$ such that $\int_u^v x_{\delta_n}''(z) - y(z)\,\mathrm{d}z \to 0$ for all $u, v \in \mathbb{R}$ with $u < v$ and some $y \in L^\infty$ w.r.t. the Lebesgue measure. By the dominated convergence theorem, we get

$$\begin{aligned} x(u+h) - x(u) &= \lim_{n\to\infty} \Big(x_{\delta_n}(u+h) - x_{\delta_n}(u)\Big) \\ &= \lim_{n\to\infty} \Big(h x_{\delta_n}'(u) + \int_u^{u+h} \int_u^v x_{\delta_n}''(z)\,\mathrm{d}z\,\mathrm{d}v\Big) \\ &= \Big(\lim_{n\to\infty} h x_{\delta_n}'(u)\Big) + \int_u^{u+h} \int_u^v y(z)\,\mathrm{d}z\,\mathrm{d}v \end{aligned}$$

for all $u \in \mathbb{R}$ and $h > 0$. In particular, x is differentiable with $x'(t) = \lim_{n\to\infty} x'_{\delta_n}(t)$ and second weak derivative $x'' = y$, i.e. $x \in W^{2,\infty}$. This shows that $D_L^s = W^{2,\infty}$. As an application of Proposition 2.7, it follows that $S(t)x \in W^{2,\infty}$ for all $t \geq 0$ and $x \in W^{2,\infty}$. Notice that we do not assume that $\underline{\sigma} > 0$, which is a standard assumption in PDE theory for obtaining regularity results in Hölder spaces (cf. Lieberman [24, Chapter XIV] and Peng [33, Appendix C, §4] for a short survey).

Appendix A. Some auxiliary results

In this section, we list some basic properties for convex operators and semigroups, which can be found, for example, in [13].

Lemma A.1 (cf. [13, Corollary A.4]). *Let $S\colon X \to X$ be a bounded and convex operator. Then, S is Lipschitz on bounded subsets, i.e., for every $r > 0$, there exists some $L > 0$ such that $\|Sx - Sy\| \leq L\|x - y\|$ for all $x, y \in B(0, r)$.*

Lemma A.2 (cf. [13, Corollary 2.4]). *Let $T > 0$ and $x_0 \in X$. Then, there exist $L \geq 0$ and $r > 0$ such that*

$$\sup_{t\in[0,T]} \|S(t)y - S(t)z\| \leq L\|y - z\|$$

for all $y, z \in B(x_0, r)$.

Corollary A.3 (cf. [13, Corollary 2.5]). *The map $[0,\infty) \to X,\ t \mapsto S(t)x$ is continuous for all $x \in X$.*

Proof. Let $t \geq 0$ and $x \in X$. Then, by Corollary A.2, there exist $L \geq 0$ and $r > 0$ such that

$$\sup_{s\in[0,t+1]} \|S(s)y - S(s)x\| \leq L\|y - x\|$$

for all $y \in B(x, r)$. Moreover, there exists some $\delta \in (0, 1]$ such that $\|S(h)x - x\| \leq r$ for all $h \in [0, \delta]$. For $s \geq 0$ with $|s - t| \leq \delta$ it follows that

$$\|S(t)x - S(s)x\| = \|S(s \wedge t)S(|t - s|)x - S(s \wedge t)x\| \leq L\|S(|t - s|)x - x\| \to 0$$

as $s \to t$. □

Proposition A.4 (cf. [13, Proposition 2.7]). *Let $x \in X$ with*

$$\sup_{h\in(0,h_0]} \left\| \frac{S(h)x - x}{h} \right\| < \infty \quad \textit{for some } h_0 > 0.$$

Then, the map $[0,\infty) \to X,\ t \mapsto S(t)x$ is locally Lipschitz continuous, i.e., for every $T > 0$, there exists some $L_T \geq 0$ such that $\|S(t)x - S(s)x\| \leq L_T|t - s|$ for all $s, t \in [0, T]$.

Appendix B. Directional derivatives of convex operators

In this section, we provide some fundamental results on directional derivatives of convex operators. Again, we are in the standard setting of the paper, i.e. X is a Banach lattice which is a Riesz subspace of a Dedekind σ-complete Riesz space $\bar{X}$. Let M be the space of all positive linear functionals $\mu\colon X_\delta \to \mathbb{R}$ which are continuous from above. We assume that M separates the points of X_δ.

Lemma B.1. *Let $(x_n)_n$ be a sequence in X. If $(y_n)_n$ and $(z_n)_n$ are decreasing sequences in X which are bounded from below such that $\|x_n - y_n\| \to 0$ and $\|x_n - z_n\| \to 0$, then $\inf_n y_n = \inf_n z_n$.*

Proof. Fix $\mu \in M$. Since μ is continuous on X, see e.g. [1, Theorem 9.6], one has

$$\mu(y_n - z_n) = \mu(y_n - x_n) + \mu(x_n - z_n) \to 0,$$

which shows that

$$\mu\Big(\inf_n y_n\Big) = \lim_{n\to\infty} \mu y_n + \lim_{n\to\infty} \mu(z_n - y_n) = \lim_{n\to\infty} \mu z_n = \mu\Big(\inf_n z_n\Big).$$

Since $\inf_n y_n, \inf_n z_n \in X_\delta$ and M separates the points of X_δ, it follows that $\inf_n y_n = \inf_n z_n$. □

Lemma B.2. *Let $S\colon X \to X$ be a convex monotone operator which is continuous from above. Then, it has a unique monotone convex extension $S\colon X_\delta \to X_\delta$ which is continuous from above.*

Proof. For each $\mu \in M$, the convex monotone functional $\mu S\colon X \to \mathbb{R}$ is continuous from above. Thus, by [12, Lemma 3.9], it has a unique extension to a convex monotone functional $\mu S\colon X_\delta \to \mathbb{R}$ which is continuous from above.

Fix $x \in X_\delta$. For $(x_n)_n$ and $(y_n)_n$ in X with $x_n \downarrow x$ and $y_n \downarrow x$, one has

$$\mu\Big(\inf_n Sx_n\Big) = \inf_n \mu S x_n = \mu S\Big(\inf_n x_n\Big) = \mu S\Big(\inf_n y_n\Big) = \inf_n \mu S y_n = \mu\Big(\inf_n S y_n\Big),$$

so that $Sx := \inf_n Sx_n$ is well defined as M separates the points of X_δ. Then, S is convex and continuous from above as

$$\mu\Big(\inf_n Sx_n\Big) = \inf_n \mu S x_n = \mu S x$$

for every $(x_n)_n$ in X_δ with $x_n \downarrow x \in X_\delta$. Moreover, if $\tilde{S}$ is another extension which is continuous from above, then $\tilde{S}x = \lim_{n\to\infty} \tilde{S}x_n = \lim_{n\to\infty} Sx_n = Sx$ for every $(x_n)_n$ in X with $x_n \downarrow x \in X_\delta$, which shows that such an extension is unique. □

Let $S\colon X \to X$ be a convex operator. Then, the function

$$\mathbb{R}\setminus\{0\} \to X, \quad h \mapsto \frac{S(x+hy) - Sx}{h}$$

is increasing for all $x, y \in X$. Hence, for all $x \in X$, the operators

$$S'_+(x)y := \inf_{h>0} \frac{S(x+hy) - Sx}{h} \quad \text{and} \quad S'_-(x)y := \sup_{h<0} \frac{S(x+hy) - Sx}{h} \tag{B.1}$$

for $y \in X$ are well-defined with values in $\bar{X}$ since

$$S'_+(x)y = \inf_{n\in\mathbb{N}} \frac{S(x+h_n y) - Sx}{h_n} \in X_\delta \quad \text{and} \quad S'_-(x)y = \sup_{n\in\mathbb{N}} \frac{Sx - S(x-h_n y)}{h_n} \in -X_\delta$$

for every sequence $(h_n)_n$ in $(0,\infty)$ with $h_n \to 0$. The following properties follow directly from the definition.

Remark B.3. For all $x, y \in X$ one has

(i) $S'_-(x)y = -S'_+(x)(-y)$,
(ii) $S'_-(x)y \le S'_+(x)y$,
(iii) $S'_+(x)y = S'_-(x)y = Sy$, if S is linear.

If $S\colon X \to X$ is a convex monotone operator which is continuous from above, then by Lemma B.2 it has a unique convex monotone extension $S\colon X_\delta \to X_\delta$ which is continuous from above. Therefore, $S(x+hy) \in X_\delta$ for all $y \in X_\delta$ and $h > 0$. Hence, $S'_+(x)$ extends to

$$S'_+(x)\colon X_\delta \to X_\delta, \quad y \mapsto \inf_{h>0} \frac{S(x+hy) - Sx}{h}$$

for all $x \in X$.

Lemma B.4. *Let $S\colon X \to X$ be a convex monotone operator which is continuous from above. For every $x \in X$, the mapping $S'_+(x)$ has the following properties:*

(i) *$S'_+(x)y \le S_x y$ for all $y \in X_\delta$,*
(ii) *$S'_+(x)\colon X_\delta \to X_\delta$ is convex and positive homogeneous,*
(iii) *$S'_+(x)$ is continuous from above,*
(iv) *$\frac{S(x+h_n y_n)-Sx}{h_n} \downarrow S'_+(x)y$, for all sequences (h_n) in $(0,\infty)$ and (y_n) in X_δ which satisfy $h_n \downarrow 0$ and $y_n \downarrow y \in X_\delta$.*

Proof. (i) For every $y \in X_\delta$, one has $S'_+(x)y \le S(x+y) - S(x) = S_x(y)$.

(ii) For $\varepsilon > 0$, $\mu \in M$, and $\lambda \in [0,1]$ there exists some $h > 0$ such that

$$\begin{aligned}
&\mu\big(\lambda S'_+(x)y_1 + (1-\lambda)S'_+(x)y_2\big) + \varepsilon \\
&\ge \lambda \frac{\mu S(x+hy_1) - \mu S(x)}{h} + (1-\lambda)\frac{\mu S(x+hy_2) - \mu S(x)}{h} \\
&\ge \frac{\mu S\big(x + h(\lambda y_1 + (1-\lambda)y_2)\big) - \mu S(x)}{h} \ge \mu S'_+(x)\big(\lambda y_1 + (1-\lambda)y_2\big).
\end{aligned}$$

This shows that $S'_+(x)$ is convex on X_δ. Moreover, for $\lambda > 0$ and $y \in X_\delta$ it holds

$$S'_+(x)(\lambda y) = \inf_{h>0} \frac{S(x+\lambda h y) - Sx}{h} = \lambda \inf_{h>0} \left(\frac{S(x+\lambda h y) - Sx}{\lambda h}\right) = \lambda S'_+(x)y.$$

(iii) For every $y_n \downarrow y$ one has

$$\inf_n S'_+(x)y_n = \inf_{h>0} \inf_n \frac{S(x+hy_n) - S(x)}{h} = \inf_{h>0} \frac{S(x+hy) - S(x)}{h} = S'_+(x)y.$$

(iv) Fix $\varepsilon > 0$, and $\mu \in M$. By definition of S'_+ and continuity from above of S, there exist $n_0, m_0 \in \mathbb{N}$ such that

$$\begin{aligned}
\mu S'_+(x)y + 2\varepsilon &\ge \frac{\mu S(x + h_{n_0} y) - \mu Sx}{h_{n_0}} + \varepsilon \\
&\ge \frac{\mu S(x + h_{n_0} y_{m_0}) - \mu Sx}{h_{n_0}} \ge \frac{\mu S(x + h_{n_1} y_{n_1}) - \mu Sx}{h_{n_1}}
\end{aligned}$$

for $n_1 := n_0 \vee m_0$. This shows that $\frac{S(x+h_n y_n)-Sx}{h_n} \downarrow S'_+(x)y$. □

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