NON-UNIQUENESS IN LAW OF STOCHASTIC 3D NAVIER–STOKES EQUATIONS

MARTINA HOFMANOVÁ, RONGCHAN ZHU, AND XIANGCHAN ZHU

ABSTRACT. We consider the stochastic Navier–Stokes equations in three dimensions and prove that the law of analytically weak solutions is not unique. In particular, we focus on two iconic examples of a stochastic perturbation: either an additive or a linear multiplicative noise driven by a Wiener process. In both cases, we develop a stochastic counterpart of the convex integration method introduced recently by Buckmaster and Vicol. This permits to construct probabilistically strong and analytically weak solutions defined up to a suitable stopping time. In addition, these solutions fail the corresponding energy inequality at a prescribed time with a prescribed probability. Then we introduce a general probabilistic construction used to extend the convex integration solutions beyond the stopping time and in particular to the whole time interval $[0, \infty)$. Finally, we show that their law is distinct from the law of solutions obtained by Galerkin approximation. In particular, non-uniqueness in law holds on an arbitrary time interval $[0, T]$, $T > 0$.

CONTENTS

Date: December 26, 2019.

2010 *Mathematics Subject Classification.* 60H15; 35R60; 35Q30.

Supported in part by NSFC (No. 11671035, No. 11771037, No.11922103). Financial support by the DFG through the CRC 1283 "Taming uncertainty and profiting from randomness and low regularity in analysis, stochastics and their applications" and support by key Lab of Random Complex Structures and Data Science, Chinese Academy of Science are gratefully acknowledged.

1. Introduction

The fundamental problems in fluid dynamics remain largely open. On the theoretical side, existence and smoothness of solutions to the three dimensional incompressible Navier–Stokes system is one of the Millennium Prize Problems. An intimately related question is that of uniqueness of solutions. Intuitively, smooth solutions are unique whereas uniqueness for less regular solutions, such as weak solutions, is very challenging and not even true for a number of models.

A revolutionary step was made through the method of convex integration by De Lellis and Székelyhidi Jr. [DLS09, DLS10, DLS13]. They were able to construct infinitely many weak solutions to the incompressible Euler system which dissipate energy and even satisfy various additional criteria such as a global or local energy inequality. After this breakthrough, an avalanche of excitement and intriguing results followed, proving existence of solutions with often rather pathological behavior. In particular, it is nowadays well understood that the compressible counterpart of the Euler system is desperately ill-posed: even certain smooth initial data give rise to infinitely many weak solutions satisfying an energy inequality, see Chiodaroli et al. [CKMS19]. Very recently, the non-uniqueness of weak solutions to the incompressible Navier–Stokes equations was obtained by Buckmaster and Vicol [BV19b], see also Buckmaster, Colombo and Vicol [BCV18].

In view of these substantial theoretical difficulties, it is natural to believe that a certain probabilistic description is indispensable and may eventually help with the non-uniqueness issue. In particular, it is essential to develop a suitable probabilistic understanding of the deterministic systems, in order to capture their chaotic and intrinsically random nature after the blow-up and loss of uniqueness. Moreover, there is evidence that a suitable stochastic perturbation may provide a regularizing effect on deterministically ill-posed problems, in particular those involving transport as shown, e.g., by Flandoli, Gubinelli and Priola [FGP10]. Also a linear multiplicative noise as treated in the present paper has a certain stabilizing effect on the three dimensional Navier–Stokes system, see Röckner, Zhu and Zhu [RZZ14].

On the other hand, an external stochastic forcing is often included in the system of governing equations, taking additional model uncertainties into account. Mathematically, this introduces new phenomena and raises basic questions of solvability of the system, i.e. existence and uniqueness of solutions, as well as their long time behavior. In particular, the question of uniqueness of the probability measures induced by solutions, the so-called uniqueness in law, has been a longstanding open problem in the field.

In the present paper, we prove that non-uniqueness in law holds for the stochastic three dimensional Navier–Stokes system posed on a periodic domain in a class of analytically weak solutions. This system governs the time evolution of the velocity u of a viscous incompressible fluid under stochastic perturbations. It reads as

$$\begin{aligned} du - \nu\Delta u dt + \mathrm{div}(u\otimes u)dt + \nabla P dt &= G(u)dB, \\ \mathrm{div} u &= 0, \end{aligned} \tag{1.1}$$

where $G(u)dB$ represents a stochastic force acting on the fluid and $\nu > 0$ is the kinematic viscosity. We particularly focus on two iconic examples of a stochastic forcing, namely, an additive noise driven by a cylindrical Wiener process B with diffusion coefficients G^i being smooth functions of the spatial variable x, i.e.,

$$G(u)dB = GdB = \sum_{i=1}^{\infty} G^i dB_i, \quad G^i = G^i(x), \tag{1.2}$$

and a linear multiplicative noise driven by a real-valued Wiener process B_1, i.e.,

$$G(u)dB = udB_1. \tag{1.3}$$

In both settings, we develop a stochastic counterpart of the convex integration method introduced by Buckmaster and Vicol [BV19a] and construct analytically weak solutions with unexpected behavior defined up to suitable stopping times. The striking feature of these solutions is that they are probabilistically strong, i.e., adapted to the given Wiener process. This severely contradicts the general belief present within the SPDEs community, namely, that probabilistically strong solutions and uniqueness in law could help with the uniqueness problem for the Navier–Stokes system.

We say that uniqueness in law holds for a system of SPDEs provided the probability law induced by the solutions is uniquely determined. On the other hand, we say that pathwise uniqueness holds true if two solutions coincide almost surely. There are explicit examples of stochastic differential equations (SDEs), where pathwise uniqueness does not hold but uniqueness in law is valid. Pathwise uniqueness for the stochastic Navier–Stokes system essentially poses the same difficulties as uniqueness in the deterministic setting. As a consequence, there has been a clear hope that showing uniqueness in law for the Navier–Stokes system might be easier than proving pathwise uniqueness. Furthermore, Yamada–Watanabe–Engelbert's theorem states that, for a certain class of SDEs, pathwise uniqueness is equivalent to uniqueness in law and existence of a probabilistically strong solution, see Kurtz [K07], Cherny [C03]. This suggests another possible way towards pathwise uniqueness, provided one could prove uniqueness in law.

Our main result proves the above hopes wrong, at least for a certain class of analytically weak solutions. However, the question of uniqueness of the so-called Leray solutions remains an outstanding open problem. In particular, we show that non-uniqueness in law for analytically weak solutions holds true on an arbitrary time interval $[0, T]$, $T > 0$. This trivially implies pathwise non-uniqueness. More precisely, we construct a deterministic divergence-free initial condition $u(0) \in L^2$ which gives rise to two solutions to the Navier–Stokes system (1.1) with distinct laws. One of the solutions is constructed by means of the convex integration method whereas the other one is a solution obtained by a classical compactness argument from a Galerkin approximation, see e.g. [FG95].

We note that the solutions obtained by Galerkin approximation are clearly more physical as they correspond to Leray solutions in the deterministic setting and satisfy the energy inequality. However, these solutions are not probabilistically strong as the adaptedness with respect to the given noise is lost within the stochastic compactness method. On the other hand, the convex integration permits to construct adapted solutions up to a stopping time but they behave in an unphysical way with respect to the energy inequality. Moreover, the spatial regularity is worse as we can only prove that they belong to H^γ for a certain $\gamma > 0$ small.

1.1. **Main results.** Even though the main result, i.e., non-uniqueness in law, is the same in both considered settings (1.2) and (1.3), the proofs are different. The additive noise case is easier and we present a direct construction of two solutions with different laws. This is not possible in the case of a linear multiplicative noise where the proof becomes more involved. For notational simplicity, we suppose from now on that $\nu = 1$.

1.1.1. *Additive noise.* Consider the stochastic Navier–Stokes system driven by an additive noise on $\mathbb{T}^3$, which reads as

$$\begin{aligned} du - \Delta u dt + \operatorname{div}(u \otimes u)dt + \nabla P dt &= dB, \\ \operatorname{div} u &= 0, \end{aligned} \tag{1.4}$$

where B is a GG^*-Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and G is a Hilbert–Schmidt operator from L^2 to L^2. Let $(\mathcal{F}_t)_{t\geq 0}$ denote the normal filtration generated by B, that is, the canonical right continuous filtration augmented by all the $\mathbf{P}$-negligible events.

Our first result in this setting is the existence of a probabilistically strong solution which is defined up to a stopping time and which violates the corresponding energy inequality.

Theorem 1.1. *Suppose that* $\operatorname{Tr}((-\Delta)^{\frac{3}{2}+2\sigma} GG^*) < \infty$ *for some* $\sigma > 0$. *Let* $T > 0$, $K > 1$ *and* $\kappa \in (0,1)$ *be given. Then there exist* $\gamma \in (0,1)$ *and a* $\mathbf{P}$*-a.s. strictly positive stopping time* $\mathfrak{t}$ *satisfying* $\mathbf{P}(\mathfrak{t} \geq T) > \kappa$ *such that the following holds true: There exists an* $(\mathcal{F}_t)_{t\geq 0}$*-adapted process* u *which belongs to* $C([0,\mathfrak{t}]; H^\gamma)$ $\mathbf{P}$*-a.s. and is an analytically weak solution to* (1.4) *with* $u(0)$ *deterministic. In addition,*

$$\operatorname*{esssup}_{\omega\in\Omega} \sup_{t\in[0,\mathfrak{t}]} \|u(t)\|_{H^\gamma} < \infty, \tag{1.5}$$

and

$$\|u(T)\|_{L^2} > K\|u(0)\|_{L^2} + K\big(T \operatorname{Tr}(GG^*)\big)^{1/2} \tag{1.6}$$

on the set $\{\mathfrak{t} \geq T\}$.

The proof of this result relies on a the convex integration method and the stopping time is employed in the construction in order to control the noise in various bounds. While this result readily implies non-uniqueness in law for solutions defined on the random time interval $[0,\mathfrak{t}]$, our main result is more general: we prove non-uniqueness in law on an arbitrary time interval or more generally up to an arbitrary stopping time.

Theorem 1.2. *Suppose that* $\operatorname{Tr}((-\Delta)^{\frac{3}{2}+2\sigma} GG^*) < \infty$ *for some* $\sigma > 0$. *Then non-uniqueness in law holds for the Navier–Stokes system* (1.4) *on* $[0,\infty)$. *Furthermore, for every given* $T > 0$, *non-uniqueness in law holds for the Navier–Stokes system* (1.4) *on* $[0,T]$.

In order to derive the result of Theorem 1.2 from Theorem 1.1, it is necessary to extend the convex integration solutions to the whole time interval $[0,\infty)$. To this end, we present a general probabilistic construction which connects the law of solutions defined up to a stopping time to a law of a solution obtained by the classical compactness argument. The principle difficulty is to allow for the concatenation of solutions at a random time. Since the stopping time $\mathfrak{t}$ is defined in terms of the solution u, we work with the notion of martingale solution which is defined as the law of a solution u. Consequently, we are able to obtain non-uniqueness in law, i.e., non-uniqueness of martingale solutions directly, as opposed to the case of a linear multiplicative noise.

1.1.2. *Linear multiplicative noise.* Consider the stochastic Navier–Stokes equation driven by linear multiplicative noise on $\mathbb{T}^3$, which reads as

$$\begin{aligned} du - \Delta u dt + \operatorname{div}(u \otimes u)dt + \nabla P dt &= u dB, \\ \operatorname{div} u &= 0, \end{aligned} \tag{1.7}$$

where B is a real-valued Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Similarly to above, we denote by $(\mathcal{F}_t)_{t\geq 0}$ the normal filtration generated by B. The main results in this case are as follows.

Theorem 1.3. *Let $T > 0$, $K > 1$ and $\kappa \in (0,1)$ be given. Then there exist $\gamma \in (0,1)$ and a $\mathbf{P}$-a.s. strictly positive stopping time $\mathfrak{t}$ satisfying $\mathbf{P}(\mathfrak{t} \geq T) > \kappa$ and the following holds true: There exists an $(\mathcal{F}_t)_{t\geq 0}$-adapted process u which belongs to $C([0,\mathfrak{t}]; H^\gamma)$ $\mathbf{P}$-a.s. and is an analytically weak solution to (1.7) with $u(0)$ deterministic. In addition,*

$$\operatorname*{esssup}_{\omega\in\Omega} \sup_{t\in[0,\mathfrak{t}]} \|u(t)\|_{H^\gamma} < \infty,$$

and

$$\|u(T)\|_{L^2} > K e^{T/2} \|u(0)\|_{L^2}$$

on the set $\{\mathfrak{t} \geq T\}$.

Theorem 1.4. *Non-uniqueness in law holds for the Navier–Stokes system (1.7) on $[0,\infty)$. Furthermore, for every given $T > 0$, non-uniqueness in law holds for the Navier–Stokes system (1.7) on $[0,T]$.*

Contrary to the additive noise setting, the stopping time $\mathfrak{t}$ in the case of the linear multiplicative noise is a function of B and not a function of the solution u. As a consequence, we are forced to work with the notion of a probabilistically weak solution which governs the joint law of (u,B). We extend our method of concatenation of two solutions to connect the probabilistically weak solution obtained through Theorem 1.3 to a probabilistically weak solution obtained by compactness. Accordingly, we first only deduce joint non-uniqueness in law, i.e., non-uniqueness of probabilistically weak solutions. Finally, we prove that joint non-uniqueness in law implies non-uniqueness in law, concluding the proof of Theorem 1.4. This relies on a generalization of the result of Cherny [C03] to the infinite dimensional setting which is interesting in its own right, see Appendix C.

1.2. **Further relevant literature.** Stochastic Navier–Stokes equations driven by a trace-class noise, have been the subject of interest of a large number of works. The reader is referred e.g. to [FG95, HM06, De13] and the reference therein. In the two dimensional case, existence and uniqueness of strong solutions was obtained if the noisy forcing term is white in time and colored in space. In the three dimensional case, existence of martingale solutions was proved in [FR08, DPD03, GRZ09]. Furthermore, ergodicity was proved if the system is driven by non-degenerate trace-class noise, see [DPD03, FR08]. Navier–Stokes equations driven by space-time white noise are also considered in [DPD02] and [ZZ15] and the system is studied in the context of rough paths theory in [HLN19a, HLN19b]. The linear multiplicative noise (1.3) can be seen as a damping term: it is shown in [RZZ14] that it prevents the system from exploding with a large probability. In a more recent work, Flandoli and Luo [FL19] proved that one kind of transport noise improves the vorticity blow-up in 3D Navier-Stokes equations with large probability. In [BR17], a global solution starting from small initial data was constructed for 3D Navier–Stokes equations in vorticity formulation driven by linear multiplicative noise. However, the solutions are not adapted to the filtration generated by the noise and the stochastic integral should be understood in a rough path sense (see [RZZ19] and [MR19] for more general noise). By the methods in [BR17, MR19], adapted solutions up to a stopping time can also be obtained. However, we note that existence of globally defined probabilistically strong solutions without any stopping time remains a challenging open problem. Finally, we note that the convex integration has already been applied in a stochastic setting, namely, to the isentropic Euler system in [BFH17] and to the full Euler system in [CFF19].

1.3. **Organization of the paper.** In Section 2, we collect the notations used throughout the paper. Section 3 and Section 4 are devoted to the proof of our first main result Theorem 1.2, the non-uniqueness in law for the case of an additive noise. First, in Section 3 we introduce the notion of martingale solution and present a general method of extending martingale solutions defined up to a stopping time to the whole time interval $[0,\infty)$. This is then applied to solutions obtained through the convex integration technique and the non-uniqueness in law is shown in Section 3.3. The convex integration solutions are constructed in Section 4, which proves Theorem 1.1. A similar structure can be found in Section 5 and Section 6 devoted to the setting of a linear multiplicative noise. This relies on the notion of probabilistically weak solution and a general concatenation procedure presented in Section 5.2. Application to the convex integration solutions together with the proof of Theorem 1.4 can be found in Section 5.3. The convex integration in this setting is applied in Section 6, where Theorem 1.3 is established. In Appendix, we collect several auxiliary results concerning stability of martingale as well as probabilistically weak solutions in Appendix A and the construction of intermittent jets needed for the convex integration in Appendix B. Finally, in Appendix C, we show that non-uniqueness in law implies joint non-uniqueness in law in a general infinite dimensional SPDE setting.

Acknowledgments

We would like to thank Michael Röckner for helpful suggestions.

2. Notations

2.1. **Function spaces.** Throughout the paper, we use the notation $a\lesssim b$ if there exists a constant $c>0$ such that $a\leq cb$, and we write $a\simeq b$ if $a\lesssim b$ and $b\lesssim a$. Given a Banach space E with a norm $\|\cdot\|_E$ and $T>0$, we write $C_TE=C([0,T];E)$ for the space of continuous functions from $[0,T]$ to E, equipped with the supremum norm $\|f\|_{C_TE}=\sup_{t\in[0,T]}\|f(t)\|_E$. We also use CE or $C([0,\infty);E)$ to denote the space of continuous functions from $[0,\infty)$ to E. For $\alpha\in(0,1)$ we define C^α_TE as the space of α-Hölder continuous functions from $[0,T]$ to E, endowed with the seminorm $\|f\|_{C^\alpha_TE}=\sup_{s,t\in[0,T],s\neq t}\frac{\|f(s)-f(t)\|_E}{|t-s|^\alpha}$. Here we use C^α_T to denote the case when $E=\mathbb{R}$. We also use $C^\alpha_{\mathrm{loc}}E$ to denote the space of functions from $[0,\infty)$ to E satisfying $f|_{[0,T]}\in C^\alpha_TE$ for all $T>0$. For $p\in[1,\infty]$ we write $L^p_TE=L^p([0,T];E)$ for the space of L^p-integrable functions from $[0,T]$ to E, equipped with the usual L^p-norm. We also use $L^p_{\mathrm{loc}}([0,\infty);E)$ to denote the space of functions f from $[0,\infty)$ to E satisfying $f|_{[0,T]}\in L^p_TE$ for all $T>0$. We use L^p to denote the set of standard L^p-integrable functions from $\mathbb{T}^3$ to $\mathbb{R}^3$. For $s>0$, $p>1$ we set $W^{s,p}:=\{f\in L^p;\|(I-\Delta)^{\frac{s}{2}}f\|_{L^p}<\infty\}$ with the norm $\|f\|_{W^{s,p}}=\|(I-\Delta)^{\frac{s}{2}}f\|_{L^p}$. Set $L^2_\sigma=\{u\in L^2;\mathrm{div}u=0\}$. For $s>0$, $H^s:=W^{s,2}\cap L^2_\sigma$. For $s<0$ define H^s to be the dual space of H^{-s}.

$\|f\|_{C^N_{t,x}}=\sum_{0\leq n+|\alpha|\leq N}\|\partial_t^nD^\alpha f\|_{L^\infty_tL^\infty}$. For a Polish space H we also use $\mathcal{B}(H)$ to denote the σ-algebra of Borel sets in H.

2.2. **Probabilistic elements.** Let $\Omega_0:=C([0,\infty),H^{-3})\cap L^\infty_{\mathrm{loc}}([0,\infty),L^2_\sigma)$ and let $\mathscr{P}(\Omega_0)$ denote the set of all probability measures on $(\Omega_0,\mathcal{B})$ with $\mathcal{B}$ being the Borel σ-algebra coming from the topology of locally uniform convergence on Ω_0. Let $x:\Omega_0\to H^{-3}$ denote the canonical process on Ω_0 given by
$$x_t(\omega)=\omega(t).$$
Similarly, for $t\geq0$ we define $\Omega_t:=C([t,\infty),H^{-3})\cap L^\infty_{\mathrm{loc}}([t,\infty),L^2_\sigma)$ equipped with its Borel σ-algebra $\mathcal{B}^t$ which coincides with $\sigma\{x(s),s\geq t\}$. Finally, we define the canonical filtration $\mathcal{B}^0_t:=\sigma\{x(s),s\leq t\}$, $t\geq0$, as well as its right continuous version $\mathcal{B}_t:=\cap_{s>t}\mathcal{B}^0_s$, $t\geq0$. For given probability measure P we use E^P to denote the expectation under P.

For a Hilbert space U, let $L_2(U,L^2_\sigma)$ be the space all Hilbert–Schmidt operators from U to L^2_σ with the norm $\|\cdot\|_{L_2(U,L^2_\sigma)}$. Let $G:L^2_\sigma\to L_2(U,L^2_\sigma)$ be $\mathcal{B}(L^2_\sigma)/\mathcal{B}(L_2(U,L^2_\sigma))$ measurable. In the following, we assume
$$\|G(x)\|_{L_2(U,L^2_\sigma)}\leq C(1+\|x\|_{L^2}),$$

for every $x \in C^\infty(\mathbb{T}^3) \cap L^2_\sigma$ and if in addition $y_n \to y$ in L^2 then

$$\lim_{n\to\infty} \|G(y_n)^* x - G(y)^* x\|_U = 0,$$

where the asterisk denotes the adjoint operator.

Suppose there is another Hilbert space U_1 such that the embedding $U \subset U_1$ is Hilbert–Schmidt. Let $\bar{\Omega} := C([0,\infty); H^{-3} \times U_1) \cap L^\infty_{\mathrm{loc}}([0,\infty); L^2_\sigma \times U_1)$ and let $\mathscr{P}(\bar{\Omega})$ denote the set of all probability measures on $(\bar{\Omega}, \bar{\mathcal{B}})$ with $\bar{\mathcal{B}}$ being the Borel σ-algebra coming from the topology of locally uniform convergence on $\bar{\Omega}$. Let $(x, y) : \bar{\Omega} \to H^{-3} \times U_1$ denote the canonical process on $\bar{\Omega}$ given by

$$(x_t(\omega), y_t(\omega)) = \omega(t).$$

For $t \geq 0$ we define σ-algebra $\bar{\mathcal{B}}^t = \sigma\{(x(s), y(s)), s \geq t\}$. Finally, we define the canonical filtration $\bar{\mathcal{B}}^0_t := \sigma\{(x(s), y(s)), s \leq t\}$, $t \geq 0$, as well as its right continuous version $\bar{\mathcal{B}}_t := \cap_{s>t} \bar{\mathcal{B}}^0_s$, $t \geq 0$.

3. Non-uniqueness in law I: the case of an additive noise

3.1. **Martingale solutions.** Let us begin with a definition of martingale solution on $[0,\infty)$. In what follows, we fix $\gamma \in (0,1)$.

Definition 3.1. Let $s \geq 0$ and $x_0 \in L^2_\sigma$. A probability measure $P \in \mathscr{P}(\Omega_0)$ is a martingale solution to the Navier–Stokes system (1.1) with the initial value x_0 at time s provided

(M1) $P(x(t) = x_0, 0 \leq t \leq s) = 1$, and for any $n \in \mathbb{N}$

$$P\left\{x \in \Omega_0 : \int_0^n \|G(x(r))\|^2_{L_2(U;L^2_\sigma)} dr < +\infty\right\} = 1.$$

(M2) For every $e_i \in C^\infty(\mathbb{T}^3) \cap L^2_\sigma$, and for $t \geq s$ the process

$$M^i_{t,s} := \langle x(t) - x(s), e_i\rangle + \int_s^t \langle \mathrm{div}(x(r) \otimes x(r)) - \Delta x(r), e_i\rangle dr$$

is a continuous square integrable $(\mathcal{B}_t)_{t\geq s}$-martingale under P with the quadratic variation process given by $\int_s^t \|G(x(r))^* e_i\|^2_U dr$, where the asterisk denotes the adjoint operator.

(M3) For any $q \in \mathbb{N}$ there exists a positive real function $t \mapsto C_{t,q}$ such that for all $t \geq s$

$$E^P\left(\sup_{r\in[0,t]} \|x(r)\|^{2q}_{L^2} + \int_s^t \|x(r)\|^2_{H^\gamma} dr\right) \leq C_{t,q}(\|x_0\|^{2q}_{L^2} + 1),$$

where E^P denotes the expectation under P.

In particular, we observe that in the context of Definition 3.1 for additive noise case, i.e. G independent of x, if $\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of L^2_σ consisting of eigenvector of GG^* then $M_{t,s} := \sum_{i\in\mathbb{N}} M^i_{t,s} e_i$ is a GG^*-Wiener process starting from s with respect to the filtration $(\mathcal{B}_t)_{t\geq s}$ under P.

Similarly, we may define martingale solutions up to a stopping time $\tau : \Omega_0 \to [0,\infty]$. To this end, we define the space of trajectories stopped at the time τ by

$$\Omega_{0,\tau} := \{\omega(\cdot \wedge \tau(\omega)); \omega \in \Omega_0\}.$$

We note that due to the Borel measurability of τ, the set $\Omega_{0,\tau} = \{\omega : x(t,\omega) = x(t\wedge\tau(\omega), \omega), \forall t \geq 0\}$ is a Borel subset of Ω_0 hence $\mathscr{P}(\Omega_{0,\tau}) \subset \mathscr{P}(\Omega_0)$.

Definition 3.2. Let $s \geq 0$ and $x_0 \in L^2_\sigma$. Let $\tau \geq s$ be a $(\mathcal{B}_t)_{t\geq s}$-stopping time. A probability measure $P \in \mathscr{P}(\Omega_{0,\tau})$ is a martingale solution to the Navier–Stokes system (1.1) on $[s,\tau]$ with the initial value x_0 at time s provided

(M1) $P(x(t) = x_0, 0 \le t \le s) = 1$, and for any $n \in \mathbb{N}$
$$P\left\{x \in \Omega_0 : \int_0^{n\wedge\tau} \|G(x(r))\|^2_{L_2(U;L_2^\sigma)} dr < +\infty\right\} = 1.$$
(M2) For every $e_i \in C^\infty(\mathbb{T}^3) \cap L^2_\sigma$, and for $t \ge s$ the process
$$M^i_{t\wedge\tau,s} := \langle x(t\wedge\tau) - x_0, e_i\rangle + \int_s^{t\wedge\tau} \langle \mathrm{div}(x(r)\otimes x(r)) - \Delta x(r), e_i\rangle dr$$
is a continuous square integrable $(\mathcal{B}_t)_{t\ge s}$-martingale under P with the quadratic variation process given by $\int_s^{t\wedge\tau} \|G(x(r))^* e_i\|^2_U dr$.
(M3) For any $q \in \mathbb{N}$ there exists a positive real function $t \mapsto C_{t,q}$ such that for all $t \ge s$
$$E^P\left(\sup_{r\in[0,t\wedge\tau]} \|x(r)\|^{2q}_{L^2} + \int_s^{t\wedge\tau} \|x(r)\|^2_{H^\gamma} dr\right) \le C_{t,q}(\|x_0\|^{2q}_{L^2} + 1),$$
where E^P denotes the expectation under P.

The following result provides the existence of martingale solutions as well as a stability of the set of all martingale solutions. A similar result can be found in [FR08, GRZ09] but in the present paper we require in addition stability with respect to the initial time. For completeness, we include the proof in Appendix A.

Theorem 3.1. *For every $(s, x_0) \in [0,\infty) \times L^2_\sigma$, there exists $P \in \mathscr{P}(\Omega_0)$ which is a martingale solution to the Navier–Stokes system (1.1) starting at time s from the initial condition x_0 in the sense of Definition 3.1. The set of all such martingale solutions with the same $C_{t,q}$ in* (M3) *of Definition 3.1 is denoted by $\mathscr{C}(s, x_0, C_{t,q})$.*

Let $(s_n, x_n) \to (s, x_0)$ in $[0,\infty) \times L^2_\sigma$ as $n \to \infty$ and let $P_n \in \mathscr{C}(s_n, x_n, C_{t,q})$. Then there exists a subsequence n_k such that the sequence $\{P_{n_k}\}_{k\in\mathbb{N}}$ converges weakly to some $P \in \mathscr{C}(s, x_0, C_{t,q})$.

For completeness, let us recall the definition of uniqueness in law.

Definition 3.3. We say that uniqueness in law holds for (1.1) if martingale solutions starting from the same initial distribution are unique.

Now, we have all in hand to proceed with the proof of our first main result, Theorem 1.2. On the one hand, by classical arguments as in Theorem 3.1 we obtain existence of a martingale solution to (1.1) which satisfies the corresponding energy inequality. On the other hand, for the case of an additive noise, Theorem 1.1 provides a stopping time $\mathfrak{t}$ such that there exists an $(\mathcal{F}_t)_{t\ge0}$-adapted analytically weak solution $u \in C([0,\mathfrak{t}]; H^\gamma)$ to (1.4), which violates the energy inequality. The main idea is to construct a martingale solution which is defined on the full interval $[0,\infty)$ and preserves the properties of the adapted solution on $[0,\mathfrak{t}]$, that is, the energy inequality is not satisfied in this random time interval. To this end, the essential point is to make use of adaptedness of solutions obtained through Theorem 1.1 and connect them to ordinary martingale solutions obtained by Theorem 3.1. The difficulty is that the connection has to happen at a random time, which only turns out to be a stopping time with respect the right continuous filtration $(\mathcal{B}_t)_{t\ge0}$. Consequently, the classical martingale theory of Stroock and Varadhan [SV79] does not apply and we are facing a number of measurability issues which have to be carefully treated.

3.2. **General construction for martingale solutions.** First, we present an auxiliary result which is then used in order to extend martingale solutions defined up a stopping time τ to the whole interval $[0,\infty)$. To this end, we denote by $\mathcal{B}_\tau$ the σ-field associated to the stopping time τ. The results of this section apply to a general form of a noise in (1.1), the restriction to an additive noise is only required in Section 3.3 below in order to apply the result of Theorem 1.1.

Proposition 3.2. *Let τ be a bounded $(\mathcal{B}_t)_{t\geq 0}$-stopping time. Then for every $\omega \in \Omega_0$ there exists $Q_\omega \in \mathscr{P}(\Omega_0)$ such that*

$$Q_\omega\big(\omega' \in \Omega_0; x(t,\omega') = \omega(t) \text{ for } 0 \leq t \leq \tau(\omega)\big) = 1, \tag{3.1}$$

and

$$Q_\omega(A) = R_{\tau(\omega),x(\tau(\omega),\omega)}(A) \qquad \text{for all } A \in \mathcal{B}^{\tau(\omega)}. \tag{3.2}$$

where $R_{\tau(\omega),x(\tau(\omega),\omega)} \in \mathscr{P}(\Omega_0)$ is a martingale solution to the Navier–Stokes system (1.1) starting at time $\tau(\omega)$ from the initial condition $x(\tau(\omega),\omega)$. Furthermore, for every $B \in \mathcal{B}$ the mapping $\omega \mapsto Q_\omega(B)$ is $\mathcal{B}_\tau$-measurable.

Proof. It is necessary to select in a measurable way from the set of all martingale solutions. To this end, we observe that as a consequence of the stability with respect to the initial time and the initial condition in Theorem 3.1, for every $(s,x_0) \in [0,\infty)\times L^2_\sigma$ the set $\mathscr{C}(s,x_0,C_{t,q})$ of all associated martingale solutions to (1.1) with the same $C_{t,q}$ is compact with respect to the weak convergence of probability measures. Let $\mathrm{Comp}(\mathscr{P}(\Omega_0))$ denote the space of all compact subsets of $\mathscr{P}(\Omega_0)$ equipped with the Hausdorff metric. Using the stability from Theorem 3.1 together with [SV79, Lemma 12.1.8] we obtain that the map

$$[0,\infty)\times L^2_\sigma \to \mathrm{Comp}(\mathscr{P}(\Omega_0)), \qquad (s,x_0) \mapsto \mathscr{C}(s,x_0,C_{t,q}),$$

is Borel measurable. Accordingly, [SV79, Theorem 12.1.10] gives the existence of a measurable selection. More precisely, there exists a Borel measurable map

$$[0,\infty)\times L^2_\sigma \to \mathscr{P}(\Omega_0), \qquad (s,x_0) \mapsto R_{s,x_0},$$

such that $R_{s,x_0} \in \mathscr{C}(s,x_0,C_{t,q})$ for all $(s,x_0) \in [0,\infty)\times L^2_\sigma$.

As the next step, we recall that the canonical process x on Ω_0 is continuous in H^{-3}, hence $x : [0,\infty)\times\Omega_0 \to H^{-3}$ is progressively measurable with respect to the canonical filtration $(\mathcal{B}^0_t)_{t\geq 0}$ and consequently it is also progressively measurable with respect to the right continuous filtration $(\mathcal{B}_t)_{t\geq 0}$. Furthermore, $L^2_\sigma \subset H^{-3}$ continuously and densely, by Kuratowski's measurable theorem we know $L^2_\sigma \in \mathcal{B}(H^{-3})$ and $\mathcal{B}(L^2_\sigma) = \mathcal{B}(H^{-3}) \cap L^2_\sigma$, which implies that $x : [0,\infty)\times\Omega_0 \to L^2_\sigma$ is progressively measurable with respect to the right continuous filtration $(\mathcal{B}_t)_{t\geq 0}$. In addition, τ is a stopping time with respect to the same filtration $(\mathcal{B}_t)_{t\geq 0}$. Therefore, it follows from [SV79, Lemma 1.2.4] that both τ and $x(\tau(\cdot),\cdot)$ is $\mathcal{B}_\tau$-measurable, where $\mathcal{B}_\tau$ is the σ-algebra associated to τ. Combining this fact with the measurability of the selection $(s,x_0) \mapsto R_{s,x_0}$ constructed above, we deduce that

$$\Omega_0 \to \mathscr{P}(\Omega_0), \qquad \omega \mapsto R_{\tau(\omega),x(\tau(\omega),\omega)} \tag{3.3}$$

is $\mathcal{B}_\tau$-measurable as a composition of $\mathcal{B}_\tau$-measurable mappings. Recall that for every $\omega \in \Omega_0$ this mapping gives a martingale solution starting at the deterministic time $\tau(\omega)$ from the deterministic initial condition $x(\tau(\omega),\omega)$. In other words,

$$R_{\tau(\omega),x(\tau(\omega),\omega)}\big(\omega' \in \Omega_0; x(\tau(\omega),\omega') = x(\tau(\omega),\omega)\big) = 1.$$

Now, we apply [SV79, Lemma 6.1.1] and deduce that for every $\omega \in \Omega_0$ there is a unique probability measure

$$Q_\omega = \delta_\omega \otimes_{\tau(\omega)} R_{\tau(\omega),x(\tau(\omega),\omega)} \in \mathscr{P}(\Omega_0), \tag{3.4}$$

such that (3.1) and (3.2) hold. This permits to concatenate, at the deterministic time $\tau(\omega)$, the Dirac mass δ_ω with the martingale solution $R_{\tau(\omega),x(\tau(\omega),\omega)}$.

In order to show that the mapping $\omega \mapsto Q_\omega(B)$ is $\mathcal{B}_\tau$-measurable for every $B \in \mathcal{B}$, it is enough to consider sets of the form $A = \{x(t_1) \in \Gamma_1, \dots, x(t_n) \in \Gamma_n\}$ where $n \in \mathbb{N}$, $0 \le t_1 < \dots < t_n$, and $\Gamma_1, \dots, \Gamma_n \in \mathcal{B}(H^{-3})$. Then by the definition of Q_ω, we have

$$
\begin{aligned}
Q_\omega(A) = &\mathbf{1}_{[0,t_1)}(\tau(\omega)) R_{\tau(\omega),x(\tau(\omega),\omega)}(A) \\
&+ \sum_{k=1}^{n-1} \mathbf{1}_{[t_k,t_{k+1})}(\tau(\omega)) \mathbf{1}_{\Gamma_1}(x(t_1,\omega)) \cdots \mathbf{1}_{\Gamma_k}(x(t_k,\omega)) \\
&\quad \times R_{\tau(\omega),x(\tau(\omega),\omega)}\big(x(t_{k+1}) \in \Gamma_{k+1}, \dots, x(t_n) \in \Gamma_n\big) \\
&+ \mathbf{1}_{[t_n,\infty)}(\tau(\omega)) \mathbf{1}_{\Gamma_1}(x(t_1,\omega)) \cdots \mathbf{1}_{\Gamma_n}(x(t_n,\omega)).
\end{aligned}
$$

Here the right hand side is $\mathcal{B}_\tau$-measurable as a consequence of the $\mathcal{B}_\tau$-measurability of (3.3) and τ. The proof is complete. □

Remark 3.3. If P as a martingale solution up to a stopping time τ, our ultimate goal is to make use of Proposition 3.2, in order to define a probability measure

$$
P \otimes_\tau R(\cdot) := \int_{\Omega_0} Q_\omega(\cdot)\, P(d\omega)
$$

and show that it is a martingale solution on $[0,\infty)$ in the sense of Definition 3.1 which coincides with P up to the time τ. However, due to the fact that τ is only a stopping time with respect to the right continuous filtration $(\mathcal{B}_t)_{t\ge0}$, (3.1) does not suffice to show that $(Q_\omega)_{\omega\in\Omega_0}$ is a conditional probability distribution of $P \otimes_\tau R$ given $\mathcal{B}_\tau$. More precisely, we cannot prove that for every $A \in \mathcal{B}_\tau$ and $B \in \mathcal{B}$

$$
P \otimes_\tau R(A \cap B) = \int_A Q_\omega(B) P(d\omega).
$$

This is the reason why the corresponding results of [SV79], namely Theorem 6.1.2 and in particular Theorem 1.2.10 leading to the desired martingale property (M2), cannot be applied. It will be seen below in Proposition 3.4 that an additional condition on Q_ω, i.e., (3.5), is necessary in order to guarantee (M1), (M2) and (M3). To conclude this remark, we note that certain measurability of the mapping $\omega \mapsto Q_\omega(B)$ is only needed to define the integral in (3.6). Since we do not show that $(Q_\omega)_{\omega\in\Omega_0}$ is a conditional probability distribution, the $\mathcal{B}_\tau$-measurability from Proposition 3.2 is actually not used in the sequel.

Proposition 3.4. *Let $x_0 \in L^2_\sigma$. Let P be a martingale solution to the Navier–Stokes system (1.1) on $[0,\tau]$ starting at the time 0 from the initial condition x_0. In addition to the assumptions of Proposition 3.2, suppose that there exists a Borel set $\mathcal{N} \subset \Omega_{0,\tau}$ such that $P(\mathcal{N}) = 0$ and for every $\omega \in \mathcal{N}^c$ it holds*

$$
Q_\omega\big(\omega' \in \Omega_0; \tau(\omega') = \tau(\omega)\big) = 1. \tag{3.5}
$$

Then the probability measure $P \otimes_\tau R \in \mathscr{P}(\Omega_0)$ defined by

$$
P \otimes_\tau R(\cdot) := \int_{\Omega_0} Q_\omega(\cdot)\, P(d\omega) \tag{3.6}
$$

satisfies $P \otimes_\tau R = P$ on $\Omega_{0,\tau}$ and it is a martingale solution to the Navier–Stokes system (1.1) on $[0,\infty)$ with initial condition x_0.

Proof. First, we observe that due to (3.5) and (3.1), it holds $P \otimes_\tau R(A) = P(A)$ for every Borel set $A \subset \Omega_{0,\tau}$. It remains to verify that $P \otimes_\tau R$ satisfies (M1), (M2) and (M3) in Definition 3.1 with $s = 0$. The first condition in (M1) follows easily since by construction $P \otimes_\tau R(x(0) = x_0) = P(x(0) = x_0) = 1$.

The second one in (M1) follows from (M3) and the assumption on G. In order to show (M3), we write

$$\begin{aligned}&E^{P\otimes_\tau R}\left(\sup_{r\in[0,t]}\|x(r)\|_{L^2}^{2q}+\int_0^t\|x(r)\|_{H^\gamma}^2dr\right)\\&\le E^{P\otimes_\tau R}\left(\sup_{r\in[0,t\wedge\tau]}\|x(r)\|_{L^2}^{2q}+\int_0^{t\wedge\tau}\|x(r)\|_{H^\gamma}^2dr\right)+E^{P\otimes_\tau R}\left(\sup_{r\in[t\wedge\tau,t]}\|x(r)\|_{L^2}^{2q}+\int_{t\wedge\tau}^t\|x(r)\|_{H^\gamma}^2dr\right).\end{aligned}$$

Here, the first term can be estimated due to the bound (M3) for P, whereas the second term can be bounded based on (M3) for R. Then by (3.5)

$$\begin{aligned}&E^{P\otimes_\tau R}\left(\sup_{r\in[0,t]}\|x(r)\|_{L^2}^{2q}+\int_0^t\|x(r)\|_{H^\gamma}^2dr\right)\\&\quad\le C(\|x_0\|_{L^2}^{2q}+1)+C(E^P\|x(\tau)\|_{L^2}^{2q}+1)\le C(\|x_0\|_{L^2}^{2q}+1).\end{aligned}$$

In the last step, we used the fact that τ is bounded together with (M3) for P.

Finally, we shall verify (M2). To this end, we recall that since P is a martingale solution on $[0,\tau]$, the process $M^i_{t\wedge\tau,0}$ is a continuous square integrable $(\mathcal{B}_t)_{t\ge0}$-martingale under P with the quadratic variation process given by $\int_0^{t\wedge\tau}\|G(x(r))^*e_i\|_U^2dr$. On the other hand, since for every $\omega\in\Omega_0$, the probability measure $R_{\tau(\omega),x(\tau(\omega),\omega)}$ is a martingale solution starting at the time $\tau(\omega)$ from the initial condition $x(\tau(\omega),\omega)$, the process $M^i_{t,t\wedge\tau(\omega)}$ is a continuous square integrable $(\mathcal{B}_t)_{t\ge\tau(\omega)}$-martingale under $R_{\tau(\omega),x(\tau(\omega),\omega)}$ with the quadratic variation process given by $\int_{t\wedge\tau(\omega)}^t\|G(x(r))^*e_i\|_U^2dr$, $t\ge\tau(\omega)$. In other words, the process $M^i_{t,0}-M^i_{t\wedge\tau(\omega),0}$ is a continuous square integrable $(\mathcal{B}_t)_{t\ge0}$-martingale under $R_{\tau(\omega),x(\tau(\omega),\omega)}$ with the quadratic variation process given by $\int_{t\wedge\tau(\omega)}^t\|G(x(r))^*e_i\|_U^2dr$.

Next, we will show that $M^i_{t,0}$ is a continuous square integrable $(\mathcal{B}_t)_{t\ge0}$-martingale under $P\otimes_\tau R$ with the quadratic variation process given by $\int_0^t\|G(x(r))^*e_i\|_U^2dr$. To this end, let $s\le t$ and $A\in\mathcal{B}_s$. We first prove that

$$E^{Q_\omega}\left[M^i_{t,0}\mathbf{1}_A\right]=E^{Q_\omega}\left[M^i_{(t\wedge\tau(\omega))\vee s,0}\mathbf{1}_A\right].\tag{3.7}$$

In fact, it is enough to consider sets of the form $A=\{x(t_1)\in\Gamma_1,\dots,x(t_n)\in\Gamma_n\}$ where $n\in\mathbb{N}$, $0\le t_1<\dots<t_n\le s$, and $\Gamma_1,\dots,\Gamma_n\in\mathcal{B}(H^{-3})$. For more general $A\in\mathcal{B}_s$ we could use the approximation and the continuity of $M^i_{\cdot,0}$ to conclude. Then by the definition of Q_ω and using the martingale property with respect to $R_{\tau(\omega),x(\tau(\omega),\omega)}$ which is valid for $t\ge\tau(\omega)$, we have

$$\begin{aligned}&E^{Q_\omega}\left[(M^i_{t,0}-M^i_{(t\wedge\tau(\omega))\vee s,0})\mathbf{1}_A\right]\\&=\mathbf{1}_{[0,t_1)}(\tau(\omega))E^{R_{\tau(\omega),x(\tau(\omega),\omega)}}\left[(M^i_{t,0}-M^i_{s,0})\mathbf{1}_A\right]\\&\quad+\sum_{k=1}^{n-1}\mathbf{1}_{[t_k,t_{k+1})}(\tau(\omega))\mathbf{1}_{\Gamma_1}(x(t_1,\omega))\cdots\mathbf{1}_{\Gamma_k}(x(t_k,\omega))\\&\qquad\times E^{R_{\tau(\omega),x(\tau(\omega),\omega)}}\left((M^i_{t,0}-M^i_{s,0})\mathbf{1}_{x(t_{k+1})\in\Gamma_{k+1},\dots,x(t_n)\in\Gamma_n}\right)\\&\quad+\mathbf{1}_{[t_n,\infty)}(\tau(\omega))\mathbf{1}_{\Gamma_1}(x(t_1,\omega))\cdots\mathbf{1}_{\Gamma_n}(x(t_n,\omega))\times E^{R_{\tau(\omega),x(\tau(\omega),\omega)}}(M^i_{t,0}-M^i_{(t\wedge\tau(\omega))\vee s,0})\\&=0.\end{aligned}$$

Now (3.7) follows.

Then it follows from (3.6) and (3.4) that

$$
\begin{aligned}
E^{P\otimes_\tau R}\left[M^i_{t,0}\mathbf{1}_A\right] &= \int_{\Omega_0} E^{Q_\omega}\left[M^i_{t,0}\mathbf{1}_A\right]P(d\omega)\\
&= \int_{\Omega_0} E^{\delta_\omega\otimes_{\tau(\omega)}R_{\tau(\omega),x(\tau(\omega),\omega)}}\left[M^i_{t,0}\mathbf{1}_A\right]P(d\omega).
\end{aligned}
$$

According to (3.7) and then using the key assumption (3.5) we further deduce that

$$
\begin{aligned}
E^{P\otimes_\tau R}\left[M^i_{t,0}\mathbf{1}_A\right] &= \int_{\Omega_0} E^{\delta_\omega\otimes_{\tau(\omega)}R_{\tau(\omega),x(\tau(\omega),\omega)}}\left[M^i_{(t\wedge\tau(\omega))\vee s,0}\mathbf{1}_A\right]P(d\omega)\\
&= E^{P\otimes_\tau R}\left[M^i_{(t\wedge\tau)\vee s,0}\mathbf{1}_A\right]\\
&= E^{P\otimes_\tau R}\left[M^i_{t\wedge\tau,0}\mathbf{1}_{A\cap\{\tau>s\}}\right] + E^{P\otimes_\tau R}\left[M^i_{s,0}\mathbf{1}_{A\cap\{\tau\le s\}}\right].
\end{aligned}
$$

Finally, using the martingale property up to τ with respect to P, we get

$$
\begin{aligned}
E^{P\otimes_\tau R}\left[M^i_{t,0}\mathbf{1}_A\right] &= E^{P\otimes_\tau R}\left[M^i_{s,0}\mathbf{1}_{A\cap\{\tau>s\}}\right] + E^{P\otimes_\tau R}\left[M^i_{s,0}\mathbf{1}_{A\cap\{\tau\le s\}}\right]\\
&= E^{P\otimes_\tau R}\left[M^i_{s,0}\mathbf{1}_A\right].
\end{aligned}
$$

Hence M^i is a $(\mathcal{B}_t)_{t\ge0}$-martingale with respect to $P\otimes_\tau R$. In order to identify its quadratic variation, we proceed similarly and write

$$
\begin{aligned}
&E^{P\otimes_\tau R}\left[\left((M^i_{t,0})^2 - \int_0^t \|G(x(r))^* e_i\|_U^2 dr\right)\mathbf{1}_A\right]\\
&= \int_{\Omega_0} E^{Q_\omega}\left[\left((M^i_{t,0} - M^i_{t\wedge\tau(\omega),0})^2 - \int_{t\wedge\tau(\omega)}^t \|G(x(r))^* e_i\|_U^2 dr\right)\mathbf{1}_A\right]P(d\omega)\\
&\quad + \int_{\Omega_0} E^{Q_\omega}\left[\left((M^i_{t\wedge\tau(\omega),0})^2 - \int_0^{t\wedge\tau(\omega)} \|G(x(r))^* e_i\|_U^2\right)\mathbf{1}_A\right]P(d\omega)\\
&\quad + 2\int_{\Omega_0} E^{Q_\omega}\left[\left(M^i_{t\wedge\tau(\omega),0}(M^i_{t,0} - M^i_{t\wedge\tau(\omega),0})\right)\mathbf{1}_A\right]P(d\omega)\\
&=: J_1 + J_2 + J_3.
\end{aligned}
$$

Here, due to the martingale property with respect to R and P similar as in (3.7), we obtain

$$
J_1 = \int_{\Omega_0} E^{Q_\omega}\left[\left((M^i_{t\wedge\tau(\omega)\vee s,0} - M^i_{t\wedge\tau(\omega),0})^2 - \int_{t\wedge\tau(\omega)}^{t\wedge\tau(\omega)\vee s} \|G(x(r))^* e_i\|_U^2 dr\right)\mathbf{1}_A\right]P(d\omega),
$$

$$
J_2 = \int_{\Omega_0} E^{Q_\omega}\left[\left((M^i_{s\wedge\tau(\omega),0})^2 - \int_0^{s\wedge\tau(\omega)} \|G(x(r))^* e_i\|_U^2 dr\right)\mathbf{1}_A\right]P(d\omega),
$$

$$
J_3 = 2\int_{\Omega_0} E^{Q_\omega}\left[M^i_{t\wedge\tau(\omega),0}\left(M^i_{t\wedge\tau(\omega)\vee s,0} - M^i_{t\wedge\tau(\omega),0}\right)\mathbf{1}_A\right]P(d\omega).
$$

Combining these calculations and using (3.5) as above we finally deduce that

$$
\begin{aligned}
&E^{P\otimes_\tau R}\left[\left((M^i_{t,0})^2 - \int_0^t \|G(x(r))^* e_i\|_U^2 dr\right)\mathbf{1}_A\right]\\
&= E^{P\otimes_\tau R}\left[\left((M^i_{s\wedge\tau,0})^2 - \int_0^{s\wedge\tau} \|G(x(r))^* e_i\|_U^2 dr\right)\mathbf{1}_A\right]\\
&\quad + E^{P\otimes_\tau R}\left[\left((M^i_{s,0} - M^i_{\tau,0})^2 - \int_\tau^s \|G(x(r))^* e_i\|_U^2 dr\right)\mathbf{1}_{A\cap\{\tau\le s\}}\right]\\
&\quad + 2E^{P\otimes_\tau R}\left[M^i_{\tau,0}\left(M^i_{s,0} - M^i_{\tau,0}\right)\mathbf{1}_{A\cap\{\tau\le s\}}\right]\\
&= E^{P\otimes_\tau R}\left[\left((M^i_{s,0})^2 - \int_0^s \|G(x(r))^* e_i\|_U^2 dr\right)\mathbf{1}_A\right],
\end{aligned}
$$

which completes the proof of (M2). □

As the next step, we present an auxiliary result which allows to show that for weakly continuous stochastic processes, hitting times of open sets are stopping times with respect to the corresponding right continuous filtration. Here we want to emphasize that the filtration $(\mathcal{B}_t)_{t\geq 0}$ used below is not the augmented one since we have to consider different probabilities. As a consequence, we have to be careful about making any conclusions about stopping times.

Lemma 3.5. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis. Let H_1, H_2 be separable Hilbert spaces such that the embedding $H_1 \subset H_2$ is continuous. Suppose that there exists $\{h_k\}_{k\in\mathbb{N}} \subset H_2^* \subset H_1^*$ to have for $f \in H_1$*

$$\|f\|_{H_1} = \sup_{k\in\mathbb{N}} h_k(f).$$

Suppose that X is an $(\mathcal{F}_t)_{t\geq 0}$-adapted stochastic process with trajectories in $C([0,\infty); H_2)$. Let $L > 0$ and $\alpha \in (0,1)$. Then

$$\tau_1 := \inf\{t \geq 0; \|X(t)\|_{H_1} > L\} \qquad \text{and} \qquad \tau_2 := \inf\{t \geq 0; \|X\|_{C_t^\alpha H_1} > L\}$$

are $(\mathcal{F}_{t+})_{t\geq 0}$-stopping times where $\mathcal{F}_{t+} = \cap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon}$.

We note that in the above result, the process X a priori does not need to take values in H_1. In other words, without additional regularity of the trajectories of X, we simply have $\tau_1 = \tau_2 = 0$. However, in the application of Lemma 3.5 in the proof of Theorem 1.2 below, additional regularity will be known a.s. under a suitable probability measure.

Proof of Lemma 3.5. In the proof we use $X^\omega(s)$ to denote $X(s,\omega)$. First, we observe that the trajectories of X are lower semicontinuous in H_1 in the following sense

$$\|X(t)\|_{H_1} = \sup_{k\in\mathbb{N}} h_k(X(t)) = \sup_{k\in\mathbb{N}} \lim_{s\to t} h_k(X(s)) \leq \liminf_{s\to t} \sup_{k\in\mathbb{N}} h_k(X(s)) \leq \liminf_{s\to t} \|X(s)\|_{H_1}, \tag{3.8}$$

where $t \geq 0$. Note that since by assumption we only know that X takes values in $H_2 \supset H_1$, the H_1-norms appearing in (3.8) may be infinite. Next, we have for $t > 0$

$$\{\tau_1 \geq t\} = \cap_{s\in[0,t]} \{\|X(s)\|_{H_1} \leq L\} = \cap_{s\in[0,t]\cap\mathbb{Q}} \{\|X(s)\|_{H_1} \leq L\} \in \mathcal{F}_t.$$

Indeed, to show the first equality, we observe that the right hand side is a subset of the left one. For the converse inclusion, we know that $\{\tau_1 > t\}$ is a subset of the right hand side. Now, we consider $\omega \in \{\tau_1 = t\}$. In this case, $\|X^\omega(s)\|_{H_1} \leq L$ for every $s \in [0,t)$. Thus, there exists a sequence $t_k \uparrow t$ such that $\|X^\omega(t_k)\|_{H_1} \leq L$ and by the lower semicontinuity of X it follows that $\|X^\omega(t)\|_{H_1} \leq L$. The second equality is also a consequence of lower semicontinuity. Indeed, if ω belongs to the right hand side, then for $s \in [0,t]$, $s \notin \mathbb{Q}$, there is a sequence $(s_k)_{k\in\mathbb{N}} \subset [0,t]\cap\mathbb{Q}$, $s_k \to s$, such that $\|X^\omega(s_k)\|_{H_1} \leq L$. Hence $\|X^\omega(s)\|_{H_1} \leq L$ and ω belongs to the let hand side as well. Therefore, we deduce that

$$\{\tau_1 \leq t\} = \cap_{\varepsilon>0}\{\tau < t+\varepsilon\} \in \mathcal{F}_{t+},$$

which proves that τ_1 is an $(\mathcal{F}_{t+})_{t\geq 0}$-stopping time.

We proceed similarly for τ_2. By the same argument as in (3.8) we obtain that also the time increments of X are lower semicontinuous in H_1. More precisely, for $t_1, t_2 \geq 0$ we have

$$\|X(t_1) - X(t_2)\|_{H_1} \leq \liminf_{s_1\to t_1, s_2\to t_2} \|X(s_1) - X(s_2)\|_{H_1}$$

and as a consequence if $t_1 \neq t_2$ then

$$\frac{\|X(t_1) - X(t_2)\|_{H_1}}{|t_1 - t_2|^\alpha} \leq \liminf_{\substack{s_1\to t_1, s_2\to t_2\\ s_1\neq s_2}} \frac{\|X(s_1) - X(s_2)\|_{H_1}}{|s_1 - s_2|^\alpha}.$$

This implies for $t>0$ that

$$\{\tau_2 \geq t\} = \left\{\|X\|_{C_t^\alpha H_1} \leq L\right\} = \cap_{s_1\neq s_2\in[0,t]\cap\mathbb{Q}} \left\{\frac{\|X(s_1)-X(s_2)\|_{H_1}}{|s_1-s_2|^\alpha} \leq L\right\} \in \mathcal{F}_t, \tag{3.9}$$

Indeed, for the first equality, we obtain immediately that the right hand side is a subset of the left one, because the process $t\mapsto \|X\|_{C_t^\alpha H_1}$ is nondecreasing. For the converse inclusion, we know that $\{\tau_2>t\}$ is a subset of the right hand side. Let $\omega\in\{\tau_2=t\}$. Then there is a sequence $t_k\uparrow t$ such that $\|X^\omega\|_{C_{t_k}^\alpha H} \leq L$ and we have

$$\begin{aligned}\sup_{s_1\neq s_2\in[0,t]} \frac{\|X^\omega(s_1)-X^\omega(s_2)\|_{H_1}}{|s_1-s_2|^\alpha} &\leq \sup_{s_1\neq s_2\in[0,t]} \liminf_{k\to\infty} \frac{\|X^\omega(s_1\wedge t_k)-X^\omega(s_2\wedge t_k)\|_{H_1}}{|s_1\wedge t_k-s_2\wedge t_k|^\alpha}\\ &\leq \sup_{k\in\mathbb{N}} \sup_{s_1\neq s_2\in[0,t_k]} \frac{\|X^\omega(s_1)-X^\omega(s_2)\|_{H_1}}{|s_1-s_2|^\alpha} \leq L.\end{aligned}$$

We deduce that $\|X^\omega\|_{C_t^\alpha H_1} \leq L$ hence ω also belongs to the set on the right hand side of the first equality in (3.9). The second equality in (3.9) follows by a similar argument. Therefore, we conclude that τ_2 is an $(\mathcal{F}_{t+})_{t\geq 0}$-stopping time. □

3.3. **Application to solutions obtained through Theorem 1.1.** As the first step, we decompose the Navier–Stokes system (1.4) into two parts, one is linear and contains the stochastic integral, whereas the second one is nonlinear but random PDE. More precisely, we consider

$$\begin{aligned} dz - \Delta z + \nabla P_1 dt &= dB,\\ \mathrm{div} z &= 0,\\ z(0) &= 0,\end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \partial_t v - \Delta v + \mathrm{div}((v+z)\otimes(v+z)) + \nabla P_2 &= 0,\\ \mathrm{div} v &= 0.\end{aligned} \tag{3.11}$$

This allows to separate the difficulties coming from the stochastic perturbation from those originating in the nonlinearity.

Now, we fix a GG^*-Wiener process B defined on a probability space $(\Omega,\mathcal{F},\mathbf{P})$ and we denote by $(\mathcal{F}_t)_{t\geq0}$ its normal filtration, i.e. the canonical filtration of B augmented by all the $\mathbf{P}$-negligible sets. This filtration is right continuous. We recall that using the factorization method it is standard to derive regularity of the stochastic convolution z which solves the linear equation (3.10) on $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq0},\mathbf{P})$. In particular, the following result follows from [De13, Proposition IV.1.2] together with the Kolmogorov continuity criterion.

Proposition 3.6. *Suppose that* $\mathrm{Tr}((-\Delta)^{\frac{3}{2}+2\sigma}GG^*)<\infty$ *for some* $\sigma>0$. *Then for all* $\delta\in(0,1)$ *and* $T>0$

$$E^{\mathbf{P}}\left[\|z\|_{C_T H^{\frac{5+\sigma}{2}}} + \|z\|_{C_T^{\frac12-\delta} H^{\frac{3+\sigma}{2}}}\right] < \infty.$$

As the next step, for every $\omega\in\Omega_0$ we define a process $M_{t,0}^\omega$ similarly to Definition 3.1, that is,

$$M_{t,0}^\omega := \omega(t)-\omega(0) + \int_0^t [\mathbb{P}\mathrm{div}(\omega(r)\otimes\omega(r)) - \Delta\omega(r)]dr \tag{3.12}$$

and for every $\omega\in\Omega_0$ we let

$$Z^\omega(t) := M_{t,0}^\omega + \int_0^t \mathbb{P}\Delta e^{(t-r)\Delta} M_{r,0}^\omega dr. \tag{3.13}$$

The idea behind these definitions is as follows. The process M is defined in terms of the canonical process x and hence its definition makes sense for every $\omega\in\Omega_0$, i.e. without the reference to any probability measure. Consequently, the same applies to Z. In addition, if P is a martingale solution to the Navier–Stokes system (1.4), the process M is a GG^*-Wiener process under P. Hence we may apply an integration by parts formula to show that, the process Z solves (3.10) with B replaced by M. In other words, under P, Z is almost surely equal to a stochastic convolution, i.e., we have

$$Z(t)=\int_0^t \mathbb{P}e^{(t-r)\Delta}dM_{r,0}\qquad P\text{-a.s.}$$

In addition, by definition of Z and M together with the regularity of trajectories in Ω_0, it follows that for every $\omega\in\Omega_0$, $Z^\omega\in C([0,\infty),H^{-3})$. For $n\in\mathbb{N}, L>0$ and for $\delta\in(0,1/12)$ to be determined below we define

$$\tau_L^n(\omega)=\inf\left\{t\geq 0, \|Z^\omega(t)\|_{H^{\frac{5+\sigma}{2}}}>\frac{(L-\frac{1}{n})^{1/4}}{C_S}\right\}\bigwedge\inf\left\{t>0, \|Z^\omega\|_{C_t^{\frac12-2\delta}H^{\frac{3+\sigma}{2}}}>\frac{(L-\frac{1}{n})^{1/2}}{C_S}\right\}\bigwedge L,$$

where C_S is the Sobolev constant for $\|f\|_{L^\infty}\leq C_S\|f\|_{H^{\frac{3+\sigma}{2}}}$ with $\sigma>0$. We observe that the sequence $(\tau_L^n)_{n\in\mathbb{N}}$ is nondecreasing and define

$$\tau_L:=\lim_{n\to\infty}\tau_L^n. \tag{3.14}$$

Note that without additional regularity of the trajectory ω, it holds true that $\tau_L^n(\omega)=0$. However, under P we may use the regularity assumption on G to deduce that $Z\in CH^{\frac{5+\sigma}{2}}\cap C^{1/2-\delta}_{\mathrm{loc}}H^{\frac{3+\sigma}{2}}$ P-a.s. By Lemma 3.5 we obtain that τ_L^n is $(\mathcal{B}_t)_{t\geq0}$-stopping time and consequently also τ_L is a $(\mathcal{B}_t)_{t\geq0}$-stopping time as an increasing limit of stopping times.

As the next step, we apply Theorem 1.1 on the stochastic basis $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq0},\mathbf{P})$. We note that the stopping time $\mathfrak{t}$ from the statement of Theorem 1.1 is given by T_L for a sufficiently large $L>1$, defined in (4.2) below. We recall that u is adapted with respect to $(\mathcal{F}_t)_{t\geq0}$ which is an essential property employed in the sequel. We denote by P the law of u and prove the following result.

Proposition 3.7. *The probability measure P is a martingale solution to the Navier–Stokes system (1.4) on $[0,\tau_L]$ in the sense of Definition 3.2, where τ_L was defined in (3.14).*

Proof. Recall that the stopping time T_L was defined in (4.2) in terms of the process z, the solution to the linear equation (3.10). Theorem 1.1 yields the existence of a solution u to the Navier–Stokes system (1.4) on $[0,T_L]$ such that $u(\cdot\wedge T_L)\in\Omega_0$ $\mathbf{P}$-a.s. We will now prove that

$$\tau_L(u)=T_L\quad \mathbf{P}\text{-a.s.} \tag{3.15}$$

To this end, we observe that due to the definition of M in (3.12) and Z in (3.13) together with the fact that u solves the Navier–Stokes system (1.4) on $[0,T_L]$, we have

$$Z^u(t)=z(t)\qquad \text{for } t\in[0,T_L]\ \mathbf{P}\text{-a.s.} \tag{3.16}$$

Since $z\in CH^{\frac{5+\sigma}{2}}\cap C^{\frac12-\delta}_{\mathrm{loc}}H^{\frac{3+\sigma}{2}}$ $\mathbf{P}$-a.s. according to Proposition 3.6, the trajectories of the processes

$$t\mapsto\|z(t)\|_{H^{\frac{5+\sigma}{2}}}\quad\text{ and }\quad t\mapsto\|z\|_{C_t^{\frac12-2\delta}H^{\frac{3+\sigma}{2}}}$$

are $\mathbf{P}$-a.s. continuous. It follows from the definition of T_L that one of the following three statements holds $\mathbf{P}$-a.s.:

$$\text{either}\ \ T_L=L\ \ \text{or}\ \ \|z(T_L)\|_{H^{\frac{5+\sigma}{2}}}\geq L^{1/4}/C_S\ \ \text{or}\ \ \|z\|_{C_{T_L}^{\frac12-2\delta}H^{\frac{3+\sigma}{2}}}\geq L^{1/2}/C_S.$$

Therefore, as a consequence of (3.16), we deduce that $\tau_L(u) \le T_L$ $\mathbf{P}$-a.s. Suppose now that $\tau_L(u) < T_L$ holds true on a set of positive probability $\mathbf{P}$. Then it holds on this set that

$$\|z(\tau_L(u))\|_{H^{\frac{5+\sigma}{2}}} = \|Z^u(\tau_L(u))\|_{H^{\frac{5+\sigma}{2}}} \ge L^{1/4}/C_S \quad \text{or} \quad \|Z^u\|_{C^{\frac12-2\delta}_{\tau_L(u)} H^{\frac{3+\sigma}{2}}} = \|z\|_{C^{\frac12-2\delta}_{\tau_L(u)} H^{\frac{3+\sigma}{2}}} \ge L^{1/2}/C_S,$$

which however contradicts the definition of T_L. Hence we have proved (3.15).

Recall that τ_L is a $(\mathcal{B}_t)_{t\ge0}$-stopping time. We intend to show that P is a martingale solution to the Navier–Stokes system (1.4) on $[0,\tau_L]$ in the sense of Definition 3.2. First, we observe that it can be seen from the construction in Theorem 1.1 that the initial value $u(0) = v(0) + z(0) = v(0)$ is indeed deterministic. Hence the condition (M1) follows. However, we note that the initial value $v(0)$ cannot be prescribed in advance. In other words, Theorem 1.1 does not yield a solution to the Cauchy problem, it only provides the existence of an initial condition for which a solution violating the energy inequality exists. For an appropriate choice of the constant $C_{t,q}$ in Definition 3.2, which has to depend on the constant C_L in (1.5) in Theorem 1.1, the condition (M3) also follows.

Let us now verify (M2). To this end, let $s \le t$ and let g be a bounded and real valued $\mathcal{B}_s$-measurable and continuous function on Ω_0. Since $u(\cdot\wedge T_L)$ is an $(\mathcal{F}_t)_{t\ge0}$-adapted process and (3.15) holds, we deduce that $u(\cdot\wedge\tau_L(u))$ is also $(\mathcal{F}_t)_{t\ge0}$-adapted. Consequently, the composition $g(u(\cdot\wedge\tau_L(u)))$ is $\mathcal{F}_s$-measurable. On the other hand, we know that under $\mathbf{P}$, $M^{u,i}_{t\wedge\tau_L(u),0} = \langle B_{t\wedge\tau_L(u)}, e_i\rangle$ is an $(\mathcal{F}_t)_{t\ge0}$-martingale. Its quadratic variation process is given by $\|Ge_i\|^2_{L^2}(t\wedge\tau_L(u))$. Therefore, we have

$$E^P[M^i_{t\wedge\tau_L,0}g] = E^{\mathbf{P}}[M^{u,i}_{t\wedge\tau_L(u),0}g(u)] = E^{\mathbf{P}}[M^{u,i}_{s\wedge\tau_L(u),0}g(u)] = E^P[M^i_{s\wedge\tau_L,0}g]$$

and by similar arguments we also obtain that

$$E^P\left[\left((M^i_{t\wedge\tau_L,0})^2 - (t\wedge\tau_L)\|Ge_i\|^2_{L^2}\right)g\right] = E^P\left[\left((M^i_{s\wedge\tau_L,0})^2 - (s\wedge\tau_L)\|Ge_i\|^2_{L^2}\right)g\right].$$

Accordingly, the process $M^i_{t\wedge\tau_L,0}$ is a continuous square integrable $(\mathcal{B}_t)_{t\ge0}$-martingale under P with the quadratic variation process given by $\|Ge_i\|^2_{L^2}(t\wedge\tau_L)$ and (M2) in Definition 3.2 follows. □

At this point, we are already able to deduce that martingale solutions on $[0,\tau_L]$ in the sense of Definition 3.2 are not unique. However, we aim at a stronger result, namely that globally defined martingale solutions on $[0,\infty)$ in the sense of Definition 3.1 are not unique. Moreover, we will prove that for an arbitrary time interval $[0,T]$, the martingale solutions on $[0,T]$ are not unique. To this end, we will extend P to a martingale solution on $[0,\infty)$ through the procedure developed in Section 3.2. More precisely, as an immediate corollary of Proposition 3.7 and the fact that τ_L is a $(\mathcal{B}_t)_{t\ge0}$-stopping time, we may apply Proposition 3.2. In particular, we construct Q_ω for all $\omega\in\Omega_0$. In view of Proposition 3.4, (M1-M3) follows once we verify the condition (3.5) for Q_ω. This will be achieved in the following result.

Proposition 3.8. *The probability measure $P\otimes_{\tau_L} R$ is a martingale solution to the Navier–Stokes system* (1.4) *on $[0,\infty)$ in the sense of Definition 3.1.*

Proof. In light of Proposition 3.2 and Proposition 3.4, it only remains to establish (3.5). Due to (3.15) and (3.16), we know that

$$\begin{aligned} P\left(\omega : Z^\omega(\cdot\wedge\tau_L(\omega)) \in CH^{\frac{5+\sigma}{2}} \cap C^{\frac12-\delta}_{\mathrm{loc}} H^{\frac{3+\sigma}{2}}\right) &= \mathbf{P}\left(Z^u(\cdot\wedge\tau_L(u)) \in CH^{\frac{5+\sigma}{2}} \cap C^{\frac12-\delta}_{\mathrm{loc}} H^{\frac{3+\sigma}{2}}\right) \\ &= \mathbf{P}\left(z(\cdot\wedge T_L) \in CH^{\frac{5+\sigma}{2}} \cap C^{\frac12-\delta}_{\mathrm{loc}} H^{\frac{3+\sigma}{2}}\right) = 1. \end{aligned}$$

This means that there exists a P-measurable set $\mathcal{N}\subset\Omega_0$ such that $P(\mathcal{N}) = 0$ and for $\omega\in\mathcal{N}^c$

$$Z^\omega_{\cdot\wedge\tau_L(\omega)} \in CH^{\frac{5+\sigma}{2}} \cap C^{\frac12-\delta}_{\mathrm{loc}} H^{\frac{3+\sigma}{2}}. \tag{3.17}$$

On the other hand, it follows from (3.13) that for every $\omega' \in \Omega_0$

$$\begin{aligned} Z^{\omega'}(t) - Z^{\omega'}(t\wedge\tau_L(\omega)) &= M^{\omega'}_{t,0} - e^{(t-t\wedge\tau_L(\omega))\Delta} M^{\omega'}_{t\wedge\tau_L(\omega),0} + \int_{t\wedge\tau_L(\omega)}^t \mathbb{P}\Delta e^{(t-s)\Delta} M^{\omega'}_{s,0} ds \\ &\quad + (e^{(t-t\wedge\tau_L(\omega))\Delta} - I)\left[M^{\omega'}_{t\wedge\tau_L(\omega),0} + \int_0^{t\wedge\tau_L(\omega)} \mathbb{P}\Delta e^{(t\wedge\tau_L(\omega)-s)\Delta} M^{\omega'}_{s,0} ds\right] \\ &= \mathbb{Z}^{\omega'}_{\tau_L(\omega)}(t) + (e^{(t-t\wedge\tau_L(\omega))\Delta} - I) Z^{\omega'}(t\wedge\tau_L(\omega)), \end{aligned}$$

with

$$\begin{aligned} \mathbb{Z}^{\omega'}_{\tau_L(\omega)}(t) &= M^{\omega'}_{t,0} - e^{(t-t\wedge\tau_L(\omega))\Delta} M^{\omega'}_{t\wedge\tau_L(\omega),0} + \int_{t\wedge\tau_L(\omega)}^t \mathbb{P}\Delta e^{(t-s)\Delta} M^{\omega'}_{s,0} ds \\ &= M^{\omega'}_{t,0} - M^{\omega'}_{t\wedge\tau_L(\omega),0} + \int_{t\wedge\tau_L(\omega)}^t \mathbb{P}\Delta e^{(t-s)\Delta}(M^{\omega'}_{s,0} - M^{\omega'}_{s\wedge\tau_L(\omega),0}) ds. \end{aligned}$$

Since $M_{\cdot,0} - M_{\cdot\wedge\tau_L(\omega),0}$ is $\mathcal{B}^{\tau_L(\omega)}$-measurable, we know that $\mathbb{Z}^{\omega'}_{\tau_L(\omega)}$ is $\mathcal{B}^{\tau_L(\omega)}$-measurable.

Using (3.1) and (3.2) it holds that for all $\omega \in \Omega_0$

$$\begin{aligned} & Q_\omega\left(\omega' \in \Omega_0; Z^{\omega'}_\cdot \in CH^{\frac{5+\sigma}{2}} \cap C^{1/2-\delta}_{\mathrm{loc}} H^{\frac{3+\sigma}{2}}\right) \\ &= Q_\omega\left(\omega' \in \Omega_0; Z^{\omega'}_{\cdot\wedge\tau_L(\omega)} \in CH^{\frac{5+\sigma}{2}} \cap C^{1/2-\delta}_{\mathrm{loc}} H^{\frac{3+\sigma}{2}}, \mathbb{Z}^{\omega'}_{\tau_L(\omega)} \in CH^{\frac{5+\sigma}{2}} \cap C^{1/2-\delta}_{\mathrm{loc}} H^{\frac{3+\sigma}{2}}\right) \\ &= \delta_\omega\left(\omega' \in \Omega_0; Z^{\omega'}_{\cdot\wedge\tau_L(\omega)} \in CH^{\frac{5+\sigma}{2}} \cap C^{1/2-\delta}_{\mathrm{loc}} H^{\frac{3+\sigma}{2}}\right) \\ &\quad \times R_{\tau_L(\omega),x(\tau_L(\omega),\omega)}\left(\omega' \in \Omega_0; \mathbb{Z}^{\omega'}_{\tau_L(\omega)} \in CH^{\frac{5+\sigma}{2}} \cap C^{1/2-\delta}_{\mathrm{loc}} H^{\frac{3+\sigma}{2}}\right). \end{aligned}$$

Here the first factor on the right hand side equals to 1 for all $\omega \in \mathcal{N}^c$ due to (3.17). Since $R_{\tau_L(\omega),x(\tau_L(\omega),\omega)}$ is a martingale solution to the Navier–Stokes system (1.4) starting at the deterministic time $\tau_L(\omega)$ from the deterministic initial condition $x(\tau_L(\omega),\omega)$, the process $\omega' \mapsto M^{\omega'}_{\cdot,0} - M^{\omega'}_{\cdot\wedge\tau_L(\omega),0}$ is a GG^*-Wiener process starting from $\tau_L(\omega)$ with respect to $(\mathcal{B}_t)_{t\geq 0}$ under the measure $R_{\tau_L(\omega),x(\tau_L(\omega),\omega)}$. Due to the regularity of its covariance we deduce that also the second factor equals to 1. Indeed, we have for $R_{\tau_L(\omega),x(\tau_L(\omega),\omega)}$-a.e. ω' that

$$\mathbb{Z}^{\omega'}_{\tau_L(\omega)}(t) = \int_0^t \mathbb{P} e^{(t-s)\Delta} d(M^{\omega'}_{s,0} - M^{\omega'}_{s\wedge\tau_L(\omega),0})$$

and the regularity of this stochastic convolution follows again from Proposition 3.6. In particular, it holds for $R_{\tau_L(\omega),x(\tau_L(\omega),\omega)}$-a.e. ω' that

$$\mathbb{Z}^{\omega'}_{\tau_L(\omega)} \in CH^{\frac{5+\sigma}{2}} \cap C^{1/2-\delta}_{\mathrm{loc}} H^{\frac{3+\sigma}{2}}.$$

To summarize, we have proved that for all $\omega \in \mathcal{N}^c$

$$Q_\omega\left(\omega' \in \Omega_0; Z^{\omega'}_\cdot \in CH^{\frac{5+\sigma}{2}} \cap C^{1/2-\delta}_{\mathrm{loc}} H^{\frac{3+\sigma}{2}}\right) = 1.$$

As a consequence, for all $\omega \in \mathcal{N}^c$ there exists a measurable set N_ω such that $Q_\omega(N_\omega) = 0$ and for all $\omega' \in N^c_\omega$ the trajectory $t \mapsto Z^{\omega'}(t)$ belongs to $CH^{\frac{5+\sigma}{2}} \cap C^{1/2-\delta}_{\mathrm{loc}} H^{\frac{3+\sigma}{2}}$. Therefore, by (3.14) we obtain that $\tau_L(\omega') = \bar{\tau}_L(\omega')$ for all $\omega' \in N^c_\omega$ where

$$\bar{\tau}_L(\omega') := \inf\left\{t \geq 0, \|Z^{\omega'}(t)\|_{H^{\frac{5+\sigma}{2}}} \geq L^{1/4}/C_S\right\} \bigwedge \inf\left\{t \geq 0, \|Z^{\omega'}\|_{C_t^{1/2-2\delta} H^{\frac{3+\sigma}{2}}} \geq L^{1/2}/C_S\right\} \bigwedge L.$$

This implies that for $t < L$

$$\begin{aligned}\big\{\omega' \in N_\omega^c, \tau_L(\omega') \le t\big\} &= \left\{\omega' \in N_\omega^c, \sup_{s\in\mathbb{Q}, s\le t} \|Z^{\omega'}(s)\|_{H^{\frac{5+\sigma}{2}}} \ge L^{1/4}/C_S\right\} \\ &\qquad \bigcup \left\{\omega' \in N_\omega^c, \sup_{s_1\neq s_2\in\mathbb{Q}\cap[0,t]} \frac{\|Z^{\omega'}(s_1) - Z^{\omega'}(s_2)\|_{H^{\frac{3+\sigma}{2}}}}{|s_1 - s_2|^{\frac{1}{2}-2\delta}} \ge L^{1/2}/C_S\right\}.\end{aligned} \tag{3.18}$$

Finally, we deduce that for all $\omega \in \mathcal{N}^c$

$$\begin{aligned}Q_\omega\big(\omega' \in \Omega_0; \tau_L(\omega') = \tau_L(\omega)\big) &= Q_\omega\big(\omega' \in N_\omega^c; \tau_L(\omega') = \tau_L(\omega)\big) \\ &= Q_\omega\big(\omega' \in N_\omega^c; \omega'(s) = \omega(s), 0 \le s \le \tau_L(\omega), \tau_L(\omega') = \tau_L(\omega)\big) = 1,\end{aligned} \tag{3.19}$$

where we used (3.1) and we used (3.18) implies the fact that $\{\omega' \in N_\omega^c; \tau_L(\omega') = \tau_L(\omega)\} \in N_\omega^c \cap \mathcal{B}^0_{\tau_L(\omega)}$. This verifies the condition (3.5) in Proposition 3.4 and as a consequence $P \otimes_{\tau_L} R$ is a martingale solution to the Navier–Stokes system (1.4) on $[0,\infty)$ in the sense of Definition 3.1. □

Remark 3.9. The property (3.19) is essential for showing that the concatenated probability measure satisfies (M1-M3). This is the reason why we had to introduce $\bar{\tau}_L$ and make use of the continuity of Z under the law of a martingale solution, which is different from the original regularity of Z which follows merely from its definition (3.13) together with the regularity of trajectories in Ω_0. Without the improved regularity, we could only prove that τ_L is a stopping time with respect to the right continuous filtration $(\mathcal{B}_t)_{t\ge0}$ and the dependence on the right limit does not allow to establish (3.19).

Finally, we have all in hand to conclude the proof of our main result, Theorem 1.2.

Proof of Theorem 1.2. Let $T > 0$ be arbitrary, let $\kappa = 1/2$ and $K = 2$. Based on Theorem 1.1 and Proposition 3.8 there exists $L > 1$ and a measure $P \otimes_{\tau_L} R$ which is a martingale solution to the Navier–Stokes system (1.4) on $[0,\infty)$ and it coincides on the random interval $[0,\tau_L]$ with the law of the solution constructed through Theorem 1.1. The martingale solution $P\otimes_{\tau_L} R$ starts from certain deterministic initial value $x_0 = v(0) \in L^2_\sigma$ dictated by the construction in Theorem 1.1. The key result is the failure of the energy inequality at time T formulated in (1.6) on the set $\{T_L \ge T\} \subset \Omega$. In view of (3.6), (3.19) and (3.15), we obtain

$$\begin{aligned}P \otimes_{\tau_L} R\big(\tau_L \ge T\big) &= \int_{\Omega_0} Q_\omega\big(\tau_L \ge T\big) P(d\omega) = \int_{\Omega_0} Q_\omega\big(\tau_L(\omega) \ge T\big) P(d\omega) \\ &= P\big(\tau_L \ge T\big) = \mathbf{P}\big(\tau_L(u) \ge T\big) = \mathbf{P}\big(T_L \ge T\big) > 1/2,\end{aligned}$$

which by (1.6) and the choice of $K = 2$ in particular implies

$$\begin{aligned}E^{P\otimes_{\tau_L} R}\big[\|x(T)\|_{L^2}^2\big] &= E^{P\otimes_{\tau_L} R}\big[\mathbf{1}_{\{\tau_L\ge T\}}\|x(T)\|_{L^2}^2\big] + E^{P\otimes_{\tau_L} R}\big[\mathbf{1}_{\{\tau_L< T\}}\|x(T)\|_{L^2}^2\big] \\ &> 2\left(\|x_0\|_{L^2}^2 + T\,\mathrm{Tr}(GG^*)\right).\end{aligned}$$

On the other hand, by a classical compactness argument based on a Galerkin approximation we may construct another martingale solution $\tilde{P}$ which starts from the same deterministic initial condition x_0 and which satisfies the energy inequality

$$E^{\tilde{P}}\big[\|x(T)\|_{L^2}^2\big] \le \|x_0\|_{L^2}^2 + T\,\mathrm{Tr}(GG^*).$$

Therefore, we can finally conclude that the two martingale solutions $P \otimes_{\tau_L} R$ and $\tilde{P}$ are distinct and non-uniqueness in law holds for the Navier–Stokes system (1.4). □

4. Proof of Theorem 1.1

In this section we fix a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and let B be a GG^*-Wiener process on $(\Omega, \mathcal{F}, \mathbf{P})$. We let $(\mathcal{F}_t)_{t\geq 0}$ be the normal filtration generated by B, that is, the canonical right continuous filtration augmented by all the $\mathbf{P}$-negligible sets. In order to verify that the solution constructed in this section is a martingale solution before a suitable stopping time, it is essential that the solution is adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$, which corresponds to a probabilistically strong solution. In the following, we construct a probabilistically strong solution before a stopping time. Furthermore, the solutions do not satisfy the energy inequality.

We intend to develop an iteration procedure leading to the proof of Theorem 1.1. More precisely, we apply the convex integration method to the nonlinear equation (3.11). The iteration is indexed by a parameter $q \in \mathbb{N}_0$. At each step q, a pair $(v_q, \mathring{R}_q)$ is constructed solving the following system

$$\begin{aligned} \partial_t v_q - \Delta v_q + \mathrm{div}((v_q + z) \otimes (v_q + z)) + \nabla p_q &= \mathrm{div}\mathring{R}_q, \\ \mathrm{div} v_q &= 0. \end{aligned} \tag{4.1}$$

We consider an increasing sequence $\{\lambda_q\}_{q\in\mathbb{N}} \subset \mathbb{N}$ which diverges to ∞, and a sequence $\{\delta_q\}_{q\in\mathbb{N}} \subset (0,1)$ which is decreasing to 0. We choose $a \in \mathbb{N}, b \in \mathbb{N}, \beta \in (0,1)$ and let

$$\lambda_q = a^{(b^q)}, \quad \delta_q = \lambda_q^{-2\beta},$$

where β will be chosen sufficiently small and a as well as b will be chosen sufficiently large. By the Sobolev embedding we know $\|f\|_{L^\infty} \leq C_S \|f\|_{H^{\frac{3+\sigma}{2}}}$ for $\sigma > 0$. Define for $L > 1$ and $0 < \delta < 1/12$

$$T_L := \inf\{t \geq 0, \|z(t)\|_{H^{\frac{5+\sigma}{2}}} \geq L^{1/4}/C_S\} \wedge \inf\{t \geq 0, \|z\|_{C_t^{1/2-2\delta} H^{\frac{3+\sigma}{2}}} \geq L^{1/2}/C_S\} \wedge L. \tag{4.2}$$

According to Proposition 3.6, the stopping time T_L is $\mathbf{P}$-a.s. strictly positive and it holds that $T_L \uparrow \infty$ as $L \to \infty$ $\mathbf{P}$-a.s. Moreover, for $t \in [0, T_L]$

$$\|z(t)\|_{L^\infty} \leq L^{1/4}, \quad \|\nabla z(t)\|_{L^\infty} \leq L^{1/4}, \quad \|z\|_{C_t^{\frac{1}{2}-2\delta} L^\infty} \leq L^{1/2}. \tag{4.3}$$

Let $M_0(t) = L^4 e^{4Lt}$. By induction on q we assume the following bounds for the iterations $(v_q, \mathring{R}_q)$: if $t \in [0, T_L]$ then

$$\begin{aligned} \|v_q\|_{C_t L^2} &\leq M_0(t)^{1/2} (1 + \sum_{1\leq r\leq q} \delta_r^{1/2}) \leq 2M_0(t)^{1/2}, \\ \|v_q\|_{C^1_{t,x}} &\leq M_0(t)^{1/2} \lambda_q^4, \\ \|\mathring{R}_q\|_{C_t L^1} &\leq M_0(t) c_R \delta_{q+1}. \end{aligned} \tag{4.4}$$

Here we defined $\sum_{1\leq r\leq 0} := 0$, $c_R > 0$ is a sufficiently small universal constant given in (4.28) and (4.37) below. In addition, we used $\sum_{r\geq 1} \delta_r^{1/2} \leq \sum_{r\geq 1} a^{-rb\beta} = \frac{a^{-\beta b}}{1-a^{-\beta b}} < 1/2$ which boils down to the requirement

$$a^{\beta b} > 3, \tag{4.5}$$

which we assume from now on. The iteration will be initiated through the following result which also establishes compatibility conditions between the parameters L, a, β, b essential for the sequel.

Lemma 4.1. *For $L > 1$ define*

$$v_0(t,x) = \frac{L^2 e^{2Lt}}{(2\pi)^{\frac{3}{2}}} (\sin(x_3), 0, 0).$$

Then the associated Reynolds stress is given by[1]

$$\mathring{R}_0(t,x) = \frac{(2L+1)L^2 e^{2Lt}}{(2\pi)^{3/2}} \begin{pmatrix} 0 & 0 & -\cos(x_3) \\ 0 & 0 & 0 \\ -\cos(x_3) & 0 & 0 \end{pmatrix} + v_0 \mathring{\otimes} z + z \mathring{\otimes} v_0 + z \mathring{\otimes} z. \tag{4.6}$$

Moreover, all the estimates in (4.4) *on the level* $q = 0$ *for* $(v_0, \mathring{R}_0)$ *as well as* (4.5) *are valid provided*

$$45 \cdot (2\pi)^{3/2} < 5 \cdot (2\pi)^{3/2} a^{2\beta b} \le c_R L \le c_R \left(\frac{(2\pi)^{3/2} a^4}{2} - 1 \right). \tag{4.7}$$

In particular, we require

$$c_R L > 45 \cdot (2\pi)^{3/2}. \tag{4.8}$$

Furthermore, the initial values $v_0(0,x)$ *and* $\mathring{R}_0(0,x)$ *are deterministic.*

Proof. The first bound in (4.4) follows immediately since

$$\|v_0(t)\|_{L^2} = \frac{L^2 e^{2Lt}}{\sqrt{2}} \le M_0(t)^{1/2}.$$

For the second bound, we have

$$\|v_0\|_{C^1_{t,x}} \le M_0(t)^{1/2} \frac{2(1+L)}{(2\pi)^{3/2}} \le M_0(t)^{1/2} \lambda_0^4 = M_0(t)^{1/2} a^4$$

provided

$$\frac{2(1+L)}{(2\pi)^{3/2}} \le a^4. \tag{4.9}$$

A direct computation implies that the corresponding Reynolds stress is given by (4.6) and we obtain

$$\|\mathring{R}_0(t)\|_{L^1} \le (2\pi)^{3/2} M_0(t)^{1/2} 2(2L+1) + 2(2\pi)^3 M_0(t)^{1/2} L^{1/4} + (2\pi)^3 L^{1/2}.$$

Therefore, the desired third bound in (4.4) holds provided

$$\|\mathring{R}_0(t)\|_{L^1} \le 5 \cdot (2\pi)^{3/2} M_0(t)/L \le M_0(t) c_R \delta_1 = M_0(t) c_R a^{-2\beta b},$$

which requires $5 \cdot (2\pi)^{3/2} L^{-1} \le c_R a^{-2\beta b}$. Here we used (4.8) in the first inequality. Combining this condition with (4.9), we obtain the requirement

$$5 \cdot (2\pi)^{3/2} a^{2\beta b} \le c_R L \le c_R \left(\frac{(2\pi)^{3/2} a^4}{2} - 1 \right). \tag{4.10}$$

In particular, we require that

$$c_R L > 5 \cdot (2\pi)^{3/2}, \tag{4.11}$$

otherwise the left inequality in (4.10) cannot be fulfilled. Under these conditions, all the estimates in (4.4) are valid on the level $q = 0$. Taking into account (4.5), the conditions (4.10) and (4.11) are strengthened to (4.7) and (4.8) from the statement of the lemma and the proof is complete. □

The key result of this section which is used to prove Theorem 1.1 is the following.

[1]We denote by $\mathring{\otimes}$ the trace-free part of the tensor product.

Proposition 4.2. (Main iteration) *Let $L>1$ satisfying (4.8) be given and let $(v_q,\mathring{R}_q)$ be an $(\mathcal{F}_t)_{t\geq 0}$-adapted solution to (4.1) satisfying (4.4). Then there exists a choice of parameters a,b,β such that (4.7) is fulfilled and there exist $(\mathcal{F}_t)_{t\geq 0}$-adapted processes $(v_{q+1},\mathring{R}_{q+1})$ which solve (4.1), obey (4.4) at level $q+1$ and for $t\in[0,T_L]$ we have*

$$\|v_{q+1}(t)-v_q(t)\|_{L^2}\leq M_0(t)^{1/2}\delta_{q+1}^{1/2}. \tag{4.12}$$

Furthermore, if $v_q(0),\mathring{R}_q(0)$ are deterministic, so are $v_{q+1}(0),\mathring{R}_{q+1}(0)$.

The proof of Proposition 4.2 is presented in Section 4.1. At this point, we take Proposition 4.2 for granted and apply it in order to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. The proof relies on the above described iteration procedure. More precisely, our goal is to prove that for $L>1$ satisfying (4.8), Lemma 4.1 and Proposition 4.2 give rise to an $(\mathcal{F}_t)_{t\geq 0}$-adapted analytically weak solution v to the transformed problem (3.11). By possibly increasing the value of L, the corresponding solution v fails a suitable energy inequality at the given time T. Finally, again by possibly making L bigger, we verify that $u:=v+z$ and $\mathfrak{t}:=T_L$ fulfill all the requirements in the statement of the theorem.

Starting from $(v_0,\mathring{R}_0)$ given in Lemma 4.1, the iteration Proposition 4.2 yields a sequence $(v_q,\mathring{R}_q)$ satisfying (4.4) and (4.12). By interpolation we deduce that the following series is summable for $\gamma\in(0,\frac{\beta}{4+\beta})$, $t\in[0,T_L]$

$$\sum_{q\geq 0}\|v_{q+1}(t)-v_q(t)\|_{H^\gamma}\lesssim\sum_{q\geq 0}\|v_{q+1}(t)-v_q(t)\|_{L^2}^{1-\gamma}\|v_{q+1}(t)-v_q(t)\|_{H^1}^{\gamma}\lesssim M_0(t)\sum_{q\geq 0}\delta_{q+1}^{\frac{1-\gamma}{2}}\lambda_{q+1}^{4\gamma}\lesssim M_0(t).$$

Thus we obtain a limiting solution $v=\lim_{q\to\infty}v_q$, which lies in $C([0,T_L],H^\gamma)$. Since v_q is $(\mathcal{F}_t)_{t\geq 0}$-adapted for every $q\geq 0$, the limit v is $(\mathcal{F}_t)_{t\geq 0}$-adapted as well. Furthermore, v is an analytically weak solution to (3.11) since it holds $\lim_{q\to\infty}\mathring{R}_q=0$ in $C([0,T_L];L^1)$. In addition, there exists a deterministic constant C_L such that

$$\|v(t)\|_{H^\gamma}\leq C_L \tag{4.13}$$

holds true for all $t\in[0,T_L]$.

Let us now show that the constructed solution v fails the corresponding energy inequality at time T. Namely, we will show

$$\|v(T)\|_{L^2}>(\|v(0)\|_{L^2}+L)e^{LT}. \tag{4.14}$$

According to (4.12), in view of $b^{q+1}\geq b(q+1)$ which holds if $b\geq 2$ and then applying (4.5), we obtain for all $t\in[0,T_L]$

$$\|v(t)-v_0(t)\|_{L^2}\leq\sum_{q\geq 0}\|v_{q+1}(t)-v_q(t)\|_{L^2}\leq M_0(t)^{1/2}\sum_{q\geq 0}\delta_{q+1}^{1/2}\leq M_0(t)^{1/2}\sum_{q\geq 0}(a^{-\beta b})^{q+1}$$

$$=M_0(t)^{1/2}\frac{a^{-\beta b}}{1-a^{-\beta b}}<\frac{1}{2}M_0(t)^{1/2}.$$

Consequently,

$$(\|v(0)\|_{L^2}+L)e^{LT}\leq(\|v_0(0)\|_{L^2}+\|v(0)-v_0(0)\|_{L^2}+L)e^{LT}\leq\left(\frac{3}{2}M_0(0)^{1/2}+L\right)e^{LT},$$

which we want to estimate (strictly) by

$$\left(\frac{1}{\sqrt{2}}-\frac{1}{2}\right)M_0(T)^{1/2}\leq\|v_0(T)\|_{L^2}-\|v(T)-v_0(T)\|_{L^2}\leq\|v(T)\|_{L^2}$$

on the set $\{T_L \geq T\} \subset \Omega$. In view of the definition of $M_0(t)$, this is indeed possible provided

$$\left(\frac{3}{2}+\frac{1}{L}\right) < \left(\frac{1}{\sqrt{2}}-\frac{1}{2}\right)e^{LT}. \tag{4.15}$$

In other words, given $T>0$ and the universal constant $c_R>0$, we can choose $L=L(T,c_R)>1$ large enough so that (4.8) as well as (4.15) holds and consequently (4.14) is satisfied. Moreover, in view of Proposition 3.6 and the definition of the stopping times (4.2), we observe that for a given $T>0$ we may possibly increase L so that the set $\{T_L \geq T\}$ satisfies $\mathbf{P}(T_L \geq T) > \kappa$.

Let us now define $u := v + z$. Then u is $(\mathcal{F}_t)_{t\geq 0}$-adapted, solves the Navier–Stokes system (1.4) and we deduce from (4.13) together with (4.3) that (1.5) holds true. To verify (1.6), we use (4.3) and apply (4.14) on $\{T_L \geq T\}$ to obtain

$$\|u(T)\|_{L^2} \geq \|v(T)\|_{L^2} - \|z(T)\|_{L^2} > (\|v(0)\|_{L^2} + L)e^{LT} - (2\pi)^{3/2}L^{1/4}.$$

Thus, since $u(0)=v(0)$ we may possibly increase the value of L depending on K and $\mathrm{Tr}(GG^*)$ in order to conclude the desired lower bound (1.6). The initial value $v(0)$ is deterministic by our construction. Finally, we set $\mathfrak{t} := T_L$ which finishes the proof. □

To summarize the above discussion, first we fix the parameter L large enough in dependence on T, c_R, κ, K and $\mathrm{Tr}(GG^*)$. Then we apply Proposition 4.2 and deduce the result of Theorem 1.1. It remains to prove Proposition 4.2 and to verify that the parameters a, b, β can be appropriately chosen.

4.1. **The main iteration – proof of Proposition 4.2.** The proof of Proposition 4.2 proceeds along the lines of [BV19a, Section 7]. We have to track the proof carefully to make the construction in each step $(\mathcal{F}_t)_{t\geq 0}$-adapted and the initial value $v(0)$ deterministic. In the course of the proof we will need to adjust the value of the parameters a, b, β as further conditions on these parameters will appear. The parameter L is given and will be kept fixed. In addition, we have to make sure that the condition (4.7), which is essential in order to prove the failure of the energy inequality in Theorem 1.1, is not violated. However, we observe that the right inequality in (4.7) remains valid if we increase the value of a. In other words, given L we find the minimal value of a for which this inequality holds and from now on we may increase a as we wish. On the other hand, increasing the value of a or b can in principle cause problems in the left inequality in (4.7), but here we may make the parameter β smaller so that the inequality remains true. To summarize, we may freely increase a or b at the cost of making β smaller.

4.1.1. *Choice of parameters.* In the sequel, additional parameters will be indispensable and their value has to be carefully chosen in order to respect all the compatibility conditions appearing in the estimations below. First, for a sufficiently small $\alpha \in (0,1)$ to be chosen below, we let $\ell \in (0,1)$ be a small parameter satisfying

$$\ell \lambda_q^4 \leq \lambda_{q+1}^{-\alpha}, \quad \ell^{-1} \leq \lambda_{q+1}^{2\alpha}, \quad 4L \leq \ell^{-1}. \tag{4.16}$$

In particular, we define

$$\ell := \lambda_{q+1}^{-\frac{3\alpha}{2}} \lambda_q^{-2}. \tag{4.17}$$

The last condition in (4.16) together with (4.7) leads to

$$45 \cdot (2\pi)^{3/2} < 5 \cdot (2\pi)^{3/2} a^{2\beta b} \leq c_R L \leq c_R \frac{a^4 \cdot (2\pi)^{3/2} - 1}{2}.$$

We remark that the reasoning from the beginning of Section 4.1 remains valid for this new condition: we may freely increase the value of a provided we make β smaller at the same time. In addition, we will require $\alpha b > 16$ and $\alpha > 8\beta b$.

In order to verify the inductive estimates (4.4) in Section 4.1.4 and Section 4.1.6, it will also be necessary to absorb various expressions including $M_0(t)^{1/2}$ for all $t \in [0, T_L]$. Since the stopping time T_L is bounded by L, this reduces to absorbing $M_0(L)^{1/2}$ and it will be seen that the strongest such requirement is

$$M_0(L)^{1/2}\lambda_{q+1}^{13\alpha-\frac{1}{7}} \leq \frac{c_R\delta_{q+2}}{10} \tag{4.18}$$

needed in Section 4.1.6. In other words,

$$L^2 e^{2L^2} a^{b(13\alpha-\frac{1}{7}+2b\beta)} \ll 1$$

and choosing $b = (7L^2) \vee (17 \cdot 14^2)$, $L \in \mathbb{N}$, (this choice is coming from the fact that with our choice of α below we want to guarantee that $\alpha b > 16$ as well as the fact that b is a multiple of 7 needed for the choice of parameters needed for the intermittent jets below, cf. Appendix B) and $e^2 \leq a^{1/14}$ leads to

$$b a^{b/14} a^{b(13\alpha-\frac{1}{7}+2b\beta)} \ll 1.$$

In view of $\alpha > 8\beta b$, this can be achieved by choosing a large enough and $\alpha = 14^{-2}$. This choice also satisfies $\alpha b > 16$ required above and the condition $\alpha > 8\beta b$ can be achieved by choosing β small. It is also compatible with all the other requirements needed below.

From now on, the parameters α and b remain fixed and the free parameters are a and β for which we already have a lower, respectively upper, bound. In the sequel, we will possibly increase a and decrease β at the same time in order to preserve all the above conditions and to fulfil further conditions appearing below.

4.1.2. *Mollification.* We intend to replace v_q by a mollified velocity field v_ℓ. To this end, let $\{\phi_\varepsilon\}_{\varepsilon>0}$ be a family of standard mollifiers on $\mathbb{R}^3$, and let $\{\varphi_\varepsilon\}_{\varepsilon>0}$ be a family of standard mollifiers with support on $\mathbb{R}^+$. We define a mollification of v_q, $\mathring{R}_q$ and z in space and time by convolution as follows

$$v_\ell = (v_q *_x \phi_\ell) *_t \varphi_\ell, \qquad \mathring{R}_\ell = (\mathring{R}_q *_x \phi_\ell) *_t \varphi_\ell, \qquad z_\ell = (z *_x \phi_\ell) *_t \varphi_\ell,$$

where $\phi_\ell = \frac{1}{\ell^3}\phi(\frac{\cdot}{\ell})$ and $\varphi_\ell = \frac{1}{\ell}\varphi(\frac{\cdot}{\ell})$. Since the mollifier φ_ℓ is supported on $\mathbb{R}^+$, it is easy to see that z_ℓ is $(\mathcal{F}_t)_{t\geq 0}$-adapted and so are v_ℓ and $\mathring{R}_\ell$. Since φ_ℓ is supported on $\mathbb{R}^+$, if the initial values $v_q(0), \mathring{R}_q(0)$ are deterministic, so are $v_\ell(0)$ and $\mathring{R}_\ell(0), \partial_t \mathring{R}_\ell(0)$. Moreover, $z(0) = 0$ implies that $z_\ell(0)$ and $R_{\mathrm{com}}(0)$ given below are deterministic as well. Then using (4.1) we obtain that $(v_\ell, \mathring{R}_\ell)$ satisfies

$$\begin{aligned} \partial_t v_\ell - \Delta v_\ell + \operatorname{div}((v_\ell + z_\ell) \otimes (v_\ell + z_\ell)) + \nabla p_\ell &= \operatorname{div}(\mathring{R}_\ell + R_{\mathrm{com}}) \\ \operatorname{div} v_\ell &= 0, \end{aligned} \tag{4.19}$$

where

$$R_{\mathrm{com}} = (v_\ell + z_\ell)\mathring{\otimes}(v_\ell + z_\ell) - ((v_q + z)\mathring{\otimes}(v_q + z)) *_x \phi_\ell *_t \varphi_\ell,$$

$$p_\ell = (p_q *_x \phi_\ell) *_t \varphi_\ell - \frac{1}{3}(|v_\ell + z_\ell|^2 - (|v_q + z|^2 *_x \phi_\ell) *_t \varphi_\ell).$$

By using (4.4) and (4.16) we know for $t \in [0, T_L]$

$$\|v_q - v_\ell\|_{C_t L^2} \lesssim \|v_q - v_\ell\|_{C^0_{t,x}} \lesssim \ell \|v_q\|_{C^1_{t,x}} \leq \ell \lambda_q^4 M_0(t)^{1/2} \leq M_0(t)^{1/2}\lambda_{q+1}^{-\alpha} \leq \frac{1}{4} M_0(t)^{1/2}\delta_{q+1}^{1/2}, \tag{4.20}$$

where we used the fact that $\alpha > \beta$ and we chose a large enough in order to absorb the implicit constant. In addition, it holds for $t \in [0, T_L]$

$$\|v_\ell\|_{C_tL^2} \leq \|v_q\|_{C_tL^2} \leq M_0(t)^{1/2}(1 + \sum_{1\leq r\leq q} \delta_r^{1/2}), \tag{4.21}$$

and for $N \geq 1$

$$\|v_\ell\|_{C^N_{t,x}} \lesssim \ell^{-N+1}\|v_q\|_{C^1_{t,x}} \leq \ell^{-N+1}\lambda_q^4 M_0(t)^{1/2} \leq M_0(t)^{1/2}\ell^{-N}\lambda_{q+1}^{-\alpha}, \tag{4.22}$$

where we have chosen a large enough to absorb implicit constant.

4.1.3. *Construction of v_{q+1}*. Let us now proceed with the construction of the perturbation w_{q+1} which then defines the next iteration by $v_{q+1} := v_\ell + w_{q+1}$. To this end, we make use of the construction of the intermittent jets [BV19a, Section 7.4], which we recall in Appendix B. In particular, the building blocks $W_{(\xi)} = W_{\xi,r_\perp,r_\parallel,\lambda,\mu}$ for $\xi \in \Lambda$ are defined in (B.3) and the set Λ is introduced in Lemma B.1. The necessary estimates are collected in (B.7). For the intermittent jets we choose the following parameters

$$\lambda = \lambda_{q+1}, \qquad r_\parallel = \lambda_{q+1}^{-4/7}, \qquad r_\perp = r_\parallel^{-1/4}\lambda_{q+1}^{-1} = \lambda_{q+1}^{-6/7}, \qquad \mu = \lambda_{q+1} r_\parallel r_\perp^{-1} = \lambda_{q+1}^{9/7}. \tag{4.23}$$

It is required that b is a multiple of 7 to ensure that $\lambda_{q+1} r_\perp = a^{(b^{q+1})/7} \in \mathbb{N}$.

In order to define the amplitude functions, let χ be a smooth function such that

$$\chi(z) = \begin{cases} 1, & \text{if } 0 \leq z \leq 1, \\ z, & \text{if } z \geq 2, \end{cases}$$

and $z \leq 2\chi(z) \leq 4z$ for $z \in (1,2)$. We then define for $t \in [0, T_L]$ and $\omega \in \Omega$

$$\rho(\omega, t, x) = 4c_R\delta_{q+1}M_0(t)\chi\left((c_R\delta_{q+1}M_0(t))^{-1}|\mathring{R}_\ell(\omega, t, x)|\right),$$

which is $(\mathcal{F}_t)_{t\geq 0}$-adapted and we have

$$\left|\frac{\mathring{R}_\ell(\omega, t, x)}{\rho(\omega, t, x)}\right| = \frac{1}{4}\frac{(c_R\delta_{q+1}M_0(t))^{-1}|\mathring{R}_\ell(\omega, t, x)|}{\chi((c_R\delta_{q+1}M_0(t))^{-1}|\mathring{R}_\ell(\omega, t, x)|)} \leq \frac{1}{2}.$$

Note that if $\mathring{R}_\ell(0, x), \partial_t\mathring{R}_\ell(0, x)$ are deterministic, so is $\rho(0, x)$ and $\partial_t\rho(0, x)$. Moreover, we have for any $p \in [1, \infty]$, $t \in [0, T_L]$

$$\|\rho\|_{C_tL^p} \leq 16\left((8\pi^3)^{1/p}c_R\delta_{q+1}M_0(t) + \|\mathring{R}_\ell\|_{C_tL^p}\right). \tag{4.24}$$

Furthermore, by mollification estimates, the embedding $W^{4,1} \subset L^\infty$ and (4.4) we obtain for $N \geq 0$ $t \in [0, T_L]$

$$\|\mathring{R}_\ell\|_{C^N_{t,x}} \lesssim \ell^{-4-N}c_R\delta_{q+1}M_0(t)$$

and by a repeated application of the chain rule (see [BDLIS16, Proposition C.1]) we obtain

$$\begin{aligned}\|\rho\|_{C^N_{t,x}} &\lesssim \ell^{-4-N}c_R\delta_{q+1}M_0(t) + (c_R\delta_{q+1}M_0(t))^{-N+1}\ell^{-5N}(c_R\delta_{q+1}M_0(t))^N \\ &\lesssim \ell^{-4-5N}c_R\delta_{q+1}M_0(t),\end{aligned} \tag{4.25}$$

where we used the fact that $\frac{d}{dt}M_0(t) = 4LM_0(t)$ as well as $4L \leq \ell^{-1}$ and the implicit constants are independent of ω.

As the next step, we define the amplitude functions

$$a_{(\xi)}(\omega, t, x) := a_{\xi,q+1}(\omega, t, x) := \rho(\omega, t, x)^{1/2}\gamma_\xi\left(\mathrm{Id} - \frac{\mathring{R}_\ell(\omega, t, x)}{\rho(\omega, t, x)}\right)(2\pi)^{-\frac{3}{4}}, \tag{4.26}$$

where γ_ξ is introduced in Lemma B.1. Since ρ and $\mathring{R}_\ell$ are $(\mathcal{F}_t)_{t\geq 0}$-adapted, we know that also $a_{(\xi)}$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted. If $\mathring{R}_\ell(0,x), \partial_t \mathring{R}_\ell(0,x)$ are deterministic, so are $a_{(\xi)}(0,x)$ and $\partial_t a_{(\xi)}(0,x)$. By (B.5) we have

$$(2\pi)^{\frac{3}{2}} \sum_{\xi\in\Lambda} a_{(\xi)}^2 \fint_{\mathbb{T}^3} W_{(\xi)} \otimes W_{(\xi)} dx = \rho \mathrm{Id} - \mathring{R}_\ell, \tag{4.27}$$

and using (4.24) for $t \in [0, T_L]$

$$\|a_{(\xi)}\|_{C_t L^2} \leq \|\rho\|_{C_t L^1}^{1/2} \|\gamma_\xi\|_{C^0(B_{1/2}(\mathrm{Id}))} \leq \frac{4c_R^{1/2}(8\pi^3+1)^{1/2} M}{8|\Lambda|(8\pi^3+1)^{1/2}} M_0(t)^{1/2} \delta_{q+1}^{1/2} \leq \frac{c_R^{1/4} M_0(t)^{1/2} \delta_{q+1}^{1/2}}{2|\Lambda|}, \tag{4.28}$$

where we choose c_R as a small universal constant to absorb M and we use M to denote the universal constant as in Lemma B.1. Furthermore, by using the fact that ρ is bounded from below by $4c_R \delta_{q+1} M_0(t)$ we obtain by similar arguments as in (4.25) that it holds for $t \in [0, T_L]$ that

$$\|a_{(\xi)}\|_{C^N_{t,x}} \leq \ell^{-2-5N} c_R^{1/4} \delta_{q+1}^{1/2} M_0(t)^{1/2}, \tag{4.29}$$

for $N \geq 0$.

With these preparations in hand, we define the principal part $w_{q+1}^{(p)}$ of the perturbation w_{q+1} as

$$w_{q+1}^{(p)} := \sum_{\xi\in\Lambda} a_{(\xi)} W_{(\xi)}. \tag{4.30}$$

If $\mathring{R}_\ell(0,x), \partial_t \mathring{R}_\ell(0,x)$ are deterministic, so are $w_{q+1}^{(p)}(0,x)$ and $\partial_t w_{q+1}^{(p)}(0,x)$. Since the coefficients $a_{(\xi)}$ are $(\mathcal{F}_t)_{t\geq 0}$-adapted and $W_{(\xi)}$ is a deterministic function we deduce that $w_{q+1}^{(p)}$ is also $(\mathcal{F}_t)_{t\geq 0}$-adapted. Moreover, according to (4.27) and (B.4) it follows that

$$w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} + \mathring{R}_\ell = \sum_{\xi\in\Lambda} a_{(\xi)}^2 \mathbb{P}_{\neq 0}(W_{(\xi)} \otimes W_{(\xi)}) + \rho \mathrm{Id}, \tag{4.31}$$

where we use the notation $\mathbb{P}_{\neq 0} f := f - \mathcal{F}f(0) = f - (2\pi)^{3/2} \fint_{\mathbb{T}^3} f$.

We also define an incompressibility corrector by

$$w_{q+1}^{(c)} := \sum_{\xi\in\Lambda} \mathrm{curl}(\nabla a_{(\xi)} \times V_{(\xi)}) + \nabla a_{(\xi)} \times \mathrm{curl} V_{(\xi)} + a_{(\xi)} W_{(\xi)}^{(c)}, \tag{4.32}$$

with $W_{(\xi)}^{(c)}$ and $V_{(\xi)}$ being given in (B.6). Since $a_{(\xi)}$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted and $W_{(\xi)}, W_{(\xi)}^{(c)}$ and $V_{(\xi)}$ are deterministic functions we know that $w_{q+1}^{(c)}$ is also $(\mathcal{F}_t)_{t\geq 0}$-adapted. If $\mathring{R}_\ell(0,x), \partial_t \mathring{R}_\ell(0,x)$ are deterministic, so are $w_{q+1}^{(c)}(0,x)$ and $\partial_t w_{q+1}^{(c)}(0,x)$. By a direct computation we deduce that

$$w_{q+1}^{(p)} + w_{q+1}^{(c)} = \sum_{\xi\in\Lambda} \mathrm{curl}\,\mathrm{curl}(a_{(\xi)} V_{(\xi)}),$$

hence

$$\mathrm{div}(w_{q+1}^{(p)} + w_{q+1}^{(c)}) = 0.$$

We also introduce a temporal corrector

$$w_{q+1}^{(t)} := -\frac{1}{\mu} \sum_{\xi\in\Lambda} \mathbb{P}\mathbb{P}_{\neq 0} \left(a_{(\xi)}^2 \phi_{(\xi)}^2 \psi_{(\xi)}^2 \xi \right), \tag{4.33}$$

where $\mathbb{P}$ is the Helmholtz projection. If $\mathring{R}_\ell(0,x), \partial_t\mathring{R}_\ell(0,x)$ are deterministic, so is $w_{q+1}^{(t)}(0,x)$. Similarly to above $w_{q+1}^{(t)}$ is $(\mathcal{F}_t)_{t\geq0}$-adapted and by a direct computation we obtain

$$\begin{aligned}\partial_t w_{q+1}^{(t)} &+ \sum_{\xi\in\Lambda}\mathbb{P}_{\neq0}\left(a_{(\xi)}^2\mathrm{div}(W_{(\xi)}\otimes W_{(\xi)})\right)\\ &= -\frac{1}{\mu}\sum_{\xi\in\Lambda}\mathbb{P}\mathbb{P}_{\neq0}\partial_t\left(a_{(\xi)}^2\phi_{(\xi)}^2\psi_{(\xi)}^2\xi\right)+\frac{1}{\mu}\sum_{\xi\in\Lambda}\mathbb{P}_{\neq0}\left(a_{(\xi)}^2\partial_t(\phi_{(\xi)}^2\psi_{(\xi)}^2\xi)\right)\\ &= (\mathrm{Id}-\mathbb{P})\frac{1}{\mu}\sum_{\xi\in\Lambda}\mathbb{P}_{\neq0}\partial_t\left(a_{(\xi)}^2\phi_{(\xi)}^2\psi_{(\xi)}^2\xi\right)-\frac{1}{\mu}\sum_{\xi\in\Lambda}\mathbb{P}_{\neq0}\left(\partial_t a_{(\xi)}^2(\phi_{(\xi)}^2\psi_{(\xi)}^2\xi)\right).\end{aligned}\tag{4.34}$$

Note that the first term on the right hand side can be viewed as a pressure term ∇p_1.

Finally, the total perturbation w_{q+1} is defined by

$$w_{q+1} := w_{q+1}^{(p)} + w_{q+1}^{(c)} + w_{q+1}^{(t)},\tag{4.35}$$

which is mean zero, divergence free and $(\mathcal{F}_t)_{t\geq0}$-adapted. If $\mathring{R}_\ell(0,x), \partial_t\mathring{R}_\ell(0,x)$ are deterministic, so is $w_{q+1}(0,x)$. The new velocity v_{q+1} is defined as

$$v_{q+1} := v_\ell + w_{q+1}.\tag{4.36}$$

Thus, it is also $(\mathcal{F}_t)_{t\geq0}$-adapted. If $\mathring{R}_q(0,x), v_q(0,x)$ are deterministic, so is $v_{q+1}(0,x)$.

4.1.4. *Verification of the inductive estimates for v_{q+1}.* Next, we verify the inductive estimates (4.4) on the level $q+1$ for v and we prove (4.12). First, we recall the following result from [BV19a, Lemma 7.4].

Lemma 4.3. *Fix integers $N,\kappa\geq1$ and let $\zeta>1$ be such that*

$$\frac{2\pi\sqrt{3}\zeta}{\kappa}\leq\frac{1}{3}\quad\text{and}\quad\zeta^4\frac{(2\pi\sqrt{3}\zeta)^N}{\kappa^N}\leq1.$$

Let $p\in\{1,2\}$ and let f be a $\mathbb{T}^3$-periodic function such that there exists a constant $C_f>0$ such that

$$\|D^j f\|_{L^p}\leq C_f\zeta^j,$$

holds for all $0\leq j\leq N+4$. In addition, let g be a $(\mathbb{T}/\kappa)^3$-periodic function. Then it holds that

$$\|fg\|_{L^p}\lesssim C_f\|g\|_{L^p},$$

where the implicit constant is universal.

This result shall be used in order to bound $w_{q+1}^{(p)}$ in L^2 whereas for the other L^p-norms we apply a different approach. By (4.28) and (4.29) we obtain for $t\in[0,T_L]$

$$\|D^j a_{(\xi)}\|_{C_tL^2}\lesssim\frac{c_R^{1/4}M_0(t)^{1/2}}{2|\Lambda|}\delta_{q+1}^{1/2}\ell^{-8j},$$

which combined with Lemma 4.3 for $\zeta=\ell^{-8}$ we obtain for $t\in[0,T_L]$

$$\|w_{q+1}^{(p)}\|_{C_tL^2}\leq\sum_{\xi\in\Lambda}\frac{1}{2|\Lambda|}c_R^{1/4}M_0(t)^{1/2}\delta_{q+1}^{1/2}\|W_{(\xi)}\|_{C_tL^2}\leq\frac{1}{2}M_0(t)^{1/2}\delta_{q+1}^{1/2},\tag{4.37}$$

where we used $c_R^{1/4}$ to absorb the universal constant and the fact that due to (B.3) together with the normalizations (B.1), (B.2) we have that $\|W_{(\xi)}\|_{L^2}\simeq1$ uniformly in all the involved parameters.

For general L^p norm we apply (B.7) and (4.29) to deduce for $t\in[0,T_L]$, $p\in(1,\infty)$

$$\|w^{(p)}_{q+1}\|_{C_tL^p}\lesssim\sum_{\xi\in\Lambda}\|a_{(\xi)}\|_{C^0_{t,x}}\|W_{(\xi)}\|_{C_tL^p}\lesssim M_0(t)^{1/2}\delta^{1/2}_{q+1}\ell^{-2}r_\perp^{2/p-1}r_\parallel^{1/p-1/2},\tag{4.38}$$

$$\begin{aligned}\|w^{(c)}_{q+1}\|_{C_tL^p}&\lesssim\sum_{\xi\in\Lambda}\left(\|a_{(\xi)}\|_{C^0_{t,x}}\|W^{(c)}_{(\xi)}\|_{C_tL^p}+\|a_{(\xi)}\|_{C^2_{t,x}}\|V_{(\xi)}\|_{C_tW^{1,p}}\right)\\&\lesssim M_0(t)^{1/2}\delta^{1/2}_{q+1}\ell^{-12}r_\perp^{2/p-1}r_\parallel^{1/p-1/2}\left(r_\perp r_\parallel^{-1}+\lambda_{q+1}^{-1}\right)\lesssim M_0(t)^{1/2}\delta^{1/2}_{q+1}\ell^{-12}r_\perp^{2/p}r_\parallel^{1/p-3/2},\end{aligned}\tag{4.39}$$

and

$$\begin{aligned}\|w^{(t)}_{q+1}\|_{C_tL^p}&\lesssim\mu^{-1}\sum_{\xi\in\Lambda}\|a_{(\xi)}\|^2_{C^0_{t,x}}\|\phi_{(\xi)}\|^2_{L^{2p}}\|\psi_{(\xi)}\|^2_{C_tL^{2p}}\\&\lesssim\delta_{q+1}M_0(t)\ell^{-4}r_\perp^{2/p-1}r_\parallel^{1/p-2}(\mu^{-1}r_\perp^{-1}r_\parallel)=M_0(t)\delta_{q+1}\ell^{-4}r_\perp^{2/p-1}r_\parallel^{1/p-2}\lambda_{q+1}^{-1}.\end{aligned}\tag{4.40}$$

We note that for $p=2$ (4.38) provides a worse bound than (4.37) which was based on Lemma 4.3. Since by (4.18) $M_0(L)^{1/2}\lambda_{q+1}^{4\alpha-\frac17}<1$ we have for $t\in[0,T_L]$

$$\begin{aligned}&\|w^{(c)}_{q+1}\|_{C_tL^p}+\|w^{(t)}_{q+1}\|_{C_tL^p}\\&\lesssim M_0(t)^{1/2}\delta^{1/2}_{q+1}\ell^{-2}r_\perp^{2/p-1}r_\parallel^{1/p-1/2}\left(\ell^{-10}r_\perp r_\parallel^{-1}+M_0(t)^{1/2}\delta^{1/2}_{q+1}\ell^{-2}r_\parallel^{-3/2}\lambda_{q+1}^{-1}\right)\\&\lesssim M_0(t)^{1/2}\delta^{1/2}_{q+1}\ell^{-2}r_\perp^{2/p-1}r_\parallel^{1/p-1/2},\end{aligned}\tag{4.41}$$

where we use (4.16) and the fact that $\lambda_{q+1}^{20\alpha-\frac27}<1$ by our choice of α. The bound (4.41) will be used below in the estimation of the Reynolds stress.

Combining (4.37), (4.39) and (4.40) we obtain for $t\in[0,T_L]$

$$\begin{aligned}\|w_{q+1}\|_{C_tL^2}&\le M_0(t)^{1/2}\delta^{1/2}_{q+1}\left(\frac12+C\ell^{-12}r_\perp r_\parallel^{-1}+CM_0(t)^{1/2}\delta^{1/2}_{q+1}\ell^{-4}r_\parallel^{-3/2}\lambda_{q+1}^{-1}\right)\\&\le M_0(t)^{1/2}\delta^{1/2}_{q+1}\left(\frac12+C\lambda_{q+1}^{24\alpha-2/7}+CM_0(t)^{1/2}\delta^{1/2}_{q+1}\lambda_{q+1}^{8\alpha-1/7}\right)\le\frac34M_0(t)^{1/2}\delta^{1/2}_{q+1},\end{aligned}\tag{4.42}$$

where by (4.18) we choose β small enough and a large enough such that

$$C\lambda_{q+1}^{24\alpha-2/7}\le1/8,\quad\text{and}\quad CM_0(L)^{1/2}\delta^{1/2}_{q+1}\lambda_{q+1}^{8\alpha-1/7}\le1/8.$$

The bound (4.42) can be directly combined with (4.21) and the definition of the velocity v_{q+1} (4.36) to deduce the first bound in (4.4) on the level $q+1$. Indeed, for $t\in[0,T_L]$

$$\|v_{q+1}\|_{C_tL^2}\le\|v_\ell\|_{C_tL^2}+\|w_{q+1}\|_{C_tL^2}\le M_0(t)^{1/2}(1+\sum_{1\le r\le q+1}\delta_r^{1/2}).$$

In addition, (4.42) together with (4.20) yields for $t\in[0,T_L]$

$$\|v_{q+1}-v_q\|_{C_tL^2}\le\|w_{q+1}\|_{C_tL^2}+\|v_\ell-v_q\|_{C_tL^2}\le M_0(t)^{1/2}\delta^{1/2}_{q+1},$$

hence (4.12) holds.

As the next step, we shall verify the second bound in (4.4). Using (4.29)and (B.7) we have for $t\in[0,T_L]$

$$\begin{aligned}\|w^{(p)}_{q+1}\|_{C^1_{t,x}}&\le\sum_{\xi\in\Lambda}\|a_{(\xi)}\|_{C^1_{t,x}}\|W_{(\xi)}\|_{C^1_{t,x}}\\&\lesssim M_0(t)^{1/2}\ell^{-7}r_\perp^{-1}r_\parallel^{-1/2}\lambda_{q+1}\left(1+\frac{r_\perp\mu}{r_\parallel}\right)\lesssim M_0(t)^{1/2}\ell^{-7}r_\perp^{-1}r_\parallel^{-1/2}\lambda_{q+1}^2,\end{aligned}\tag{4.43}$$

$$
\begin{aligned}
\|w_{q+1}^{(c)}\|_{C^1_{t,x}} &\lesssim \sum_{\xi\in\Lambda}\Big(\|a_{(\xi)}\|_{C^1_{t,x}}\|W_{(\xi)}^{(c)}\|_{C^1_{t,x}}+\|a_{(\xi)}\|_{C^3_{t,x}}(\|V_{(\xi)}\|_{C^1_{t,x}}+\|V_{(\xi)}\|_{C_tC^2_x})\Big)\\
&\lesssim M_0(t)^{1/2}\ell^{-17}r_{\|}^{-3/2}\lambda_{q+1}\left(1+\frac{r_\perp\mu}{r_\|}\right)\lesssim M_0(t)^{1/2}\ell^{-17}r_\|^{-3/2}\lambda_{q+1}^2,
\end{aligned}\tag{4.44}
$$

and

$$
\begin{aligned}
\|w_{q+1}^{(t)}\|_{C^1_{t,x}} &\le \frac{1}{\mu}\sum_{\xi\in\Lambda}[\|a^2_{(\xi)}\phi^2_{(\xi)}\psi^2_{(\xi)}\|_{C_tW^{1+\alpha,p}}+\|a^2_{(\xi)}\phi^2_{(\xi)}\psi^2_{(\xi)}\|_{C^1_tW^{\alpha,p}}]\\
&\le \frac{1}{\mu}\sum_{\xi\in\Lambda}\Big(\|a_{(\xi)}\|_{C^0_{t,x}}\|a_{(\xi)}\|_{C^{1+\alpha}_{t,x}}\|\phi_{(\xi)}\|^2_{L^\infty}\|\psi_{(\xi)}\|^2_{C_tL^\infty}\\
&\quad+\|a_{(\xi)}\|^2_{C^0_{t,x}}\|\phi_{(\xi)}\|_{L^\infty}\|\phi_{(\xi)}\|_{W^{1+\alpha,\infty}}\|\psi_{(\xi)}\|^2_{C_tL^\infty}\\
&\quad+\|a_{(\xi)}\|^2_{C^0_{t,x}}\|\phi_{(\xi)}\|^2_{L^\infty}\big(\|\psi_{(\xi)}\|_{C_tL^\infty}\|\psi_{(\xi)}\|_{C_tW^{1+\alpha,p}}+\|\psi_{(\xi)}\|_{C^1_tL^\infty}\|\psi_{(\xi)}\|_{C_tW^{\alpha,p}}\\
&\quad+\|\psi_{(\xi)}\|_{C_tL^\infty}\|\psi_{(\xi)}\|_{C^1_tW^{\alpha,p}}\big)\Big)\\
&\lesssim M_0(t)\ell^{-9}r_\perp^{-1}r_\|^{-2}\lambda_{q+1}^{1+\alpha}\lambda_{q+1}^{-1}\left(1+\frac{r_\perp\mu}{r_\|}\right)\lesssim M_0(t)\ell^{-9}r_\perp^{-1}r_\|^{-2}\lambda_{q+1}^{1+\alpha},
\end{aligned}\tag{4.45}
$$

where we chose p large enough and applied the Sobolev embedding in the first inequality in (4.45) needed because $\mathbb{P}\mathbb{P}_{\neq 0}$ is not a bounded operator on C^0; in the last inequality we used interpolation and an extra λ_{q+1}^α appeared. Combining (4.22) and (4.43), (4.44), (4.45) with (4.16) we obtain for $t\in[0,T_L]$

$$
\begin{aligned}
\|v_{q+1}\|_{C^1_{t,x}} &\le \|v_\ell\|_{C^1_{t,x}}+\|w_{q+1}\|_{C^1_{t,x}}\\
&\le M_0(t)^{1/2}\Big(\lambda_{q+1}^\alpha+C\lambda_{q+1}^{14\alpha+22/7}+C\lambda_{q+1}^{34\alpha+20/7}+CM_0(t)^{1/2}\lambda_{q+1}^{19\alpha+3}\Big)\le M_0(t)^{1/2}\lambda_{q+1}^4,
\end{aligned}
$$

where we used (4.18) to have the fact that $CM_0(L)^{1/2}\le\frac12\lambda_{q+1}^{1-19\alpha}$. Thus, the second estimate in (4.4) holds true on the level $q+1$.

We conclude this part with further estimates of the perturbations $w_{q+1}^{(p)}$, $w_{q+1}^{(c)}$ and $w_{q+1}^{(t)}$, which will be used below in order to bound the Reynolds stress $\mathring{R}_{q+1}$ and to establish the final estimate in (4.4) on the level $q+1$. By a similar approach as in (4.38), (4.39), (4.40), we derive the following estimates: for $t\in[0,T_L]$ by using (4.16), (4.29) and (B.7)

$$
\begin{aligned}
\|w_{q+1}^{(p)}+w_{q+1}^{(c)}\|_{C_tW^{1,p}} &\le \sum_{\xi\in\Lambda}\|\mathrm{curl}\,\mathrm{curl}(a_{(\xi)}V_{(\xi)})\|_{C_tW^{1,p}}\\
&\lesssim \sum_{\xi\in\Lambda}\Big(\|a_{(\xi)}\|_{C^3_{t,x}}\|V_{(\xi)}\|_{C_tL^p}+\|a_{(\xi)}\|_{C^2_{t,x}}\|V_{(\xi)}\|_{C_tW^{1,p}}\\
&\quad+\|a_{(\xi)}\|_{C^1_{t,x}}\|V_{(\xi)}\|_{C_tW^{2,p}}+\|a_{(\xi)}\|_{C^0_{t,x}}\|V_{(\xi)}\|_{C_tW^{3,p}}\Big)\\
&\lesssim M_0(t)^{1/2}r_\perp^{2/p-1}r_\|^{1/p-1/2}\left(\ell^{-17}\lambda_{q+1}^{-2}+\ell^{-12}\lambda_{q+1}^{-1}+\ell^{-7}+\ell^{-2}\lambda_{q+1}\right)\\
&\lesssim M_0(t)^{1/2}r_\perp^{2/p-1}r_\|^{1/p-1/2}\ell^{-2}\lambda_{q+1},
\end{aligned}\tag{4.46}
$$

and

$$
\begin{aligned}
\|w_{q+1}^{(t)}\|_{C_tW^{1,p}} &\leqslant \frac{1}{\mu}\sum_{\xi\in\Lambda}\Big(\|a_{(\xi)}\|_{C^0_{t,x}}\|a_{(\xi)}\|_{C^1_{t,x}}\|\phi_{(\xi)}\|^2_{L^{2p}}\|\psi_{(\xi)}\|^2_{C_tL^{2p}}\\
&\quad+\|a_{(\xi)}\|^2_{C^0_{t,x}}\|\phi_{(\xi)}\|_{L^{2p}}\|\nabla\phi_{(\xi)}\|_{L^{2p}}\|\psi_{(\xi)}\|^2_{C_tL^{2p}}\\
&\quad+\|a_{(\xi)}\|^2_{C^0_{t,x}}\|\phi_{(\xi)}\|^2_{L^{2p}}\|\nabla\psi_{(\xi)}\|_{C_tL^{2p}}\|\psi_{(\xi)}\|_{C_tL^{2p}}\Big)\\
&\lesssim \frac{M_0(t)}{\mu}r_\perp^{2/p-2}r_\parallel^{1/p-1}\left(\ell^{-9}+\ell^{-4}\lambda_{q+1}\right)\lesssim M_0(t)r_\perp^{2/p-2}r_\parallel^{1/p-1}\ell^{-4}\lambda_{q+1}^{-2/7}.
\end{aligned}
\tag{4.47}
$$

4.1.5. *Definition of the Reynolds stress $\mathring{R}_{q+1}$.* Subtracting from (4.1) at level $q+1$ the system (4.19), we obtain

$$
\begin{aligned}
\mathrm{div}\mathring{R}_{q+1}-\nabla p_{q+1} &= \underbrace{-\Delta w_{q+1}+\partial_t(w_{q+1}^{(p)}+w_{q+1}^{(c)})+\mathrm{div}((v_\ell+z_\ell)\otimes w_{q+1}+w_{q+1}\otimes(v_\ell+z_\ell))}_{\mathrm{div}(R_{\mathrm{lin}})+\nabla p_{\mathrm{lin}}}\\
&\quad+\underbrace{\mathrm{div}\Big((w_{q+1}^{(c)}+w_{q+1}^{(t)})\otimes w_{q+1}+w_{q+1}^{(p)}\otimes(w_{q+1}^{(c)}+w_{q+1}^{(t)})\Big)}_{\mathrm{div}(R_{\mathrm{cor}})+\nabla p_{\mathrm{cor}}}\\
&\quad+\underbrace{\mathrm{div}(w_{q+1}^{(p)}\otimes w_{q+1}^{(p)}+\mathring{R}_\ell)+\partial_t w_{q+1}^{(t)}}_{\mathrm{div}(R_{\mathrm{osc}})+\nabla p_{\mathrm{osc}}}\\
&\quad+\underbrace{\mathrm{div}\left(v_{q+1}\otimes z-v_{q+1}\otimes z_\ell+z\otimes v_{q+1}-z_\ell\otimes v_{q+1}+z\otimes z-z_\ell\otimes z_\ell\right)}_{\mathrm{div}(R_{\mathrm{com1}})+\nabla p_{\mathrm{com1}}}\\
&\quad+\mathrm{div}(R_{\mathrm{com}})-\nabla p_\ell.
\end{aligned}
\tag{4.48}
$$

We recall the inverse divergence operator $\mathcal{R}$ as in [BV19a, Section 5.6], which acts on vector fields v with $\int_{\mathbb{T}^3}v dx=0$ as

$$
(\mathcal{R}v)^{kl}=(\partial_k\Delta^{-1}v^l+\partial_l\Delta^{-1}v^k)-\frac{1}{2}(\delta_{kl}+\partial_k\partial_l\Delta^{-1})\mathrm{div}\Delta^{-1}v,
$$

for $k,l\in\{1,2,3\}$. Then $\mathcal{R}v(x)$ is a symmetric trace-free matrix for each $x\in\mathbb{T}^3$, and $\mathcal{R}$ is a right inverse of the div operator, i.e. $\mathrm{div}(\mathcal{R}v)=v$. By using $\mathcal{R}$ we define

$$
\begin{aligned}
R_{\mathrm{lin}} &:= -\mathcal{R}\Delta w_{q+1}+\mathcal{R}\partial_t(w_{q+1}^{(p)}+w_{q+1}^{(c)})+(v_\ell+z_\ell)\mathring{\otimes}w_{q+1}+w_{q+1}\mathring{\otimes}(v_\ell+z_\ell),\\
R_{\mathrm{cor}} &:= (w_{q+1}^{(c)}+w_{q+1}^{(t)})\mathring{\otimes}w_{q+1}+w_{q+1}^{(p)}\mathring{\otimes}(w_{q+1}^{(c)}+w_{q+1}^{(t)}),\\
R_{\mathrm{com1}} &:= v_{q+1}\mathring{\otimes}z-v_{q+1}\mathring{\otimes}z_\ell+z\mathring{\otimes}v_{q+1}-z_\ell\mathring{\otimes}v_{q+1}+z\mathring{\otimes}z-z_\ell\mathring{\otimes}z_\ell.
\end{aligned}
$$

We observe that if $\mathring{R}_q(0,x),v_q(0,x)$ are deterministic, the same is valid for the above defined error terms $R_{\mathrm{lin}}(0,x)$, $R_{\mathrm{cor}}(0,x)$, $R_{\mathrm{com1}}(0,x)$.

In order to define the remaining oscillation error from the third line in (4.48), we apply (4.31) and (4.34) to obtain

$$
\begin{aligned}
&\mathrm{div}(w_{q+1}^{(p)}\otimes w_{q+1}^{(p)}+\mathring{R}_\ell)+\partial_t w_{q+1}^{(t)}\\
&\quad=\sum_{\xi\in\Lambda}\mathrm{div}\left(a_{(\xi)}^2\mathbb{P}_{\neq0}(W_{(\xi)}\otimes W_{(\xi)})\right)+\nabla\rho+\partial_t w_{q+1}^{(t)}\\
&\quad=\sum_{\xi\in\Lambda}\mathbb{P}_{\neq0}\left(\nabla a_{(\xi)}^2\mathbb{P}_{\neq0}(W_{(\xi)}\otimes W_{(\xi)})\right)+\nabla\rho+\sum_{\xi\in\Lambda}\mathbb{P}_{\neq0}\left(a_{(\xi)}^2\mathrm{div}(W_{(\xi)}\otimes W_{(\xi)})\right)+\partial_t w_{q+1}^{(t)}
\end{aligned}
$$

$$= \sum_{\xi\in\Lambda} \mathbb{P}_{\neq 0}\left(\nabla a_{(\xi)}^2 \mathbb{P}_{\neq 0}(W_{(\xi)}\otimes W_{(\xi)})\right) + \nabla\rho + \nabla p_1 - \frac{1}{\mu}\sum_{\xi\in\Lambda}\mathbb{P}_{\neq 0}\left(\partial_t a_{(\xi)}^2(\phi_{(\xi)}^2\psi_{(\xi)}^2\xi)\right)$$

Therefore,

$$R_{\mathrm{osc}} := \sum_{\xi\in\Lambda}\mathcal{R}\left(\nabla a_{(\xi)}^2\mathbb{P}_{\neq 0}(W_{(\xi)}\otimes W_{(\xi)})\right) - \frac{1}{\mu}\sum_{\xi\in\Lambda}\mathcal{R}\left(\partial_t a_{(\xi)}^2(\phi_{(\xi)}^2\psi_{(\xi)}^2\xi)\right) =: R_{\mathrm{osc}}^{(x)} + R_{\mathrm{osc}}^{(t)},$$

which is also deterministic at time 0. Finally we define the Reynolds stress on the level $q+1$ by

$$\mathring{R}_{q+1} := R_{\mathrm{lin}} + R_{\mathrm{cor}} + R_{\mathrm{osc}} + R_{\mathrm{com}} + R_{\mathrm{com1}}.$$

We note that by construction $\mathring{R}_{q+1}(0,x)$ is deterministic.

4.1.6. *Verification of the inductive estimate* (4.4) *for* $\mathring{R}_{q+1}$. To conclude the proof of Proposition 4.2, we shall verify the third estimate in (4.4). To this end, we estimate each term in the definition of $\mathring{R}_{q+1}$ separately.

In the following we choose $p = \frac{32}{32-7\alpha} > 1$ so that it holds in particular that $r_\perp^{2/p-2} r_\parallel^{1/p-1} \leq \lambda_{q+1}^{\alpha}$. For the linear error we apply (4.4) to obtain for $t\in[0,T_L]$

$$\begin{aligned}\|R_{\mathrm{lin}}\|_{C_tL^p} &\lesssim \|\mathcal{R}\Delta w_{q+1}\|_{C_tL^p} + \|\mathcal{R}\partial_t(w_{q+1}^{(p)} + w_{q+1}^{(c)})\|_{C_tL^p} + \|(v_\ell+z_\ell)\mathring{\otimes} w_{q+1} + w_{q+1}\mathring{\otimes}(v_\ell+z_\ell)\|_{C_tL^p}\\ &\lesssim \|w_{q+1}\|_{C_tW^{1,p}} + \sum_{\xi\in\Lambda}\|\partial_t\mathrm{curl}(a_{(\xi)}V_{(\xi)})\|_{C_tL^p} + M_0(t)^{1/2}(\lambda_q^4+1)\|w_{q+1}\|_{C_tL^p},\end{aligned}$$

where by (B.7) and (4.29)

$$\begin{aligned}\sum_{\xi\in\Lambda}\|\partial_t\mathrm{curl}(a_{(\xi)}V_{(\xi)})\|_{C_tL^p} &\leq \sum_{\xi\in\Lambda}\left(\|a_{(\xi)}\|_{C_tC_x^1}\|\partial_t V_{(\xi)}\|_{C_tW^{1,p}} + \|\partial_t a_{(\xi)}\|_{C_tC_x^1}\|V_{(\xi)}\|_{C_tW^{1,p}}\right)\\ &\lesssim M_0(t)^{1/2}\ell^{-7}r_\perp^{2/p}r_\parallel^{1/p-3/2}\mu + M_0(t)^{1/2}\ell^{-12}r_\perp^{2/p-1}r_\parallel^{1/p-1/2}\lambda_{q+1}^{-1}\end{aligned}$$

In view of (4.46), (4.47) as well as (4.38), (4.41), we deduce for $t\in[0,T_L]$

$$\begin{aligned}\|R_{\mathrm{lin}}\|_{C_tL^p} &\lesssim M_0(t)^{1/2}\ell^{-2}r_\perp^{2/p-1}r_\parallel^{1/p-1/2}\lambda_{q+1} + M_0(t)\ell^{-4}r_\perp^{2/p-2}r_\parallel^{1/p-1}\lambda_{q+1}^{-2/7}\\ &\quad + M_0(t)^{1/2}\ell^{-7}r_\perp^{2/p}r_\parallel^{1/p-3/2}\mu + M_0(t)^{1/2}\ell^{-12}r_\perp^{2/p-1}r_\parallel^{1/p-1/2}\lambda_{q+1}^{-1} + M_0(t)\ell^{-2}r_\perp^{2/p-1}r_\parallel^{1/p-1/2}\lambda_q^4\\ &\lesssim M_0(t)^{1/2}\lambda_{q+1}^{5\alpha-1/7} + M_0(t)\lambda_{q+1}^{9\alpha-2/7} + M_0(t)^{1/2}\lambda_{q+1}^{15\alpha-1/7} + M_0(t)^{1/2}\lambda_{q+1}^{25\alpha-15/7}\\ &\leq \frac{M_0(t)c_R\delta_{q+2}}{5}.\end{aligned}$$

Here, we have taken a sufficiently large and β sufficiently small.

The corrector error is estimated using (4.38), (4.39), (4.40), (4.41) for $t\in[0,T_L]$ as

$$\begin{aligned}\|R_{\mathrm{cor}}\|_{C_tL^p} &\leq \|w_{q+1}^{(c)} + w_{q+1}^{(t)}\|_{C_tL^{2p}}\|w_{q+1}\|_{C_tL^{2p}} + \|w_{q+1}^{(c)} + w_{q+1}^{(t)}\|_{C_tL^{2p}}\|w_{q+1}^{(p)}\|_{C_tL^{2p}}\\ &\lesssim M_0(t)\left(\ell^{-12}r_\perp^{1/p}r_\parallel^{1/(2p)-3/2} + \ell^{-4}M_0(t)^{1/2}r_\perp^{1/p-1}r_\parallel^{1/(2p)-2}\lambda_{q+1}^{-1}\right)\ell^{-2}r_\perp^{1/p-1}r_\parallel^{1/(2p)-1/2}\\ &\lesssim M_0(t)\left(\ell^{-14}r_\perp^{2/p-1}r_\parallel^{1/p-2} + \ell^{-6}M_0(t)^{1/2}r_\perp^{2/p-2}r_\parallel^{1/p-5/2}\lambda_{q+1}^{-1}\right)\\ &\lesssim M_0(t)\left(\lambda_{q+1}^{29\alpha-2/7} + M_0(t)^{1/2}\lambda_{q+1}^{13\alpha-1/7}\right) \leq \frac{M_0(t)c_R\delta_{q+2}}{5}.\end{aligned}$$

Here we use (4.18) to have $M_0(L)^{1/2}\lambda_{q+1}^{13\alpha-1/7} \leq \frac{c_R\delta_{q+2}}{10}$.

Finally, we proceed with the oscillation error $R_{\rm osc}$ and we focus on $R_{\rm osc}^{(x)}$ first. Since $W_{(\xi)}$ is $(\mathbb{T}/(r_\perp\lambda_{q+1}))^3$ periodic, we deduce that

$$\mathbb{P}_{\neq 0}(W_{(\xi)}\otimes W_{(\xi)}) = \mathbb{P}_{\geq r_\perp\lambda_{q+1}/2}(W_{(\xi)}\otimes W_{(\xi)}),$$

where $\mathbb{P}_{\geq r} = \mathrm{Id} - \mathbb{P}_{<r}$ and $\mathbb{P}_{<r}$ denotes a Fourier multiplier operator, which projects a function onto its Fourier frequencies $< r$ in absolute value. We also recall the following results from [BV19a, Lemma 7.5].

Lemma 4.4. *Fix parameters $1 \leq \zeta < \kappa, p \in (1,2]$, and assume there exists $N \in \mathbb{N}$ such that $\zeta^N \leq \kappa^{N-2}$. Let $a \in C^N(\mathbb{T}^3)$ be such that there exists $C_a > 0$ with*

$$\|D^j a\|_{C^0} \leq C_a \zeta^j,$$

for all $0 \leq j \leq N$. Assume that $f \in L^p(\mathbb{T}^3)$ such that $\int_{\mathbb{T}^3} a(x)\mathbb{P}_{\geq\kappa} f(x)dx = 0$. Then we have

$$\||\nabla|^{-1}(a\mathbb{P}_{\geq\kappa} f)\|_{L^p} \lesssim C_a \frac{\|f\|_{L^p}}{\kappa},$$

where the implicit constant depends only on p and N.

Using Lemma 4.4 with $a = \nabla a_{(\xi)}^2$ for $C_a = M_0(t)\ell^{-9}$, $\zeta = \ell^{-5}$, $\kappa = r_\perp\lambda_{q+1}$ and any $N \geq 3$, we have

$$\begin{aligned}
\|R_{\rm osc}^{(x)}\|_{C_tL^p} &\leq \sum_{\xi\in\Lambda} \left\|\mathcal{R}\left(\nabla a_{(\xi)}^2 \mathbb{P}_{\geq r_\perp\lambda_{q+1}/2}(W_{(\xi)}\otimes W_{(\xi)})\right)\right\|_{C_tL^p}\\
&\lesssim M_0(t)\ell^{-9}\frac{\|W_{(\xi)}\otimes W_{(\xi)}\|_{C_tL^p}}{r_\perp\lambda_{q+1}} \lesssim M_0(t)\ell^{-9}\frac{\|W_{(\xi)}\|_{C_tL^{2p}}^2}{r_\perp\lambda_{q+1}}\\
&\lesssim M_0(t)\ell^{-9} r_\perp^{2/p-2} r_\parallel^{1/p-1}(r_\perp^{-1}\lambda_{q+1}^{-1}) \lesssim M_0(t)\ell^{-9}\lambda_{q+1}^{\alpha}(r_\perp^{-1}\lambda_{q+1}^{-1})\\
&\lesssim M_0(t)\lambda_{q+1}^{19\alpha-1/7} \leq \frac{M_0(t)c_R\delta_{q+2}}{10}.
\end{aligned}$$

For the second term $R_{\rm osc}^{(t)}$ we use Fubini's theorem to integrate along the orthogonal directions of $\phi_{(\xi)}$ and $\psi_{(\xi)}$ and use (B.7) to deduce

$$\begin{aligned}
\|R_{\rm osc}^{(t)}\|_{C_tL^p} &\leq \mu^{-1}\sum_{\xi\in\Lambda}\|\partial_t a_{(\xi)}^2\|_{C^0_{t,x}}\|\phi_{(\xi)}\|_{C_tL^{2p}}^2\|\psi_{(\xi)}\|_{C_tL^{2p}}^2\\
&\lesssim M_0(t)\mu^{-1}\ell^{-9} r_\perp^{2/p-2} r_\parallel^{1/p-1} \lesssim M_0(t)\lambda_{q+1}^{19\alpha-9/7} \leq \frac{M_0(t)c_R\delta_{q+2}}{10}.
\end{aligned}$$

In view of the standard mollification estimates we have that for $t\in[0,T_L]$

$$\begin{aligned}
\|R_{\rm com}\|_{C_tL^1} &\lesssim \ell(\|v_q\|_{C^1_{t,x}} + \|z\|_{C_tC^1})(\|v_q\|_{C_tL^2} + \|z\|_{C_tL^\infty})\\
&\quad + \ell^{\frac12-2\delta}(\|z\|_{C_t^{\frac12-2\delta}L^\infty} + \|v\|_{C^1_{t,x}})(\|v_q\|_{C_tL^2} + \|z\|_{C_tL^\infty})\\
&\lesssim 2\ell\lambda_q^4 M_0(t) + \ell^{\frac12-2\delta}\lambda_q^4 M_0(t) \leq \frac{M_0(t)c_R\delta_{q+2}}{5},
\end{aligned}$$

where $\delta < \frac{1}{12}$ and we require that $\ell^{\frac12-2\delta}\lambda_q^4 < \frac{c_R\delta_{q+2}}{10}$. With the choice of ℓ in (4.17) and since we postulated that $\alpha > 8\beta b$ and $\alpha b > 16$, this can indeed be achieved by possibly increasing a and consequently decreasing β. Finally, we obtain for $t\in[0,T_L]$

$$\|R_{\rm com1}\|_{C_tL^1} \lesssim M_0(t)^{1/2}\|z_\ell - z\|_{L^\infty} \leq M_0(t)\ell^{\frac12-2\delta} \leq \frac{M_0(t)c_R\delta_{q+2}}{5}.$$

Here we used $\alpha > 8\beta b$ in the last inequality. Summarizing all the above estimates we obtain

$$\|\mathring{R}_{q+1}\|_{C_t L^1} \leq M_0(t) c_R \delta_{q+2},$$

which is the desired last bound in (4.4). The proof of Proposition 4.2 is complete.

5. Non-uniqueness in law II: the case of a linear multiplicative noise

5.1. **Probabilistically weak solutions.** In the case of an additive noise, the stopping times employed in the convex integration can be regarded as functions of the solution u. This does not follow a priori from their definition (4.2), but can be seen from (3.13) and (3.16). Accordingly, it was possible to prove non-uniqueness of martingale solutions in the sense of Definition 3.1 directly. However, the situation is rather different in the case of a linear multiplicative noise. Indeed, the stopping times are functions of the driving noise B, which is not a function of u, and therefore it is necessary to work with the extended canonical space $\bar{\Omega}$ including trajectories of both the solution u and the noise B. To this end, we define the notion of probabilistically weak solution. In the first step, we then establish joint non-uniqueness in law: we show that the joint law of (u, B) is not unique. In the second step, we extend the finite-dimensional result of Cherny [C03] to a general SPDE setting (see Appendix C), proving that uniqueness in law implies joint uniqueness in law. This permits us to conclude the desired non-uniqueness of martingale solutions stated in Theorem 1.4.

To avoid confusion, we point out that the two notions of solution, i.e. martingale solution and probabilistically weak solution, are equivalent. The only reason why the proof of non-uniqueness in law from Section 3 does not apply to the case of linear multiplicative noise is the different definition of stopping times. Conversely, the proof of the present section applies to the additive noise case as well. However, it is more complicated than the direct proof in Section 3 which does not rely on the generalization Cherny's result, Theorem C.1.

Definition 5.1. Let $s \geq 0$ and $x_0 \in L^2_\sigma$, $y_0 \in U_1$. A probability measure $P \in \mathscr{P}(\bar{\Omega})$ is a probabilistically weak solution to the Navier–Stokes system (1.1) with the initial value (x_0, y_0) at time s provided

(M1) $P(x(t) = x_0, y(t) = y_0, 0 \leq t \leq s) = 1$ and for any $n \in \mathbb{N}$

$$P\left\{(x, y) \in \bar{\Omega} : \int_0^n \|G(x(r))\|^2_{L_2(U; L_2^\sigma)} dr < +\infty\right\} = 1.$$

(M2) Under P, y is a cylindrical $(\bar{\mathcal{B}}_t)_{t \geq s}$-Wiener process on U starting from y_0 at time s and for every $e_i \in C^\infty(\mathbb{T}^3) \cap L^2_\sigma$, and for $t \geq s$

$$\langle x(t) - x(s), e_i \rangle + \int_s^t \langle \mathrm{div}(x(r) \otimes x(r)) - \Delta x(r), e_i \rangle dr = \int_s^t \langle e_i, G(x(r)) dy_r \rangle.$$

(M3) For any $q \in \mathbb{N}$ there exists a positive real function $t \mapsto C_{t,q}$ such that for all $t \geq s$

$$E^P\left(\sup_{r \in [0,t]} \|x(r)\|^{2q}_{L^2} + \int_s^t \|x(r)\|^2_{H^\gamma} dr\right) \leq C_{t,q}(\|x_0\|^{2q}_{L^2} + 1).$$

For the application to the Navier–Stokes system, we will again require a definition of probabilistically weak solutions defined up to a stopping time τ. To this end, we set

$$\bar{\Omega}_\tau := \{\omega(\cdot \wedge \tau(\omega)); \omega \in \bar{\Omega}\}.$$

Definition 5.2. Let $s \geq 0$ and $x_0 \in L^2_\sigma$, $y_0 \in U_1$. Let $\tau \geq s$ be a $(\bar{\mathcal{B}}_t)_{t \geq s}$-stopping time. A probability measure $P \in \mathscr{P}(\bar{\Omega}_\tau)$ is a probabilistically weak solution to the Navier–Stokes system (1.1) on $[s, \tau]$ with the initial value (x_0, y_0) at time s provided

(M1) $P(x(t)=x_0, y(t)=y_0, 0\le t\le s)=1$ and for any $n\in\mathbb{N}$

$$P\left\{(x,y)\in\bar\Omega:\int_0^{n\wedge\tau}\|G(x(r))\|^2_{L_2(U;L^2_\sigma)}dr<+\infty\right\}=1.$$

(M2) Under P, $\langle y(\cdot\wedge\tau),l_i\rangle_U$ is a continuous square integrable $(\bar{\mathcal{B}}_t)_{t\ge s}$-martingale starting from y_0 at time s with quadratic variation process given by $(t\wedge\tau-s)\|l_i\|_U^2$, where $\{l_i\}_{i\in\mathbb{N}}$ is an orthonormal basis in U. For every $e_i\in C^\infty(\mathbb{T}^3)\cap L^2_\sigma$, and for $t\ge s$

$$\langle x(t\wedge\tau)-x(s),e_i\rangle+\int_s^{t\wedge\tau}\langle \mathrm{div}(x(r)\otimes x(r))-\Delta x(r),e_i\rangle dr=\int_s^{t\wedge\tau}\langle e_i,G(x(r))dy_r\rangle.$$

(M3) For any $q\in\mathbb{N}$ there exists a positive real function $t\mapsto C_{t,q}$ such that for all $t\ge s$

$$E^P\left(\sup_{r\in[0,t\wedge\tau]}\|x(r)\|^{2q}_{L^2}+\int_s^{t\wedge\tau}\|x(r)\|^2_{H^\gamma}dr\right)\le C_{t,q}(\|x_0\|^{2q}_{L^2}+1).$$

Similarly to Theorem 3.1 we obtain the following existence and stability result. The proof is presented in Appendix A.

Theorem 5.1. *For every $(s,x_0,y_0)\in[0,\infty)\times L^2_\sigma\times U_1$, there exists $P\in\mathscr{P}(\bar\Omega)$ which is a probabilistically weak solution to the Navier–Stokes system (1.1) starting at time s from the initial condition (x_0,y_0) in the sense of Definition 5.1. The set of all such probabilistically weak solutions with the same implicit constant $C_{t,q}$ in Definition 5.1 is denoted by $\mathscr{W}(s,x_0,y_0,C_{t,q})$.*

Let $(s_n,x_n,y_n)\to(s,x_0,y_0)$ in $[0,\infty)\times L^2_\sigma\times U_1$ as $n\to\infty$ and let $P_n\in\mathscr{W}(s_n,x_n,y_n,C_{t,q})$. Then there exists a subsequence n_k such that the sequence $(P_{n_k})_{k\in\mathbb{N}}$ converges weakly to some $P\in\mathscr{W}(s,x_0,y_0,C_{t,q})$.

As in the case of additive noise, the non-uniqueness in law stated in Theorem 1.4 means non-uniqueness of martingale solutions in the sense of Definition 3.1. Non-uniqueness of probabilistically weak solutions corresponds to the joint non-uniqueness in law.

Definition 5.3. We say that joint uniqueness in law holds for (1.1) if probabilistically weak solutions starting from the same initial distribution are unique.

5.2. **General construction for probabilistically weak solutions.** The overall strategy is similar to Section 3.2: in the first step, we shall extend probabilistically weak solutions defined up a $(\bar{\mathcal{B}}_t)_{t\ge0}$-stopping time τ to the whole interval $[0,\infty)$. We denote by $\bar{\mathcal{B}}_\tau$ the σ-field associated to τ.

Proposition 5.2. *Let τ be a bounded $(\bar{\mathcal{B}}_t)_{t\ge0}$-stopping time. Then for every $\omega\in\bar\Omega$ there exists $Q_\omega\in\mathscr{P}(\bar\Omega)$ such that*

$$Q_\omega\big(\omega'\in\bar\Omega;(x,y)(t,\omega')=(x,y)(t,\omega)\ \text{for}\ 0\le t\le\tau(\omega)\big)=1,\tag{5.1}$$

and

$$Q_\omega(A)=R_{\tau(\omega),x(\tau(\omega),\omega),y(\tau(\omega),\omega)}(A)\qquad\text{for all}\ A\in\mathcal{B}^{\tau(\omega)}.\tag{5.2}$$

where $R_{\tau(\omega),x(\tau(\omega),\omega),y(\tau(\omega),\omega)}\in\mathscr{P}(\bar\Omega)$ is a probabilistically weak solution to the Navier–Stokes system (1.1) starting at time $\tau(\omega)$ from the initial condition $(x(\tau(\omega),\omega),y(\tau(\omega),\omega))$. Furthermore, for every $B\in\bar{\mathcal{B}}$ the mapping $\omega\mapsto Q_\omega(B)$ is $\bar{\mathcal{B}}_\tau$-measurable.

Proof. The proof of this result is identical to the proof of Proposition 3.2 applied to the extended path space $\bar\Omega$ instead of Ω_0 and making use of Theorem 5.1 instead of Theorem 3.1. $\square$

We proceed with a result which is analogous to Proposition 3.4.

Proposition 5.3. *Let $x_0 \in L^2_\sigma$. Let P be a probabilistically weak solution to the Navier–Stokes system (1.1) on $[0,\tau]$ starting at the time 0 from the initial condition $(x_0,0)$. In addition to the assumptions of Proposition 5.2, suppose that there exists a Borel set $\mathcal{N} \subset \bar{\Omega}_\tau$ such that $P(\mathcal{N}) = 0$ and for every $\omega \in \mathcal{N}^c$ it holds*

$$Q_\omega\big(\omega' \in \bar{\Omega}; \tau(\omega') = \tau(\omega)\big) = 1. \tag{5.3}$$

Then the probability measure $P \otimes_\tau R \in \mathscr{P}(\bar{\Omega})$ defined by

$$P \otimes_\tau R(\cdot) := \int_{\bar{\Omega}} Q_\omega(\cdot)\, P(d\omega)$$

satisfies $P \otimes_\tau R = P$ on $\bar{\Omega}_\tau$ and is a probabilistically weak solution to the Navier–Stokes system (1.1) on $[0,\infty)$ with initial condition $(x_0,0)$.

Proof. The fact that $P \otimes_\tau R(A) = P(A)$ holds for every Borel set $A \subset \bar{\Omega}_\tau$ as well as the property (M1) follows directly from the construction together with (5.3). In order to show (M3), we write

$$\begin{aligned}
&E^{P\otimes_\tau R}\left(\sup_{r\in[0,t]} \|x(r)\|_{L^2}^{2q} + \int_0^t \|x(r)\|_{H^\gamma}^2 dr\right)\\
&\leq E^{P\otimes_\tau R}\left(\sup_{r\in[0,t\wedge\tau]} \|x(r)\|_{L^2}^{2q} + \int_0^{t\wedge\tau} \|x(r)\|_{H^\gamma}^2 dr\right) + E^{P\otimes_\tau R}\left(\sup_{r\in[t\wedge\tau,t]} \|x(r)\|_{L^2}^{2q} + \int_{t\wedge\tau}^t \|x(r)\|_{H^\gamma}^2 dr\right)\\
&\leq C(\|x_0\|_{L^2}^{2q}+1) + C(E^P\|x(\tau)\|_{L^2}^{2q}+1) \leq C(\|x_0\|_{L^2}^{2q}+1),
\end{aligned}$$

where we used (M3) for P and for R, (5.3) and the boundedness of the stopping time τ.

For (M2), we first recall that since P is a probabilistically weak solution on $[0,\tau]$, the process $\langle y_{t\wedge\tau}, l_i\rangle_U$ is a continuous square integrable $(\bar{\mathcal{B}}_t)_{t\geq0}$-martingale under P with the quadratic variation process given by $(t\wedge\tau)\|l_i\|_U^2$. On the other hand, since for every $\omega \in \bar{\Omega}$, the probability measure $R_{\tau(\omega),x(\tau(\omega),\omega),y(\tau(\omega),\omega)}$ is a probabilistically weak solution starting at the time $\tau(\omega)$ from the initial condition $(x(\tau(\omega),\omega), y(\tau(\omega),\omega))$, the process $\langle y_t - y_{t\wedge\tau(\omega)}, l_i\rangle_U$ is a continuous square integrable $(\bar{\mathcal{B}}_t)_{t\geq\tau(\omega)}$-martingale under $R_{\tau(\omega),x(\tau(\omega),\omega),y(\tau(\omega),\omega)}$ with the quadratic variation process given by $(t-\tau(\omega))\|l_i\|_U^2$, $t \geq \tau(\omega)$. Then by the same arguments as in the proof of Proposition 3.4 we deduce that under $P\otimes_\tau R$, the process $\langle y, l_i\rangle_U$ is a continuous square integrable $(\bar{\mathcal{B}}_t)_{t\geq0}$-martingale with the quadratic variation process given by $t\|l_i\|_U^2$, $t\geq0$, which implies that y is an cylindrical $(\bar{\mathcal{B}}_t)_{t\geq0}$-Wiener process on U.

Furthermore, under P it holds for every $e_i \in C^\infty(\mathbb{T}^3)\cap L^2_\sigma$ and for $t\geq0$

$$M^{x,y,i}_{t\wedge\tau,0} := \langle x(t\wedge\tau) - x(0), e_i\rangle + \int_0^{t\wedge\tau} \langle \mathrm{div}(x(r)\otimes x(r)) - \Delta x(r), e_i\rangle dr = \int_0^{t\wedge\tau} \langle e_i, G(x(r))dy_r\rangle.$$

On the other hand, for $\omega\in\bar{\Omega}$, under $R_{\tau(\omega),x(\tau(\omega),\omega),y(\tau(\omega),\omega)}$ it holds for $t\geq\tau(\omega)$

$$M^{x,y,i}_{t,t\wedge\tau} := \langle x(t) - x(\tau(\omega)), e_i\rangle + \int_{\tau(\omega)}^t \langle \mathrm{div}(x(r)\otimes x(r)) - \Delta x(r), e_i\rangle dr = \int_{\tau(\omega)}^t \langle e_i, G(x(r))dy_r\rangle.$$

Therefore, we obtain

$$\begin{aligned}
&P\otimes_\tau R\left\{M^{x,y,i}_{t,0} = \int_0^t \langle e_i, G(x(r))dy_r\rangle, e_i\in C^\infty(\mathbb{T}^3)\cap L^2_\sigma, t\geq0\right\}\\
&= \int_{\bar{\Omega}} dP(\omega) Q_\omega\bigg\{M^{x,y,i}_{t,t\wedge\tau(\omega)} = \int_{t\wedge\tau(\omega)}^t \langle e_i, G(x(r))dy_r\rangle,\\
&\qquad\qquad M^{x,y,i}_{t\wedge\tau(\omega),0} = \int_0^{t\wedge\tau(\omega)} \langle e_i, G(x(r))dy_r\rangle, e_i\in C^\infty(\mathbb{T}^3)\cap L^2_\sigma, t\geq0\bigg\}.
\end{aligned}$$

Now, using (5.3) and (5.2) we obtain

$$\begin{aligned}&\int_{\bar{\Omega}} dP(\omega)Q_\omega\Big\{M^{x,y,i}_{t,t\wedge\tau(\omega)} = \int_{t\wedge\tau(\omega)}^{t}\langle e_i, G(x(r))dy_r\rangle, e_i\in C^\infty(\mathbb{T}^3)\cap L^2_\sigma, t\geq 0\Big\}\\ &= \int_{\bar{\Omega}} dP(\omega)R_{\tau(\omega),x(\tau(\omega),\omega),y(\tau(\omega),\omega)}\Big\{M^{x,y,i}_{t\wedge\tau(\omega),0} = \int_0^{t\wedge\tau(\omega)}\langle e_i, G(x(r))dy_r\rangle, e_i\in C^\infty(\mathbb{T}^3)\cap L^2_\sigma, t\geq 0\Big\}\\ &= 1,\end{aligned}$$

and using (5.3) and (5.1) we deduce

$$\begin{aligned}&\int_{\bar{\Omega}} dP(\omega)Q_\omega\Big\{M^{x,y,i}_{t\wedge\tau(\omega),0} = \int_0^{t\wedge\tau(\omega)}\langle e_i, G(x(r))dy_r\rangle, e_i\in C^\infty(\mathbb{T}^3)\cap L^2_\sigma, t\geq 0\Big\}\\ &= P\Big\{M^{x,y,i}_{t\wedge\tau,0} = \int_0^{t\wedge\tau}\langle e_i, G(x(r))dy_r\rangle, e_i\in C^\infty(\mathbb{T}^3)\cap L^2_\sigma, t\geq 0\Big\} = 1.\end{aligned}$$

In view of the elementary inequality for probability measures $Q_\omega(A\cap B)\geq 1-Q_\omega(A^c)-Q_\omega(B^c)$, we finally deduce that $P\otimes_\tau R$-a.s.

$$M^{x,y,i}_{t,0} = \int_0^t\langle e_i, G(x(r))dy_r\rangle \qquad \text{for all}\ \ e_i\in C^\infty(\mathbb{T}^3)\cap L^2_\sigma,\ t\geq 0,$$

hence the condition (M2) follows. □

5.3. **Application to solutions obtained through Theorem 1.3.** The general construction presented in Section 5.2 applies to a general infinite dimensional stochastic perturbation of the Navier–Stokes system. From now on, we restrict ourselves to the setting of a linear multiplicative noise. In particular, the driving Wiener process is real-valued and consequently $U = U_1 = \mathbb{R}$.

For $n\in\mathbb{N}$, $L>1$ and $\delta\in(0,1/12)$ we define

$$\tau^n_L(\omega) = \inf\Big\{t\geq 0, |y(t,\omega)| > (L-\frac{1}{n})^{1/4}\Big\}\bigwedge \inf\Big\{t>0, \|y(t,\omega)\|_{C_t^{\frac{1}{2}-2\delta}} > (L-\frac{1}{n})^{1/2}\Big\}\bigwedge L.$$

Then the sequence $\{\tau^n_L\}_{n\in\mathbb{N}}$ is nondecreasing and we define

$$\tau_L := \lim_{n\to\infty}\tau^n_L. \tag{5.4}$$

Without additional regularity of the process y, it holds true that $\tau^n_L(\omega) = 0$. By Lemma 3.5 we obtain that τ^n_L is $(\bar{\mathcal{B}}_t)_{t\geq 0}$-stopping time and consequently also τ_L is a $(\bar{\mathcal{B}}_t)_{t\geq 0}$-stopping time as an increasing limit of stopping times.

Now, we fix a real-valued Wiener process B defined on a probability space $(\Omega,\mathcal{F},\mathbf{P})$ and we denote by $(\mathcal{F}_t)_{t\geq 0}$ its normal filtration. On this stochastic basis, we apply Theorem 1.3 and denote by u the corresponding solution to the Navier–Stokes system (1.7) on $[0,T_L]$, where the stopping time T_L is defined in (6.3). We recall that u is adapted with respect to $(\mathcal{F}_t)_{t\geq 0}$ which is an essential property employed to prove the martingale property in the proof of Proposition 5.4. We denote by P the law of (u,B) and obtain the following result by similar arguments as in the proof of Proposition 3.7.

Proposition 5.4. *The probability measure P is a probabilistically weak solution to the Navier–Stokes system (1.7) on $[0,\tau_L]$ in the sense of Definition 5.2, where τ_L was defined in (5.4).*

Proof. The proof is similar as the proof of Proposition 3.7 once we note that

$$y(t,(u,B)) = B(t) \quad \text{for } t\in[0,T_L] \quad \mathbf{P}\text{-a.s.}$$

In particular, the property (M2) in Definition 5.2 follows since (u,B) satisfies (1.7). □

Proposition 5.5. *The probability measure $P\otimes_{\tau_L} R$ is a probabilistically weak solution to the Navier–Stokes system (1.7) on $[0,\infty)$ in the sense of Definition 5.1.*

Proof. In light of Proposition 5.2 and Proposition 5.3, it only remains to establish (5.3), which follows by similar arguments as in the proof of Proposition 3.8. □

Finally, we have all in hand to conclude the proof of our main result Theorem 1.4.

Proof of Theorem 1.4. Let $T>0$ be arbitrary. Let $\kappa=1/2$ and $K=2$ and apply Theorem 1.3 and Proposition 5.5. As in the proof of Theorem 1.2 it follows that the constructed probability measure $P\otimes_{\tau_L} R$ satisfies

$$P\otimes_{\tau_L} R(\tau_L\geq T)=\mathbf{P}(T_L\geq T)>1/2,$$

and consequently

$$E^{P\otimes_{\tau_L} R}\big[\|x(T)\|_{L^2}^2\big]>2e^T\|x_0\|_{L^2}^2.$$

The initial value $x_0=v(0)\in L^2_\sigma$ is given through the construction in Theorem 1.3. However, based on a Galerkin approximation one can construct a probabilistically weak solution $\tilde{P}$ to (1.7) starting from the same initial value as $P\otimes_{\tau_L} R$. In addition, this solution satisfies the usual energy inequality, that is,

$$E^{\tilde{P}}\big[\|x(T)\|_{L^2}^2\big]\leq e^T\|x_0\|_{L^2}^2.$$

Therefore, the two probabilistically weak solutions are distinct and as a consequence joint non-uniqueness in law, i.e. non-uniqueness of probabilistically weak solutions, holds for the Navier–Stokes system (1.7). In view of Theorem C.1 we finally deduce that the desired non-uniqueness in law, i.e., non-uniqueness of martingale solutions, holds as well. □

6. Proof of Theorem 1.3

As the first step, we transform (1.7) to a random PDE. To this end, we consider the stochastic process

$$\theta(t)=e^{B_t},\ t\geq 0,$$

and define $v:=\theta^{-1}u$. Then, by Itô's formula we obtain

$$\begin{aligned}\partial_t v+\frac{1}{2}v-\Delta v+\theta\mathrm{div}(v\otimes v)+\theta^{-1}\nabla P&=0,\\ \mathrm{div}v&=0.\end{aligned}\tag{6.1}$$

Our aim is to develop a similar induction argument as in Section 4 and apply it to (6.1). At each step $q\in\mathbb{N}_0$, a pair $(v_q,\mathring{R}_q)$ is constructed solving the following system

$$\begin{aligned}\partial_t v_q+\frac{1}{2}v_q-\Delta v_q+\theta\mathrm{div}(v_q\otimes v_q)+\nabla p_q&=\mathrm{div}\mathring{R}_q,\\ \mathrm{div}v_q&=0.\end{aligned}\tag{6.2}$$

We choose suitable parameters $a\in\mathbb{N}$ and $b\in\mathbb{N}$ sufficiently large and a parameter $\beta\in(0,1)$ sufficiently small and define

$$\lambda_q=a^{(b^q)},\quad \delta_q=\lambda_q^{-2\beta}.$$

The necessary stopping times T_L are now defined in terms of the Wiener Process B as

$$T_L:=\inf\{t>0,|B(t)|\geq L^{1/4}\}\wedge\inf\{t>0,\|B\|_{C_t^{1/2-2\delta}}\geq L^{1/2}\}\wedge L\tag{6.3}$$

for $L>1$ and $\delta\in(0,1/12)$. As a consequence, it holds for $t\in[0,T_L]$

$$|B(t)|\leq L^{1/4},\quad \|B\|_{C_t^{1/2-2\delta}}\leq L^{1/2},\tag{6.4}$$

which implies

$$\|\theta\|_{C_t^{\frac{1}{2}-2\delta}} + |\theta(t)| + |\theta^{-1}(t)| \leq 3L^{1/2}e^{L^{1/4}} =: m_L^2. \tag{6.5}$$

We also define

$$M_0(t) := e^{4Lt+2L}. \tag{6.6}$$

For the induction, we will assume the following bounds for $(v_q, \mathring{R}_q)$ which are valid for $t \in [0, T_L]$

$$\begin{aligned} \|v_q\|_{C_tL^2} &\leq m_L M_0(t)^{1/2}\Big(1 + \sum_{1\leq r\leq q} \delta_r^{1/2}\Big) \leq 2m_L M_0(t)^{1/2}, \\ \|v_q\|_{C^1_{t,x}} &\leq m_L M_0(t)^{1/2}\lambda_q^4, \\ \|\mathring{R}_q\|_{C_tL^1} &\leq c_R M_0(t)\delta_{q+1}. \end{aligned} \tag{6.7}$$

Here $\sum_{1\leq r\leq 0} \delta_r^{1/2} := 0$, $c_R > 0$ is a sufficiently small universal constant given in (6.22), (6.24) and we used the fact that $\sum_{r\geq 1} \delta_r^{1/2} \leq \sum_{r\geq 1} a^{-rb\beta} = \frac{a^{-\beta b}}{1-a^{-\beta b}} < 1/2$ and

$$a^{\beta b} > 3 \tag{6.8}$$

in the first inequality. The following result sets the starting point of our iteration procedure and gives the key compatibility conditions between the parameters L, a, β, b.

Lemma 6.1. *Let $L > 1$ and define*

$$v_0(t,x) := \frac{m_L e^{2Lt+L}}{(2\pi)^{\frac{3}{2}}} \left(\sin(x_3), 0, 0\right).$$

Then the associated Reynolds stress is given by

$$\mathring{R}_0(t,x) = \frac{m_L(2L+3/2)e^{2Lt+L}}{(2\pi)^{3/2}} \begin{pmatrix} 0 & 0 & -\cos(x_3) \\ 0 & 0 & 0 \\ -\cos(x_3) & 0 & 0 \end{pmatrix}.$$

The initial values $v_0(0,x)$ and $\mathring{R}_0(0,x)$ are deterministic. Moreover, all the estimates in (6.7) on the level $q = 0$ for $(v_0, \mathring{R}_0)$ as well as (6.8) are valid provided

$$18 \cdot (2\pi)^{3/2}\sqrt{3} < 2 \cdot (2\pi)^{3/2}\sqrt{3}a^{2\beta b} \leq \frac{c_R e^L}{L^{1/4}(2L+\frac{3}{2})e^{\frac{1}{2}L^{1/4}}}, \qquad 4L \leq a^4. \tag{6.9}$$

In particular, the minimal lower bound for L is given through

$$18 \cdot (2\pi)^{3/2}\sqrt{3} < \frac{c_R e^L}{L^{1/4}(2L+\frac{3}{2})e^{\frac{1}{2}L^{1/4}}}. \tag{6.10}$$

Proof. We observe that for $t \in [0, T_L]$

$$\|v_0(t)\|_{L^2} = \frac{m_L e^{2Lt+L}}{\sqrt{2}} \leq m_L M_0(t)^{1/2}, \qquad \|v_0\|_{C^1_{t,x}} \leq 4Lm_L e^{2Lt+L} \leq m_L M_0(t)^{1/2}\lambda_0^4,$$

provided

$$4L \leq a^4. \tag{6.11}$$

The associated Reynolds stress can be directly computed and admits the bound

$$\|\mathring{R}_0(t)\|_{L^1} \leq 2 \cdot (2\pi)^{\frac{3}{2}} m_L(2L+3/2)e^{2Lt+L} \leq M_0(t)c_R\delta_1,$$

provided

$$2\cdot(2\pi)^{\frac{3}{2}}\sqrt{3}L^{1/4}(2L+3/2)e^{\frac{1}{2}L^{1/4}}\leq e^{L}c_Ra^{-2\beta b}. \tag{6.12}$$

Under the conditions (6.11) and (6.12) all the estimates in (6.7) are valid on the level $q=0$. Combining (6.11), (6.12) with (6.8) we arrive at (6.9), (6.10) from the statement of the lemma. □

We note that the compatibility conditions (6.9), (6.10) are similar in spirit to the corresponding conditions in the additive noise case, i.e. (4.7), (4.8). In other words, (6.10) gives the minimal admissible lower bound for L. Then based on the second condition in (6.9) we obtain a minimal admissible lower bound for a. Whenever we need to increase a or b in the course of the main iteration proposition below, we have to decrease the value of β simultaneously so that the first condition in (6.9) is not violated.

Proposition 6.2. (Main iteration) *Let $L>1$ satisfying (6.10) be given and let $(v_q,\mathring{R}_q)$ be an $(\mathcal{F}_t)_{t\geq0}$-adapted solution to (6.2) satisfying (6.7). Then there exists a choice of parameters a,b,β such that (6.9) is fulfilled and there exist $(\mathcal{F}_t)_{t\geq0}$-adapted processes $(v_{q+1},\mathring{R}_{q+1})$ which solve (6.2), obey (6.7) at level $q+1$ and for $t\in[0,T_L]$ we have*

$$\|v_{q+1}(t)-v_q(t)\|_{L^2}\leq m_LM_0(t)^{1/2}\delta_{q+1}^{1/2}. \tag{6.13}$$

Furthermore, if $v_q(0),\mathring{R}_q(0)$ are deterministic, so are $v_{q+1}(0),\mathring{R}_{q+1}(0)$.

The proof of Proposition 6.13 is presented in Section 6.1 below. Based on this result, we are able to conclude the proof of Theorem 1.3.

Proof of Theorem 1.3. Starting from $(v_0,\mathring{R}_0)$ given in Lemma 6.1 and using Proposition 6.2 we obtain a sequence $(v_q,\mathring{R}_q)$ satisfying (6.7) and (6.13). By interpolation, it follows for $\gamma\in(0,\frac{\beta}{4+\beta})$, $t\in[0,T_L]$

$$\sum_{q\geq0}\|v_{q+1}(t)-v_q(t)\|_{H^\gamma}\lesssim\sum_{q\geq0}\|v_{q+1}(t)-v_q(t)\|_{L^2}^{1-\gamma}\|v_{q+1}(t)-v_q(t)\|_{H^1}^{\gamma}\lesssim m_LM_0(t)^{1/2}.$$

Therefore, the sequence v_q converges to a limit $v\in C([0,T_L],H^\gamma)$ which is $(\mathcal{F}_t)_{t\geq0}$-adapted. Furthermore, we know that v is an analytically weak solution to (6.1) with a deterministic initial value, since due to (6.7) it holds $\lim_{q\to\infty}\mathring{R}_q=0$ in $C([0,T_L];L^1)$. According to (6.13) and (6.8), it follows for $t\in[0,T_L]$

$$\|v(t)-v_0(t)\|_{L^2}\leq\sum_{q\geq0}\|v_{q+1}(t)-v_q(t)\|_{L^2}\leq m_LM_0(t)^{1/2}\sum_{q\geq0}\delta_{q+1}^{1/2}\leq\frac{1}{2}m_LM_0(t)^{1/2}.$$

Now, we show that for a given $T>0$ we can choose $L=L(T)>1$ large enough so that v fails the corresponding energy inequality at time T, namely, it holds

$$\|v(T)\|_{L^2}>e^{2L^{1/2}}\|v(0)\|_{L^2} \tag{6.14}$$

on the set $\{T_L\geq T\}$. To this end, we observe that

$$e^{2L^{1/2}}\|v(0)\|_{L^2}\leq e^{2L^{1/2}}\left(\|v_0(0)\|_{L^2}+\|v(0)-v_0(0)\|_{L^2}\right)\leq e^{2L^{1/2}}\frac{3}{2}m_LM_0(0)^{1/2}.$$

On the other hand, we obtain on $\{T_L\geq T\}$

$$\|v(T)\|_{L^2}\geq\left(\|v_0(T)\|_{L^2}-\|v(T)-v_0(T)\|_{L^2}\right)\geq\left(\frac{1}{\sqrt{2}}-\frac{1}{2}\right)m_LM_0(T)^{1/2}>e^{2L^{1/2}}\frac{3}{2}m_LM_0(0)^{1/2}$$

provided

$$\left(\frac{1}{\sqrt{2}}-\frac{1}{2}\right)e^{2LT}>\frac{3}{2}e^{2L^{1/2}}. \tag{6.15}$$

Hence (6.14) follows for a suitable choice of L satisfying additionally (6.15). Furthermore, for a given $T>0$ we could possibly increase L so that $\mathbf{P}(T_L\geq T)>\kappa$.

To conclude the proof, we define $u:=\theta v$ and observe that $u(0)=v(0)$. In addition, u is $(\mathcal{F}_t)_{t\geq0}$-adapted and solves the original Navier–Stokes system (1.4). Then in view of (6.14) and the fact that due to (6.4) it holds true $|\theta_T|\geq e^{-L^{1/4}}$ on the set $\{T_L\geq T\}$, we obtain

$$\|u(T)\|_{L^2}=|\theta(T)|\,\|v(T)\|_{L^2}>e^{L^{1/2}}\|u(0)\|_{L^2}$$

on $\{T_L\geq T\}$. Choosing L sufficiently large in dependence on K and T from the statement of the theorem, the desired lower bound follows. Finally, setting $\mathfrak{t}:=T_L$ completes the proof. □

6.1. **The main iteration – proof of Proposition 6.2.** The overall strategy of the proof is similar to Section 4.1 but modifications are required since the approximate system on the level q has a different form. As in Section 4.1, we have to make sure that the construction is $(\mathcal{F}_t)_{t\geq0}$-adapted at each step.

6.1.1. *Choice of parameters.* We choose a small parameter $\ell\in(0,1)$ as in Section 4.1.1: for a sufficiently small $\alpha\in(0,1)$ to be chosen below, we let $\ell\in(0,1)$ be a small parameter defined in (4.17) and satisfying (4.16). We note that the compatibility conditions (6.9), (6.10) as well and the last condition in (4.16) can all be fulfilled provided we make a large enough and β small enough at the same time. In addition, we will require $\alpha b>16$ and $\alpha>8\beta b$.

In order to verify the inductive estimates (6.7) we need to absorb various expressions including $m_L^4M_0(t)^{1/2}$ for all $t\in[0,T_L]$. To this end, we need to change the condition (4.18) in Section 4.1.1 to

$$Cm_L^4\ell^{1/3}\lambda_q^4\leq\frac{c_R\delta_{q+2}}{5},\quad m_L^4M_0(L)^{1/2}\lambda_{q+1}^{13\alpha-\frac{1}{7}}\leq\frac{c_R\delta_{q+2}}{10},\quad m_L\leq\ell^{-1}. \tag{6.16}$$

In other words, we need

$$\begin{aligned}9Le^{2L^{1/4}}a^{b(-\frac{\alpha}{2}+\frac{10}{3b}+2b\beta)}&\ll1,\\ 9Le^{2L^{1/4}}e^{2L^2+L}a^{b(13\alpha-\frac{1}{7}+2b\beta)}&\ll1,\\ \sqrt{3}L^{1/4}e^{1/2L^{1/4}}&\leq a^{2+\frac{3\alpha}{2}\cdot7L^2}.\end{aligned}$$

Choosing $b=(7L^2)\vee(17\cdot14^2)$, in view of $\alpha>8\beta b$, (6.16) can be achieved by choosing a large enough and $\alpha=14^{-2}$. This choice also satisfies $\alpha b>16$ required above and the condition $\alpha>8\beta b$ can be achieved by choosing β small. It is also compatible with all the other requirements needed below.

6.1.2. *Mollification.* As the next step, we define space-time mollifications of v_q and $\mathring{R}_q$ and a time mollification of θ as follows

$$v_\ell=(v_q*_x\phi_\ell)*_t\varphi_\ell,\qquad\mathring{R}_\ell=(\mathring{R}_q*_x\phi_\ell)*_t\varphi_\ell,\qquad\theta_\ell=e^B*_t\varphi_\ell.$$

By choosing time mollifiers that are compactly supported in $\mathbb{R}^+$, the mollification preserves $(\mathcal{F}_t)_{t\geq0}$-adaptedness. If the initial data $v_q(0),\mathring{R}_q(0)$ are deterministic, so are $v_\ell(0)$ and $\mathring{R}_\ell(0),\partial_t\mathring{R}_\ell(0)$. Then using (6.2) we obtain that $(v_\ell,\mathring{R}_\ell)$ satisfies

$$\begin{aligned}\partial_tv_\ell+\frac{1}{2}v_\ell-\Delta v_\ell+\theta_\ell\mathrm{div}(v_\ell\otimes v_\ell)+\nabla p_\ell&=\mathrm{div}(\mathring{R}_\ell+R_{\mathrm{com}})\\ \mathrm{div}v_\ell&=0,\end{aligned}$$

where

$$R_{\text{com}} = \theta_\ell(v_\ell \mathring{\otimes} v_\ell) - (\theta v_q \mathring{\otimes} v_q) *_x \phi_\ell *_t \varphi_\ell,$$

$$p_\ell = (p_q *_x \phi_\ell) *_t \varphi_\ell - \frac{1}{3}(\theta_\ell |v_\ell|^2 - (\theta |v_q|^2 *_x \phi_\ell) *_t \varphi_\ell).$$

With this setting, the counterparts of the estimates (4.20), (4.21) and (4.22) are obtained the same way only replacing $M_0(t)^{1/2}$ by $m_L M_0(t)^{1/2}$. In particular,

$$\|v_q - v_\ell\|_{C_t L^2} \leq \frac{1}{4} m_L M_0(t)^{1/2} \delta_{q+1}^{1/2}, \tag{6.17}$$

$$\|v_\ell\|_{C_t L^2} \leq m_L M_0(t)^{1/2}(1 + \sum_{1 \leq r \leq q} \delta_r^{1/2}) \leq 2 m_L M_0(t)^{1/2}, \tag{6.18}$$

$$\|v_\ell\|_{C^N_{t,x}} \leq m_L M_0(t)^{1/2} \ell^{-N} \lambda_{q+1}^{-\alpha}. \tag{6.19}$$

6.1.3. *Construction of* v_{q+1}. We recall that the intermittent jets $W_{(\xi)}$ and the corresponding estimates are summarized in Appendix B. The parameters $\lambda, r_\|, r_\perp, \mu$ are chosen as in (4.23) and we define χ and ρ be the same functions as in Section 4.1.3 with $M_0(t)$ given by (6.6). Now, we define the modified amplitude functions

$$\begin{aligned} \bar{a}_{(\xi)}(\omega, t, x) := \bar{a}_{\xi, q+1}(\omega, t, x) &:= \theta_\ell^{-1/2} \rho(\omega, t, x)^{1/2} \gamma_\xi \left(\mathrm{Id} - \frac{\mathring{R}_\ell(\omega, t, x)}{\rho(\omega, t, x)} \right) (2\pi)^{-\frac{3}{4}} \\ &= \theta_\ell^{-1/2} a_{\xi, q+1}(\omega, t, x), \end{aligned} \tag{6.20}$$

where γ_ξ is introduced in Lemma B.1 and $a_{\xi,q+1}$ is as in Section 4.1.3 with $M_0(t)$ given in (6.6). Since ρ, θ_ℓ and $\mathring{R}_\ell$ are $(\mathcal{F}_t)_{t\geq 0}$-adapted, we know $\bar{a}_{(\xi)}$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted. Note that since $\theta_\ell(0)$ and $\partial_t \theta_\ell(0)$ are deterministic, if $\mathring{R}_\ell(0), \partial_t \mathring{R}_\ell(0)$ are deterministic, so are $\bar{a}_\xi(0)$ and $\partial_t \bar{a}_\xi(0)$. By (B.5) we have

$$(2\pi)^{3/2} \sum_{\xi \in \Lambda} \bar{a}_{(\xi)}^2 \fint_{\mathbb{T}^3} W_{(\xi)} \otimes W_{(\xi)} dx = \theta_\ell^{-1}(\rho \mathrm{Id} - \mathring{R}_\ell), \tag{6.21}$$

and for $t \in [0, T_L]$

$$\begin{aligned} \|\bar{a}_{(\xi)}\|_{C_t L^2} &\leq \|\theta_\ell^{-1/2}\|_{C_t} \|\rho\|_{C_t L^1}^{1/2} \|\gamma_\xi\|_{C^0(B_{1/2}(\mathrm{Id}))} \\ &\leq \frac{4 c_R^{1/2} (8\pi^3 + 1)^{1/2} M}{8|\Lambda| (8\pi^3 + 1)^{1/2}} m_L M_0(t)^{1/2} \delta_{q+1}^{1/2} \leq \frac{c_R^{1/4} m_L M_0(t)^{1/2} \delta_{q+1}^{1/2}}{2|\Lambda|}, \end{aligned} \tag{6.22}$$

where we choose c_R as a small universal constant to absorb M and M denotes the universal constant from Lemma B.1 and we apply the bound $|\theta_\ell^{-1}| \leq m_L^2$. Furthermore, since ρ is bounded from below by $4 c_R \delta_{q+1} M_0(t)$ we obtain for $t \in [0, T_L]$

$$\|\bar{a}_{(\xi)}\|_{C^N_{t,x}} \lesssim \ell^{-2-5N} c_R^{1/4} m_L M_0(t)^{1/2} \delta_{q+1}^{1/2}, \tag{6.23}$$

for $N \geq 0$, where we used (6.5) and $4L \leq \ell^{-1}$ and the derivative of $\theta_\ell^{-1/2}$ gives extra $\ell^{-1} m_L^4$ and $m_L \leq \ell^{-1}$

As the next step, we define w_{q+1} similarly as in Section 4.1.3. In particular, first we define the principal part $w_{q+1}^{(p)}$ of w_{q+1} as (4.30) with $a_{(\xi)}$ replaced by $\bar{a}_{(\xi)}$ given in (6.20). Then it follows from (6.21)

$$\theta_\ell w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} + \mathring{R}_\ell = \theta_\ell \sum_{\xi \in \Lambda} \bar{a}_{(\xi)}^2 \mathbb{P}_{\neq 0}(W_{(\xi)} \otimes W_{(\xi)}) + \rho \mathrm{Id}.$$

The incompressible corrector $w_{q+1}^{(c)}$ is therefore defined as in (4.32) again with $a_{(\xi)}$ replaced by $\bar{a}_{(\xi)}$. The temporal corrector $w_{q+1}^{(t)}$ is now defined as in (4.33) with $a_{(\xi)}$ given in (4.26) for $M_0(t)$ from (6.6). Note that for the temporal corrector we use the original amplitude functions $a_{(\xi)}$ from Section 4.1.3 (only using a different function $M_0(t)$), since we need the extra θ_ℓ to obtain a suitable cancelation. The total velocity increment w_{q+1} and the new velocity v_{q+1} are then given by

$$w_{q+1} := w_{q+1}^{(p)} + w_{q+1}^{(c)} + w_{q+1}^{(t)}, \qquad v_{q+1} := v_\ell + w_{q+1}.$$

Both are $(\mathcal{F}_t)_{t\geq 0}$-adapted, divergence free and w_{q+1} is mean zero. If $v_q(0)$, $\mathring{R}_\ell(0)$ are deterministic, so is $v_{q+1}(0)$.

6.1.4. *Verification of the inductive estimates for v_{q+1}.* For the counterparts of the estimates (4.37)-(4.47), the main difference now is the extra m_L appearing in the bounds (6.22) and (6.23) for $\bar{a}_{(\xi)}$. Therefore, many of the estimates remain valid with $M_0(t)^{1/2}$ replaced by $m_L M_0(t)^{1/2}$, only the bounds for the temporal corrector $w_{q+1}^{(t)}$ do not change. More precisely, we obtain for $t \in [0, T_L]$

$$\|w_{q+1}^{(p)}\|_{C_tL^2} \leq \frac{1}{2} m_L M_0(t)^{1/2} \delta_{q+1}^{1/2}, \tag{6.24}$$

$$\|w_{q+1}^{(p)}\|_{C_tL^p} \lesssim m_L M_0(t)^{1/2} \delta_{q+1}^{1/2} \ell^{-2} r_\perp^{2/p-1} r_\parallel^{1/p-1/2}, \tag{6.25}$$

$$\|w_{q+1}^{(c)}\|_{C_tL^p} \lesssim m_L M_0(t)^{1/2} \delta_{q+1}^{1/2} \ell^{-12} r_\perp^{2/p} r_\parallel^{1/p-3/2}, \tag{6.26}$$

$$\|w_{q+1}^{(t)}\|_{C_tL^p} \lesssim M_0(t) \delta_{q+1} \ell^{-4} r_\perp^{2/p-1} r_\parallel^{1/p-2} \lambda_{q+1}^{-1}. \tag{6.27}$$

Combining (6.24), (6.26) and (6.27) then leads to

$$\|w_{q+1}\|_{C_tL^2} \leq m_L M_0(t)^{1/2} \delta_{q+1}^{1/2} \left(\frac{1}{2} + C\lambda_{q+1}^{24\alpha-2/7} + C M_0(t)^{1/2} \delta_{q+1}^{1/2} \lambda_{q+1}^{8\alpha-1/7} \right) \leq \frac{3}{4} m_L M_0(t)^{1/2} \delta_{q+1}^{1/2}, \tag{6.28}$$

where we used (6.16) to bound $C M_0(L)^{1/2} \delta_{q+1}^{1/2} \lambda_{q+1}^{8\alpha-1/7} \leq 1/8$.

As a consequence of (6.28) and (6.18), the first bound in (6.7) on the level $q+1$ readily follows. In addition, (6.28) together with (6.17) implies (6.13) from the statement of the proposition. In order to verify the second bound in (6.7), we observe that similarly to (4.43)-(4.45) it holds for $t \in [0, T_L]$

$$\|w_{q+1}^{(p)}\|_{C^1_{t,x}} \lesssim m_L M_0(t)^{1/2} \ell^{-7} r_\perp^{-1} r_\parallel^{-1/2} \lambda_{q+1}^2, \tag{6.29}$$

$$\|w_{q+1}^{(c)}\|_{C^1_{t,x}} \lesssim m_L M_0(t)^{1/2} \ell^{-17} r_\parallel^{-3/2} \lambda_{q+1}^2, \tag{6.30}$$

$$\|w_{q+1}^{(t)}\|_{C^1_{t,x}} \lesssim M_0(t) \ell^{-9} r_\perp^{-1} r_\parallel^{-2} \lambda_{q+1}^{1+\alpha}. \tag{6.31}$$

Combining (6.29), (6.30), (6.31) with (6.19) and taking (6.16) into account, the second bound in (6.7) follows.

In order to control the Reynolds stress below, we observe that similarly to (4.46), (4.47), the following bounds hold true for $t \in [0, T_L]$, $p \in (1, \infty)$

$$\|w_{q+1}^{(p)} + w_{q+1}^{(c)}\|_{C_tW^{1,p}} \leq m_L M_0(t)^{1/2} r_\perp^{2/p-1} r_\parallel^{1/p-1/2} \ell^{-2} \lambda_{q+1}, \tag{6.32}$$

$$\|w_{q+1}^{(t)}\|_{C_tW^{1,p}} \leq M_0(t) r_\perp^{2/p-2} r_\parallel^{1/p-1} \ell^{-4} \lambda_{q+1}^{-2/7}. \tag{6.33}$$

6.1.5. *Definition of the Reynolds Stress $\mathring{R}_{q+1}$.* Similar as before we know

$$\begin{aligned}
\mathrm{div}\mathring{R}_{q+1} - \nabla p_{q+1} = &\underbrace{\frac{1}{2}w_{q+1} - \Delta w_{q+1} + \partial_t(w^{(p)}_{q+1} + w^{(c)}_{q+1}) + \theta_\ell \mathrm{div}(v_\ell \otimes w_{q+1} + w_{q+1} \otimes v_\ell)}_{\mathrm{div}(R_{\mathrm{lin}})+\nabla p_{\mathrm{lin}}} \\
&+ \underbrace{\theta_\ell \mathrm{div}\Big((w^{(c)}_{q+1} + w^{(t)}_{q+1}) \otimes w_{q+1} + w^{(p)}_{q+1} \otimes (w^{(c)}_{q+1} + w^{(t)}_{q+1})\Big)}_{\mathrm{div}(R_{\mathrm{cor}}+\nabla p_{\mathrm{cor}})} \\
&+ \underbrace{\mathrm{div}(\theta_\ell w^{(p)}_{q+1} \otimes w^{(p)}_{q+1} + \mathring{R}_\ell) + \partial_t w^{(t)}_{q+1}}_{\mathrm{div}(R_{\mathrm{osc}})+\nabla p_{\mathrm{osc}}} \\
&+ \underbrace{(\theta - \theta_\ell)\mathrm{div}\,(v_{q+1} \mathring{\otimes} v_{q+1})}_{\mathrm{div}(R_{\mathrm{com1}})+\nabla p_{\mathrm{com1}}} + \mathrm{div}(R_{\mathrm{com}}) - \nabla p_\ell.
\end{aligned}$$

Therefore, applying the inverse divergence operator $\mathcal{R}$ we define

$$R_{\mathrm{lin}} := \frac{1}{2}\mathcal{R}w_{q+1} - \mathcal{R}\Delta w_{q+1} + \mathcal{R}\partial_t(w^{(p)}_{q+1} + w^{(c)}_{q+1}) + \theta_\ell v_\ell \mathring{\otimes} w_{q+1} + \theta_\ell w_{q+1} \mathring{\otimes} v_\ell,$$

$$R_{\mathrm{cor}} := \theta_\ell(w^{(c)}_{q+1} + w^{(t)}_{q+1}) \mathring{\otimes} w_{q+1} + \theta_\ell w^{(p)}_{q+1} \mathring{\otimes} (w^{(c)}_{q+1} + w^{(t)}_{q+1}),$$

$$R_{\mathrm{com1}} := (\theta_\ell - \theta)(v_{q+1} \mathring{\otimes} v_{q+1}).$$

And similarly to Section 4.1.5 we have

$$R_{\mathrm{osc}} := \sum_{\xi\in\Lambda} \mathcal{R}\left(\nabla a^2_{(\xi)} \mathbb{P}_{\neq 0}(W_{(\xi)} \otimes W_{(\xi)})\right) - \frac{1}{\mu}\sum_{\xi\in\Lambda} \mathcal{R}\left(\partial_t a^2_{(\xi)} (\phi^2_{(\xi)} \psi^2_{(\xi)} \xi)\right),$$

with $a_{(\xi)}$ given in (4.26) for $M_0(t)$ from (6.6). Hence the bounds for R_{osc} are the same as in Section 4.1.6. The Reynolds stress on the level $q+1$ is then defined as

$$\mathring{R}_{q+1} := R_{\mathrm{lin}} + R_{\mathrm{osc}} + R_{\mathrm{cor}} + R_{\mathrm{com}} + R_{\mathrm{com1}}.$$

6.1.6. *Verification of the inductive estimate for $\mathring{R}_{q+1}$.* In the following we estimate the remaining terms in $\mathring{R}_{q+1}$ separately. We choose $p = \frac{32}{32-7\alpha} > 1$.

For the linear error, have for $t \in [0, T_L]$

$$\begin{aligned}
\|R_{\mathrm{linear}}\|_{C_t L^p} &\lesssim \|w_{q+1}\|_{C_t W^{1,p}} + \|\mathcal{R}\partial_t(w^{(p)}_{q+1} + w^{(c)}_{q+1})\|_{C_t L^p} + m_L^2 \|v_\ell \mathring{\otimes} w_{q+1} + w_{q+1} \mathring{\otimes} v_\ell\|_{C_t L^p} \\
&\lesssim \|w_{q+1}\|_{C_t W^{1,p}} + \sum_{\xi\in\Lambda} \|\partial_t \mathrm{curl}(\bar{a}_{(\xi)} V_{(\xi)})\|_{C_t L^p} + \lambda_q^4 m_L^3 M_0(t)^{1/2} \|w_{q+1}\|_{C_t L^p}.
\end{aligned}$$

Hence using (6.32), (6.33), (6.23), (B.7), (6.25), (6.26) and (6.27) we have for $t \in [0, T_L]$

$$\begin{aligned}
\|R_{\mathrm{linear}}\|_{C_t L^p} \lesssim\, & m_L M_0(t)^{1/2} \ell^{-2} r_\perp^{2/p-1} r_\parallel^{1/p-1/2} \lambda_{q+1} + M_0(t) \ell^{-4} r_\perp^{2/p-2} r_\parallel^{1/p-1} \lambda_{q+1}^{-2/7} \\
&+ m_L M_0(t)^{1/2} \ell^{-7} r_\perp^{2/p} r_\parallel^{1/p-3/2} \mu + m_L M_0(t)^{1/2} \ell^{-12} r_\perp^{2/p-1} r_\parallel^{1/p-1/2} \lambda_{q+1}^{-1} \\
&+ m_L^4 M_0(t) \ell^{-2} r_\perp^{2/p-1} r_\parallel^{1/p-1/2} \lambda_q^4 \\
\lesssim\, & m_L M_0(t)^{1/2} \lambda_{q+1}^{5\alpha-1/7} + M_0(t) \lambda_{q+1}^{9\alpha-2/7} + m_L M_0(t)^{1/2} \lambda_{q+1}^{15\alpha-1/7} \\
&+ m_L M_0(t)^{1/2} \lambda_{q+1}^{25\alpha-15/7} + m_L^4 M_0(t) \lambda_{q+1}^{7\alpha-8/7} \\
\leq\, & \frac{M_0(t) c_R \delta_{q+2}}{5},
\end{aligned}$$

where we used the fact that a is sufficiently large and β is suffiently small, in particular, (6.16) holds.

The corrector error is estimated by (6.25), (6.26) and (6.27) for $t\in[0,T_L]$ as follows

$$\begin{aligned}
\|R_{\rm cor}\|_{C_tL^p} &\leq m_L^2\|w^{(c)}_{q+1}+w^{(t)}_{q+1}\|_{C_tL^{2p}}\|w_{q+1}\|_{C_tL^{2p}}+m_L^2\|w^{(c)}_{q+1}+w^{(t)}_{q+1}\|_{C_tL^{2p}}\|w^{(p)}_{q+1}\|_{C_tL^{2p}}\\
&\lesssim m_L^4M_0(t)\left(\ell^{-12}r_\perp^{1/p}r_\parallel^{1/(2p)-3/2}+\ell^{-4}M_0(t)^{1/2}r_\perp^{1/p-1}r_\parallel^{1/(2p)-2}\lambda_{q+1}^{-1}\right)\ell^{-2}r_\perp^{1/p-1}r_\parallel^{1/(2p)-1/2}\\
&\lesssim m_L^4M_0(t)\left(\ell^{-14}r_\perp^{2/p-1}r_\parallel^{1/p-2}+\ell^{-6}M_0(t)^{1/2}r_\perp^{2/p-2}r_\parallel^{1/p-5/2}\lambda_{q+1}^{-1}\right)\\
&\leq m_L^4M_0(t)\left(\lambda_{q+1}^{29\alpha-2/7}+M_0(t)^{1/2}\lambda_{q+1}^{13\alpha-1/7}\right)\leq\frac{M_0(t)c_R\delta_{q+2}}{5},
\end{aligned}$$

where we used again (6.16) to have $m_L^4\lambda_{q+1}^{29\alpha-2/7}+m_L^4M_0(L)^{1/2}\lambda_{q+1}^{13\alpha-1/7}\leq\frac{c_R\delta_{q+2}}{5}$.

In view of a standard mollification estimate we deduce that for $t\in[0,T_L]$

$$\begin{aligned}
|\theta_\ell(t)-\theta(t)|&\leq\ell^{1/2-2\delta}L^{1/2}e^{L^{1/2}}\leq\ell^{1/2-2\delta}m_L^2,\\
\|R_{\rm com}\|_{C_tL^1}&\lesssim m_L^2\ell\|v_q\|_{C^1_{t,x}}\|v_q\|_{C_tL^2}+\ell^{\frac12-2\delta}m_L^4M_0(t)\lambda_q^4\\
&\lesssim\ell^{\frac12-2\delta}m_L^4M_0(t)\lambda_q^4\leq\frac{M_0(t)c_R\delta_{q+2}}{5},
\end{aligned}$$

where $\delta\in(0,1/12)$ and we choose a large enough to have

$$C\ell^{\frac12-2\delta}m_L^4\lambda_q^4<\frac{c_R}{5}\lambda_{q+2}^{-2\beta}. \tag{6.34}$$

With the choice of ℓ and since we postulated that $\alpha>8\beta b$ and $\alpha b>16$, this can indeed be achieved by possibly increasing a and consequently decreasing β.

The second commutator error can be estimated for $t\in[0,T_L]$ as follows

$$\|R_{\rm com1}\|_{C_tL^1}\leq\ell^{1/2-2\delta}m_L^4M_0(t)\leq\frac{M_0(t)c_R\delta_{q+2}}{5},$$

where we used (6.34) to have $\ell^{1/2-2\delta}m_L^4<\frac{c_R}{5}\delta_{q+2}$.

Thus, collecting the above estimates we obtain the desired third bound in (6.7) and the proof of Proposition 6.2 is complete.

Appendix A. Proof of Theorem 3.1 and Theorem 5.1

Let us begin with the following tightness result.

Lemma A.1. *Let $\{(s_n,x_n)\}_{n\in\mathbb{N}}\subset[0,\infty)\times L^2_\sigma$ such that $(s_n,x_n)\to(s,x_0)$. Let $\{P_n\}_{n\in\mathbb{N}}$ be a family of probability measures on Ω_0 satisfying for all $n\in\mathbb{N}$*

$$P_n(x(t)=x_n,0\leq t\leq s_n)=1 \tag{A.1}$$

and for some $\gamma,\kappa>0$ and any $T>0$

$$\sup_{n\in\mathbb{N}}E^{P_n}\left(\sup_{t\in[0,T]}\|x(t)\|_{L^2}+\sup_{r\neq t\in[0,T]}\frac{\|x(t)-x(r)\|_{H^{-3}}}{|t-r|^\kappa}+\int_{s_n}^T\|x(r)\|^2_{H^\gamma}dr\right)<\infty. \tag{A.2}$$

Then $\{P_n\}_{n\in\mathbb{N}}$ is tight in $\mathbb{S}:=C_{\rm loc}([0,\infty);H^{-3})\cap L^2_{\rm loc}([0,\infty);L^2_\sigma)$.

Proof. In view of the uniform bound (A.2), the canonical process under the measure P_n is bounded in $L^\infty_{\rm loc}([0,\infty);L^2)\cap C^\kappa_{\rm loc}([0,\infty);H^{-3})\cap L^2_{\rm loc}([s_n,\infty);H^\gamma)$ and the bounds are uniform in n. We recall that a set $K\subset\mathbb{S}$ is compact provided

$$K_T:=\{f|_{[0,T]};f\in K\}\subset C([0,T];H^{-3})\cap L^2(0,T;L^2_\sigma)$$

is compact for every $T>0$. In addition, for every $T>0$, the following embedding

$$L^\infty(0,T;L^2)\cap C^\kappa([0,T];H^{-3})\cap L^2([0,T];H^\gamma)\subset C([0,T];H^{-3})\cap L^2(0,T;L^2_\sigma)$$

is compact, see e.g. [BFH18, Section 1.8.2]. This implies that also the embedding of the local-in-time spaces

$$L^\infty_{\rm loc}([0,\infty);L^2)\cap C^\kappa_{\rm loc}([0,\infty);H^{-3})\cap L^2_{\rm loc}([0,\infty);H^\gamma)\subset \mathbb{S}$$

is compact as well. This result, however, cannot be applied directly in order to prove the claim of the lemma due to the fact that the uniform H^γ regularity in (A.2) only holds on the respective time intervals $[s_n,T]$. The idea is instead to use (A.1) which says that under each measure P_n the canonical process is constant on $[0,s_n]$ and its value equals to x_n. Together with the fact that $(s_n,x_n)\to(s,x_0)$ in $[0,\infty)\times L^2_\sigma$, the desired compactness then follows.

To be more precise, we fix $\epsilon>0$ and any $k\in\mathbb{N}$, $k\geq k_0:=\sup_{n\in\mathbb{N}}s_n$, we may choose $R_k>0$ sufficiently large such that

$$P_n\left(x\in\Omega_0:\sup_{t\in[0,k]}\|x(t)\|_{L^2}+\sup_{r\neq t\in[0,k]}\frac{\|x(t)-x(r)\|_{H^{-3}}}{|t-r|^\kappa}+\int_{s_n}^k\|x(r)\|^2_{H^\gamma}dr>R_k\right)\leq\epsilon/2^k.$$

Now, we set $\Omega_n:=\{x\in\Omega_0;x(t)=x_n,0\leq t\leq s_n\}$ and define

$$K:=\bigcup_{n\in\mathbb{N}}\bigcap_{\substack{k\in\mathbb{N}\\k\geq k_0}}\left\{x\in\Omega_n;\sup_{t\in[0,k]}\|x(t)\|_{L^2}+\sup_{r\neq t\in[0,k]}\frac{\|x(t)-x(r)\|_{H^{-3}}}{|t-r|^\kappa}+\int_{s_n}^k\|x(r)\|^2_{H^\gamma}dr\leq R_k\right\}. \tag{A.3}$$

By Chebyshev's inequality together with (A.2), it follows that

$$\sup_{n\in\mathbb{N}}P_n(\overline{K}^c)\leq\sup_{n\in\mathbb{N}}P_n(K^c)\leq\epsilon,$$

so it only remains to show that the set $\overline{K}$ is a compact set in $\mathbb{S}$. As mentioned above, it is sufficient to prove that for every $k\in\mathbb{N}$, $k\geq k_0$, the restriction of functions in K to $[0,k]$ is relatively compact in $\mathbb{S}_k:=C([0,k],H^{-3})\cap L^2(0,k;L^2_\sigma)$.

To this end, let $\{x_m\}_{m\in\mathbb{N}}$ be a sequence in K. If there exists $N\in\mathbb{N}$ so that for infinitely many m it holds $x_m\in\Omega_N$, the result can be obtained by a standard argument based on the compact embedding discussed above. If this is not true, we may assume without loss of generality that $x_m\in\Omega_m$. The compactness in $C([0,k];H^{-3})$ is a direct consequence of the bound

$$\sup_{t\in[0,k]}\|x_m(t)\|_{L^2}+\sup_{r\neq t\in[0,k]}\frac{\|x_m(t)-x_m(r)\|_{H^{-3}}}{|t-r|^\kappa}\leq R_k$$

and the compact embedding

$$L^\infty(0,k;L^2)\cap C^\kappa([0,k];H^{-3})\subset C([0,k];H^{-3}).$$

Consequently, we can find a subsequence x_{m_l} such that

$$\lim_{l,n\to\infty}\sup_{t\in[0,k]}\|x_{m_l}(t)-x_{m_n}(t)\|_{H^{-3}}=0. \tag{A.4}$$

With this in hand, we deduce

$$\begin{aligned}\int_0^k \|x_{m_l}(t)-x_{m_n}(t)\|_{L^2}^2dt &\leq \int_0^{s_{m_l}\wedge s_{m_n}} \|x_{m_l}(t)-x_{m_n}(t)\|_{L^2}^2dt\\ &\quad+\int_{s_{m_l}\wedge s_{m_n}}^{s_{m_l}\vee s_{m_n}} \|x_{m_l}(t)-x_{m_n}(t)\|_{L^2}^2dt+\int_{s_{m_l}\vee s_{m_n}}^{k} \|x_{m_l}(t)-x_{m_n}(t)\|_{L^2}^2dt\\ &\leq k\|x_{m_l}(0)-x_{m_n}(0)\|_{L^2}^2+4R_k^2(s_{m_l}\vee s_{m_n}-s_{m_l}\wedge s_{n_m})\\ &\quad+\varepsilon\int_{s_{m_l}\vee s_{m_n}}^{k} \|x_{m_l}(t)-x_{m_n}(t)\|_{H^\gamma}^2dt+C_\varepsilon k\sup_{t\in[0,k]}\|x_{m_l}(t)-x_{m_n}(t)\|_{H^{-3}}^2\\ &\leq k\|x_{m_l}(0)-x_{m_n}(0)\|_{L^2}^2+4R_k^2(s_{m_l}\vee s_{m_n}-s_{m_l}\wedge s_{m_n})\\ &\quad+4\varepsilon R_k+C_\varepsilon k\sup_{t\in[0,k]}\|x_{m_l}(t)-x_{m_n}(t)\|_{H^{-3}}^2\to 0,\end{aligned}$$

as $m_l, m_n \to \infty$ and we used interpolation and Young's inequality in the second step and we used (A.4) in the last step. Now the proof is complete. □

Proof of Theorem 3.1. The first result giving existence of a martingale solution can be easily deduced by Galerkin approximation and the same arguments as in [FR08, GRZ09]. The second result giving the stability of martingale solutions with respect to the initial time and initial condition will be proved in the sequel based on Lemma A.1.

First, we prove that the set $\{P_n\}_{n\in\mathbb{N}}$ is tight in $\mathbb{S}:=C_{\rm loc}([0,\infty);H^{-3})\cap L^2_{\rm loc}([0,\infty);L^2_\sigma)$. To this end, we denote $F(x):=-\mathbb{P}{\rm div}(x\otimes x)+\Delta x$. Since for every $n\in\mathbb{N}$, the measure P_n is a martingale solution to (1.1) starting from the initial condition x_n at time s_n in the sense of Definition 3.1, we know that for $t\in[s_n,\infty)$

$$x(t)=x_n+\int_{s_n}^t F(x(r))dr+M^x_{t,s_n},\quad P_n\text{-a.s.},$$

where $t\mapsto M^{x,i}_{t,s_n}=\langle M^x_{t,s_n},e_i\rangle$, $x\in\Omega_0$, is a continuous square integrable martingale with respect to $(\mathcal{B}_t)_{t\geq s_n}$ with the quadratic variation process given by $t\mapsto\int_{s_n}^t\|G(x(r))^*e_i\|_U^2dr$. Moreover, according to (M3) it holds for every $p>1$

$$\begin{aligned}E^{P_n}\left[\sup_{r\neq t\in[s_n,T]}\frac{\|\int_r^tF(x(l))dl\|_{H^{-3}}^p}{|t-r|^{p-1}}\right]&\leq E^{P_n}\left[\int_{s_n}^t\|F(x(r))\|_{H^{-3}}^pdr\right]\\ &\lesssim\|x_n\|_{L^2}^{2p}+1,\end{aligned}$$

where the implicit constant is universal and therefore independent of n since all P_n share the same $C_{t,q}$. By the condition on G we have for every $p>1$

$$\begin{aligned}E^{P_n}\|M_{t,s_n}-M_{r,s_n}\|_{L^2}^{2p}&\leq C_pE^{P_n}\left(\int_r^t\|G(x(l))\|^2_{L_2(U,L^2_\sigma)}dl\right)^p\leq C_p|t-r|^{p-1}E^{P_n}\int_r^t\|G(x(l))\|^{2p}_{L_2(U,L^2_\sigma)}dl\\ &\leq C_p|t-r|^{p-1}E^{P_n}\int_r^t(\|x(l)\|_{L^2}^{2p}+1)dl\leq C_p|t-r|^{p-1}(\|x_n\|_{L^2}^{2p}+1).\end{aligned}$$

By Kolmogorov's criterion, for any $\alpha\in(0,\frac{p-1}{2p})$ we get

$$E^{P_n}\left[\sup_{r\neq t\in[0,T]}\frac{\|M_{t,s_n}-M_{r,s_n}\|_{L^2}}{|t-r|^{p\alpha}}\right]\leq C_p(\|x_n\|_{L^2}^{2p}+1).$$

Combining the above estimates, we conclude for all $\kappa \in (0,1/2)$ that

$$\sup_{n\in\mathbb{N}} E^{P_n}\left[\sup_{r\neq t\in[0,T]}\frac{\|x(t)-x(r)\|_{H^{-3}}}{|t-r|^{\kappa}}\right]<\infty. \tag{A.5}$$

Combining (A.5), (M3) and using Lemma A.1 it follows that the set $\{P_n\}_{n\in\mathbb{N}}$ is tight in $\mathbb{S}$.

Without loss of generality, we may assume that P_n converges weakly to some probability measure $P \in \mathscr{P}(\Omega_0)$. It remains to prove that $P \in \mathscr{C}(s, x_0, C_{t,q})$. By Skorohod's representation theorem, there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and $\mathbb{S}$-valued random variables $\tilde{x}_n$ and $\tilde{x}$ such that

(i) $\tilde{x}_n$ has the law P_n for each $n \in \mathbb{N}$,

(ii) $\tilde{x}_n \to \tilde{x}$ in $\mathbb{S}$ $\tilde{P}$-a.s., and $\tilde{x}$ has the law P.

Since the initial conditions x_n as well as the initial times s_n are deterministic, we obtain by (i), (ii), and (M1) applied to P_n that

$$\begin{aligned}P(x(t)=x_0, 0\leq t\leq s) &= \tilde{P}(\tilde{x}(t)=x_0, 0\leq t\leq s) = \lim_{n\to\infty}\tilde{P}(\tilde{x}_n(t)=x_n, 0\leq t\leq s_n)\\ &= \lim_{n\to\infty} P_n(x(t)=x_n, 0\leq t\leq s_n)=1.\end{aligned}$$

As the next step, we verify (M2) for P. We know that under $\tilde{P}$ it holds according to the convergence in (ii) that for every $e_i \in C^\infty(\mathbb{T}^3)$

$$\langle \tilde{x}_n(t), e_i\rangle \to \langle \tilde{x}(t), e_i\rangle, \qquad \int_{s_n}^t \langle F(\tilde{x}_n(r)), e_i\rangle dr \to \int_s^t \langle F(\tilde{x}(r)), e_i\rangle dr \qquad \tilde{P}\text{-a.s.}$$

This implies for every $t \in [s,\infty)$ and every $p>1$

$$\sup_{n\in\mathbb{N}} E^{\tilde{P}}\big[|M^{\tilde{x}_n,i}_{t,s_n}|^{2p}\big] \leq C\sup_{n\in\mathbb{N}} E^{P_n}\left[\left(\int_{s_n}^t \|G(x(r))\|^2_{L_2(U,L^2_\sigma)}ds\right)^p\right]<\infty,$$

$$\lim_{n\to\infty} E^{\tilde{P}}\big[|M^{\tilde{x}_n,i}_{t,s_n} - M^{\tilde{x},i}_{t,s}|\big]=0. \tag{A.6}$$

Let $t > r \geq s$ and g be any bounded and real-valued $\mathcal{B}_r$-measurable continuous function on $\mathbb{S}$. Using (A.6) we know

$$\begin{aligned}E^P\big[(M^{x,i}_{t,s}-M^{x,i}_{r,s})g(x)\big] &= E^{\tilde{P}}\big[(M^{\tilde{x},i}_{t,s}-M^{\tilde{x},i}_{r,s})g(\tilde{x})\big] = \lim_{n\to\infty}E^{\tilde{P}}\big[(M^{\tilde{x}_n,i}_{t,s_n}-M^{\tilde{x}_n,i}_{r,s_n})g(\tilde{x}_n)\big]\\ &= \lim_{n\to\infty}E^{P_n}\big[(M^{x,i}_{t,s_n}-M^{x,i}_{r,s_n})g(x)\big]=0.\end{aligned}$$

Consequently, we deduce

$$E^P\big[M^{x,i}_{t,s}|\mathcal{B}_r\big] = M^{x,i}_{r,s}$$

hence $t\mapsto M^i_{t,s}$ is a $(\mathcal{B}_t)_{t\geq s}$-martingale under P. Similarly, we have

$$\lim_{n\to\infty} E^{\tilde{P}}\big[|M^{\tilde{x}_n,i}_{t,s_n} - M^{\tilde{x},i}_{t,s}|^2\big]=0,$$

which gives

$$E^P\big[(M^{x,i}_{t,s})^2 - \int_s^t \|G(x(l))^*e_i\|_U^2 dl|\mathcal{B}_r\big] = (M^{x,i}_{r,s})^2 - \int_r^t \|G(x(l))^*e_i\|_U^2 dl|$$

and accordingly (M2) follows.

Finally, we verify (M3). Define

$$S(t,s,x) := \sup_{r\in[0,t]}\|x(r)\|^{2q}_{L^2} + \int_s^t \|x(r)\|^2_{H^\gamma}dr,$$

It is easy to see that $x \mapsto S(t,s,x)$ is lower semicontinuous on $\mathbb{S}$. Hence, by Fatou's lemma

$$E^P[S(t,s,x)] = E^{\tilde{P}}[S(t,s,\tilde{x})] \le \liminf_{n\to\infty} E^{\tilde{P}}[S(t,s_n,\tilde{x}_n)] \le C_{t,q} \liminf_{n\to\infty}(\|x_n\|_{L^2}^{2q}+1) < \infty.$$

The proof is complete. □

Proof of Theorem 5.1. The existence of a probabilistically weak solution can be easily deduced from Theorem 3.1 and the martingale representation theorem, see [DPZ92]. The stability of weak solutions with respect to the initial time and initial condition will be proved in a similar way as in Theorem 3.1. First, we prove that the set $(P_n)_{n\in\mathbb{N}}$ is tight in

$$\bar{\mathbb{S}} := C_{\text{loc}}([0,\infty); H^{-3}\times U_1) \cap L^2_{\text{loc}}([0,\infty); L^2_\sigma \times U_1).$$

To this end, we denote $F(x) := -\mathbb{P}\text{div}(x\otimes x) + \Delta x$ and recall that for every $n\in\mathbb{N}$, the measure P_n is a probabilistically weak solution to (1.1) starting from the initial condition x_n at time s_n in the sense of Definition 5.1. Thus, for $t\in[s_n,\infty)$

$$x(t) = x_n + \int_{s_n}^t F(x(r))dr + \int_{s_n}^t G(x_r)dy_r, \quad P_n\text{-a.s.}$$

where under P_n the process y is a cylindrical Wiener process on U starting from y_n at time s_n. In other words, under P_n the process $t\mapsto y(t+s_n) - y_n$ is a cylindrical Wiener process on U starting at time 0 from the initial value 0. Since the law of the Wiener process is unique and tight, for a given $\epsilon>0$ there exists a compact set $K_1 \subset C([0,\infty);U_1)\cap L^2_{\text{loc}}([0,\infty);U_1)$ such that

$$\sup_{n\in\mathbb{N}} P_n\big(y(\cdot+s_n) - y_n \in K_1^c\big) \le \epsilon.$$

Let us now define

$$K_2 := \bigcup_{n\in\mathbb{N}} \big\{y\in C([0,\infty);U_1); \\ y(t+s_n)-y_n \in K_1 \text{ for } t\in[0,\infty),\ y(t)=y_n \text{ for } t\in[0,s_n]\big\}.$$

Then

$$\sup_{n\in\mathbb{N}} P_n(\overline{K_2}^c) \le \sup_{n\in\mathbb{N}} P_n\big(y(\cdot+s_n)-y_n\in K_1^c\big) \le \epsilon \tag{A.7}$$

and we claim that K_2 is relatively compact in $C([0,\infty);U_1)\subset L^2_{\text{loc}}([0,\infty);U_1)$. Indeed, let $\{y^m\}_{m\in\mathbb{N}}$ be a sequence in K_2. Then for every $m\in\mathbb{N}$ there exists $n_m\in\mathbb{N}$ and $y^{m,n_m}\in K_1$ so that

$$y^m(t+s_{n_m}) - y_{n_m} = y^{m,n_m}(t) \quad \text{for } t\in[0,\infty), \qquad y^m(t) = y_{n_m} \quad \text{for } t\in[0,s_{n_m}].$$

If there exists $N\in\mathbb{N}$ such that $n_m = N$ for infinitely many $m\in\mathbb{N}$ then the relative compactness of $\{y^m\}_{m\in\mathbb{N}}$ follows directly from the fact that the corresponding sequence $\{y^{m,n_m}\}_{m\in\mathbb{N}}$ is relatively compact due to compactness of K_1. If such an N does not exist, then by passing to a subsequence and relabelling we can assume without loss of generality that $n_m = m$. In addition, it holds for $t\in[s_n,\infty)$

$$y^m(t) = y^{m,m}(t-s_m) + y_m.$$

Hence using the relative compactness of

$$\{y^{m,m}\}_{m\in\mathbb{N}} \subset K_1 \quad \text{and} \quad \{(s_m,y_m)\}_{m\in\mathbb{N}} \subset [0,\infty)\times U_1,$$

we finally deduce that the given sequence $\{y^m\}_{m\in\mathbb{N}}$ is relatively compact.

Now, we recall that the set K defined in the course of the proof of Theorem 3.1 in (A.3) is relatively compact in $C_{\text{loc}}([0,\infty);H^{-3})\cap L^2_{\text{loc}}([0,\infty);L^2_\sigma)$. Chebyshev's inequality again shows that

$$\sup_{n\in\mathbb{N}} P_n(\overline{K}^c) \le \sup_{n\in\mathbb{N}} P_n(K^c) \le \epsilon. \tag{A.8}$$

Hence the set $K \times K_2$ is relatively compact in $\bar{\mathbb{S}}$ and the desired tightness follows from (A.7), (A.8).

Without loss of generality, we may assume that P_n converges weakly to some probability measure P. It remains to prove that $P \in \mathscr{W}(s, x_0, y_0, C_{t,q})$. By Skorokhod's representation theorem, there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and $\bar{\mathbb{S}}$-valued random variables $(\tilde{x}_n, \tilde{y}_n)$ and $(\tilde{x}, \tilde{y})$ such that

(i) $(\tilde{x}_n, \tilde{y}_n)$ has the law P_n for each $n \in \mathbb{N}$,

(ii) $(\tilde{x}_n, \tilde{y}_n) \to (\tilde{x}, \tilde{y})$ in $\bar{\mathbb{S}}$ $\tilde{P}$-a.s., and $(\tilde{x}, \tilde{y})$ has the law P.

Let $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ be the $\tilde{P}$-augmented canonical filtration of the process $(\tilde{x}, \tilde{y})$. Then it is easy to see that $\tilde{y}$ is a cylindrical Wiener process on U with respect to $(\tilde{\mathcal{F}}_t)_{t\geq 0}$. The conditions (M1) and (M3) follow similarly as in the proof of Theorem 3.1. Finally, we shall verify (M2) for P. We know that under $\tilde{P}$ it holds according to the convergence in (ii) that for every $e_i \in C^\infty(\mathbb{T}^3)$

$$\langle \tilde{x}_n(t), e_i\rangle \to \langle \tilde{x}(t), e_i\rangle, \quad \int_{s_n}^t \langle F(\tilde{x}_n(r)), e_i\rangle dr \to \int_s^t \langle F(\tilde{x}(r)), e_i\rangle dr \qquad \tilde{P}\text{-a.s.}$$

Define

$$M_{t,s}^{x,i} = \langle x(t) - x(s) - \int_s^t F(x(r))dr, e_i\rangle.$$

Then we have for every $t \in [s, \infty)$ and every $p \in (1, \infty)$

$$\sup_{n\in\mathbb{N}} E^{\tilde{P}}\big[|M_{t,s_n}^{\tilde{x}_n,i}|^{2p}\big] \leq C \sup_{n\in\mathbb{N}} E^{P_n}\left[\left(\int_{s_n}^t \|G(x(r))\|^2_{L_2(U,L^2_\sigma)} ds\right)^p\right] < \infty,$$

$$\lim_{n\to\infty} E^{\tilde{P}}\big[|M_{t,s_n}^{\tilde{x}_n,i} - M_{t,s}^{\tilde{x},i}|^2\big] = 0. \tag{A.9}$$

Let $t > r \geq s$ and g be any bounded continuous function on $\bar{\mathbb{S}}$. Using (A.9) we know

$$\begin{aligned}
&E^P\big[(M_{t,s}^{x,i} - M_{r,s}^{x,i})g(x|_{[0,r]}, y|_{[0,r]})\big] = E^{\tilde{P}}\big[(M_{t,s}^{\tilde{x},i} - M_{r,s}^{\tilde{x},i})g(\tilde{x}|_{[0,r]}, \tilde{y}|_{[0,r]})\big]\\
&= \lim_{n\to\infty} E^{\tilde{P}}\big[(M_{t,s_n}^{\tilde{x}_n,i} - M_{r,s_n}^{\tilde{x}_n,i})g(\tilde{x}_n|_{[0,r]}, \tilde{y}_n|_{[0,r]})\big] = \lim_{n\to\infty} E^{P_n}\big[(M_{t,s_n}^{x,i} - M_{r,s_n}^{x,i})g(x|_{[0,r]}, y|_{[0,r]})\big] = 0.
\end{aligned}$$

Consequently, we deduce that $t \mapsto M_{t,s}^i$ is a $(\bar{\mathcal{B}}_t)_{t\geq s}$-martingale under P. Similarly, we obtain

$$E^P\left[(M_{t,s}^{x,i})^2 - \int_s^t \|G(x)^* e_i\|_U^2 dl \Big| \bar{\mathcal{B}}_r\right] = (M_{r,s}^{x,i})^2 - \int_s^r \|G(x)^* e_i\|_U^2 dl,$$

which identifies the quadratic variation of $t \mapsto M_{t,s}^i$. It remains to identify the cross variation of this process with the cylindrical Wiener process y under P. To this end, we let $\{l_j\}_{j\in\mathbb{N}}$ be an orthonormal basis of U and define $y_j = \langle y, l_j\rangle_U$. Then we deduce that

$$E^P\left[M_{t,s}^{x,i}(y_j(t) - y_j(s)) - \int_s^t \langle G^*(x)e_i, l_j\rangle_U dl \Big| \bar{\mathcal{B}}_r\right] = M_{r,s}^{x,i}(y_j(r) - y_j(s)) - \int_s^r \langle G^*(x)e_i, l_j\rangle_U dl.$$

Thus, the quadratic variation process of $M_{t,s}^{x,i} - \int_s^t \langle e_i, G(x)dy\rangle$ is 0 which implies (M2). The proof is complete. □

Appendix B. Intermittent jets

In this part we recall the construction of intermittent jets from [BV19a, Section 7.4]. We point out that the construction is entirely deterministic, that is, none of the functions below depends on ω. Let us begin with the following geometric lemma which can be found in [BV19a, Lemma 6.6].

Lemma B.1. *Denote by* $\overline{B_{1/2}}(\mathrm{Id})$ *the closed ball of radius* $1/2$ *around the identity matrix* Id*, in the space of* 3×3 *symmetric matrices. There exists* $\Lambda\subset\mathbb{S}^2\cap\mathbb{Q}^3$ *such that for each* $\xi\in\Lambda$ *there exists a* C^∞*-function* $\gamma_\xi : B_{1/2}(\mathrm{Id})\to\mathbb{R}$ *such that*

$$R=\sum_{\xi\in\Lambda}\gamma_\xi^2(R)(\xi\otimes\xi)$$

for every symmetric matrix satisfying $|R-\mathrm{Id}|\leq 1/2$*. For* $C_\Lambda = 8|\Lambda|(1+8\pi^3)^{1/2}$*, where* $|\Lambda|$ *is the cardinality of the set* Λ*, we define the constant*

$$M=C_\Lambda\sup_{\xi\in\Lambda}(\|\gamma_\xi\|_{C^0}+\sum_{|j|\leq N}\|D^j\gamma_\xi\|_{C^0}).$$

For each $\xi\in\Lambda$ *let us define* $A_\xi\in\mathbb{S}^2\cap\mathbb{Q}^3$ *to be an orthogonal vector to* ξ*. Then for each* $\xi\in\Lambda$ *we have that* $\{\xi,A_\xi,\xi\times A_\xi\}\subset\mathbb{S}^2\cap\mathbb{Q}^3$ *form an orthonormal basis for* $\mathbb{R}^3$*. We label by* n_* *the smallest natural such that*

$$\{n_*\xi,n_*A_\xi,n_*\xi\times A_\xi\}\subset\mathbb{Z}^3$$

for every $\xi\in\Lambda$*.*

Let $\Phi:\mathbb{R}^2\to\mathbb{R}^2$ be a smooth function with support in a ball of radius 1. We normalize Φ such that $\phi=-\Delta\Phi$ obeys

$$\frac{1}{4\pi^2}\int_{\mathbb{R}^2}\phi^2(x_1,x_2)dx_1dx_2=1. \tag{B.1}$$

By definition we know $\int_{\mathbb{R}^2}\phi dx=0$. Define $\psi:\mathbb{R}\to\mathbb{R}$ to be a smooth, mean zero function with support in the ball of radius 1 satisfying

$$\frac{1}{2\pi}\int_{\mathbb{R}}\psi^2(x_3)dx_3=1. \tag{B.2}$$

For parameters $r_\perp,r_\|>0$ such that

$$r_\perp\ll r_\|\ll 1,$$

we define the rescaled cut-off functions

$$\phi_{r_\perp}(x_1,x_2)=\frac{1}{r_\perp}\phi\left(\frac{x_1}{r_\perp},\frac{x_2}{r_\perp}\right),\quad \Phi_{r_\perp}(x_1,x_2)=\frac{1}{r_\perp}\Phi\left(\frac{x_1}{r_\perp},\frac{x_2}{r_\perp}\right),\quad \psi_{r_\|}(x_3)=\frac{1}{r_\|^{1/2}}\psi\left(\frac{x_3}{r_\|}\right).$$

We periodize $\phi_{r_\perp},\Phi_{r_\perp}$ and $\psi_{r_\|}$ so that they are viewed as periodic functions on $\mathbb{T}^2,\mathbb{T}^2$ and $\mathbb{T}$ respectively.

Consider a large real number λ such that $\lambda r_\perp\in\mathbb{N}$, and a large time oscillation parameter $\mu>0$. For every $\xi\in\Lambda$ we introduce

$$\begin{aligned}
\psi_{(\xi)}(t,x)&:=\psi_{\xi,r_\perp,r_\|,\lambda,\mu}(t,x):=\psi_{r_\|}(n_*r_\perp\lambda(x\cdot\xi+\mu t))\\
\Phi_{(\xi)}(x)&:=\Phi_{\xi,r_\perp,\lambda}(x):=\Phi_{r_\perp}(n_*r_\perp\lambda(x-\alpha_\xi)\cdot A_\xi,n_*r_\perp\lambda(x-\alpha_\xi)\cdot(\xi\times A_\xi))\\
\phi_{(\xi)}(x)&:=\phi_{\xi,r_\perp,\lambda}(x):=\phi_{r_\perp}(n_*r_\perp\lambda(x-\alpha_\xi)\cdot A_\xi,n_*r_\perp\lambda(x-\alpha_\xi)\cdot(\xi\times A_\xi)),
\end{aligned}$$

where $\alpha_\xi\in\mathbb{R}^3$ are shifts to ensure that $\{\Phi_{(\xi)}\}_{\xi\in\Lambda}$ have mutually disjoint support.

The intermittent jets $W_{(\xi)}:\mathbb{T}^3\times\mathbb{R}\to\mathbb{R}^3$ are defined as in [BV19a, Section 7.4].

$$W_{(\xi)}(t,x):=W_{\xi,r_\perp,r_\|,\lambda,\mu}(t,x):=\xi\psi_{(\xi)}(t,x)\phi_{(\xi)}(x). \tag{B.3}$$

By the choice of α_ξ we have that

$$W_{(\xi)}\otimes W_{(\xi')}\equiv 0,\ \text{for } \xi\neq\xi'\in\Lambda, \tag{B.4}$$

and by the normalizations (B.1) and (B.2) we obtain

$$\fint_{\mathbb{T}^3} W_{(\xi)}(t,x)\otimes W_{(\xi)}(t,x)dx = \xi\otimes\xi.$$

These facts combined with Lemma B.1 imply that

$$\sum_{\xi\in\Lambda}\gamma_\xi^2(R)\fint_{\mathbb{T}^3} W_{(\xi)}(t,x)\otimes W_{(\xi)}(t,x)dx = R, \tag{B.5}$$

for every symmetric matrix R satisfying $|R-\mathrm{Id}|\le 1/2$. Since $W_{(\xi)}$ are not divergence free, we introduce the corrector term

$$W_{(\xi)}^{(c)} := \frac{1}{n_*^2\lambda^2}\nabla\psi_{(\xi)}\times\operatorname{curl}(\Phi_{(\xi)}\xi) = \operatorname{curl}\operatorname{curl}V_{(\xi)} - W_{(\xi)}. \tag{B.6}$$

with

$$V_{(\xi)}(t,x) := \frac{1}{n_*^2\lambda^2}\xi\psi_{(\xi)}(t,x)\Phi_{(\xi)}(x).$$

Thus we have

$$\operatorname{div}\left(W_{(\xi)} + W_{(\xi)}^{(c)}\right)\equiv 0.$$

Finally, we recall the key bounds from [BV19a, Section 7.4]. For $N,M\ge 0$ and $p\in[1,\infty]$ the following holds

$$\begin{aligned}
\|\nabla^N\partial_t^M\psi_{(\xi)}\|_{C_tL^p} &\lesssim r_\|^{1/p-1/2}\left(\frac{r_\perp\lambda}{r_\|}\right)^N\left(\frac{r_\perp\lambda\mu}{r_\|}\right)^M,\\
\|\nabla^N\phi_{(\xi)}\|_{L^p} + \|\nabla^N\Phi_{(\xi)}\|_{L^p} &\lesssim r_\perp^{2/p-1}\lambda^N,\\
\|\nabla^N\partial_t^M W_{(\xi)}\|_{C_tL^p} + \frac{r_\|}{r_\perp}\|\nabla^N\partial_t^M W_{(\xi)}^{(c)}\|_{C_tL^p} + \lambda^2\|\nabla^N\partial_t^M V_{(\xi)}\|_{C_tL^p} &\lesssim r_\perp^{2/p-1}r_\|^{1/p-1/2}\lambda^N\left(\frac{r_\perp\lambda\mu}{r_\|}\right)^M,
\end{aligned} \tag{B.7}$$

where the implicit constants may depend on p,N and M, but are independent of $\lambda, r_\perp, r_\|, \mu$.

Appendix C. Uniqueness in law implies joint uniqueness in law

In this part we will extend the result of Cherny [C03] to a general infinite dimensional setting. A generalization to a semigroup framework in Banach spaces was proved by Ondreját in [On04]. Let U, U_1, H and H_1 be separable Hilbert spaces and suppose that the embedding $U\subset U_1$ is Hilbert–Schmidt and the embedding $H\subset H_1$ is continuous. Consider the SPDE of the form

$$dX = F(X)dt + G(X)dB, \qquad X(0)=x\in H, \tag{C.1}$$

where $F: H\to H_1$ is $\mathcal{B}(H)/\mathcal{B}(H_1)$ measurable and B is a cylindrical Wiener process on the Hilbert space U which is defined on a stochastic basis $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge0},P)$. In other words, B can be viewed as a continuous process taking values in U_1 and we assume that for $x\in H$, $G(x)$ is an Hilbert–Schmidt operator from U to H. Solutions to (C.1) are then understood in the following sense.

Definition C.1. A pair (X,B) is a solution to (C.1) provided there exists a stochastic basis $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge0},P)$ such that
(H1) B is a cylindrical $(\mathcal{F}_t)_{t\ge0}$-Wiener process on U;
(H2) X is an $(\mathcal{F}_t)_{t\ge0}$-adapted process in $C([0,\infty);H_1)$ P-a.s.;
(H3) $F(X)\in L^1_{\mathrm{loc}}([0,\infty);H_1)$ and $G(X)\in L^2_{\mathrm{loc}}([0,\infty);L_2(U,H))$ P-a.s.;
(H4) P-a.s. it holds for all $t\in[0,\infty)$

$$X_t = x + \int_0^t F(X_s)ds + \int_0^t G(X_s)dB_s.$$

Let us now recall the definition of uniqueness in law as well as joint uniqueness in law.

Definition C.2. We say that uniqueness in law holds for (C.1) if for any two solutions (X, B) and $(\tilde{X}, \tilde{B})$ starting from the same initial distribution, one has $\mathrm{Law}(X) = \mathrm{Law}(\tilde{X})$. We say that joint uniqueness in law holds for (C.1) if for any two solutions (X, B) and $(\tilde{X}, \tilde{B})$ starting from the same initial distribution, one has $\mathrm{Law}(X, B) = \mathrm{Law}(\tilde{X}, \tilde{B})$.

Clearly, joint uniqueness in law implies uniqueness in law. The following result shows that the two notions are in fact equivalent for SPDEs of the form (C.1).

Theorem C.1. *Suppose that uniqueness in law holds for (C.1). Then joint uniqueness in law holds for (C.1).*

Set $E = L_2(U, H)$. Since E is separable, it follows that $C([0,t], E)$ is dense in $L^2([0,t], E)$. By the same argument as Lemma 3.2 in [C03], we can prove the following result.

Lemma C.2. *Let $t > 0$ and $f \in L^2([0,t], E)$. For $k \in \mathbb{N}$, set*

$$f^{(k)}(s) = \begin{cases} 0, & \text{if } s \in [0, \frac{t}{k}], \\ \frac{k}{t}\int_{(i-1)t/k}^{it/k} f(r)dr, & \text{if } s \in (\frac{it}{k}, \frac{(i+1)t}{k}], \end{cases} \qquad (i = 1, \dots, k-1).$$

Then $f^{(k)} \to f$ in $L^2([0,t], E)$.

By Lemma C.2 and the same argument as in Lemma 3.3 in [C03], we obtain the following.

Lemma C.3. *Let (X, B) be a solution to (C.1) defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. Let $(Q_\omega)_{\omega\in\Omega}$ be a conditional probability distribution of (X, B) given $\mathcal{F}_0$[2]. Let Y be the coordinate process with values in H_1 and let Z be the coordinate process with values in U_1. Let $(\mathcal{H}_t)_{t\geq 0}$ be the canonical filtration on $C([0,\infty), H_1 \times U_1)$ and denote $\mathcal{H} = \bigvee_{t\geq 0} \mathcal{H}_t$. Then for P-a.e. $\omega \in \Omega$ the pair (Y, Z) is a solution to (C.1) on the stochastic basis $(C([0,\infty); H_1 \times U_1), \mathcal{H}, (\mathcal{H}_t)_{t\geq 0}, Q_\omega)$.*

Proof of Theorem C.1. Let (X, B) be a solution to (C.1) on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. Let $\{\beta^k\}_{k\in\mathbb{N}}$ and $\{\bar{\beta}^k\}_{k\in\mathbb{N}}$ be two families of independent real-valued Wiener processes defined on another stochastic basis $(\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\geq 0}, P')$ and set

$$(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{P}) = \big(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', (\mathcal{F}_t \otimes \mathcal{F}'_t)_{t\geq 0}, P \otimes P'\big).$$

All the processes $X, B, \beta^k, \bar{\beta}^k$, $k \in \mathbb{N}$, can be defined on $\tilde{\Omega}$ in an obvious way. Assume that the cylindrical Wiener process B admits the decomposition $B = \sum_{k=1}^\infty \alpha^k l_k$, where $\{\alpha^k\}_{k\in\mathbb{N}}$ is a family of independent real-valued Wiener processes and $\{l_k\}_{k\in\mathbb{N}}$ is an orthonormal basis in U. Let $\varphi(x)$ be the orthogonal projection from U to $(\ker G(x))^\perp$ and $\psi(x)$ be the orthogonal projection from U to $\ker G(x)$. Then set

$$\varphi_s := \varphi(X_s), \quad \psi_s := \psi(X_s),$$
$$V_t = \sum_{k=1}^\infty \Big[\int_0^t \varphi_s d\alpha_s^k\, l_k + \int_0^t \psi_s d\beta_s^k\, l_k\Big], \qquad \bar{V}_t = \sum_{k=1}^\infty \Big[\int_0^t \varphi_s d\bar{\beta}_s^k\, l_k + \int_0^t \psi_s d\alpha_s^k\, l_k\Big].$$

In the following, $\langle\!\langle \cdot, \cdot \rangle\!\rangle_t$ denotes the cross-variation process at time t. We obtain

$$\begin{aligned}
\langle\!\langle \langle V, l_i\rangle_U, \langle V, l_j\rangle_U \rangle\!\rangle_t &= \sum_{k=1}^\infty \Big[\int_0^t \langle \varphi_s l_k, l_i\rangle_U \langle \varphi_s l_k, l_j\rangle_U ds + \int_0^t \langle \psi_s l_k, l_i\rangle_U \langle \psi_s l_k, l_j\rangle_U ds\Big] \\
&= \int_0^t [\langle \varphi_s l_i, \varphi_s l_j\rangle_U + \langle \psi_s l_i, \psi_s l_j\rangle_U]\, ds = \int_0^t \langle (\varphi_s + \psi_s) l_i, (\varphi_s + \psi_s) l_j\rangle_U ds \\
&= \int_0^t \langle l_i, l_j\rangle_U ds = \delta_{ij} t.
\end{aligned}$$

[2]Here, we consider (X, B) as a $C([0,\infty); H_1 \times U_1)$-valued process.

Similarly, we obtain

$$\langle\!\langle \langle V, l_i\rangle_U, \langle \bar V, l_j\rangle_U \rangle\!\rangle_t = 0, \qquad \langle\!\langle \langle \bar V, l_i\rangle_U, \langle \bar V, l_j\rangle_U \rangle\!\rangle_t = \delta_{ij} t.$$

As a consequence, under $\tilde P$ the process $(V, \bar V)$ is an $(\tilde{\mathcal{F}}_t)_{t\geq 0}$-cylindrical Wiener process on $U\times U$. Moreover, for any $t\geq 0$, we have

$$\int_0^t G(X_s)dB_s = \int_0^t G(X_s)\varphi_s dB_s = \int_0^t G(X_s)dV_s.$$

Hence (X,V) is a solution to (C.1) on $(\tilde\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde P)$.

Consider now the filtration

$$\mathcal{G}_s = \tilde{\mathcal{F}}_s \vee \sigma\big(\bar V_t; t\geq 0\big) = \tilde{\mathcal{F}}_s \vee \sigma\big(\bar V_t - \bar V_s; t\geq s\big), \quad s\geq 0.$$

Since $\tilde{\mathcal{F}}_s$ and $\sigma(V_t - V_s; t\geq s)\vee\sigma(\bar V_t - \bar V_s; t\geq s)$ are independent, the process V is a cylindrical $(\mathcal{G}_t)_{t\geq 0}$-Wiener process on U under $\tilde P$. Thus (X,V) is a solution to (C.1) on $(\tilde\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{G}}_t)_{t\geq 0}, \tilde P)$.

Let $(Q_{\tilde\omega})_{\tilde\omega\in\tilde\Omega}$ be a conditional probability distribution of (X,V) given $\mathcal{G}_0$. By Lemma C.3, for $\tilde P$-a.e. $\tilde\omega\in\tilde\Omega$, the pair (Y,Z) is a solution to (C.1) on $(C([0,\infty); H\times U), \mathcal{H}, (\mathcal{H}_t)_{t\geq 0}, Q_{\tilde\omega})$. As the uniqueness in law holds for (C.1), the probability law induced by Y on each of these stochastic bases, i.e. $Q_{\tilde\omega}\circ Y^{-1}$, is the same for $\tilde P$-a.e. $\tilde\omega\in\tilde\Omega$. Since this is the conditional probability distribution of X given $\mathcal{G}_0$, it follows that the process X is independent of $\mathcal{G}_0$. In particular, we deduce that X and $\bar V$ are independent. Let $\chi(x)$ be the pseudo-inverse of $G(x)$ (see e.g. [LR15, Appendix C] for more details), then $\chi(x)G(x) = \varphi(x)$. Set $\chi_s := \chi(X_s)$. Thus,

$$\int_0^t \varphi_s dB_s = \int_0^t \chi_s G(X_s)dB_s = \int_0^t \chi_s dM_s,$$

where

$$M_t = \int_0^t G(X_s)dB_s = X_t - x - \int_0^t F(X_s)ds.$$

Accordingly, we obtain

$$B_t = \int_0^t \varphi_s dB_s + \int_0^t \psi_s dB_s = \int_0^t \chi_s dM_s + \int_0^t \psi_s d\bar V_s.$$

The process M is a measurable functional of X while $\bar V$ is independent of X. Thus the distribution $\mathrm{Law}(X,B)$ is unique. □

References

[BFH17] D. Breit, E. Feireisl, and M. Hofmanová. On solvability and ill-posedness of the compressible Euler system subject to stochastic forces. *Analysis and PDE, to appear*, arxiv:1705.08097, 2017.

[BFH18] D. Breit, E. Feireisl, M. Hofmanová, Stochastically forced compressible fluid flows, De Gruyter Series in Applied and Numerical Mathematics. De Gruyter, Berlin/Munich/Boston, 2018.

[BCV18] T. Buckmaster, M. Colombo and V. Vicol, Wild solutions of the Navier–Stokes equations whose singular sets in time have hausdorff dimension strictly less than 1, arXiv:1809.00600, 2018.

[BDLIS16] T. Buckmaster, C. De Lellis, P. Isett, and L. Székelyhidi, Jr., Anomalous dissipation for 1/5-Hölder Euler flows, *Annals of Mathematics* **182** (1), 127-172, 2015.

[BR17] V. Barbu and M. Röckner, Global solutions to random 3D vorticity equations for small initial data, *Journal of Differential Equations*, **263**, 5395?5411, 2017.

[BV19a] T. Buckmaster, V. Vicol, Convex integration and phenomenologies in turbulence, arXiv:1901.09023, 2019.

[BV19b] T. Buckmaster, V. Vicol, Nonuniqueness of weak solutions to the Navier–Stokes equation. *Ann. of Math.* **189** (1), 101-144, 2019.

[C03] A. S. Cherny. On the uniqueness in law and the pathwise uniqueness for stochastic differential equations. *Theory Probab. Appl.* **46** (3), 406-419, 2003.

[CFF19] E. Chiodaroli, E. Feireisl, F. Flandoli, Ill-posedness for the full Euler system driven by multiplicative white noise, arXiv:1904.07977, 2019.

[CKMS19] E. Chiodaroli, O. Kreml, V. Mácha, and S. Schwarzacher. Non–uniqueness of admissible weak solutions to the compressible Euler equations with smooth initial data. arXiv:1812.09917, 2019.

[De13] A. Debussche, Ergodicity Results for the Stochastic Navier–Stokes Equations: An Introduction, Topics in Mathematical Fluid Mechanics Lecture Notes in Mathematics, 23-108, 2013.

[DLS09] C. De Lellis and L. Székelyhidi, Jr. The Euler equations as a differential inclusion. *Ann. of Math. (2)* **170** (3), 1417-1436, 2009.

[DLS10] C. De Lellis and L. Székelyhidi, Jr. On admissibility criteria for weak solutions of the Euler equations. *Arch. Ration. Mech. Anal.* **195** (1), 225-260, 2010.

[DLS13] C. De Lellis and L. Székelyhidi, Jr. Dissipative continuous Euler flows. *Invent. Math.* **193** (2), 377-407, 2013.

[DPD02] G. Da Prato, A. Debussche, 2D Navier–Stokes equations driven by a space-time white noise. *J. Funct. Anal.* **196** (1), 180-210, 2002.

[DPD03] G. Da Prato, A. Debussche, Ergodicity for the 3D stochastic Navier–Stokes equations, *J. Math. Pures Appl.* (9) **82** (8), 877-947, 2003.

[DPZ92] G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions, Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge University Press, Cambridge, 1992.

[FG95] F. Flandoli, D. Gatarek, Martingale and stationary solutions for stochastic Navier–Stokes equations, *Probab. Theory Relat. Fields* **102**, 367-391, 1995.

[FGP10] F. Flandoli, M. Gubinelli, and E. Priola. Well-posedness of the transport equation by stochastic perturbation. *Invent. Math.* **180** (1), 1-53, 2010.

[FL19] F. Flandoli, D. Luo. High mode transport noise improves vorticity blow-up control in 3D Navier-Stokes equations , arXiv:1910.05742, 2019.

[FR08] F. Flandoli, M. Romito, Markov selections for the 3D stochastic Navier–Stokes equations, *Probab. Theory Relat. Fields* **140**, 407-458, 2008.

[GRZ09] B. Goldys, M. Röckner and X. C. Zhang, Martingale solutions and Markov selections for stochastic partial differential equations, *Stochastic Processes Appl.* **119**, 1725-1764, 2009.

[HM06] M. Hairer, J. C. Mattingly, Ergodicity of the 2D Navier–Stokes equations with degenerate stochastic forcing *Annals of Math.* **164**, 993-1032, 2006.

[HLN19a] M. Hofmanová, J.-M. Leahy, and T. Nilssen. On a rough perturbation of the Navier-Stokes system and its vorticity formulation. arXiv:1902.09348, 2019.

[HLN19b] M. Hofmanová, J.-M. Leahy, and T. Nilssen. On the Navier-Stokes equation perturbed by rough transport noise. *J. Evol. Equ.* **19** (1), 203-247, 2019.

[K07] T. G. Kurtz, The Yamada–Watanabe–Engelbert theorem for general stochastic equations and inequalities, *Electron. J. Probab.* **12**, 951-965, 2007.

[LR15] W. Liu, M. Röckner, Stochastic Partial Differential Equations: An Introduction, Universitext, Springer, 2015.

[MR19] I. Munteanu, M. Röckner, Global solutions for random vorticity equations perturbed by gradient dependent noise, in two and three dimensions, arXiv:1905.02437, 2019.

[On04] M. Ondreját, Uniqueness for stochastic evolution equations in Banach spaces, *Dissertationes Mathematicae* **426**, 2004.

[RZZ14] M. Röckner, R. Zhu, X. Zhu, Local existence and non-explosion of solutions for stochastic fractional partial differential equations driven by multiplicative noise, *Stochastic Processes and their Applications* **124**, 1974-2002, 2014.

[RZZ19] M. Röckner, R. Zhu, X. Zhu, A remark on global solutions to random 3D vorticity equations for small initial data, *Discrete Continuous Dynamical Systems - B* 2019 24(8): 4021-4030 , 2019

[SV79] D. W. Stroock, S. R. S. Varadhan, Multidimensional Diffusion Processes, Springer-Verlag, Berlin, 1979.

[ZZ15] R. Zhu, X. Zhu, Three-dimensional Navier–Stokes equations driven by space-time white noise, *Journal of Differential Equations* **259** (9), 4443-4508, 2015.

(M. Hofmanová) Fakultät für Mathematik, Universität Bielefeld, D-33501 Bielefeld, Germany
E-mail address: hofmanova@math.uni-bielefeld.de

(R. Zhu) Department of Mathematics, Beijing Institute of Technology, Beijing 100081, China; Fakultät für Mathematik, Universität Bielefeld, D-33501 Bielefeld, Germany
E-mail address: zhurongchan@126.com

(X. Zhu) Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; Fakultät für Mathematik, Universität Bielefeld, D-33501 Bielefeld, Germany
E-mail address: zhuxiangchan@126.com