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On averaged expected cost control for 1D ergodic diffusions with switching

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Abstract

Ergodic Bellman's (HJB) equation is proved for a 1D controlled diffusion with switching with variable diffusion and drift coefficients both depending on control, and convergence of the iteration improvement algorithm to its (unique) solution is established.

1 Introduction

The paper is a continuation of the earlier publication about ergodic control for the 1D diffusion without switching [1] and of its short version [2]. Partly the proofs repeat

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the calculus in [1], but new serious technical difficulties arise due to the switching component, which, in particular, make Bellman's equation a system, and which do not allow automatic extension to this case and which difficulties are overcome here.

On a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ with a one-dimensional $(\mathcal{F}_t)$ Wiener process $W = (W_t)_{t\geq 0}$ on it, a one-dimensional SDE with coefficients b, σ with switching and a stationary Markov control function α is considered,

$$dX_t^\alpha = b(\alpha(X_t^\alpha, Z_t), X_t^\alpha, Z_t)\,dt + \sigma(\alpha(X_t^\alpha, Z_t), X_t^\alpha, Z_t)\,dW_t, \quad t \geq 0, \tag{1}$$
$$X_0^\alpha = x,\ Z_0 = z,$$

where Z_t is a continuous-time Markov process on a finite state space $\mathcal{S} = (1, \ldots, N)$ with (positive) intensities λ_{ij}, $1 \leq i, j \leq N$, $j \neq i$. In the first instant throughout the paper we assume that these intensities do not depend on the control.

Let a non-empty compact set U be a range of possible control values. Let $b : U \times \mathbb{R} \times \mathcal{S} \to \mathbb{R}$, $\sigma : U \times \mathbb{R} \times \mathcal{S} \to \mathbb{R}$, $\alpha : \mathbb{R} \times \mathcal{S} \to U$ be given Borel functions; some more regularity assumptions will be presented later on.

Denote by L^α the (extended) generator, which corresponds to the equation (1) with a fixed function $\alpha(\cdot)$ depending on (x, z):

$$L^\alpha h(x,z) = b(\alpha(x,z),x,z)\frac{dh}{dx}(x,z) \ + \frac{1}{2}\sigma^2(\alpha(x,z),x,z)\frac{dh}{dx^2}(x,z) \tag{2}$$
$$\tag{3}$$
$$+ \sum_{j\in\mathcal{S}\setminus z} \lambda_{zj}\left(h(x,j) - h(x,z)\right), \quad x \in \mathbb{R},\ z \in \mathcal{S}. \tag{4}$$

Given a running cost function $f : U \times \mathbb{R} \times \mathcal{S} \to \mathbb{R}$ from a suitable function class $\mathcal{K}$ – which will be defined in the sequel – our goal is to choose an optimal, or nearly-optimal stationary Markov control strategy $\alpha : \mathbb{R} \times \mathcal{S} \to U$ such that the corresponding solution X^α minimizes the averaged cost function

$$\rho^\alpha(x,z) := \limsup_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{E}_{x,z} f(\alpha(X_t^\alpha, Z_t), X_t^\alpha, Z_t)\,dt. \tag{5}$$

The class of such strategies with a weak solution of the equation (1) will be denoted by $\mathcal{A}$; they will be called admissible. In the sequel, assumptions will be assumed such that any Borel funciton of the variables (x, z) with values in U belongs to $\mathcal{A}$. For every $\alpha \in \mathcal{A}$ we define the function $f^\alpha : \mathbb{R} \to \mathbb{R}$, $f^\alpha(x,z) = f(\alpha(x,z),x,z)$, $x \in \mathbb{R}, z \in \mathcal{S}$. Now, the definition (5) is equivalent to

$$\rho^\alpha(x,z) = \limsup_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{E}_{x,z} f^\alpha(X_t^\alpha, Z_t)\,dt. \tag{6}$$

Finally, the cost function is defined as

$$\rho(x,z) := \inf_{\alpha\in\mathcal{A}} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{E}_{x,z} f^\alpha(X_t^\alpha, Z_t)\,dt. \tag{7}$$

Suppose that for every $\alpha \in \mathcal{A}$ the solution X^α of the equation (1) is a Markov ergodic process, that is, there exists a unique limiting and invariant distribution μ^α of the pair $(X_t^\alpha, Z_t),\ t\to\infty$, the same for all initial conditions $X_0 = x \in \mathbb{R}, Z_0 = z \in \mathcal{S}$,

$$\rho^\alpha(x,z) \equiv \rho^\alpha := \int f^\alpha(x,z)\,\mu^\alpha(dxdz) =: \langle f^\alpha, \mu^\alpha\rangle, \tag{8}$$

and

$$\rho(x,z) \equiv \rho := \inf_{\alpha\in\mathcal{A}} \int f^\alpha(x,z)\,\mu^\alpha(dxdz) = \inf_{\alpha\in K}\langle f^\alpha, \mu^\alpha\rangle. \tag{9}$$

Note that under our assumptions below on the growth bounds of f and on the ergodicity properties of the process with any strategy, the value of ρ will not depend on x, z. Let us define auxiliary functions

$$v^\alpha(x,z) := \int_0^\infty E_{x,z}(f^\alpha(X_t^\alpha, Z_t) - \rho^\alpha)\,dt, \qquad \alpha\in\mathcal{A}. \tag{10}$$

Under the assumptions below the integral in (10) will converge.

The first goal of this paper is to prove that the cost ρ is the component of the pair (V,ρ), which is a unique in the appropriate class solution of the *ergodic HJB or Bellman's equation,*

$$\inf_{u\in U}[L^u V(x,z) + f^u(x,z) - \rho] = 0, \quad x\in\mathbb{R},\ z\in\mathcal{S}, \tag{11}$$

where similarly to (2) L^u for $u \in U$ is defined by the formula

$$\begin{aligned} L^u h(x,z) &:= b(u,x,z)\frac{dh}{dx}(x,z) \,+ \frac{1}{2}\sigma^2(u,x,z)\frac{d^2h}{dx^2}(x,z) \\ &+ \sum_{j\in\mathcal{S}\setminus z} \lambda_{zj}\left(h(x,j) - h(x,z)\right), \quad x\in\mathbb{R},\ z\in\mathcal{S}. \end{aligned} \tag{12}$$

Note that (11) may be treated as one equation, or as a system of (nonlinear) differential equations of the second order linked one to another via the zero order terms.

The uniqueness of the component V will be shown up to an additive constant, while ρ will be unique in the standard sense. The meaning of the function V is that it coincides with v^α for the optimal strategy α if the latter exists. The class where solution (V, ρ) will be studied is the family of all Borel functions V on $\mathbb{R} \times \mathcal{S}$ and constants $\rho \in \mathbb{R}$ such that V has two Sobolev derivatives in x for each z which are all locally integrable in any power, and such that the function V has a no more than some polynomial growth in x. The equation (11) is to be understood initially almost everywhere in x; however, eventually it will be shown that this equation is satisfied for all $x \in \mathbb{R}$ and $z \in \mathcal{S}$.

The second goal is to show convergence of the "reward improvement algorithm", or, "RIA", or, in some papers, "PIA" (for "policy improvement algorithm") to the cost ρ, see the details of this algorithm below.

For ergodicity, we will assume recurrence conditions, which provide a uniform recurrence for *any* strategy. However, it is likely that such restrictions may be relaxed to the "near monotonicity" type conditions as in [3] and other sources.

Remarks and comments about the history of the problem the reader may find in [1], [3], and in the references therein. In most of the works on the topic, measurability of the optimal or improved strategy (see below) is assumed. Some account about it can be found in [1].

The paper consists of three sections: 1 – Introduction, 2 – Assumptions and some auxiliaries, 3 – Main result and its proof.

2 Assumptions and some auxiliaries

We assume the following.

(A1) The function b is bounded, $|b(u, x, z)| \le C_b$, and

$$\lim_{|x|\to\infty} \sup_{u\in U, z\in\mathcal{S}} x\, b(u, x, z) = -\infty. \tag{13}$$

(A2) The function σ is bounded, $|\sigma(u, x, z)| \le C_\sigma$, uniformly non-degenerate, $|\sigma(u, x, z)|^{-1} \le C_\sigma$, all intensities are bounded and bounded away from zero: $0 < \min_{ij} \lambda_{ij} \le \max_{ij} \lambda_{ij} < \infty$.

(A3) The function f belongs to the class $\mathcal{K}$ of functions which admit a uniform in u (and z) polynomial bound: there exist constants $C_1, m_1 > 0$ such that for any x,

$$\sup_{u\in U, z\in\mathcal{S}} |f^u(x,z)| \le C_1(1+|x|^{m_1}).$$

(A4) The functions $\sigma(u,x,z)$, $b(u,x,z)$, $f^u(x,z)$ are continuous in u for every x, z.

(A5) The set U is compact.

We will need the following four lemmata.

Lemma 1. *Weak solution (X,Z) for the equation (1) exists for any strategy α, is unique in distribution and, hence, is Markov and strong Markov.*

Proof. This proof is provided for the completeness of this presentation and for its self-sufficiency; in principle, the result is simple and should be known, although, the authors were unable to find a proper reference for this particuar model. We mention that under the condition of continuity of all coefficients with respect to the variable x such existence is stated in [13, Section 2.2.1]; for strong solutions (which are not used in the present paper, but which can be used nevertheless) under some additional regularity conditions on the coefficients see [10].

1. Under the additional assumption of the positivity of σ existence of solution follows by a mollification of $\bar\sigma(\cdot,z) := \sigma(\alpha(\cdot,z),\cdot,z)$ with respect to $\cdot$ which stands for x, from tightness of the family of solutions related to different smoothed σ and from Skorokhod's techniques of the unique probability space where some subsequence of solutions converges in probability to some limit, which will be a required solution. This step is analogous to the proof of a weak solution in [7, Chapter 2, Theorem 2.6.1] Note that both functions $\bar\sigma(x,z)$ and $\bar b(x,z) := b(\alpha(x,z),x,z)$ are automatically continuous with respect to the discrete component z, which helps to justify the limiting procedure similarly to [7, Chapter 2] where there is no discrete component. In the limit, clearly, the second component Z is a Markov process with intensities λ_{ij} as for the prelimiting processes Z^n, and the first component X solves the equation (1), which follows by the same lines of arguments as in [7, Chapter 2, Theorem 2.6.1].

2. Further, for a general σ which can change its sign, there is no mollification which keeps the smoothed non-degenerate. So, we firstly find a solution to the equation

$$X_t = x + \int_0^t b(\alpha(X_s,Z_s),X_s,Z_s)\,ds + \int_0^t |\sigma(\alpha(X_s,Z_s),X_s,Z_s)|\,dW_s.$$

Note that the same process is a solution to the equation

$$X_t = x + \int_0^t b(\alpha(X_s, Z_s), X_s, Z_s)\, ds + \int_0^t \sigma(\alpha(X_s, Z_s), X_s, Z_s)\, d\tilde{W}_s,$$

with a new Wiener process

$$\tilde{W}_t = \int_0^t \mathrm{sign}(\sigma(\alpha(X_s, Z_s), X_s, Z_s))\, dW_s,$$

as required.

3. Weak uniqueness for 1D homogeneous diffusion without switching is known for one SDE; below we recall one possible way to establish it. Then, weak uniqueness for the SDE with switching follows from its uniqueness between the jumps of the component Z (which jumps do not accumulate a.s.) and from the fact that these moments of jumps are defined independently of the trajectory of X.

One way to show weak uniqueness for the 1D homogeneous SDE without switching is to change time making the diffusion coefficient a constant and to use uniqueness (weak and strong) for the resulting SDE in a new time scale and the possibility of the inverse time change given the non-degeneracy of σ^2. This approach will be presented in details in the proof of the lemma 4; note that, of course, the proof of the present lemma is independent of the lemma 4.

4. The Markov and strong Markov properties follow both from weak uniqueness and from the results in [8]. The Lemma 1 is proved.

$\square$

Lemma 2. *Let the assumptions (A1) – (A3) hold true. Then*

- *For any $C_1, \ell_1 > 0$ there exist $C_2, \ell_2 > 0$ such that for any strategy $\alpha \in \mathcal{A}$ and for any function g growing no faster than $C_1(1 + |x|^{\ell_1})$ uniformly with respect to z,*

$$\sup_t |\mathbb{E}_{x,z} g(X_t^\alpha, Z_t)| \le C(1 + |x|^{\ell_2}). \tag{14}$$

- *For any $\alpha \in \mathcal{A}$, the invariant measure μ^α integrates any polynomial and, moreover,*

$$\sup_{\alpha \in \mathcal{A}} \int |x|^k \, \mu^\alpha(dxdz) < \infty, \quad \forall\, k > 0. \tag{15}$$

- *For any strategy $\alpha \in \mathcal{A}$ the function ρ^α is a constant, and*

$$\sup_{\alpha\in\mathcal{A}} |\rho^\alpha| \le C < \infty; \tag{16}$$

- *Moreover, for any $k > 0$ and $f \in \mathcal{K}$,*

$$\sup_{\alpha\in\mathcal{A}} t^k \left|\mathbb{E}_{x,z} f^\alpha(X_t^\alpha, Z_t) - \rho^\alpha\right| \to 0, \quad t \to \infty, \tag{17}$$

and

$$\sup_{\alpha\in\mathcal{A}} \left| \frac{1}{T}\int_0^T \mathbb{E}_x f^\alpha(X_t^\alpha, Z_t)\, dt - \rho^\alpha \right| \to 0, \quad T \to \infty. \tag{18}$$

Proof. Since existence and weak uniqueness for the solution of the SDE (7) is known from the previous lemma, the proof of the present lemma follows from [15] and [12] with some minimal adjustments due to the presence of switching. Here we remind briefly the main arguments. For a compact $K = B_R \subset \mathbb{R}$ define a stopping time

$$\tau = \tau(K) = \inf(t \ge 0 : |X_t| \in K).$$

From [15] (as well as from some other sources) it follows via Lyapunov functions like $|x|^\ell$ and $(1+t)^k|x|^\ell$ that for any $k > 0$ there exist $C > 0$ and $\ell > 0$ such that

$$E_x \tau^k \le C(1 + |x|^\ell), \tag{19}$$

and in addition that

$$\sup_{t\ge 0} E_x 1(t < \tau)|X_t|^\ell \le C(1 + |x|^\ell). \tag{20}$$

Along with the irreducibility of the process due to the non-degeneracy of σ, these are the main tools for establishing all statements of the Lemma.

To prove (14) let us define by induction: $\tau_1 = \tau$;

$$T_n := \inf(t > \tau_n : |X_t| \ge R + 1), \quad \tau_{n+1} := \inf(t > T_n : |X_t| \le R), \quad n = 1, 2, \ldots$$

Then it follows from the general properties of homogeneous Markov processes that there exists $\alpha > 0$ such that

$$\sup_\omega \sup_{|x|\le R} E_x(\exp(\alpha(T_n - \tau_n))|\mathcal{F}_{\tau_n}) < \infty. \tag{21}$$

Also, due to the bound (19) above we have,

$$E_x(\tau_{n+1}-T_n)^k|\mathcal{F}_{T_n}) \le C. \tag{22}$$

Then the bound (14) follows form the representation

$$\mathbb{E}_x|X_t|^\ell = \mathbb{E}_x|X_t|^\ell 1(t<\tau_1) + \sum_{n\ge 1}\mathbb{E}_x|X_t|^\ell 1(\tau_n \le t < T_n) + \sum_{n\ge 1}\mathbb{E}_x|X_t|^\ell 1(T_n \le t < \tau_{n+1})$$

and from (21) and (22) in a standard way (cf. [15]).

Further, the estimates (15) and (16) follow from (14), ergodicity and Fatou lemma; and (17) and (18) are corollaries of (19) similarly to [15] due to the coupling method. The Lemma 2 is proved. □

In the next Lemma we use Sobolev spaces $W^2_{p,loc}$ with any $p>1$; recall that this notation (see [9]) stands for the class of (equivalent) functions $g(x,z)$, $x\in\mathbb{R}, z\in S$, which are locally in x integrable in L_p (denoted as $L_{p,loc}$) and such that u and its first and second generalised derivatives u_x and u_{xx} with respect to x are approximated in $L_{p,loc}$ by smooth (infinitely differentiable) functions.

Lemma 3. *Let the assumptions (A1) – (A4) be satisfied. Then the cost function v^α has the following properties:*

- $v^\alpha \in W^2_{p,loc}$ *for any* $p\ge 1$ *and satisfies in the Sobolev sense a Poisson equation (system) in the whole space,*

  $$L^\alpha v^\alpha + f^\alpha - \langle f^\alpha, \mu^\alpha\rangle = 0. \tag{23}$$

  In particular, for almost every $x\in\mathbb{R}$ and every $z\in\mathcal{S}$,

  $$L^\alpha(x,z)v^\alpha(x,z) + f^\alpha(x,z) - \langle f^\alpha, \mu^\alpha\rangle = 0, \tag{24}$$

  or, equivalently,

  $$\inf_{u\in U}\left[\frac{1}{\sigma^2(u,x,z)}L^u V(x,z) + \frac{f^u(x,z)}{\sigma^2(u,x,z)} - \frac{\rho}{\sigma^2(u,x,z)}\right] = 0, \quad x\in\mathbb{R},\ z\in\mathcal{S}. \tag{25}$$

- *Solution of the equation (23) is unique up to an additive constant in the class of Sobolev solutions $W^2_{p,loc}$ in x for each z with a no more than some (any) polynomial growth.*

- $\langle v^\alpha, \mu^\alpha\rangle = \sum_z \int v^\alpha(x,z)\mu^\alpha(dx,z) = 0.$
- *For any strategy* α *the function* v^α *is continuous as well as* $(v^\alpha)'_x$, *and there exist* $C, m > 0$ *both depending only on the constants in (A1)–(A3) such that*

$$\sup_\alpha (|v^\alpha(x,z)| + |v^\alpha(x,z)'_x|) \le C(1+|x|^\ell). \qquad (26)$$

- $v^\alpha \in C^{1,Lip}$ *in* x *(i.e.,* $(v^\alpha)'_x$ *is locally Lipschitz).*

Proof. Denote

$$\bar L^u = \frac{1}{\sigma^2(u,x,z)} L^u, \quad \bar L^\alpha = \frac{1}{\sigma^2(\alpha(x,z),x,z)} L^\alpha.$$

1. Due to the Lemma 1 the solution of the SDE (1) exists and is weakly unique. Hence, in particular, invariant measure μ^α is unique. Therefore, the value ρ^α is also uniquely determined, which was, actually, already used in the previous lemma.

2. The random time change ([11, Chapter 2.5], [4, Theorem 15.5]) allows to rewrite the definition of v^α as follows,

$$v^\alpha(x,z) = \int_0^\infty E_{x,z}(f^\alpha(X_t^\alpha, Z_t) - \rho^\alpha)\,dt$$

$$= \int_0^\infty E_{x,z}\bar f^\alpha(\bar X_t^\alpha, \bar Z_t)\,dt, \qquad (27)$$

with

$$\bar f^\alpha(x,z) = \frac{f^\alpha(x,z) - \rho^\alpha}{\sigma^2(\alpha(x,z),x,z)},$$

and where the process $\bar X_t^\alpha$ satisfies the following SDE

$$d\bar X_t^\alpha = \bar b(\alpha(\bar X_t, \bar Z_t), \bar X_t, \bar Z_t)\,dt + d\bar W_t, \quad \bar X_0^\alpha = x,$$

where $t'(t)$ is the inverse function for the mapping $t \mapsto \displaystyle\int_0^t \sigma^2(\alpha(X_s^\alpha, Z_s), X_s^\alpha, Z_s)\,ds$, with a new Wiener process $\bar W_t = \displaystyle\int_0^{t'(t)} \sigma(\alpha(X_s^\alpha, Z_s), X_s^\alpha, Z_s)\,dW_s$ and with $\bar b^\alpha(x,z) = b^\alpha(x,z)\sigma^{-2}(\alpha(x,z),x,z)$, and where $\bar Z_t$ is a jump process on the state space S with intensities

$$\bar\lambda^\alpha_{\bar Z_t,j}(\bar X_t) = \frac{\lambda_{\bar Z_t,j}}{\sigma^2(\alpha(\bar X_t, \bar Z_t), \bar X_t, \bar Z_t)}.$$

So, it is reasonable to introduce new intensities at (x,z) by the expression $\bar\lambda^\alpha_{z,j}(x) = \lambda_{z,j}\sigma^{-2}(\alpha(x,z),x,z)$. Note that this couple $(\bar X, \bar Z)$ is well-defined as a unique in distribution strong Markov process by its SDE and the intensities (as well as the old pair (X,Z) in a changed time). The process $(\bar X,\bar Z)$ is unique in distribution, ergodic with a unique invariant measure $\bar\mu^\alpha(dx,dz)$ and there is a convergence better than any polynomial to it in total variation. Hence, the only option for the integral in (27) to converge is that $\bar f^\alpha$ satisfies a centering condition

$$\langle \bar f^\alpha, \bar\mu^\alpha\rangle = 0.$$

Now it follows from (27) that the function v^α is a solution of the Poisson equation

$$\bar L^\alpha v^\alpha + \bar f^\alpha = 0; \tag{28}$$

recall that here

$$\bar L^u g(x,z) := \bar b(u,x,z)\frac{dg}{dx}(x,z) + \frac{1}{2}\frac{d^2 g}{dx^2}(x,z)$$

$$+ \sum_{j\in\mathcal{S}\setminus z} \bar\lambda^u_{zj}(x)\left(g(u,x,j) - g(u,x,z)\right), \quad x\in\mathbb{R},\ z\in\mathcal{S},$$

and

$$\bar L^\alpha g(x,z) := \bar b(\alpha(x,z),x,z)\frac{dg}{dx}(x,z) + \frac{1}{2}\frac{d^2 g}{dx^2}(x,z)$$

$$+ \sum_{j\in\mathcal{S}\setminus z} \bar\lambda^\alpha_{zj}(x)\left(g(\alpha(x,j),x,j) - g(\alpha(x,z),x,z)\right), \quad x\in\mathbb{R},\ z\in\mathcal{S},$$

3. Let us show existence of derivatives v^α_x and v^α_{xx} in a Sobolev sense in the spaces $W^2_{p,loc}$ with any $p>1$ (see notations in [9]). For this aim let us denote

$$v^{(T)}(s,x,z) := \int_0^{T-s} E_{x,z}\bar f^\alpha(\bar X^\alpha_t, \bar Z_t)\,dt, \qquad 0\le s\le T,$$

for any $T>0$. Actually, only $v^{(T)}(0,x,z)$ will be analysed in the sequel; however, it is important for us to define $v^{(T)}$ as a function of three variables (s,x,z) because we will use the fact that with respect to this triple $v^{(T)}$ is a solution of the backward

Kolmogorov system in Sobolev spaces. In fact, it turns out to be a bit easier to prove a formally stronger claim than just existence of v_x^α and v_{xx}^α, namely, that

$$(v_x^{(T)}, v_{xx}^{(T)})|_{t=0} \underset{T\to\infty}{\to} (v_x^\alpha, v_{xx}^\alpha) \quad \text{in the } L_{p,loc} \text{ sense,}$$

or, a bit more precisely, that the left hand side here converges in $L_{p,loc}$, and the limit turns out to be $(v_x^\alpha, v_{xx}^\alpha)$. The functions v^T have been introduced for this aim. In the sequel the following short notations will be used: $\bar T_t \bar f^\alpha(x,z) := \mathbb{E}_{x,z}(\bar f^\alpha(\bar X_t^\alpha, \bar Z_t))$. (On the one hand, there is a slight abuse of notations here: T is time and $\bar T_t$ is a semigroup; on the other hand, the semigroup is never used without a lower index, so there is no danger to get one for the other.) We have,

$$\begin{aligned} v_x^{(T)}(0,x,z) = \partial_x \int_0^T \bar T_t \bar f^\alpha(x,z))dt = \partial_x \int_0^1 \bar T_t \bar f^\alpha(x,z)\, dt + \partial_x \int_1^T \bar T_t \bar f^\alpha(x,z))dt \\ = \partial_x \int_0^1 \bar T_t \bar f^\alpha(x,z)dt + \partial_x \int_1^T \bar T_1 \bar T_{t-1} \bar f^\alpha(x,z)\, dt \\ = \partial_x \int_0^1 \bar T_t \bar f^\alpha(x,z)dt + \int_1^T \partial_x \bar T_1(\bar T_{t-1}\bar f^\alpha)(x,z)dt. \end{aligned} \tag{29}$$

The first term here does not change with time (as far as $T \geq 1$) and is well-defined in $L_{p,loc}$ Sobolev sense along with $\partial_x^2 \displaystyle\int_0^1 \bar T_t \bar f^\alpha(x,z))dt$, due to [14, Theorem 5.5, 5.7].

The integrand in the second term admits the bound (see [14, Theorems 5.7 & 5.5] with $T_1 = \epsilon > 0$),

$$\|\partial_x \bar T_\cdot(\bar T_{t-1}\bar f^\alpha)\|_{W_p^2((\epsilon,1)\times B_R)} \leq C_\epsilon(\|\bar T_\cdot(\bar T_{t-1}\bar f^\alpha)\|_{L_p((\epsilon,1)\times B_{R+1?})})$$

It follows from ergodic bounds of the SDE solutions similar to those in [15] that the right hand side here decreases to zero faster than any polynomial in time, and, hence, the second integral in the representation (29) converges as $T \to \infty$; this convergence is locally uniform with respect to the initial value x and, of course, uniform with respect to z, as the latter variable takes values from a finite set. The same is also true for the second derivative $\partial_x^2 \displaystyle\int_1^\infty \bar T_t \bar f^\alpha(x,z))dt$. Thus, we obtain,

$$\partial_x \int_0^T \bar T_t \bar f^\alpha(x,z))dt \to \bar v_1(x,z), \quad \partial_x^2 \int_0^T \bar T_t \bar f^\alpha(x,z))dt \to \bar v_2(x,z)$$

as $T \to \infty$, both locally uniformly in x and, hence, also in the $L_{p,loc}$ sense.

In a standard manner by integration over x it follows that $\bar{v}_1(x,z)$ and $\bar{v}_2(x,z)$ serve as v^α_x and v^α_{xx}, respectively. Namely, for any x_1, x_2 and for each z we have by the first theorem of the calculus,

$$v^T(0,x_1,z) - v^T(0,x_2,z) = \int_{x_1}^{x_2} v^T_x(0,x',z)\,dx'$$

and in the limit as $T \to \infty$ we obtain

$$v^\alpha(x_1,z) - v^\alpha(x_2,z) = \int_{x_1}^{x_2} \bar{v}_1(x',z)\,dx',$$

which means exactly that $\bar{v}_1 = v^\alpha_x$. Similarly,

$$v^T_x(0,x_1,z) - v^T_x(0,x_2,z) = \int_{x_1}^{x_2} v^T_{xx}(0,x',z)\,dx'$$

and in the limit as $T \to \infty$ we obtain

$$v^\alpha_x(x_1,z) - v^\alpha_x(x_2,z) = \int_{x_1}^{x_2} \bar{v}_2(x',z)\,dx',$$

which means that $\bar{v}_2 = v^\alpha_{xx}$, as required. So, indeed, $v^\alpha \in W^2_{p,loc}$ (and even a bit better, with classical derivatives almost everywhere (a.e.), as usual in $\mathbb{R}^1$).

4. Now let us show that in the generalised sense the function u from the previous step satisfies the equation (28) and, hence, also to the equivalent one (23). Indeed, for any smooth test function $g(x,z)$ with a compact support in x we have,

$$\begin{aligned}
\langle \bar{L}^\alpha v^\alpha, g\rangle = \langle v^\alpha, \bar{L}^* g\rangle = \lim_{T\to\infty} \langle v^{(T)}|_{t=0}, (\bar{L}^\alpha)^* g\rangle = \lim_{T\to\infty} \langle \bar{L}^\alpha v^{(T)}|_{t=0}, g\rangle \\
= \lim_{T\to\infty} \langle \int_0^T \bar{L}^\alpha E_{x,z} \bar{f}^\alpha(\bar{X}^\alpha_t, \bar{Z}_t)dt, g\rangle \\
= \lim_{T\to\infty} \langle \int_0^T \partial_t E_{x,z} \bar{f}^\alpha(\bar{X}^\alpha_t, \bar{Z}_t)dt, g\rangle \\
= \lim_{T\to\infty} \langle E_{x,z} \bar{f}^\alpha(\bar{X}^\alpha_T, \bar{Z}_T) - \bar{f}^\alpha(x,z), g\rangle \\
= -\langle \bar{f}^\alpha, g\rangle,
\end{aligned}$$

which means that u is a generalised solution of (28); the last equality in this calculus is because of the ergodic properties of the process $(\bar{X}_t^\alpha, \bar{Z}_t)$ from the Lemma 2 and due to the cenering property of $\bar{f}^\alpha$. Now, since v^α, actually, possesses two Sobolev derivatives in x, this function is not just a generalised, but a true Sobolev solution to the equation (23).

5. Now let us show uniquenes of solution of the linear Poisson equation (system)

$$L^\alpha v(x,z) + f^\alpha(x,z) - \rho^\alpha = 0$$

in the Sobolev sense up to a constant in the class of functions growing no faster than some polynomial in x. (Clearly, if v is a solution, then $v + C$ is also a solution for any constant C.) Suppose there is a solution v and let us add a constant to it so that to make it centered, i.e.,

$$\langle v, \mu^\alpha \rangle = 0.$$

Let us use Itô – Krylov's formula, which is applicable between the jumps of the component Z, and at the moments of jumps the zero order term $\displaystyle\int \sum_{j\in\mathcal{S}\setminus Z_t} \lambda_{zj}\left(v(X_t^\alpha, j) - v(X_t^\alpha, Z_t)\right) dt$ arises (see [13, Lemma 2.2.3] for the classical derivatives; in the case of Sobolev derivatives it follows due to Krylov's estimates [7, Chapter 2] in a standard way). We obtain

$$\mathbb{E}_{x,z} v(X_T^\alpha, Z_T) - v(x,z) = \mathbb{E}_{x,z} \int_0^T L^\alpha v(X_t^\alpha, Z_t) dt.$$

Since $\mathbb{E}_{x,z} v(X_T^\alpha, Z_T) \to \langle v, \mu^\alpha \rangle = 0$ as $T \to \infty$, then in the limit we get

$$v(x,z) = \int_0^\infty \mathbb{E}_{x,z}(f^\alpha(X_t^\alpha, Z_t) - \rho^\alpha)\, dt. \tag{30}$$

This shows uniqueness of solution of the Poisson system (23) in the described class of functions.

6. Now when it has been established that the components of the vector-solution v of the equation (23) are the components of the vector-function v^α from (10), we can treat the system as a single equation on each component where all other components become the parts of the right hand side. Since this right hand side satisfies polynomial growth restrictions, the property (26) follows from a similar lemma for a single equation without switching (see [1, Lemma 1]).

We have already seen that the centered version of the solution to (23) is equal to v^α defined in (10); so, v^α itself is μ^α-centered. Equivalently the equality $\langle v^\alpha, \mu^\alpha \rangle = 0$ follows from integration of the right hand side in the definition of v^α with respect to μ^α, due to the centering property of the function $\bar{f}^\alpha$. The Lemma 3 follows. $\square$

Denote $\lambda_- = \min_{i,j} \lambda_{ij}$, $\lambda_+ = \max_{i,j} \lambda_{ij}$, $\Lambda_i = \sum_{j:j\neq i} \lambda_{ij}$, $\Lambda_+ = \max_i \Lambda_i$; the notations $Q^{\alpha,z}(x', dx)$ and $\mu^{\alpha,z}(dx)$ stand for the conditional transition kernel of the process $X^{\alpha,z}$ satisfying the equation with a fixed variable z,

$$dX_t^{\alpha,z} = b(\alpha(X_t^\alpha, z), X_t^\alpha, z)\, dt + \sigma(\alpha(X_t^\alpha, z), X_t^\alpha, z)\, dW_t, \quad t \geq 0,$$

$$X_0^{\alpha,z} = x,$$

and for its unique invariant measure, respectively, both given $\alpha(\cdot)$ and z.

Lemma 4. *Let the assumptions (A1) – (A3) hold true. Then for any $k > 0$, and any $N > 0$ there exist $C(N), C > 0$ such that uniformly in $\alpha(\cdot)$ for any function $g \geq 0$ growing not faster than some polynomial and for every $m \in \mathbb{N}$*

$$\sum_z \int g(x,z) 1(|x| \leq N) dx \leq C(N) \lambda_-^{-1} \left(\exp(+\Lambda_+ m) \langle g, \mu^\alpha \rangle + \lambda_+ \exp(\Lambda_+) \frac{C}{m^k} \right). \tag{31}$$

Proof. According to the invariance equation equivalent to Chapman – Kolmogorov equation in this case, for any $n > 0$ we have,

$$\mu^\alpha(dx, z) = \sum_{z'} \int \mu^\alpha(dx', z') Q_n(x', z'; dx, z),$$

where the integration is over x' and where Q_t is the transition kernel over t units of time from the state (x', z') to (dx, z); where necessary to highlight the dependence of the strategy we will use notation Q_t^α.

For any function $0 \leq g(x, z) \in \mathcal{K}$ and such that $g(x, z) \equiv 0$ for $|x| > N$, and such that $g(x, z) = g(x, z)\delta_{z,z_0}$. The latter restriction will be later dropped; in the sequel g will be ψ, or ψ^2. We estimate using Chapman – Kolmogorov's equation and integration over $0 \leq t \leq 1$,

$$\langle g, \mu^\alpha \rangle$$

$$= \int \int \sum_{z'} \mu^\alpha(dx', z') \int \sum_{z''} Q_t^\alpha(x', z'; dx'', z'') \int \sum_{z} Q_{m-t}^\alpha(x'', z''; dx, z) g(x, z)$$

$$\geq \int \int \sum_{z'} \mu^\alpha(dx', z') \int_0^1 \lambda_{z',z} dt \int Q_t^{\alpha,z'}(x', dx'') \exp(-\Lambda_{z'} t)$$

$$\times \exp(-\Lambda_{z_0}(m-t)) Q_{m-t}^{\alpha,z_0}(x'', dx) g(x, z_0)$$

$$= \int \int \sum_{z'} \mu^\alpha(dx', z') \int_0^1 \lambda_{z',z} dt \int Q_t^{\alpha,z'}(x', dx'') \exp(-\Lambda_{z'} t)$$

$$\times \exp(-\Lambda_{z_0}(m-t)) \mu^\alpha(dx, z_0) g(x, z_0)$$

$$- \int \int \sum_{z'} \mu^\alpha(dx', z') \int_0^1 \lambda_{z',z} dt \int Q_t^{\alpha,z'}(x', dx'') \exp(-\Lambda_{z'} t)$$

$$\times \exp(-\Lambda_{z_0}(m-t)) (\mu^\alpha(dx, z_0) - Q_{m-t}^{\alpha,z_0}(x'', dx)) g(x, z_0)$$

$$= \int\int\sum_{z'}\mu^{\alpha}(dx',z')\int_0^1\lambda_{z',z_0}dt\int Q_t^{\alpha,z'}(x',dx'')\exp(-\Lambda_{z'}t)$$

$$\times\exp(-\Lambda_{z_0}(m-t))\mu^{\alpha}(dx,z_0)g(x,z_0)$$

$$-\int\int\sum_{z'}\mu^{\alpha}(dx',z')\int_0^1\lambda_{z',z_0}dt\int Q_t^{\alpha,z'}(x',dx'')\exp(-\Lambda_{z'}t)$$

$$\times\exp(-\Lambda_{z_0}(m-t))(\mu^{\alpha}(dx,z_0)-Q_{m-t}^{\alpha,z_0}(x'',dx))g(x,z_0)$$

$$=: J_1^g - J_2^g.$$

For J_1^g we have the following lower bound:

$$J_1^g = \int\int\sum_{z'}\mu^\alpha(dx',z')\int_0^1 \lambda_{z',z}dt\int Q_t^{\alpha,z'}(x',dx'')\exp(-\Lambda_{z'}t)$$

$$\times\exp(-\Lambda_{z_0}(m-t))\mu^{\alpha,z_0}(dx)g(x,z_0)$$

$$\geq \lambda_-\exp(-\Lambda_+)\exp(-\Lambda_{z_0}m)\int\int\sum_{z'}\mu^\alpha(dx',z')$$
$$\times\int\int_0^1 dt\, Q_t^{\alpha,z'}(x',dx'')\mu^{\alpha,z_0}(dx)g(x,z_0)$$

$$= \lambda_-\exp(-\Lambda_+)\exp(-\Lambda_{z_0}m)\int_0^1 dt$$
$$\times\int\int\sum_{z'}\mu^\alpha(dx',z')\int Q_t^{\alpha,z'}(x',dx'')\mu^{\alpha,z_0}(dx)g(x,z_0)$$

$$= \lambda_-\exp(-\Lambda_+)\exp(-\Lambda_{z_0}m)\int\mu^{\alpha,z_0}(dx)g(x,z_0)$$

$$\geq c(N)\lambda_-\exp(-\Lambda_+)\exp(-\Lambda_{z_0}m)\int g(x,z_0)dx.$$

We used the identity $\int\sum_{z'}\mu^\alpha(dx',z')\int Q_t^{\alpha,z'}(x',dx'') = 1$ and the explicit formula for the invariant measure $\mu^{\alpha,z}$ given $\alpha(\cdot)$ and z:

$$p^{\alpha,z}(x) = \frac{\mu^{\alpha,z}(dx)}{dx} = C_{\alpha,z}\frac{1}{\sigma^2(\alpha(x,z),x,z)}\exp\left(2\int_0^x \frac{b(\alpha(y,z),y,z)}{\sigma^2(\alpha(y,z),y,z)}\,dy\right), \quad (32)$$

where $C_{\alpha,z}$ is a normed constant – see, for example, [5, Lemma 4.16, equation (4.70)] – or, more precisely, its straightforward corollary due to the assumptions on the coefficients that for any $N > 0$ there exists a constant $c(N) > 0$ such that

$$\inf_{\alpha,z}\inf_{|x|\leq N} p^{\alpha,z}(x) \geq c(N).$$

For J_2^g we have an upper bound,

$$0 \le J_2^g = \int\int\sum_{z'}\mu^{\alpha}(dx',z')\int_0^1\lambda_{z',z}dt\int Q_t^{\alpha,z'}(x',dx'')\exp(-\Lambda_{z'}t)$$

$$\times\exp(-\Lambda_{z_0}(m-t))(\mu^{\alpha,z_0}(dx)-Q_{m-t}^{\alpha,z_0}(x'',dx))g(x,z_0)$$

$$\le \lambda_+\exp(-\Lambda_{z_0}(m-1))\int\int\sum_{z'}\mu^{\alpha}(dx',z')$$

$$\times\int\int_0^1 dt\,Q_t^{\alpha,z'}(x',dx'')|\mu^{\alpha,z_0}(dx)-Q_{m-t}^{\alpha,z_0}(x'',dx)|g(x,z_0)$$

$$= \lambda_+\exp(\Lambda_+)\exp(-\Lambda_{z_0}m)\int\int\sum_{z'}\mu^{\alpha}(dx',z')$$

$$\times\int_0^1 dt\,Q_t^{\alpha,z'}(x',dx'')\int|\mu^{\alpha,z_0}(dx)-Q_{m-t}^{\alpha,z_0}(x'',dx)|g(x,z_0)$$

$$\le \lambda_+\exp(\Lambda_+)\int_0^1 dt\;\exp(-\Lambda_{z_0}m)\int\int\sum_{z'}\mu^{\alpha}(dx',z')Q_t^{\alpha,z'}(x',dx'')$$

$$\times(\int|\mu^{\alpha,z_0}(dx)-Q_{m-t}^{\alpha,z_0}(x'',dx)|)^{1/2}(\int g(x,z_0)^2|\mu^{\alpha,z_0}(dx)-Q_{m-t}^{\alpha,z_0}(x'',dx)|)^{1/2}$$

$$\le \lambda_+\exp(\Lambda_+)\int_0^1 dt\;\exp(-\Lambda_{z_0}m)\int\int\sum_{z'}\mu^{\alpha}(dx',z')Q_t^{\alpha,z'}(x',dx'')$$

$$\times(\int|\mu^{\alpha,z_0}(dx)-Q_{m-t}^{\alpha,z_0}(x'',dx)|)^{1/2}(\int g(x,z_0)^2|\mu^{\alpha,z_0}(dx)+Q_{m-t}^{\alpha,z_0}(x'',dx)|)^{1/2}$$

$$\le \lambda_+\exp(\Lambda_+)\exp(-\Lambda_{z_0}m)\int_0^1 dt\int\sum_{z'}\mu^{\alpha}(dx',z')\int Q_t^{\alpha,z'}(x',dx'')\frac{C(x'')^{1/2}}{(1+m-t)^{k/2}}.$$

Here due to the inequality (14) of the Lemma 2, and since $C(x'') \le 1+|x''|^{\ell_1}$ with some ℓ_1,

$$\int C(x'')^{1/2} Q_t^{\alpha,z'}(x', dx'') \leq C_1(x'),$$

where $C_1(x') = C(1 + |x'|^{\ell_2})$ with some $\ell_2, C > 0$, and because $k > 0$ may be chosen arbitrarily large, and due to (15), we continue,

$$\begin{aligned} J_2^g &\leq \lambda_+ \exp(\Lambda_+) \exp(-\Lambda_{z_0} m) \int_0^1 dt \int_A \int \sum_{z'} \mu^\alpha(dx', z') \frac{C_1(x')}{(1+m-t)^k} \\ &\leq \lambda_+ \exp(\Lambda_+) \exp(-\Lambda_{z_0} m) \frac{C}{m^k}, \end{aligned}$$

with some new $C < \infty$. Combining the bounds on J_1^g and J_2^g, we obtain,

$$\langle g, \mu^\alpha \rangle = J_1^g - J_2^g \geq c(N) \lambda_- \exp(-\Lambda_{z_0} m) \int g(x,z) dx - \lambda_+ \exp(\Lambda_+) \exp(-\Lambda_{z_0} m) \frac{C}{m^k}.$$

It follows that

$$c(N) \lambda_- \exp(-\Lambda_{z_0} m) \int g(x,z) dx \leq \langle g, \mu^\alpha \rangle + \lambda_+ \exp(\Lambda_+) \exp(-\Lambda_{z_0} m) \frac{C}{m^k},$$

or,

$$\begin{aligned} \int g(x, z_0) dx &\leq c(N)^{-1} \lambda_-^{-1} \exp(+\Lambda_{z_0} m) (\langle g, \mu^\alpha \rangle + \lambda_+ \exp(\Lambda_+) \exp(-\Lambda_{z_0} m) \frac{C}{m^k}) \\ &= c(N)^{-1} \lambda_-^{-1} \left(\exp(+\Lambda_{z_0} m) \langle g, \mu^\alpha \rangle + \lambda_+ \exp(\Lambda_+) \frac{C}{m^k} \right) \\ &\leq c(N)^{-1} \lambda_-^{-1} \left(\exp(+\Lambda_+ m) \langle g, \mu^\alpha \rangle + \lambda_+ \exp(\Lambda_+) \frac{C}{m^k} \right). \end{aligned}$$

So, due to linearity of integration, for any $g \geq 0$ equal zero outside $|x| > N$ (not only for $g = g1(z = z_0)$) we obtain for any m the desired bound (31),

$$\sum_z \int g(x,z) dx \leq C c(N)^{-1} \lambda_-^{-1} \left(\exp(+\Lambda_+ m) \langle g, \mu^\alpha \rangle + \lambda_+ \exp(\Lambda_+) \frac{C}{m^k} \right).$$

The Lemma 4 is proved. $\square$

NB: In the proof of the Theorem in (44) it will follow that for any $\epsilon > 0$ there exists $n(\epsilon)$ such that

$$\sup_{n\geq n(\epsilon)} \sum_z \int_{|x|\leq N} \psi_n^2(x,z)\, dx < \epsilon^2/2.$$

Indeed, choose there ψ_n^2 so that $\langle \psi_n^2, \mu^\alpha\rangle \leq \epsilon^2 \exp(-\Lambda_+ m)$; then we obtain

$$\sum_z \int_{|x|\leq N} \psi_n^2(x,z)dx \leq Cc(N)^{-1}\lambda_-^{-1}\left(\exp(+\Lambda_+ m)\langle \psi_n^2, \mu^\alpha\rangle + \lambda_+ \exp(\Lambda_+)\frac{C}{m^k}\right)$$

$$\leq C(N)(\epsilon^2 + m^{-k}).$$

So, taking m large enough here we get,

$$\sum_z \int_{|x|\leq N} \psi_n^2(x,z)dx \leq 2C(N)\epsilon^2.$$

It remains to redenote ϵ by $\epsilon/\sqrt{2C(N)}$.

Corollary 1. *Lebesgue's measure $\Lambda(dx)$ is absolutely continuous with respect to $\mu^\alpha(dx,z)$ for each α and for every z.*

Proof. If $g \geq 0$ (for example, $g(x) = 1(x \in A)$ for some $A \subset \mathbb{R}$) and $\langle g, \mu^\alpha\rangle = 0$, then by choosing m large enough we can make the right hand side in (31) arbitrarily small, as required. □

This Corollary will be used in the end of the proof of the Theorem.

3 Main results

Recall that the state space dimension is $D = 1$. We accept in this section that solution of the SDE with any Markov strategy may be *weak*; however, we want it to be unique in distribution, strong Markov and ergodic. All of these follow from [8] and from the assumptions (A1) and (A2) (see [15] about ergodicity and the Lemma 1).

The exact RIA reads as follows. We start with some homogeneous Markov strategy α_0, which uniquely determines $\rho_0 = \rho^{\alpha_0} \equiv \langle f^{\alpha_0}, \mu^{\alpha_0}\rangle$ and $v_0 = v^{\alpha_0}$. Next, for any couple $(v,\rho):\ v \in \bigcap_{p>1} W^2_{p,loc}$, $\rho \in \mathbb{R}$, define

$$F[v,\rho](x,z) := \inf_{u\in U} \left[L^u v(x,z) + f^u(x,z) - \rho\right].$$

Of course, due to the embedding theorems (see [9, Chapter 2]), we may consider v and v' continuous, Hölder continuous, and absolutely continuous; however, this cannot be applied directly to v''. Respectively, the function $F[v,\rho](\cdot)$ is defined by the formula above as a function of the class $L_{p,loc}$ for any $p>1$; in particular, it is Lebesgue measurable and as such it is defined only a.e. x. Further, we may and will take a (any) Borel measurable version of this Lebesgue measurable function.

By induction, given α_n, ρ_n and v_n, the next "improved" strategy α_{n+1} is defined as follows: for a.e. x and any z the function α_{n+1} is chosen so that

$$L^{\alpha_{n+1}}v_n(x,z)+f^{\alpha_{n+1}}(x,z)-\rho_n = F[v_n,\rho_n](x,z) \equiv \inf_{u\in U}\left[L^u v_n(x,z)+f^u(x,z)-\rho_n\right], \tag{33}$$

which is equivalent to

$$L^{\alpha_{n+1}}v_n(x,z)+f^{\alpha_{n+1}}(x,z) = \inf_{u\in U}\left[L^u v_n(x,z)+f^u(x,z)\right] =: G[v_n](x,z),$$

and also to saying that

$$\alpha_{n+1}(x,z) = \operatorname{arginf}_{u\in U}\left[L^u v_n(x,z)+f^u(x,z)\right].$$

We assume that a Borel measurable version of such strategy may be chosen; see the reference in [1, Appendix]. The value ρ_{n+1} is then defined as

$$\rho_{n+1} := \langle f^{\alpha_{n+1}}, \mu^{\alpha_{n+1}}\rangle,$$

where in turn, $\mu^{\alpha_{n+1}}$ is the (unique) invariant measure, which corresponds to the strategy α_{n+1}. Also, recall that

$$v_n(x,z) = \int_0^\infty \mathbb{E}_{x,z}(f^{\alpha_n}(X_t^{\alpha_n},Z_t)-\rho_n)\,dt,$$

and

$$v_{n+1}(x,z) = \int_0^\infty \mathbb{E}_{x,z}(f^{\alpha_{n+1}}(X_t^{\alpha_{n+1}},Z_t)-\rho_{n+1})\,dt.$$

Theorem 1. *Let the assumptions (A1) – (A5) be satisfied. Then:*

1. For any n, $\rho_{n+1}\le\rho_n$, and there is a limit $\rho_n\downarrow\tilde\rho$.

2. The sequence (v_n) is pre-compact in $C^1[-N,N]$ in x for each $N>0$.

3. If $\tilde v$ is any limiting point of the family $(v_n, n\ge 1)$ in C^1_{loc} in the variable x, then the couple $(\tilde v,\tilde\rho)$ solves the equation (11).

4. This solution $(\tilde v,\tilde\rho)$ is unique in the class of functions growing no faster than some polynomial and belonging to the class $W^2_{p,loc}$ in x for any $p>0$ and any z.

5. The component $\tilde\rho$ in the couple $(\tilde v,\tilde\rho)$ coincides with ρ.

Proof. 1. Due to (33) and (23), for almost every (a.e.) $x \in \mathbb{R}$,

$$\rho_n = \rho^{\alpha_n} = \langle f^{\alpha_n}, \mu^{\alpha_n} \rangle = L^{\alpha_n} v_n(x,z) + f^{\alpha_n}(x,z)$$

$$\geq \inf_u [L^u v_n(x,z) + f^u(x,z)] = L^{\alpha_{n+1}} v_n(x,z) + f^{\alpha_{n+1}}(x,z)$$

and also for a.e. $x \in \mathbb{R}$ and any z,

$$\rho_{n+1} = L^{\alpha_{n+1}} v_{n+1}(x,z) + f^{\alpha_{n+1}}(x,z).$$

So,

$$\begin{aligned} \rho_n - \rho_{n+1} &\overset{a.e.}{\geq} (L^{\alpha_{n+1}} v_n + f^{\alpha_{n+1}})(x,z) - (L^{\alpha_{n+1}} v_{n+1} + f^{\alpha_{n+1}})(x,z) \\ &= (L^{\alpha_{n+1}} v_n - L^{\alpha_{n+1}} v_{n+1})(x,z). \end{aligned} \tag{34}$$

Now let us apply Dynkin's formula to $(v_n - v_{n+1})(X_t^{\alpha_{n+1}}, Z_t)$ (recall that $v \in C$ due to the embedding theorems): we have for any $x \in \mathbb{R}, z$,

$$\mathbb{E} v_n(X_t^{\alpha_{n+1}}, Z_t) - \mathbb{E} v_{n+1}(X_t^{\alpha_{n+1}}, Z_t) - v_n(x,z) + v_{n+1}(x,z)$$

$$= \mathbb{E} \int_0^t (L^{\alpha_{n+1}} v_n - L^{\alpha_{n+1}} v_{n+1})(X_s^{\alpha_{n+1}}, Z_s)\, ds$$

$$\leq \mathbb{E} \int_0^t (\rho_n - \rho_{n+1})\, ds = (\rho_n - \rho_{n+1})\, t. \tag{35}$$

It can be justified by Itô–Krylov's formula (with switching) where the expectation of the martingale term is zero because of the a priori moment bounds for the solution (X_t). Here we also used the fact that the distribution of $X_s^{\alpha_{n+1}}$ for almost all $s > 0$ is absolutely continuous with respect to the Lebesgue measure due to the non-degeneracy and by virtue of Krylov's estimates [7]; due to this reason and because $v_n, v_{n+1} \in C$, the a.e. inequality (34) implies (35) for every x. Further, since the left hand side in (35) is bounded for a fixed x, we divide all terms of the latter inequality by t and let $t \to \infty$ to get,

$$0 \leq \rho_n - \rho_{n+1},$$

as required.

Hence, $\rho_n \geq \rho_{n+1}$, so that $\rho_n \downarrow \tilde{\rho}$ (since ρ_n is bounded for $f^\alpha \in \mathcal{K}$, see (16) in the Lemma 2) with some $\tilde{\rho}$. Thus, the RIA does converge, although, so far we do not know whether $\tilde{\rho} = \rho$.

Note that clearly $\tilde{\rho} \geq \rho$, since ρ is the inf over all Markov strategies, while $\tilde{\rho}$ is the inf over some countable subset (a sequence) of them. *We shall see later that they do coincide:* $\tilde{\rho} = \rho$.

Recall that now we want to show that $v_n \to \tilde{v}$, so that the couple $(\tilde{v}, \tilde{\rho})$ satisfies the HJB equation (11), and that $\tilde{\rho}$ – as well as $\tilde{v}$ in some sense – here is unique.

2. Let us show local pre-compactness of the family of functions (v_n) in C^1. Note that the equation (11) is equivalent to the following:

$$v''(x,z) + \inf_{u \in U} \left[\frac{b(u,x,z)}{a(u,x,z)} v'(x,z) + \frac{f(u,x,z)}{a(u,x,z)} - \frac{\rho}{a(u,x,z)} \right] + \frac{1}{a(u,x,z)} \sum_{j \in \mathcal{S} \setminus z} \lambda_{zj} \left(v(x,j) - v(x,z) \right) = 0, \tag{36}$$

while the equation

$$L^{\alpha_{n+1}} v_{n+1}(x,z) + f^{\alpha_{n+1}}(x,z) - \rho_{n+1} \stackrel{a.e.}{=} 0, \tag{37}$$

is equivalent to

$$v''_{n+1}(x,z) + \frac{b(\alpha_{n+1}(x),x,z)}{a(\alpha_{n+1}(x),x,z)} v'_{n+1}(x,z) + \frac{f(\alpha_{n+1}(x),x,z)}{a(\alpha_{n+1}(x),x,z)} - \frac{\rho_{n+1}}{a(\alpha_{n+1}(x),x,z)}$$

$$+ \frac{1}{a(\alpha_{n+1}(x),x,z)} \sum_{j \in \mathcal{S} \setminus z} \lambda_{zj} \left(v(x,j) - v(x,z) \right) = 0.$$

According to the Lemma 3, the functions v'_{n+1} are uniformly locally bounded. Since the sequence ρ_{n+1} is bounded and due to the uniform local boundedness of the functions $f(\alpha_{n+1}(x),x)$ and uniform nondegeneracy of a, it follows that (v''_n) are locally uniformly bounded and satisfy the uniform in n growth bounds similar to (26) for the function itself and for its first derivative due to the equation (e.g., due to (36)). This guarantees the pre-compactness of (v_n) in C^1 locally.

3. Due to the (local) compactness property showed in the previous step, by the diagonal procedure from any infinite sub-family of functions v_n it is possible to choose

a converging in C^1_{loc} subsequence. We want to show that up to a constant the limit is unique. For this aim, first of all we shall see in a minute that if some $v_{n_j}(x,z)$ has a limit, say, $\tilde v(x,z)$ (locally in C) then $v_{n_j+1}(x,z)+\beta_{n_j}$ has the same limit, where β_n is some bounded sequence of real values. (In fact, what will be established is a little bit more complicated but still enough for our purposes.) We have,

$$L^{\alpha_{n+1}}v_{n+1}(x,z)+f^{\alpha_{n+1}}(x,z)-\rho_{n+1}\stackrel{a.e.}{=}0,$$

and

$$L^{\alpha_{n+1}}v_n(x,z)+f^{\alpha_{n+1}}(x,z)-\rho_n=:-\psi_{n+1}(x,z)\stackrel{a.e.}{\le}0. \tag{38}$$

Let us rewrite it as follows,

$$L^{\alpha_{n+1}}v_n(x,z)+f^{\alpha_{n+1}}(x,z)-\rho_n+\psi_{n+1}(x,z)\stackrel{a.e.}{=}0.$$

In other words, the function v_n solves the Poisson equation with the second order operator $L^{\alpha_{n+1}}$ and the "right hand side" $-(f^{\alpha_{n+1}}(x,z)+\psi_{n+1}(x,z)-\rho_n)$. This is only possible if the expression $f^{\alpha_{n+1}}(x,z)+\psi_{n+1}(x,z)-\rho_n$ is centered with respect to the invariant measure μ^{n+1} because Poisson equations in the whole space have no solutions for non-centered right hand sides (cf., e.g., [12]). This implies that

$$\langle f^{\alpha_{n+1}}+\psi_{n+1}-\rho_n,\mu^{n+1}\rangle=0.$$

So,

$$\langle\psi_{n+1},\mu^{n+1}\rangle=\rho_n-\rho_{n+1}. \tag{39}$$

Now denote

$$w_n(x,z):=v_n(x,z)-v_{n+1}(x,z).$$

We have,

$$L^{\alpha_{n+1}}w_n(x,z)+\psi_{n+1}(x,z)-(\rho_n-\rho_{n+1})\stackrel{a.e.}{=}0.$$

So, there is a constant $\beta_n=\langle w_n,\mu^{n+1}\rangle$ such that

$$w_n(x,z)-\beta_n=\int_0^\infty \mathbb{E}_x(\psi_{n+1}(X^{n+1}_t,Z_t)-(\rho_n-\rho_{n+1}))\,dt. \tag{40}$$

Let us show that for any $N>0$,

$$\int_{-N}^{N}\psi^2_n(x,z)\,dx\to 0,\quad n\to\infty. \tag{41}$$

First of all, note that all functions ψ_n and, hence, ψ_n^2 are uniformly locally bounded and may only grow polynomially fast,

$$(0 \le \,)\ \psi_n(x,z) \le C(1+|x|^m), \tag{42}$$

with some C, m the same for all values of n. which follows from the definition (38), and the properties of derivatives v_n' and v_n'', and from the Lemma 4, and due to

$$\langle \psi_{n+1}, \mu^{n+1}\rangle = \rho_n - \rho_{n+1} \to 0, \quad n \to \infty.$$

Now let us rewrite the equation (40) via a stationary version of the pair (X, Z), say, $(\tilde X_t^{n+1}, \tilde Z_t)$:

$$w_n(x,z) - \beta_n = \int_0^\infty \mathbb{E}_x(\psi_{n+1}(X_t^{n+1}, Z_t) - E_{\mu^{n+1}}(\psi_{n+1}(\tilde X_t^{n+1}, \tilde Z_t))\, dt.$$

(Note that if we knew that w_n were centered with respect to the invariant measure μ^{n+1} then we would have $\beta_n = 0$; however, the functions v_n and v_{n+1} are both centered with respect to two different measures, and this is the reason why their difference is not just small, but small up to some additive constant; this very constant is denoted by β_n.) Using the coupling idea (cf., e.g., [15]), let us consider the independent processes (X_t^{n+1}, Z_t^{n+1}) and $(\tilde X_t^{n+1}, \tilde Z_t^{n+1})$ on the same probability space (just considering the product space) and denote the moment of the first meeting

$$\tau := \inf(t \ge 0 :\ X_t^{n+1} = \tilde X_t^{n+1}\ \&\ Z_t^{n+1} = \tilde Z_t^{n+1}),$$

which is a stopping time and is (apparently) finite with probability one. It is known (see [15] and the proof of the Lemma 3) that under our recurrence assumptions for any $k > 0$ there are some constants C_k, m such that uniformly with respect to n,

$$\mathbb{E}_{x,\mu^{n+1}} \tau^k \le C_k(1+|x|^m).$$

Denote

$$\hat X_t^{n+1} := 1(t<\tau) X_t^{n+1} + 1(t \ge \tau)\tilde X_t^{n+1}, \quad \hat Z_t^{n+1} := 1(t<\tau) Z_t^{n+1} + 1(t \ge \tau)\tilde Z_t^{n+1}.$$

Since τ is a stopping time and because the quadruple $(X_t^{n+1}, Z_t^{n+1}, \tilde X_t^{n+1}, \tilde Z_t^{n+1})$ is strong Markov (see [6]), the couple $(\hat X_t^{n+1}, \hat Z_t^{n+1})$ is also strong Markov equivalent to (X_t^{n+1}, Z_t^{n+1}) in the sense of distributions of trajectories. Therefore, it is possible to rewrite,

$$w_n(x,z) - \beta_n = \int_0^\infty \mathbb{E}_{x,\mu}(\psi_{n+1}(\hat X_t^{n+1}, \hat Z_t^{n+1}) - \psi_{n+1}(\tilde X_t^{n+1}, \tilde Z_t^{n+1})\, dt.$$

Hence, using the fact that after τ the pairs $(\hat{X}_t^{n+1}, \hat{Z}_t^{n+1})$ and $(\tilde{X}_t^{n+1}, \tilde{Z}_t^{n+1})$ coincide, we obtain

$$w_n(x,z) - \beta_n = \int_0^\infty \mathbb{E}_{x,\mu} 1(t < \tau)(\psi_{n+1}(\hat{X}_t^{n+1}, \hat{Z}_t^{n+1}) - \psi_{n+1}(\tilde{X}_t^{n+1}, \tilde{Z}_t^{n+1}))\, dt$$

$$= \int_0^\infty \mathbb{E}_{x,\mu} \sum_{i=0}^\infty 1(i \le \tau < i+1) 1(t < \tau)(\psi_{n+1}(\hat{X}_t^{n+1}, \hat{Z}_t^{n+1}) - \psi_{n+1}(\tilde{X}_t^{n+1}, \tilde{Z}_t^{n+1}))\, dt$$

$$= \sum_{i=0}^\infty \mathbb{E}_{x,\mu} \int_0^\infty 1(i \le \tau < i+1) 1(t < \tau)(\psi_{n+1}(\hat{X}_t^{n+1}, \hat{Z}_t^{n+1}) - \psi_{n+1}(\tilde{X}_t^{n+1}, \tilde{Z}_t^{n+1}))\, dt.$$

Thus, using Cauchy-Buniakovsky-Schwarz inequality and Fubini Theorem, we have,

$$|w_n(x) - \beta_n| \le \sum_{i=0}^\infty \mathbb{E}_{x,\mu} \int_0^{i+1} 1(\tau > i)|\psi_{n+1}(\hat{X}_t^{n+1}, \hat{Z}_t^{n+1}) - \psi_{n+1}(\tilde{X}_t^{n+1}), \tilde{Z}_t^{n+1}|\, dt$$

$$\le \sum_{i=0}^\infty \int_0^{i+1} \mathbb{E}_{x,\mu} 1(\tau > i)(|\psi_{n+1}(\hat{X}_t^{n+1}, \hat{Z}_t^{n+1})| + |\psi_{n+1}(\tilde{X}_t^{n+1}, \tilde{Z}_t^{n+1})|)\, dt$$

$$\le \sum_{i=0}^\infty \int_0^{i+1} (\mathbb{E}_{x,\mu} 1(\tau > i))^{1/2} (\mathbb{E}_{x,\mu}(|\psi_{n+1}(\hat{X}_t^{n+1}, \hat{Z}_t^{n+1})| + |\psi_{n+1}(\tilde{X}_t^{n+1}, \tilde{Z}_t^{n+1})|)^2)^{1/2}\, dt$$

$$\le 2 \sum_{i=0}^\infty (\mathbb{E}_{x,\mu} 1(\tau > i))^{1/2} \int_0^{i+1} (\mathbb{E}_{x,\mu}|\psi_{n+1}(\hat{X}_t^{n+1}, \hat{Z}_t^{n+1})|^2 + \mathbb{E}_{x,\mu}|\psi_{n+1}(\tilde{X}_t^{n+1}, \tilde{Z}_t^{n+1})|^2)^{1/2}\, dt$$

$$\le 2 \sum_{i=0}^\infty (\mathbb{E}_{x,\mu} 1(\tau > i))^{1/2} \int_0^{i+1} [(\mathbb{E}_{x,\mu}(\psi_{n+1}(\hat{X}_t^{n+1}, \hat{Z}_t^{n+1}))^2)^{1/2} + (\mathbb{E}_{x,\mu}(\psi_{n+1}(\tilde{X}_t^{n+1}, \tilde{Z}_t^{n+1})^2)^{1/2}]\, dt.$$

Now, let us take any $\epsilon > 0$ and use the inequality $\sqrt{a} \le \frac{\epsilon}{2} + \frac{a}{2\epsilon}$. We estimate,

$$\int_0^{i+1} [(\mathbb{E}_{x,\mu}(\psi_{n+1}(\hat{X}_t^{n+1}, \hat{Z}_t^{n+1}))^2)^{1/2} + (\mathbb{E}_{x,\mu}(\psi_{n+1}(\tilde{X}_t^{n+1}, \tilde{Z}_t^{n+1}))^2)^{1/2}]\, dt$$

$$\le \epsilon(i+1) + \frac{1}{2\epsilon} \int_0^{i+1} [\mathbb{E}_{x,\mu}\psi_{n+1}^2(\hat{X}_t^{n+1}, \hat{Z}_t^{n+1}) + \mathbb{E}_{x,\mu}\psi_{n+1}^2(\tilde{X}_t^{n+1}, \tilde{Z}_t^{n+1})]\, dt.$$

Let us first consider the stationary term. We have,

$$\frac{1}{2\epsilon} \int_0^{i+1} \mathbb{E}_{x,\mu}\psi_{n+1}^2(\tilde{X}_t^{n+1}, \tilde{Z}_t^{n+1})\, dt + \frac{1}{2\epsilon} \int_0^{i+1} \mathbb{E}_{x,\mu}\psi_{n+1}^2(\hat{X}_t^{n+1}, \hat{Z}_t^{n+1})\, dt$$

$$= \frac{1}{2\epsilon} \int_0^{i+1} \mathbb{E}_{x,\mu}\psi_{n+1}^2 1_{[-N,N]}(\tilde{X}_t^{n+1}, \tilde{Z}_t^{n+1})\, dt + \frac{1}{2\epsilon} \int_0^{i+1} \mathbb{E}_{x,\mu}\psi_{n+1}^2 1_{\mathbb{R}\setminus[-N,N]}(\tilde{X}_t^{n+1}, \tilde{Z}_t^{n+1})\, dt$$

$$+ \frac{1}{2\epsilon} \int_0^{i+1} \mathbb{E}_{x,\mu}\psi_{n+1}^2 1_{[-N,N]}(\hat{X}_t^{n+1}, \hat{Z}_t^{n+1})\, dt + \frac{1}{2\epsilon} \int_0^{i+1} \mathbb{E}_{x,\mu}\psi_{n+1}^2 1_{\mathbb{R}\setminus[-N,N]}(\hat{X}_t^{n+1}, \hat{Z}_t^{n+1})\, dt.$$

Given (42) and because any stationary measure integrates uniformly any power function, let us find such N that uniformly with respect to n,

$$\langle C(1 + |x|^{2m}) 1_{\mathbb{R}\setminus[-N,N]}, \mu^{n+1} \rangle < \epsilon^2/2, \tag{43}$$

which is possible due to the Lemmata 2 and 4, and also such that $N > \epsilon^{-2}$. Then choose $n(\epsilon)$ such that

$$\sup_{n \ge n(\epsilon)} \int_{|x| \le N} \psi_n^2(x)\, dx < \epsilon^2/2. \tag{44}$$

This is possible since $\langle \psi_{n+1}, \mu^{n+1} \rangle$ and hence also $\langle \psi_{n+1}^2, \mu^{n+1} \rangle$ is small (since ψ_{n+1} has a limited polynomial growth), and because of the bounds (43) above and of (31) from the Lemma 4.

Now we evaluate with $n \ge n(\epsilon)$ due to Krylov's estimate [7, 8],

$$\frac{1}{2\epsilon} \int_0^{i+1} \mathbb{E}_{x,\mu}\psi_{n+1}^2 1_{[-N,N]}(\tilde{X}_t^{n+1})\, dt$$

$$= \frac{1}{2\epsilon} \sum_{k=0}^{i} \mathbb{E}_x \int_k^{k+1} \psi_{n+1}^2 1_{[-N,N]}(\tilde{X}_t^{n+1})\, dt \le \frac{i+1}{2\epsilon} K \|\psi_{n+1}^2 1_{[-N,N]}\|_{L^2} \le \frac{(i+1)K\epsilon}{2}.$$

This argument works for the non-stationary process as well: due to Krylov's estimate,

$$\frac{1}{2\epsilon}\int_0^{i+1} \mathbb{E}_{x,\mu}\psi_{n+1}^2 1_{[-N,N]}(\hat{X}_t^{n+1}, \hat{Z}_t^{n+1})\, dt$$

$$= \frac{1}{2\epsilon}\sum_{k=0}^{i}\mathbb{E}\int_k^{k+1} \psi_{n+1}^2 1_{[-N,N]}(\hat{X}_t^{n+1}, \hat{Z}_t^{n+1})\, dt \le \frac{i+1}{2\epsilon}K\|\psi_{n+1}^2 1_{[-N,N]}\|)_{L^2} \le \frac{(i+1)K\epsilon}{2}.$$

Further,

$$\frac{1}{2\epsilon}\int_0^{i+1} \mathbb{E}_{x,\mu}\psi_{n+1}^2 1_{\mathbb{R}\setminus[-N,N]}(\hat{X}_t^{n+1}, \hat{Z}_t^{n+1})\, dt \le \frac{i+1}{2\epsilon}\times\frac{\epsilon^2}{2} = \frac{(i+1)\epsilon}{4}.$$

Finally, using (44), we obtain with some ℓ,

$$\frac{1}{2\epsilon}\int_0^{i+1} \mathbb{E}_{x,\mu}\psi_{n+1}^2 1_{\mathbb{R}\setminus[-N,N]}(\hat{X}_t^{n+1}, \hat{Z}_t^{n+1})\, dt = \frac{1}{2\epsilon}\int_0^{i+1} \mathbb{E}_{x,\mu}\psi_{n+1}^2 1_{\mathbb{R}\setminus[-N,N]}(X_t^{n+1}, Z_t^{n+1})\, dt$$

$$\le C\frac{i+1}{2\epsilon}\frac{(1+|x|^\ell)}{N} \le C(i+1)(1+|x|^\ell)\epsilon.$$

Overall, this shows that with the appropriately chosen (uniformly bounded) β_n,

$$|w_n(x)-\beta_n| \le C(1+|x|^\ell)\epsilon\sum_{i=0}^{\infty}(i+1)(\mathbb{E}_{x,\mu}1(\tau>i))^{1/2}, \quad n\ge n(\epsilon). \qquad (45)$$

By virtue of the results in [15], for any $k>0$ there are $C, \ell(k)>0$ such that

$$\mathbb{P}_{x,\mu}1(\tau>i) \le C\frac{1+|x|^{\ell(k)}}{1+i^k}.$$

Therefore, taking $k>1$, we have that the series in (45) converges providing us an estimate

$$|w_n(x)-\beta_n| \le C(1+|x|^\ell)\epsilon, \quad n\ge n(\epsilon), \qquad (46)$$

with some new ℓ. In other words, the difference $w_n(x)-\beta_n = v_n - v_{n+1} - \beta_n$ is locally uniformly converging to zero as $n\to\infty$. Naturally, it also implies that for any subsequence n_j such that v_{n_j} converges locally uniformly in C^1 we have that v'_{n_j} and v'_{n_j+1} may only converge to the same limit, i.e., derivatives $v'_{n_j} - v'_{n_j+1} \to 0$

(locally uniformly) as $j \to \infty$. Indeed, otherwise we just integrate to show that the limits of v_{n_j} and $v_{n_j+1} + \beta_{n_j}$ are different, which contradicts to what was established earlier.

4. To start with, what we want to do is to pass to the limit of equations as $n \to \infty$ in the system

$$L^{\alpha_{n+1}} v_{n+1}(x,z) + f^{\alpha_{n+1}}(x,z) - \rho_{n+1} \stackrel{a.e.}{=} 0,$$

$$G[v_n](x,z) - \rho_n = L^{\alpha_{n+1}} v_n(x,z) + f^{\alpha_{n+1}}(x,z) - \rho_n \le 0,$$

after having showed the pre-compactness of the family of vector-functions (v_n) in C^1 in x. Denote

$$G[v_n](x,z) := \inf_{u \in U} (L^u v_n + f^u)(x,z) \quad (= L^{\alpha_{n+1}} v_n(x,z) + f^{\alpha_{n+1}}(x,z)),$$

and

$$F_1[x,z,v,v',\rho] := \inf_{u \in U} [\hat{b}^u v' + \hat{f}^u - \hat{\rho}](x,z) + \sum_{j \in \mathcal{S} \setminus z} \lambda_{zj} \left(v(x,j) - v(x,z) \right),$$

where

$$\begin{aligned} a^u(x,z) &= \frac{1}{2}(\sigma^u(x,z))^2, \quad \hat{b}^u(x,z) = b^u(x,z)/a^u(x,z), \\ \hat{f}^u(x,z) &= f^u(x,z)/a^u(x,z), \quad \hat{\rho}^u(x,z) = \rho/a^u(x,z). \end{aligned}$$

Now, the system of equations

$$L^{\alpha_{n+1}} v_{n+1}(x,z) + f^{\alpha_{n+1}}(x,z) - \rho_{n+1} \stackrel{a.e.}{=} 0, \tag{47}$$

is equivalent to

$$v''_{n+1}(x,z) + \frac{b(\alpha_{n+1}(x,z),x,z)}{a(\alpha_{n+1}(x,z),x,z)} v'_{n+1}(x,z) - \Lambda_z(x) v_{n+1}(x,z)$$

$$+ \frac{f(\alpha_{n+1}(x,z),(x,z))}{a(\alpha_{n+1}(x,z),x,z)} - \frac{\rho_{n+1}}{a(\alpha_{n+1}(x,z),x,z)} + \sum_{j: j \neq z} \bar{\lambda}_{z,j}(x) v_{n+1}(x,j) = 0,$$

with $\Lambda_z(x) = \dfrac{\sum_{j\in\mathcal{S}\setminus z}\lambda_{zj}}{a(\alpha_{n+1}(x,z),x,z)}$. Now the equation on the function v can be rewritten as follows,

$$v''_{n+1}(x,z) + \frac{b(\alpha_{n+1}(x,z),x,z)}{a(\alpha_{n+1}(x,z),x,z)}v'_{n+1}(x,z) - \Lambda_z(x)v_{n+1}(x,z) \\ + \bar f(\alpha_{n+1}(x,z),(x,z)) = 0, \tag{48}$$

with

$$\bar f(\alpha_{n+1}(x,z),(x,z)) := \frac{f(\alpha_{n+1}(x,z),(x,z))}{a(\alpha_{n+1}(x,z),x,z)} - \frac{\rho_{n+1}}{a(\alpha_{n+1}(x,z),x,z)} + \sum_{j:j\neq z}\bar\lambda_{z,j}(x)v_{n+1}(x,j).$$

According to the Lemma 3, the functions v'_{n+1} and, hence, also the function $\bar f$ are all uniformly locally bounded. It follows that (v''_n) are locally uniformly bounded. This guarantess a precompactness of the family (v_n) in C^1 locally with respect to x.

5. Further, from

$$G[v_n](x,z) - \rho_n = L^{\alpha_{n+1}}v_n(x,z) + f^{\alpha_{n+1}}(x,z) - \rho_n$$

$$(= \inf_{u\in U}[L^u v_n(x,z) + f^u(x,z) - \rho_n] \overset{a.e.}{\le} 0),$$

by subtracting zero a.e. (47), we obtain a.e.,

$$G[v_n](x,z) - \rho_n = L^{\alpha_{n+1}}(v_n(x,z) - v_{n+1}(x,z)) - (\rho_n - \rho_{n+1}). \tag{49}$$

Now we want to show for the (any) limiting function v that

$$\tilde v'_x(x,z) - \tilde v'_r(r,z) + \int_r^x F_1[s,z,\tilde v(s),\tilde v'(s),\tilde\rho]\,ds = 0, \tag{50}$$

which in turn implies by differentiation the equation equivalent to (11),

$$\tilde v''(x,z) + F_1[x,z,\tilde v,\tilde v',\tilde\rho](x,z) = 0, \tag{51}$$

for any x, z, with the note that each component of $\tilde v'$ is absolutely continuous in x.

Let us show why (49), indeed, implies (50). Recall that $G[v_n](x,z) - \rho_n \le 0$ (a.e.). Denote

$$\sum_{j:j\neq z} c_{x,j}v_{n+1}(x,j) =: \chi(x,z).$$

Let us divide (49) by $a_{n+1} = a^{\alpha_{n+1}}(x,z)$ and use $\delta := \inf_{u,x,z} a^u(x,z) > 0$: we get a.e.,

$$0 \geq \frac{(G[v_n](x,z) - \rho_n)}{a_{n+1}(x,z)}$$

$$= (v_n''(x,z) - v_{n+1}''(x,z)) + (\hat{b}^{\alpha_{n+1}}(v_n' - v_{n+1}'))(x,z)$$

$$-\frac{(\rho_n - \rho_{n+1})}{a_{n+1}}(x,z) + \sum_{j\neq z} \bar{\lambda}_{zj}(x)(v_n(x,j) - v_n(x,z) - v_{n+1}(x,j) + v_{n+1}(x,z))$$

$$\geq (v_n''(x,z) - v_{n+1}''(x,z)) - \frac{K}{\delta}|v_n'(x,z) - v_{n+1}'(x,z)|$$

$$-\frac{1}{\delta}(\rho_n - \rho_{n+1}) - \sum_{j\neq z} |\bar{\lambda}_{zj}(x)||v_n(x,j) - v_n(x,z) - v_{n+1}(x,j) + v_{n+1}(x,z)|. \quad (52)$$

So, we have just shown that a.e.,

$$0 \geq (v_n''(x,z) - v_{n+1}''(x,z)) - \frac{K}{\delta}|v_n'(x) - v_{n+1}'(x)|$$

$$-\frac{\rho_n - \rho_{n+1}}{\delta} - C\sum_{j\neq z} |v_n(x,j) - v_n(x,z) - v_{n+1}(x,j) + v_{n+1}(x,z)|. \quad (53)$$

The next trick is to note that again due to (52) and $\rho_n \geq \rho_{n+1}$, and since $\delta \leq a \leq C$,

$$0 \overset{a.e.}{\geq} G[v_n](x,z) - \rho_n \geq a_{n+1}(v_n'' - v_{n+1}'')(x,z) - C'|v_n' - v_{n+1}'|(x,z)$$

$$-C''\|v_n - v_{n+1} - \beta_n\|(x) - (\rho_n - \rho_{n+1}),$$

which implies that

$$0 \overset{a.e.}{\geq} v_n'' + F_1[x,z,v_n,v_n',\rho_n] \geq ((v_n'' - v_{n+1}'')(x,z) - C|v_n' - v_{n+1}'|)(x,z)$$
$$-C\|v_n - v_{n+1} - \beta_n\|)(x,z) - c(\rho_n - \rho_{n+1}). \quad (54)$$

(Recall the notations: $G[v_n] := \inf_\alpha(L^\alpha v_n + f^\alpha)(= L^{\alpha_{n+1}} v_n + f^{\alpha_{n+1}})$, and $F_1[x,z,v,v',\rho] := \inf_u[\hat{b}^u v' + \hat{f}^u - \hat{\rho}^u](x) + \sum_{j\in\mathcal{S}\setminus z} \bar{\lambda}_{zj}(x)\,(v(x,j) - v(x,z))\,.$) Now, since v_n' is absolutely continuous, we can integrate (54) so as to get the following: for any (not a.e.!) x and r with $x > r$,

$$0 \ge v'_{n_k}(x,z) - v'_{n_k}(r,z) + \int_r^x F_1[s,z,v_{n_k}(s,z), v'_{n_k}(s,z), \rho_{n_k}]\,ds$$

$$\ge \int_r^x ((v''_{n_k} - v''_{n_k+1})(s,z) - C|v'_{n_k} - v'_{n_k+1}|(s,z)$$

$$-C\|v_{n_k} - v_{n_k+1} - \beta_{n_k}\|(s) - c(\rho_{n_k} - \rho_{n_k+1}))\,ds$$

$$= v'_{n_k}(x,z) - v'_{n_k}(r,z) - v'_{n_k+1}(x,z) + v'_{n_k+1}(r,z) - C\int_r^x |v'_{n_k} - v'_{n_k+1}|(s,z)ds$$

$$-C\int_r^x \|v_{n_k} - v_{n_k+1} - \beta_{n_k}\|(s)ds - c(x-r)(\rho_{n_k} - \rho_{n_k+1}). \quad (55)$$

As we have seen, we may assume that there exists a sub-sequence $n_k \to \infty$, $k \to \infty$ such that $v_{n_k} \to \tilde{v}$ and $v'_{n_k} \to \tilde{v}'$ (locally in C) for some $\tilde{v} \in C^1$, and that also $v_{n_{k+1}} \to \tilde{v}$ and $v'_{n_{k+1}} \to \tilde{v}'$ (locally in C), too. Note that $\tilde{v}'$ is absolutely continuous. Therefore, it is possible to get to the limit in the inequality (55) as $k \to \infty$: for any $x > r$,

$$0 \ge \tilde{v}'(x,z) - \tilde{v}'(r,z) + \lim_k \int_r^x F_1[s,z,v_{n_k}(s,z), v'_{n_k}(s,z), \rho_{n_k}]\,ds \ge 0,$$

since the right hand side in (55) clearly goes to zero.

Here

$$F_1[s,z,v_{n_k}(s,z),v'_{n_k}(s,z),\rho_{n_k}]$$

$$= \inf_u \left[\frac{b^u}{a^u}v'_{n_k}(s,z) + \frac{f^u}{a^u}(s,z) - \frac{\rho_{n_k}}{a^u}(s,z)\right] + \sum_{j\in\mathcal{S}\setminus z}\bar{\lambda}_{zj}(x)\left(v_{n_k}(x,j) - v_{n_k}(x,z)\right)$$

$$\to \inf_{u\in U}\left[\frac{b^u}{a^u}\tilde{v}'(s,z) + \frac{f^u}{a^u}(s,z) - \frac{\tilde{\rho}}{a^u}(s,z)\right] + \sum_{j\in\mathcal{S}\setminus z}\bar{\lambda}_{zj}(x)\left(\tilde{v}(x,j) - \tilde{v}(x,z)\right)$$

$$= F_1[s,z,\tilde{v}(s,z),\tilde{v}'(s,z),\tilde{\rho}], \quad n_k\to\infty.$$

So, from (55) we obtain the desired equation (50)

$$\tilde{v}'(x,z) - \tilde{v}'(r,z) + \int_r^x F_1[s,z,\tilde{v}(s,z),\tilde{v}'(s,z),\tilde{\rho}]\,ds = 0.$$

In turn, since $F_1[s,z,\tilde{v}(s,z),\tilde{v}'(s,z),\tilde{\rho}]$ is continuous and absolutely continuous in s, it implies $\tilde{v}\in C^2$ in s, and by (well-defined) differentiation we get the equation (51) for every $x\in\mathbb{R}, z\in\mathcal{S}$. Recall that (51) is equivalent to the HJB equation (11).

6. *Uniqueness for ρ in (11).* Suppose there are two solutions of the (HJB) equation, v^1,ρ^1 and v^2,ρ^2 with a polynomial growth for v^i. Earlier it was shown that both v^1 and v^2 are classical solutions with locally Lipschitz second derivatives. Denote $v(x,z) := v^1(x,z) - v^2(x,z)$ and consider two strategies $\alpha_1(x,z) = \mathrm{argsup}_{u\in U}(L^u v(x,z))$ and $\alpha_2(x,z) = \mathrm{arginf}_{u\in U}(L^u v(x,z))$, and let X^i_t be a (weak) solution of the SDE (1) corresponding to each strategy α_i; by μ_i we denote the corresponding stationary measures relates to the strategy α_i, $i=1,2$. Note that due to the measurable choice arguments – see, e.g., the Appendix in [1] – such Borel strategies exist; corresponding weak solutions also exist and are weakly unique given the strategies.

Let us denote

$$h_1(x,z) := \sup_{u\in U}(L^u v(x,z) + \rho^1 - \rho^2), \quad h_2(x,z) := \inf_{u\in U}(L^u v(x,z) + \rho^1 - \rho^2).$$

Then,

$$h_2(x,z) = \inf_{u\in U}(L^u v^1(x,z) + f^u(x,z) + \rho^1 - (L^u v^2(x,z) + f^u(x,z) + \rho^2))$$

$$\le \inf_{u\in U}(L^u v^1(x,z) + f^u(x,z) + \rho^1) - \inf_{u\in U}(L^u v^2(x,z) + f^u(x,z) + \rho^2) = 0,$$

and similarly,

$$h_1(x,z) = -\inf_u(L^u(-v)(x,z) + \rho^2 - \rho^1)$$

$$= -\inf_u(L^u v^2(x) + f^u(x,z) + \rho^2 - (L^u v^1(x,z) + f^u(x,z) + \rho^1))$$

$$\ge -[\inf_u(L^u v^2(x,z) + f^u(x,z) + \rho^2) - \inf_u(L^u v^1(x,z) + f^u(x,z) + \rho^1)] = 0.$$

We have,

$$L^{\alpha_2} v(x,z) = h_2(x,z) + \rho^2 - \rho^1,$$

and

$$L^{\alpha_1} v(x,z) = h_1(x,z) + \rho^2 - \rho^1.$$

Further, Dynkin's (Itô–Krylov's with expectations) formula is applicable. So,

$$\mathbb{E}_{x,z} v(X^1_t, Z_t) - v(x,z) = \mathbb{E}_{x,z}\int_0^t L^{\alpha_1} v(X^1_s, Z_s)\, ds$$

$$= \mathbb{E}_{x,z}\int_0^t h_1(X^1_s, Z_s)\, ds + (\rho^2 - \rho^1)\, t \overset{(h_1\ge 0)}{\ge} (\rho^2 - \rho^1)\, t.$$

Here the left hand side is bounded (x fixed) due to the Lemma 2, so, we obtain,

$$\rho_1 - \rho_2 \ge 0.$$

Similarly, considering α_2 and using $h_2 \le 0$ we conclude that

$$\mathbb{E}_{x,z} v(X^2_t, Z_t) - v(x,z) = \mathbb{E}_{x,z}\int_0^t L^{\alpha_2} v(X^2_s, Z_s)\, ds$$

$$= \mathbb{E}_{x,z}\int_0^t h_2(X^2_s, Z_s)\, ds + (\rho^2 - \rho^1)\, t.$$

From here, due to the boundedness of the left hand side (Lemma 2) we get,

$$\rho^1 - \rho^2 = \liminf_{t\to 0}(t^{-1}\mathbb{E}_{x,z}\int_0^t h_2(X_s^2, Z_s)\,ds) \stackrel{(h_2\le 0)}{\le} 0.$$

Thus, eventually,

$$\rho_1 = \rho_2.$$

7. *Let us show that* $\rho = \tilde\rho$. We have seen that for any initial strategy α_0, the sequence ρ_n converges monotonically decreasing to $\tilde\rho$, which is a unique component of solution of the equation (11). Hence, given some (any) $\epsilon > 0$, let us take any initial strategy α_0 such that

$$\rho_0 = \rho^{\alpha_0} < \rho + \epsilon.$$

Then, clearly, the corresponding limit $\tilde\rho$ will satisfy the same inequality,

$$\tilde\rho = \lim_{n\to\infty} \rho_n < \rho + \epsilon.$$

Due to uniqueness of $\tilde\rho$ as a component of solution of the equation (11) and since $\epsilon > 0$ is arbitrary, we conclude that

$$\tilde\rho \le \rho.$$

But also $\tilde\rho \ge \rho$ since $\tilde\rho$ is the infimum of the cost function values over a smaller – just countable – family of strategies. So,

$$\tilde\rho = \rho.$$

8. *Uniqueness for* V *up to an additive constant.* Let us have another look at the earlier equations, replacing $\rho^2 - \rho^1$ by zero as we know that the second component in the solution is unique:

$$\mathbb{E}_{x,z} v(X_t^1, Z_t) - v(x,z) = \mathbb{E}_x \int_0^t h_1(X_s^1, Z_s)\,ds. \tag{56}$$

Clearly, $h_1 \ge 0$ with $h_1 \ne 0$ – i.e., with $\Lambda(x : h_1(x) > 0) > 0$ – would imply that $\langle h_1, \mu_1\rangle > 0$, which contradicts to the zero left hand side (after division by t with $t \to \infty$). So, we conclude that

$$h_1(x,z) = 0, \quad \mu_1 - \text{a.s. for each } z$$

By virtue of the Corollary 1, $\Lambda(\cdot) \ll \mu_1(\cdot, z)$ for any z. Therefore,

$$h_1(x, z) = 0, \quad \Lambda - \text{a.s. for each } z.$$

So, by Krylov's estimate we have between any two subsequent stopping times of jumps of the component Z, say, $T_1 < T_2$ with the value $Z_s = \bar{z}$ between them,

$$\begin{aligned}(0 \le)\mathbb{E}_{x,z} \int_{T_1\wedge t}^{T_2\wedge t} h_1(X_s^1, Z_s)\, ds = \mathbb{E}_{x,z} \left(\mathbb{E}_{x,z} \int_{T_1\wedge t}^{T_2\wedge t} h_1(X_s^1, \bar{z})\, ds | \mathcal{F}_{T_1\wedge t} \right) |_{\bar{z}=Z_{T_1\wedge t}} \\ \le N_t \mathbb{E}_{x,z} \|h_1(\cdot, \bar{z})\|_{L_1}|_{\bar{z}=Z_{T_1\wedge t}} = 0.\end{aligned} \tag{57}$$

So, also

$$(0 \le)\mathbb{E}_{x,z} \int_0^t h_1(X_s^1, Z_s)\, ds = 0, \tag{58}$$

Hence, in fact, the equality (56) may be rewritten as

$$\mathbb{E}_{x,z} v(X_t^1, Z_t) - v(x, z) = 0. \tag{59}$$

Further, (59) and due to the Lemma 2 it follows that

$$v(x, z) = \lim_{t\to\infty} \mathbb{E}_{x,z} v(X_t^1, Z_t) = \langle v, \mu_1 \rangle.$$

Hence, $v(x, z)$ is a constant. Recall that uniqueness of the first component V is stated up to a constant, and it was just established that

$$v^1(x, z) - v^2(x, z) = \text{const}.$$

So, the last claim of the Theorem is proved. □

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